INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313-761-4700 800-521-0600
Linkage by generically Gorenstein Cohen-Macaulay ideals

Martin, Heath Mayall, Ph.D.

The Louisiana State University and Agricultural and Mechanical Col., 1993
LINKAGE BY GENERICALLY GORENSTEIN COHEN–MACAULAY IDEALS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Heath M. Martin
B.S., University of Texas at Austin, 1988
August, 1993
ACKNOWLEDGEMENTS

It is with great pleasure that I thank my advisor, Bernard Johnston. His constant encouragement and enthusiasm for mathematics are unfailingly communicated to those around him. The benefits I have gained from our association are almost without measure. I also owe a debt of thanks to Dan Katz, who is always free with his ideas and suggestions.

My friends and colleagues at LSU deserve recognition: my teachers for their time and energy spent in instructing and guiding me, and my friends, for many happy times. I also extend special thanks to the Department of Mathematics at Florida Atlantic University, and especially its chairman, Dr. James Brewer, for its generous hospitality and warmth.

Finally, my parents and family, to whom I affectionately dedicate this work, always give me their unconditional love and support. Their quiet and gentle ways have instilled in me a set of values I shall always treasure.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>I  INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II DEFINITIONS AND BASIC RESULTS</td>
<td>8</td>
</tr>
<tr>
<td>III ON A GENERAL NOTION OF LINKAGE</td>
<td>17</td>
</tr>
<tr>
<td>IV ON THE CANONICAL MODULE OF A DETERMINANTAL RING</td>
<td>50</td>
</tr>
<tr>
<td>V  NOTES ON QUADRATIC SEQUENCES</td>
<td>62</td>
</tr>
<tr>
<td>VI SUMMARY AND OPEN QUESTIONS</td>
<td>80</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>84</td>
</tr>
<tr>
<td>VITA</td>
<td>89</td>
</tr>
</tbody>
</table>
ABSTRACT

In a Gorenstein local ring $R$, two ideals $A$ and $B$ are said to be linked by an ideal $I$ if the two relations $A = (I : B)$ and $B = (I : A)$ hold. In the case that $I$ is a complete intersection, or a Gorenstein ideal, it is known that linkage preserves the Cohen–Macaulay property. That is, if $A$ is a Cohen–Macaulay ideal, then so is $B$. However, if $I$ is allowed to be a generically Gorenstein, Cohen–Macaulay ideal, easy examples show that this type of linkage does not preserve the Cohen–Macaulay property. The primary purpose of this work is to investigate how much of the Cohen–Macaulay property this more general kind of linkage does preserve.

By associating to $I$ an auxiliary ideal $J$, for which $J/I$ is isomorphic to the canonical module $K_{R/I}$ of $R/I$, we are able to give complete conditions for various types of Cohen–Macaulay conditions that $B$ possesses, when $B$ is linked by $I$ to a Cohen–Macaulay ideal $A$. In particular, we give a criterion for $B$ to be a Cohen–Macaulay ideal, and when it is not, for $R/B$ to have high depth. We also give a description in some cases of the non-Cohen–Macaulay locus of $R/B$, including a calculation of its dimension. In these cases, there is an interesting relationship between the depth of $R/B$ and the dimension of the non-Cohen–Macaulay locus. Finally, we give some remarks on a construction of a free resolution of $R/B$ from given resolutions.
CHAPTER I

INTRODUCTION

The notion of linkage, at least in a geometric context, goes back at least to M. Noether, where he used it to classify space curves. Since then, it has proven to be a powerful tool both in geometry and in commutative algebra. The classical notion of linkage allowed only the class of complete intersection ideals, or varieties, as possible linking ideals. Until quite recently, though this classical linkage has been very well researched, there has not been a concerted attempt to widen the class of linking ideals. In this thesis, our primary efforts are devoted to the study of a more general notion of linkage, by allowing as linking ideals the class of generically Gorenstein, Cohen–Macaulay ideals. See Chapter 2 for precise definitions.

A general problem in commutative algebra is to identify the Cohen–Macaulay rings. In the context of classical linkage, that is, linkage by a complete intersection, if an ideal \( A \) is Cohen–Macaulay and linked to an ideal \( B \), then \( B \) is also Cohen–Macaulay, at least when the base ring is regular local, or even Gorenstein. This was first shown by C. Peskine and L. Szpiro in their fundamental paper on linkage, [PS]. On the other hand, if the hypotheses are weakened, this property may fail. For instance, Peskine and Szpiro give an example to show that if the base ring is only Cohen–Macaulay, then linkage by a complete intersection may not preserve the Cohen–Macaulay property. In response to this, C. Huneke developed the notion of strongly Cohen–Macaulay, and showed in [Hun1] that, in a Cohen–Macaulay ring
\( R \), if \( A \) is strongly Cohen–Macaulay and linked to \( B \) by a complete intersection, then \( B \) is Cohen–Macaulay.

In a slightly different vein, if we require our base ring to be Gorenstein, but allow a wider class of possible linking ideals, easy examples show that this, too, need not preserve the Cohen–Macaulay property. In particular, the subject of this paper is to examine this question for linkage by the class of generically Gorenstein, Cohen–Macaulay ideals. We attack the question from two overlapping viewpoints: first, when does such linkage preserve the Cohen–Macaulay property, and second, when it does not, how much of the Cohen–Macaulay property does pass along the linkage? Both of these viewpoints are answered in terms of the linking ideal, or more precisely, in terms of an ideal closely associated to the linking ideal. For instance, we are able to prove the following result:

**Theorem.** Suppose \( R \) is a local Gorenstein ring; let \( A \) and \( B \) be ideals of \( R \) linked by the generically Gorenstein, Cohen–Macaulay ideal \( I \). Write \( K_{R/I} \cong J/I \), the canonical module \( R/I \). If \( A \) is Cohen–Macaulay, then \( B \) is Cohen–Macaulay if and only if the ideal \( A + J \) is Cohen–Macaulay with \( \dim R/(A + J) = \dim R/A - 1 \).

By localizing, we are able to obtain precise statements about the non-Cohen-Macaulay locus of \( B \), when \( B \) is not Cohen–Macaulay. It is always easily described in terms of \( A \) and \( J \), and in some cases we can get a lower bound in terms of the depth of \( R/B \) (Theorem 3.1.15). This lower bound is quite interesting; it shows that even if the linkage has “good” Cohen-Macaulay properties, in the sense that the non-Cohen-Macaulay locus of \( R/B \) is small, then also the linkage has “bad” Cohen-Macaulay properties, in the sense that depth \( R/B \) is also small.
Our other primary thrust in this research was to discover when the linkage preserves high depth. That is, if \( B \) is linked in our general sense to a Cohen-Macaulay ideal \( A \), when does \( R/B \) have high depth, and in particular, when is there an inequality \( \text{depth} \, R/B \geq \dim R/B - 1 \)? Whereas our characterization of linkage preserving the entire Cohen-Macaulay property involved the ideal \( A + J \), with the notation as in the Theorem above, the characterization of when this “depth inequality” is satisfied involves the unmixed part of \( A + J \). Namely, we have the following result:

**Theorem.** Suppose the Cohen-Macaulay ideal \( A \) is linked to \( B \) by the generically Gorenstein, Cohen-Macaulay ideal \( I \), and write \( K_{R/I} \cong J/I, \) where neither \( A \) nor \( B \) contains a non-zero-divisor for \( R/J \), and where \( \text{depth} \, R/(A+J) \geq \dim R/(A+J) - 1 \). Then \( (A + J)' \) is a Cohen-Macaulay ideal if and only if \( R/B \) satisfies the depth inequality.

The assumptions on non-zero-divisors are technical, which can always be satisfied. Unfortunately, the assumption on depth \( R/(A + J) \) seems to be essential, and we have been unable to remove it, even though it seems likely to always hold. In any case, we have no example where it does not. It is a condition which we have had to assume in several of our results.

We note that the proof of the above result is accomplished by “lifting” the linkage of \( A \) and \( B \) to a linkage by a Gorenstein ideal, a situation which is much better understood. Hopefully, this technique should prove useful in other situations as well.
Our original intent when we began this research was to generalize the classical notion of linkage to linkage by determinantal ideals. Though this did not materialize, we note that the class of determinantal ideals and the class of generically Gorenstein, Cohen-Macaulay ideals has a large intersection. Hence, it is natural to look in this intersection for easily computed examples. This is essentially the purpose of Chapter 4. Though the results contained there are not, strictly speaking, new, our proofs are very concrete in nature, and lend themselves well to computation, especially with the computer algebra program MACAULAY.

The main result of Chapter 4 extends to the non-generic case a well-known result about the canonical modules of certain kinds of determinantal ideals. Precisely, for a generically Gorenstein ideal $I$, not necessarily determinantal, the canonical module $K_{R/I}$ is isomorphic to an ideal $J/I$ of $R/I$. When $I$ is also determinantal, our Theorem 4.9 shows how to obtain an explicit set of generators for $J$ in terms of the matrix whose maximal minors generate $I$. We do this by adapting a proof of Y. Yoshino, and including a general position argument. This general position argument is interesting in itself. It is essentially a prime avoidance lemma for minors of matrices, the usual prime avoidance lemma being the one-rowed case. Again, our proof is concrete, though a more general result follows from the theory of basic elements.

The present work seems to be among the few which have explicitly allowed a general class of linking ideals besides the complete intersections. As such, it represents only a beginning of many possible research directions. It is hoped that a more general notion of linkage, such as the one discussed here, or perhaps even more general,
might prove useful in attacking some other questions from commutative algebra. It would perhaps be appropriate to review briefly, for the reader not well-acquainted with the theory of linkage, some of the extant literature. However, we do not claim that this is a complete list. All of the cited papers contain their own bibliographies, which the interested reader may consult.

The fundamental paper of Peskine and Szpiro [PS] is widely regarded as the modern beginning of the study of linkage. It contains many important algebraic results, which are then applied to geometric contexts. For instance, in codimension 2 in a regular local ring, every Cohen-Macaulay ideal is in the linkage class of a complete intersection ideal. This kind of result fails in higher codimension, but this type of condition is important enough to have deserved a name, licci, for Linkage Class of a Complete Intersection, and it has been highly researched. If anything, it is one of the most well-researched subjects in the algebraic theory of linkage. See for instance [HU1, 2, 5, 7], [KM5] and [U2,3].

Still dealing with the classical theory of linkage by a complete intersection, C. Huneke and B. Ulrich have developed a very sophisticated and powerful notion of “generic linkage” in [HU1, 4, 7], and of a closely related idea of “generic residual intersection,” [HU3, 6]. This is perhaps the most general study of the algebraic theory of linkage.

On the geometric side, the structure of linkage, or liaison as it is commonly called in this context, is best known only in codimension 2, primarily because the codimension 2 locally Cohen-Macaulay varieties are well-known. The papers of A. Rao [R1, 2] established important invariance properties of liaison in codimension 2. Recent
work by G. Bolondi, A. Geramita, J. Migliore, and others has also further clarified the structure of liaison in codimension 2. See for instance [BBM], [BM1, 2], [GM], and [M1, 2].

There are many papers where the notion of linkage has been used to prove theorems about other subjects. Most notable of these is the concerted effort of A. Kustin and M. Miller to describe the Gorenstein ideals of codimension 4, [KM1–5]. In particular, in [KM2, 3], they explicitly defined and used a notion of Gorenstein linkage. Linkage apparently also has some usefulness in work on finding lower bounds for Betti numbers. See [CEM] and [EG1].

We would like to mention the papers which we know of that explicitly allow a class of linking ideals more general than the complete intersections. In [G], E. Golod extended Peskine and Szpiro's result on preservation of the Cohen–Macaulay property to linkage by Gorenstein ideals, when the base ring is regular local. P. Schenzel in [Sc2] also considered Gorenstein linkage, giving a new proof of the Peskine–Szpiro result, and extending the invariance properties of Rao mentioned above. We have benefited greatly from this paper. R. Sjögren used Gorenstein linkage in [Sj] to prove the Cayley–Bacharach Theorem. As mentioned above, Kustin and Miller used Gorenstein linkage in their work on the Gorenstein ideals of codimension 4. A recent paper of C. Walter [W] defined Cohen–Macaulay linkage, and showed that every ideal is in the Cohen–Macaulay linkage class of a Cohen–Macaulay ideal, in a quite general base ring, including the Gorenstein rings. Finally, the papers [Hun1, 2, 5] of C. Huneke considered linkage by a complete intersection in a Cohen–Macaulay ring; naturally, this is quite closely connected with Cohen–Macaulay linkage. These
papers arose, in part, to correct a mistake in M. Artin and M. Nagata's paper [AN], which looked at residual intersections in Cohen–Macaulay rings.

Chapter 5 is independent of the others, and can be read separately. It deals with the concept of quadratic sequences, a generalization by K. Raghavan of Huneke's notion of weak d-sequences, which are themselves generalizations of the well-known regular sequences. Our main result is the following theorem:

**Theorem.** Let \( x_1, \ldots, x_s \) be a d-sequence. Then the set of monomials \( \{ x_{i_1} \cdots x_{i_n} : (i_1, \ldots, i_n) \in S(n) \} \) is a quadratic sequence, for each \( n \geq 1 \).

Thus, we have new natural examples of quadratic sequences. Moreover, it follows immediately that every power of an ideal generated by a d-sequence has relation type at most 2. We include in this chapter some remarks about the connection between quadratic sequences and d-sequences, which can be considered as the linearly ordered quadratic sequences.

Finally, in Chapter 6, we summarize the main results of this work, and indicate some open questions. In contrast to questions we ask in the main body of this paper, the remarks in Chapter 6 are intended to lay out directions for further research.
Chapter II
Definitions and Basic Results

In this chapter, we collect together the definitions and notation to be used in the following two chapters, and state some of the basic results of linkage theory. Most, or all, of this material is well-known; sometimes, however, due to lack of a good reference, we will indicate a proof. We will include in this chapter a short discussion of canonical modules, which are our primary tool. For the reader who is unfamiliar with canonical modules, we provide a motivation for their study by indicating the connection with local cohomology via the Local Duality Theorem. This will all be presented without proofs; we do, however, give references to the literature.

Our setting will always be in a Gorenstein local ring \( R \). At times, we may also require \( R \) to be regular local. An ideal \( I \) of \( R \) is said to be a Cohen–Macaulay ideal, or a Gorenstein ideal, if the residue ring \( R/I \) has the corresponding property. In particular, an ideal \( I \) is generically Gorenstein provided \( I_p \) is a Gorenstein ideal of \( R_p \), for each minimal prime \( p \) of \( I \).

Our basic definition is:

**Definition 2.1.** Let \( R \) be a local Gorenstein ring, and let \( A \) and \( B \) be ideals of \( R \). Then \( A \) and \( B \) are said to be linked by an ideal \( I \) if \( I \subseteq A \cap B \), and if \( A = (I : B) \) and \( B = (I : A) \). Equivalently, \( A \) and \( B \) are linked by \( I \) if

\[
A/I \cong \text{Hom}_R(R/B, R/I) \quad \text{and} \quad B/I \cong \text{Hom}_R(R/A, R/I).
\]
We note that this is a very general definition. When $I$ is a complete intersection, this definition coincides with the classical notion of linkage, e.g., in [PS, Sect. 2]. When $I$ is a Gorenstein ideal, the definition corresponds to Gorenstein linkage defined in [Sc2, Def. 2.1] and [KM2, Def. 1.3(2)]. In this paper, we will restrict the class of linking ideals to the generically Gorenstein, Cohen–Macaulay ideals, and, unless otherwise stated, by \textit{linkage}, we will always mean linkage by such an ideal.

The next Proposition shows how the colon ideals behave with respect to primary decompositions. It is well-known.

\textbf{Proposition 2.2.} Let $I = q_1 \cap \ldots \cap q_n$ be an irredundant primary decomposition of $I$, where $q_i$ is associated to $p_i$. Let $A$ be another ideal. By renumbering the $q_i$, suppose that $A \subseteq p_i$ for $i = 1, \ldots, k$, that $q_i \subseteq A \subseteq p_i$, for $i = k + 1, \ldots, l$, and that $A \subseteq q_i$, for $i = l + 1, \ldots, n$. Then a primary decomposition (not necessarily irredundant) for $(I : A)$ is given by

\[(I : A) = \bigcap_{i=1}^{k} q_i \cap \bigcap_{i=k+1}^{l} (q_i : A).\]

\textit{Proof.} The proof of this is easy. First note that

\[(I : A) = (\bigcap_{i} q_i : A) = \bigcap_{i} (q_i : A).\]

For those $q_i$ which contain $A$, $(q_i : A) = R$; if $A \subseteq p_i$, then $(q_i : A) = p_i$; and if $q_i \subseteq A \subseteq p_i$, then $(q_i : A)$ is primary to $p_i$. □

In the context of linkage, this has the following consequence:

\textbf{Corollary 2.3.} If $A$ and $B$ are ideals which are linked by $I$, then

\[\text{Ass}(R/A) \cup \text{Ass}(R/B) = \text{Ass}(R/I).\] □
In particular, since in this paper, the linking ideal $I$ is always assumed to be Cohen–Macaulay, it is unmixed of pure height $g$, say. Hence also $A$ and $B$ are unmixed of pure height $g$.

One of the primary reasons we are restricting this study to linkage by a generically Gorenstein, Cohen–Macaulay ideal is that this kind of linkage has a symmetry property. In other words, one of the relations $A = (I : B)$, $B = (I : A)$ implies the other. Although different versions of this appear in [Hun2, Remark 0.2] and [Sc2, Prop. 2.2], neither statement is exactly what we need. So we state it formally here, and combine the proofs from those two references. See also [PS, Prop. 2.1].

**Lemma 2.4.** Let $R$ be a Cohen-Macaulay local ring, and $I$ a Cohen-Macaulay ideal. Suppose $A$ is an unmixed ideal containing $I$, with height $A = \text{height } I$. Put $B = (I : A)$. If $R_p$ is Gorenstein for each minimal prime $p$ of $A$ and $I_p$ is a Gorenstein ideal, then $A = (I : B)$.

**Proof.** By considering the Cohen-Macaulay ring $R/I$ in place of $R$, we may assume that height $A = 0$ and $B = (0 : A)$. We need to show that $A = (0 : B) = (0 : (0 : A))$, and since the forward containment always holds, we need only see that $(0 : (0 : A)) \subseteq A$. For this, it suffices to show $(0 : (0 : A_p)) \subseteq A_p$ for each minimal prime $p$ over $A$. But $R_p$ is zero-dimensional Gorenstein for each such prime by assumption, and in these rings, every ideal $J$ satisfies $J = (0 : (0 : J))$, [HK, Satz 1.44]. In particular, this is true when $J = A_p$. □

It is in general an interesting question when $A = (I : B)$ implies $B = (I : A)$. As above, if $I$ is generically Gorenstein, this statement does hold. More generally,
if $A$ is an unmixed ideal containing a Cohen–Macaulay ideal $I$, in some sense the number of possible $B$ so that $A = (I : B)$ depends on the Cohen–Macaulay type of $I_p$, for the minimal primes $p$ not associated to $A$. More precisely, we have the following result.

**Proposition 2.5.** Let $R$ be a Gorenstein local ring, and let $I$ be a Cohen–Macaulay ideal of height $g$. Suppose $A$ is an ideal of height $g$ containing $I$, and let $B_i$, for $i = 1, ..., s$ be unmixed ideals of height $g$ containing $I$, so that $A = (I : B_i)$ for each $i$. If $p \in \text{Ass}(R/I) \setminus \text{Ass}(R/A)$, then $I_p = \bigcap_i B_{ip}$.

**Proof.** Let $I = q_1 \cap \cdots \cap q_k$ be an irredundant primary decomposition of $I$. Suppose $q_1$ is primary to $p$, where $p$ is not minimal over $A$. Thus, we have

$$I \subseteq \bigcap_i B_i \subseteq (I : A) = (q_1 : A) \cap \cdots \cap (q_k : A).$$

On localizing at $p$, and using that $q_1 \subseteq A$ but $A \not\subseteq p$, we obtain

$$I_p \subseteq \bigcap_i B_{ip} \subseteq (q_1 : A)_p = q_1p = I_p.$$

Hence equality holds throughout, and we obtain the result. □

**Remark.** Thus, if $r := r(R_p/I_p)$ denotes the Cohen–Macaulay type of $R_p/I_p$, then if $s > r$, the set $B_i$ of ideals is redundant at $p$. That is, there are containment relations among the $B_{ip}$. In this sense, the number of $B_i$ possible is bounded above at $p$ by $r$. In particular, if $I$ is Gorenstein at $p$, all the $B_{ip}$ are equal.

**Example.** We cannot improve this in general to include associated primes of $A$. In $R = k[x, y]$, take $I = (x^3, x^2y, xy^2, y^3)$, $B_1 = (x^2, y)$, and $B_2 = (x, y^2)$. Then
we have \((I : B_1) = (I : B_2) = (x^2, xy, y^2)\). Note that \(p = (x, y) \in \text{Ass}(R/A) = \text{Ass}(R/I)\) and \(I_p = (x^3, x^2y, xy^2, y^3)\), while \(B_1p \cap B_2p = (x^2, xy, y^2)\).

In the last part of the next chapter, we shall very briefly be concerned with geometric linkage; hence, we define it here, and show that it is just linkage with an additional assumption on the associated primes.

**Definition 2.6.** Ideals \(A\) and \(B\) of the Gorenstein local ring \(R\) are *geometrically linked* if they are each unmixed of pure height \(g\), have no common primary components, and \(I := A \cap B\) is generically Gorenstein, Cohen–Macaulay.

Note that Proposition 2.2 implies that if \(A\) and \(B\) are geometrically linked by \(I\), then they are linked by \(I\). Conversely, we have the following well-known result. Its proof is easy, and appears in [Sc2, Lemma 2.3] for instance.

**Lemma 2.7.** If \(A\) and \(B\) are unmixed ideals, linked by the generically Gorenstein, Cohen–Macaulay ideal \(I\), and if \(A\) and \(B\) have no common components, then \(A\) and \(B\) are geometrically linked by \(I\), i.e., \(I = A \cap B\).

Our primary tool as we study linkage by generically Gorenstein, Cohen–Macaulay ideals will be the theory of canonical modules. At least for Cohen–Macaulay rings, this is developed in [HK]. Another excellent reference for material on canonical modules, especially for non–Cohen–Macaulay rings, is the book of Schenzel, [Sc1]. We will give below the general definition of a canonical module; because we will generally work over a Gorenstein ring, we can specialize the definition, and extend it, as in [Sc1], to modules.
In the following remarks, let $R$ be a local ring, without any \textit{a priori} restrictions. Denote by $H^i_m(-)$ the local cohomology functors, and by $E(M)$ the injective hull of the module $M$.

Over complete local rings, the following Theorem defines canonical modules:

**Theorem 2.8.** [HK, Satz 5.2] Let $R$ be a complete local ring, $\dim R = n$. Then the functor on modules defined by $M \mapsto H^n_m(M)$ is representable; that is, there exists a module $K_R$ such that for each module $M$, there is a functorial isomorphism

$$H^n_m(M)^\vee \cong \Hom_R(M, K_R),$$

where, for a module $N$, $N^\vee$ denotes the Matlis dual $\Hom_R(N, E(R/m))$ of $N$.

**Definition 2.9.** [HK, Def. 5.6] A canonical module for a local ring $R$ is a module $K_R$ such that $K_R \otimes \widehat{R} \cong K_{\widehat{R}}$, where $\widehat{R}$ is the completion of $R$, and $K_{\widehat{R}}$ is the module in Theorem 2.8.

We note that for arbitrary local rings, a canonical module may not exist. When it does, however, it is unique [HK, Bemerkung 5.7]. For sufficiently good rings, the canonical module does exist, and takes a particularly nice form. We summarize these in the following:

**Proposition 2.10.**

(1) [HK, Satz 5.9] If $R$ is a Gorenstein ring, then a canonical module $K_R$ for $R$ exists, and $K_R \cong R$. Conversely, if $R$ is Cohen-Macaulay, and if it has a canonical module $K_R$ with $K_R \cong R$, then $R$ is Gorenstein.
(2) [HK, Satz 5.12] Suppose $S \to R$ is a local, surjective homomorphism of rings, and suppose a canonical module $K_S$ exists for $S$. Then a canonical module exists for $R$, and $K_R \cong \text{Ext}^d_S(R, K_S)$, where $d = \dim S - \dim R$.

(3) In particular, if $R$ is a Gorenstein ring, and if $I$ is an ideal of height $g$, then the residue class ring $R/I$ has a canonical module, and $K_{R/I} \cong \text{Ext}^g_R(R/I, R)$.

Since we will always be working over a Gorenstein base ring, we will take (3) as our working definition for canonical modules. Following Schenzel [Sc1], we extend this definition to an arbitrary module:

**Definition 2.11.** Let $R$ be a local Gorenstein ring, and let $M$ be a module; put $d = \dim R - \dim M$. Then the canonical module of $M$ is $K_M = \text{Ext}^d_R(M, R)$.

In particular, we can speak of the canonical module of a canonical module, which we will denote by $K_{K_{R/I}}$ for an ideal $I$ of $R$.

An important property of canonical modules which we will require is

**Proposition 2.12.** Suppose $M$ is a Cohen-Macaulay module over a Gorenstein local ring $R$. Then $K_M$ is also Cohen-Macaulay.

*Proof.* If $M = R/I$, this is well-known; see for example, from [HK, Satz 6.1(d)]. For arbitrary modules, it appears in the proof of [Sc1, Lemma 3.1.1(c)]. □

We note that even in the case $M = R/I$, if $R/I$ is not Cohen–Macaulay, the depth of $K_{R/I}$ is quite difficult to get a handle on. See, for instance, [A], where there are constructed examples showing that the canonical module may have any possible
depth. We also note that the converse of Proposition 2.12 does not hold. On the other hand, $K_M$ always satisfies the Serre condition $S_2$ [Sc1, Lemma 3.1.1(c)], so for $M = R/I$, the first case where $K_{R/I}$ is not Cohen–Macaulay is for $\dim R/I = 3$. In $R = k[u, v, x, y, z]_{(u, v, x, y, z)}$, one such ideal is $I = (u^3, u^2v, uv^2, v^3, u^2x - uvy - v^2z)$.

The importance of canonical modules in commutative algebra arises from the Local Duality Theorem, which expresses the somewhat mysterious local cohomology modules in a more concrete form. Since this theorem provides a sufficient motivation for the study of canonical modules, we state it, though we will not need it explicitly. For a partial converse, see [Sc1, Kor. 3.5.3].

**Local Duality Theorem.** [HK, Satz 5.5] Let $R$ be a local ring, $\dim R = n$, with a canonical module $K_R$. If $R$ is Cohen–Macaulay then for each module $M$, there is an isomorphism

$$H^i_m(M)^\vee \cong \text{Ext}^{n-i}_R(M, K_R), \quad i = 0, \ldots, n.$$}

We already gave in Lemma 2.4 one reason for restricting this study to linkage by generically Gorenstein, Cohen–Macaulay ideals. Our other main reason for this restriction is that if $I$ is such an ideal in a Gorenstein ring, the canonical module of $R/I$ is embeddable as an ideal in $R/I$. Indeed, we have the following result:

**Lemma 2.13.** [HK, Kor. 6.7, 6.13] Let $R$ be a Cohen–Macaulay ring of dimension at least 1, which possesses a canonical module $K_R$. Then $K_R$ is isomorphic to an ideal of $R$ if and only if $R_p$ is Gorenstein, for each minimal prime $p$. Furthermore, if $K_R \cong J$ is an ideal of $R$, then $J$ is a height 1, Gorenstein ideal of $R$.  

Thus, in particular, when $I$ is a generically Gorenstein, Cohen–Macaulay ideal of height $g$ in the Gorenstein ring $R$, then $K_{R/I}$ is isomorphic to $J/I$ for some height $g + 1$ Gorenstein ideal $J$ containing $I$.

In light of the usefulness of this result in our study, it seems natural to ask the following:

**Question.** Suppose $R$ is a Gorenstein local ring, and $I$ is a Cohen–Macaulay ideal of $R$. Does there exist a nice submodule $M$ of $K_{R/I}$ so that $K_{R/I}/M$ is isomorphic to an ideal of $R/I$?

Finally, in the proofs of many of our results, we will chase depths along exact sequences. For this, we will use the Depth Lemma, which is well-known and easy to prove.

**Depth Lemma.** Suppose there is a short exact sequence of modules

$$ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 $$

over a Noetherian ring $R$. Then one of the following holds:

- $\text{depth } A \geq \text{depth } B = \text{depth } C$
- $\text{depth } B \geq \text{depth } A = \text{depth } C + 1$
- $\text{depth } C > \text{depth } A = \text{depth } C$. □
3.1. The Cohen–Macaulay Property

Our starting point in this chapter is the following theorem:

Theorem 3.1.1. Let $R$ be a local Gorenstein ring. Suppose $A$ and $B$ are linked by an ideal $I$, where $I$ is a complete intersection, or a Gorenstein ideal. If $A$ is a Cohen–Macaulay ideal, so is $B$.

The case that $I$ is a complete intersection was first shown in [PS, Prop. 1.3]. For $I$ a perfect Gorenstein ideal, this was shown in [G]. Also, Schenzel gave a different proof in [Sc2], which works for any Gorenstein ideal. We note that easy examples show that Theorem 3.1.1 is false if we allow $I$ to be a generically Gorenstein, Cohen–Macaulay ideal. The following example is perhaps the easiest; it was given in [Sc2].

Example 3.1.2. Let $R = k[x, y, z, w]_{(x, y, z, w)}$ be the ring of polynomials in four variables, localized at the origin. Put $A = (y, z)$ and $B = (x, y) \cap (z, w) = (xz, yz, xw, yw)$. Then $A$ and $B$ are height 2 ideals linked by $I = (x, y) \cap (y, z) \cap (z, w) = (xy, yz, yw)$. Evidently, $I$ is generically Gorenstein (in fact, generically a complete intersection). To see that it is Cohen–Macaulay, just note that it is the ideal of maximal minors of the matrix

$$
\begin{pmatrix}
    x & y & 0 \\
    w & 0 & z
\end{pmatrix},
$$

17
whence it is Cohen–Macaulay by the results of [EN]. Here, though, \( A \) is a complete intersection, hence in particular, \( R/A \) is Cohen–Macaulay, but \( R/B \) has depth 1. Indeed, it is easy to verify that \( x + w \) is a maximal regular sequence on \( R/B \).

Our purpose in this chapter is to investigate how much of the Cohen–Macaulay property is preserved along linkage by generically Gorenstein, Cohen–Macaulay ideals. In particular, if \( A \) and \( B \) are linked by a generically Gorenstein, Cohen–Macaulay ideal \( I \), and \( A \) is Cohen–Macaulay, we give conditions for \( B \) to be a Cohen–Macaulay ideal, and when it is not Cohen–Macaulay, we can describe its non-Cohen–Macaulay locus, and give a criterion for \( R/B \) to have high depth.

Recall that since \( I \) is generically Gorenstein, Cohen–Macaulay, Proposition 2.13 shows that there is a Gorenstein ideal \( J \), containing \( I \), so that \( J/I \) is a height 1 ideal of \( R/I \) and the canonical module \( K_{R/I} \) of \( R/I \) is isomorphic to \( J/I \). We note that \( J \) is highly non-unique. Also, for future reference, we note that if a prime \( p \) contains \( I \), then \( I_p \) is a Gorenstein ideal of \( R_p \) if and only if either \( J \) is not contained in \( p \) or \( J_p = (I_p, c) \), for some element \( c \in R_p \) which is a non-zero-divisor on \( R_p/I_p \). This is because in either case, \( K_{R_p/I_p} = (K_{R/I})_p = J_p/I_p \cong R_p/I_p \), whence \( R_p/I_p \) is Gorenstein, by Proposition 2.10(1).

All of our results will depend on the interaction of \( A \) and \( J \), where \( K_{R/I} \cong J/I \), for some choice of \( J \). As such, we begin with some preliminary information on this interaction.

**Lemma 3.1.3.** Let \( R \) be a local ring for which the zero ideal is unmixed, \( A \) any ideal of height \( \leq 1 \), and \( J \) any ideal of height 1. Then there exists a non-zero-divisor \( c \) of \( R \) such that \( A \) is contained in a minimal prime of \( cJ \).
Proof. By the height condition on $A$, there certainly exists a prime $p$ containing $A$ with height $p = 1$. Thus $p$ is minimal over some non-zero-divisor $c$. Clearly $cJ \subseteq p$. Since $J$ is height 1, it contains a non-zero-divisor; hence, $cJ$ also contains a non-zero-divisor, so is of height at least 1. Since it is contained in the height 1 prime $p$, $p$ must be minimal over $cJ$. By construction, $p$ contains $A$, so we are done. □

Corollary 3.1.4. Suppose $I \subseteq A$, and that $I$ is generically Gorenstein and Cohen-Macaulay. Then $K_{R/I} \cong J/I$, where $J$ is a Gorenstein ideal of height one more than $I$, and where $A$ does not contain a non-zero-divisor for $R/J$.

Proof. Since $I$ is generically Gorenstein, Cohen-Macaulay, there exists a Gorenstein ideal $J$ of height one over $I$ such that $K_{R/I} \cong J/I$. But the ideals $A/I$ and $J/I$ of $R/I$ satisfy the conditions of Lemma 3.1.3, so for some non-zero-divisor $\bar{c}$ of $R/I$, $A$ is contained in a minimal prime for $\bar{c}J/I$. However, as $R/I$ modules, we have $J/I \cong \bar{c}J/I$; hence, letting $J'$ be the complete pre-image in $R$ of $\bar{c}J/I$, we see that $K_{R/I} \cong J'/I$ and $A$ is contained in a minimal prime for $J'$. □

Note that the minimal primes of $J$ are also minimal primes of $J'$. Thus, if $A$ and $B$ are linked by a generically Gorenstein, Cohen-Macaulay ideal $I$, then we can choose a $J$ such that both $A$ and $B$ are contained in some minimal prime of $J$. In particular, $\dim R/(A + J) = d - 1$, where $d := \dim R/A$.

On the other hand, it may sometimes be possible to choose an ideal $J$ so that height $A + J >$ height $J$, i.e., $A$ contains a non-zero-divisor for $R/J$. Our next lemma gives a necessary and sufficient condition for this to hold, and we will see later that often, such a choice of $J$ may not be possible (Corollary 3.1.12).
Lemma 3.1.5. Let $A$ and $B$ be linked by the generically Gorenstein, Cohen-Macaulay ideal $I$ and suppose $J$ is such that $K_{R/I} \cong J/I$. Then $A$ contains a non-zero-divisor for $R/J$ if and only if $B \subseteq J$.

Proof. First, suppose $x \in A$ is a non-zero-divisor for $R/J$. If $b \in B$, then $xb \in I \subseteq J$, whence $b \in J$. Thus $B \subseteq J$.

Conversely, suppose $B \subseteq J$. Then since $R/A$ is a homomorphic image of $R/I$, and of the same dimension, by Proposition 2.10 we have

$$K_{R/A} \cong \text{Hom}(R/A, K_{R/I}) = \text{Hom}(R/I, J/I) = ((I : A) \cap J)/I = (B \cap J)/I = B/I.$$ 

Next, if we apply the functor $\text{Hom}(R/A, -)$ to the short exact sequence

$$0 \longrightarrow J/I \overset{i}{\longrightarrow} R/I \longrightarrow R/J \longrightarrow 0,$$

we obtain a long exact sequence

$$0 \longrightarrow \text{Hom}(R/A, J/I) \overset{i^*}{\longrightarrow} \text{Hom}(R/A, R/I) \longrightarrow \text{Hom}(R/A, R/J) \longrightarrow \ldots.$$

As above $\text{Hom}(R/A, J/I) = K_{R/I} \cong B/I$, and by the linkage $\text{Hom}(R/A, R/I) = B/I$. It is now easily checked that the map $i^*$ commutes with these isomorphisms, which shows that it is an isomorphism, and hence $\text{Hom}(R/A, R/J) = 0$. This means that $A$ contains a non-zero-divisor for $R/J$. \qed

We would like to show that, when $A$ is a Cohen-Macaulay ideal, and $J$ is chosen so that $\text{height}(A + J) = \text{height} J$, then $A + J$ is nearly a Cohen-Macaulay ideal, in the sense that $\text{depth} R/(A + J) \geq \dim R/(A + J) - 1$. However, we have been unable to prove this, or to find a counterexample. On the other hand, this property is definitely independent of the choice of the ideal $J$, as the next proposition shows. This result will be used often throughout the remainder of this chapter.
Proposition 3.1.6. Suppose $A$ and $B$ are linked by the generically Gorenstein, Cohen–Macaulay ideal $I$. Put $d := \dim R/I$, and write $K_{R/I} \cong J/I$. If $R/A$ is Cohen–Macaulay, then

1. if $R/(A + J)$ is Cohen–Macaulay of dimension $d - 1$, then $K_{R/B}$ is Cohen–Macaulay;

2. otherwise, $\text{depth} K_{R/B} = \text{depth} R/(A + J) + 2$.

Proof. First, since $A$ and $B$ are linked, we have

$$K_{R/B} \cong \text{Hom}(R/B, K_{R/I}) = \text{Hom}(R/B, J/I) = ((I : B) \cap J)/I = (A \cap J)/I.$$ 

Now, since $A \cap J$ has embedded components (namely, the primary components of $J$ not containing $A$), it cannot be a Cohen–Macaulay ideal. Hence, from the depth lemma applied to the short exact sequence

$$0 \longrightarrow (A \cap J)/I \longrightarrow R/I \longrightarrow R/(A \cap J) \longrightarrow 0$$

we see that $\text{depth}(A \cap J)/I = \text{depth} R/(A \cap J) + 1$.

Next, note that there is an isomorphism $A/(A \cap J) \cong (A + J)/J$. The depth lemma applied to the sequence

$$0 \longrightarrow A/(A \cap J) \longrightarrow R/(A \cap J) \longrightarrow R/A \longrightarrow 0,$$

along with the fact that $R/(A \cap J)$ is not Cohen–Macaulay, shows that $\text{depth} R/(A \cap J) = \text{depth} A/(A \cap J)$. A third application of the depth lemma on the sequence

$$0 \longrightarrow (A + J)/J \longrightarrow R/J \longrightarrow R/(A + J) \longrightarrow 0$$
shows that if depth $R/(A + J) = \text{depth } R/J$, then depth$(A + J)/J = \text{depth } R/(A + J)$; otherwise, depth$(A + J)/J = \text{depth } R/(A + J) + 1$.

Finally, note that height $A+J \geq \text{height } J$; hence if depth $R/(A+J) = \text{depth } R/J$, then $R/(A+J)$ is Cohen–Macaulay of dimension $d - 1$. Thus, we have the desired conclusion. \(\square\)

Note that because $K_{R/B}$ does not depend on $J$, we have some invariance properties for the ideal $A + J$ as $J$ runs among the ideals for which $K_{R/I} \cong J/I$. One useful such property is isolated in the following statement.

**Corollary 3.1.7.** Let $d := \dim R/I$. If $K_{R/B}$ is Cohen–Macaulay, then for every choice of $J$ so that $K_{R/I} \cong J/I$,

1. if $A$ contains a non-zero-divisor for $R/J$, then height $A + J = \text{height } J + 1$, and $R/(A + J)$ is Cohen–Macaulay;

2. otherwise, depth $R/(A + J) \geq \dim R/(A + J) - 1$.

**Proof.** For (1) note that certainly height $A + J \geq \text{height } J + 1$. Thus, $\dim R/(A + J) \leq d - 2$, and so from Proposition 3.1.6, we see that

$$d = \text{depth } K_{R/B} = \text{depth } R/(A + J) + 2 \leq \text{dim } R/(A + J) + 2 \leq d.$$  

Hence, $\text{depth } R/(A + J) = \dim R/(A + J) = d - 2$, showing that $R/(A + J)$ is Cohen–Macaulay, with height $A + J = \text{height } J + 1$.

The second statement follows similarly: the assumption that $A$ not contain a non-zero-divisor for $R/J$ implies that $\dim R/(A + J) = d - 1$. If $R/(A + J)$ is not Cohen–Macaulay, then Proposition 3.1.6(2) shows that depth $R/(A + J) = d - 2$. In either case, then, we have depth $R/(A + J) \geq \dim R/(A + J) - 1$. \(\square\)
Proposition 3.1.6 also allows us to obtain our first Cohen–Macaulay criterion for $R/B$, when $B$ is linked to a Cohen–Macaulay ideal:

**Corollary 3.1.8.** If $\text{depth } R/(A+J) \geq \dim R/J - 1$, then $R/B$ is Cohen–Macaulay if and only if $R/B$ satisfies the Serre condition $S_2$.

**Proof.** Since under the hypothesis, $K_{R/B}$ is Cohen–Macaulay, this follows immediately from [Sc1, Satz 3.2.3]. □

In some sense, Corollary 3.1.8 is unsatisfactory, for it characterizes the Cohen–Macaulay property for $R/B$ in terms of $R/B$. Ideally, we want to use conditions on $A$ and $I$ to give a Cohen–Macaulay criterion for $R/B$; this is accomplished in Corollary 3.1.11 below. To do this, we need to extend a result of Schenzel, [Sc2, Theorem 3.1], to our situation of more general linkage.

Since $R$ is a Gorenstein local ring, it possesses a finite injective resolution, say $D^\bullet$. This complex $D^\bullet$ is a *dualizing complex* for $R$. That is, it is a bounded complex of injective modules, with finitely generated cohomology, such that for every bounded complex $G^\bullet$ with finitely generated cohomology, the natural map of complexes

$$G^\bullet \rightarrow \text{Hom}(\text{Hom}(G^\bullet, D^\bullet), D^\bullet)$$

induces isomorphisms on cohomology, cf. [Sh] or [Ha]. For a module $M$ the first non-vanishing cohomology of the complex $\text{Hom}(M, D^\bullet)$ is $\text{Ext}^g(M, R) = K_M$, $(g := \dim R - \dim M)$, by Local Duality and the characterization of dimension in terms of the local cohomologies of $M$. Thus, we have an exact sequence of complexes in the derived category

\[(*) \quad 0 \longrightarrow K_M[-g] \longrightarrow \text{Hom}(M, D^\bullet) \longrightarrow \mathcal{F}^\bullet(M) \longrightarrow 0,\]
where $\mathcal{J}^*(M)$, the truncated dualizing complex of $M$, is the factor complex of the left-hand embedding. Note that from the long exact sequence on cohomology associated to (*),

$$H^i(\mathcal{J}^*(M)) = \begin{cases} 0 & \text{if } i \leq g \\ \text{Ext}^i(M, R) & \text{otherwise}. \end{cases}$$

**Theorem 3.1.9.** Suppose $A$ and $B$ are linked by a generically Gorenstein, Cohen-Macaulay ideal $I$. Write $K_{R/I} \cong J/I$, for a Gorenstein ideal $J$ of height 1 over $I$.

Then there exists a quasi-isomorphism

$$\mathcal{J}^*(J/(A \cap J))[g] \to \text{Hom}(\mathcal{J}^*(R/B), D^*).$$

**Proof.** First, recall that $K_{R/B} \cong (A \cap J)/I$, as in the proof of Proposition 3.1.6. Hence we have a short exact sequence

$$0 \to K_{R/B} \to J/I \to J/(A \cap J) \to 0.$$

But since $R/I$ is Cohen-Macaulay, then in the derived category, there is an isomorphism $J/I[-g] \cong \text{Hom}(R/I, D^*)$. Hence the exact sequence gives rise to an exact sequence in the derived category:

$$0 \to K_{R/B}[-g] \to \text{Hom}(R/I, D^*) \to J/(A \cap J)[-g] \to 0.$$

Using the morphism $\text{Hom}(R/B, D^*) \to \text{Hom}(R/I, D^*)$ induced by the surjection $R/I \to R/B$ we obtain a commutative diagram of exact sequences in the derived category:

$$\begin{array}{cccccccccc}
0 & \to & K_{R/B}[-g] & \to & \text{Hom}(R/I, D^*) & \to & J/(A \cap J)[-g] & \to & 0 \\
| & & \| & & \uparrow & & \phi^* \uparrow & & \\
0 & \to & K_{R/B}[-g] & \to & \text{Hom}(R/B, D^*) & \to & \mathcal{J}^*(R/B) & \to & 0
\end{array}$$
where $\phi^*$ is induced from the first two maps. On applying the dualizing functor $\Hom(-, D^*)$, we obtain the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Hom(J/(A \cap J), D^*))[g] & \longrightarrow & R/I & \longrightarrow & \Hom(K_{R/B}, D^*))[g] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & & & & \\
0 & \longrightarrow & \Hom(J^*(R/B), D^*) & \longrightarrow & R/B & \longrightarrow & \Hom(K_{R/B}, D^*))[g] & \longrightarrow & 0
\end{array}
\]

where $\psi^*$ is the dual of $\phi^*$. We just need to show that $H^0(\Hom(J^*(R/B), D^*)) = 0$ and that $\psi^i$ induces isomorphisms on the homology level, for $i > 0$. Thus taking the long exact sequences on homology associated to the diagram (*), we obtain commutative diagrams

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Ext^g_R(J/(A \cap J), R) & \longrightarrow & R/I & \longrightarrow \\
\downarrow & & \downarrow & & & & & & & \\
0 & \longrightarrow & H^0(\Hom(J^*(R/B), D^*)) & \longrightarrow & R/B & \longrightarrow \\
\hspace{1cm} & & \hspace{1cm} & & \hspace{1cm} & & \hspace{1cm} & & \hspace{1cm} & \\
\Ext^g_R(K_{R/B}, R) & \longrightarrow & \Ext^{g+1}_R(J/(A \cap J), R) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & & & & \\
\Ext^g_R(K_{R/B}, R) & \longrightarrow & H^1(\Hom(J^*(R/B), D^*)) & \longrightarrow & 0
\end{array}
\]

and

\[
\begin{array}{cccccc}
\Ext^{g+i}_R(K_{R/B}, R) & \longrightarrow & \Ext^{g+i+1}_R(J/(A \cap J), R) & \longrightarrow \\
\downarrow & & \downarrow & & & & & & & \\
\Ext^{g+i}_R(K_{R/B}, R) & \longrightarrow & H^{i+1}(\Hom(J^*(R/B), D^*))
\end{array}
\]

The five-lemma applied to the first diagram shows the last map is an isomorphism, and the second diagram has the right-hand maps isomorphisms. These maps are just the maps induced on homology by $\psi^i$, $i > 0$. But since $R/B$ is unmixed, it follows from [Sc1, Satz 3.2.2] that $H^0(\Hom(J^*(R/B), D^*)) = 0$, and so we are finished with the proof. □
Remark. We recover the result of [Sc2, Theorem 3.1] when $I$ is Gorenstein, for then $K_{R/I} \cong J/I$ is isomorphic to $R/I$, whence $(A + J)/J \cong R/A$. Thus, $J^*(R/A)[g] \cong \text{Hom}(J^*(R/B), D^*)$.

**Corollary 3.1.10.** If $A$ is Cohen–Macaulay and linked to $B$ by the generically Gorenstein ideal $I$, where $K_{R/I} = J/I$, then $\text{Ext}^{g+2}_R(R/(A + J), R)$ is isomorphic to the cokernel of the natural map

$$R/B \to \text{Ext}^g_R(K_{R/B}, R).$$

**Proof.** The proof of the above Theorem 3.1.9 shows that there is an exact sequence

$$0 \to R/B \to \text{Ext}^g_R(K_{R/B}, R) \to \text{Ext}^{g+1}_R(J/(A \cap J), R) \to 0.$$

But clearly $J/(A \cap J) \cong (A + J)/A$. Using that $R/A$ is Cohen–Macaulay and taking the long exact sequence on Ext associated to the short exact sequence

$$0 \to (A + J)/A \to R/A \to R/(A + J) \to 0$$

we see that $\text{Ext}^{g+1}_R((A + J)/A, R) \cong \text{Ext}^{g+2}_R(R/(A + J), R)$, which finishes the proof. □

The next two Corollaries establish the main Cohen–Macaulay properties of $R/B$. In the first, we give a strong characterization for $R/B$ to be Cohen–Macaulay, which in contrast to our earlier characterization of Corollary 3.1.8, depends only on $A$ and $J$. The second Corollary uses this characterization to get some information on the non-Cohen–Macaulay locus of $R/B$. 
Corollary 3.1.11. Suppose $A$ is Cohen–Macaulay and linked to $B$ by the generically Gorenstein ideal $I$. Write $K_{R/I} \cong J/I$. Then $R/B$ is Cohen–Macaulay if and only if $A + J$ is a Cohen–Macaulay ideal with height $A + J = \text{height } J$.

Proof. Put $g = \text{height } I$ and $d = \text{dim } R/I$. Suppose that $R/B$ is Cohen–Macaulay. Then the natural map $R/B \to K_{K_{R/B}}$ is an isomorphism [Sc2, Satz 3.2.2]. Thus from Corollary 3.1.10, we see that $\text{Ext}^{g+2}(R/(A + J), R) = 0$. Next, from Proposition 3.1.6, we have depth $R/(A + J) \geq \text{depth } K_{R/B} - 1$. Since $R/B$ is Cohen–Macaulay, so is $K_{R/B}$, necessarily of the same dimension. Hence there is only one non-vanishing Ext module for $R/(A + J)$ and this is $\text{Ext}^{g+1}(R/(A + J), R)$. This shows that $R/(A + J)$ is Cohen–Macaulay of dimension $d - 1$.

Conversely, suppose $R/(A + J)$ is Cohen–Macaulay of dimension $d - 1$. Then in particular, $\text{Ext}^{g+2}(R/(A+J), R) = 0$, showing that the natural map $R/B \to K_{K_{R/B}}$ is an isomorphism. But Proposition 3.1.6(1) shows that $K_{R/B}$ is Cohen–Macaulay; hence, $K_{K_{R/B}} \cong R/B$ is also Cohen–Macaulay. □

Remark. We recover the fact that Gorenstein linkage preserves the Cohen–Macaulay property because if $I$ is Gorenstein, then $J = (I, c)$, for some non-zero-divisor $c$ over $R/I$. Hence $A + J = (A, c)$; since $c$ is a non-zero-divisor for $R/I$, it is also a non-zero-divisor for $R/A$. Thus, $\text{height}(A, c) = \text{height } A + 1$, and $R/(A, c)$ is Cohen–Macaulay. By the Corollary, this implies $R/B$ is Cohen–Macaulay.

The next result is immediate.

Corollary 3.1.12. If there is an ideal $J$ with $K_{R/I} \cong J/I$ and so that $A$ contains a non-zero-divisor for $R/J$, then $R/B$ is not Cohen–Macaulay. □
For the following results, define the *non-Cohen-Macaulay locus* of a module $M$ to be \( \text{NCM}(M) := \{ p \in \text{Supp}(M) : M_p \text{ is not Cohen-Macaulay} \} \).

**Corollary 3.1.13.** Suppose \( \text{height}\ A + J = \text{height}\ J \). Then

\[
\text{NCM}(R/B) = \text{NCM}(R/(A + J)).
\]

**Proof.** Let \( p \in \text{NCM}(R/(A + J)) \). First, we claim \( p \in \text{Supp}(R/B) \). For if not, then \( R_p = B_p = (I_p : A_p) \), and hence \( A_p = I_p \). Since \( I_p \subseteq J_p \), we have \( A_p + J_p = I_p + J_p = J_p \), and this is a Cohen-Macaulay ideal (even Gorenstein), which contradicts that \( p \in \text{NCM}(R/(A + J)) \). Thus in the ring \( R_p \), we have the following situation: the proper ideal \( B_p \) is linked to the Cohen-Macaulay ideal \( A_p \) by the generically Gorenstein, Cohen-Macaulay ideal \( I_p \), and \( K_{R_p/I_p} = J_p/I_p \). But since \( p \in \text{NCM}(R/(A + J)) \), \( (A + J)_p \) is a non-Cohen-Macaulay ideal. Applying Corollary 3.1.11 to this situation, we see that \( B_p \) is not a Cohen-Macaulay ideal.

Conversely, suppose \( p \in \text{NCM}(R/B) \), i.e., \( p \in \text{Supp}(R/B) \) and \( R_p/B_p \) is not Cohen-Macaulay. Then we claim that \( p \) must contain both \( A \) and \( J \). If \( A \nsubseteq p \), then \( B_p = (I_p : A_p) = I_p \) would be Cohen-Macaulay, a contradiction. If \( J \nsubseteq p \), then \( I_p \) is a Gorenstein ideal. Thus \( B_p \) is Gorenstein linked to the Cohen-Macaulay ideal \( A_p \), and by Theorem 3.1.1, this means \( B_p \) is a Cohen-Macaulay ideal, again a contradiction. In particular, this means that \( p \in \text{Supp}(R/(A + J)) \). By Corollary 3.1.11 again, since \( R_p/B_p \) is not Cohen-Macaulay, we have that \( (A + J)_p \) is not a Cohen-Macaulay ideal, which shows that \( p \in \text{NCM}(R/(A + J)) \). □

Though the proof of the previous result only covers the case that \( A \) does not contain a non-zero-divisor for \( R/J \), a similar result holds when \( A \) does contain a
non-zero-divisor for $R/J$. Though the basic ideas of the previous proof, with an application of Corollary 3.1.7, would suffice to prove this, we offer below a somewhat different proof.

**Proposition 3.1.14.** If there exists $J$ such that $\text{height}(A + J) > \text{height} J$ (i.e., if $A$ contains a non-zero-divisor for $J$), then $A + J$ defines the non-Cohen–Macaulay locus of $R/B$. That is, $\text{NCM}(R/B) = \text{Supp}(R/(A + J))$.

**Proof.** Since $A$ contains a non-zero-divisor for $J$, $B$ must be contained in $J$. Thus,

$$K_{R/A} = \text{Hom}_R(R/A, J/I) = ((I : A) \cap J)/I = (B \cap J)/I = B/I,$$

and in particular, $B/I$ is Cohen–Macaulay of dimension $d := \dim R/I$. So from the short exact sequence

$$(1) \quad 0 \to B/I \to R/I \to R/B \to 0$$

we obtain a long exact sequence on $\text{Ext}(\text{-}, R)$:

$$(2) \quad 0 \to K_{R/B} \to K_{R/I} \to K_{B/I} \to \text{Ext}^{g+1}_R(R/B, R) \to 0.$$

But applying the depth lemma to (1), we see that $\text{depth} R/B \geq d - 1$. Hence, $\text{NCM}(R/B) = \text{Supp}(\text{Ext}^{g+1}_R(R/B, R))$. Since $K_{R/B} \cong (A \cap J)/I$, and $K_{B/I} = K_{K_{R/A}} = R/A$, the sequence (2) shows that $\text{Ext}^{g+1}_R(R/B, R) \cong R/(A + J)$. So $\text{NCM}(R/B) = \text{Supp}(R/(A + J))$. □

**Remark.** Note that the above proof shows that $R/(A+J) \cong \text{Ext}^{g+1}_R(R/B, R)$. Since the module $\text{Ext}^{g+1}_R(R/B, R)$ is unchanged as $J$ is varied among the ideals so that
$K_{R/I} \cong J/I$ and $A$ contains a non-zero-divisor for $R/J$, then also $R/(A + J)$, and with it the ideal $A + J$, is unchanged.

Our next two results give some information on the dimension of the non-Cohen–Macaulay locus of $R/B$, under the additional hypothesis that $K_{R/B}$ is Cohen–Macaulay. Of course, if $A$ contains a non-zero-divisor for $R/J$, then Proposition 3.1.14 shows, in particular, that $\dim \text{NCM}(R/B) = d - 2$, where $d = \dim R/B$. Thus we need only be concerned with the case that no such $J$ exists.

**Theorem 3.1.15.** Suppose $A$ and $B$ are linked by $I$, and that $K_{R/B}$ is Cohen–Macaulay. Choose $J$ so that $K_{R/I} \cong J/I$ and height $A + J = \text{height } J$. Then $\dim \text{NCM}(R/B) \geq \text{depth } R/B - 1$, with equality holding if and only if the module $\text{Ext}^{g+2}_R(R/(A + J), R)$ is Cohen–Macaulay.

**Proof.** First, from Corollary 3.1.10 there is a short exact sequence

$$0 \longrightarrow R/B \longrightarrow K_{K_{R/B}} \longrightarrow \text{Ext}^{g+2}_R(R/(A + J), R) \longrightarrow 0.$$ 

Moreover, the hypothesis that $K_{R/B}$ is Cohen–Macaulay implies that $K_{K_{R/B}}$ is also Cohen–Macaulay; hence, using the depth lemma on the exact sequence above, we see that $\text{depth } R/B = \text{depth } \text{Ext}^{g+2}_R(R/(A + J), R) + 1$.

Now, we also know from Corollary 3.1.7 that $\text{depth } R/(A + J) \geq \dim R/(A + J) - 1$. Thus, $\text{Ext}^{g+2}_R(R/(A + J), R)$ is the only non-vanishing higher Ext module; in particular, $\text{NCM}(R/(A + J)) = \text{Supp } \text{Ext}^{g+2}_R(R/(A + J), R)$, and thus using Proposition 3.1.13, we have the following:
\[
\dim \text{NCM}(R/B) = \dim \text{NCM}(R/(A + J)) = \dim \text{Ext}_{R}^{q+2}(R/(A + J), R) \\
\geq \text{depth} \text{Ext}_{R}^{q+2}(R/(A + J), R) \\
= \text{depth} R/B - 1.
\]

The inequality in the middle is an equality if and only if \(\text{Ext}_{R}^{q+2}(R/(A + J), R)\) is Cohen–Macaulay, and this finishes the proof. □

**Example 3.1.16.** We may have a strict inequality in Theorem 3.1.15. Let \(R = k[u, v, x, y, z]\), and put

\[
I = (u^3v - uvxz - u^2yz, u^2v^2 - v^2xz - uvyz, u^3y - v^2z^2, u^2xy - uv^2z + vyz^2), \\
A = (vz, u^2y, u^2v).
\]

It is easily checked that \(I\) is generically Gorenstein, Cohen–Macaulay, and \(A\) is Cohen–Macaulay, both of height 2. On the other hand \(B := (I : A)\) has depth 1, but \(\text{NCM}(R/B) = \mathcal{V}(a_3a_4)\), where \(a_i = \text{ann} \text{Ext}_{R}^{i}(R/B, R)\), [Sc1, Satz 2.4.6], and this has dimension 1. Hence, \(\dim \text{NCM}(R/B) = 1 > 0 = \text{depth} R/B - 1.\)

**Remark.** Though we do not as yet have an example, it seems plausible that the hypothesis in Theorem 3.1.15 that \(B\) be linked to a Cohen–Macaulay ideal \(A\) cannot be dropped, in general. As part of a general question about non-Cohen–Macaulay ideals, it would be interesting to investigate the properties of such ideals which have Cohen–Macaulay canonical modules. For instance, how much of the above result can be proven without the assumption of linkage?
Theorem 3.1.17. Let $R$ be an $n$-dimensional Gorenstein ring, and $A$ and $B$ height $g$ ideals linked by a generically Gorenstein, Cohen–Macaulay ideal $I$, with $K_{R/I} \cong J/I$, and $R/A$ Cohen–Macaulay. Suppose $K_{R/B}$ is Cohen–Macaulay, and also that $(A+J)/J$ satisfies the Serre condition $S_r$, but not $S_{r+1}$. Then $\dim \text{NCM}(R/B) = n - g - r - 1$.

Proof. As in the proof of Theorem 3.1.15, we have

\[ \dim \text{NCM}(R/B) = \dim \text{Ext}^{g+2}_R(R/(A+J), R). \]

Now, from the short exact sequence

\[ 0 \longrightarrow (A+J)/A \longrightarrow R/A \longrightarrow R/(A+J) \longrightarrow 0, \]

and using that $R/A$ is Cohen–Macaulay, we see that $\text{Ext}^{g+i}_R((A+J)/A, R) \cong \text{Ext}^{g+i}_R(R/(A+J), R)$, for $i \geq 1$. Also, since $K_{R/B}$ is Cohen–Macaulay, by Corollary 3.1.7, depth $R/(A+J) \geq \dim R/(A+J) - 1$; hence, all the modules $\text{Ext}^i_R(R/(A+J), R)$ vanish, for $i > g + 2$, and so the corresponding Ext modules for $(A+J)/A$ also vanish. Furthermore, the module $(A+J)/A$ is equidimensional, since its minimal primes are minimal primes of the Cohen–Macaulay ideal $A$. Thus we can apply [Sc1, Lemma 3.2.1]. This says that $\dim \text{Ext}^{g+1}_R((A+J)/A, R) = n - g - 1 - r$, where $r$ is the largest integer so that $(A+J)/A$ satisfies $S_r$. \(\square\)

3.2. ON A LOWER BOUND FOR THE DEPTH OF A LINKED IDEAL

Suppose that $A$ and $B$ are linked by a generically Gorenstein, Cohen–Macaulay ideal $I$, and that $R/A$ is Cohen–Macaulay. In the previous section, we concentrated,
for the most part, on when $R/B$ was Cohen–Macaulay, i.e., when $R/B$ had maximal depth. In this section, we are concerned with when $R/B$ has high depth. By this, we mean that depth $R/B \geq \dim R/B - 1$; thus, $R/B$ is nearly Cohen–Macaulay. We will say in this case that $R/B$ (or $B$) "satisfies the depth inequality." One of the conditions which imply this will lead naturally to further consideration, and we are able to show that a result of Peskine and Szpiro on the sum of $A$ and $B$ continues to be valid in many cases. On the other hand, we give an example where it fails.

We have already seen one case when the above depth inequality holds, namely, when $K_{R/I} \cong J/I$, where $J$ is an of height 1 over $I$ and $A$ contains a non-zero-divisor for $J$. For in this case, we must have $B \subseteq J$, and hence

$$K_{R/A} = \text{Hom}_R(R/A, J/I) = ((I : A) \cap J)/I = B/I$$

is Cohen–Macaulay. Thus, the depth lemma applied to the exact sequence

$$0 \to B/I \to R/I \to R/B \to 0$$

shows that depth $R/B \geq \dim R/B - 1$.

The next example shows that this is not a necessary condition:

**Example 3.2.1.** Let $R = k[u,v,x,y,z,w]_R$ be the localized polynomial ring in 6 variables. Let $I$ be the ideal generated by the $2 \times 2$ minors of the matrix

$$
\begin{pmatrix}
  u-v & w^2 & y & wx \\
  x-w & 0 & x+z & yz
\end{pmatrix}.
$$

Thus, $I$ has height 3 and is Cohen–Macaulay. Also, $K_{R/I} \cong J/I$, where $J = (u-v, x-w)^3 + I$, so $R/I$ is generically Gorenstein. If we let $B$ be the unmixed
part of the ideal \((x, w, z)^2 + I\), and put \(A = (I : B)\), then it is easily checked, by MACAULAY for example, that \(A\) is a Cohen–Macaulay ideal, \(B\) satisfies the depth inequality, but \(K_{R/A} \neq B/I\). Thus, \(A\) does not contain a non-zero-divisor for \(J\), for any \(J\) with \(J/I \cong K_{R/I}\).

We note also, that the depth inequality is not always satisfied. The following example was communicated to me by C. Walter. The methods of his paper [W], especially those of Remark 1.1, allow many other similar examples to be easily computed.

**Example 3.2.2.** Let \(R = k[u, v, w, y, z]\) be the polynomial ring in five indeterminates. Let \(B\) be the ideal defining the Veronesean surface in \(\mathbb{P}^4\). That is, \(B\) is the kernel of the obvious map

\[
k[u, v, x, y, z] \to k[\alpha^2, \beta^2, \gamma^2, \beta(\alpha - \gamma), \gamma(\alpha - \beta)].
\]

Put \(A = (x - z, vz - yz)\), and \(I = A \cap B\). Then it is easy to check that \(I\) is a generically Gorenstein, Cohen–Macaulay ideal. Thus \(A\) and \(B\) are linked by \(I\), \(R/A\) is Cohen–Macaulay, but \(R/B\) has depth 1. In particular, \(B\) does not satisfy the depth inequality.

Recall from Corollary 3.1.11 that \(R/B\) is Cohen–Macaulay if and only if \(R/(A + J)\) is Cohen–Macaulay of dimension \(d - 1\), where \(d = \dim R/A\). Our characterization of when \(R/B\) satisfies the depth inequality is similar, except that it only involves the unmixed part of \(A + J\). Unfortunately, we have to make the additional assumption that \(K_{R/B}\) is Cohen–Macaulay; as stated previously, we believe that this is already implied by the linkage, but have been unable to prove it.
Notation. For an ideal $G$, we denote by $G'$ the unmixed part of $G$. That is, $G'$ is the intersection of the primary components of $G$ of highest dimension. Since we are in a Gorenstein local ring, note that $G' = \text{ann} \Ext^n_R (R/G, R)$, where $g = \text{height} G$.

We first need a lemma, which is well-known; cf. [Mat, Exercise 6.4]:

**Lemma 3.2.3.** Let $R$ be a Cohen-Macaulay ring, and $A$ and $B$ ideals of the same height with $B \subseteq A$. Denote by $A'$, resp. $B'$, the intersection of the minimal primary components of $A$, resp. $B$. Then $B' = A'$ if and only if $B_p = A_p$ for each minimal prime $p$ of $B$.

**Proof.** The forward direction is trivial, so assume that $B_p = A_p$ for each minimal prime of $B$. Let $B' = q_1 \cap \cdots \cap q_n$ and $A' = Q_1 \cap \cdots \cap Q_m$ be primary decompositions. Then it will suffice to show that $n = m$, and that $q_i = Q_i$, perhaps after renumbering. First, since $A$ and $B$ have the same height, if $P$ is a minimal prime of $A$, it is also minimal over $B$. Thus, $m \leq n$. On the other hand, if $p$ is minimal over $B$, then since $A_p = B_p$ is a proper ideal of $R_p$, we see that $p$ is minimal over $A$, as well, and that the primary components of $A$ and $B$ corresponding to $p$ are equal. Thus $m = n$ and $q_i = Q_i$ for each $i$, up to renumbering. □

**Proposition 3.2.4.** Let $R$ be a local Cohen-Macaulay ring of dimension $n$. Suppose the ideals $A$ and $B$ have height $n - 1$, and are linked by the generically Gorenstein, Cohen-Macaulay ideal $I$, where $R/I$ has canonical module isomorphic to $J/I$, for an ideal $J$ of height 1 over $I$. Then $A + J$ and $B + J$ are linked by $J$.

**Proof.** By working modulo $I$, we reduce to the case that $R$ is a 1-dimensional Cohen–Macaulay ring, $A$ and $B$ are height 0 ideals such that $A = (0 : B)$ and
$B = (0 : A)$, and $J$ is isomorphic to the canonical module of $R$. We need to show that $\text{Hom}_R(R/B, R/J) \cong (A + J)/J$ and $\text{Hom}_R(R/A, R/J) \cong (B + J)/J$. It will clearly suffice to prove just one of these, as the proof of the other will be similar, and, in any case, follows from Lemma 2.4.

Now, there is a natural short exact sequence

$$0 \to J \to R \to R/J \to 0,$$

which gives rise to the long exact sequence

$$(1) \quad 0 \to \text{Hom}_R(R/B, J) \to \text{Hom}_R(R/B, R) \to \text{Hom}_R(R/B, R/J) \to \text{Ext}^1_R(R/B, J) \to \ldots.$$

By the linkage of $A$ and $B$, we have natural isomorphisms

$$\text{Hom}_R(R/B, J) \cong A \cap J \quad \text{and} \quad \text{Hom}_R(R/B, R) \cong A.$$

Also, since $R/B$ is unmixed in the 1-dimensional CM ring $R$, it is Cohen–Macaulay; in particular, the local cohomology $H^0_m(R/B) = 0$. But $R$ is Cohen–Macaulay with canonical module $J$; thus, by the Local Duality Theorem and the homological characterization of depth, $\text{Ext}^1_R(R/B, J) \cong H^0_m(R/B)^\vee = 0$, where $M^\vee$ is the Matlis dual of $M$. Thus, the sequence (1) gives rise to a short exact sequence

$$0 \to A \cap J \to A \to \text{Hom}_R(R/B, R/J) \to 0,$$

where the first map is the inclusion. That is, $\text{Hom}_R(R/B, R/J) \cong A/(A \cap J) \cong (A + J)/J$, as required. □
Theorem 3.2.5. Suppose $A$ and $B$ are linked by a generically Gorenstein, Cohen-Macaulay ideal $I$ in the local Gorenstein ring $R$. Write $K_{R/I} \cong J/I$ for a Gorenstein ideal $J$ of height 1 over $I$, and suppose neither $A$ nor $B$ contains a non-zero-divisor for $R/J$. Then $(A + J)'$ and $(B + J)'$ are linked by $J$.

Proof. First, we have $(J : (B + J)') = (J : B + J)$, by considering a primary decomposition of $B + J$, and using that $J$ is unmixed. Next, since $A$ and $B$ are linked, we clearly have $A + J \subseteq (J : B + J)$. Thus, by Lemma 3.2.3, we have only to show that $(A + J)_p = (J : B + J)_p$, for each minimal prime $p$ of $A + J$.

Now, such a minimal prime $p$ will always contain $A$ and $J$. If $B \not\subseteq p$, then choosing an element $x \in B \setminus p$, for each $a \in A$, we have $ax \in I \subseteq J$, showing that $A$ is contained in the $p$-primary component of $J$. Thus, on localizing at $p$, we have $A_p \subseteq J_p$, and hence $(A + J)_p = J_p = (J_p : R_p) = (J_p : B_p + J_p)$, so the equality holds in this case.

Thus, we are reduced to the case that $p$ contains $B$. By localizing at $p$ we are in the situation described by Proposition 3.2.4, and the proof of the theorem is finished. □

With these preliminaries, we are able to prove the main result of this section. It establishes a necessary and sufficient condition in terms of $A$ and $J$ for $R/B$ to have high depth. As we have already noted, this condition is quite similar in spirit to the one of Corollary 3.1.11.

Proposition 3.2.6. Suppose the Cohen-Macaulay ideal $A$ is linked to $B$ by the generically Gorenstein, Cohen-Macaulay ideal $I$, and write $K_{R/I} \cong J/I$, where
neither $A$ nor $B$ contains a non-zero-divisor for $R/J$. Suppose that $K_{R/B}$ is Cohen-Macaulay. Then $(A + J)'$ is a Cohen-Macaulay ideal if and only if $R/B$ satisfies the depth inequality.

**Proof.** Note that Corollary 3.1.11 shows that $R/B$ is Cohen-Macaulay if and only if $R/(A + J)$ is Cohen-Macaulay. Thus, for the rest of the proof, we may assume that neither of these properties hold.

Now, from Corollary 3.1.7(ii) and the depth lemma applied to the sequence

\[
(*) \quad 0 \longrightarrow (A + J)/J \longrightarrow R/J \longrightarrow R/(A + J) \longrightarrow 0,
\]

we see that $(A + J)/J$ is Cohen-Macaulay of the same dimension as $R/J$. In particular, on applying the functor $\text{Hom}(-, R)$ to $(*)$, we obtain a long exact sequence

\[
(1) \quad 0 \longrightarrow \text{Ext}^{g+1}_R(R/(A + J), R) \longrightarrow \text{Ext}^{g+1}_R(R/J, R) \longrightarrow \text{Ext}^{g+2}_R((A + J)/J, R) \longrightarrow \text{Ext}^{g+2}_R(R/(A + J), R) \longrightarrow 0.
\]

Now, the cokernel of the first map in this sequence is isomorphic to $R/(B + J)'$.

This follows from the commutative diagram

\[
\begin{array}{cccccc}
\text{Ext}^{g+1}_R(R/(A + J), R) & \longrightarrow & \text{Ext}^{g+1}_R(R/(A + J)', R) & \longrightarrow \\
\downarrow & & \downarrow & & \\
\text{Ext}^{g+1}_R(R/J, R) & \longrightarrow & \text{Ext}^{g+1}_R(R/J, R) & \longrightarrow \\
& & \text{Hom}(R/(A + J)', R/J) & \longrightarrow & (B + J)'/J \\
& & \downarrow & & \\
& & \text{Hom}(R/J, R/J) & \longrightarrow & R/J
\end{array}
\]
The first square of (2) follows from applying $\text{Hom}(-, R)$ to the diagram of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & (A + J)/J & \longrightarrow & R/J & \longrightarrow & R/(A + J) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (A + J)/J & \longrightarrow & R/J & \longrightarrow & R/(A + J)' & \longrightarrow & 0
\end{array}
\]

The second square of (2) follows from [G, Lemma 2]. Finally, the last square follows from Theorem 3.2.5.

As the vertical map on the left of (2) is induced from the natural surjection of $R/J$ onto $R/(A + J)'$, and all the horizontal maps are natural isomorphisms, the cokernel of this map is isomorphic to the cokernel of the natural injection of $(B + J)'/J$ into $R/J$, which is what we wanted to show.

Thus, the exact sequence (1) gives rise to an exact sequence

(3)

\[
0 \rightarrow R/(B + J)' \rightarrow \text{Ext}^{g+1}_R((A + J)/J, R) \rightarrow \text{Ext}^{g+2}_R(R/(A + J), R) \rightarrow 0
\]

Now, the depth lemma applied to the exact sequence of Corollary 3.1.10 shows that $\text{depth } R/B = \text{depth } \text{Ext}^{g+2}_R(R/(A + J), R) + 1$, since we have assumed that $R/B$ is not Cohen–Macaulay, and that $K_{R/B}$ is Cohen–Macaulay. Also, note that since $(A + J)/J$ is Cohen–Macaulay, so too is $\text{Ext}^{g+1}_R((A + J)/J, R)$.

Thus, assume that $R/(A + J)'$ is Cohen–Macaulay. Since $(A + J)'$ is linked to $(B + J)'$ by the Gorenstein ideal $J$, then $R/(B + J)'$ is also Cohen–Macaulay. Hence, the previous remark together with the depth lemma applied to (3) shows that $\text{depth } R/B = \text{depth } \text{Ext}^{g+2}_R(R/(A + J), R) + 1 = \text{depth } R/(B + J)' = d - 1$, so $R/B$ satisfies the depth inequality. Conversely, if $R/B$ satisfies the depth inequality,
this forces depth $\text{Ext}^{g+2}_R(R/(A + J), R) = d - 2$, and hence $R/(B + J)'$ is Cohen–Macaulay, from the depth lemma applied to (3), again. But now the linkage shows that $R/(A + J)'$ is also Cohen–Macaulay (Theorem 3.1.1). □

The next proposition gives a criterion for the depth inequality in terms of the sum of $A$ and $B$, when $A$ and $B$ are geometrically linked:

**Proposition 3.2.7.** Suppose $A$ and $B$ are ideals of height $g$ of a Gorenstein ring $R$, with $A$ a Cohen–Macaulay ideal geometrically linked to $B$ by a generically Gorenstein ideal $I$. Then $B$ satisfies the depth inequality if and only if $A + B$ is a Cohen–Macaulay ideal of height $g + 1$.

**Proof.** First, application of the $\text{Hom}(k, -)$ functor to the short exact sequence

$$0 \longrightarrow R/B \longrightarrow R/A \oplus R/B \longrightarrow R/A \longrightarrow 0,$$

shows that $\text{Ext}^i_R(k, R/B) \cong \text{Ext}^i_R(k, R/A \oplus R/B)$ for all $i < \dim R/A = d$.

Since $A$ and $B$ are geometrically linked by $I$, there is an exact sequence

$$0 \longrightarrow R/I \longrightarrow R/A \oplus R/B \longrightarrow R/(A + B) \longrightarrow 0.$$

Again, applying the functor $\text{Hom}(k, -)$, we obtain

$$\text{Ext}^i_R(k, R/(A + B)) \cong \text{Ext}^i_R(k, R/A \oplus R/B),$$

for $j < d - 1$. Thus we see that

$$\text{Ext}^i_R(k, R/(A + B)) \cong \text{Ext}^i_R(k, R/B), \quad \text{for all } i < d - 1.$$
Now note that since $A$ and $B$ are geometrically linked, they have no common primary components, and hence $\dim R/(A + B) \leq d - 1$. If $R/(A + B)$ is Cohen–Macaulay of dimension $d - 1$, then the above isomorphism shows that $\text{depth } R/B \geq d - 1 = \dim R/B - 1$. Conversely, if this inequality holds, the isomorphism shows $\text{depth } R/(A + B) \geq d - 1 \geq \dim R/(A + B)$; thus equality holds and $R/(A + B)$ is Cohen–Macaulay of dimension $d - 1$. □

Motivation for the above characterization is provided by [PS, Remarque 1.4], where it is shown that when $A$ and $B$ are height $g$ ideals geometrically linked by a complete intersection (or more generally, a Gorenstein ideal) $I$, and if $A$, and hence also $B$, is a Cohen–Macaulay ideal, then in fact $A + B$ is a Gorenstein ideal of height $g + 1$. The next example shows that this fails, in general, for linkage by generically Gorenstein, Cohen–Macaulay ideals.

**Example 3.2.8.** Let $R = k[x, y, z, w](x, y, z, w)$ be the polynomial ring in four variables, localized at the origin. Let $I = (w^2, x^2) \cap (z, y) \cap (w^2, y^2)$. Then $I$ is a height 2, generically Gorenstein ideal. To see that $I$ is Cohen–Macaulay, note that it is generated by the $2 \times 2$ minors of the matrix

\[
\begin{pmatrix}
    z & y & 0 \\
    x^2 y & 0 & w^2
\end{pmatrix}.
\]

Let $A = (z, y)$ and $B = (w^2, x^2) \cap (w^2, y^2)$. Thus $A$ and $B$ are geometrically linked by $I$, and $A$ is Cohen–Macaulay. But it is easily checked, by MACAULAY, for example, that $A + B$ is a height 3 Cohen–Macaulay ideal which is not Gorenstein.

Note, however, that $K_{R/A} \cong R/A \not\cong B/I$. The next result shows that this is a consequence of $A + B$ not being Gorenstein.
**Proposition 3.2.9.** Suppose $A$ and $B$ are height $g$ ideals of the Gorenstein ring $R$, with $A$ Cohen–Macaulay and geometrically linked by a generically Gorenstein, Cohen–Macaulay ideal $I$ to $B$. If $K_{R/A} \cong B/I$, then $A + B$ is a Gorenstein ideal of height $g + 1$.

**Proof.** We have

$$K_{R/A} \cong B/I = B/(A \cap B) \cong (A + B)/A \subseteq R/A.$$  

By Lemma 2.12, since $R/A$ is Cohen–Macaulay, this shows that $(A + B)/A$ is a height 1 Gorenstein ideal of $R/A$. Since $A \subseteq A + B$, this is equivalent to saying that $A + B$ is a height $g + 1$ Gorenstein ideal of $R$, as required. □

**Remark.** We recover the result of [PS, Remarque 1.4], for when $A$ and $B$ are linked by a Gorenstein ideal $I$, then automatically, $K_{R/A} \cong B/I$. For a comprehensive study of this situation, see B. Ulrich’s paper [U3].

### 3.3. On the Generators and Free Resolutions of Linked Ideals

The results in this section arose in an attempt to generalize to the notion of linkage being considered here the following result of Peskine and Szpiro:

**Theorem 3.3.1.** Let $R$ be a Gorenstein local ring, and let $A$ and $B$ be ideals linked by a complete intersection ideal $I$. Suppose $A$ is perfect; that is, $R/A$ is Cohen–Macaulay of finite projective dimension. Let $F$ be a minimal free resolution of $R/A$, $K$ the Koszul complex resolving $R/I$, and let $\alpha : K \rightarrow F$ be a comparison map induced by the inclusion of $I$ into $A$. Let $C$ be the mapping cylinder of the dual map $\alpha^* : F^* \rightarrow K^*$. Then $C$ is a free resolution of $R/B$. 
It is easy to see that if we replace $I$ by a Gorenstein ideal of finite projective dimension, and, correspondingly, $K$ by a minimal free resolution of $R/I$, then the conclusion of Theorem 3.3.1 continues to hold. Essential for this is the functorial isomorphism

$$\text{Ext}^q_R(R/A, R) \cong \text{Hom}_R(R/A, R/I),$$

which follows from [G, Lemma 2].

Our two main results of this section are as follows. First, we wish to see what happens if we take the naive approach, and just work through this mapping cylinder construction, when $A$ and $B$ are linked by a generically Gorenstein, Cohen–Macaulay ideal $I$ of finite projective dimension and $A$ is perfect. We obtain a free resolution of a quotient of canonical modules, and this quotient has the unmixed part of its first Fitting invariant equal to $B$. That is, its presentation matrix has maximal minors which generate an ideal whose unmixed part is $B$. Moreover, such a presentation matrix is easy to come by; it arises from resolutions of $R/A$ and $R/I$, and the comparison map between these. Our second aim in this section is to find an alternative method of constructing a resolution of $R/B$. Although we cannot do this in general, we can give a construction for one easy case.

We first need to recall the notion of a mapping cylinder of a map of complexes. Suppose $(C, d^C)$ and $(D, d^D)$ are complexes, and let $\alpha : C \to D$ be a morphism of complexes. Then the mapping cylinder $C := C(\alpha)$ of $\alpha$ is the complex with component modules $C_n = C_{n-1} \oplus D_n$ and differentials $d_n : C_n \to C_{n-1}$ defined by

$$d_n(x, y) = (d^C_{n-1}(x), \alpha_{n-1}(x) + (-1)^n d^D_n(y)).$$
There is an obvious short exact sequence of complexes

\[ 0 \longrightarrow D \longrightarrow C \longrightarrow C[-1] \longrightarrow 0 \]

and taking the long exact sequence on homology we obtain:

**Lemma 3.3.2.** With the notation as above, if \( C \) and \( D \) are acyclic, then \( H_i(C) = 0 \) for \( i \geq 2 \). Furthermore, if the morphism \( \alpha \) induces an injection \( H_0(C) \to H_0(D) \), then \( H_1(C) = 0 \), i.e., \( C \) is acyclic. \( \Box \)

For future reference, recall that if \( M \) is a finitely generated module over a Noetherian ring \( R \), and if \( p \in \text{Spec} \, R \) is a prime ideal, then there is a functorial isomorphism

\[ \text{Ext}^i_R(M, N)_p \cong \text{Ext}^i_{R_p}(M_p, N_p). \]

We state the next lemma in somewhat more generality than we have used so far in this paper. Specifically, we suppose that \( A \) and \( B \) are ideals linked by the Cohen–Macaulay ideal \( I \); we do not assume \( I \) to be generically Gorenstein.

**Lemma 3.3.3.** Let \( A, B \) and \( I \) be as noted above. Let \( i : A/I \to R/I \) be the canonical inclusion, and let \( M = \text{im} \, i^* \), where \( i^* : \text{Hom}(R/A, K_{R/I}) \to \text{Hom}(R/I, K_{R/I}) \). Then \( \text{ann} \, M = B \).

**Proof.** The map \( i \) fits into an exact sequence

\[ 0 \longrightarrow A/I \overset{i}{\longrightarrow} R/I \overset{\pi}{\longrightarrow} R/A \longrightarrow 0. \]

Applying the functor \( \text{Hom}(\_, K_{R/I}) \), we obtain a long exact sequence

\[ 0 \to \text{Hom}(R/A, K_{R/I}) \overset{\pi^*}{\longrightarrow} \text{Hom}(R/I, K_{R/I}) \overset{i^*}{\longrightarrow} \text{Hom}(A/I, K_{R/I}) \to \ldots. \]
Since clearly $B \subseteq \text{Hom}(A/I, K_{R/I})$, also $B \subseteq \text{ann } M$. For the opposite inclusion, suppose $b \in \text{ann } M$. Let $a \in A$. For each $\phi \in \text{Hom}(R/I, K_{R/I})$, we see that $i^*(b\phi) = bi^*(\phi) = 0$, so $b\phi = \pi^*(\psi)$ for some $\psi$ in $\text{Hom}(R/A, K_{R/I})$. Thus $ab\phi = a\pi^*(\psi) = \pi^*(a\psi) = 0$, since $a \in A = \text{ann } \text{Hom}(R/A, K_{R/I})$. Since $\phi$ was arbitrary, $ab \in \text{ann } \text{Hom}(R/I, K_{R/I}) = I$, and since $a \in A$ was arbitrary, $b \in (I : A) = B$. □

From the elementary properties of the Fitting invariants, the next Corollary is immediate.

**Corollary 3.3.4.** Let $\mathcal{F} := \mathcal{F}_0(M)$ denote the first Fitting invariant of $M$. Then $\text{rad } \mathcal{F} = \text{rad } B$. □

This corollary essentially says that $B$ and $\mathcal{F}$ have the same minimal primes. When we require $I$ to be generically Gorenstein, then they also have the same minimal primary components:

**Proposition 3.3.5.** Suppose $A$ and $B$ are linked by the generically Gorenstein, Cohen–Macaulay ideal $I$. With the notation as above, $\mathcal{F}' = B$.

**Proof.** By Lemma 3.3.3 and the elementary properties of Fitting invariants, we have $\mathcal{F} \subseteq B$. Hence, by Lemma 3.2.3, we need only show that for each minimal prime $p$ of $\mathcal{F}$, $\mathcal{F}_p = B_p$. Note that by Corollary 3.3.4, such a prime $p$ is also minimal over $B$, hence also minimal over $I$.

Now, if we localize the exact sequence

$$0 \longrightarrow \text{Hom}(R/A, K_{R/I}) \overset{\pi^*}{\longrightarrow} \text{Hom}(R/I, K_{R/I}) \longrightarrow M \longrightarrow 0,$$

and use the functorial isomorphism above, together with the functorial identifica-
tions $\text{Hom}(R/A, K_{R/I}) \cong K_{R/A}$ and $\text{Hom}(R/I, K_{R/I}) \cong K_{R/I}$, we obtain the short exact sequence

$$0 \longrightarrow K_{R_p/A_p} \overset{\pi_p^*}{\longrightarrow} K_{R_p/I_p} \longrightarrow M_p \longrightarrow 0.$$ 

But $I$ is generically Gorenstein, so $I_p$ is a Gorenstein ideal. Hence $K_{R_p/I_p} \cong R_p/I_p$, and $K_{R_p/A_p} \cong B_p/I_p$, and $\pi_p^*$ is the inclusion of $B_p/I_p$ into $R_p/I_p$. Hence, $M_p \cong R_p/B_p$, and this has Fitting invariant $\mathcal{F}_p = \mathcal{F}(M_p) = B_p$, which finishes the proof. □

The advantage of this description of $B$ is that the Fitting invariant can be computed quite easily, at least when both $I$ and $A$ are perfect ideals. Indeed, let $F$ and $G$ be minimal finite free resolutions of $R/I$ and $R/A$, respectively. Then as in [St, p. 57] or [G, Lemma 1], the dual complexes $F^*$ and $G^*$ are free resolutions of $K_{R/I}$ and $K_{R/A}$, respectively. Moreover, if $\alpha : F \to G$ is the comparison map covering the surjection $R/I \to R/A$, then $\alpha^*$ induces the injection $\pi^* : K_{R/A} \to K_{R/I}$ which appears in the proof of Proposition 3.3.5. By Lemma 3.3.2, then, the mapping cylinder of $\alpha^*$ is a free resolution of $M$. In particular, the first map in this resolution is a presentation matrix for $M$, and the Fitting invariant $\mathcal{F}$ is the ideal generated by the maximal minors of this matrix.

As an immediate corollary, when the Fitting invariant $\mathcal{F}$ is unmixed, we have an upper bound for the number of generators of $B$. For this, recall that for a Cohen–Macaulay ideal $H$, $r(R/H) = \mu(K_{R/H})$, where $r(R/H)$ denotes the Cohen–Macaulay type of $R/H$, [HK, Kor 6.11].
Proposition 3.3.6. With the notation as above, suppose \( \mathcal{F} \) is unmixed. Let \( t := \beta_g = \mu(K_{R/I}) = \tau(R/I) \) and \( \beta_{g-1} \) be the last two non-zero Betti numbers of \( R/I \), and let \( s = \tau(R/A) = \mu(K_{R/A}) \). Then \( \mu(B) \leq (\beta_{g-1} + s) \).

Proof. The presentation matrix for \( M \) obtained as the previous paragraph has size \((\beta_{g-1} + s) \times t \). Since \( \mathcal{F} \) is unmixed, \( \mathcal{F} = B \), and this is generated by the maximal minors of the presentation matrix of \( M \). There are exactly \((\beta_{g-1} + s) \) of these. \( \square \)

Our next objective is to construct, in certain cases, a resolution of \( R/B \) in terms of resolutions of \( R/I \) and \( R/A \). We need to make the following assumption: write \( K_{R/I} \cong J/I \), and assume that \( B \subseteq J \). This is equivalent, by Lemma 3.1.5, to saying that \( A \) contains a non-zero-divisor for \( R/J \). Note that in this case, by the remarks preceding Example 3.2.1, \( B \) satisfies the depth inequality of Section 2; that is, \( \text{depth} R/B \geq \dim R/B - 1 \). Thus, if \( \text{height} B = g \), the best resolution we can hope for will have length \( g + 1 \). This is indeed what we obtain.

Now, let \( F \to R/I \to 0 \) be a minimal free resolution of \( R/I \). Since \( R/I \) is Cohen–Macaulay, the dual complex \( F^* \) is a resolution of \( K_{R/I} \cong J/I \). Thus, the mapping cylinder construction applied to the diagram

\[
\begin{array}{ccc}
F^* & \longrightarrow & J/I \\
\downarrow \alpha & & \downarrow \\
F & \longrightarrow & R/I \\
\end{array}
\]

where \( \alpha \) is a comparison map induced by the inclusion of \( J/I \) into \( R/I \), produces a resolution of \( R/J \). Note that this resolution has the "right" length: \( \text{height} J = \text{height} I + 1 \), and the resolution produced in the above construction has height \( I + 1 \) terms. On the other hand, if \( \alpha \) is not a minimal lifting of the embedding \( J/I \to R/I \),
then the resolution will not be a minimal resolution of \( R/J \). Also, note that in this case, the mapping cylinder construction is a kind of "symmetrization" process; that is, if \( C \) is the mapping cylinder of \( \alpha \), then \( C \cong C^* \).

Next, let \( G \to R/A \to 0 \) be a minimal free resolution of \( R/A \). Again, since \( R/A \) is Cohen–Macaulay, the dual complex \( G^* \) is a free resolution of \( K_{R/A} \). However, \( K_{R/A} \cong \text{Hom}_R(R/A,K_{R/I}) = ((I : A) \cap J)/I = (B \cap J)/I = B/I \), the last equality following from our assumption that \( B \subseteq J \). Also, from the short exact sequence

\[
0 \longrightarrow A/I \longrightarrow R/I \longrightarrow R/A \longrightarrow 0
\]

and the fact that \( R/A \) is Cohen–Macaulay, by dualizing, we obtain another short exact sequence

\[
(*) \quad 0 \longrightarrow K_{R/A} \longrightarrow K_{R/I} \longrightarrow K_{A/I} \longrightarrow 0.
\]

Again, the mapping cylinder construction now produces a free resolution of \( K_{A/I} \) from the two complexes

\[
G^* \to K_{R/A} \to 0
\]

\[
F^* \to K_{R/I} \to 0
\]

But using the isomorphisms \( K_{R/I} \cong J/I \) and \( K_{R/A} \cong B/I \), the sequence \((*)\) shows that \( K_{A/I} \cong J/B \).

Hence, we have constructed resolutions of \( J/B \) and \( R/J \). Finally, the horseshoe construction applied to the short exact sequence

\[
0 \to J/B \to R/B \to R/J \to 0
\]
produces a resolution of $R/B$. We note that under our assumption that $B \subseteq J$, we know that depth $R/B \geq \dim R/B - 1$, and that the resolution of $R/B$ produced has exactly height $B + 1$ terms. So in general, it is the shortest possible. On the other hand, it is definitely not a minimal resolution, as $R/B$ is cyclic, but the rightmost free module has rank equal to $\mu(J/B) + \mu(R/J) > 1$. 
CHAPTER IV

ON THE CANONICAL MODULE OF A DETERMINANTAL RING

Let $R$ be a local Cohen–Macaulay ring, which is Gorenstein at each of its minimal primes. Then Lemma 2.13 shows that the canonical module of $R$, when it exists, is isomorphic to an ideal of $R$; however, an explicit set of generators of such an ideal is usually difficult to give. In this chapter, our interest is to compute an explicit set of generators for the canonical module of the class of determinantal rings which are generically Gorenstein; that is, rings of the form $R/I$, where $I$ is an ideal generated by the maximal minors of a matrix over $R$, and $I$ is a Gorenstein ideal at each of its minimal primes. This is accomplished in Theorem 4.9. To do this, we needed to prove an extended version of the prime avoidance lemma, which shows that prime avoidance holds for minors of matrices of arbitrary size. This result is interesting in its own right; we have set it aside as the first main result of this chapter (Theorem 4.1).

We note that most of the results of this chapter are not new, and are probably well-known to experts. However, the proofs given here are very concrete, and useful computationally. In particular, the results of this chapter allow for the construction of examples which complement, in a particularly nice class of ideals, the theory of the previous chapter. Indeed, it was the need for examples in the development of that theory that led to the computations of this chapter.
We begin with the prime avoidance result mentioned above. The statement of Theorem 4.1 also follows from the theory of basic elements. See [B2, Hilfssatz 2.2]. Because in our later computations we will need to know which submatrix of $M$ has the property described by Theorem 4.1, it seems useful to have this explicit proof.

**Theorem 4.1.** Let $M$ be an $m \times n$ matrix over a commutative ring $R$, and let $p_1, \ldots, p_s$ be a finite set of prime ideals. Let $0 < t \leq \min\{m, n\}$ be a fixed integer and suppose that the property (*) holds: for each prime $p_i$, there exists a $t \times t$ minor of $M$ not contained in $p_i$. Then after performing elementary row and column operations on $M$, there exists a $t \times t$ minor not contained in any $p_i$.

**Remark.** Thus the usual prime avoidance lemma is the case $m = t = 1$.

Denote by $(i|j) = (i_1, \ldots, i_t|j_1, \ldots, j_t)$ the $t \times t$ minor of $M$ involving the rows $i_1 < \cdots < i_t$ and the columns $j_1 < \cdots < j_t$. To make the proof of the Theorem smoother, in the following lemma we isolate and leave to the reader a straightforward, if tedious, calculation.

**Lemma 4.2.** Let $r$ be a ring element, and choose $t - l$ fixed columns $\alpha_{l+1} < \cdots < \alpha_t$. Let $M'$ be the matrix obtained from $M$ by adding $r$ times the column $\alpha_k$ to the column $k$, for each $k = l + 1, \ldots, t$. For a minor $(i|j)$ of $M$, denote by $(i|j)'$ the corresponding minor of $M'$. Then

$$(i|1, \ldots, t)' = \sum (-1)^{g(\beta)} f(\beta) (i|1, \ldots, l, \beta_{l+1}, \ldots, \beta_t),$$

the sum ranging over all $(t - l)$-tuples $(\beta_{l+1}, \ldots, \beta_t)$ where $\beta_k = k$ or $\beta_k = \alpha_k$, and where $f(\beta)$ is the number of $\beta_k$ with $\beta_k = \alpha_k$, and $g(\beta)$ is a sign function. A similar result holds if the rows are transformed instead of the columns.
Proof of the Theorem. We may clearly assume that the primes $p_i$ are irredundant, that is, $p_i \not\subseteq \cap_{i \neq j} p_j$. By permuting the rows and columns, and by renumbering the prime ideals, we may assume that the minor $m := (1, \ldots, t|1, \ldots, t) \not\subseteq p_1, \ldots, p_k$, but $m \in p_{k+1}, \ldots, p_s$, for some $k = 1, \ldots, s$. If $k = s$, then $m$ is the required minor, and we are done. Thus, we may suppose that $k < s$. We will show that by performing elementary row and column operations on $M$, we may transform $m$ so that $m \not\subseteq p_1, \ldots, p_{k+1}$, and that these transformations do not change the property (*) of $M$. Thus, we will be done by induction on $k$.

By hypothesis, there is a minor $(i|j)$ not contained in $p_{k+1}$, Choose this minor so that it shares the maximal number of rows with $m$. That is, if we write $(i|j) = (i_1, \ldots, i_t|j_1, \ldots, j_t)$, with $i_1 = 1, \ldots, i_t = l, i_{t+1} \neq l + 1, \ldots, i_t \neq t$ and $j_1 = 1, \ldots, j_{l'} = l', j_{l'+1} \neq l' + 1, \ldots, j_t \neq t$, then $l$ is the maximum possible among all minors not in $p_{k+1}$. Also, choose $r \in \cap_{j \neq k+1} p_j \setminus p_{k+1}$.

First, perform the following elementary column operations on $M$: add $r$ times the column $j_h$ to the column $h$, for each $h = l'+1, \ldots, t$. In the new matrix, by Lemma 4.2 and the choice of $r$, we see that the minor $(i_1, \ldots, i_t|1, \ldots, t)$ is not in $p_{k+1}$. Also, by maximality of $l$ and the choice of $r$, we still have $(1, \ldots, t|1, \ldots, t) \not\subseteq p_1, \ldots, p_k$, but in $p_{k+1}, \ldots, p_s$. Finally, again by the choice of $r$ and Lemma 4.2, if a minor $n$ of the original matrix $M$ was not in some $p_i$, then the corresponding minor $n'$ in the new matrix is still not in $p_i$. Thus we have reduced to the case that the minor $(i|j)$ comes from the first $t$ columns.

Next, perform the following elementary row operations: add $r$ times the row $i_h$ to the row $h$, for each $h = l + 1, \ldots, t$. Now, in the new matrix, we have
by the choice of $r$ and Lemma 4.2, that $(1,\ldots , t|1,\ldots , t) \notin p_1,\ldots , p_{k+1}$, but in $p_{k+2},\ldots , p_s$. Also, if a minor $n$ of the original matrix $M$ were not in some $p_i$, then the corresponding minor $n'$ of the new matrix is still not in $p_i$. Thus the property $(\ast)$ of $M$ is unchanged, and we have produced a minor of $M$ not contained in $k+1$ of the given primes. This completes the proof. □

Throughout the remainder of this section, except where noted, we will assume $R$ to be local Gorenstein. Also, by a determinantal ideal, we mean an ideal $I$ generated by the maximal minors of an $m \times n$ matrix $M$, where $m \leq n$, and height $I = m-n+1$, the maximum possible.

The following Theorem is well-known; see [St, p. 57], or [G, Lemma 1].

**Theorem 4.3.** Let $R$ be a local Gorenstein ring of dimension $n$, and $I$ a perfect ideal of $R$ of height $d$. Let $F \to R/I \to 0$ be a minimal free resolution of $R/I$. Then the dual complex $F^*$ is a minimal free resolution of the canonical module $K_{R/I}$. □

Now, we recall the construction of the Eagon-Northcott complex in [EN]. Let $R$ be any ring, and $M = (a_{ij})$ an $m \times n$ matrix, $m \leq n$, with entries in $R$. Each row of $M$ determines a Koszul differential $\Delta_i : \wedge R^n \to \wedge R^n$ given by

$$\Delta_i(e_{j_1} \wedge \ldots \wedge e_{j_k}) = \sum_{s=1}^{k} (-1)^{s+1} a_{is} e_{j_1} \wedge \ldots \wedge \widehat{e_{j_s}} \wedge \ldots \wedge e_{j_k},$$

where $e_1,\ldots , e_n$ is the canonical basis for $R^n$. Next, let $X_1,\ldots , X_m$ be new indeterminates over $R$ and for each $t$, let $\Phi_t$ be the $R$-submodule of $R[X_1,\ldots , X_m]$ of homogeneous elements of degree $t$. Then the Eagon-Northcott complex $A := A^M$
associated to $M$ has component modules

$$A_0 = R$$

$$A_{t+1} = \bigwedge^{m+t} R^n \otimes \Phi_t$$

for $t = 0, \ldots, n - m$,

and differential maps $d_{t+1} : A_{t+1} \to A_t$, for $t = 1, \ldots, n - m$, defined by

$$d_{t+1}(e_{j_1} \wedge \ldots \wedge e_{j_{m+t}} \otimes X_1^{\nu_1} \ldots X_m^{\nu_m}) =$$

$$\sum_{i=1}^{n} (-1)^{i+1} \Delta_i (e_{j_1} \wedge \ldots \wedge e_{j_{m+t}}) \otimes X_1^{\nu_1} \ldots X_i^{\nu_i-1} \ldots X_m^{\nu_m},$$

where $\nu_1 + \ldots + \nu_m = t$, and where we sum over those indices $i$ for which $\nu_i > 0$. The differential $d_1 : A_1 \to A_0$ is given by $d_1(e_{j_1} \wedge \ldots \wedge e_{j_m}) = \det(M')$, where $M'$ is the $m \times m$ submatrix of $M$ consisting of the columns $j_1, \ldots, j_m$. It is straightforward to check that $A$ is a complex. Also, if we represent the map $d_i$ by a matrix, then each entry of this matrix is either zero, or an entry from $M$. In particular, if $R$ is local, and $M$ has no entries which are units, then $A$ is a minimal complex.

**Theorem 4.4.** [EN] If the maximal minors of $M$ generate an ideal $I$ of maximal height, i.e., of height $n - m + 1$, then the Eagon-Northcott complex $A := A^M$ associated to $M$ is acyclic and hence a free resolution of $R/I$. □

Consequently, $I$ is a Cohen-Macaulay ideal, and the dual complex $A^*$ gives a free resolution of $K_{R/I}$.

Using the Eagon-Northcott complex, we give a characterization of the determinantal ideals $I$ which are Gorenstein. Recall that for a Cohen-Macaulay ideal $J$, the Cohen-Macaulay type $\tau(R/J)$ of $R/J$ is the rank of the last module in a minimal free resolution of $R/J$. Moreover, $R/J$ is Gorenstein if and only if $\tau(R/J) = 1$. 
Lemma 4.5. Let $R$ be a Gorenstein local ring, and $I$ a determinantal ideal of $R$. Then $R/I$ is Gorenstein if and only if $I$ is a complete intersection.

Proof. The sufficiency is clear. For necessity, suppose $I$ is determinantal for the $m \times n$ matrix $M$. Thus, in particular, height $I = n - m + 1$. Furthermore, we may suppose that no entry of $M$ is a unit in $R$. Thus, the Eagon-Northcott complex is a minimal free resolution of $R/I$, and the rank of the last module is $r(R/I) = \binom{n-1}{m-1}$. Hence, $r(R/I) = 1$ if and only if either $m = n$, in which case $I$ is principal of height 1, or $m = 1$, and so $I$ is generated by $n$ elements, and height $I = n$. In both cases $I$ is a complete intersection. □

Proposition 4.6. Let $R$ be Gorenstein local, and $I$ determinantal for the $m \times n$ matrix $M$. Suppose $m < n$. Then the following are equivalent:

1. $I_p$ is a Gorenstein ideal of $R_p$ for each minimal prime $p$ of $I$.
2. $I_p$ is a complete intersection for each minimal prime $p$ of $I$.
3. For each minimal prime $p$ of $I$, there exists an $m - 1 \times m - 1$ submatrix $N$ of $M$ with $\det N \not\in p$.

Proof. Since $I_p$ is determinantal for the matrix $M_p$ consisting of the canonical images of the entries of $M$, the equivalence of (1) and (2) is a consequence of Lemma 4.5.

Now, suppose that there exists a minimal prime $p$ for which every $m - 1 \times m - 1$ submatrix of $M$ has determinant contained in $p$. Let $k$ be the largest integer for which there exists a $k \times k$ submatrix $N$ of $M$ with $\det N \not\in p$. Thus, $k < m - 1$,
and, after localizing at \( p \),

\[
\begin{pmatrix}
M_0 & * \\
* & N
\end{pmatrix}
\]

for some \( m - k \times n - k \) matrix \( M_0 \). But \( M_0 \) cannot contain any units of \( R_p \), for otherwise we could adjoin more rows and columns to \( N \) and still have a \( \det N \) a unit, which would contradict maximality of \( k \). Now, by elementary row and column operations on \( M \), we see that \( I_p \) is determinantal for the matrix \( M_0 \), and hence \( \tau(R_p/I_p) = \binom{n-k-1}{m-k-1} \). Since \( m - k - 1 > 0 \), \( \tau(R_p/I_p) > 1 \), contradicting that \( I_p \) is a Gorenstein ideal. This shows (1) implies (3).

Now suppose (3) holds. Let \( p \) be a minimal prime of \( I \). Then \( I_p \) is determinantal for the matrix

\[
\begin{pmatrix}
M_0 & * \\
* & N
\end{pmatrix}
\]

where \( M_0 \) is a \( 1 \times n - m + 1 \) matrix, and \( \det N \) is a unit in \( R_p \). Thus \( I_p \) is also determinantal for \( M_0 \), hence is generated by \( n - m + 1 = \text{height } I_p \) elements. This shows that \( I_p \) is a complete intersection. \( \square \)

Remark. The results of Proposition 4.6 were essentially obtained by J. Eagon in [E], where he worked with a matrix of indeterminates.

Next, we make some remarks about fractional ideals of a ring \( R \). Let \( Q(R) \) be the total ring of fractions of \( R \); that is, \( Q(R) \) is \( R \) localized at the set of non-zero-divisors of \( R \). Then a fractional ideal \( I \) of \( R \) is an \( R \)-submodule of \( Q(R) \) such that there exists a non-zero-divisor \( a \in Q(R) \) with \( aI \subseteq R \). Note that the map \( I \to aI \) given by \( x \mapsto ax \) is an \( R \)-module isomorphism, and \( aI \) is an ordinary ideal of \( R \). Furthermore, each ordinary ideal of \( R \) is a fractional ideal. The following lemmas
seem to be well-known, at least in the case where \( R \) is an integral domain; however, we lack a reference when \( R \) is not a domain, and for the sake of completeness, supply proofs.

**Lemma 4.7.** Suppose \( I \) is a fractional ideal of \( R \), and \( \varphi : I \to R \) is an \( R \)-module map. Then for each \( a, b \in I \), \( a\varphi(b) = b\varphi(a) \).

*Proof.* Write \( a = r/s \), \( b = u/v \) where \( r, s, u, v \in R \), and \( u, v \) are non-zero-divisors. Then \( sa = r \) and \( vb = u \); hence \( sab = rb \in I \) and \( vab = ua \in I \). Thus also \( s\varphi(vab) = \varphi(svab) = v\varphi(sab) \). But then \( a\varphi(b) = sa\varphi(b)/s = \varphi(sab)/s = \varphi(vab)/v = v\varphi(a)/v = b\varphi(a) \), as required. \( \Box \)

**Lemma 4.8.** Suppose \( I \) and \( J \) are fractional ideals of \( R \), with \( J \) containing a non-zero-divisor of \( R \), and \( \varphi : I \to J \) is a surjective map. Then \( \varphi \) is injective.

*Proof.* Since \( J \) is a fractional ideal of \( R \), there is a non-zero-divisor \( \alpha \) in \( \mathbb{Q}(R) \) such that \( \alpha J \subseteq R \). By considering \( \alpha J \) in place of \( J \), we may assume that \( J \subseteq R \), and thus \( \varphi : I \to R \) with image \( J \). Let \( r \in J \) be a non-zero-divisor of \( R \); then since \( \varphi \) is surjective there is \( a \in I \) with \( \varphi(a) = r \). Now, suppose \( b \in I \) satisfies \( \varphi(b) = 0 \). Then, by Lemma 4.7, we have

\[ 0 = a\varphi(b) = b\varphi(a) = br. \]

Writing \( b = u/v \), with \( u, v \) in \( R \) and \( v \) a non-zero-divisor, we obtain \( ru = 0 \) in \( R \). But \( r \) a non-zero-divisor in \( R \) implies that \( u \) and hence also \( b \) is zero. \( \Box \)
With the preliminary results established above, we can prove the main result of this chapter by imitating the proof given in [Y].

**Theorem 4.9.** Let $I$ be a determinantal ideal of a local Gorenstein ring $R$, generated by the maximal minors of the $m \times n$ matrix $M$, $m < n$. Suppose $\dim R/I \geq 1$, and $I_p$ is a Gorenstein ideal in $R_p$ for every minimal prime $p$ of $I$. Then, perhaps after elementary row and column operations on $M$, there is an $m \times m - 1$ submatrix $N$ of $M$ such that the following statements hold:

1. The ideal $J/I$ of $R/I$ generated by the images of the maximal minors of $N$ has height at least 1.

2. The canonical module $K_{R/I}$ of $R/I$ is isomorphic to $J^{n-m}/I$.

**Proof.** The first statement follows from Theorem 4.1 applied to the $m - 1 \times m - 1$ minors of $M$ and the set of minimal primes of $I$. For by Proposition 4.6, for each of these primes $p$, there is such a minor not contained in $p$. Then just take $N$ to be the $m \times m - 1$ submatrix of $M$ containing the $m - 1 \times m - 1$ minor of $M$ produced by Theorem 4.1.

By permuting the columns of $M$, we may assume that the submatrix $N$ is obtained from the first $m - 1$ columns of $M$. Let $A := A^M$ be the Eagon-Northcott complex associated to $M$. Thus the dual complex $A^*$ is a free resolution of $K_{R/I}$. Hence, the following sequence is exact:

$$A_{m-n}^* \xrightarrow{d_{n-m+1}^*} A_{n-m+1}^* \longrightarrow K_{R/I} \longrightarrow 0$$

Next, note that the sets

$$\{ \alpha(\nu_1, \ldots, \nu_m) := e_1 \wedge \ldots \wedge e_n \otimes X_1^{\nu_1} \cdots X_m^{\nu_m} : \nu_1 + \ldots + \nu_m = n - m \}$$
\{ \beta(j; \mu_1, \ldots, \mu_m) := e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_n \otimes X_{1}^{\mu_1} \cdots X_{m}^{\mu_m} : \\
\mu_1 + \ldots + \mu_m = n - m - 1 \}

are free bases of \( A_{n-m} \) and \( A_{n-m-1} \), respectively. Hence, if we define

\[ \alpha^*(\nu_1, \ldots, \nu_m)(\alpha(\nu'_1, \ldots, \nu'_m)) = \delta_{\nu_1 \nu'_1} \cdots \delta_{\nu_m \nu'_m}, \]

\[ \beta^*(j; \mu_1, \ldots, \mu_m)(\beta(k; \mu'_1, \ldots, \mu'_m)) = \delta_{jk} \delta_{\mu_1 \mu'_1} \cdots \delta_{\mu_m \mu'_m}, \]

then the sets \{ \alpha^*(\nu_1, \ldots, \nu_m) : \nu_1 + \ldots + \nu_m = n - m \} and \{ \beta^*(j; \mu_1, \ldots, \mu_m) : \mu_1 + \ldots + \mu_m = n - m - 1 \} are free bases for the duals \( A^*_{n-m} \) and \( A^*_{n-m-1} \), respectively. Moreover, we have

\[ d^*_{n-m+1}(\beta^*(j; \mu_1, \ldots, \mu_m)) = (-1)^{j+1} \sum_{\nu=1}^{m} a_{sj} \alpha^*(\mu_1, \ldots, \mu_s + 1, \ldots, \mu_m). \]

Now, for each \( i = 1, \ldots, m \), let \( b_i \) be \((-1)^{i+1}\) times the residue class modulo \( I \) of the minor of \( N \) obtained by deleting the \( i \)-th row of \( N \). Thus we have \( \sum_{i=1}^{m} b_i a_{ij} = 0 \), in \( R/I \), for each \( j = 1, \ldots, n \). Also, the ideal \( J^{n-m} \) is generated by the elements \( b_1^{\nu_1} \cdots b_m^{\nu_m} \), where \( \nu_1 + \ldots + \nu_m = n - m \). So if we define \( \pi : A^*_{n-m} \rightarrow J^{n-m} \) by \( \pi(\alpha^*(\nu_1, \ldots, \nu_m)) = b_1^{\nu_1} \cdots b_m^{\nu_m} \), then \( \pi \) is a surjective morphism. Moreover, we have

\[ \pi(d^*_{n-m+1}(\beta^*(j; \mu_1, \ldots, \mu_m))) = \pi((-1)^{j+1} \sum_{\nu=1}^{m} a_{sj} \alpha^*(\mu_1, \ldots, \mu_s + 1, \ldots, \mu_m)) \]

\[ = (-1)^{j+1} \sum_{\nu=1}^{m} a_{sj} b_1^{\mu_1} \cdots b_1^{\mu_{s+1}} \cdots b_m^{\mu_m} \]

\[ = (-1)^{j+1} b_1^{\mu_1} \cdots b_m^{\mu_m} \sum_{\nu=1}^{m} a_{sj} b_s \]

\[ = 0. \]
Thus in the commutative diagram

\[
\begin{array}{ccc}
A_{n-m} & \xrightarrow{d_{n-m+1}} & A_{n-m+1} \\
| & \downarrow{id} & | \\
A_{n-m} & \xrightarrow{d_{n-m+1}} & A_{n-m+1} \\
\end{array}
\rightarrow \begin{array}{c}
K_{R/I} \\
\downarrow{id} \\
J^{n-m} \\
\end{array}
\rightarrow 0
\]

the top row is exact, the bottom row is a complex, and \( \pi \) surjective. So there is an induced surjection \( \psi : K_{R/I} \to J^{n-m} \). However, by Lemma 2.13, \( K_{R/I} \) is a fractional ideal of \( R/I \), and since \( J \) has height at least 1 in the Cohen-Macaulay ring \( R/I \), it contains a non-zero-divisor. Thus Lemma 4.8 shows that \( \psi \) must be an isomorphism. □

Remarks. Some remarks concerning this result are in order. First, in the case that \( R = k[X] \) is a graded ring on the indeterminates \( X_{ij} \) in the matrix \( X \), W. Bruns [B1] and Y. Yoshino [Y] have obtained results similar to ours. Indeed, with the necessary modifications, our method of proof is taken from [Y]. Next, since \( K_{R/I} \cong J^{n-m}/I \), then by Lemma 2.13, in fact height \( J/I = 1 \) in \( R/I \), whereas we could a priori assume only that height \( J/I \) was 1 or 2. Finally, again by Lemma 2.13, the theorem provides a concrete way of constructing Gorenstein ideals in a local ring, since the pre-image in \( R \) of \( J^{n-m} \) will be a Gorenstein ideal generated by \( I \) and the maximal minors of \( N \). It would be interesting to see to what extent these ideals encompass all the Gorenstein ideals.

Finally, we show that for a given choice of generating matrix for \( I \), elementary row and column operations may be necessary.

Example 4.10. Put

\[
M = \begin{pmatrix}
y & 0 & z \\
z - y & x - w & 0
\end{pmatrix}
\]
over the polynomial ring \( k[x, y, z, w] \) localized at the origin. Then \( I = (yx - yw, z^2 - yz, zx - zw) = (x - w, y - z) \cap (y, z) \cap (x - w, z) \) is a prime decomposition of \( I \), the ideal of maximal minors of \( M \). Since each primary component of \( I \) is a complete intersection, \( I \) is Gorenstein at each of its minimal primes. But note that the ideal generated by any single column is contained in some minimal prime of \( I \), hence cannot extend to a height 1 ideal in \( R/I \).

One possible way to transform \( M \) so as to conform to Theorem 4.1 is

\[
M' = \begin{pmatrix}
(x - w)(z - y + x - w) + y(x - w)^2 & z \\
z - y + x - w & x - w
\end{pmatrix}.
\]

Thus, Theorem 4.9 shows that \( K_{R/I} \cong J/I \), where \( J \) is generated modulo \( I \) by the elements in the first column of \( M' \).
Sequential conditions on a set of generators for an ideal $I$ in a commutative ring $R$ with identity have long proven useful for studying the properties of $I$. Regular sequences are perhaps the best, and most well-known, examples of such conditions. More recently, C. Huneke introduced the notion of a $d$-sequence, [Hun7], and its partially ordered companion, weak $d$-sequence, [Hun4]. These sequences have been shown to have many nice properties; for instance, $d$-sequences have linear relation type, [Hun3]; the asymptotic value of depth $R/I^n$, and the symbolic powers $I^{(n)}$, are both well-behaved when $I$ is generated by a weak $d$-sequence, [Hun4] and [Hun6]. Quadratic sequences were introduced by K. N. Raghavan in [R3] as a generalization and simplification of weak $d$-sequences. He proved that many of the important properties of weak $d$-sequences are shared by quadratic sequences. The simplification made even more clear the similarity of quadratic sequences with linearly ordered $d$-sequences.

In this chapter, our primary concern is to explore this connection between $d$-sequences and quadratic sequences. Our main philosophy is to find conditions which force elements from one type of sequence to form the other type. Among our main results are the facts that the monomials consisting of elements from a $d$-sequence form a quadratic sequence in a natural partial order, and that systems of parameters
in a local ring which are quadratic sequences are in fact $d$-sequences in some linear order. We draw some interesting corollaries from these facts, including that all powers of an ideal generated by a $d$-sequence have relation type at most 2.

We begin with definitions and some preliminary results. Unless noted otherwise, $R$ is a commutative ring with identity.

**Definition 5.1.1.** A sequence $x_1, \ldots, x_s$ of elements of $R$ is a $d$-sequence if for each $i = 0, \ldots, s - 1$ we have

\[ ((x_1, \ldots, x_i) : x_{i+1}) \cap I = (x_1, \ldots, x_i), \]

where $I = (x_1, \ldots, x_s)$.

Before we define quadratic sequences, we fix some notation which will remain in effect for the rest of this paper. Let $(\Lambda, \leq)$ be a finite partially ordered set, or poset for short. A subset $\Sigma$ of $\Lambda$ is called a poset ideal if $\alpha \in \Sigma$ and $\beta \leq \alpha$ implies $\beta \in \Sigma$. An element $\lambda \in \Lambda$ is said to be just above $\Sigma$ if $\alpha < \lambda$ implies $\alpha \in \Sigma$. If $\{ x_\lambda : \lambda \in \Lambda \}$ is a set of elements indexed by $\Lambda$, we put $X := (x_\lambda : \lambda \in \Lambda)$, and for each ideal $\Sigma \subseteq \Lambda$, $X_\Sigma = (x_\sigma : \sigma \in \Sigma)$.

**Definition 5.1.2.** Let $\Lambda$ be a poset. A set $\{ x_\lambda : \lambda \in \Lambda \}$ of elements indexed by $\Lambda$ is a quadratic sequence if for each ideal $\Sigma$ of $\Lambda$ and each element $\lambda \in \Lambda$ just above $\Sigma$, there exists an ideal $\Theta := \Theta_{\Sigma, \lambda}$ of $\Lambda$ such that

1. $X_\Sigma : x_\lambda \cap X \subseteq X_\Theta$
2. $x_\lambda X_\Theta \subseteq X_\Sigma X$.

There are many natural examples of these sequences. For $d$-sequences, see [Hun7]; for quadratic sequences, see [Hun4] and [R3, Sect. 10]. Note, of course, that $d$-
sequences are linearly ordered examples of quadratic sequences. Besides this, we will be content to note that the widest class of quadratic sequences arises from straightening closed ideals in algebras with straightening law. See [BV] for the relevant definitions. One of the most important examples is the following. Let $k$ be field, and $X$ an $m \times n$ matrix of indeterminates over $k$, with $m \leq n$. Let $R = k[X]$. Then the maximal minors of $X$ are partially ordered in a natural manner: denote the maximal minor involving columns $j_1, \ldots, j_m$ as $(j_1, \ldots, j_m)$, where $j_1 < j_2 \ldots < j_m$. Then $(j_1, \ldots, j_m) \leq (i_1, \ldots, i_m)$ if and only if $j_1 \leq i_1, \ldots, j_m \leq i_m$. With this order, the maximal minors form a quadratic sequence. More precisely, for an ideal $\Sigma$ of the set of maximal minors, and an element $\lambda$ just above $\Sigma$, the ideal required by Definition 5.1.2 is $\Theta_{\Sigma, \lambda} = \{ \alpha : \alpha \not\geq \lambda \}$. This follows from the theory of algebras with straightening law, [Hun4, Prop. 1.3 and 1.19]. We further note that when $X$ is an $m \times m + 1$ matrix, then the maximal minors form a $d$-sequence in any linear order, [Hun7, Prop 1.1].

For ease of reference, we state some easy results about these sequences.

**Lemma 5.1.3.** [Hun7] If $x_1, \ldots, x_s$ is a $d$-sequence in $R$, then $\overline{x_{k+1}}, \ldots, \overline{x_s}$ form a $d$-sequence in the ring $R/(x_1, \ldots, x_k)$. □

**Lemma 5.1.4.** [R3, Remark 9.4] If $\{ x_\lambda : \lambda \in \Lambda \}$ is a quadratic sequence and $\Psi$ is an ideal of $\Lambda$, then in $R/X_\Psi$, the images $\{ \overline{x_\lambda} : \lambda \in \Lambda \setminus \Psi \}$ form a quadratic sequence. Moreover, if $\Sigma$ is an ideal of $\Lambda \setminus \Psi$ and $\lambda \in \Lambda \setminus \Psi$ is just above $\Sigma$, then $\Theta_{\Sigma, \lambda} = \Theta_{\Sigma \cup \Psi, \lambda \setminus \Psi}$. □
We also have a partial converse to 5.1.3. To make its statement easier, we say that 
\( x_1, \ldots, x_s \in R \) is a \( d \)-sequence with respect to the ideal \( J \), where \( (x_1, \ldots, x_s) \subseteq J \), if 
\( ((x_1, \ldots, x_{i-1} : x_i) \cap J = (x_1, \ldots, x_{i-1}) \), for each \( i = 0, \ldots, s \).

**Lemma 5.1.5.** Suppose \( x_1, \ldots, x_s \in R \) with \( x_1, \ldots, x_k \) a \( d \)-sequence with respect to the ideal \( (x_1, \ldots, x_s) \), such that the images \( \overline{x_{k+1}}, \ldots, \overline{x_s} \) form a \( d \)-sequence in 
\( R/(x_1, \ldots, x_k) \). Then \( x_1, \ldots, x_s \) is a \( d \)-sequence in \( R \).

**Proof.** We must show that 
\( ((x_1, \ldots, x_i) : x_{i+1}) \cap (x_1, \ldots, x_s) = (x_1, \ldots, x_i) \), for each \( i = 0, \ldots, s - 1 \). For \( i = 0, \ldots, k - 1 \) this is true by hypothesis, so suppose 
\( i \geq k \). If \( r \in ((x_1, \ldots, x_i) : x_{i+1}) \cap (x_1, \ldots, x_s) \), then in \( R/(x_1, \ldots, x_k) \),

\( r \in ((\overline{x_{k+1}}, \ldots, \overline{x_i}) : \overline{x_{i+1}}) \cap (\overline{x_{k+1}}, \ldots, \overline{x_s}) = (\overline{x_{k+1}}, \ldots, \overline{x_i}) \).

Thus, lifting back to \( R \), we have \( r \in (x_1, \ldots, x_i) \), as required. \( \square \)

**Remark.** We will most often use Lemma 5.1.5 for the case \( k = 1 \).

For future reference, we make some remarks concerning the relation type of an ideal. Let \( I = (a_1, \ldots, a_n) \) be an ideal in the ring \( R \). The Rees algebra associated to \( I \) is the subalgebra \( \mathcal{R}(I) := R[It] = R[a_1t, \ldots, a_nt] \) of \( R[t] \), where \( t \) is an indeterminate. If \( X_1, \ldots, X_n \) are new indeterminates, then there is a canonical surjection 
\( R[X_1, \ldots, X_n] \to \mathcal{R}(I) \) given by \( X_i \mapsto a_i t \). If we let \( J \) be the kernel of this map, then \( J \) is a homogeneous ideal and of course \( \mathcal{R}(I) \cong R[X_1, \ldots, X_n]/J \). We say that 
\( I \) has relation type \( r \) if \( r \) is the least integer so that \( J \) has a generating set whose elements have degree at most \( r \). Note that when \( R \) is Noetherian, the integer \( r \) always exists, since then \( J \) is finitely generated. The integer \( r \) does not depend on the generating set chosen for \( I \). An elementary proof of this is in [R3, Def. 7.9]; see
also \cite{Huc}. We remark that the relation type of $I$ is at most 1 if and only if the symmetric algebra $\text{Sym}(I)$ of $I$ is isomorphic to $\mathcal{R}(I)$. Such ideals are also said to have linear relation type. If an ideal has relation type at most 2, we say, briefly, that it has quadratic relation type. A major class of ideals of linear relation type are those generated by $d$-sequences, see \cite{Hun3} and also \cite{C} and \cite{R2}. Quadratic sequences are known to have quadratic relation type, \cite[Cor. 9.8]{R3}. As a partial converse, it is shown in \cite[Thm. 2.2.2]{BST} that quadratic sequences arising from straightening closed ideals in algebras with straightening law are $d$-sequences if they have linear relation type. To our knowledge, this is not known in general.

5.2. Powers of $d$-sequences are quadratic sequences

Fix an integer $s \geq 1$ and let $S = \{1, \ldots, s\}$ be the linearly ordered poset on $s$ elements. For each $n \geq 1$, define $S^{(n)} := \{(i_1, \ldots, i_n) : 1 \leq i_1 \leq \cdots \leq i_n \leq s\}$. Then $S^{(n)}$ is partially ordered by considering the total order componentwise. More precisely, $(i_1, \ldots, i_n) \leq (j_1, \ldots, j_n)$ if and only if $i_1 \leq j_1, \ldots, i_n \leq j_n$. We fix this order on $S^{(n)}$ for the remainder of this section. Also, if $X$ is the ideal generated by $x_1, \ldots, x_s$, then the ideal $X^n$ is generated by all the products $x_{i_1} \cdots x_{i_n}$, where $(i_1, \ldots, i_n) \in S^{(n)}$. We will denote by $X^n_{\Sigma}$ the ideal $(x_{i_1} \cdots x_{i_n} : (i_1, \ldots, i_n) \in \Sigma)$, where $\Sigma \subseteq S^{(n)}$ is a poset ideal.

**Lemma 5.2.1.** Let $\Sigma$ be an ideal of $S^{(n)}$, and $\lambda = (i_1, \ldots, i_n)$ an element of $S^{(n)}$ just above $\Sigma$. If $b < i_n$, and if $k$ is the least integer so that $i_1 \leq \cdots \leq i_k \leq b \leq i_{k+1} \cdots \leq i_n$, then $\alpha := (i_1, \ldots, i_k, b, i_{k+1}, \ldots, i_{n-1}) \in \Sigma$. 

Proof. The inequalities $i_1 \leq i_1, \ldots, i_k \leq i_k, b \leq i_{k+1}, i_{k+1} \leq i_{k+2}, \ldots, i_{n-1} \leq i_n,$ show $\alpha \leq \lambda$. But $b < i_n$ implies that $\alpha \neq \lambda$, and since $\lambda$ is just above $\Sigma$, we have $\alpha \in \Sigma$. □

Lemma 5.2.2. Let $\Sigma$ and $\lambda$ be as in Lemma 1. Let

$$\Sigma' := \{(a_1, \ldots, a_{n-1}) \in S^{(n-1)} : (a_1, \ldots, a_{n-1}, a) \in \Sigma \text{ for some } a \in S\}.$$ 

Then $\Sigma'$ is an ideal of $S^{(n-1)}$ and $\lambda' := (i_1, \ldots, i_{n-1})$ is just above $\Sigma'$.

Proof. We first show that $\Sigma'$ is an ideal. Suppose $(a_1, \ldots, a_{n-1}) \in \Sigma'$, and that $(b_1, \ldots, b_{n-1}) \leq (a_1, \ldots, a_{n-1})$. Then for some $a \in S$, $(a_1, \ldots, a_{n-1}, a) \in \Sigma$, and since $(b_1, \ldots, b_{n-1}, a) \leq (a_1, \ldots, a_{n-1}, a)$, then $(b_1, \ldots, b_{n-1}, a) \in \Sigma$, showing that $(b_1, \ldots, b_{n-1}) \in \Sigma'$.

Now, for the second statement, if $(a_1, \ldots, a_{n-1}) < (i_1, \ldots, i_{n-1})$, then also $(a_1, \ldots, a_{n-1}, i_n) < (i_1, \ldots, i_n) = \lambda$. So since $\lambda$ is just above $\Sigma$, then we have $(a_1, \ldots, a_{n-1}, i_n) \in \Sigma$, showing that $(a_1, \ldots, a_{n-1}) \in \Sigma'$. Thus $\lambda'$ is just above $\Sigma'$. □

Theorem 5.2.3. Let $x_1, \ldots, x_s$ be a $d$-sequence. Then for each fixed integer $n$, the set of monomials $\{x_{i_1} \cdots x_{i_n} : (i_1, \ldots, i_n) \in S^{(n)}\}$ of length $n$ is a quadratic sequence.

Proof. For each ideal $\Sigma$ of $S^{(n)}$, and each element $\lambda = (i_1, \ldots, i_n)$ just above $\Sigma$, define an ideal $\Theta_{\Sigma, \lambda}$ of $S^{(n)}$ by $\Theta_{\Sigma, \lambda} = \{(a_1, \ldots, a_n) : a_1 < i_n\}$. Note that $X_{\Theta_{\Sigma, \lambda}}^n = X_{i_{n-1}}X_{n-1}^n$. We show by induction on $n$ that this ideal satisfies the conditions of Definition 2.
For \( n = 1 \), this is clear, for then \( S^{(n)} = S \), \( \Sigma = \{1, \ldots, k\} \) for some \( k \), \( \lambda = k + 1 \) and \( \Theta_{\Sigma, \lambda} = \Sigma \). Since \( x_1, \ldots, x_s \) is a d-sequence, \( \Theta_{\Sigma, \lambda} \) is the required ideal.

Now suppose \( n > 1 \). If \( (a_1, \ldots, a_n) \in \Theta_{\Sigma, \lambda} \), then by definition \( a_1 < i_n \) and so by Lemma 1,

\[
x_{i_1} \cdots x_{i_n} x_{a_1} \cdots x_{a_n} = (x_{i_1} \cdots x_{i_k} x_{a_1} x_{i_{k+1}} \cdots x_{i_{n-1}})(x_{i_n} x_{a_2} \cdots x_{a_n}) \in X^{n}_\Sigma X^n.
\]

This shows that (2) of Definition 2 is satisfied.

For (1), suppose \( r \in (X^n_{\Sigma} : x_{i_1} \cdots x_{i_n}) \cap X^n \). With notation as in Lemma 2, this means

\[
rx_{i_n} \in (X^{n-1}_{\Sigma'} : x_{i_1} \cdots x_{i_{n-1}}) \cap X^{n-1} \subseteq X^{n-1}_{\Theta_{\Sigma', \lambda'}},
\]

where the inclusion follows from the induction hypothesis.

Thus, we can write

\[
rx_{i_n} = \sum r_j x_{j_1} \cdots x_{j_{n-1}},
\]

the summation running over all \( j = (j_1, \ldots, j_{n-1}) \in \Theta_{\Sigma', \lambda'} \).

But \( (j_1, \ldots, j_{n-1}) \in \Theta_{\Sigma', \lambda'} \) if and only if \( j_1 < i_{n-1} \). Since \( i_{n-1} \leq i_n \), we also have \( j_1 < i_n \). Thus the right-hand side of (*) is in \( X_{i_{n-1}} \), so

\[
r \in (X_{i_{n-1}} : x_{i_n}) \cap X = X_{i_{n-1}}.
\]

From [R3, Cor. 7.8], \( X_{i_{n-1}} \cap X^n = X_{i_{n-1}} X^n \), since \( x_1, \ldots, x_s \) is a d-sequence. Thus

\[
r \in X_{i_{n-1}} \cap X^n = X_{i_{n-1}} X^{n-1} = X^n_{\Theta_{\Sigma, \lambda}}.
\]

This shows (1) of Definition 5.1.2, and the proof is complete. \( \Box \)
Corollary 5.2.4. If $I$ is generated by a $d$-sequence, then for each $n > 0$, $I^n$ has relation type at most 2.

Proof. For by the theorem, $I^n$ is generated by a quadratic sequence, and such ideals are known to have quadratic relation type, [R3, Cor. 9.8]. □

Example. For instance, if $X$ is an $m \times m + 1$ matrix of indeterminates over a ring $R$, then the maximal minors $\mu_1, \ldots, \mu_{m+1}$ form a $d$-sequence. Hence, the ideal $(\mu_1, \ldots, \mu_{m+1})^n$ has relation type at most 2, for all $n > 0$.

Remark. It has recently been shown in [JK] that if $I$ is any ideal in a Noetherian ring $R$, then the relation type of $I^n$ is 2, for all $n$ sufficiently large.

5.3. SEQUENCES AND RELATION TYPES

In this section, we want to look more closely at how the structure of certain sequences affects the relation type of ideal they generate. Our results further generalize the discussion in [R3, Sect. 8], where conditions on a linearly ordered sequence were shown to determine an upper bound for relation type. Taking as our model the generalization of $d$-sequences to quadratic sequences, we formulate appropriate definitions, and show that the properties retained in the passage from $d$-sequence to quadratic sequence are also retained for these more general sequences.

Definition 5.3.1. A linearly ordered sequence $x_1, \ldots, x_s$ satisfies condition (*) for a fixed $m \geq 1$ if for each $i = 1, \ldots, s$,

$$(I_{i-1}I^{m-1} : x_i) \cap I^m = I_{i-1}I^{m-1},$$

where $I = (x_1, \ldots, x_s)$, and $I_i = (x_1, \ldots, x_i)$. 
**Definition 5.3.2.** Let $\Lambda$ be a finite poset. The set $\{x_\lambda : \lambda \in \Lambda\}$ satisfies condition (**) for a fixed $m \leq 1$, if for each ideal $\Sigma$ of $\Lambda$ and each element $\lambda \in \Lambda$ just above $\Sigma$, there exists an ideal $\Theta$ of $\Lambda$ such that

1. $(X_\Sigma X^{m-1} : x_\lambda) \cap X^m \subseteq X_\Theta X^{m-1}$
2. $x_\lambda X_\Theta X^{m-1} \subseteq X_\Sigma X^m$,

where $X = (x_\sigma : \sigma \in \Lambda)$ and $X_\Sigma = (x_\sigma : \sigma \in \Sigma)$.

**Remark 5.3.3.** It is precisely condition (*) that is considered in [R3, Sect. 8]. Also, note that Definition 5.3.1 specifies a $d$-sequence when $m = 1$, and Definition 5.3.2 specifies a quadratic sequence when $m = 1$.

**Definition 5.3.4.** Let $S = R[U_\lambda : \lambda \in \Lambda]$. If $F \in S$, and $\Sigma$ is an ideal of $\Lambda$, then $\Sigma$ divides $F$ (written $\Sigma | F$) if every monomial term of $F$ contains an element $U_\sigma$ for some $\sigma \in \Sigma$. That is, $\Sigma | F$ if and only if $F \in (U_\sigma : \sigma \in \Sigma)$.

We first prove an analogue of [R3, Prop. 8.2] using the techniques found in [R3, Thm. 9.6].

**Theorem 5.3.5.** Suppose $\{x_\lambda : \lambda \in \Lambda\}$ satisfies condition (**) for some fixed $m \geq 1$. Let $N$ be the ideal of $S$ generated by the relations on $\{x_\lambda : \lambda \in \Lambda\}$ of degree at most $m + 1$. Let $\Sigma$ be an ideal of $\Lambda$ and $F \in S$ a form of degree $p \geq m$ such that $F(\underline{x}) \in X_\Sigma X^{m-1}$. Then there exists a form $G \in S$ of degree $p$ such that $F - G \in N$ and $\Sigma$ divides $G$.

**Proof.** We do a double induction, first on the degree $p$ and then on $#\Lambda \setminus \Sigma$. If $p = m$, then since $F(\underline{x}) \in X_\Sigma X^{m-1}$, there exists a form $G$ of degree $m$ with $\Sigma$ dividing $G$ and $F(\underline{x}) = G(\underline{x})$. Thus $G$ is the required polynomial.
Next, assume $p > m$. If $\# \Lambda \setminus \Sigma = 0$, then take $F = G$. Otherwise, $\Sigma \subseteq \Lambda$.

Let $\lambda \in \Lambda$ be an element just above $\Sigma$. Then since $\Sigma \subseteq \Sigma \cup \{ \lambda \}$, by the second induction hypothesis, there exists a form $G_1$ of degree $p$, such that $\Sigma \cup \{ \lambda \}$ divides $G_1$ and $F - G \in N$. Write $G_1 = U_\lambda G_1' + H$, where $\deg G_1' = p - 1$, $\deg H = p$, and $\Sigma$ divides $H$. Note that $x_\lambda G_1'(x) = G_1(x) + H(x) \in X_\Sigma X^{m-1}$, since $F - G_1 \in N$ implies $G_1(x) = F(x) \in X_\Sigma X^{m-1}$, and $H(x) \in X_\Sigma X^{p-1} \subseteq X_\Sigma X^{m-1}$. Thus $G_1'(x) \in (X_\Sigma X^{m-1} : x_\lambda) \cap X^m \subseteq X_\Theta X^{m-1}$. But the induction hypothesis on the degree shows there exists a form $G_2$ of degree $p - 1$, such that $\Theta$ divides $G_2$ and $G_1' - G_2 \in N$. Thus, $G_2(x) \in X_\Theta X^{p-2} = X_\Theta X^{m-1} X^{p-m-1}$, so there are forms $h_i$ of degree $m$, with $\Theta$ dividing each $h_i$, and forms $H_i$ of degree $p - m - 1$ so that $G_2(x) = \sum h_i(x) H_i(x)$.

Now, condition (2.) of property $\text{(**)}$ says that there are forms $h_i'$ of degree $m + 1$ with $\Sigma$ dividing each $h_i'$, so that $h_i'(x) = x_\lambda h_i(x)$. Set $G = H + \sum h_i' H_i$. Since each $h_i' - U_\lambda h_i$ is a generator of $N$, we have

\[
F - G = (F - G_1) + (G_1 - G) \\
= (F - G_1) + U_\lambda G_1' + H - G \\
= (F - G_1) + U_\lambda G_1' - \sum h_i' H_i \\
= (F - G_1) + U_\lambda (G_1' - G_2) + U_\lambda G_2 - \sum h_i' H_i \\
= (F - G_1) + U_\lambda (G_1' - G_2) + \sum H_i (U_\lambda h_i - h_i')
\]

in $N$, since each term is in $N$. This completes the proof. $\square$
Corollary 5.3.6. If \((**\) holds for some fixed \(m \geq 1\), then \((**\) holds for all \(p \geq m\).

Proof. Let \(\Sigma \subseteq \Lambda\) be an ideal and \(\lambda\) just above \(\Sigma\). Since \((**\) holds for \(m\), let \(\Theta\) be the ideal satisfying the conditions of \((**\). We show that \(\Theta\) also satisfies the conditions of \((**\) for exponent \(p \geq m\).

First, let \(f(\underline{x}) \in (X_\Sigma X^{p-1} : x_\lambda) \cap X^p\), where \(f\) is a form of degree \(p\). Since \(X^p \subseteq X^m\) and \(X_\Sigma X^{p-1} \subseteq X_\Sigma X^{m-1}\), then \(f(\underline{x}) \in (X_\Sigma X^{m-1} : x_\lambda) \cap X^m = X_\Theta X^{m-1}\).

Now, by Theorem 5.3.5, there is a form \(g\) of degree \(p\) such that \(\Theta\) divides \(g\) and \(f(\underline{x}) = g(\underline{x})\); hence \(f(\underline{x}) = g(\underline{x}) \in X_\Theta X^{p-1}\). This shows condition (1.) of property \((**\).

For (2.), suppose \(x_\lambda f(\underline{x}) \in x_\lambda X_\Theta X^{p-1}\), where \(f\) is a form of degree \(p\) and \(\Theta\) divides \(f\). Thus also \(x_\lambda f(\underline{x}) \in x_\lambda X_\Theta X^{m-1}\), and so \(x_\lambda f(\underline{x}) \in X_\Sigma X^{m-1}\). Again, by Theorem 5.3.5, there is a form \(g\) of degree \(p\) with \(\Sigma\) dividing \(g\), and \(g(\underline{x}) = x_\lambda f(\underline{x})\). Thus \(x_\lambda f(\underline{x}) \in X_\Sigma X^{p-1}\) as required \(\Box\)

Hence, by Remark 5.3.3, a quadratic sequence satisfies \((**\) for all \(m \geq 1\).

Corollary 5.3.7. If \((**\) holds for some fixed \(m \geq 1\), then the relation type of \(X\) is at most \(m + 1\).

Proof. Let \(F\) be a relation of degree \(p \geq m + 1\). Apply Theorem 5.3.5 with \(\Sigma = \emptyset\). Then there exists a form \(G\) of degree \(p\) such that \(\Sigma\) divides \(G\) and \(F - G \in N\). But the empty set divides only the zero polynomial, and hence we have \(F \in N\), as required. \(\Box\)
Lemma 5.3.8. If (*) holds for some fixed $m \geq 1$, then for all $n \geq 1$, and all $i = 1, \ldots, s$, $I_{i-1}I^{m-1} \cap I^n \subseteq I_{i-1}I^{n-1}$.

Proof. If $n < m$, then $I_{i-1}I^{m-1} \subseteq I^n$, so $I_{i-1}I^{m-1} \cap I^n = I_{i-1}I^{m-1} \subseteq I_{i-1}I^n$.

Thus, we may suppose $n \geq m$. Let $a \in I_{i-1}I^{m-1} \cap I^n$. Then there exists a form $F \in R[U_1, \ldots, U_s]$ of degree $n$ such that $F(\underline{x}) = a$. Thus $F(\underline{x}) \in I_{i-1}I^{m-1}$, so by [R3, Prop. 8.2], there is a form $G$ of degree $n$, with $G \in (U_1, \ldots, U_{i-1})$, such that $F - G$ is a relation. That is, $G(\underline{x}) = F(\underline{x}) = a$. But then $a = G(\underline{x}) \in I_{i-1}I^{n-1}$. □

Lemma 5.3.9. If (**) holds for some fixed $m \geq 1$, then for all $n \geq 1$, and all ideals $\Sigma \subseteq \Lambda$, $X_\Sigma X^{m-1} \cap X^n \subseteq X_\Sigma X^{n-1}$.

Proof. This is the same proof as for Lemma 5.3.8. If $n < m$, the inclusion is clear. Hence suppose $n \geq m$, and let $a \in X_\Sigma X^{m-1} \cap X^n$. Thus there is a form $F \in S$ of degree $n$ so that $F(\underline{x}) = a \in X_\Sigma X^{m-1}$. By Theorem 5.3.5, there is a form $G$ of degree $n$ with $\Sigma$ dividing $G$ and $F - G$ a relation. In particular, $a = F(\underline{x}) = G(\underline{x}) \in X_\Sigma X^{n-1}$. □

Remark. Note that we have equality in both 5.3.8 and 5.3.9 when $n \geq m$, for then the reverse inclusion is clear.

Our next result extends Theorem 5.2.3 to these more general sequences.

Theorem 5.3.10. If the sequence $x_1, \ldots, x_s$ satisfies condition (*) for some fixed $m \geq 1$, then the set $\{ x_{i_1} \cdots x_{i_n} : (i_1, \ldots, i_n) \in S^{(n)} \}$ satisfies condition (**) for $m$.

Proof. We proceed in much the same way as in the proof of Theorem 5.2.3. First, as a matter of notation, we will denote the ideal generated by the $x_i$, for $i \leq k$,
by \( I_k \), and if \( \Sigma \) is a subset of \( S^{(n)} \), \( X_\Sigma \) will denote the ideal generated by all the monomials \( x_{i_1} \cdots x_{i_n} \), where \((i_1, \ldots, i_n) \in \Sigma \). In particular, note that \( X_{S^{(n)}} = I^n \).

Now, for an ideal \( \Sigma \) of \( S^{(n)} \) and an element \( \lambda = (a_1, \ldots, a_n) \in S^{(n)} \) just above \( \Sigma \), define an ideal \( \Theta = \Theta_{\Sigma, \lambda} := \{(a_1, \ldots, a_n) \in S^{(n)} : a_1 < a_n\} \). Note that \( X_\Theta = I_{a_n - 1}I^{n-1} \). We will prove by induction on \( n \) that this ideal satisfies the conditions of Definition 5.3.2.

The hypothesis on the sequence \( x_1, \ldots, x_s \) takes care of the case \( n = 1 \). Hence, we may assume that \( n \geq 2 \), and that the theorem has been proven for all smaller positive integers. We first show that condition (1.) of the definition is satisfied. For this, suppose \( r \in (X_\Sigma X^{m-1} : x_{a_1} \cdots x_{a_n}) \cap X_{S^{(n)}}^{m-1} \). Using the notation of Lemma 5.2.2, this implies

\[
r \in (X_\Sigma X^{m-1} : x_{a_1} \cdots x_{a_{n-1}}) \cap X_{S^{(n-1)}}^{m-1} \subseteq X_{\Theta'} X^{m-1},
\]

where the second inclusion follows from the inductive hypothesis. Thus we can write

\[
r x_{a_n} = \sum r_j x_{j_1} \cdots x_{j_{n-1}} \tag{*}
\]

where \( j = (j_1, \ldots, j_{n-1}) \) runs through \( \Theta_{\Sigma', \lambda'} \), and \( r_j \in X_{S^{(n)}}^{m-1} \). But \( j \in \Theta_{\Sigma', \lambda'} \) if and only if \( j_1 < a_{n-1} \leq a_n \), and hence the right side of (\( * \)) is in \( I_{a_n - 1}X_{S^{(n)}}^{m-1} \cap I_{a_n - 1}X^{m-1} = I_{a_n - 1}I_{m-1} \). This means that \( r \in (I_{a_n - 1}I^{m-1} : x_{a_n}) \cap I^m \subseteq I_{a_n - 1}I^{m-1} \), since the sequence satisfies condition (\( * \)). Now by Lemma 5.3.8, we obtain

\[
r \in I_{a_n - 1}I^{m-1} \cap X_{S^{(n)}} = I_{a_n - 1}I^{m-1} \cap I^{mn} \subseteq I_{a_n - 1}I^{mn} = X_\Theta X^{m-1}.
\]

This shows condition (1.) of the definition.
For condition (2.), suppose \( j = (j_1, \ldots, j_n) \in \Theta \), and \( r \in X_{S(n)}^{m-1} \). Then by Lemma 5.2.1

\[
x_{a_1} \cdots x_{a_n} x_{j_1} \cdots x_{j_n}^r = (x_{a_1} \cdots x_{a_k} x_{j_1} x_{a_{k+1}} \cdots x_{a_{n-1}})(x_{a_n} x_{j_2} \cdots x_{j_n})^r,
\]

which is evidently in \( X_\Sigma X^m \). This finishes the proof. □

Corollary 5.3.11. If the sequence \( x_1, \ldots, x_s \) satisfies condition (*) for some fixed \( m \geq 1 \), then for each \( n \geq 1 \), the relation type of \( I^n \) is at most \( m + 1 \).

Proof. Use Theorem 5.3.10 and Corollary 5.3.7. □

5.4. CONSTRAINTS ON QUADRATIC SEQUENCES.

A quadratic sequence comes equipped with a partial order on its constituent elements. It is natural to wonder about the uniqueness of this order, and whether a different order on the elements will give rise to another quadratic sequence. In particular, since \( d \)-sequences are just certain linearly ordered quadratic sequences, under what conditions can a quadratic sequence be given a linear order, and when does this linear order actually specify a \( d \)-sequence? We give in this section various results in this direction, each of which assumes some extra structure on the sequence.

Our first result shows that a quadratic sequence with an especially simple structure is a \( d \)-sequence in some linear order. Precisely, let us call a quadratic sequence \( \{x_\lambda : \lambda \in \Lambda \} \) a \textit{partially ordered \( d \)-sequence} if the ideal \( \Theta \) of Definition 5.1.2 can always be chosen to be \( \Sigma \). Thus \( \{x_\lambda : \lambda \in \Lambda \} \) is a partially ordered \( d \)-sequence if and only if for each ideal \( \Sigma \) of \( \Lambda \) and each element \( \lambda \) just above \( \Sigma \), we have \( (X_\Sigma : x_\lambda) \cap X = X_\Sigma \). The second condition of Definition 5.1.2 is superfluous in
this case. In particular, a $d$-sequence is a partially ordered $d$-sequence in which the partial order is linear.

**Theorem 5.4.1.** Let $\{ x_\lambda : \lambda \in \Lambda \}$ be a partially ordered $d$-sequence. Then there is a linear order on $\Lambda$ so that, with this order, $\{ x_\lambda : \lambda \in \Lambda \}$ is a $d$-sequence.

**Proof.** We prove the theorem by induction on the size $\# \Lambda$ of $\Lambda$. If $\Lambda = \{ \lambda \}$ is a singleton, then the hypothesis shows that $(0 : x_\lambda) \cap (x_\lambda) = 0$, and so $\{x_\lambda\}$ is a $d$-sequence.

Now suppose $\# \Lambda > 1$. Let $\alpha$ be a minimal element of $\Lambda$. Then the hypothesis shows that $(0 : x_\alpha) \cap X = (0)$, so $x_\alpha$ is a $d$-sequence with respect to the ideal $X$. Next, in the ring $R/(x_\alpha)$, the set $\{ x_\lambda : \lambda \in \Lambda \setminus \{ \alpha \} \}$ is a partially ordered $d$-sequence, by Lemma 5.1.4. Thus by the induction hypothesis, there is a linear order on $\Lambda \setminus \{ \alpha \}$ so that this is a $d$-sequence in $R/(x_\alpha)$. Now give $\Lambda$ the linear order induced from $\Lambda \setminus \{ \alpha \}$, with $\alpha$ the initial element. Then by Lemma 5.1.5, with this linear order, the elements $\{x_\lambda\}$ form a $d$-sequence. □

We next look at systems of parameters which form a quadratic sequence. Recall that a local ring $R$ is a *Buchsbaum ring* if every system of parameters of $R$ is a $d$-sequence. In particular, a system of parameters forms a $d$-sequence in any linear order. It is natural to ask if there is a different class of rings whose systems of parameters are quadratic sequences. First, we might require that every system of parameters forms a quadratic sequence in every partial order. Note that this class of rings is contained in the Buchsbaum rings. In fact, we show in Corollary 5.4.3 that this class is exactly the class of Buchsbaum rings.
On the other hand, we may require instead that every system of parameter forms a quadratic sequence in some particular partial order. This class of rings contains the Buchsbaum rings. We show in Corollary 5.4.6 that a system of parameters which is a quadratic sequence in some partial order, is in fact a $d$-sequence in some linear order. Thus the class of rings whose s.o.p.'s are quadratic sequences is the same class whose s.o.p.'s form $d$-sequences in some linear order.

**Theorem 5.4.2.** Let $x_1, \ldots, x_n$ be a set of elements in a ring $R$ which forms a $d$-sequence in any linear order. Then under any partial order, the set $x_1, \ldots, x_n$ forms a quadratic sequence.

**Proof.** Let $\Lambda$ be a partially ordered set of $n$ elements, and let it index the set $x_1, \ldots, x_n$. Let $\Sigma$ be a poset ideal of $\Lambda$ and let $\lambda \in \Lambda$ be just above $\Sigma$. It will suffice to show that $(X_\Sigma : x_\lambda) \cap X = X_\Sigma$. To see this, order the $x_i$ in any linear order, so that the elements indexed by $\Sigma$ form an initial subset, and so that $x_\lambda$ is just above this initial subset. Then in this linear order, the $x_i$ form a $d$-sequence, and so $(X_\Sigma : x_\lambda) \cap X = X_\Sigma$, as required. □

**Remark.** Indeed, the proof shows that the $x_i$ form a partially ordered $d$-sequence in any partial order. Thus Theorem 5.4.2 is a converse to Theorem 5.4.1.

**Corollary 5.4.3.** Let $R$ be a local ring. Then $R$ is a Buchsbaum ring if and only if every system of parameters forms a quadratic sequence in any partial order. □

Next, we examine those ring whose systems of parameters form a quadratic sequence in some particular partial order.
Theorem 5.4.4. Let $R$ be an $n$-dimensional local ring, $n \geq 1$, and let $x_1, \ldots, x_n$ be a system of parameters for $R$. Suppose there is a partial ordering on $x_1, \ldots, x_n$ so that this is a quadratic sequence. Then there is a linear order on $x_1, \ldots, x_n$ so that this is a $d$-sequence.

Proof. We induct on $n = \dim R$. When $n = 1$, the element $x_1$ is a quadratic sequence. Thus, we have either $(0 : x_1) \cap (x_1) = (0)$, in which case $x_1$ is a $d$-sequence, or $(0 : x_1) \cap (x_1) = (x_1)$. But if the latter occurs, then in particular $x_1x_1 = 0$, which contradicts that $x_1$ is a system of parameters.

Now, suppose $n \geq 2$, and let $\Lambda$ be a partially ordered set which makes $x_1, \ldots, x_n$ a quadratic sequence. Let $\alpha \in \Lambda$ be any minimal element, and without loss of generality, suppose $\alpha$ indexes the element $x_1$. Then in $R/(x_1)$ the images of $x_2, \ldots, x_n$ form a system of parameters which is a quadratic sequence indexed by the poset $\Lambda \setminus \alpha$. Thus, by the induction hypothesis, there is a linear order on $x_2, \ldots, x_n$ so that with this order, the images in $R/(x_1)$ form a $d$-sequence. We may suppose, by renumbering, that the order is just $x_2 \leq x_3 \leq \ldots \leq x_n$.

Now, suppose $\Theta$ is an ideal of $\Lambda$ such that $(0 : x_1) \cap I = I_{\Theta}$, where $I$ (resp., $I_{\Theta}$) is the ideal generated by the $x_i$ (resp., $x_i \in \Theta$). If $\Theta \neq \emptyset$, then there is $x_i \in \Theta$ with $x_1x_i = 0$. But choosing a minimal prime $p$ of $R$ with $\dim R/p = \dim R$, we have that $x_1x_i \in p$, and hence either $x_1 \in p$, or $x_i \in p$. Both possibilities contradict that $x_1$ and $x_i$ are parts of a system of parameters. Hence, we must have $\Theta = \emptyset$, and $(0 : x_1) \cap I = (0)$. By Lemma 5.1.5, $x_1, \ldots, x_n$ is a $d$-sequence. $\square$
Lemma 5.4.5. Let $R$ be a ring, not necessarily local. Suppose $x_1, \ldots, x_n$ is a linearly ordered quadratic sequence in $R$. If, for each $i = 0, \ldots, n-1$, $x_{i+1} \notin \text{rad}(I)$, then $x_1, \ldots, x_n$ is a $d$-sequence in this order.

Proof. We need to show that for each $i = 0, \ldots, n-1$,

$$(x_1, \ldots, x_i : x_{i+1}) \cap (x_1, \ldots, x_n) = (x_1, \ldots, x_i).$$

Suppose that this equality does not hold for some fixed $i$. Then since $x_1, \ldots, x_n$ is a quadratic sequence, we have

$$(x_1, \ldots, x_i) : x_{i+1} \subseteq (x_1, \ldots, x_n) : x_{i+1} = (x_1, \ldots, x_j).$$

where $j \geq i + 1$. In particular, $x_{i+1}^2 \in (x_1, \ldots, x_j)$, showing that $x_i \in \text{rad}(I)$, contradicting the hypothesis. □

Example. Let $x_1, \ldots, x_n$ be a $d$-sequence in the ring $R$. Let $y \in R$ be idempotent modulo the ideal $(x_1, \ldots, x_n)$; that is, $y \notin (x_1, \ldots, x_n)$, but $y^2 \in (x_1, \ldots, x_n)$. Then $x_1, \ldots, x_n, y$ is a linearly ordered quadratic sequence which is not a $d$-sequence.

Corollary 5.4.6. Let $R$ be an $n$-dimensional local ring, $n \geq 1$, and suppose $x_1, \ldots, x_n$ is a system of parameters in $R$ which form a quadratic sequence in this linear order. Then $x_1, \ldots, x_n$ is a $d$-sequence in this order.

Proof. A system of parameters in a local ring satisfies the hypothesis of Lemma 5.4.5. □
CHAPTER VI
SUMMARY AND OPEN QUESTIONS

In this chapter, we summarize the main results of the previous chapters and list some questions that remain to be answered. The first section summarizes Chapters 3 and 4; the second section summarizes Chapter 5. We will state explicitly some questions that arose as this research progressed, and also indicate some directions of research that our approach suggests might be useful.

6.1. Summary and Questions on Linkage

Throughout this section, we keep the notation of Chapter 3; that is, in the Gorenstein local ring $R$, the ideals $A$ and $B$ are linked by a generically Gorenstein, Cohen–Macaulay ideal $I$, and $A$ is a Cohen–Macaulay ideal. Our main results of Chapter 3 showed essentially that though linkage by generically Gorenstein, Cohen–Macaulay ideals does not preserve the Cohen–Macaulay property in its entirety, nonetheless, at least for direct linkage, quite a bit can be said. By using the properties of an ideal $J$, related to the linking ideal $I$ by $K_{R/I} \cong J/I$, we were able to give a complete description of when $B$ is a Cohen–Macaulay ideal. This lead easily to a description of the non-Cohen–Macaulay locus of $R/B$, when $B$ is not a Cohen–Macaulay ideal, and in certain cases we were able to compute the dimension of the non-Cohen–Macaulay locus. We further investigated when $R/B$ had nearly maximal depth, obtaining a characterization similar in spirit to the characterization for when $R/B$
is Cohen–Macaulay. In the case when $A$ and $B$ are geometrically linked, we gave a different characterization for high depth in terms of the Cohen–Macaulayness of the sum $A + B$. The third section of Chapter 3 gave, for some cases, a method for finding generators for $B$ in terms of the generators of $A$, $J$, and $I$, and more generally, for a method for constructing a (non-minimal) free resolution for $R/B$. Finally, Chapter 4 was devoted to showing explicitly how generators for $J$ can be found, for a specific class of ideals $I$, the generically Gorenstein, determinantal ideals.

Our first question was what originally motivated many of the results in Chapter 3; unfortunately, we have been unable to answer it as yet.

**Question 6.1.1.** With the notation as above, is the canonical module $K_{R/B}$ of $R/B$ always Cohen–Macaulay? The evidence suggests a positive answer; indeed, it is likely that the converse also holds. That is, if $K_{R/B}$ is Cohen–Macaulay, is $B$ linked to a Cohen–Macaulay ideal by a generically Gorenstein, Cohen–Macaulay ideal?

A positive answer to Question 6.1.1 would make the statements of our Theorem 3.1.15 and Theorem 3.1.17 stronger. On the other hand, the conclusion of Theorem 3.1.15 may hold independently of the linkage assumption:

**Question 6.1.2.** Suppose $B$ is a non-Cohen–Macaulay ideal, for which the canonical module $K_{R/B}$ is Cohen–Macaulay. In general, what kinds of properties does $B$ possess? For instance, does the inequality of Theorem 3.1.15 hold: $\dim \text{NCM}(R/B) \geq \text{depth } R/B - 1$?
Also, Question 6.1.1 is related to the depth of the module $\text{Ext}_R^{g+1}(R/B, R)$, where height $B = g$. This leads us to ask about the behavior of these higher Ext modules:

**Question 6.1.3.** What properties do the $\text{Ext}^i(R/B, R)$ possess? For instance, when do they vanish, or have reasonably high depth or dimension? It seems reasonable to expect relatively good behavior; for instance, we have no example where $\dim R/B \geq 3$ and $\text{depth} \text{Ext}_R^{g+1}(R/B, R) = 0$. Thus we conjecture that in this case $\text{depth} \text{Ext}_R^{g+1}(R/B, R) \geq 1$, and this would imply a positive answer to Question 6.1.1.

Finally, given the usefulness of the ideal $J$ in this study, in trying to extend these results by dropping the assumption that $I$ be generically Gorenstein, it seems natural to want to associate to $I$ an ideal similar to $J$. One such way might be the following:

**Question 6.1.4.** Suppose $I$ is a Cohen–Macaulay ideal. Is there a “nice” submodule $M$ of $K_{R/I}$ so that $K_{R/I}/M$ is isomorphic to an ideal of $R$? Note that if $I$ is generically Gorenstein, $M = 0$. By “nice”, we mean homologically simple; an optimistic hope is that $M$ is free or at least Cohen–Macaulay.

### 6.2. Summary and Questions on Quadratic Sequences

Our main result of Chapter 5 was that the set of monomials of fixed length whose terms come from a $d$-sequence form a quadratic sequence in a natural order. Thus it is natural to ask if a similar result holds for the monomials whose terms come from a quadratic sequence:
Question 6.2.1. Do the monomials of fixed length whose terms come from a quadratic sequence in turn form a quadratic sequence? The main difficulty here is that the quadratic sequence is not linearly ordered; in particular, there are incomparable elements. This is the obstruction to extending the proof of Theorem 5.2.3 to this case.

Also in Chapter 5, we defined a new types of sequences, a linear version (Definition 5.3.1) and a partially ordered version (Definition 5.3.2) and showed analogous statements as those in section 5.2 hold. In particular, there are upper bounds on the relation types of such sequences. However, we are unable to give any examples. Hence, we ask:

Question 6.2.2. Are there any natural examples of sequences satisfying the conditions of Definitions 5.3.1 and 5.3.2?

Finally, in connection with sequences and relation type, D. Costa defined and investigated the notion of a “sequence of linear type” [C]. It might be interesting to look at different kinds of sequences defined by relation types. For instance, we might call a sequence $x_1,\ldots,x_n$ a sequence of type $(r_1,\ldots,r_n)$ if the ideal $I_i = (x_1,\ldots,x_i)$ has relation type $r_i$. Similarly, we could allow partially ordered indexing sets. Preliminary calculations using MACAULAY suggest that the generic determinantal ideals, using the natural partial order on the minors, have a rather interesting relation type structure, in the sense as above.
BIBLIOGRAPHY


VITA

Heath Mayall Martin was born on July 18, 1966, and grew up in central Texas. After receiving a B. S. in Mathematics with Honors from the University of Texas at Austin in December, 1988, he began graduate studies in Mathematics at Louisiana State University as a Board of Regents Fellow.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Heath M. Martin
Major Field: Mathematics
Title of Dissertation: Linkage by Generically Gorenstein Cohen-Macaulay Ideals

Approved:

[Signatures]

Major Professor and Chairman
Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination: June 25, 1993