3-17-2021

Connectivity of Matroids and Polymatroids

Zachary R. Gershkoff
Louisiana State University and Agricultural and Mechanical College

CONNECTIVITY OF MATROIDS AND POLYMATROIDS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Zachary R. Gershkoff
B.S., University of Florida, 2009
M.S., Louisiana State University, 2015
May 2021

All shall be well, and all shall be well,
and all manner of thing shall be well.

—*Julian of Norwich*

Acknowledgments

This dissertation is a product of the circumstances under which it was written. That is the most true and complete acknowledgment; every person I met, everything that happened to me, and everything that I did somehow led me here. There are only some that I wish to honor by acknowledging them in these pages.

It is easy to feel gratitude in the sense that one can think of a person and decide objectively that, yes, this person had a positive impact. But sometimes, for some people, it can be hard to truly comprehend the breadth of that impact, and the help that was given selflessly and without obligation. That is my best attempt to describe the gratitude that I owe to my advisor James Oxley, who has been more reliable, kind, patient, and understanding than I have any right to expect. I hope that, in some small way, my gratitude shows in the work that I've put in to bring this document up to his standards.

I wish to acknowledge my friends who helped me get here, and I am reluctant to name names because I don't want to leave anyone out. My roommates, who kept me grounded. My first-year graduate student cohort, who worked with me to prepare for quals and made me feel like we were all in this together. My friends in South Africa, both the ones that went there with me, and the ones who made me feel welcome in their home country. My friends in undergrad, who would collaborate on homework with me, or visit me in the hospital. My mother, for raising me. And Judith, who makes me want to be half as capable as she thinks I am.

Table of Contents

Abstract

This dissertation is a collection of work on matroid and polymatroid connectivity. Connectivity is a useful property of matroids that allows a matroid to be decomposed naturally into its connected components, which are like blocks in a graph. The Cunningham-Edmonds tree decomposition further gives a way to decompose matroids into 3-connected minors. Much of the research below concerns alternate senses in which matroids and polymatroids can be connected.

After a brief introduction to matroid theory in Chapter 1, the main results of this dissertation are given in Chapters 2 and 3. Tutte proved that, for an element e of a 2-connected matroid M, either the deletion or the contraction of e for M is 2-connected. In Chapter 2, a new notion of matroid connectivity is defined and it is shown that this new notion only enjoys the above inductive property when it agrees with the usual notion of 2-connectivity. Another result is proved to reinforce the special importance of this usual notion. In Chapter 3, a result of Brylawski and Seymour is considered. That result extends Tutte's theorem by showing that if the element e is chosen to avoid a 2-connected minor N of M, then the deletion or contraction of e form M is not only 2-connected but maintains N as a minor. The main result of Chapter 3 proves an analogue of this result for 2-polymatroids, a natural extension of matroids. Chapter 4 describes a class of binary matroids that generalizes cubic graphs. Specifically, attention is focused on binary matroids having a cocircuit basis where every cocircuit in the basis, as well as the symmetric difference of all these cocircuits, has precisely three elements.

Chapter 1. Introduction

1.1. Basic definitions

The terminology here will follow Oxley [14] except where otherwise stated. The reader is assumed to have basic familiarity with matroid theory. Graph theory is not used extensively in this dissertation; the only graphs used here are graphs that yield cubic binary matroids in Chapter 4, or graphs that come from the Cunningham-Edmonds tree decomposition [5] of a matroid in Chapter 2. Terminology for graph theory will follow Diestel [6], except that we use *graph* to mean what Diestel calls a *multigraph*.

Matroids can be defined with many different axiom systems. The definition in terms of rank axioms is given below because these axioms are used to define polymatroids in Chapter 3. Let E be a finite set. A function r from the power set of E to the nonnegative integers is the *rank function* of a *matroid* M on E if and only if r has the following properties:

(R1) If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

(R3) If X and Y are subsets of E, then

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

The set E in the above definition is the *ground set* of the matroid M. The ground set of a matroid M may be denoted $E(M)$ to indicate which matroid is being considered. We say that a matroid M *uses* an element e or a set Z if $e \in E(M)$ or $Z \subseteq E(M)$. Moreover, we may use e to denote the singleton set $\{e\}$ when the context is clear, such as when using set union and difference.

1.2. Important classes of matroids

Let M_1 and M_2 be matroids having ground sets E_1 and E_2, and rank functions r_1 and r_2, respectively. We say M_1 and M_2 are *isomorphic* if there is a bijective map ϕ from E_1 to E_2 such that, for every $X \subseteq E_1$, we have that $r_1(X) = r_2(\phi(X))$. Elements of a matroid M are *clones* if the bijection on $E(M)$ that interchanges e and f and fixes every other element is an isomorphism. A *clonal class* of M is a maximal subset of $E(M)$ in which every two members are clones. A clonal class is *trivial* if it has just one element.

Given a graph $G = (V, E)$, we can derive a matroid M by letting the edge set E of G be the ground set of M. The rank of a set $X \subseteq E(M)$ is the number of vertices covered by X in G minus the number of connected components in the subgraph induced by X. We refer to this matroid as the *cycle matroid* $M(G)$ of G. A *graphic matroid* is a matroid that is isomorphic to the cycle matroid of some graph. Graphs that have isomorphic cycle matroids are 2-*isomorphic*.

Given a collection E of vectors over a field $\mathbb{F}$, the rank of a set $X \subseteq E$ is the rank of X in the usual sense of linear algebra, that is, the dimension of the space spanned by X. Matroids that can be derived from vectors over a field F are *F-representable matroids*, or more generally, if we do not wish to specify the field, *representable matroids*. In particular, GF(2)-representable matroids are known as *binary matroids*.

1.3. Matroid minors and duality

Given a matroid M with ground set E and rank function r, there is another matroid M^* having the same ground set and a rank function r^*, where $r^*(X) = r(E - X) + |X| - r(M)$ for all subsets X of E. We say that M^* is the *dual* of M and r^* is the *corank*

function of M.

When an element e of M is *deleted*, we obtain the matroid $M\backslash e$, which is simply the matroid having ground set $E(M) - e$ and the same rank function r_M on sets without e. The *contraction* M/e of e from M is obtained by deleting e from M^* and taking the dual of the resulting matroid; that is, $M/e = (M^*\backslash e)^*$. A matroid N that can be obtained from a series of deletions and contractions of M is a *minor* of M. An N*-minor* of M is a minor of M that is isomorphic to N.

1.4. Matroid connectivity

A matroid is *connected* if, for every distinct pair of elements e and f, there is a circuit with $\{e, f\}$ as a subset. In terms of the rank axioms given above, a matroid is connected if there is no partition (X, Y) of the ground set such that $r(X) + r(Y) = r(E(M))$.

For an integer n exceeding one, the notion of n-connectivity (not to be confused with N-connectivity to be defined in Chapter 2) comes from the matroid connectivity function λ. Let M be a matroid with ground set E. If $X \subseteq E$, then

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$.

A *k-separation* of M is a pair $(X, E - X)$ for which $\lambda_M(X) < k$ and $\min\{|X|, |E - X|\} \geq k$. A matroid is *$n$-connected* if it has no k-separations for all positive integers $k < n$. It is important to note that a matroid is 2-connected if and only if it is connected in the sense defined above. It is straightforward to show that if a matroid is n-connected and has at least $2(n - 1)$ elements, then every minor obtained by removing $n - 2$ or fewer elements will be connected.

Let M_1 and M_2 be matroids such that $E(M_1) \cap E(M_2) = \{p\}$. Assume that $r(p) > 0$ and $r*(p) > 0$ in M_1 and M_2. Then there is a matroid $P(M_1, M_2)$ on $E(M_1) \cap E(M_2)$ having as its set of circuits $\mathcal{C}(M_1) \cup \mathcal{C}(M_2) \cup \{(C_1 \cup C_2) - p : p \in C_1 \in \mathcal{C}(M_i)\}$. This matroid is called the *parallel connection* of M_1 and M_2 with *basepoint* p. When M_1 and M_2 each have at least three elements, the matroid $P(M_1, M_2)\backslash p$ is the 2-sum of M_1 and M_2.

1.5. Cunningham-Edmonds tree decomposition

Cunningham and Edmonds's decomposition [5] of matroids allows us to decompose matroids that are 2-connected but not 3-connected. More complete details can be found in [14, Section 8.3]. First recall that, when (X, Y) is a 2-separation of a connected matroid M, we can write M as $M_X \oplus_2 M_Y$ where M_X and M_Y are matroids having ground sets $X \cup p$ and $Y \cup p$. A *matroid-labeled tree* is a tree T with vertex set $\{M_1, M_2, \ldots, M_n\}$ such that each M_i is a matroid and, for distinct vertices M_j and M_k, the sets $E(M_j)$ and $E(M_k)$ are disjoint if M_j and M_k are non-adjacent, whereas if M_j and M_k are joined by an edge e, then $E(M_j) \cap E(M_k) = \{e\}$, and $\{e\}$ does not have rank or corank equal to 0 in either M_j or M_k.

When f is an edge of a matroid-labeled tree T joining vertices M_i and M_j, if we contract the edge f, we obtain a new matroid-labeled tree T/f by relabeling the composite vertex that results from this contraction as $M_i \oplus_2 M_j$, with every other vertex retaining its original label.

A *tree decomposition* of a 2-connected matroid M is a matroid-labeled tree T such that if $V(T) = \{M_1, M_2, \ldots, M_n\}$ and $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$, then

(i) $E(M) = (E(M_1) \cup E(M_2) \cup \cdots \cup E(M_n)) - \{e_1, e_2, \ldots, e_{n-1}\}$;

(ii) $|E(M_i)| \geq 3$ for all i unless $|E(M)| < 3$, in which case, $n = 1$ and $M_1 = M$; and

(iii) the label of the single vertex of $T/\{e_1, e_2, \ldots, e_{n-1}\}$ is M.

We call the members of $\{e_1, e_2, \ldots, e_{n-1}\}$ *basepoints* since each member of this set is the basepoint of a 2-sum when we construct M. Cunningham and Edmonds (in [5]) proved the following (see also [14, Theorem 8.3.10]).

Theorem 1.5.1. *Let M be a 2-connected matroid. Then M has a tree decomposition T in which every vertex label that is not a circuit or a cocircuit is 3-connected, and there are no adjacent vertices that are both labeled by circuits or are both labeled by cocircuits. Moreover, T is unique up to relabeling of its edges.*

The tree decomposition T whose existence is guaranteed by the last theorem is called the *canonical tree decomposition* of M. Although circuits and cocircuits with at most three elements are 3-connected matroids, when we refer to a 3-*connected vertex*, we shall mean one that is labeled by a matroid with at least four elements. Clearly, for each edge p of T, the graph $T \backslash p$ has two components. Thus p induces a partition of $V(T)$ and a corresponding partition (X_p, Y_p) of $E(M)$. The latter partition is a 2-separation of M; we say that it is *displayed* by the edge p. Moreover, $M = M_{X_p} \oplus_2 M_{Y_p}$ where M_{X_p} and M_{Y_p} have ground sets $X_p \cup p$ and $Y_p \cup p$, respectively. We shall refer to this 2-sum decomposition of M as having been *induced* by the edge p of T.

Given a 2-separation of M, we can say that a vertex M_i of T is on a particular side of the separation, even if M_i has no elements of $E(M)$.

1.6. Polymatroids

For a positive integer k, a *k-polymatroid* M is a pair (E, r) consisting of a finite *ground set* E and a *rank function* r, from the power set of E into the integers, satisfying the following conditions:

(i) $r(\emptyset) = 0$;

(ii) if $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$;

(iii) if X and Y are subsets of E, then $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$; and

(iv) $r(\{e\}) \leq k$ for all $e \in E$.

Axioms (i), (ii), and (iii) tell us that the rank function is normalized, increasing, and submodular, respectively. The reader will note the similarity with the matroid axioms given in Section 1.1.

A matroid is just a 1-polymatroid, so every matroid is a 2-polymatroid. We call M a *polymatroid* if M is a k-polymatroid for some k. Our focus here will be mainly on 2-polymatroids. Elements of a polymatroid of ranks 0, 1, and 2 are called *loops*, *points*, and *lines*, respectively. Non-loop elements p and q are *parallel* if $r(\{p, q\}) = r(\{p\}) = r(\{q\})$.

Many matroid concepts that are stated in terms of the rank function can be extended to polymatroids. In particular, for a polymatroid $M = (E, r)$ and a subset T of E, the *deletion* $M \backslash T$ and the *contraction* M/T of T from M are the polymatroids with ground set $E - T$ and rank functions $r_{M \backslash T}$ and $r_{M/T}$ where $r_{M \backslash T}(X) = r(X)$ and $r_{M/T}(X) = r(X \cup T) - r(T)$ for all subsets X of $E - T$. A *minor* of M is any polymatroid that can be obtained from M by a sequence of deletions and contractions. A polymatroid M is *connected*, or equivalently 2-*connected*, if there is no non-empty proper subset X of

its ground set E such that $r(X) + r(E - X) = r(E)$. As with matroids, we sometimes use $E(M)$ and r_M to denote the ground set and rank function of M.

1.7. Cocircuit spaces

Let M be a binary matroid with ground set $\{1, 2, \ldots, n\}$. The *cocircuit space* of M is the subspace of the n-dimensional vector space over GF(2) generated by the incidence vectors of the cocircuits of M. The following two propositions found in Oxley[14] are useful for understanding cocircuit spaces.

Proposition 1.7.1 ([14], 9.2.2)**.** *Let A be a binary representation of a rank-r binary matroid M. Then the cocircuit space of M equals the row space of A. Moreover, this space has dimension r.*

Proposition 1.7.2 ([14], 9.2.4)**.** *Let A be a binary representation of a matroid M. Then the set of cocircuits of M coincides with the set of minimal non-empty supports of vectors from the row space of A.*

It follows that, in a binary representation of a matroid M, the support of every row is a disjoint union of cocircuits. If $r(M) = r$ and A is a binary representation of M with r rows, then the r rows of A are linearly independent. The set S of $r + 1$ vectors consisting of the rows of A along with their sum forms a Hamiltonian circuit in the matroid whose ground set consists of vectors in the cocircuit space of M. Thus any subset of r vectors from S is also a basis for the cocircuit space of M.

Chapter 2. N-connectivity

2.1. Introduction

A matroid M with $|E(M)| \geq 2$ is *N-connected* if, for every pair of distinct elements e, f of $E(M)$, there is a minor of M that is isomorphic to N and uses $\{e, f\}$.

We will assume, unless otherwise stated, that the matroids discussed in this chapter have at least two elements. Note that $U_{1,2}$-connectivity coincides with the usual notion of connectivity for matroids. Hence, relying on a well-known inductive property of matroid connectivity [23], we have that if a matroid M is $U_{1,2}$-connected, $e \in E(M)$, and $|E(M)| \geq 3$, then $M \backslash e$ or M/e is $U_{1,2}$-connected. A consequence of the following theorem is that $U_{1,2}$ is the unique connected matroid with this property, which we prove in Section 2.3.

Theorem 2.1.1. *Let N be a matroid. If, for every N-connected matroid M with $|E(M)| > |E(N)|$ and, for every e in $E(M)$, at least one of $M \backslash e$ or M/e is N-connected, then N is isomorphic to one of $U_{1,2}$, $U_{0,2}$, or $U_{2,2}$.*

One attractive property of matroid connectivity is that elements can be assigned to components. We say that a matroid N has the *transitivity property* if, for every matroid M and every triple $\{e, f, g\} \subseteq E(M)$, if e is in an N-minor with f, and f is in an N-minor with g, then e is in an N-minor with g. Let $M(\mathcal{W}_2)$ be the rank-2 wheel. In Section 2.6, we prove the following result.

Theorem 2.1.2. *The only matroids with the transitivity property are $U_{1,2}$ and $M(\mathcal{W}_2)$.*

On combining the last two theorems, we get the following result, which indicates how special the usual matroid connectivity is.

This chapter is adapted from Z. Gershkoff and J. Oxley, A notion of minor-based matroid connectivity, *Adv. in Appl. Math.* **100** (2018), 163-178. It is reprinted with permission from Advances in Applied Mathematics.

Corollary 2.1.3. *Let N be a matroid with the transitivity property such that, whenever M is an N-connected matroid, $e \in E(M)$, and $|E(M)| > |E(N)|$, at least one of $M\backslash e$ and M/e is N-connected. Then $N \cong U_{1,2}$.*

The concept of N-connectivity can also convey interesting information when N is disconnected, as the next result indicates.

Theorem 2.1.4. *A matroid M is $U_{0,1} \oplus U_{1,1}$-connected if and only if every clonal class of M is trivial.*

In the next section, we note some basic results that will be needed for the proof of main theorems of the chapter. Sections 2.3, 2.4, and 2.5 treat the cases of N-connected matroids when N is 3-connected, connected, and disconnected, respectively. In particular, we prove Theorems 2.1.1, and 2.1.2 in Section 2.6 and Theorem 2.1.4 in Section 2.5. Finally, in Section 2.7, we consider what can be said when every set of three elements occurs in some minor. Moss [12] showed that 3-connected matroids can be characterized as those in which every set of four elements is contained in a minor isomorphic to a member of $\{\mathcal{W}^2, \mathcal{W}^3, \mathcal{W}^4, M(\mathcal{W}_3), M(\mathcal{W}_4), Q_6\}$, where each of these matroids is defined in the appendix of [14].

2.2. Preliminaries

The concept of N-connectivity is closely related to roundedness, which is exemplified by Bixby's [1] result that if e is an element of a 2-connected non-binary matroid M, then M has a $U_{2,4}$-minor using e. Formally, let t be a positive integer and let $\mathcal{N}$ be a class of matroids. A matroid M *has an $\mathcal{N}$-minor* if M has a minor isomorphic to a member of $\mathcal{N}$. Seymour [21] defined $\mathcal{N}$ to be *t-rounded* if, for every $(t+1)$-connected matroid M with

an $\mathcal{N}$-minor and every subset X of $E(M)$ with at most t elements, M has an $\mathcal{N}$-minor using X. Thus Bixby's result shows that $\{U_{2,4}\}$ is 1-rounded. Seymour [20] extended this result as follows.

Theorem 2.2.1. *Let M be a 3-connected matroid having a $U_{2,4}$-minor, and let e and f be elements of M. Then M has a $U_{2,4}$-minor using $\{e, f\}$.*

For disjoint subsets A, B of $E(M)$, define

$$\kappa_M(A, B) = \min\{\lambda_M(X) : A \subseteq X \subseteq E(M) - B\}.$$

Lemma 2.2.2. *If N is a minor of M and A, B are disjoint subsets of $E(N)$, then $\kappa_N(A, B) \leq \kappa_M(A, B)$.*

We shall frequently use the following well-known result, which appears, for example, as [18, Lemma 2.15].

Lemma 2.2.3. *Let M_1 and M_2 label distinct vertices in a tree decomposition T of a connected matroid M. Let P be the path in T joining M_1 and M_2, and let p_1 and p_2 be the edges of P meeting M_1 and M_2, respectively. Then M has a minor that uses $(E(M_1) \cup E(M_2)) \cap E(M)$ and is isomorphic to the 2-sum of M_1 and M_2, with respect to the basepoints p_1 and p_2.*

We will often use the next result, another consequence of Theorem 1.5.1.

Lemma 2.2.4. *Let (X, Y) be a 2-separation displayed by an edge p in a 2-connected matroid M. Suppose $y \in Y$. Then M has, as a minor, the matroid $M_X(y)$ that is obtained from M_X be relabeling p by y. In particular, let N be a 3-connected minor of M with $|E(N)| \geq 4$ and $|E(N) \cap Y| \leq 1$. If $|E(N) \cap Y| = 1$, let $y \in E(N) \cap Y$; otherwise let y be an arbitrary element of Y. Then $M_X(y)$ has N as a minor.*

Let T be the canonical tree decomposition of a 2-connected matroid M, and let M_0 label a vertex of T. Let $p_1, p_2, \ldots, p_d$ be the edges of T that meet M_0. For each p_i, let (X_i, Y_i) be the 2-separation of M displayed by p_i, where M_0 is on the X_i-side of the 2-separation. For each i, let $y_i \in Y_i$. Then, by repeated application of Lemma 2.2.4, we deduce that M has, as a minor, the matroid that is obtained from M_0 by relabeling p_i by y_i for all i in $\{1, 2, \ldots, d\}$. We denote this matroid by $M_0(y_1, y_2, \ldots, y_d)$ and call it a *specially relabeled* M_0-minor of M.

The following result, which is straightforward to prove by repeated application of Lemma 2.2.2, is well known.

Lemma 2.2.5. *Let N be a 3-connected matroid with $|E(N)| \geq 3$. Let M be a 2-connected matroid with canonical tree decomposition T. Then there is a unique vertex M' of T such that, for each edge p of T, the partition of $V(T)$ induced by p has the vertex M' on the same side as at least $|E(N)| - 1$ elements of N. Moreover, there is a specially relabeled M'-minor of M that has N as a minor.*

2.3. 3-connected matroids

Let $\mathcal{N}$ be a set of matroids. A matroid M is $\mathcal{N}$-*connected* if, for every two distinct elements e and f of M, there is an N-minor of M that uses $\{e, f\}$ for some N in $\mathcal{N}$. A consequence of [14, Proposition 4.3.6] is that a matroid with at least three elements is $\{U_{1,3}, U_{2,3}\}$-connected if and only if it is connected. The first result in this section characterizes $U_{2,3}$-connected matroids. One may hope for a characterization of 3-connectivity in terms of $\mathcal{N}$-connectivity, but no such characterization exists. To see this, note that if M is $\mathcal{N}$-connected, then so is $M \oplus_2 M$. A characterization of 3-connectivity in terms of minors

containing 4-element sets, as opposed to the 2-element sets currently under consideration, is given in [12].

Proposition 2.3.1. *A matroid M is $U_{2,3}$-connected if and only if M is connected and simple.*

Proof. Suppose M is $U_{2,3}$-connected. Clearly M is connected and simple. Conversely, if M is connected and simple, and e and f are distinct elements of M, then M has a circuit C containing $\{e, f\}$ and $|C| \geq 3$. Hence M has a $U_{2,3}$-minor using $\{e, f\}$, so M is $U_{2,3}$-connected. □

Corollary 2.3.2. *A matroid M is $U_{1,3}$-connected if and only if M is connected and cosimple.*

We will describe N-connectivity for a 3-connected matroid N by first considering the case when N is $U_{2,4}$. We will refer to binary and non-binary matroids that label vertices of a canonical tree decomposition as *binary* and *non-binary vertices*.

Theorem 2.3.3. *A matroid M is $U_{2,4}$-connected if and only if M is connected and non-binary, and, in the canonical tree decomposition of M,*

(i) every binary vertex has at most one element that is not a basepoint; and

(ii) on every path between two binary vertices that each contain a unique element of $E(M)$, there is a non-binary vertex.

Proof. Suppose M is non-binary and connected, and the canonical tree decomposition T of M satisfies the above conditions. Suppose e and f are distinct elements of M. If e and f are in the same 3-connected vertex M_0 of T, then, by (i), M_0 is non-binary. Thus, by Theorem 2.2.1, M has a $U_{2,4}$-minor using $\{e, f\}$.

Next suppose e belongs to a binary vertex M_1 of T, and f belongs to a non-binary vertex M_0 of degree d. By Lemma 2.2.4, M contains a specially labeled M_0-minor $M_0(e, y_2, y_3, \ldots, y_d)$ using $\{e, f\}$. Similarly, let e and f belong to binary vertices M_1 and M_2, and let M_0 be a non-binary vertex on the path between them in T. Then M contains a specially labeled M_0-minor $M_0(e, f, y_3, y_4, \ldots, y_d)$. Thus, by Theorem 2.2.1, M has a $U_{2,4}$-minor using $\{e, f\}$.

Suppose now that M is $U_{2,4}$-connected. Clearly M is non-binary and connected. If a binary vertex M_1 in T contains two non-basepoints e and f, then, by Lemma 2.2.5, a $U_{2,4}$-minor of M using $\{e, f\}$ must be a minor of M_1, a contradiction.

Now suppose e and f are the unique non-basepoints of binary vertices M_1 and M_2, respectively, in T, and let N be a $U_{2,4}$-minor of M using $\{e, f\}$. By Lemma 2.2.5, T has a nonbinary vertex M_0 such that, for every edge p of T, the partition of $V(T)$ induced by p has M_0 on the same side as at least $|E(N)| - 1$ elements of N. Let p_1 be the edge incident with M_0 such that M_1 and M_0 are on opposite sides of the induced partition of $V(T)$. Then M_2 must be on the same side of this partition as M_0. Hence M_0 lies on the path in T between M_1 and M_2. $\square$

The last theorem can be generalized as follows.

Theorem 2.3.4. *Let N be a 3-connected matroid with at least four elements. A matroid M is N-connected if and only if M is connected, has N as a minor, and, in the canonical tree decomposition of M,*

(i) every vertex that is not N-connected has at most one element that is not a basepoint; and

(ii) on every path between two vertices that are not N-connected and that each have unique non-basepoints, there is an N-connected vertex.

2.4. Connected matroids

In this section, we consider N-connected matroids when N is connected but not 3-connected.

Theorem 2.4.1. *A matroid M is $M(\mathcal{W}_2)$-connected if and only M is connected and non-uniform.*

Proof. If M is $M(\mathcal{W}_2)$-minor, then it is clearly both connected and non-uniform. To prove the converse, suppose M is connected and non-uniform. We argue by induction that M is $M(\mathcal{W}_2)$-connected. This is immediate if $|E(M)| = 4$, since $M(\mathcal{W}_2)$ is the unique 4-element connected, non-uniform matroid. Assume it holds for $|E(M)| < n$ and let $|E(M)| = n > 4$. Distinguish two elements x and y of $E(M)$.

Suppose there is an element e of $E(M) - \{x, y\}$ such that M/e is disconnected. Then M is the parallel connection, with basepoint e, of two matroids M_1 and M_2. Now $M\backslash e$ is connected. We may assume that it is uniform; otherwise, by the induction assumption, $M\backslash e$ and hence M has an $M(\mathcal{W}_2)$-minor using $\{x, y\}$. Now $r(E(M_1)-e)+r(E(M_2)-e) - r(M\backslash e) = 1$. Suppose each of $|E(M_1) - e|$ and $|E(M_2) - e|$ has at least two elements. Then $M\backslash e$ has a 2-separation. Since $M\backslash e$ is uniform, it follows that $M\backslash e$ is a circuit or a cocircuit. In the latter case, M is also a cocircuit, a contradiction. If $M\backslash e$ is a circuit, then M is the parallel connection of two circuits, and M is easily seen to have an $M(\mathcal{W}_2)$-minor using $\{x, y\}$.

Now suppose that $|E(M_1) - e| = 1$. Thus M has a circuit, $\{e, f\}$ say, containing e.

As $M\backslash e$ is uniform but M is not, $r(M) \geq 2$, so $M\backslash e$ has a circuit containing $\{f, x, y\}$. It follows that M has an $M(\mathcal{W}_2)$-minor with ground set $\{e, f, x, y\}$.

We may now assume that M/e is connected for all e in $E(M) - \{x, y\}$. Moreover, by replacing M with M^* in the argument above, we may also assume that $M\backslash e$ is connected for all such e. If $M\backslash e$ or M/e is non-uniform, then, by the induction assumption, M has an $M(\mathcal{W}_2)$-minor using $\{x, y\}$. Thus both $M\backslash e$ and M/e are uniform. Let $r(M\backslash e) = r$. Then every circuit of $M\backslash e$ has $r + 1$ elements. Since M is not uniform, it has a circuit containing e that has at most r elements. Contracting e from M produces a rank-$(r - 1)$ matroid having a circuit with at most $r - 1$ elements. Since M/e is uniform, this is a contradiction. □

Lemma 2.4.2. *If M, N, and N' are matroids such that M is N-connected and N is N'-connected, then M is N'-connected.*

Proof. As M is N-connected, for every two elements $x, y \in E(M)$, there is an N-minor of M containing x and y. Since N is N'-connected, x and y are also in an N'-minor of N, which is also a minor of M. □

If we wish to describe the class of N-connected matroids for a 3-connected matroid N, it suffices to describe the N-connected matroids that are 3-connected and then apply Theorem 2.3.4. If N is not 3-connected, the task of describing N-connected matroids becomes harder, and we omit any attempt to provide a general theorem for N-connectivity in this case. We will instead give characterizations for two specific matroids that are not 3-connected, namely $U_{1,4}$ and its dual $U_{3,4}$. We will use the following theorem of Oxley [13].

Theorem 2.4.3. *Let M be a 3-connected matroid having rank and corank at least three, and suppose that $\{x, y, z\} \subseteq E(M)$. Then M has a minor isomorphic to one of $U_{3,6}, P_6, Q_6, \mathcal{W}^3$, or $M(K_4)$ that uses $\{x, y, z\}$.*

Proposition 2.4.4. *A 3-connected matroid M is $U_{1,4}$-connected if and only if either $M \cong U_{2,n}$ for some $n \geq 5$, or M has rank and corank at least three.*

Proof. Clearly if $n \geq 5$, then $U_{2,n}$ is $U_{1,4}$-connected. Now assume that $r(M) \geq 3$ and $r^*(M) \geq 3$. Suppose $\{x, y\} \subseteq E(M)$. Then, by Theorem 2.4.3, M has an $\mathcal{N}$-minor using $\{x, y\}$ where $\mathcal{N}$ is $\{U_{3,6}, P_6, Q_6, \mathcal{W}^3, M(K_4)\}$. One easily checks that each member of $\mathcal{N}$ is $U_{1,4}$-connected. Hence, by Lemma 2.4.2, M is $U_{1,4}$-connected.

To prove the converse, assume that M is $U_{1,4}$-connected. Since $r^*(U_{1,4}) = 3$, it follows that $r^*(M) \geq 3$. The required result holds if $r(M) \geq 3$. But, since M is 3-connected and $U_{1,4}$-connected, $r(M) \geq 2$. Moreover, if $r(M) = 2$, then $M \cong U_{2,n}$ for some $n \geq 5$. □

Duality gives a corresponding result for $U_{3,4}$-connectivity.

Corollary 2.4.5. *A 3-connected matroid M is $U_{3,4}$-connected if and only if either $M \cong U_{n-2,n}$ where $n \geq 5$, or M has rank and corank at least 3.*

Observe that this fails to fully characterize $U_{3,4}$-connectivity for if we let $M = M(K_{2,3})$, then M is $U_{3,4}$-connected but none of the matroids in its canonical tree decomposition is $U_{3,4}$-connected. We can instead describe $U_{3,4}$-connectivity in terms of forbidden configurations of matroids in the canonical tree decomposition.

Proposition 2.4.6. *Suppose M is not 3-connected. Then M is $U_{3,4}$-connected if and only if M is connected and simple, and, in the canonical tree decomposition T of M, there is no vertex of degree at most two that is labeled by some $U_{2,n}$ such that its only neighbors in T*

are cocircuits that use elements of $E(M)$.

Proof. Let T be the canonical tree decomposition of M. Assume M is $U_{3,4}$-connected. Then, by Lemma 2.4.2, M is $U_{2,3}$-connected, so M is connected and simple. Suppose that T has a vertex M_0 whose degree d is at most two such that M_0 is labeled by some $U_{2,n}$ and has its only neighbors $M_1, \ldots, M_d$ labeled by cocircuits that use elements of $E(M)$. For each i in $\{1, \ldots, d\}$, suppose $f_i \in E(M_i) \cap E(M)$. Then M can be obtained from a copy of $U_{2,n}$ using $\{f_1, \ldots, f_d\}$ by, for each i, adjoining some matroid via parallel connection across the basepoint f_i. If $d = 1$, let f_2 be an element of M_0 other than f_1. Clearly M has no circuit using $\{f_1, f_2\}$ that has more than three elements.

Now assume that M is connected and simple and that T satisfies the specified conditions. Let $\{e, f\}$ be a subset of $E(M)$ that is not contained in a $U_{3,4}$-minor. Assume first that e and f belong to the same vertex M_1 of T. As M is simple, M_1 is not a cocircuit. Now M has a specially relabeled M_1-minor using $\{e, f\}$. Thus, by Corollary 2.4.5, $M_1 \cong U_{2,n}$ for some $n \geq 3$. Let p be an edge of T that meets M_1. Consider the 2-sum $N_1 \oplus_2 N_2$ induced by p where $\{e, f\} \subseteq E(N_1)$. Certainly N_1 has a circuit containing $\{e, f, p\}$, and N_2 has a circuit of size at least three containing p. Thus M has a $U_{3,4}$-minor containing $\{e, f\}$, a contradiction.

We may now know that e and f belong to distinct vertices M_1 and M_2 of T. Each edge p of the path P in T joining M_1 and M_2 induces a 2-sum decomposition of M into two matroids, N_{1p} and N_{2p}. Moreover, an element x_i of $E(N_{ip})$ is in a circuit of N_{ip} of size at least three containing p unless x_i is parallel to p in N_{ip}. Thus e or f is parallel to p in N_{1p} or N_{2p}, respectively. Let the edges of P, in order, be $p_1, p_2, \ldots, p_k$ where p_1 meets M_1.

We may assume that e is parallel to p_1 in N_{1p_1}. Then the vertex M_1 of T containing e is a cocircuit.

Suppose $k \geq 3$. As no two adjacent vertices of T are cocircuits, neither e nor f is parallel to p_2 in N_{1p_2} or N_{2p_2}. Hence M has a $U_{3,4}$-minor using $\{e, f\}$. This contradiction implies that $k \in \{1, 2\}$. Suppose $k = 2$. Then f is parallel to p_2 in N_{2p_2}. Thus M_2 is a cocircuit. Since M has no $U_{3,4}$-minor using $\{e, f\}$, the vertex M_3 of T that is adjacent to both M_1 and M_2 is isomorphic to some $U_{2,n}$. By assumption, M_3 must have another neighbor in T to which it is joined by the edge q, say. Then, for the 2-sum decomposition $Q_1 \oplus_2 Q_2$ of M induced by q, there is a circuit of Q_1 containing $\{e, f, q\}$ and a circuit of Q_2 of size at least three containing q. Thus M has a $U_{3,4}$-minor using $\{e, f\}$. This contradiction implies that $k = 1$. Then $M = N_{1p_1} \oplus_2 N_{2p_1}$. Thus the specially relabeled minor $N_{2p_1}(e)$ uses $\{e, f\}$. Now the canonical tree decomposition T' of $N_{2p_1}(e)$ can be obtained from the component of $T \backslash p_1$ using N_{2p_1} by replacing M_2 by $M_2(e)$. As e and f are contained in the same vertex of T', we deduce from the second paragraph that $N_{2p_1}(e)$, and hence M, has a $U_{3,4}$-minor using $\{e, f\}$, a contradiction. $\square$

2.5. Disconnected matroids

We now turn our attention to N-connectivity where N is disconnected. The following is essentially immediate.

Proposition 2.5.1. *Let n be an integer exceeding one. A matroid M is $U_{n,n}$-connected if and only if M is simple with rank at least n.*

Recall that elements x and y of a matroid M are *clones* if the bijection on $E(M)$ that interchanges x and y but fixes every other element yields the same matroid. Next we

prove Theorem 2.1.4, showing that a matroid is $U_{0,1} \oplus U_{1,1}$-connected if and only if no element has a clone. The proof will use the well-known fact (see, for example, [2]) that two elements in a matroid are clones if and only if they are in precisely the same cyclic flats.

Proof of Theorem 2.1.4. Suppose every clonal class of M is trivial and let x and y be distinct elements of M. Then M has a cyclic flat F that contains exactly one of x and y, say x. In $M/(F-x)$, the element x is a loop but y is not. Thus M has a $U_{0,1} \oplus U_{1,1}$-minor using $\{x, y\}$, so M is $U_{0,1} \oplus U_{1,1}$-connected.

Conversely, assume M is $U_{0,1} \oplus U_{1,1}$-connected, but M has elements x and y that are in the same cyclic flats. Suppose that $M/C\backslash D \cong U_{0,1} \oplus U_{1,1}$ and $E(M/C\backslash D) = \{x, y\}$. Let x be the loop of $M/C\backslash D$. Then $x \in \mathrm{cl}_M(C)$. Thus $y \in \mathrm{cl}_M(C)$, so y is a loop in $M/C\backslash D$, a contradiction. □

Recall, for the next result, that an element is *free* in a matroid if it is not a coloop and every circuit that contains it is spanning.

Theorem 2.5.2. *A matroid M is $U_{1,2} \oplus U_{1,1}$-connected if and only if M is loopless, has at most one coloop, and has at most one free element.*

Proof. Clearly if M is $U_{1,2} \oplus U_{1,1}$-connected, then it obeys the specified conditions. Conversely, suppose M is loopless, has at most one coloop, and has at most one free element. Let e and f be elements of M. Suppose first that M is disconnected. If e and f are in the same component, then they are in a $U_{1,2}$-minor of that component, so M has a $U_{1,2} \oplus U_{1.1}$-minor using $\{e, f\}$. If e and f are in different components, then one of these components is not a coloop. That component has a $U_{1,2}$-minor using e or f. It follows that M has a

$U_{1,2} \oplus U_{1,1}$-minor using $\{e, f\}$.

Now suppose M is connected. Suppose that e is free in M. Then f is in some non-spanning circuit, C_f. Choose g in $C_f - f$. Contracting $C_f - \{f, g\}$ and deleting every other element of M yields a $U_{1,2} \oplus U_{1,1}$-minor of M using $\{e, f\}$.

Suppose neither e nor f is free in M. If there is a non-spanning circuit C containing $\{e, f\}$, we can find a $U_{1,2} \oplus U_{1,1}$-minor by contracting every element of C except e and f, and deleting every other element except for one. Now suppose every circuit containing $\{e, f\}$ is spanning. Since e is not free, there is a non-spanning circuit C containing e. Clearly $f \notin \mathrm{cl}(C)$ otherwise $M|\mathrm{cl}(C)$ is a connected matroid of rank less than $r(M)$ so it contains a circuit containing $\{e, f\}$, a contradiction. Therefore, after we contract all of C except for e and one other element, we see that f will not be a loop. Thus we can find a $U_{1,2} \oplus U_{1,1}$-minor using $\{e, f\}$. □

Corollary 2.5.3. *A matroid M is $U_{1,2} \oplus U_{0,1}$-connected if and only if M is coloopless and has at most one element that is in every dependent flat.*

2.6. N-connectivity as compared to connectivity

Before proving Theorem 2.1.1, we state and prove its converse.

Proposition 2.6.1. *If $N \in \{U_{1,2}, U_{0,2}, U_{2,2}\}$, then, for every N-connected matroid M with $|E(M)| \geq 3$ and for every e in $E(M)$, at least one of $M\backslash e$ or M/e is N-connected.*

Proof. The result is immediate if $N \cong U_{1,2}$. By duality, it suffices to deal with the case when $N \cong U_{2,2}$. Suppose M is $U_{2,2}$-connected, and $|E(M)| \geq 3$. By Proposition 2.5.1, M is simple with rank at least two. Therefore if M is $U_{2,2}$-connected and $r(M) > 2$, we can delete any element e of M and still have an N-connected matroid. Observe that if $r(M) =$

2, then M must be connected since it is simple. Therefore M has no coloops, so $r(M\backslash e) = 2$ for all e of $E(M)$. Thus $M\backslash e$ is $U_{2,2}$-connected. $\square$

Proof of Theorem 2.1.1. First we consider the case when N is connected. Then N is $U_{1,2}$-connected. Thus, by Lemma 2.4.2, every N-connected matroid is $U_{1,2}$-connected and so is connected. Suppose M is an N-connected matroid with $|E(M)| > |E(N)|$.

Assume N is simple. Then, by Proposition 2.3.1 and Lemma 2.4.2, N, and hence M, is $U_{2,3}$-connected. Let M_1 and M_2 be isomorphic copies of M with disjoint ground sets. Pick arbitrary elements g_1 and g_2 in M_1 and M_2, and let M_3 be the parallel connection of M_1 and M_2 with respect to the basepoints g_1 and g_2, which we relabel as g in M_3. Then one easily sees that M_3 is N-connected. Let $e, f \in E(M_1) - g$. By assumption, we can remove all the elements of $E(M_1) - \{e, f, g\}$ from M_3 via deletion or contraction to obtain a matroid M_4 that is still N-connected. Since M_4 is $U_{2,3}$-connected, it follows that $\{e, f, g\}$ is a triangle in M_4. Moreover, $\{e, f\}$ is a series pair in M_4. However, neither $M_4\backslash e$ nor M_4/e is $U_{2,3}$-connected since $M_4\backslash e$ is disconnected, and M_4/e has f and g in parallel. We deduce that N is not simple. Dually, N is not cosimple. The only uniform matroid that is neither simple nor cosimple is $U_{1,2}$, so either $N \cong U_{1,2}$, or N is non-uniform.

Next we show that N cannot be non-uniform. Suppose, instead, that N is non-uniform. Then, as N is connected, by Theorem 2.4.1, N is $M(\mathcal{W}_2)$-connected.

Recall that M is N-connected with $|E(M)| > |E(N)|$. Let $n = |E(N)| + 1$ and distinguish elements e, f of $E(M)$. Let each of $M_1, M_2, \ldots, M_n$ be a copy of M and let e_i and f_i be the elements of M_i corresponding to e and f. Let M' be the parallel connection of

$M_1, M_2, \dots, M_n$ with respect to the basepoints $e_1, e_2, \dots, e_n$ where these elements are relabeled as e in M'. By assumption, for each M_i, we can remove $E(M_i) - \{e, f_i\}$ from M' in such a way that the resulting matroid M'' is N-connected. Since M'' is connected, it must be isomorphic to $U_{1,n+1}$, which is clearly not $M(\mathcal{W}_2)$-connected, a contradiction. We conclude that N cannot be non-uniform, and hence the theorem holds when N is connected.

Next we consider the case when N is disconnected, first showing the following.

2.6.1.1. *If each element of N is a loop or a coloop, then $N \cong U_{0,2}$ or $U_{2,2}$.*

Suppose $n \geq 3$ and let $N \cong U_{n,n}$. Let $M = U_{2,3} \oplus U_{n-2,n-2}$. Then M is N-connected, but if e is a coloop of M, then neither $M \backslash e$ nor M/e has a $U_{n,n}$-minor. Therefore $N \not\cong U_{n,n}$; dually, $N \not\cong U_{0,n}$.

If $N = U_{0,1} \oplus U_{1,1}$, then let $M = M(K_4)$. By Theorem 2.1.4, M is N-connected, but, for every e of $E(M)$, both $M \backslash e$ and M/e have nontrivial clonal classes and are therefore not N-connected. Now assume $N \cong U_{0,n} \oplus U_{m,m}$ for some $n \geq 2$ and $m \geq 1$. Then $U_{0,n+1} \oplus U_{m,m}$ is an N-connected matroid, say M. But if e is a coloop, then neither $M \backslash e$ nor M/e has an N-minor. On combining this contradiction with duality, we conclude that 2.6.1.1 holds.

Now assume that N has $k + s$ components $N_1, N_2, \dots, N_{k+s}$ where those with at least two elements are $N_1, N_2, \dots, N_k$. Then $k \geq 1$. For each i in $\{1, 2, \dots, k\}$, choose an element e_i of N_i and relabel it as p. Let M' be the parallel connection of $N_1, N_2, \dots, N_k$ with respect to the basepoint p where we take $M' = N_1$ if $k = 1$. Let N' be a copy of N whose ground set is disjoint from $E(N)$, and let n_i' be the component of N' corresponding to N_i. Let $M_1 = N' \oplus M'$. We show next that

2.6.1.2. *M_1 is N-connected.*

Suppose $\{e, f\} \subseteq E(M_1)$. Certainly M_1 has an N-minor using $\{e, f\}$ if $\{e, f\} \subseteq E(N')$. Next suppose that $e \in E(M')$. Then, since M' is a connected parallel connection, we see that, for each i in $\{1, 2, \ldots, k+s\}$, there is an N_i-minor of M' using e. Thus, if $f \in E(N')$, say $f \in E(N'_j)$, then we can choose $i \neq j$ and get an N-minor of M_1 using $\{e, f\}$ unless $k = 1 = j$. In the exceptional case, M' has an N_2-minor with ground set $\{e\}$ and again we get an N-minor of M_1 using $\{e, f\}$. We may now assume that $f \in E(M')$, say $f \in E(N_j)$. Then M' has an N_j-minor using $\{e, f\}$, so M_1 has an N-minor using $\{e, f\}$. Thus 2.6.1.2 holds.

Since M_1 is N-connected, by assumption, we may delete or contract elements of M_1 until we obtain an N-connected matroid M_2 with $|E(M_2)| = |E(N)| + 1$. In particular, we may remove elements from M' in M_1 until a single element g remains. Now choose e in $E(N'_1)$. Then $M_2 \backslash e$ or M_2 / e is isomorphic to N. But both $M_2 \backslash e$ and M_2 / e have more one-element components than N', a contradiction. $\square$

Recall that we say that a matroid N has the transitivity property if, for every matroid M and every triple $\{e, f, g\} \subseteq E(M)$, if e is in an N-minor with f, and f is in an N-minor with g, then e is in an N-minor with g. Clearly N has the transitivity property if and only if N^* has the transitivity property.

Lemma 2.6.2. *Suppose N is a matroid having the transitivity property. Let N' be obtained from N by adding an element f in parallel to a non-loop element e of N. Then there is an element g of $E(N')$ such that $N' \backslash g$ is isomorphic to N and has $\{e, f\}$ as a 2-circuit. Moreover, g is in a 2-circuit in N.*

Proof. The transitivity property implies that $\{e, f\}$ is in an N-minor of N'. Since $r^*(N') > r^*(N)$, there must be an element g of $E(N') - \{e, f\}$ such that $N' \backslash g \cong N$. Since we have introduced a new 2-circuit in constructing N', when we delete g, we must destroy a 2-circuit. □

By the last lemma and duality, we obtain the following result.

Corollary 2.6.3. *If N is a matroid having the transitivity property, then N has a component having more than one element.*

The following elementary observation and its dual will be used repeatedly in the proof of Theorem 2.1.2.

Lemma 2.6.4. *Suppose N is a matroid having the transitivity property. Let N_0 be a component of N having the largest number of elements. Suppose f is added in parallel to an element e of N_0. Let N_0' and N' be the resulting extensions of N_0 and N, respectively. Suppose $g \in E(N')$ such that $N' \backslash g \cong N$. Then $g \in E(N_0')$.*

Recall that a set S of elements of a matroid M is a *fan* if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of the elements of S such that, for all i in $\{1, 2, \ldots, n-2\}$,

(i) $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle or a triad; and

(ii) when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad; and when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

Note that the above extends the definition given in [14] by eliminating the requirement that M be simple and cosimple. We shall follow the familiar practice here of blurring the distinction between a fan and a fan ordering.

Lemma 2.6.5. *Let $(s_1, s_2, \ldots, s_n)$ be a fan X in a matroid M such that each of $\{s_1, s_2\}$*

and $\{s_{n-1}, s_n\}$ is a circuit or a cocircuit. Then X is a component of M.

Proof. By switching to the dual if necessary, we may assume that $\{s_1, s_2, s_3\}$ is a triangle of M. Thus $\{s_1, s_2\}$ is a cocircuit. Observe that $\{s_i : i \text{ is odd}\}$ spans X. This is immediate if n is odd, while it follows if n is even from the fact that $\{s_{n-1}, s_n\}$ is a circuit in that case. By duality, $\{s_i : i \text{ is even}\}$ spans X in M^*. Hence $r(X) + r^*(X) \leq |X|$; that is, $\lambda(X) \leq 0$, so X is a component of M. $\square$

We define a *special fan* to be a fan $(s_1, s_2, \dots s_k)$ such that $\{s_1, s_2\}$ is a cocircuit of M. We will now show that $U_{1,2}$ and $M(\mathcal{W}_2)$ are the only connected matroids with the transitivity property.

Proof of Theorem 2.1.2. It is clear that $U_{1,2}$ has the transitivity property. By Theorem 2.4.1, two elements of M are in an $M(\mathcal{W}_2)$-minor together if and only if they are in a connected, non-uniform component together. It follows that $M(\mathcal{W}_2)$ has the transitivity property.

Suppose that N has the transitivity property. Assume that N is not isomorphic to $U_{1,2}$ or $M(\mathcal{W}_2)$. Next we show the following.

2.6.5.1. *Let N_0 be a largest component of N. Then N_0 is isomorphic to $U_{1,2}$ or $M(\mathcal{W}_2)$.*

Assume that this assertion fails. Then, by Corollary 2.6.3, N_0 has at least two, and hence at least three, elements. Take an element e of N_0 and add an element f in series with it. Let the resulting coextensions of N_0 and N be N_0' and N', respectively. Then, by the transitivity property, $N'/a \cong N$ for some element a of $E(N') - \{e, f\}$. Furthermore, by the dual of Lemma 2.6.4, $a \in E(N_0)$. We deduce that N_0 has a 2-cocircuit, say $\{a, b\}$.

In N_0, add an element c in parallel to a to get N_1. Then, by transitivity and Lemma 2.6.4, there is an element s_1 of $E(N_1) - \{a, c\}$ such that $N_1 \backslash s_1 \cong N_0$. Since $N_1 \backslash b$ has $\{a, c\}$ as a component, the component sizes of $N_1 \backslash b$ and N_0 do not match, so $s_1 \neq b$. Thus $s_1 \in E(N) - \{a, b, c\}$, so $N_1 \backslash s_1$ has $\{c, a, b\}$ as a cocircuit. Next add an element d to $N_1 \backslash s_1$, putting it in series with c. Let the resulting matroid be N_2. By the dual of Lemma 2.6.4, there is an element s_2 of $E(N_2) - \{c, d\}$ such that $N_2/d \cong N_0$. Moreover, s_2 must be in a 2-cocircuit of N_2, and s_2 is in a triangle in N_2 as N_2/s_2 must have a 2-circuit that is not present in N_2 since adding d destroyed the 2-circuit $\{a, c\}$. Now $s_2 \neq a$ since N_2/a has $\{c, d\}$ as a component.

Suppose $s_2 = b$. Then b is in a 2-cocircuit $\{b, e\}$ in N_2. Moreover, N_2 has a triangle T containing b. By orthogonality, $T = \{b, e, a\}$. Then (d, c, a, e) is a fan X in N_2/b having $\{c, d\}$ as a cocircuit and $\{a, e\}$ as a circuit. By Lemma 2.6.5, $X = E(N_2/b)$, so $N_0 \cong N_2/b \cong M(\mathcal{W}_2)$, a contradiction.

We now know that $s_2 \neq b$, so $s_2 \notin \{a, b, c, d\}$. Thus N_0 has (d, c, a, b) as a special fan. Among all the special fans of N_0 and N_0^*, take one, $(a_1, a_2, \ldots, a_k)$, with the maximum number of elements. Then $k \geq 4$. First assume $\{a_{k-2}, a_{k-1}, a_k\}$ is a triad. Suppose $\{a_{k-1}, a_k\}$ is a 2-circuit of N_0. Then, by Lemma 2.6.5, the special fan is the whole component N_0. As $N_0 \ncong M(\mathcal{W}_2)$, we see that $k \geq 6$. Add an element f in parallel to a_3 to form a new matroid N_0'. Then $\{a_1, a_3\}$ is in an N_0-minor of N_0', and so is $\{a_1, f\}$. By the transitivity property, N_0' has $\{a_3, f\}$ in an N_0-minor. Since N_0' has $\{a_3, f\}$ and $\{a_{k-1}, a_k\}$ as its only 2-circuits, while N_0 has a single 2-circuit, we deduce that $N_0' \backslash a_k \cong N_0$. But every element of N_0 is in a cocircuit of size at most three, yet f is in no such cocircuit of $N_0' \backslash a_k$, a contradiction.

It remains to deal with the cases when, in N_0, either $\{a_{k-2}, a_{k-1}, a_k\}$ is a triad and $\{a_{k-1}, a_k\}$ is not a circuit, or $\{a_{k-2}, a_{k-1}, a_k\}$ is a triangle. In these cases, add a_0 in parallel with a_1 to produce N_3. To obtain an N_0-minor of N_3 using $\{a_0, a_1\}$, we must delete an element z of N_3 that belongs to a 2-circuit. Now z is not in $\{a_2, a_3, \dots, a_k\}$ as none of these elements is in a 2-circuit, so $N_3 \backslash z$ is isomorphic to N_0 and has $(a_0, a_1, \dots, a_k)$ as a special fan. This contradicts our assumption that a special fan in N_0 or N_0^* has at most k elements. We conclude that 2.6.5.1 holds.

2.6.5.2. *N has no single-element component.*

To see this, let N_0 be a largest component of N. By 2.6.5.1, N_0 is isomorphic to $U_{1,2}$ or $M(\mathcal{W}_2)$. Assume that N has a single-element component N_1 with $E(N_1) = \{a\}$. By replacing N by its dual if necessary, we may assume that a is a coloop of N. Let c be an element that is in a 2-cocircuit of N_0. Now let N' be obtained from N by adding an element b so that N' has $\{a, b, c\}$ as a triangle and $\{a, b\}$ as a cocircuit. Then, by the transitivity property, $N' \backslash g \cong N$ for some element g not in $\{a, b\}$. By the choice of N_0, we deduce that g must be in the same component N_0' of N' as $\{a, b, c\}$. Moreover, g must be in a 2-cocircuit of N_0'. But N_0' contains no such element. Hence 2.6.5.2 holds.

2.6.5.3. *N has a single component of maximum size.*

Assume that this fails, letting N_0 and N_1 be components of N of maximum size. Let $\{a_i, b_i\}$ be a 2-circuit of N_i. Let N_i' be obtained from N_i by adding c_i in series with b_i. Now take a copy of $U_{2,3}$ with ground set $\{c_0, z, c_1\}$ and adjoin N_0' and N_1' via parallel connection across c_0 and c_1, respectively. Truncate the resulting matroid to get N_{01}. Then $r(N_{01}) = r(N_0) + r(N_1) + 1$. Let N' be obtained from N by replacing $N_0 \oplus N_1$ by N_{01}. Now

N_{01}/c_0 and N_{01}/c_1 have $(N_0 \oplus N_1)$-minors using $\{z, c_1\}$ and $\{z, c_0\}$, respectively. Hence N'/c_0 and N'/c_1 have N-minors using $\{z, c_1\}$ and $\{z, c_0\}$. Thus, by transitivity, N' has an N-minor $\tilde{N}$ using $\{c_0, c_1\}$. As $r(N') = r(N) + 1$, there are elements e, f, and g of $E(N') - \{c_0, c_1\}$ such that $\tilde{N} = N'/e\backslash f, g$. Now N'/e must have two disjoint 2-circuits that are not in N'. Thus $e \in E(N_{01})$. As $e \notin \{c_0, c_1\}$, it follows that $N_0 \cong M(\mathcal{W}_2) \cong N_1$ and, by symmetry, we may assume that $e = a_0$. But N_{01}/a_0 does not have an $(M(\mathcal{W}_2) \oplus M(\mathcal{W}_2))$-minor. Thus 2.6.5.3 holds.

By 2.6.5.1 and 2.6.5.3, N has a single largest component N_0 and it is isomorphic to $M(\mathcal{W}_2)$. As N is disconnected, we may assume by duality that N has a component N_1 that is isomorphic to $U_{1,k}$ for some k in $\{2, 3\}$. Now take a copy of $U_{2,3}$ with ground set $\{c_0, z, c_1\}$ and adjoin copies of $U_{2,k+1}$ via parallel connection across c_0 and c_1, letting the resulting matroid be N_{01}. Replacing $N_0 \oplus N_1$ by N_{01} in N to give N', we see that $r(N') = r(N)+1$. Moreover, N'/c_0 and N'/c_1 have N-minors using $\{c_1, z\}$ and $\{c_0, z\}$, respectively. But c_0 and c_1 are the only elements e of N' such that N'/e has two disjoint 2-circuits that are not in N'. Thus N' has no N-minor using $\{c_0, c_1\}$. This contradiction completes the proof of the theorem. $\square$

We conclude this section by proving Corollary 2.1.3, which demonstrates how two of the basic properties of matroid connectivity are enough to characterize it.

Proof of Corollary 2.1.3. Assume that $N \ncong U_{1,2}$. Then, by Theorem 2.1.1 and duality, we may assume that $N \cong U_{2,2}$. But $U_{2,2}$ does not have the transitivity property as the matroid $U_{1,2} \oplus U_{1,1}$ shows. $\square$

2.7. Three-element sets

The notion of N-connectivity defined here relies on sets of two elements. Sets of size three have already been an object of some study. Seymour asked whether every 3-element set in a 4-connected non-binary matroid belongs to a $U_{2,4}$-minor but Kahn [10] and Coullard [4] answered this question negatively. Seymour [22] characterized the internally 4-connected binary matroids that are $U_{2,3}$-connected, but the problem of completely characterizing when every triple of elements in an internally 4-connected matroid is in a $U_{2,3}$-minor remains open [14, Problem 15.9.7].

For a 3-connected binary matroid M having rank and corank at least three, Theorem 2.4.3 shows that every triple of elements of M is in an $M(K_4)$-minor. The next result extends this theorem to connected binary matroids.

Proposition 2.7.1. *Let M be a connected binary matroid. For every triple $\{x, y, z\} \subseteq E(M)$, there is an $M(K_4)$-minor using $\{x, y, z\}$ if and only if every matroid that labels a vertex in the canonical tree decomposition of M has rank and corank at least* 3.

Proof. Suppose every matroid that labels a vertex in the canonical tree decomposition T of M has rank and corank at least 3. Consider a triple $\{x, y, z\}$ of elements of $E(M)$ and let M_1, M_2 and M_3 be the respective matroids of T that use them. If $M_1 = M_2 = M_3$, then $\{x, y, z\}$ is in an $M(K_4)$-minor by restricting Theorem 2.4.3 to the binary case. If two of $\{x, y, z\}$ are in the same vertex of T, say $M_1 = M_2$, and M_1 has degree d in T, then, by repeated application of Lemma 2.2.4, M has a specially relabeled M_1-minor $M_1(z, y_2, y_3, \ldots, y_d)$, and again by Theorem 2.4.3, we can find an $M(K_4)$-minor using $\{x, y, z\}$.

Now suppose that M_1, M_2, M_3 are distinct vertices of T. Suppose without loss of generality that M_2 is on the path P between M_1 and M_3 in T. Let p_1 be the basepoint of M_2 closest to M_1 on P, and let p_3 be the basepoint of M_2 closest to M_3 on P. By Lemma 2.2.4, x and z can replace distinct basepoints of M_2 in a specially relabeled M_2-minor $M_2(x, z, y_3, y_4, \ldots, y_d)$ of M. Thus $\{x, y, z\}$ is in an M_2-minor of M. By restricting Theorem 2.4.3 to the binary case, we see that M has an $M(K_4)$-minor using $\{x, y, z\}$.

Conversely, suppose M has a matroid M' that labels a vertex in its canonical tree decomposition T and has rank or corank less than 3. In the case when $E(M')$ contains two elements x, y of $E(M)$, observe, by Lemma 2.2.5, that if $\{x, y\}$ is in an $M(K_4)$-minor N of M, then N is a minor of M'. However, M' has no $M(K_4)$-minor.

Suppose $E(M')$ does not contain two elements of $E(M)$. Then M' has degree at least two in T. Choose a triple $\{x, y, z\}$ of elements in $E(M)$ in distinct matroids M_1, M_2, and M_3 such that either $M_2 = M'$ and M_2 is on the path between M_1 and M_3 in T, or $M_2 \neq M'$ and the subgraph induced by the union of all the paths between M_1, M_2, and M_3 in T forms a subdivision of a star $K_{1,3}$ with M' in the center. In either case, consider an edge p of T incident with M'. In the 2-separation displayed by p, there will be at most one of $\{x, y, z\}$ that is not on the same side as M'. Therefore, by Lemma 2.2.5, any $M(K_4)$-minor of M using $\{x, y, z\}$ must be a minor of a specially relabeled M'-minor. Since M' has rank or corank less than 3, this is a contradiction. $\square$

Chapter 3. 2-polymatroids

3.1. Introduction

Tutte [23] proved that, whenever e is an element of a connected matroid M, at least one of $M\backslash e$ and M/e is connected. Brylawski [3] and Seymour [19] independently extended this theorem by showing that if N is a connected minor of M and e is in $E(M)-E(N)$, then $M\backslash e$ or M/e is connected having N as a minor. Here we prove a similar result for 2-polymatroids. The following is the main result of this chapter.

Theorem 3.1.1. *Let M be a connected 2-polymatroid and let N be a connected minor of M. When $N \neq M$, there is an element e of $E(M) - E(N)$ such that $M\backslash e$ or M/e is connected having N as a minor.*

Unlike in the matroid case, it is not true that, for every element e of $E(M)-E(N)$, at least one of $M\backslash e$ and M/e is connected having N as a minor. For example, let $E(M) = \{x, y, z\}$ where x, y, and z are lines, $r(\{x, y\}) = r(\{y, z\}) = 3$, and $r(\{x, z\}) = 4$. Let N be the 2-polymatroid consisting of a single line z. Then both $M\backslash y$ and M/y are disconnected, as the former consists of two lines in rank 4, and the latter is isomorphic to the matroid $U_{2,2}$.

Theorem 3.1.1 will be proved in Section 3.3. The next section includes a number of preliminaries needed for this proof. In Section 3.4, we consider what can be said about the uniqueness of the element e in Theorem 3.1.1.

This chapter is adapted from Z. Gershkoff and J. Oxley, A note on the connectivity of 2-polymatroid minors, *Electron. J. Combin.* **26** (2019), Paper 4.21, 11pp. It is reprinted with permission from The Electronic Journal of Combinatorics.

3.2. Preliminaries

Much of the notation from matroid theory carries over to polymatroids. For instance, when M is the polymatroid (E, r) and $T \subseteq E$, the deletion $M\backslash(E-T)$ is also denoted by $M|T$. Moreover, we frequently write $r(M)$ for $r(E)$. A subset S of E *spans* a subset T if $r(S \cup T) = r(S)$. A *component* of M is a maximal non-empty subset X of E such that $M|X$ is connected. As for matroids, the *connectivity function* λ_M or λ of M is defined for all subsets X of $E(M)$ by $\lambda_M(X) = r(X) + r(E-X) - r(E)$. For a positive integer j and a subset Z of $E(M)$, we call Z and $(Z, E(M)-Z)$ *j-separating* if $\lambda_M(Z) < j$.

The *local connectivity* $\sqcap(X, Y)$ between subsets X and Y of E is given by $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$. Thus $\sqcap(X, E-X) = \lambda(X)$. The following useful results for local connectivity and connectivity are proved for matroids in [14, Lemma 8.2.3]; the proofs there extend to polymatroids.

Lemma 3.2.1. *Let (E, r) be a polymatroid and let X_1, X_2, Y_1, and Y_2 be subsets of E with $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. Then*

$$\sqcap(Y_1, Y_2) \leq \sqcap(X_1, X_2).$$

Lemma 3.2.2. *Let (E, r) be a polymatroid M and let X, C, and D be disjoint subsets of E. Then*

$$\lambda_{M\backslash D/C}(X) \leq \lambda_M(X).$$

Moreover, equality holds if and only if

$$r(X \cup C) = r(X) + r(C)$$

and

$$r(E-X) + r(E-D) = r(E) + r(E-(X \cup D)).$$

Next we note a useful consequence of Lemma 3.2.1.

Corollary 3.2.3. *Let X and Y be sets in a polymatroid M such that $X \cap Y \neq \emptyset$ and both $M|X$ and $M|Y$ are connected. Then $M|(X \cup Y)$ is connected.*

Proof. Suppose that $M|(X \cup Y)$ is disconnected letting Z be a component of it. Let $W = (X \cup Y) - Z$. By Lemma 3.2.1, $\sqcap(Z \cap X, W \cap X) \leq \sqcap(Z, W) = 0$. As $M|X$ is connected, $Z \cap X$ or $W \cap X$ is empty. By symmetry, $Z \cap Y$ or $W \cap Y$ is empty. As neither Z nor W is empty, we may assume that both $Z \cap X$ and $W \cap Y$ are empty. It follows that $X \cap Y$ is empty; a contradiction. □

The following generalization of a matroid result was noted in [16, Lemma 3.12(ii)].

Lemma 3.2.4. *Let A, B, and C be subsets of the ground set of a polymatroid. Then*

$$\sqcap(A \cup B, C) + \sqcap(A, B) = \sqcap(A \cup C, B) + \sqcap(A, C).$$

We omit the proof of the next result, which follows easily from Lemma 3.2.2.

Lemma 3.2.5. *If Z is a component of a minor of a polymatroid M, then Z is contained in a component of M, and $M \backslash Z = M/Z$.*

As noted in [14, p.409], with every 2-polymatroid M, we can associate a matroid as follows. Let L be the set of lines of M. For each ℓ in L, freely add two points to ℓ letting M^+ be the resulting 2-polymatroid. Then M', the *natural matroid derived from M*, is $M^+ \backslash L$. Oxley, Semple, and Whittle [16, Lemma 3.3] noted the following straightforward result.

Lemma 3.2.6. *Let M be a 2-polymatroid with $|E(M)| \geq 2$ and let M' be the natural matroid derived from M. Then M is connected if and only if M' is connected.*

The proof of our main theorem will use the operations of parallel connection and 2-sum of polymatroids as introduced by Matúš [11] and Hall [9]. For a positive integer k, let M_1 and M_2 be k-polymatroids (E_1, r_1) and (E_2, r_2). Suppose first that $E_1 \cap E_2 = \emptyset$. The *direct sum* $M_1 \oplus M_2$ of M_1 and M_2 is the k-polymatroid $(E_1 \cup E_2, r)$ where, for all subsets A of $E_1 \cup E_2$, we have $r(A) = r(A \cap E_1) + r(A \cap E_2)$. Clearly a 2-polymatroid is connected if and only if it cannot be written as the direct sum of two non-empty 2-polymatroids. Now suppose that $E_1 \cap E_2 = \{p\}$ and $r_1(\{p\}) = r_2(\{p\})$. Let $P(M_1, M_2)$ be $(E_1 \cup E_2, r)$ where r is defined for all subsets A of $E_1 \cup E_2$ by

$$r(A) = \min\{r_1(A \cap E_1) + r_2(A \cap E_2), r_1((A \cap E_1) \cup p) + r_2((A \cap E_2) \cup p) - r_1(\{p\})\}.$$

Hall [9] notes that it is routine to check that $P(M_1, M_2)$ is a k-polymatroid. We call it the *parallel connection* of M_1 and M_2 with respect to the *basepoint* p. When M_1 and M_2 are both matroids, this definition coincides with the usual definition of the parallel connection of matroids.

Now let M_1 and M_2 be 2-polymatroids having at least two elements. Suppose that $E(M_1) \cap E(M_2) = \{p\}$, that neither $\lambda_{M_1}(\{p\})$ nor $\lambda_{M_2}(\{p\})$ is 0, and that $r_1(\{p\}) = r_2(\{p\}) = 1$. We define the 2-*sum*, $M_1 \oplus_2 M_2$, of M_1 and M_2 to be $P(M_1, M_2)\backslash p$. This definition [16] extends Hall's definition since the latter requires each of M_1 and M_2 to have at least three elements. Weakening that requirement does not alter the validity of Hall's proof of the following result [9, Proposition 3.6].

Proposition 3.2.7. *Let M be a 2-polymatroid (E, r) having a partition (X_1, X_2) of E such that $r(X_1) + r(X_2) = r(E) + 1$. Then there are 2-polymatroids M_1 and M_2 with ground sets $X_1 \cup p$ and $X_2 \cup p$, where p is a new element not in E, such that $M = P(M_1, M_2)\backslash p$.*

In particular, for all $A \subseteq X_1 \cup p$,

$$r_1(A) = \begin{cases} r(A), & \text{if } p \notin A; \\ r((A-p) \cup X_2) - r(X_2) + 1, & \text{if } p \in A. \end{cases}$$

The following was shown by Hall [9, Corollary 3.5].

Proposition 3.2.8. *Let M_1 and M_2 be 2-polymatroids (E_1, r_1) and (E_2, r_2) where $E_1 \cap E_2 = \{p\}$. Suppose $r_1(\{p\}) = r_2(\{p\}) = 1$ and each of M_1 and M_2 has at least two elements. Then the following are equivalent.*

(i) *M_1 and M_2 are both 2-connected;*

(ii) *$M_1 \oplus_2 M_2$ is 2-connected; and*

(iii) *$P(M_1, M_2)$ is 2-connected.*

The next theorem, a special case of a result of Hall [9, Theorem 4.3], will play a crucial role in the proof of our main theorem.

Theorem 3.2.9. *Every connected 2-polymatroid M having at least two elements has distinct elements x and y such that each of $\{M\backslash x, M/x\}$ and $\{M\backslash y, M/y\}$ contains a connected 2-polymatroid.*

The following lemma holds for polymatroids in general and will be useful in Section 3.4.

Lemma 3.2.10. *Let M be a connected polymatroid and let M/e be disconnected. If Z is a component of M/e, then $\sqcap(Z, \{e\}) > 0$.*

Proof. Let $Y = E(M) - (Z \cup e)$. Then

$$r_{M/e}(Z) + r_{M/e}(Y) = r(M/e) = r(M) - r_M(\{e\}). \tag{3.1}$$

Moreover, since M is connected,

$$r_M(Z) + r_M(Y \cup e) > r(M). \tag{3.2}$$

By the definition of local connectivity, $r_M(Y) + r_M(\{e\}) - \sqcap(Y, \{e\}) = r(Y \cup e)$, so we can rewrite (3.2) as

$$r_M(Z) + r_M(Y) > r(M) - r_M(\{e\}) + \sqcap(Y, \{e\}). \tag{3.3}$$

By subtracting (3.1) from (3.3), we obtain

$$(r_M(Z) - r_{M/e}(Z)) + (r_M(Y) - r_{M/e}(Y)) > \sqcap(Y, \{e\}). \tag{3.4}$$

The differences on the left-hand side can be rewritten as local connectivities. Thus $\sqcap(Z, \{e\}) + \sqcap(Y, \{e\}) > \sqcap(Y, \{e\})$, so $\sqcap(Z, \{e\}) > 0$. $\square$

3.3. A splitter theorem for connected 2-polymatroids

This section is devoted to proving the main result of this chapter.

Proof of Theorem 3.1.1. Assume that the theorem fails. Then it follows from Theorem 3.2.9 that N is non-empty. Hence, as M is connected, it has no loops. Next we note the following.

3.3.1.1. *If x is an element of M such that both $M\backslash x$ and M/x have N as a minor, then x is a line of M.*

Clearly neither $M\backslash x$ nor M/x is connected. It follows, by a result of Oxley and Whittle [17, Theorem 3.1], that $r_M(\{x\}) \neq 1$. Hence x is a line of M.

Take e in $E(M) - E(N)$. Then, for some M_0 in $\{M\backslash e, M/e\}$, the 2-polymatroid M_0 has N as a minor. By assumption, M_0 is not connected. Take an element f of a com-

ponent of M_0 that avoids $E(N)$. Then both $M\backslash f$ and M/f have N as a minor. Thus, by 3.3.1.1, f is a line of M. Moreover,

3.3.1.2. $r(M\backslash f) = r(M)$.

To see this, suppose $r(E - f) < r(E)$. Then, as M is connected, $r(E - f) = r(E) - 1$. Let M' be the natural matroid derived from M and let f_1 and f_2 be the points of M' corresponding to f. Then $M'\backslash f_1$ has f_2 as a coloop, so M'/f_1 is connected. Now M/f is disconnected, so, by Lemma 3.2.6, $M'/f_1, f_2$ is disconnected. Therefore, $M'/f_1\backslash f_2$ is connected. But $M'\backslash f_2$ has f_1 as a coloop, so $M'\backslash f_2/f_1 = M'\backslash f_2\backslash f_1$. As this matroid is connected, by Lemma 3.2.6, $M\backslash f$ is too; a contradiction.

3.3.1.3. *Let K be a component of M/f. Then $M|(K \cup f)$ is connected.*

Suppose $M|(K \cup f)$ is disconnected. Then K is the disjoint union of sets X and Y such that $r(X \cup f) + r(Y) = r(K \cup f)$. As Y is 1-separating in $M|(K \cup f)$, it is 1-separating in $(M|(K \cup f))/f$, that is, in $(M/f)|K$. But K is a component of the last matroid, so $K = Y$. Thus $X = \emptyset$, so $r(K \cup f) = r(K) + r(\{f\})$. It follows that K is 1-separating in M; a contradiction. Hence 3.3.1.3 holds.

Now let F be a component of M/f that avoids $E(N)$. By 3.3.1.1, every element of F is a line in M. Let $G = E(M) - f - F$. By 3.3.1.3, $M|(F \cup f)$ is connected. Next we show the following.

3.3.1.4. *There is a line g in F such that $(M|(F \cup f))\backslash g$ or $(M|(F \cup f))/g$ is connected. Moreover, $\sqcap(\{f\}, \{g\}) < 2$.*

By Theorem 3.2.9, F contains an element g such that $(M|(F \cup f))\backslash g$ or $(M|(F \cup$

$f))/g$ is connected. As g is in F, we see that g is a line. Since the theorem fails, $M\backslash g$ is not connected, so $\sqcap(\{f\},\{g\}) < 2$. Thus 3.3.1.4 holds.

3.3.1.5. *$M|(G \cup f)$ is connected; $(M|(F \cup f))/g$ is connected; and $\sqcap(\{f\},\{g\}) = 1$.*

To see this, first note that, by 3.3.1.3, $M|(K \cup f)$ is connected for each component K of M/f. Then, by Corollary 3.2.3, $M|(G \cup f)$ is connected. The same argument shows that $(M|(F \cup f))\backslash g$ is disconnected for if it is connected, then so is $M\backslash g$; a contradiction. Thus, by 3.3.1.4, $(M|(F \cup f))/g$ is connected.

Now suppose that $\sqcap_M(\{f\},\{g\}) = 0$. Then, as $\sqcap_{M/f}(G,\{g\}) = 0$, one easily checks that $\sqcap_M(G \cup f, \{g\}) = 0$. Hence $M|(G \cup f) = (M|(G \cup f \cup g))/g = (M/g)|(G \cup f)$. Thus, as $(M|(F \cup f))/g$ and $M|(G \cup f)$ are connected and both contain f, Corollary 3.2.3 implies that M/g is connected; a contradiction. Hence 3.3.1.5 holds.

Recall that f is a line of M such that $M\backslash f$ and M/f are disconnected. Moreover, F is a component of M/f and $E(N) \subseteq G = E(M) - f - F$. Let A be a component of $M\backslash f$ avoiding $E(N)$ and let $B = E(M) - f - A$. The next two observations follow because M is connected.

3.3.1.6. *Neither A nor B spans f.*

3.3.1.7. *$r(G \cup f) < r(G) + 2$ and $r(F \cup f) < r(F) + 2$.*

Next we show the following.

3.3.1.8. *At least one of $A \cap G$, $A \cap F$, $B \cap F$, and $B \cap G$ is empty.*

Suppose that all four intersections are non-empty. By 3.3.1.2, $r(E - f) = r(E)$. Thus $r(A) + r(B) = r(E)$ and $r(F \cup f) + r(G \cup f) = r(E) + 2$. Adding these two equations

and applying submodularity to the left-hand side gives

$$r(A \cup F \cup f) + r(A \cap F) + r(B \cup G \cup f) + r(B \cap G) \leq 2r(E) + 2,$$

so

$$[r(A \cup F \cup f) + r(B \cap G)] + [r(B \cup G \cup f) + r(A \cap F)] \leq 2r(E) + 2. \tag{3.5}$$

As $(A \cup F \cup f, B \cap G)$ and $(B \cup G \cup f, A \cap F)$ are partitions of $E(M)$, we deduce, since M is connected, that equality holds in (3.5). Hence the two specified partitions are 2-separating in M. By symmetry, so are $(A \cup G \cup f, B \cap F)$ and $(B \cup F \cup f, A \cap G)$. By Propositions 3.2.7 and 3.2.8, M can be written as the 2-sum with basepoint p_{AF} of two connected 2-polymatroids, one with ground set $(A \cap F) \cup p_{AF}$ and the other, Q_0, with ground set $(E(M) - (A \cap F)) \cup p_{AF}$. By arguing in terms of the natural matroid derived from M, it is straightforward to check that, in Q_0, each of $A \cap G, B \cap F$, and $B \cap G$ is 2-separating. Hence we can decompose Q_0 as a 2-sum of two connected 2-polymatroids one with ground set $(A \cap G) \cup p_{AG}$. Repeating this process twice more, we obtain a connected 2-polymatroid Q with ground set $\{f, p_{AF}, p_{AG}, p_{BF}, p_{BG}\}$ where M is obtained from Q by attaching, via 2-sums, connected 2-polymatroids with ground sets $(A \cap F) \cup p_{AF}$, $(A \cap G) \cup p_{AG}$, $(B \cap F) \cup p_{BF}$, and $(B \cap G) \cup p_{BG}$.

As $M|A$ is connected, Proposition 3.2.8 implies that p_{AG} and p_{AF} are parallel in Q. Since $(M/f)|F$ is connected, p_{BF} and p_{AF} are parallel in Q/f. But p_{AG} and p_{AF} are also parallel in Q/f unless they are loops. In the exceptional case, $A \cap F$ contains a component of M/f; a contradiction. We deduce that the component of M/f containing F also contains $A \cap G$; a contradiction. Thus 3.3.1.8 holds.

By 3.3.1.8, A or B is contained in F or G, and F or G is contained in A or B. We

know that $B \cap G$ is non-empty because it contains $E(N)$.

Suppose both F and G span f. Then A or B spans f; a contradiction to 3.3.1.6. By 3.3.1.7, there are two remaining cases to consider:

(i) $r(F \cup f) = r(F) + 1$; and

(ii) $r(F \cup f) = r(F)$ and $r(G \cup f) = r(G) + 1$.

By 3.3.1.5, $(M|(F \cup f))/g$ is connected and $\sqcap(\{f\}, \{g\}) = 1$. Thus $r(\{f, g\}) = 3$. Assume (i) holds. Then $\sqcap_{M/g}(F - g, \{f\}) = r(F) + r(\{f, g\}) - r(F \cup f) - r(\{g\}) = 0$. Thus $\{f\}$ is a component of $(M|(F \cup f))/g$. As the last matroid is connected, we deduce that $F = \{g\}$. Thus $M \backslash g = M|(G \cup f)$ so, by 3.3.1.5, $M \backslash g$ is connected; a contradiction.

We now know that (ii) holds. As neither A nor B spans f, neither has F as a subset. Thus both $A \cap F$ and $B \cap F$ are non-empty. As $B \cap G$ is non-empty, 3.3.1.8 implies that $A \cap G$ is empty. Then $G \subseteq B$. But $r(G \cup f) = r(G) + 1$. Therefore $r(B \cup f) \leq r(B) + 1$. Since $r(B \cup f) \neq r(B)$, it follows that $r(B \cup f) = r(B) + 1$.

We have $r(A) + r(B) = r(M \backslash f)$, and, by 3.3.1.2, $r(M \backslash f) = r(M)$. As $r(B \cup f) = r(B) + 1$, we deduce that $r(A) + r(B \cup f) = r(M) + 1$, so M can be written as a 2-sum with basepoint p of two connected 2-polymatroids with ground sets $A \cup p$ and $B \cup f \cup p$. Let the former be M_1.

Suppose M_1 has at least three elements. Then, by Theorem 3.2.9, M_1 has an element q such that $q \neq p$ and $M_1 \backslash q$ or M_1/q is connected. Observe that if M_1/q is connected, then $\sqcap(\{q\}, \{p\}) = 0$ otherwise p is a loop in the contraction; a contradiction. It follows, by [16, Lemma 4.3], that $M \backslash q$ or M/q is the 2-sum of two connected 2-polymatroids each with at least two elements. Thus, by Proposition 3.2.8, $M \backslash q$ or M/q is connected. As q is in A and hence in F, both $M \backslash q$ and M/q have N has a minor and so

we obtain a contradiction.

We may now assume that M_1 consists of a single line a through p.

3.3.1.9. *M/a is connected.*

Assume M/a is disconnected. Then its ground set has a partition (V, W) such that $r_{M/a}(V) + r_{M/a}(W) = r(M/a)$. Now we may assume that f is in V. Thus $W \subseteq B$ since $A = \{a\}$. As $\sqcap_M(A, B) = 0$, it follows that $r_{M/a}(W) = r_M(W)$. Hence $r_M(V \cup a) + r_M(W) = r(M)$; a contradiction. We conclude that 3.3.1.9 holds.

As $a \in F$, we know that M/a has N as a minor. Thus we have a contradiction that completes the proof of the theorem. $\square$

The argument above relies heavily on the fact that we have a 2-polymatroid. However, we believe that the main theorem also holds for k-polymatroids for all $k > 2$.

Conjecture 3.3.2. *Let M be a connected k-polymatroid and let N be a connected minor of M. When $N \neq M$, there is an element e of $E(M) - E(N)$ such that $M\backslash e$ or M/e is connected having N as a minor.*

3.4. Uniqueness

By Theorem 3.1.1, for every connected 2-polymatroid M and every connected proper minor N of M, we can remove the elements of $E(M) - E(N)$ one at a time so that we stay connected and maintain N as a minor. In this section, we consider what can be said about the uniqueness of this sequence of element removals

Now let M be a connected polymatroid and N be a connected proper minor of M. An *admissible ordering* of $E(M) - E(N)$ is an ordering $(a_1, a_2, \ldots, a_n)$ of the set $E(M) - E(N)$ such that, for each k in $\{1, 2, \ldots, n\}$, there is a connected minor M_k of M with

ground set $E(M) - \{a_1, a_2, \ldots, a_k\}$ such that M_k is a minor of M_{k-1}, where $(M_0, M_n) = (M, N)$. We give an example below to show that an admissible ordering may be unique. We shall show, however, that we always retain some flexibility with respect to the way in which the elements are removed unless $|E(M) - E(N)| = 1$. Formally, a *constrained admissible ordering* is an ordering $((\alpha_1, a_1), (\alpha_2, a_2), \ldots, (\alpha_n, a_n))$ such that $E(M) - E(N) = \{a_1, a_2, \ldots, a_n\}$ where each α_i is a deletion or contraction operation, and, for each k in $\{1, 2, \ldots, n\}$, there is a connected minor M_k of M with ground set $E(M) - \{a_1, a_2, \ldots, a_k\}$ where M_k is obtained from M_{k-1} by removing a_k by the operation designated by α_k, and $(M_0, M_n) = (M, N)$.

To construct a 2-polymatroid with a unique admissible ordering, let N be a simple non-empty connected matroid. Take $N \oplus U_{n,n}$ where the ground set of $U_{n,n}$ is $\{b_0, b_1, \ldots, b_{n-1}\}$. Take $b_n \in E(N)$ and consider the 2-polymatroid M whose ground set is $E(N) \cup \{f_i : 1 \leq i \leq n\}$ where $f_i = \{b_{i-1}, b_i\}$ for all i, and the rank function of M is induced by that of $N \oplus U_{n,n}$. Then M is connected, $M \backslash f_1, f_2, \ldots, f_k$ is connected for all k in $\{1, 2, \ldots, n\}$, and $M \backslash f_1, f_2, \ldots, f_n = N$. Thus $(f_1, f_2, \ldots, f_n)$ is an admissible ordering of $E(M) - E(N)$. It is not difficult to check that the admissible ordering is unique. Note, however, that $M \backslash f_1 \backslash f_2 = M / f_1 \backslash f_2$, so this example does not give us a unique constrained admissible ordering. Indeed, as the next result shows, except in the trivial case, there can never be such a unique ordering.

Theorem 3.4.1. *Let M be a connected 2-polymatroid and N be a connected proper minor of M. Then there is a unique constrained admissible ordering of $E(M) - E(N)$ if and only if $|E(M) - E(N)| = 1$.*

The next two lemmas contain the core of the proof of this theorem.

Lemma 3.4.2. *Let each of $\dagger$ and $\ddagger$ denote a deletion or contraction operation. Suppose both $M\dagger e$ and $M\dagger e\ddagger f$ are connected, but $M\ddagger f$ is not. Then $\{e\}$ and Z are the components of $M \ddagger f$. Moreover,*

$$M \ddagger f\backslash e = M \ddagger f/e.$$

Proof. Let (X, Y) be a 1-separating partition of $E(M \ddagger f)$ with Y minimal and non-empty avoiding e. Then $\lambda_{M\ddagger f}(Y) = 0$. Thus, by Lemma 3.2.2, $\lambda_{M\ddagger f\dagger e}(Y) = 0$. As $M \dagger e \ddagger f$ is connected, $X - e = \emptyset$. Thus $\{e\}$ and Z are components of $M \ddagger f$. Hence, by Lemma 3.2.5, $M \ddagger f\backslash e = M \ddagger f/e$. □

Lemma 3.4.3. *Let M be a connected polymatroid and N be a connected minor of M. Let e and f be distinct elements of $E(M) - \{e, f\}$ and let $Z = E(M) - \{e, f\}$. Suppose that $\{\alpha, \beta\} = \{\backslash, /\} = \{\gamma, \delta\}$. Assume that $M\alpha e$ and $M\alpha e\gamma f$ are connected.*
Then

(i) *$M\gamma f$ is connected and $M\alpha e\gamma f = M\gamma f\alpha e$; or*

(ii) *$M\gamma f$ is disconnected, $M\alpha e\gamma f = M\beta e\gamma f$, and*

(a) *$M\beta e$ is connected; or*

(b) *$M\beta e$ is disconnected, $M\alpha e\gamma f = M\delta f\beta e$, and $M\delta f$ is connected; or*

(c) *$M\beta e$ and $M\delta f$ are disconnected, and $M\alpha e\gamma f = M\alpha e\delta f$.*

Proof. We may assume that $M\gamma f$ is disconnected otherwise (i) holds. As $M\alpha e\gamma f$ is connected, Lemma 3.4.2 implies that $\{e\}$ is a component of $M\gamma f$, and $M\gamma f\backslash e = M\gamma f/e$. Thus

$$M\alpha e\gamma f = M\gamma f\alpha e = M\gamma f\beta e = M\beta e\gamma f.$$

We may assume that $M\beta e$ is disconnected otherwise (ii)(a) holds. Then $M\beta e$ has $\{f\}$ as a

component and $M\beta e\backslash f = M\beta e/f$. Thus

$$M\alpha e\gamma f = M\beta e\gamma f = M\beta e\delta f = M\delta f\beta e.$$

We may now assume that $M\delta f$ is disconnected otherwise (ii)(b) holds. Then $M\delta f\backslash e = M\delta f/e$. Thus $M\delta f\beta e = M\delta f\alpha e$. Hence $M\alpha e\gamma f = M\delta f\beta e = M\delta f\alpha e = M\alpha e\delta f$, and (ii)(c) holds. $\square$

We are now able to prove the main result of this section.

Proof of Theorem 3.4.1. We may assume that $|E(M) - E(N)| \geq 2$. Let $((\alpha, e), (\gamma, f), (\alpha_3, a_3) \ldots, (\alpha_k, a_k))$ be a constrained admissible ordering of $E(M) - E(N)$. We use Lemma 3.4.3 to show that $E(M) - E(N)$ has a constrained admissible ordering $((\alpha_1, a_1), (\alpha_2, a_2), (\alpha_3, a_3), \ldots, (\alpha_k, a_k))$ in which $((\alpha, e), (\gamma, f)) \neq ((\alpha_1, a_1), (\alpha_2, a_2))$ and $\{e, f\} = \{a_1, a_2\}$. If $M\gamma f$ is connected, then we can take $((\alpha_1, a_1), (\alpha_2, a_2))$ to be $((\gamma, f), (\alpha, e))$. Using the notation of Lemma 3.4.3, if $M\gamma f$ is disconnected but $M\beta e$ is connected, then we can take $((\alpha_1, a_1), (\alpha_2, a_2))$ to be $((\beta, e), (\gamma, f))$. Now suppose that $M\gamma f$ and $M\beta e$ are disconnected. If $M\delta f$ is connected, then we can take $((\alpha_1, a_1), (\alpha_2, a_2))$ to be $((\delta, f), (\beta, e))$. Finally, if $M\delta f$ is disconnected, then we can take $((\alpha_1, a_1), (\alpha_2, a_2))$ to be $((\alpha, e), (\delta, f))$. $\square$

Chapter 4. Cubic Binary Matroids

4.1. Introduction

In this chapter, we introduce the notion of a cubic binary matroid. Recall from Section 1.7 that the cocircuit space of a binary matroid M is the set of vectors generated by the rows of a representation of M. A connected binary matroid is *cubic* if the cocircuit space has a basis consisting of vectors with support of size three such that the sum of these basis vectors also has a support of size three. Equivalently, a cubic binary matroid is a connected matroid having a binary representation A where every row has exactly three ones, and the row obtained from the GF(2)-sum of the other rows has exactly three ones.

As will be shown in Section 4.5, requiring cubic binary matroids to be connected, we have that the members of the basis of the cocircuit space along with their symmetric difference all correspond to triads. We will refer to this set as the set of *distinguished triads*.

The cycle matroid of a 2-connected cubic graph is always a cubic binary matroid, because the vertex-edge incidence matrix of the graph certifies that we have the required basis for the cocircuit space.

4.2. Preliminaries

A *matrix* 3*-certificate* of a rank-r matroid M is a matrix over GF(2) having $r + 1$ rows, where each row has precisely three ones, the first r rows give a representation of M, and the last row is the sum of the others. The existence of a matrix 3-certificate for M shows that M is a cubic binary matroid, provided that M is connected.

Theorem 4.2.1. *Let M be a cubic binary matroid having rank r and size n. Then*

$$2n \leq 3(r+1).$$

Proof. Let A be a matrix certificate of M. Then A has $r+1$ rows and n columns. Since A has three ones per row, the number of ones in A is $3(r+1)$. Each column of A has a positive even number of ones, since the columns sum to zero over GF(2), and M has no loops. Thus the number of ones in A is at least $2n$, and the result follows. □

Corollary 4.2.2. *Let $G = (V, E)$ be a connected graph for which $M(G)$ is a cubic binary matroid. Then $2|E| \leq 3|V|$. Equivalently, the average degree of G is at most 3.*

Proof. Recall that, for the cycle matroid of a graph, the edges correspond to the elements, and the rank is the number of vertices minus the number of graph components. Since the sum of all the degrees in the graph is $2|E|$, the average degree, $\frac{2|E|}{|V|}$, cannot exceed 3. □

Lemma 4.2.3. *Let A be a matrix 3-certificate of a cubic binary matroid M. Suppose A has m ones. Then every matrix 3-certificate of M has m ones.*

Proof. This is a simple consequence of the fact that every matrix 3-certificate has the same number of rows, and every row has exactly three ones. □

Theorem 4.2.4. *[24, Theorem 2, Section 10.5] If a matrix 3-certificate of a cubic binary matroid M has exactly two ones in every column, then M is graphic.*

4.3. Cubic binary matroid connectivity

Since every cubic binary matroid has a cocircuit of size at most three, the upper bound on the connectivity of a cubic binary matroid is 3. We therefore wish to characterize which cubic binary matroids have connectivity 3 and which have connectivity 2.

$$
\begin{array}{c}
\begin{array}{ccc} E(M_1)-p & p & E(M_2)-p \end{array}\\
\left[\begin{array}{c|c|c}
 & 0 & \\
 & 0 & \\
A_1 & \vdots & 0\\
 & 0 & \\ \cline{2-3}
 & 1 & \\ \cline{1-2}
 & 0 & \\
0 & \vdots & A_2\\
 & 0 & \\
 & 0 &
\end{array}\right]
\end{array}
$$

Figure 4.1. A representation for $P(M_1, M_2)$.

Lemma 4.3.1. *If a cubic binary matroid M is not 3-connected, then M has a pair of elements in series.*

Proof. Since M is connected but not 3-connected, it can be expressed as the 2-sum of two connected matroids M_1 and M_2. Denoting the parallel connection of M_1 and M_2 along the basepoint p as $P(M_1, M_2)$, we have $P(M_1, M_2)\backslash p = M$. Proposition 7.1.24 of [14] proves that $P(M_1, M_2)$ has a representation A as shown in Figure 4.1 where M_1 is represented by the first $|E(M_1)|$ columns of A, and M_2 is represented by the last $|E(M_2)|$ columns of A.

Since the row space of a representation of a matroid is equal to its cocircuit space, we can perform elementary row operations on the rows of A to obtain a representation B of $P(M_1, M_2)$ such that, when the column p is deleted, the result is a matrix 3-certificate for M. The columns of B can be partitioned similarly to those of A. Because the support of the sum of the rows of B is either empty or $\{p\}$, but $\{p\}$ is not a cocircuit of $P(M_1, M_2)$, we deduce that this support is the empty set. Thus p is in the support of a nonzero even number of rows of B. For each i in $\{1, 2\}$, let B_i be the set of columns of B corresponding to the elements of $E(M_i) - p$, and let C_i denote B_i along with the column of

B representing p. Then C_1 and C_2 are representations of M_1 and M_2, respectively, because they are derived from linear combinations of the rows of our previous representations of M_1 and M_2.

Since a cocircuit of $P(M_1, M_2)$ contains p if and only if it meets both $E(M_1) - p$ and $E(M_2) - p$, every non-zero row of C_1 and C_2 will have support of size 2 or 3. Note that if the size of the support of a row is 2, then the two elements in the support are in series in M_1 or M_2. We may assume that one of these elements is p; otherwise, the lemma holds. Suppose one of C_1 or C_2 has two rows with support of size 2. Then the corresponding matroid has two elements in series with p, so these elements are in series with each other, and they are also a series pair in M. Moreover, if we assume that every row in C_1 has support of size 3, then in C_2, every row with p in its support has support of size 2, and since there are at least two such rows, there is a series pair of M with elements from $E(M_2) - p$.

We now assume that both C_1 and C_2 have a row with support of size 2. Suppose $\{e_1, p\}$ is the support of a row in C_1, and $\{e_2, p\}$ is the support of a row in C_2. Then $\{e_i, p\}$ is a cocircuit of M_i, so $\{e_1, e_2, p\}$ is a cocircuit of $P(M_1, M_2)$. Thus $\{e_1, e_2\}$ is a cocircuit of $P(M_1, M_2)\backslash p$, that is, of M. $\square$

We will use $M(C_4')$ to denote the cycle matroid of the graph obtained by taking two non-adjacent edges of C_4 and adding an edge in parallel to each. The matroid $P(U_{2,3}, U_{2,3})$ can also be viewed as the cycle matroid of $K_4\backslash e$. Observe that when M is $U_{1,3}$, the unique basis of the cocircuit space corresponds to the triad $E(M)$. Moreover, the symmetric difference of the members of the basis of the cocircuit space also corre-

sponds to $E(M)$. In this aces, the set of distinguished triads of M is actually the multiset $\{E(M), E(M)\}$, but we will continue to refer to it as the set of distinguished triads of M.

Proposition 4.3.2. *A cubic binary matroid that is not* 3*-connected will not have a unique set of distinguished triads, unless it is* $U_{1,3}$*,* $M(C_4')$*, or* $P(U_{2,3}, U_{2,3})$*.*

Proof. Let M be a cubic binary matroid that is not 3-connected. By Lemma 4.3.1, M has series elements e_1 and e_2. If M has fewer than four elements, then it must be isomorphic to $U_{1,3}$. Otherwise, consider the matrix A' obtained by exchanging the supports of e_1 and e_2. The matrix A' is also a matrix 3-certificate of M. If $A' = A$, then we deduce that if $\{e_i, f_1, f_2\}$ is a distinguished triad, then so is $\{e_j, f_1, f_2\}$, for some f_1, f_2 in $E(M)$, and $\{i, j\} = \{1, 2\}$. Since every element must be in at least two distinguished triads, we have four triads $\{e_1, f_1, f_2\}, \{e_2, f_1, f_2\}, \{e_1, f_3, f_4\}, \{e_2, f_3, f_4\}$ represented in A for some f_3, f_4 in $E(M)$. These four triads are a circuit in the matroid derived from the cocircuit space of M, so in order for M to be connected, these four triads must comprise the whole matroid. If $\{f_1, f_2\} \cap \{f_3, f_4\} = \emptyset$, then $M \cong M(C_4')$. If $\{f_1, f_2\} \cap \{f_3, f_4\} \neq \emptyset$, we may assume that $f_4 = f_1$. Then $M \cong P(U_{2,3}, U_{2,3})$. $\square$

Conjecture 4.3.3. *A cubic binary matroid* M *has a unique set of distinguished triads if and only if* M *is* 3*-connected or* M *is isomorphic to* $U_{1,3}$*,* $M(C_4')$*, or* $P(U_{2,3}, U_{2,3})$*.*

4.4. Graphic cubic binary matroids

It is of interest to characterize graphs that are not cubic but have cubic binary matroids as their cycle matroids. Clearly any graph that is 2-isomorphic to a cubic graph will have a cubic binary matroid as its cycle matroid. It is also possible to have a graph that is not 2-isomorphic to a cubic graph whose cycle matroid is cubic binary, as we will demon-

strate in this section.

Theorem 4.4.1. *If M is 3-connected, graphic, and cubic binary, then every graph G such that M is isomorphic to the cycle matroid of G is cubic.*

Proof. If G has a vertex of degree 1 or 2, then M is not 3-connected unless $G \cong K_3$, as M will have a coloop or a series pair. Therefore the minimum degree of G is 3. By Corollary 4.2.2, the average degree of G is at most 3. Therefore every vertex must have degree exactly 3. □

It is possible for a cubic binary matroid that is not 3-connected to not be 2-isomorphic to a graph with maximum degree 3. This can be achieved by starting with a graph with sufficiently many vertex triads, where every edge is in at least one triad, and taking subdivisions of the edges until every edge is in an even number of triads such that we can form a set of distinguished triads. Figure 4.2 illustrates the process of subdividing edges of $\mathcal{W}_4$ until the resulting graph has a cubic binary matroid as a cycle matroid. Each dotted line passes through three edges representing a triad used to form a row in the representation, and the bold lines are the elements with an odd number of dotted lines through them. These bold lines correspond to the row representing the sum of all the other rows. The final graph in (d) has eight distinguished triads and its cycle matroid has rank 7.

The following open problem remains for the characterization of graphic cubic binary matroids.

Problem 4.4.2. *Which graphs permit subdivisions after which the resulting graphs have cubic binary matroids as their cycle matroids?*

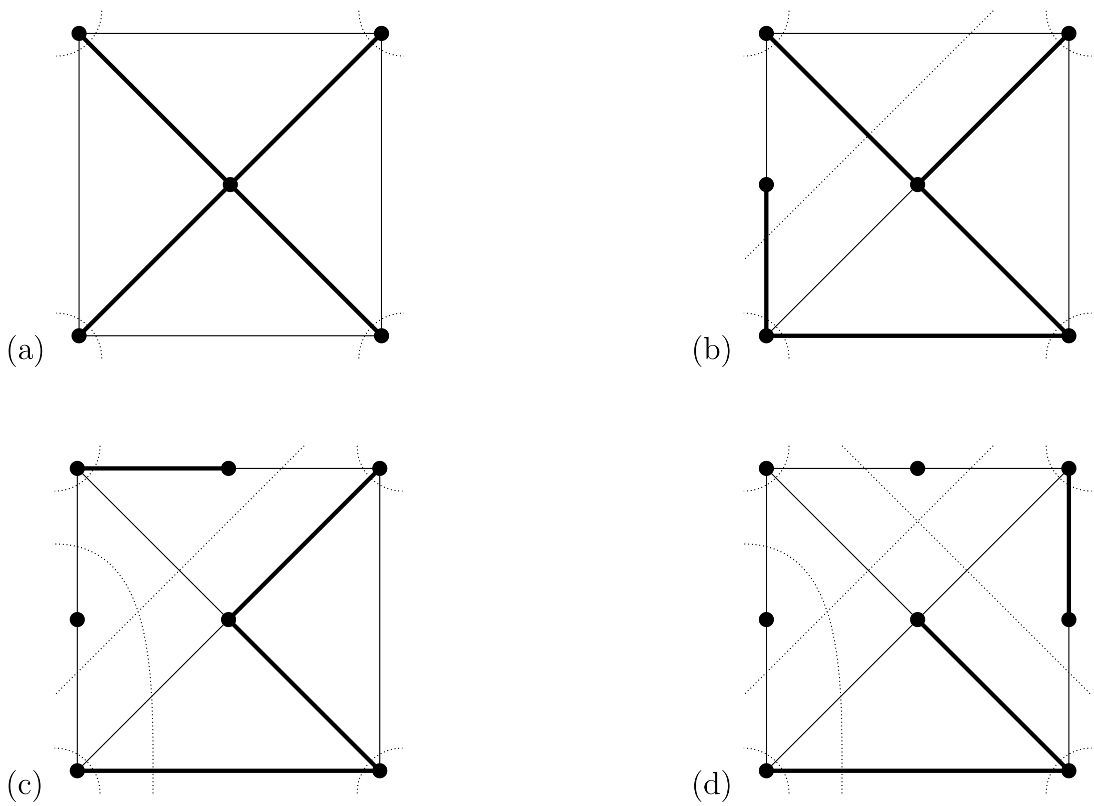

Figure 4.2. The process of subdividing $M(\mathcal{W}_4)$ to obtain a graph with a cubic binary cycle matroid.

4.5. Disconnected matroids

As a cubic graph may be connected and not 2-connected, it is natural to consider disconnected matroids that otherwise meet the criteria for being cubic binary matroids. A disconnected binary matroid is *disconnected cubic* if it has a representation where the support of every row has size three, as does the symmetric difference of these supports.

Lemma 4.5.1. *A disconnected cubic binary matroid must have a coloop.*

Proof. Let M be a disconnected cubic binary matroid and let A be a matrix 3-certificate for M. Every row of A has support of size three, and therefore, the row represents either a triad, three coloops, or a coloop and a series pair. Suppose that M has no coloops. Since every row of A represents a triad, the support of every row has elements from exactly one component of M. We can therefore consider the collection S of rows of A whose supports meet a particular component of M. Since the GF(2)-sum of the rows in S must be the zero vector, S is a circuit of size less than $r + 1$ in the matroid that is represented by A^T. This contradicts the fact that this matroid is an $(r + 1)$-element circuit. $\square$

4.6. Element bifurcation and other constructions

Let M be a cubic binary matroid having a matrix 3-certificate A. Let e be an element of M where the support of the corresponding column of A has size greater than 2. We define a *bifurcation* of e to construct a new matroid M' with representation A' as follows: partition the support of the column e into two sets S_1 and S_2 with positive even cardinality, and add columns e_1 and e_2 to A, where S_i is the support of e_i for i in $\{1, 2\}$.

Bifurcating an element e of M into elements e_1 and e_2 has the effect of placing e_1 and e_2 in the matroid such that $\{e, e_1, e_2\}$ is a triangle before e is deleted to make M'.

Moreover, bifurcation does not change the rank of the matroid, as the rank of the resulting matrix A' is equal to the rank of the original matrix A. Observe that the C is a circuit of M' containing $\{e_1, e_2\}$ if and only if $(C - \{e_1, e_2\}) \cup \{e\}$ is a circuit of M.

Lemma 4.6.1. *If M' is a matroid obtained by bifurcating an element of a cubic binary matroid M, then M' is also a cubic binary matroid. Moreover, if M is a disconnected cubic binary matroid, then bifurcating an element will result in a disconnected cubic binary matroid.*

Proof. Let e be an element of M such that in the matrix 3-certificate A of M, there are greater than two nonzero entries in the column corresponding to e. We bifurcate into elements e_1 and e_2 to obtain a new matroid M' with matrix 3-certificate A'. Adding e_1 and e_2 to M does not raise the rank; nor does deleting e from the resulting matroid so $r(M') = r(M)$. As with A, each row of A' has support of size three, and the sum of the rows in A' is the zero vector. Therefore M' is either a cubic binary matroid or a disconnected cubic binary matroid.

Assume that M' is disconnected. By Lemma 4.5.1, M' has a coloop f. Observe that e_1 and e_2 are not coloops of M' otherwise $\{e, e_1\}$ is a cocircuit of the extension of M' by e, so e is a coloop of M, a contradiction. As M is connected, it has a circuit C containing f. As C is not a circuit of M', it must contain e. Then, in the extension of M' by e, the set $(C - e) \cup \{e_1, e_2\}$ is a disjoint union of circuits that contains f but not e. Thus f is not a coloop, a contradiction. □

Theorem 4.6.2. *Iteratively applying the bifurcation operation to elements of a cubic binary matroid will result in a graphic cubic binary matroid.*

Proof. Let M be a cubic binary matroid with matrix 3-certificate A. We will proceed by bifurcating elements corresponding to columns of A with supports of size exceeding two. Since the support of every column must be even, we can apply the bifurcation operation described above to create a new matrix where every column has even support. By Lemma 4.6.1, bifurcating an element will create a matroid that is either cubic or disconnected cubic.

We deduce that bifurcation of an element in a cubic binary matroid results in a cubic binary matroid. By iterating the process, we obtain a matrix where every column has exactly two nonzero entries, which is the vertex-edge incidence matrix of a graph by Theorem 4.2.4. □

Define the operation of *adding a handle* as follows: Take a collection $\{e_1, e_2, \ldots, e_j\}$ of elements of a binary matroid M and place a new element e_i' in series with each e_i. Finally, add a new element f to form a new binary matroid in which $\{e_i, e_i', f\}$ is a triad for each i.

Theorem 4.6.3. *Let M be a cubic binary matroid. If a handle is added to an even number of elements, then the resulting matroid is a cubic binary matroid.*

Proof. Let A be a matrix 3-certificate of M, and let $E(M) = \{e_1, e_2, \ldots, e_n\}$. For some positive even integer j, we construct matrices A_i for each i in $\{1, 2, \ldots, j\}$ as follows: Begin with $A_0 = A$. To obtain A_i from A_{i-1}, we place an element e_i' in series with e_i by adding a new column e_i' with only zeros, and then adding a new row r_i whose support is only e_i and e_i'. Finally, add r_i to one of the original rows of A having e_i in its support. This maintains the GF(2)-sum of the rows of A_i as the zero vector, while keeping the size

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} e_1 & e_2 & \dots & e_j & \dots & e_n & e'_1 & e'_2 & \dots & e'_j & f \end{array} \\
\begin{array}{c} \\ \\ \\ \\ r_1 \\ r_2 \\ \vdots \\ r_j \end{array} &
\left[\begin{array}{cccc|cccc|c}
\multicolumn{4}{c|}{B} & \multicolumn{4}{c|}{C} & 0 \\
 & & & & & & & & 0 \\
 & & & & & & & & \vdots \\
 & & & & & & & & 0 \\ \hline
1 & & & & 1 & & & & 1 \\
 & 1 & & & & 1 & & & 1 \\
 & & \ddots & & & & \ddots & & \vdots \\
 & & & 1 & & & & 1 & 1
\end{array}\right]
\end{array}
$$

Figure 4.3. The matrix A'.

of the supports of the original rows as three. Note that, in the matroid defined on the rows of A_{i-1}, this operation places r_i in series with an existing element corresponding to a row of A_{i-1}. Therefore, by induction, the rows of each A_i form a Hamiltonian circuit in the matroid on the cocircuit space of $M[A_i]$.

We can append a column f to A_j to make a matrix A', where f is zero everywhere except for the rows of A_j that have exactly two ones. These are precisely the rows of A_j that are not in A, and there are j such rows. Since every row of A' has exactly three nonzero entries, and the rows form a Hamiltonian circuit, A' is a matrix 3-certificate of a cubic binary matroid M'.

Figure 4.3 shows the completed matrix A'. In this figure, the matrix C is a matrix with j ones, and each entry B_{ik} is $A_{ik} - C_{ik}$ if C_{ik} exists, and A_{ik} otherwise. $\square$

Appendix A. SageMath Code for Cubic Binary Matroids

The following SageMath 8.8 code was used in working with examples of cubic binary matroids.

```
def triangles(M):
    """
    returns a list of all triangles of a matroid


    INPUT:  M - A matroid
    OUTPUT: A list of frozensets of elements
    """
    Ind = M.independent_r_sets(2)
    C = set()
    for I in Ind:
        for e in M.groundset().difference(I):
            X = I.union(set([e]))
            if M.rank(X) == 2:
                C.add(X)
    return list(C)
from collections import defaultdict


def is_cubic_binary(M, certificate=True):
    """
    checks whether or not a given matroid is a
```

```
cubic binary matroid

INPUT:  M - A matroid
        certificate - whether or not the set of triads
        is returned
OUTPUT: If certificate is False, a boolean.
        Otherwise, a tuple (boolean, set) where the set
        consists of frozensets of three elements each.
"""
size = len(M)
r = rank(M)
T = triangles(M.dual())

def is_impossible(edict, tnum):
    # this checks if it's impossible to complete the set of
    # triads into a basis
    # by counting how many elements need to be covered by
    # a triad
    uncovered = size - len(edict)
    odd = sum(1 for e in edict if edict[e] == 1)
    remaining = r + 1 - tnum
    return remaining * 3 < uncovered + odd
```

```
def search(idx, edict, tset):

    # if we have enough triads for a basis, check if valid

    if len(tset) == r + 1:

        if (len(edict) == size and

            all([count == 0 for count in edict.values()])):

            return True, tset

        return False, set()

    # if there are not enough triads left, return False

    if len(T) - idx + len(tset) < r + 1:

        return False, set()

    # another check if possible

    if is_impossible(edict, len(tset)):

        return False, set()


    # skip triad in list and continue search

    check, cert = search(idx + 1, edict, tset)

    if check:

        return check, cert

    # add triad and continue search

    triad = T[idx]

    for e in triad:

        edict[e] = (edict[e] + 1) % 2

    tset.add(triad)
```

```
    return search(idx + 1, edict, tset)


res, cert = search(0, defaultdict(int), set())

if certificate:

    return res, cert

return res
```

Appendix B. Permissions

These are the journal permissions to reproduce previously published work.

A note on the connectivity of 2-polymatroid minors

Zachary Gershkoff   James Oxley

Department of Mathematics
Louisiana State University
Baton Rouge, U.S.A.

{zgersh2, oxley}@math.lsu.edu

Submitted: Dec 6, 2018; Accepted: Aug 6, 2019; Published: Oct 25, 2019

RIGHTS & ACCESS

Elsevier Inc.

Article:	**A notion of minor-based matroid connectivity**
Corresponding author:	**Mr. Zachary Gershkoff**
E-mail address:	**zgersh2@math.lsu.edu**
Journal:	**Advances in Applied Mathematics**
Our reference	**YAAMA1850**
PII:	**S0196-8858(18)30075-7**
DOI:	**10.1016/j.aam.2018.07.001**

YOUR STATUS

I am one author signing on behalf of all co-authors of the manuscript

ASSIGNMENT OF COPYRIGHT

I hereby assign to Elsevier Inc. the copyright in the manuscript identified above (where Crown Copyright is asserted, authors agree to grant an exclusive publishing and distribution license) and any tables, illustrations or other material submitted for publication as part of the manuscript (the "Article"). This assignment of rights means that I have granted to Elsevier Inc., the exclusive right to publish and reproduce the Article, or any part of the Article, in print, electronic and all other media (whether now known or later developed), in any form, in all languages, throughout the world, for the full term of copyright, and the right to license others to do the same, effective when the Article is accepted for publication. This includes the right to enforce the rights granted hereunder against third parties.

SUPPLEMENTAL MATERIALS

"Supplemental Materials" shall mean materials published as a supplemental part of the Article, including but not limited to graphical, illustrative, video and audio material.

With respect to any Supplemental Materials that I submit, Elsevier Inc. shall have a perpetual worldwide, non-exclusive right and license to publish, extract, reformat, adapt, build upon, index, redistribute, link to and otherwise use all or any part of the Supplemental Materials in all forms and media (whether now known or later developed), and to permit others to do so.

RESEARCH DATA

"Research Data" shall mean the result of observations or experimentation that validate research findings and that are published separate to the Article, which can include but are not limited to raw data, processed data, software, algorithms, protocols, and methods.

With respect to any Research Data that I wish to make accessible on a site or through a service of Elsevier Inc., Elsevier Inc. shall have a perpetual worldwide, non-exclusive right and license to publish, extract, reformat, adapt, build upon, index, redistribute, link to and otherwise use all or any part of the Research Data in all forms and media (whether now known or later developed) and to permit others to do so. Where I have selected a specific end user license under which the Research Data is to be made available on a site or through a service, the publisher shall apply that end user license to the Research Data on that site or service.

REVERSION OF RIGHTS

Articles may sometimes be accepted for publication but later rejected in the publication process, even in some cases after public posting in "Articles in Press" form, in which case all rights will revert to the author (see https://www.elsevier.com/about/our-business/policies/article-withdrawal).

REVISIONS AND ADDENDA

I understand that no revisions, additional terms or addenda to this Journal Publishing Agreement can be accepted without Elsevier Inc.'s express written consent. I understand that this Journal Publishing Agreement supersedes any previous agreements I have entered into with Elsevier Inc. in relation to the Article from the date hereof.

AUTHOR RIGHTS FOR SCHOLARLY PURPOSES

I understand that I retain or am hereby granted (without the need to obtain further permission) the Author Rights (see description below), and that no rights in patents, trademarks or other intellectual property rights are transferred to Elsevier Inc..

The Author Rights include the right to use the Preprint, Accepted Manuscript and the Published Journal Article for Personal Use and Internal Institutional Use. They also include the right to use these different versions of the Article for Scholarly Sharing purposes, which include sharing:

the Preprint on any website or repository at any time;
the Accepted Manuscript on certain websites and usually after an embargo period;
the Published Journal Article only privately on certain websites, unless otherwise agreed by Elsevier Inc..

In the case of the Accepted Manuscript and the Published Journal Article the Author Rights exclude Commercial Use (unless expressly agreed in writing by Elsevier Inc.), other than use by the author in a subsequent compilation of the author's works or to extend the Article to book length form or re-use by the author of portions or excerpts in other works (with full acknowledgment of the original publication of the Article).

AUTHOR REPRESENTATIONS / ETHICS AND DISCLOSURE / SANCTIONS

I affirm the Author Representations noted below, and confirm that I have reviewed and complied with the relevant Instructions to Authors, Ethics in Publishing policy, Declarations of Interest disclosure and information for authors from countries affected by sanctions (Iran, Cuba, Sudan, Burma, Syria, or Crimea). Please note that some journals may require that all co-authors sign and submit Declarations of Interest disclosure forms. I am also aware of the publisher's policies with respect to retractions and withdrawal (https://www.elsevier.com/about/our-business/policies/article-withdrawal).
For further information see the publishing ethics page at https://www.elsevier.com/about/our-business/policies/publishing-ethics and the journal home page. For further information on sanctions, see https://www.elsevier.com/about/our-business/policies/trade-sanctions

Author representations

The Article I have submitted to the journal for review is original, has been written by the stated authors and has not been previous
The Article was not submitted for review to another journal while under review by this journal and will not be submitted to any otl
The Article and the Supplemental Materials do not infringe any copyright, violate any other intellectual property, privacy or other entity, or contain any libellous or other unlawful matter.
I have obtained written permission from copyright owners for any excerpts from copyrighted works that are included and have cre the Article or the Supplemental Materials.
Except as expressly set out in this Journal Publishing Agreement, the Article is not subject to any prior rights or licenses and, if my authors' institution has a policy that might restrict my ability to grant the rights required by this Journal Publishing Agreement (ta Author Rights permitted hereunder, including Internal Institutional Use), a written waiver of that policy has been obtained.
If I and/or any of my co-authors reside in Iran, Cuba, Sudan, Burma, Syria, or Crimea, the Article has been prepared in a personal, capacity and not as an official representative or otherwise on behalf of the relevant government or institution.
If I am using any personal details or images of patients, research subjects or other individuals, I have obtained all consents requir and complied with the publisher's policies relating to the use of such images or personal information. See https://www.elsevier.cor /policies/patient-consent for further information.
Any software contained in the Supplemental Materials is free from viruses, contaminants or worms.
If the Article or any of the Supplemental Materials were prepared jointly with other authors, I have informed the co-author(s) of th Publishing Agreement and that I am signing on their behalf as their agent, and I am authorized to do so.

GOVERNING LAW AND JURISDICTION

This Agreement will be governed by and construed in accordance with the laws of the country or state of Elsevier Inc. ("the Governing State"), without regard to conflict of law principles, and the parties irrevocably consent to the exclusive jurisdiction of the courts of the Governing State.

For information on the publisher's copyright and access policies, please see http://www.elsevier.com/copyright.
For more information about the definitions relating to this agreement click here.

☑ **I have read and agree to the terms of the Journal Publishing Agreement.**

9th July 2018 T-copyright-v22/2017

Bibliography

[1] R. E. Bixby, l-matrices and a characterization of non-binary matroids, *Discrete Math.* **8** (1974), 139–145.

[2] J. E. Bonin and W. R. Schmitt, Splicing matroids, *European J. Combin.* **32** (2011), 722–744.

[3] T. H. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171** (1972), 235–282.

[4] C. R. Coullard, Counterexamples to conjectures on 4-connected matroids, *Combinatorica* **6** (1986), 315–320.

[5] W. H. Cunningham, *A combinatorial decomposition theory*, Ph.D. thesis, University of Waterloo, 1973.

[6] R. Diestel, *Graph Theory*, Third edition, Springer-Verlag, New York, 2006.

[7] Z. Gershkoff and J. Oxley, A notion of minor-based matroid connectivity, *Adv. in Appl. Math.* **100** (2018), 163-178.

[8] Z. Gershkoff and J. Oxley, A note on the connectivity of 2-polymatroid minors, *Electron. J. Combin.* **26** (2019), Paper 4.21, 11pp.

[9] D. Hall, Essential elements in connected k-polymatroids, *Adv. in Appl. Math.* **50** (2013), 281–291.

[10] J. Kahn, A problem of P. Seymour on nonbinary matroids, *Combinatorica* **5** (1985), 319–323.

[11] F. Matúš, Adhesivity of polymatroids, *Discrete Math.* **307** (2007), 2464–2477.

[12] T. Moss, A minor-based characterization of matroid 3-connectivity, *Adv. in Appl. Math.* **50** (2013), 132–141.

[13] J. G. Oxley, On nonbinary 3-connected matroids, *Trans. Amer. Math. Soc.* **300** (1987), 663–679.

[14] J. Oxley, *Matroid Theory*, Second edition, Oxford University Press, New York, 2011.

[15] J. Oxley, C. Semple, and G. Whittle, The structure of the 3-separations of 3-connected matroids, *J. Combin. Theory Ser. B* **92** (2004), 257–293.

[16] J. Oxley, C. Semple, and G. Whittle, A splitter theorem for 3-connected 2-

polymatroids, *Electron. J. Combin.* **26** (2019), Paper 2.37, 95pp.

[17] J. Oxley and G. Whittle, Connectivity of submodular functions, *Discrete Math.* **105** (1992), 173–184.

[18] J. Oxley and J. Taylor, On two classes of nearly binary matroids, *European J. Combin.* **36** (2014), 251–260.

[19] P. D. Seymour, A note on the production of matroid minors, *J. Combin. Theory Ser. B* **22** (1977), 289–295.

[20] P. D. Seymour, On minors of non-binary matroids, *Combinatorica* **1** (1981), 387–394.

[21] P. D. Seymour, Minors of 3-connected matroids, *European J. Combin.* **6** (1985), 375–382.

[22] P. D. Seymour, Triples in matroid circuits, *European J. Combin.* **7** (1986), 177–185.

[23] W. T. Tutte, Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.

[24] D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976. Reprinted by Dover Publications, New York, 2010.

Vita

Zachary Gershkoff was born in Los Angeles, California, and grew up in DeFuniak Springs, Florida. He earned a Bachelor's degree in Mathematics from the University of Florida in December 2009. After working in South Africa as a Mathematics Specialist with the Peace Corps, he came to Louisiana State University in August 2014 to pursue graduate studies in mathematics, supported by a GAANN Fellowship from January 2016 to August 2018 and from January 2020 to August 2020. He earned a Master's degree in mathematics and is currently a candidate for the degree of Doctor of Philosophy in mathematics, to be awarded May 2021.