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Knots and Links in Overtwisted Contact Manifolds

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KNOTS AND LINKS IN OVERTWISTED CONTACT MANIFOLDS

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This dissertation is dedicated to all those women who were told “Math is not for women”.

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Abstract

Suppose (M, ξ) be an overtwisted contact 3-manifold. We prove that any Legendrian and transverse link in (M, ξ) having overtwisted complement can be coarsely classified by their classical invariants. Next, we defined an invariant called the support genus for transverse links and extended the definition of support genus of Legendrian knots to Legendrian links and prove that any coarse equivalence class of Legendrian and transverse loose links has support genus zero. Further, we show that the converse is not true by explicitly constructing an example. We also find a relationship between the support genus of the transverse link and its Legendrian approximation. As a corollary to this, we show that loose, null-homologous, transverse knots have support genus zero and also give a condition when non-loose Legendrian knots have non-loose transverse push offs.

Chapter 1. Introduction

Knot theory associated to contact 3-manifolds has been a very interesting field of study. We say a knot in a contact 3-manifold is *Legendrian* if it is tangent everywhere to the contact planes and *transverse* if it is everywhere transverse. The classification of Legendrian and transverse knots has always been an interesting and difficult problem in contact geometry. Two Legendrian knots are said to be *Legendrian isotopic* if they are isotopic through Legendrian knots. A knot or link type is said to be *Legendrian simple* if it can be classified by its classical invariants up to Legendrian isotopy. There are only a few knot types that are known to be Legendrian simple in $(\mathrm{S}^3, \xi_{std})$. For example topologically trivial knots in [7], the torus knots and figure eight knots in [11] are all Legendrian simple. While there is no reason to believe all knots should be Legendrian simple, it has been historically difficult to prove otherwise. Chekanov [3] and independently, Eliashberg [6] developed invariants of Legendrian knots that show that $m(5_2)$ has Legendrian representatives that are not distinguised by their classical invariants.

Since Eliashberg's classification of overtwisted contact structures [5], the study of overtwisted contact structures and the knots and links in them, has been minimal. However, in recent years overtwisted contact structures have played central roles in many interesting applications such as building achiral Lefchetz fibration [7], near symplectic structures on 4-manifolds [13] and many more. Thus the overtwisted manifolds and the knot theory associated to them has generated significant interest. There are two types of knots/links in overtwisted contact structures, namely loose and non-loose (Also known as non-exceptional and exceptional respectively). A link in an overtwisted contact manifold is loose if its complement is overtwisted and non-loose otherwise. The first explicit example of a non-loose knot is given by Dymara in [4]. In general, non-loose knots appear to be rare. It is still not known if every knot type has a non-loose representative. We have another notion of classification of knots and links in contact manifolds known as *coarse equivalence*. We say knots/links are coarsely classified if they are classified up to orientation preserving contactomorphism, smoothly isotopic to the identity. Observe that, though classification by Legendrian isotopy and coarse equivalence are equivalent in $(\mathrm{S}^3, \xi_{std})$, they are not the same in general. Eliashberg and Fraser gave a coarse classification of Legendrian unknots in overtwisted contact structure in S^3 [7]. Later, Geiges and Onaran gave a complete coarse classification of the non-loose left handed trefoil knots in [16] and non-loose Legendrian Hopf links in [15]. Recently, all non-loose negative torus knots are coarsely classified by Matkovič in [19]. Note that, all of these classification results have been proved in overtwisted S^3.

This dissertation studies loose links in all overtwisted contact manifolds. In [8], Etnyre proved that loose Legendrian and transverse knots can be coarsely classified by their classical invariants. We extended his result for loose links. Our main theorem is the following:

Theorem 1.0.1. *([2]) Suppose L_1 and L_2 are two loose null-homologous Legendrian links with same classical invariants. Then, L_1 and L_2 are coarsely equivalent.*

Remark 1.0.1. *Here by a null-homologous link, we assume that every link component is null-homologous.*

The above theorem tells us that there is only a unique loose link with any fixed classical invariants in any overtwisted contact structure up to contactomorphism.

Remark 1.0.2. *In an overtwisted contact manifold* (M, ξ)*, classification up to contac-*

tomorphism and classification upto Legendrian isotopy are not equaivalent. Our result doesn't say anything about the Legendrian simpleness of a loose link. Dymara in [4] proved that two Legendrian knots having same classical invariants in any contact 3-manifold (M, ξ) *are Legendrian isotopic if if there exists an overtwisted disk disjoint from both of them. Obviously, this does not apply to all loose knots.*

As a corollary we proved the following result for loose transverse links.

Corollary 1.0.1. *([2]) Suppose T and T' are two transverse loose null-homologous links with same classical invariants. Then T and T' are coarsely equivalent.*

In other words, there is a unique loose null-homologous transverse link with every component having a fixed self-linking number up to contactomorphism.

Remark 1.0.3. *In [8], the theorem was proved for null-homologous knots and it was hinted that these might be extended to non-null homologous knots using Tchernov's definition of relative rotation number and relative Thurston–Benniquin number [22] with some extra conditions on the underlying manifold. It seems plausible that the same idea can be extended to links as well.*

After classifying the Legendrian and transverse loose links, we associate them with the open book decomposition of the manifold. First, we extended the definition of the support genus of a Legendrian knot defined in [21] to the support genus of a Legendrian link and proved the following theorem about coarse equivalence class of loose Legendrian links.

Theorem 1.0.2. *([2]) Suppose $[L]$ denotes the coarse equivalence class of loose, null-homologous Legendrian links with in any contact 3-manifold.Then $\mathrm{sg}([L]) = 0$.*

Like non-loose knots, non-loose links appear to be rare. The above theorem suggests, if we can find a Legendrian link L with $\mathrm{sg}(L) > 0$ that will immediately tell us that

L is non-loose. We also show that the converse of the theorem is not true.

Theorem 1.0.3. *([2]) There are examples of non-loose links with support genus zero.*

Next, we prove that we can associate any transverse link T in (M, ξ) with an open book (Σ, ϕ) supporting the contact manifold.

Theorem 1.0.4. *([2]) Suppose T be any transverse link in (M, ξ). Then T is transversely isotopic to the sub-binding of some open book (B, π) supporting (M, ξ).*

1.0.4 allows us to define the support genus of a transverse link T following the support genus of a Legendrian link.

Using the well known relation of Legendrian and transverse links, we could relate the support genus of a transverse link with the support genus of its Legendrian approximation.

Theorem 1.0.5. *([2]) Suppose T be a transverse link in (M, ξ) and L be its Legendrian approximation. Then $\mathrm{sg}(T) = \mathrm{sg}(L)$.*

As a corollary to this result, we could prove the following.

Corollary 1.0.2. *([2]) Suppose T be a loose, null-homologous transverse knot in (M, ξ). Then $\mathrm{sg}(T) = 0$.*

Also, we have a similar result for coarse equivalence class of loose transverse links.

Corollary 1.0.3. *([2]) Suppose $[T]$ be a coarse equivalence class of loose, null-homologous loose transverse links. Then $\mathrm{sg}[T] = 0$*

While it is known that non-loose transverse knots have non-loose Legendrian approximation, non-loose Legendrian knots may or may not have transverse push offs. We proved that the support genus gives a condition when we can find non-loose transverse push off of a non-loose Legendrian knot.

Corollary 1.0.4. *([2]) Suppose L be a non-loose Legendrian knot with* $\mathrm{sg}(L) > 0$. *Then its transverse push off must be non-loose.*

This dissertation is organized as follows: in chapter 2, we briefly discuss some of the basic definitions and theorems from contact geometry. We prove Theorem 1.0.1 in Chapter 3 and Theorem 1.0.2 and Theorem 1.0.3 in chapter 4. Finally, we proved Theorem 1.0.4 and Theorem 1.0.5 in Chapter 5.

Chapter 2. Background

Here, we discuss the basic definitions and results that are used throughout this dissertation.

2.1. Contact structures

Definition 2.1.1. *A contact structure* ξ *on an oriented* 3*-manifold* M *is a nowhere integrable* 2*-plane field and we call (*M, ξ*) a contact manifold.*

The non-integrability condition implies that ξ is not everywhere tangent to any surface in M. We assume that the plane fields are co-oriented, so ξ can be expressed as the kernel of some global one form α. In this case, the non-integrability condition is equivalent to $\alpha \wedge d\alpha > 0$. This one form α will be called a contact form and (M, ξ) will be called a contact manifold.

Definition 2.1.2. *Two contact manifolds* (M, ξ) *and* (M', ξ') *are said to be* contactomorphic *if there exists a diffeomorphism* $\phi\colon \mathrm{M} \to \mathrm{M}'$ *such that* $\phi^*(\xi') = \xi$*. Two contact structures* ξ *and* ξ' *on a* 3*-manifold are* M *are isotopic if there is a contactomorphism* $\phi\colon (\mathrm{M}, \xi) \to (\mathrm{M}', \xi')$ *such that phi is isotopic to the identity.*

There are two types of contact structures on a 3-manifold namely tight and overtwisted.

Definition 2.1.3. *We call a contact manifold overtwisted, if there exits an embedded overtwisted disk. Otherwise we call it tight.*

Definition 2.1.4. *An overtwisted disk is a disk embedded in a contact manifold* (M, ξ) *such that* ξ *is tangent to the boundary of the disk.*

Example 2.1.1. *Let* $\alpha = dz - ydx$*. The contact structure* $\xi_{std} = \ker \alpha$ *is the standard* *tight contact structure on* $\mathbb{R}^3$*. On the other hand, the contact structure defined as* $\xi_{OT} = \ker(\cos r dz - r \sin r d\theta)$ *is an overtwisted contact structure on* $\mathbb{R}^3$*.*

The following theorem tells us that all contact structure look same locally.

Theorem 2.1.1 (Darboux's theorem)**.** *For a given contact* 3*-manifold* (M, ξ) *and a point* $x \in \mathrm{M}$*, there is a neighborhood* U *of* x *in* M *such that* $(U, \xi|_U)$ *is contactomorphic to* $(V, \xi_{std}|_V)$ *for some open set* V *in* $(\mathbb{R}^3, \xi_{std})$*.*

Though only few results are knows about classifying tight contact structures on manifolds, overtwisted contact structures are completely classified by Eliashberg.

Theorem 2.1.2. *(Eliashberg, [5])Two overtwisted contact structures are isotopic if and only if they are homotopic as plane fields. Moreover, every homotopy class of oriented 2-plane field contains an overtwisted contact structure.*

In general, for two oriented 2-plane fields to be homotopic we have.

Theorem 2.1.3 (Gompf, [18])**.** *Two oriented 2-plane fields are homotopic if and only if their* 2*-dimensional invariants* d_2 *and* 3*-dimensional invariants* d_3 *agree.*

2.1.1. Convex surfaces in contact 3-manifold

A vector field v in a contact manifold (M, ξ) is called *contact vector field* if its flow preserves the contact structure. A surface Σ is called *convex* if there exits a contact vector field transverse to Σ. The *characteristic foliation* on Σ is the singular line field $T\Sigma \cap \xi$, which we will denote as Σ_ξ. Now if Σ is convex, then near Σ we can write the contact form as $\alpha = \beta + u dt$ where β is any one form on Σ and u is a real valued function. Then the multicurve $\Gamma_\Sigma = \{x \in \Sigma : u(x) = 0\}$ is called the dividing set of Σ.

Let Σ be a convex surface and γ a Legendrian arc in Σ that intersects the dividing

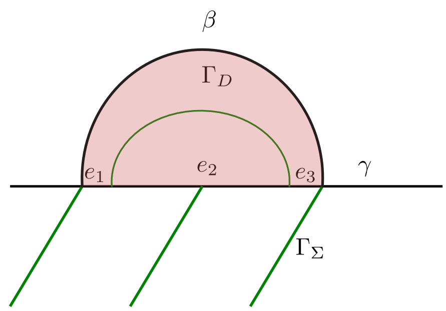

Figure 2.1. Bypass along γ.

curves Γ_Σ at three points$\{e_1, e_2, e_3\}$. Then a bypass for Σ (see 2.1) is a convex disk D with Legendrian boundary such that

1. $\mathrm{D} \cap \Sigma = \gamma$.
2. $\mathrm{tb}(\partial \mathrm{D}) = -1$.
3. $\gamma \cap \beta = \partial \mathrm{D}$.
4. along γ there are three elliptic singularities of D_ξ, two with the same sign occurring at the end points and one in the interior with opposite sign.
5. along β all the singulaties have the same sign.

Notice, all the singulaties of D_ξ are of the same sign except one. The sign of this singularity will be called the *sign of the bypass*. In other words, if the disk D has a natural orientation, then the sign of the singularity in the interior of γ decides the sign of the bypass. There is a relation between bypasses and stabilizing disks. Suppose we have a Legendrian knot K and D be a stabilizing disk for K. If K′ is the stabilized knot then D can be thought as a bypass for K′.

2.2. Legendrian links and classical invariants

Definition 2.2.1. *A link L smoothly embedded in (M, ξ) is said to be Legendrian if it is everywhere tangent to ξ, that is, $T_pL \subset \xi_p$ for every $p \in L$.*

For the purpose of this thesis, by classical invariants of a link we refer to the classical invariants of its components. The classical invariants of a Legendrian knot are the topological knot type, *Thurston–Benniquin invariant* $\mathrm{tb}(L)$ and *rotation number* $\mathrm{rot}(L)$. $\mathrm{tb}(L)$ measures the twisting of the contact framing relative to the framing given by the Seifert surface of L. In other words, choose a vector field v along L transverse to ξ and define a parallel link L' by pushing L along v, then $\mathrm{tb}(L)$ equals the linking number of L with L'. Observe that, $\mathrm{tb}(L)$ does not depend on the choice of orientation of the link. The other classical invariant $\mathrm{rot}(L)$ is defined as follows: as $\xi|_\Sigma$ is trivial (where Σ is the Seifert surface of the link L), $\mathrm{rot}(L)$ is defined to be the winding number of the link with respect to this trivialization. In other words, it is defined as the obstruction of extending a non-vanishing vector field on L to all of Σ.

2.2.1. Front diagrams for Legendrian knots

Let L be a Legendrian knot in $\mathbb{R}^3$ with the standard contact structure ξ_{std} given by the the kernal of the 1-form $dz - ydx$. The front projection of L is the image of L, $\pi(L)$, under the map $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2\colon (x, y, z) \to (x, z)$.

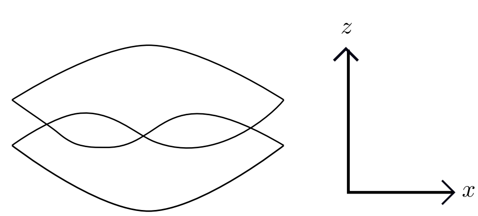

Figure 2.2. Front projection of a Legendrian right handed trefoil.

The front projections of a Legendrian knot follow these two rules:

1. No vertical tangencies.
2. Only one kind of crossing is allowed. The arc with more negative slope will be on the top of the less negatively sloped arc.

2.2.2. Standard neighborhood of Legendrian knots and links

The regular neighborhood theorem for Legendrian submanifolds tell us that given a Legendrian knot L in (M, ξ), we can always find an appropriate neighborhood of $N(L)$ such that $\mathrm{N}(L)$ is contactomorphic to the neighborhood N_0 of the image of x axis in $\mathbb{R}^3/(x \to x+1)$ in $(\mathbb{R}^3, \xi_{std})$. Using this model, we can see that $\partial\mathrm{N}$ is a torus with two dividing curves of slope $\frac{1}{n}$ where $\mathrm{tb}(L) = n$. We call $\partial\mathrm{N}$ to be in standard form. As a link is a disjoint union of S^1's this can be easily extended to a standard neighborhood of a link which is a disjoint union of solid tori each having two dividing curves of slope $\frac{1}{n_i}$ on its boundary where $\mathrm{tb}(L_i) = n_i$.

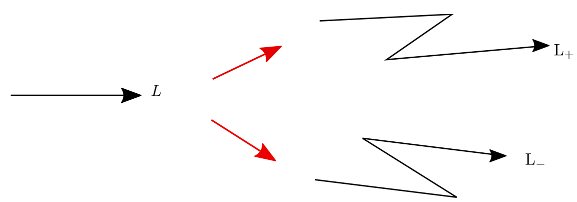

Figure 2.3. Stabilizations of a Legendrian knot.

2.2.3. Stabilization

Stabilization of a link can be done by stabilizing any of the link component. By standard neighborhood theorem of the Legendrian knot, one can identify any Legendrian link component L locally with the x axis. Stabilization is a local operation as shown in 2.3. The modification on the top right-side is called the positive stabilization and denoted as L_+. The modification on the bottom right-side is known as negative stabilizations and denoted as L_-. It does not matter which order the stabilizations are being done, it just matters where those are being done. The effect of the stabilizations on the classical invariants are as follows:

$$\mathrm{tb}(L_{\pm}) = \mathrm{tb}(L) - 1 \quad \text{and} \quad \mathrm{rot}(L_{\pm}) = \mathrm{rot}(L) \pm 1.$$

2.3. Transverse link and its relationship with a Legendrian link

Definition 2.3.1. *A link T in (M, ξ) is called transverse (positively) if it intersects the contact planes transversely with each intersection positive.*

By classical invariant of a transverse link, we will refer to the classical invariants of its components. There are two classical invariants for transverse knot, the topological knot type and the *self-linking number* $\mathrm{sl}(T)$. Self-linking number is defined for null-homologous knots. Suppose Σ be a Seifert surface of a transverse knot. As $\Sigma|_{\xi}$ is trivial, we can find a non-zero vector field v over Σ in ξ. Let T' be a copy of T obtained by pushing T slightly in the direction of v. The self-linking number $\mathrm{sl}(T)$ is defined to be the linking no of T with T'. Legendrian and transverse links are related by the operations known as transverse push off and Legendrian approximation (componentwise). Let L be a Legendrian knot in (M, ξ). Let $\mathrm{A} = \mathrm{S}^1 \times [-1, 1]$ where $L = \mathrm{S}^1 \times \{0\}$ and A is transverse to ξ. Given such an A, $L_+ = \mathrm{S}^1 \times \{\frac{1}{2}\}$ is a positive transverse knot. L_+ is called the positive transverse push-off $L_-$of L. We can similarly define a negative transverse push-off. One can check that $L_{\pm}$ is well-defined and the classical invariants are related by The classical invariants of a Legendrian link component and its transverse push off are related as follows:

$$\mathrm{sl}(L_{\pm}) = \mathrm{tb}(L) \mp \mathrm{rot}(L)$$

where $L_{\pm}$ denotes the positive and negative transverse push offs. In this dissertation, if we mention transverse push-off it is always the positive transverse pushoff unless explicitly stated otherwise. Note that, while a transverse push off is well defined, a Legendrian approximation is only well defined up to negative stabilizations. Similarly for a transverse knot, we have a Legendrian approximation. Check [10] to see how to find a Legendrian approximation for a transverse knot. Unlike transverse push-offs Legendrian approximations are not well- defined. They are only well-defined upto negative stabilization [11].

2.3.1. Types of Classification of links in contact manifolds

Here we only discuss the types of classification for Legendrian links. All of them are also true for transverse links. One can classify a Legendrian link up to Legendrian isotopy. Two Legendrian links L and L' are said to be *Legendrian isotopic* if they are isotopic through Legendrian links. Two Legendrian links L and L' in (M, ξ) are said to be *ambient contact isotopic* if there exists a one parameter family of contactomorphisms $\phi_t : \mathrm{M} \to \mathrm{M}$ such that $\phi_0 = id$ and $\phi_1(L) = L'$. This is well known that these two types of classifications are equivalent in any contact manifold (M, ξ). There is another type of classification of Legendrian links known as *coarse equivalence*. We say two Legendrian links are *coarsely classified* if they are classified up to orientation preserving contactomorphism, isotopic to the identity. In $(\mathrm{S}^3, \xi_{std})$ these two types of classification are equivalent . But in general a coarse equivalence does not imply Legendrian isotopy.

2.4. Open book decomposition and supporting contact structures

An *open book decomposition* of M is a pair (B, π) where B is an oriented link called the binding of the open book and $\pi\colon \mathrm{M} \setminus \mathrm{B} \to S^1$ is a fibration of the complement of B such that $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma_\theta \subset \mathrm{M}$ and $\partial\Sigma_\theta = \mathrm{B}$ for all $\theta \in \mathrm{S}^1$. The surface $\Sigma = \Sigma_\theta$ is called the page of the open book.

There is another way to specify open books known as the abstract open book decomposition. An *abstract open book decomposition* of a closed, oriented 3-manifold M is a pair (Σ, ϕ) such that

1. Σ is an oriented compact surface with boundary and
2. $\phi\colon \Sigma \to \Sigma$ is a diffeomorphism such that ϕ is the identity in a neighborhood of $\partial\Sigma$. The map ϕ is called the monodromy.

Remark 2.4.1. *One can go back and forth between the two definitions of open books. In this thesis, we will not distinguish them and use a combined notation* $(\mathrm{B}, \Sigma, \phi)$ *to denote an open book.*

Definition 2.4.1. *A positive (resp. negative) stabilization of an abstract open book* (Σ, ϕ) *is the open book with page* $\Sigma' = \Sigma\cup$ *1-handle and monodromy* $\phi' = \phi \circ \tau_c$ *where* τ_c *is the right (resp. left) handed Dehn twist with along a curve c which intersects the co-core of the 1-handle exactly once.*

We say a contact structure $\xi = \ker\alpha$ on M is supported by an open book decomposition (B, π) of M if

1. $d\alpha$ is a positive area form on the page of the open book.
2. $\alpha(v) > 0$, for each oriented tangent vector to B.

Given an open book decomposition of a 3-manifold M, Thurston and Winkelnkemper [23] showed how one can produce a compatible contact structure. Giroux proved that two contact structures which are compatible with the same open book are isotopic as contact structures [17]. Giroux also proved that two contact structures are isotopic if and only if they are compatible with open books which are related by positive stabilizations.

It is well known that every closed oriented 3-manifold has an open book decomposition. We can perform an operation called *Murasugi sum* to connect sum two open books and produce a new open book.

Definition 2.4.2. *Given two abstract open books* (Σ_i, ϕ_i), $i = 0, 1$, *let* c_i *be a properly embedded arc in* Σ_i *and* R_i, *a rectangular neighborhood of* c_i, $R_i = c_i \times [-1, 1]$. *The Murasugi sum of* (Σ_0, ϕ_0) *and* (Σ_1, ϕ_1) *is the open book* $(\Sigma_0, \phi_0) * (\Sigma_1, \phi_1)$ *with page*

$$\Sigma_0 * \Sigma_1 = \Sigma_0 \cup_{R1=R2} \Sigma_1$$

and monodromy $\phi_0 \circ \phi_1$.

In fact, Gabai proved that $\mathrm{M}_{(\Sigma_0,\phi_0)} \sharp \mathrm{M}_{(\Sigma_1,\phi_1)}$ is diffeomorphic to $M_{(\Sigma_0,\phi_0)*(\Sigma_1,\phi_1)}$ [12]. Also the contact structures compatible with the open books are additive under Murasugi sum and $\xi \sharp \xi'$ is compatible with the new open book.

Theorem 2.4.1. *(Gabai, Torisu) Let* (Σ_0, ϕ_0) *and* (Σ_1, ϕ_1) *are open book decompositions compatible with contact 3- manifolds* (M_0, ξ_0) *and* (M_1, ξ_1) *respectively. Then the murasugi sum of the open books* $(\Sigma_0 * \Sigma_1, \phi \circ \phi_1)$ *is compatible with the contact 3-manifold* $(M_0 \sharp M_1, \xi_0 \sharp \xi_1)$.

2.5. Homotopy classes of 2-plane fields

In this section, we review the homotopy theory of plane fields in the complement of a link. Specifically, we will study homotopy classes of 2-plane fields on manifolds with boundary. We start by recalling, Pontyragin-Thom construction associated with manifolds with boundary (For Pontyragin-Thom construction for closed manifolds see [20])

2.5.1. Pontyragin-Thom construction for manifolds with boundary

Suppose M be an oriented manifold with boundary. The space of oriented plane-fields on M will be denoted as $\mathcal{P}(\mathrm{M})$. On the other hand, if η is a plane-field defined on the boundary of M, then the set of all plane fields that extend η to all of M will be denoted by $\mathcal{P}(\mathrm{M},\eta)$. $\mathcal{V}(M)$ will be the set of all unit vector fields and $\mathcal{V}(M,v)$ will denote the set of all unit vector fields which extend v to all of M. Here v is the unit vector field defined along ∂M. Also observe the sets $\mathcal{P}(\mathrm{M},\eta)$ and $\mathcal{V}(\mathrm{M},v)$ can be empty depending on η and v.

After choosing a Riemannian metric on M we can associate a unit vector field to an oriented plane field in the following way: We send a unit vector field v to the plane field η such that v followed by the oriented basis of η orients TM. Thus there is a one-to-one correspondence between $\mathcal{P}(\mathrm{M})$ and $\mathcal{V}(M)$. Similarly for $\mathcal{P}(\mathrm{M},\eta)$ and $\mathcal{V}(\mathrm{M},v)$ where v is the unit vector field along the boundary associated to η by a choice of metric and orientation. Notice both the correspondences only depend on a choice of metrics.

We know that any 3-manifold has trivial tangent bundle. Thus fixing some trivialization we can write $\mathrm{TM} \simeq \mathrm{M}\times\mathbb{R}^3$. So the unit tangent bundle UTM can be identified with $\mathrm{M}\times\mathrm{S}^2$. Any unit vector field on M can be defined as a section of this bundle and can be associated to a map $\mathrm{M}\to\mathrm{S}^2$. We can identify $\mathcal{V}(\mathrm{M})$ with $[\mathrm{M},\mathrm{S}^2]$. Similarly if v is a unit vector field on ∂M, we can associate it with a map $f_v\colon \partial\mathrm{M} \to \mathrm{S}^2$. Thus $\mathcal{V}(\mathrm{M},v)$ can be identified with the maps from M to S^2 which coincides with f_v on the boundary, denoted by $[\mathrm{M},\mathrm{S}^2;f_v]$.

Now Suppose $f_v\colon \partial M \to \mathrm{S}^2$ misses the north pole p. Now given any $f\in[\mathrm{M},\mathrm{S}^2;f_v]$ we can homotope it so that it is transverse to the north pole (Thus p will be a regular value for f). Then $f^{-1}(p) = L_f$ will be in the interior of M with framing $\mathbf{f_f}$ given by $f^*(T\mathrm{S}^2|_p)$. As f homotopes through maps in $[\mathrm{M},\mathrm{S}^2;f_v]$ the link $(L_f,\mathbf{f_f})$ changes by framed cobordism. Thus any v defined on ∂M which extends to M can be associated to framed cobordism classes of link. This gives us the relative version of Pontyragin-Thom construction.

Remark 2.5.1. *Notice, this construction works fine if* M *has multiple boundary components.*

Lemma 2.5.1. *Assume that η is a plane field defined along the boundary of* M *that in some trivialization of* TM *corresponds to a function that misses the north pole of* S^2*. There is a one-to-one correspondence between homotopy classes of plane fields on* M *that extend η on* M *and the set of framed links in the interior of* M *up to framed cobordism.*

For the closed case, the following proposition was proved in [18].

Proposition 2.5.1. *Let* M *be a closed, connected 3-manifold. Then any trivialization τ of the tangent bundle of* M *determines a function Γ_τ sending homotopy classes of oriented 2-plane fields ξ on* M *into* $\mathrm{H}_1(\mathrm{M}, \mathbb{Z})$ *and for any ξ, $2\Gamma_\tau(\xi)$ is Poincaré dual to $c_1(\xi) \in \mathrm{H}^2(\mathrm{M}, \mathbb{Z})$. For any fixed $x \in \mathrm{H}_1(\mathrm{M}, \mathbb{Z})$, the set $\Gamma^{-1}(x)$ of classes of 2 plane-fields ξ mapping to x has a canonical $\mathbb{Z}$ action and is isomorphic to $\mathbb{Z}/d(\xi)$, where d is the divisibility of the chern class.*

Now suppose M is a manifold with boundary and $\mathcal{F}(\mathrm{M})$ denotes the set of all cobordism classes of framed link in the interior of M. Then there is a homomorphism

$$\phi\colon \mathcal{F} \to \mathrm{H}_1(\mathrm{M}, \mathbb{Z})$$

such that

$$(L_f, f) \to [L].$$

This map is clearly surjective. We want to compute the preimage of this map. First notice, there is a natural intersection pairing between $\mathrm{H}_1(\mathrm{M})$ and $\mathrm{H}_2(\mathrm{M}, \partial M)$. Let

$i\colon (\mathrm{M}, \emptyset) \to (\mathrm{M}, \partial\mathrm{M})$ induces the map $i_*\colon \mathrm{H}_2(\mathrm{M}, \mathbb{Z}) \to \mathrm{H}_2(\mathrm{M}, \partial\mathrm{M}, \mathbb{Z})$. For $L \in \mathrm{H}(\mathrm{M}, \mathbb{Z})$, set

$$D_L = \{L \cdot [\Sigma] : \ where\ \Sigma \in i_*(\mathrm{H}_2(\mathrm{M}, \mathbb{Z}))\}$$

where $L \cdot \Sigma$ denotes the intersection pairing. Clearly this is a subset of $\mathbb{Z}$. Suppose $d(L)$ is the smallest non-negative integer in D_L.

Lemma 2.5.2. *With the notations above,*

$$\phi^{-1}(L) = \mathbb{Z}/d(2L).$$

2.5.2. Lutz twist

Lutz twist is a standard technique in contact geometry to alter a contact structure. Suppose (M, ξ) be any contact manifold and K be a transverse knot. Now every transverse knot has a standard neighborhood $\mathrm{S}^1 \times \mathrm{D}^2$ with the contact structure defined by $\xi = \ker(d\theta + r^2 d\phi)$. Consider another contact structure $\xi' = \alpha' = f(r)d\theta + g(r)d\phi$ on $\mathrm{S}^1 \times \mathrm{D}^2$, where $(f(r), g(r)) = (1, r^2)$ along the boundary of D^2 and $f'g - g'f > 0$ everywhere else. Now replacing ξ by ξ' in a neighborhood of K is called *performing a Lutz twist* along K. A Lutz twist has the following effect on the contact structure:

Lemma 2.5.3. *A Lutz twist along a positive transverse knot* K *in* (M, ξ) *changes the Euler class of* ξ *in the following way*

$$e(\xi') - e(\xi) = -2PD[K]$$

Remark 2.5.2. *Here, observe that the relative Euler class in a manifold with boundary changes similarly after a Lutz twist along a knot in the interior of the manifold.*

Chapter 3. Coarse Classification of Legendrian and Transverse Links

In this chapter we will coarsely classify null-homologous loose Legendrian and transverse links as mentioned in 3.2.1.

3.1. Types of links in an overtwisted manifold

There are two types of links in an overtwisted contact manifold, namely loose (also known as non-exceptional) and loose (also known as exceptional).

Definition 3.1.1. *A Legendrian link L is called loose if the contact structure restricted to its complement is overtwisted. Otherwise, it is called non-loose. In other words, a loose link must have an overtwisted disk disjoint from it.*

Remark 3.1.1. *Note that, for a loose Legendrian link, all of its components must be loose. But a non-loose link can have loose components. In fact, a non-loose link can have all its components loose.*

3.2. Classification of loose Legendrian links

The following is our main theorem in this section.

Theorem 3.2.1. *Suppose L and L' are two Legendrian n-component links in (M, ξ) with all of their components null-homologous. We fix their Seifert surfaces. If L and L' are topologically isotopic, $\mathrm{tb}(L_i) = \mathrm{tb}(L'_i)$ and $\mathrm{rot}(L_i) = \mathrm{rot}(L'_i)$ for $i = 1 \ldots n$ (where the classical invariants are defined using the fixed Seifert surfaces), then L and L' are coarsely equivalent.*

In other words, there is a unique loose Legendrian link with the components having fixed tb and rot up to contactomorphism. Before we begin proving this, we need the following lemma:

Lemma 3.2.1. *Suppose L and L' be two Legendrian n-component links in (M, ξ) with each of their components being null-homologous. Suppose they are topologically isotopic, $\mathrm{tb}(L_i) = \mathrm{tb}(L'_i)$ and $\mathrm{rot}(L_i) = \mathrm{rot}(L'_i)$ for $i = 1 \dots n$, then $\xi|_{\mathrm{M} \setminus \mathrm{N}(L)}$ is homotopic to $\xi|_{\mathrm{M} \setminus \mathrm{N}(L')}$ rel boundary as plane fields.*

Proof. We will use techniques similar to [8]. As L and L' are topologically isotopic, there is an ambient isotopy of M which takes L to L'. We will assume that the Seifert surfaces of the link components are also related by this ambient isotopy (So after applying the ambient isotopy we assume the Seifert surfaces of the components agree).

As L and L' are topologically isotopic there is an ambient isotopy of M, ϕ_t such that $\phi_0 = id$ and $\phi_1(L) = L'$. Using this isotopy we push forward the underlying contact structure ξ. Thus we now have a new contact structure $\phi_{1*}^{-1}\xi$ and call it ξ'. Observe ξ and ξ' are homotopic as plane fields in M. After we apply the isotopy we can assume $L = L'$ and N be their standard neighborhood. Note that, tb measures the twisting of the contact framing with respect to the surface framing. As the components have the same tb, this allows us to identify the neighborhoods. Now by standard neighborhood theorem of Legendrian links, ξ and ξ' agree on N. We need to show that $\xi|_{\mathrm{M} \setminus \mathrm{N}}$ is homotopic to $\xi'|_{\mathrm{M} \setminus \mathrm{N}}$ rel boundary as plane fields. We know that homotopy class of plane fields are in one-to-one correspondence with framed links up to framed cobordism. Now using Pontyragin-Thom construction for manifolds with boundary, we will associate these plane fields with (L_ξ, f_ξ) and $(L_{\xi'}, f_{\xi'})$. We need to show that these links are homologous in $\mathrm{M} \setminus \mathrm{N}$ and that their framing differs by $2d[L_\xi]$ where d is the divisibility of the euler class of ξ.

To do this, first we will fix a trivialization of TM. Note that, Pontyragin–Thom construction works for any trivialization, but we would like to use a convenient one. Suppose v_1 be the Reeb vector field of ξ. Now we choose a Riemannian metric such that v_1 is positively orthogonal to ξ with respect to this metric. Thus v_1 defines ξ in M. To avoid ambiguity, from now on we will call the contact structure ξ, ξ_{v_1} and start making alterations to ξ_{v_1} which do not affect ξ or ξ'. Next choose v_2 in the following way:

1. Choose v_2 to be the tangent vector field along L_i , if $\mathrm{rot}(L_i)$ is even.
2. Choose v_2 to be the tangent vector field along L_i with an extra negative twist with respect to the fixed Seifert surface of the component, if $\mathrm{rot}(L_i)$ is odd.

Observe that the tangent vector field v_2 along $L = L'$ agrees as all the components have same rot (rot measures the winding of the tangent vector field along the component) Notice that as we know ξ in N, we can extend v_2 to all of N. Now we need to extend v_2 to all of M. In general, this might not be possible. The relative Euler class $e(\xi_{V_1}, \mathrm{v}_2)$ is the obstruction to this extension. So our goal is to make this obstruction vanish.

By using Lefchetz duality and Mayer–Vietoris sequence, we have

$$\mathrm{H}^2(\mathrm{X}, \partial\mathrm{X}; \mathbb{Z}) \simeq \mathrm{H}_1(\mathrm{X}; \mathbb{Z}) \simeq \mathrm{H}_1(\mathrm{M}) \oplus \mathbb{Z}^n \tag{1}$$

where each of the $\mathbb{Z}$ factors are generated by the meridian of the link components.

$$\langle e(\xi_{\mathrm{v}_2},), [\Sigma_i]\rangle = \mathrm{rot}(L_i) \quad \text{or} \quad \mathrm{rot}(L_i) + 1$$

In both the cases, this is always even for each i. So the relative Euler class is a $n + 1$ vector with every co-ordinate even. Let us rename this as α. Next we will apply half Lutz twist to alter the relative Euler class. Now choose a transverse knot K in X (that is $[\mathrm{K}] \in \mathrm{H}_1(\mathrm{X}, \mathbb{Z})$) such that $PD[\mathrm{K}] = \frac{1}{2}(\alpha)$ (We can always find such knot). If we apply half Lutz twist in X along K, we get a new contact structure $\xi_{\mathrm{v}_2'}$ such that

$$e(\xi'_{\mathrm{v}_1}, \mathrm{v}_2) - e(\xi_{\mathrm{v}_1}, \mathrm{v}_2) = -2PD[\mathrm{K}]$$

By our choice of K, $e(\xi'_{\mathrm{v}_1'}, \mathrm{v}_2)$ becomes zero. Thus we can extend v_2 as a section of ξ'_{v_1} on all of X. Now choose an almost contact structure J on M and say $\mathrm{v}_3 = J\mathrm{v}_2$. We use the vector fields $-\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ to trivialize TM and TX. Notice here v_1 is mapped to the south pole p^*. We will call this trivialization τ.

Using this trivialization, we find framed links (L_ξ, f_ξ) and $(L_{\xi'}, f_{\xi'})$ associated to ξ and ξ' by Pontyragin–Thom construction on X. As M is trivialized by τ, both L_ξ and $L_{\xi'}$ are oriented cycles. Next we need to show that L_ξ and $L_{\xi'}$ are homologous in X. As $\mathrm{H}_1(\mathrm{X}, \mathbb{Z})$ splits in $n+1$ components,we need to check if they agree in each of them. First we will show they agree in $\mathrm{H}_1(\mathrm{M}, \mathbb{Z})$. Now notice, v_1 is the vector field that defines ξ in N and also it is mapped to the south pole. So we can define a map from N to S^2 where N is collapsed to the south pole p^*. Now we can extend the map f_ξ in the following way:

$$F_\xi(x) = \begin{cases} f_\xi(x) & \text{if} \quad x \in X \\ p^* & \text{if} \quad x \in \mathrm{N} \end{cases}$$

Now $F^{-1}(p) = f^{-1}(p) = L_\xi$. Similarly for $L_{\xi'}$. Thus L_ξ and $L_{\xi'}$ are also associated to ξ and ξ' in M. Now as ξ and ξ' are homotopic as plane fields in M, the components must agree in $\mathrm{H}_1(\mathrm{M}, \mathbb{Z})$.

Next we need to verify if $L_\xi \cap \Sigma_i = L'_\xi \cap \Sigma_i$ for each i. Note that here we can take the same Seifert surfaces for each link components L_i and L'_i as they are related by the ambient isotopy. As the tangent vector v_2 gives the framing to the link L_ξ (as framing of

L_ξ is given by the pull back of $T_p\mathrm{S}^2$ and this is exactly equal to ξ along L_ξ), we have

$$\langle e(\xi, \mathrm{v}_2), \Sigma_i\rangle = L_\xi \cap \Sigma_i.$$

Same argument works for $L_{\xi'}$. Now if $\mathrm{rot}(L_i)$ is even, the definition of v_2 gives us $\mathrm{rot}(L_i) = \langle e(\xi, \mathrm{v}_2), \Sigma_i\rangle$. Thus if $\mathrm{rot}(L_i)$ is even, we have,

$$L_\xi \cap \Sigma_i = \langle e(\xi, \mathrm{v}_2), [\Sigma]\rangle = \mathrm{rot}(L_i) = \mathrm{rot}(L'_i) = \langle e(\xi', \mathrm{v}_2), [\Sigma]\rangle = L_{\xi'} \cap \Sigma_i$$

Similarly for $\mathrm{rot}(L_j)$ odd,

$$L_\xi \cap \Sigma_i = \langle e(\xi, \mathrm{v}_2), [\Sigma]\rangle = \mathrm{rot}(L_j) + 1 = \mathrm{rot}(L'_j) + 1 = L'_\xi \cap \Sigma_i.$$

Thus L_ξ and $L_{\xi'}$ are homologous in $\mathrm{H}_1(X, \mathbb{Z})$.

Next we want to show that the framing differs by $2d([L_\xi])$. Now notice that ξ and ξ' are homotopic as plane fields in M. Thus the framings of L_ξ and L'_ξ associated to ξ and ξ' must differ by $d(\xi)$ where $d(\xi)$ is the divisibility of $e(\xi)$ [18]. In other words, its the same as the divisibility of the Poincaré dual of $e(\xi)$. We will show this is exactly $2d[L_\xi]$. We know $\xi = f_\xi^*(TS^2)$.

$$e(\xi) = e(f_\xi^*(TS^2)) = f_\xi^*(e(TS^2)) = f_\xi^*(2[S^2]).$$

Now $p = PD[S^2]$ as p is a regular value. So

$$f_\xi^*(2[S^2]) = f_\xi^*(2PD[p]) = 2PD(f_\xi^{-1}(p)) = 2[L_\xi].$$

For the second equality check [14]. So the framing differs by $2d[L\xi]$. Thus by 2.5.1, $\xi_{M\setminus N}$ and $\xi'_{M\setminus N}$ are homotopic rel boundary. $\square$

Proof of 3.2.1. As L and L' are loose, they have overtwisted complements. Now by Eliashberg's classification of overtwisted contact structures we know that isotopy classes of overtwisted contact structures are in one to one correspondence with the homotopy class of plane fields [6]. Thus if each of the components of L and L' have same Thurston–Benniquin and rotation number, by 3.2.1 they have contactomorphic complements rel boundary. As we can extend this contactomorphism over the standard neighborhood of L (disjoint union of solid tori), this proves L and L' are coarsely equaivalent. □

3.3. Classification of loose transverse links

Corollary 3.3.1. *Suppose T and T' are two loose n-component transverse links with each of their components being null-homologous (i.e each of the components bounds a Seifert surface). Fix these Seifert surfaces and with respect to these surfaces suppose* $\mathrm{sl}(T_i) = \mathrm{sl}(T'_i)$*, then T and T' are coarsely equivalent.*

Proof. Suppose T and T' be two loose transverse links with each of their components being nullhomologous (i.e each component bounds a Seifert surface) and $\mathrm{sl}(T_i) = \mathrm{sl}(T'_i)$ for each i. Now we Legendrian realize T and T' component by component and call them L and L'. We can do the Legendrian approximation in a small enough neighborhood so that the Legendrian links remain loose. After this step, we can have the following two cases:

Case 1

Suppose $\mathrm{tb}(L_i) = \mathrm{tb}(L'_i)$ and $\mathrm{rot}(L_i) = \mathrm{rot}(L'_i)$ for all i. Then we have two loose Legendrian links with each component null homologous and with same classical invariants. Thus by 3.2.1, they have contactomorphic complements. Now we take the transverse push-off of L and L'. As transverse push-off is well-defined, we get back T and T'. This proves that T and T' are coarsely equivalent.

Case 2

Suppose $\mathrm{tb}(L_j) \neq \mathrm{tb}(L'_j)$ for some j. We may assume $\mathrm{tb}(L_j) > \mathrm{tb}(L'_j)$. So we start by negatively stabilizing L_j. As we can do a negative stabilization in a small enough Darboux ball, this does not effect any other link component and thus without changing the transverse link type. So we can negatively stabilize each of the link components locally one by one till $\mathrm{tb}(L_i) = \mathrm{tb}(L'_i)$ for each i. As $\mathrm{sl}(T_i) = \mathrm{sl}(T'_i)$, we must have $\mathrm{rot}(L_i) = \mathrm{rot}(L'_i)$ for each i as well. So we are back in case 1. $\square$

Chapter 4. Open Book Decompositions and Legendrian Links

In this chapter, we associate links in (M, ξ) with an open book decomposition supporting (M, ξ). We extend the idea of support genus of a Legendrian knot [21] to the support genus of a link and define a new invariant for a transverse link. Further, we prove that every equivalence class of loose null-homologous Legendrian links have support genus zero.

4.1. Support genus and loose Legendrian links

We can always associate a Legendrian link in (M, ξ) with an open book supporting the underlying manifold by including the link in the 1-skeleton of the contact cell decomposition of the contact manifold. Thus we define the support genus of a Legendrian link in (M, ξ) as follows:

Definition 4.1.1. *The support genus* $\mathrm{sg}(L)$ *of a Legendrian link* L *in a contact* 3*-manifold* (M, ξ) *is the minimal genus of a page of the open book decomposition of* M *supporting* ξ *such that* L *lies on the page of the open book and the framings given by* ξ *and the page agree.*

In [21], Onaran proved the following theorem.

Theorem 4.1.1. *Any link in a 3-manifold* M *is planar.*

The above theorem tells us that that any link in M can be put on a planar open book $(\mathrm{B}, \Sigma, \phi)$ for M. For details of the proof see [21]. Now before we proceed to the main theorem of this section, we will need the following lemmas.

Lemma 4.1.1. *Suppose* L *be a Legendrian link sitting on a planar open book as shown in 4.1. Then positive/negative stabilization of any of the link component* L_i *can be done fixing*

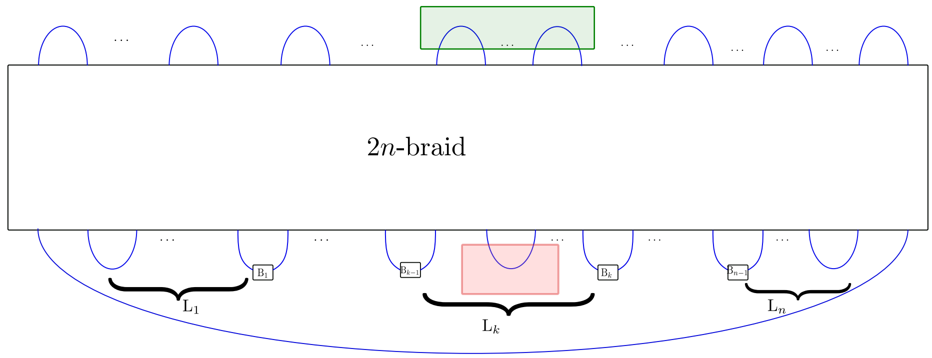

Figure 4.1. Page of a planar open book where the link lies. The blue outline shows the outer boundary component of the punctured disk. The box depicts the boundary area where we want to do the stabilization or detabilization of L_k.

the Legendrian isotopy type of the other link components.

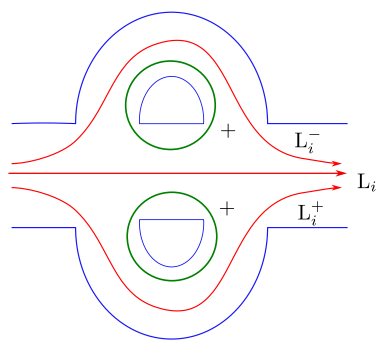

Figure 4.2. Positive and Negative stabilization of the link sitting on the page of an open book.

Proof. Suppose L be a Legendrian link sitting on the page of a planar open book. Fix an orientation of the link. Suppose B_i is the outer most binding component. Now choose a particular region of B_i which is closest to L_i and far from other components. The shaded region in 4.1 shows us where we will do the stabilizations. We do a positive stabilization along B_i and push the link component L_i along the attaching 1-handle as shown in 4.2. We call it L_i'. by our choice of attaching region, this operation is local and thus does not affect any other link component sitting on the page of the open book. Clearly D is a disk with tb $= -1$ and a single dividing curve. Thus we assume it to be convex. Therefore, L'_i is the stabilization with D being the stabilizing disk. Also D can be thought as bypass disk along L'. The sign of the stabilization will depend on the orientation of the bound-

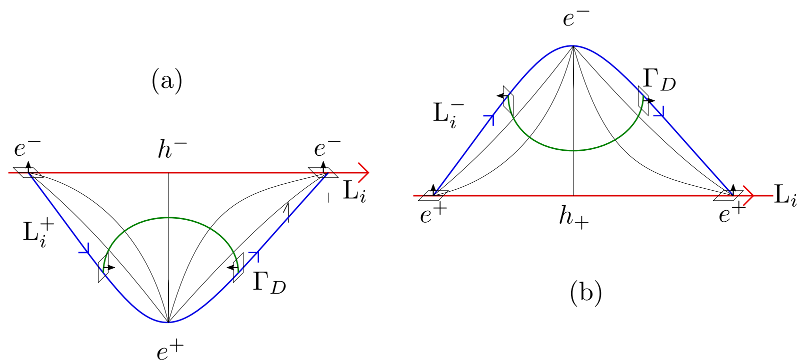

Figure 4.3. The positive and negative stabilization of L_i and the signs of bypass disks.

ary of the disk.. The orientation of the boundary of the disk is inherited by the Legendrian knot L'. The sign of the singularity of D_ξ is determined by the contact planes. We will call a singularity along ∂D positive or negative according to if the contact plane takes a right handed or a left handed turn along ∂D. See 4.3. Now clearly we have chosen to do this operation away from the other link components. Thus all other link components remain unaltered during the operation and so are their Legendrian knot types. Observe that, L_i has a fixed orientation. So we can perform any number of positive or negative stabilization of any link component away from the other components.

□

The next lemma tells us that de-stabilization of any component of a loose link can be done in the complement of other components.

Lemma 4.1.2. *Suppose L be a link sitting on the page of a planar open book* $(\mathrm{B}, \Sigma, \phi)$ *as shown in 4.1. Suppose* B_i *be the outer most boundary component. Now suppose we do* *a negative stabilization of* $(\mathrm{B}, \Sigma, \phi)$ *along* B_i*. The new open book does not support* (M, ξ) *and we get a new link* L_{new} *in the new contact structure. Now if we push* L_{new} *along the attaching handle, this will destabilize the link component and it can be performed in a way that it does not affect the Legendrian type of any other link components.*

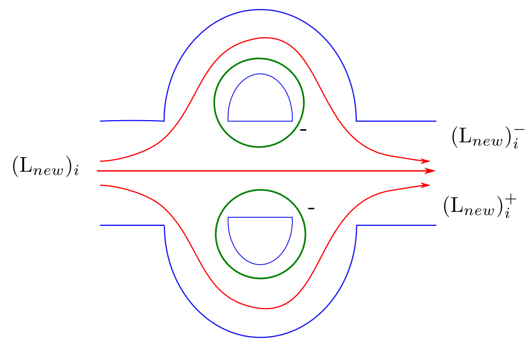

Figure 4.4. Negative stabilization of the open book and the de-stabilized link component sitting on the page

Proof. In [21], a similar version of this lemma has been proved for knots. We give a slightly different proof. Our proof relies on the fact that null-homologous Legendrian knots having same classical invariants are Legendrian isotopic in S^3 if there is an overtwisted disk disjoint from them [4].

Suppose L be a Legendrian link sitting on the page of a planar open book $(\mathrm{B}, \Sigma, \phi)$. Fix an orientation of L. Pick a link component L_i, we want to destabilize. Now we choose a particular region of the outer most boundary component near L_i and away from all other L_j's. This can be done as shown in 4.1.

Now do a negative stabilization along that region and push the link component L_i along the attaching 1-handle. By our choice of attaching region, this operation is away from the other link components. The new open book $(\mathrm{B}', \Sigma', \phi')$ doesn't support the underlying contact structure anymore. We will call the link L_{new} in the new contact structure and show that $(L'_{new})_i$ is a destabilization of $(L_{new})_i$ as shown in 4.4. Here

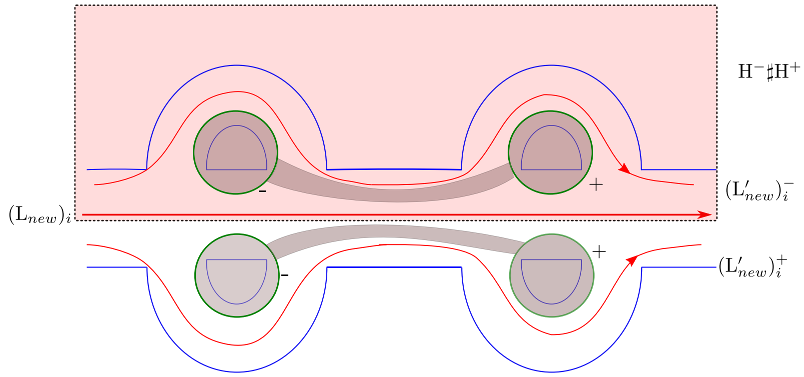

Figure 4.5. Negative stabilization followed by a positive stabilization of the open book near L_k and away from other components.

the disk D has tb $=$ 1 and thus cannot be made convex. So we stabilize the open book along the same boundary component as shown in 4.5. Now positive and negative stabilization of (Σ, ϕ) can also be thought as Murasugi summing with $(\mathrm{H}^{\pm}, \pi^{\pm})$. Also notice $(\mathrm{H}^{+}, \pi^{+}) \sharp (\mathrm{H}^{-}, \pi^{-})$ is an open book for (S^3, ξ_{-1}). As the link components are identical outside the neighborhood of the boundary, we can assume the local operation to be entirely in the overtwisted S^3. Now we push $(L'_{new})_i$ along the new attaching handle. And by 4.1.1, we get $(L'_{new})_i^{\pm}$ according to the orientation of the link component. Also we found an overtwisted disk D in the complement of $(L_{new})_i$ and $(L'_{new})_i^{\pm}$. Now by [4], $(L_{new})_i$ and $(L'_{new})_i^{\pm}$ must be Legendrian isotopic. As $(L'_{new})_i^{\pm}$ is a stabilization of $(L'_{new})_i$, clearly $(L'_{new})_i$ is the destabilization of $(L_{new})_i$. Nothing changed outside the overtwisted S^3. Thus all other link components remain unaltered and so their Legendrian isotopy class.

□

Thus 4.1.2 together with 4.1.1 proves that if a link lies on an open book as shown in 4.1, any number of positive (resp. negative) stabilization and de-stabilization of a particular link component can be done in the complement of the other link components. We will use these lemmas in the proof of our main theorem in this section.

Definition 4.1.2. *Suppose $[L]_n$ denotes the class of all the n-component links with each component having fixed* tb *and* rot*. For any two links in this class there exists a contactomorphism that takes one to the other. We call this* the coarse equivalence class *of a link.*

Theorem 4.1.2. *Suppose $[L]_n$ be the coarse equivalence class of null-homologous, loose Legendrian link in* (M, ξ)*. Then* $\mathrm{sg}([L]_n) = 0$*.*

Proof. As every link is planar, we can put L on a planar open book $(\mathrm{B}, \Sigma, \phi)$ for M. Now $(\mathrm{B}, \Sigma, \phi)$ does not necessarily support the underlying contact structure. But we can always negatively stabilize the open book and assume the contact structure it supports is overtwisted and call it ξ'. As overtwisted contact structures can be identified using their d_2 and d_3 invariant, we start making alterations to the open book so that the invariants match with those of ξ. By Lutz twist and Murasugi summing in an appropriate way we can make the d_2 and d_3 invarants agree. Note that, d_3 invariant are additive under connected sum operation. Also none of these operations change the genus of the open book. For details of these operations check [9]. Now we have a planar open book which supports a contact structure whose d_2 and d_3 invariants agree with ξ. By Eliashberg's classification of overtwisted contact structures, these contact structures are isotopic. Next we can Legendrian realize the link on the page and call it L'. Suppose we want to realize the following classical invariants, $\mathrm{tb} = (t_1, t_2, \dots t_n)$ and $\mathrm{rot} = (r_1, r_2, \dots r_n)$. If the classical invariants of L' agree with that of L, we are done. Suppose not. Then we can have the following cases:

4.1.1. Case 1

Suppose tb agrees but rot does not. Let L_j be a link component with $\mathrm{tb}(L_j) = t_j$ and $\mathrm{rot}(L_j) = r'_j \neq r_j$. Now we will negatively or positively stabilize the link component L_j to increase or decrease r'_j. We know by 4.1.1, this operation can be done fixing other link components. Notice, this will change t_j to $t_j - 1$. So we need to destabilize the link component in an appropriate way so that we do not reverse the change in r_j. This can be done in the following way, if we positively stabilize the link component, we will negatively destabilize it. This can be done fixing all other link components as stated in 4.1.2. Now this will keep the tb fixed and increase rot by 2. Similarly doing a negative stabilization and a positive destabilization will keep tb fixed and decreases rot by 2. As tb + rot is always odd for a Legendrian knot, we can achieve any possible rotation number for a link component. Now we can do this any number of time to achieve r_j while fixing the Legendrian type of all other link components. Here note that, we might end up in a contact structure different from the one we started as negative stabilization alters a contact structure. But then we can always alter it by Murasugi summing with appropriate open books of S^3. In this way, we will find a link sitting on the page of an open book supporting the contact structure ξ with $\mathrm{tb} = (t_1, t_2, \ldots t_n)$ and $\mathrm{rot} = (r_1, r_2, \ldots r_n)$. By 3.2.1, L must be in the same coarse equivalence class . This proves the theorem.

Case 2

Suppose $\mathrm{tb}(L_j) = t'_j \neq t_j$. In this case we need to stabilize or destabilize the link component L_j to decrease and increase the tb till it agrees with t_j and this can be done keeping the other components fixed by lemmas 4.1.1 and 4.1.2. Now we can do this local operation for all the link components one by one till we get the tb we desire. So we are in Case 1. □

4.2. Support genus and non-loose Legendrian link

The immediate question that arises from the previous section is if the converse is true. If it is true, then support genus can be used to distinguish between loose and non-loose links. As Onaran showed in her thesis, that non-loose knots can have support genus zero. Similarly in this section, we show that non-loose links can have support genus zero as well. Thus the converse of theorem 4.1.2 is not true.

Theorem 4.2.1. *There are examples of non-loose links with support genus zero.*

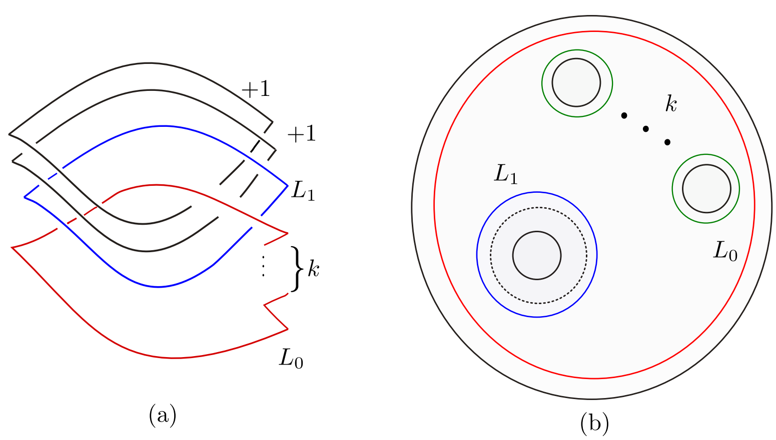

Figure 4.6. (a) Non-loose Hopf link in $(\mathrm{S}^3, \xi_{\frac{1}{2}})$. (b) Planar open book supporting the contact structure where the Hopf link sits. We do a right handed Dehn twist along the green curves and left-handed Dehn twist along the dashed one.

Proof. 4.6(a) shows a non-loose positive Hopf link in (S^3, ξ_{-1}). To see this, we do a -1 surgery along L_1 which will cancel one of the +1- surgeries and we will be left with one +1- surgery on tb $= -1$ unknot in $(\mathrm{S}^3, \xi_{std})$ which produces the unique tight $\mathrm{S}^1 \times \mathrm{S}^2$. For details, check [15]. Next, we constructed a planar open book compatible with $(\mathrm{S}^3, \xi_{\frac{1}{2}})$

where the non-loose Hopf link sits. We start with the annular open book that supports $(\mathrm{S}^3, \xi_{std})$ where we can put the positive Hopf link and used the well known stabilization method we used previously in 4.1.1. The monodromy of this open book can be computed from the Dehn twists coming from the stabilizations and the Dehn twists defined by the surgery curves. One of the left-handed Dehn twist coming from the +1 surgery will cancel the right handed Dehn twist of the annular open book we started with. We perform right handed Dehn twist along the solid green curves and the left handed Dehn twist along the dashed curve. This clearly shows $\mathrm{sg}(L_0 \sqcup L_1) = 0$. $\square$

Chapter 5. Open Books and Transverse Links

In this chapter, we associate any transverse link with an open book decomposition and define the support genus for transverse links. Next, we show that coarse equivalence class of null-homologous, loose transverse links have support genus zero. We also show as a corollary that support genus gives a condition when a non-loose Legendrian knot has a non-loose transverse push-off.

5.1. Associating an open book with transverse links

Theorem 5.1.1. *Suppose T be any transverse link in* (M, ξ)*. Then T is transversely isotopic to the sub-binding of some open book* $(\mathrm{B}, \Sigma, \pi)$ *supporting* (M, ξ)*.*

We will use a lemma from [1] to prove our result. For notational purpose, we will be using the word "to the right" in the following sense: an arc "to the right" of a link component will imply that the orientation of the link component followed by the orientation of the arc agrees with the orientation of their intersection point on the page. A set of arc $\Gamma = \{\gamma_1, \gamma_2, \dots \gamma_n\}$ is "to the right of L" means that each of the γ_i lies to the right of L_i where L_i is the i^{th} link component.

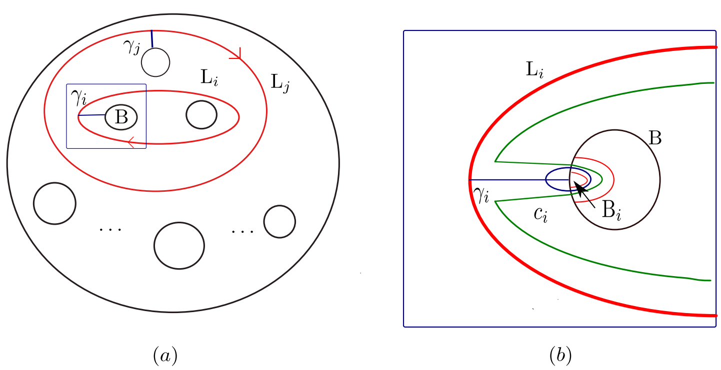

Figure 5.1. (a) A planar open book and the disjoint arcs. (b) Enlarged view of the region inside the box in disjoint arcs lying "to the right".

Proof. Suppose T be any transverse link in (M, ξ) and L be its Legendrian approximation.

Now we put L on the page of an open book supporting (M, ξ). Observe that, all of the components have a fixed orientation. There are several cases to consider here and we will prove the theorem for each of the cases.

5.1.1. Case 1(a)

[$(\mathrm{B}, \Sigma, \phi)$ is planar and all the link components are linearly independent] If each of the link components bounds exactly one binding component, then clearly that binding component is its transverse push off and we are done. If not, we start by finding a set of disjoint arcs $\{\gamma_1, \dots \gamma_n\}$ such that each of them runs from the closest boundary component of the link component L_i and stays "to the right". Clearly, if all the link components are linearly independent, we can easily find such a set where each of the arcs γ_i is disjoint from L_j for $j \neq i$ irrespective of the orientation of the link components. See 5.1(a). Now we have c_i that comes from B_i along γ_i till it hits L_i and then follows L_i and comes back to B_i following γ_i for each i. Next we positively stabilize B_i along c_i and find a link component L'_i that runs around the attaching 1-handle exactly once and is topologically isotopic to the link component L_i is the whole manifold. Legendrian realize L'_i. Clearly the new boundary component B'_i is the transverse push-off of L'_i. See 5.1(b). By our choice of γ_i, L'_i is the negative stabilization of L_i [1]. Thus they have transversely isotopic transverse push-offs. So B'_i is L_i's transverse push-off as well. The new open book will be $(\mathrm{B} \sqcup \mathrm{B}'_i, \Sigma', \phi \circ \mathrm{D}_{c_i})$. After doing the Dehn twist along all such c_i's we find a n-component link B' such that B is the transverse push-off of L. The new monodromy will be given by $\phi \circ \mathrm{D}_{c_1} \circ \dots \mathrm{D}_{c_{n-1}} \circ \mathrm{D}_{c_n}$. By the well-definedness of transverse push-off, B' is transversely isotopic to T. Observe that, as all of the c_i's are disjoint from each other, the order of Dehn twist does not matter.

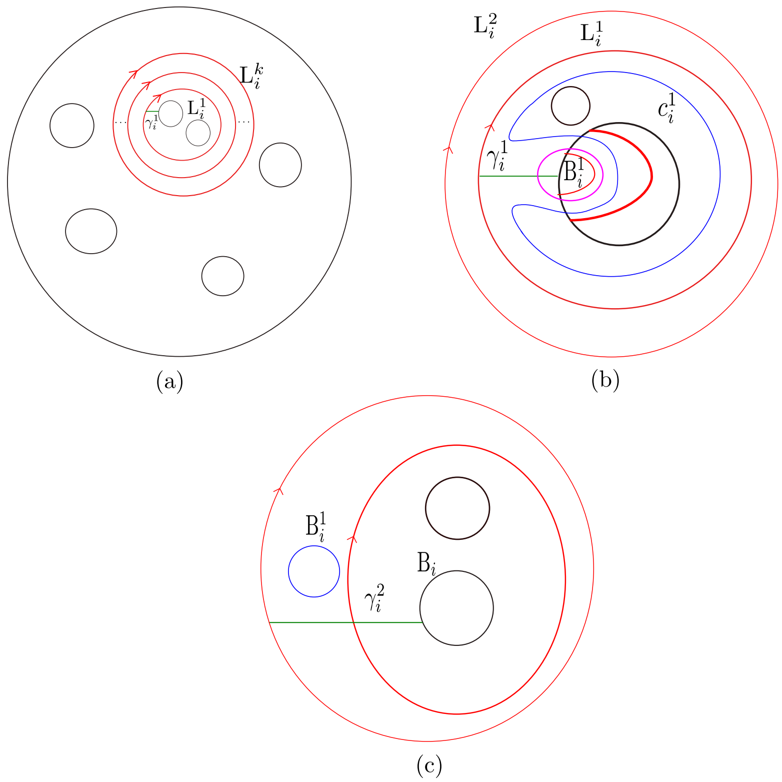

Figure 5.2. (a)A planar open book where L_i has k parallel copies. (b) A local picture with two parallel components. (c) The new boundary component B_i^1. γ_i^2 on the new open book.

5.1.2. Case 1(b)

[$(\mathrm{B}, \Sigma, \phi)$ is planar and some of the link components are parallel copies of one another] Suppose L_i has k parallel copies and we will call them L_i^j for $j = 1, 2, \ldots k$. We will treat this case differently. In this case either all the link components are oriented similarly or some of them can have opposite orientations.

Case 1(b)(i)

First we consider the case where all L_i^k's are oriented similarly. Check the local picture 5.2. Choose the link component that has a boundary component closest to it so that the arc γ_i^1 lies "to the right". Call it L_i^1. It can be the outer most or the innermost L_i^k according to the orientation. Without loss of generality, we assume it to be the innermost component and call it L_i^1. Now we find c_i^1 using γ_i^1 as before and positively stabilize the open book along B_i and find B_i^1 as the transverse push-off of the negatively stabilized L_i^1 as mentioned in 5.1.1. Once we find B_i^1, we follow the same procedure on the new open book $(\mathrm{B} \sqcup \mathrm{B}_i^1, \Sigma', \phi \circ D_{c_i^1})$ and find an arc γ_i^2 that runs from B_i to L_i^2 and stays to the right. Now notice here, this arc could possibly only intersect L_i^1 and no other link component. Repeat the same procedure as before to find B_i^2. Inductively, doing so we find a k-component link $\mathrm{B}_i^1 \sqcup \ldots \mathrm{B}_i^k$ such that $L_i^1 \sqcup \cdots \sqcup L_i^k$ has transverse pushoff transversly isotopic to $\mathrm{B}_i^1 \sqcup \ldots \mathrm{B}_i^k$. We can do this locally for all the link components which are not linearly independent. Notice, as the curves intersect each other, we strictly need to maintain the order of Dehn twist in this case.

Case 1(b)(ii)

In this case, some of the link components are oriented opposite and thus we cannot use the same boundary component for all of them. First we choose a link component L_i^k such that there exists an arc that joins L_i^k to its closest boundary component B_i, lies to the right and is disjoint from all other L_i^j's for $j \neq k$. It can be the innermost or outermost link component depending on the orientation. Without loss of generality, suppose it is the outer most component and call it L_i^1 and the arc γ_i^1. Check 5.3. Now we find c_i^1 as before and positively stabilize B_i along it. Now we have a new boundary component B_i^1 which will be transversely isotopic to the transverse push off of L_k^1. Now for the next parallel component L_k^2, if it has the same orientation, we can easily find an arc γ_i^2 that lies "to the right" and intersects L_i^1 and no other link component. We repeat this process step

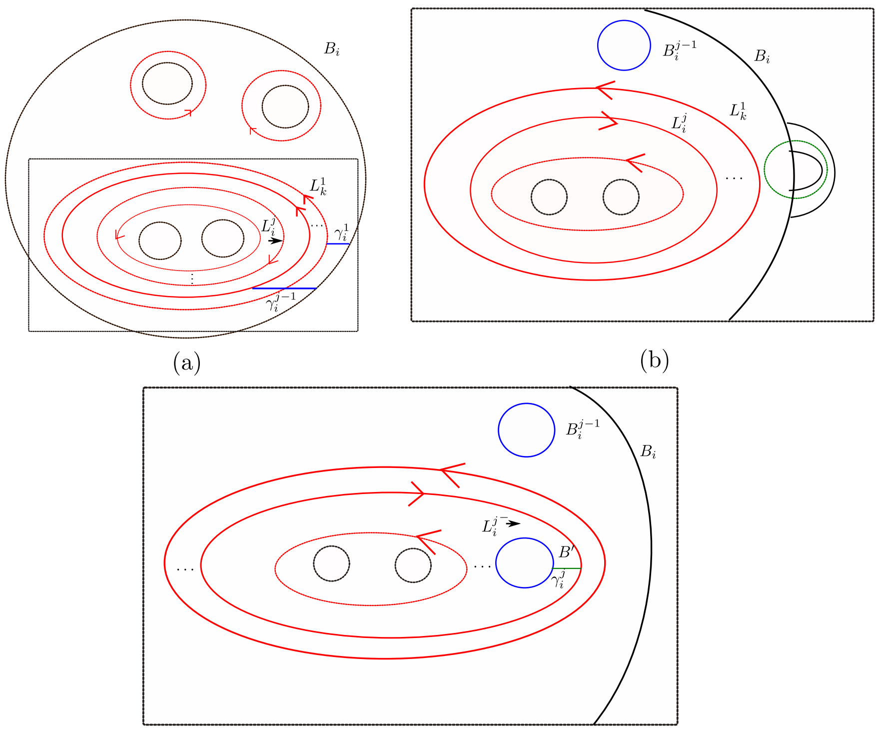

Figure 5.3. (a)Parallel copies of link component on planar open book with distinct orientation. (b)Once stabilized open book. (c)After pushing all L_k^i over the attaching one handle. Now we use the new boundary component B'.

by step till we find a component which has a different orientation. Suppose L_i^j is the link component with different orientation. Notice, now we can not find an arc that lies to the right and is disjoint from L_i^l where $l > j$. To fix this problem, we first positively stabilize the open book along the boundary component B_i along a boundary parallel curve and push all L_i^k's over the attaching 1-handle where $k = 1, 2 \ldots j$. By 4.1.1, this negatively stabilizes L_i^j. So L_i^j and ${L_i}^{j-}$ will have isotopic transverse push off. Now using the new boundary component B_i' coming from the stabilization, we can find an arc γ_i^j that lies "to the right" of L_i^j and is disjoint from all L_i^l for $l > j$. Check 5.3(c). Now we can continue using B_i' for all the link components till we find a link component that is oriented differently. Inductively, doing so will give us a transverse link that is a sub-binding of the open book.

5.1.3. Case 2(a)

[$(\mathrm{B}, \Sigma_g, \phi)$ has $g > 0$ and all the link components are linearly independent]

This is an easy case to deal with. Here we can have all link components of the same type (5.4) or different type (5.5). But irrespective of the types and orientation clearly there exists a set of disjoint arcs $\Gamma = \{\gamma_1, \ldots \gamma_n\}$ that lies on the "right" of L and runs from L_i to B as shown in 5.4(a) and 5.5(a). Now we will use Γ to find a set of disjoint closed curves $\{c_1, c_2, \ldots c_n\}$ like before and stabilize B along c_i's for every $i = 1, 2, \ldots n$. Finally we will find an n component link $\mathrm{B}' = \sqcup_{i=1}^n \mathrm{B}_i$ as before which is the transverse push-off of L for the same reason and thus B' is transversely isotopic to T. The new open book will have monodromy $\phi \circ \mathrm{D}_{c_1} \circ \ldots \mathrm{D}_{c_{n-1}} \circ \mathrm{D}_{c_n}$ and as all the curves are disjoint from the link components the order of Dehn twist does not matter in this case.

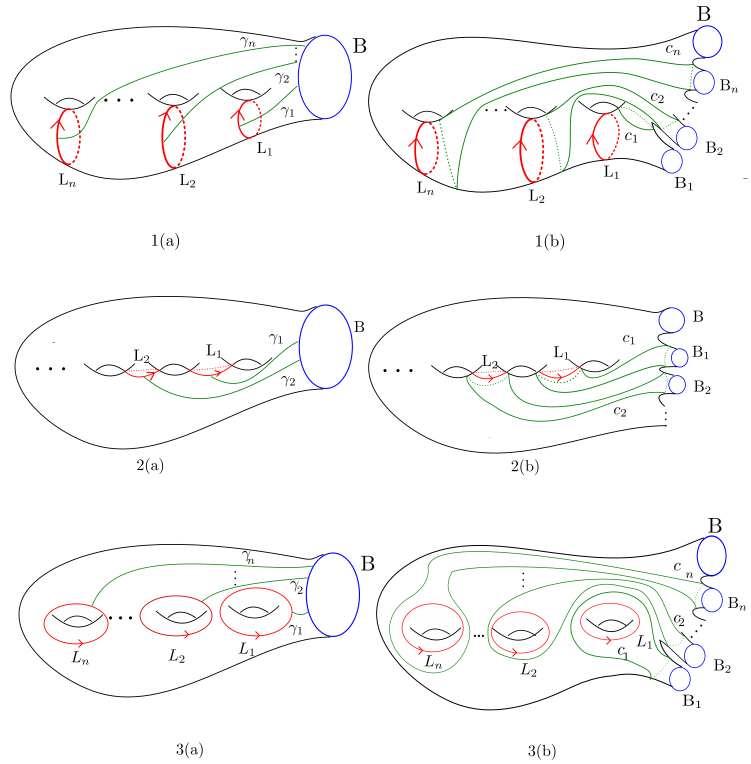

Figure 5.4. All possible cases where the link components are of same type. On the left, the set of disjoint arcs that lie "to the right". On the right the resulting open book with the transverse link $\sqcup_{i=1}^{n} B_i$.

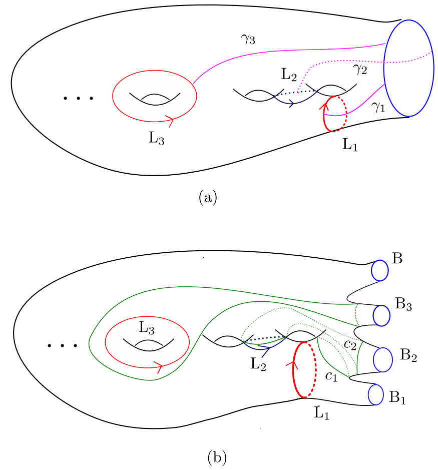

Figure 5.5. (a)Example of an open book where we have mixed type of link components. (b) The resulting open book after we do a Dehn twist along c_1, c_2 and c_3.

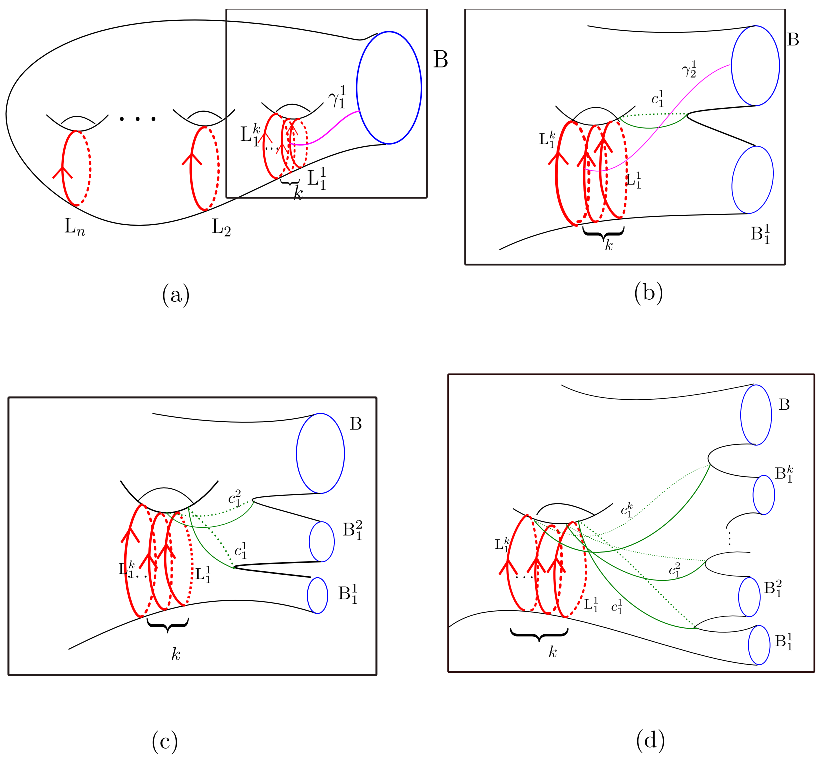

Figure 5.6. (a) L_1 has k parallel copies. (b) Enlarged view of the region near L_1. (c) The once stabilized open book along c_1^1. Find γ_1^2 and iterate this process. (d) The final result after k iterated stabilizations of B.

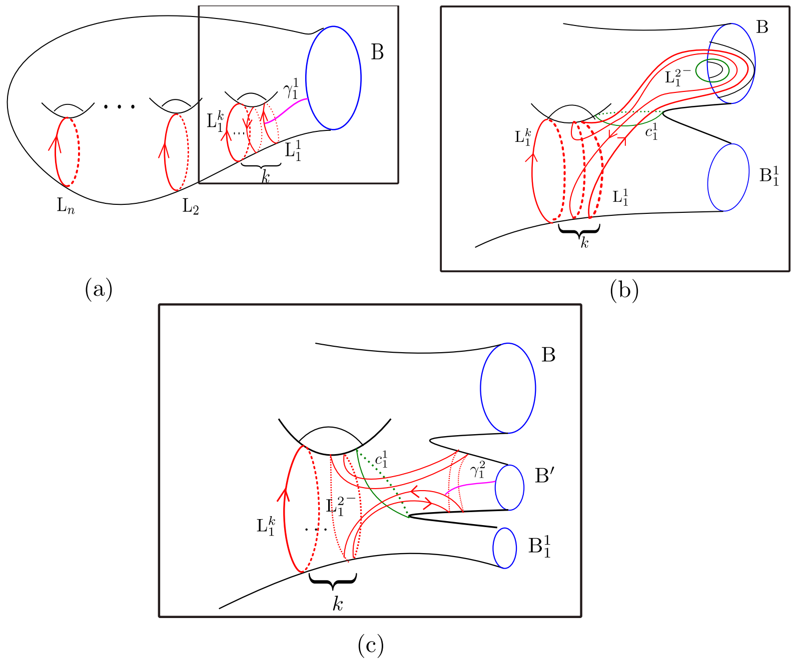

Figure 5.7. (a) Linearly dependent link components with L_1^2 having opposite orientation. (b)We stabilize B along a boundary parallel curve and move the components across the attaching 1-handle. This negatively stabilizes Ł$_1^2$. (c) Find γ_1^2.

5.1.4. Case 2(b)

[$(\mathrm{B}, \Sigma_g, \phi)$ has $g > 0$ and not all the link components are linearly independent]

Suppose L_i has k parallel copies and we call them L_i^j for $j = 1, 2, \ldots k$. This can have the following two sub cases.

Case 2(b)(i)

[If all the components are oriented similarly]

We consider the local picture 5.6(b) where are link component has the same orientation. Choose the L_i^1 which is closest to B and find an arc γ_i^1 that stays on the right. Notice, this arc does not intersect any of the L_i^k's. Do the same procedure as before and find a new boundary component B_i^1. We have the new open book $(\mathrm{B} \sqcup \mathrm{B}_i^1, \Sigma', \phi \circ D_{c_i^1})$. On this new open book, choose an arc γ_i^2 which can only possibly intersect c_i^1 (the closed curve we found using γ_i^1) and L_i^1. We iterate this process step by step. This will allow us to find a ordered set of simple closed curves $\{c_i^1, c_i^2, \ldots, c_i^k\}$ where c_i^k only possibly intersects c_i^j and L_i^j for $j = 1, 2, \ldots, j-1$. Thus if we maintain the order and do the Dehn twist step by step that will not change the other Link components we haven not dealt with in the previous steps. We finally find a k component link which is the transverse push-off of the L_k's. We can do this process locally for all link components which are not linearly independent. Combining this with all the cases from 5.1.3 gives the desired result for every posibble cases.

Case 2(b)(ii)

[If some of the link components have different orientation] To deal with this case, we follow the same procedure as in 5.1.2. Check 5.7.

Remark 5.1.1. *Note that, in 5.1.4 we assumed the components to be meridinal. The same idea also works for the other cases i.e if the linearly dependent components bound the genus or go between the genus.*

□

Now we are ready to define the support genus of a transverse link.

Definition 5.1.1. *The support genus* $\mathrm{sg}(T)$ *of a transverse link* T *in a contact* 3*-manifold* (M, ξ) *is the minimal genus of a page of the open book decomposition of* M *supporting* ξ *such that* T *can be realized as a sub-binding of that open book.*

Our next theorem finds a relationship between the support genus of a transverse link and the support genus of its Legendrian approximation.

Theorem 5.1.2. *Suppose T be a transverse link in (M, ξ) and L be its Legendrian approximation. Then* $\mathrm{sg}(T) = \mathrm{sg}(L)$.

Proof. We start with a transverse link T in (M, ξ) and suppose $\mathrm{sg}(T) = g$. So we can realize it as a sub-binding of some open book $(\mathrm{B}, \Sigma, \phi)$ with minimum genus g. Now take the Legendrian approximation of each of the components. This will give us a Legendrian link sitting on an open book with genus g. Thus $\mathrm{sg}(L) \leq g$. Now take the Legendrian link and put it on an open book with genus=$\mathrm{sg}(L)$. Now apply the algorithm we used in 5.1.1 to find a transverse push off T' which is also a sub-binding of the underlying open book. Thus $\mathrm{sg}(T') \leq \mathrm{sg}(L)$. By the well-definedness of transverse push-off T' must be transversely isotopic to T. As the support genus is an invariant of transverse links, we must have $\mathrm{sg}(T') = g$. Thus $\mathrm{sg}(T) = \mathrm{sg}(L)$. □

The following theorem was proved in [21].

Theorem 5.1.3 ([21])**.** *Suppose L be a loose, null-homologous Legendrian knot in* (M, ξ). *Then* $\mathrm{sg}(L) = 0$.

The following corollary follows immediately from the above theorem and 5.1.2.

Corollary 5.1.1. *Suppose T be a loose, null-homologous transverse knot in* (M, ξ). *Then* $\mathrm{sg}(T) = 0$.

Proof. Suppose T be a loose, null-homologous transverse knot in (M, ξ). We can always do the Legendrian approximation in a small enough neighborhood so that that Legendrian approximation is also loose. The result follows immediately from 5.1.2. □

A similar result is true for a coarse equivalence class of transverse links.

Definition 5.1.2. *Suppose* $[T]$ *denotes the class of all transverse links with all of the components null-homologous and having a fixed self linking number. Any two transverse links in this class are related by contactomorphism. We call this class* coarse equivalence class *of transverse links.*

Corollary 5.1.2. *Suppose* $[T]$ *denotes the coarse equivalence class of loose transverse links. Then* $\mathrm{sg}([T] = 0$*.*

Proof. Proof follows from 4.1.2. $\square$

5.2. Support genus and non-loose knots

It is known that a Legendrian approximation of a non-loose transverse knot is always non-loose [8]. But the transverse push off of a non-loose Legendrian knot may or may not be non-loose. As an example, there exists no non-loose representative of the transverse unknot in (S^3, ξ_{-1}) where as there exists non-loose Legendrian unknot in the same contact structure. The following corollary gives a condition, where we can find a non-loose transverse push off.

Corollary 5.2.1. *Suppose* L *be a non-loose Legendrian knot with* $\mathrm{sg}(L) > 0$*. Then its transverse push off must be non-loose.*

Thus every Legendrian knot with $\mathrm{sg}(L) > 0$ gives rise to a non-loose transverse representative of the same knot type. Note that, non-loose unknot has no non-loose transverse representative in (S^3, ξ_{-1}). But non-loose Legendrian unknot has support genus zero as it can be put on an annular open book supporting (S^3, ξ_{-1}). Thus the above corollary cannot be applied.

Proof. The proof follows immediately from 5.1.2 and 5.1.1. $\square$

Bibliography

[1] Kenneth L. Baker, John B. Etnyre, and Van Horn-Morris, Jeremy. Cabling, contact structures and mapping class monoids. *J. Differential Geom.*, 90(1):1–80, 01 2012.

[2] Rima Chatterjee. Links in overtwisted contact manifolds, 2020.

[3] Yuri Chekanov. Differential algebra of legendrian links. *Inventiones mathematicae*, 150(3):441–483, 2002.

[4] Katarzyna Dymara. Legendrian knots in overtwisted contact structures, 2004.

[5] Yakov Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Inventiones mathematicae*, 98(3):623–637, 1989.

[6] Yakov Eliashberg. Invariants in contact topology. In *Proceedings of the International Congress of Mathematicians*, volume 2, pages 327–338. Berlin, 1998.

[7] Yakov Eliashberg and Maia Fraser. Topologically trivial Legendrian knots, 2008.

[8] John Etnyre. On knots in overtwisted contact structures. *Quantum Topology*, 4, 12 2010.

[9] John B Etnyre. Planar open book decompositions and contact structures. *International Mathematics Research Notices*, 2004(79):4255–4267, 2004.

[10] John B Etnyre. Legendrian and transversal knots. In *Handbook of knot theory*, pages 105–185. Elsevier, 2005.

[11] John B. Etnyre and Ko Honda. Knots and contact geometry I: Torus knots and the figure eight knot. *J. Symplectic Geom.*, 1(1):63–120, 12 2001.

[12] David Gabai. The murasugi sum is a natural geometric operation, h. *Contemp. Math*, 44:93–100, 1985.

[13] David T Gay and Robion Kirby. Constructing symplectic forms on 4–manifolds which vanish on circles. *Geometry & Topology*, 8(2):743–777, 2004.

[14] Hansjörg Geiges. *An introduction to contact topology*, volume 109. Cambridge University Press, 2008.

[15] Hansjörg Geiges and Sinem Onaran. Exceptional Legendrian torus knots, 2018.

[16] Hansjörg Geiges and Sinem Onaran. Legendrian hopf links, 2019.

[17] Emmanuel Giroux. Géométrie de contact: de la dimension trois vers les dimensions supérieures. *Proceedings of the International Congress of Mathematicians*, II:405–414, 2002.

[18] Robert E Gompf. Handlebody construction of stein surfaces. *Annals of mathematics*, pages 619–693, 1998.

[19] Irena Matkovič. Non-loose negative torus knots, 2020.

[20] John Milnor and David W Weaver. *Topology from the differentiable viewpoint*. Princeton university press, 1997.

[21] Sinem Çelik Onaran. Invariants of Legendrian knots from open book decompositions. *International Mathematics Research Notices*, 2010(10):1831–1859, 2010.

[22] Vladimir Tchernov. Vassiliev invariants of legendrian, transverse, and framed knots in contact three-manifolds. *Topology*, 42(1):1 – 33, 2003.

[23] William P Thurston and Horst E Winkelnkemper. On the existence of contact forms. *Proceedings of the American Mathematical Society*, pages 345–347, 1975.

Vita

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