The Congruence Extension Property and Related Topics in Semigroups.

Jill Ann Dumesnil

Louisiana State University and Agricultural & Mechanical College

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The congruence extension property and related topics in semigroups

Dumesnil, Jill Ann, Ph.D.
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THE CONGRUENCE EXTENSION PROPERTY
AND RELATED TOPICS IN SEMIGROUPS

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in

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by

Jill Ann Dumesnil
B.S., Lamar University, 1988
M.S., Louisiana State University, 1990
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Preface

Motivation for this work was provided by the 1988 Doctoral Dissertation of Josefa Garcia concerning the congruence extension property for algebraic semigroups. Other authors greatly influencing this work are B. Biró, E. Kiss, P. Pálfy, A. Stralka, T. Tamura, and M. Yamada.

The objectives of this research are:

(1) Characterize classes of semigroups with the congruence extension property.
(2) Explore topics related to and consequences of the congruence extension property.
(3) Study these concepts from a topological perspective.

A characterization is given in Chapter 2 for semigroups with the congruence extension property (CEP) in terms of a condition on the lattice of congruences. A similar result is obtained for semigroups with the ideal extension property (IEP).

In Chapter 3, t-semigroups (semigroups in which the relation “is an ideal of” is transitive) are defined and discussed. That each semigroup having CEP or IEP is a t-semigroup is shown. Additionally, it is obtained that for a cyclic semigroup $S$, the notions of CEP, IEP, t-semigroup, and $\text{index}(S) \leq 3$ are equivalent.

Semigroups with the property that every subsemigroup is an ideal of some ideal (m-semigroups) are studied in Chapter 4. It is obtained that m-semigroups are periodic semigroups with zero and index less than or equal to 5. Those commutative m-semigroups whose index is less than or equal to 3 are characterized.

Bands of groups with CEP are examined in Chapter 5. It is shown that a semilattice of abelian groups has CEP if and only if each of the groups has CEP. It is observed that if $S$ is a semilattice of abelian groups, then $S$ is commutative,
\( \mathcal{H} \) is a congruence on \( S \), and \( S/\mathcal{H} \cong E(S) \) is a semilattice which has CEP. Thus, the question of whether moding the congruence \( \mathcal{H} \) out of a commutative semigroup preserves CEP arises. A reduction of this problem is obtained.

Chapter 6 deals with completely simple semigroups with CEP. A construction is given which yields an alternative proof for the known result in the algebraic case and is amenable to direct extension to the topological result. In the process of obtaining the construction, subsemigroups of completely simple semigroups with torsion subgroups are studied. It is obtained that all subsemigroups of a completely simple semigroup \( S \) are themselves completely simple if and only if \( S \) has torsion subgroups. Additionally, it is demonstrated that homomorphic images of completely simple semigroups with CEP retain CEP.

These concepts are explored from a topological perspective in Chapter 7. Topological analogues of previous results are given for compact semigroups. Other related results are provided and discussed.

Chapter 8 summarizes the results of the other chapters and lists some questions which remain open to future research on the congruence extension property and related topics for both algebraic and topological semigroups.
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Abstract

A semigroup has the congruence extension property (CEP) provided that each congruence on each subsemigroup can be extended to a congruence on the semigroup. This property, the ideal extension property (IEP), and other related concepts are studied from both an algebraic and a topological perspective in this work.

A characterization of semigroups with CEP is given in terms of the lattice of congruences. A similar result is obtained for IEP.

Semigroups in which the relation "is an ideal of" is transitive (t-semigroups) are explored. It is shown that each of CEP and IEP implies this condition and that these are all equivalent for cyclic semigroups. Semigroups in which every subsemigroup is an ideal of some ideal (m-semigroups) are discussed. It is obtained that m-semigroups are periodic semigroups with zero and index less than or equal to 5. Those commutative m-semigroups with index less than or equal to 3 are characterized. The majority of these results are topologized for compact semigroups.

Compact completely simple semigroups with CEP are characterized. A construction is given which yields an alternative proof for the known result in the algebraic case and is amenable to direct extension to the topological result. Additionally, subsemigroups of a completely simple semigroup with torsion subgroups are characterized. It is shown that CEP is retained by continuous homomorphic images of compact completely simple semigroups with CEP.
The primary purpose of this first chapter is to establish the necessary basic notions and to present previous results that are fundamental to the remainder of this work.

A groupoid is a non-empty set $S$ together with a binary operation from $S \times S$ into $S$, denoted as multiplication $(x, y) \mapsto xy$. A semigroup is a groupoid such that the multiplication is associative.

Notation. We will denote the set of idempotents of a semigroup $S$ by either $E(S)$ or $E_S$ or simply $E$ when the semigroup $S$ is understood. Similarly, if $S$ has a minimal ideal, we will denote the minimal ideal of $S$ by either $M(S)$ or $M_S$ or simply $M$. We will also denote by $S^1$ ($S^0$) the semigroup obtained by adjoining an identity (zero) element to $S$.

A congruence $\sigma$ on a semigroup $S$ is defined to be an equivalence relation on $S$ which is compatible with the multiplication in $S$, i.e.,

1. $\Delta_S \subseteq \sigma$, where $\Delta_S = \{(x, x) : x \in S\}$ is the diagonal of $S$ ($\sigma$ is reflexive);
2. $\sigma^{-1} \subseteq \sigma$, where $\sigma^{-1} = \{(b, a) : (a, b) \in \sigma\}$ ($\sigma$ is symmetric);
3. $\sigma \circ \sigma \subseteq \sigma$, where $\sigma \circ \sigma$ is the composition of $\sigma$ with itself ($\sigma$ is transitive); and
4. If $(a, b) \in \sigma$ and $s \in S$, then $(as, bs) \in \sigma$ and $(sa, sb) \in \sigma$ ($\sigma$ is compatible).

It is well-known that in the presence of conditions (1) and (3), we have that condition (4) is equivalent to: (4)' If $(a, b) \in \sigma$ and $(c, d) \in \sigma$, then $(ac, bd) \in \sigma$. 

1
If \( S \) is any semigroup, the **diagonal** \( \Delta_S = \{(s, s) : s \in S\} \) is a minimal congruence on \( S \) and \( S \times S \) is a maximal congruence on \( S \). The congruence \( S \times S \) is known as the **universal** congruence on \( S \).

For a semigroup \( S \), let \( \mathcal{C}(S) = \{\sigma : \sigma \) is a congruence on \( S\} \). Then we have that \( \bigcap \{\sigma : \sigma \in \mathcal{C}(S)\} = \Delta_S \) and \( \bigcup \{\sigma : \sigma \in \mathcal{C}(S)\} = S \times S \). If \( \rho \) is any subset of \( S \times S \), then the congruence on \( S \) generated by \( \rho \) (denoted \( \langle \rho \rangle_S \)) is the minimal congruence on \( S \) containing \( \rho \). It is immediate that \( \langle \rho \rangle_S = \bigcap \{\sigma : \sigma \in \mathcal{C}(S) \text{ and } \rho \subseteq \sigma\} \).

For a semigroup \( S \), two binary operations are defined on \( \mathcal{C}(S) \). Let \( \sigma, \rho \in \mathcal{C}(S) \).

The **meet** (denoted \( \wedge \)) of \( \sigma \) and \( \rho \) is defined by \( \sigma \wedge \rho = \sigma \cap \rho \). Thus, \( \sigma \wedge \rho \in \mathcal{C}(S) \).

The **join** (denoted \( \vee \)) of \( \sigma \) and \( \rho \) is defined by \( \sigma \vee \rho = \langle \sigma \cup \rho \rangle_S \), the congruence on \( S \) generated by \( \sigma \cup \rho \). We have that \( \sigma \vee \rho \in \mathcal{C}(S) \), even though \( \sigma \cup \rho \) may not be a congruence on \( S \).

A **partial ordering** is a reflexive, anti-symmetric, transitive relation. A partial ordering \( \leq \) on a set \( X \) is called a **total ordering** or chain provided for \( x, y \in X \) either \( x \leq y \) or \( y \leq x \). A lattice is a partially ordered set \( X \) such that every two-element subset has a meet and join in \( X \).

Let \( S \) be a semigroup. The set of congruences \( \mathcal{C}(S) \) is partially ordered by inclusion. Hence, under the operations meet and join defined above, the set of congruences \( \mathcal{C}(S) \) of \( S \) forms a lattice.

Notice the above discussion of congruences holds verbatim for a groupoid \( S \).

If \( S \) is a semigroup, \( T \) is a subsemigroup of \( S \), and \( \sigma \) is a congruence on \( T \), then an **extension** of \( \sigma \) to a subsemigroup \( R \) of \( S \) where \( T \subseteq R \) is a congruence \( \tilde{\sigma} \) on \( R \) such that \( \tilde{\sigma} \cap (T \times T) = \sigma \).
A semigroup $S$ is said to have the **congruence extension property** (CEP) provided that for each subsemigroup $T$ of $S$ and each congruence $\sigma$ on $T$, there exists an extension $\bar{\sigma}$ of $\sigma$ to $S$.

The following results concerning congruences and the congruence extension property are well-known and may be found in [Garcia, 1988]. For $n \in \mathbb{N}$, $\rho^{(n)}$ denotes the $n$-fold composition of a relation $\rho$ with itself.

1.1 A semigroup $S$ has the congruence extension property if and only if each subsemigroup of $S$ has the congruence extension property.

1.2 Let $S$ be a semigroup, $T$ a subsemigroup of $S$, $\sigma$ a congruence on $T$, and $\rho = \{(xay, xby) : (a, b) \in \Delta_S, x, y \in S^1\}$. Then the congruence on $S$ generated by $\sigma$ is given by

$$\langle \sigma \rangle_S = \bigcup_{n \in \mathbb{N}} \rho^{(n)}.$$

1.3 Let $S$ be a semigroup, $T$ a subsemigroup of $S$, and $\sigma$ a congruence on $T$. Then $\sigma$ has an extension to $S$ if and only if $\langle \sigma \rangle_S$ is an extension of $\sigma$ to $S$.

1.4 A semigroup $S$ has the congruence extension property if and only if for each subsemigroup $T$ of $S$ and each congruence $\sigma$ on $T$, $\langle \sigma \rangle_S \cap (T \times T) \subseteq \sigma$.

1.5 These are equivalent for a semigroup $S$:

(1) $S$ has the congruence extension property;

(2) $S^1$ has the congruence extension property; and

(3) $S^0$ has the congruence extension property.

It is also well-known and shown in [Garcia, 1988] that congruence extensions need not be unique, $\mathbb{N}$ does not have CEP, and CEP is not productive.
A congruence \( \alpha \) on a semigroup \( S \) is called a principal congruence if \( \alpha \) is generated by a single pair \((a, b) \in S \times S\).

If \( S \) is a semigroup and \((a, b) \in S \times S\), then the minimal congruence on \( S \) containing the pair \((a, b)\) is denoted by \( \alpha^S(a, b) \). The congruence \( \alpha^S(a, b) \) can be constructed as follows:

If \( S \) is a semigroup and \((a, b) \in S \times S\), let

\[
\alpha_0 = \{(a, b), (b, a)\},
\]

\[
\alpha_1 = \alpha_0 \cup \Delta_S, \quad \text{and}
\]

\[
\alpha_2 = \{(x, y, x, y) : (c, d) \in \alpha_1, \; x, y \in S^1\}.
\]

Then

\[
\alpha^S(a, b) = \bigcup_{n \in \mathbb{N}} \alpha_2^{(n)}.
\]

A semigroup \( S \) is said to have the principal congruence extension property (PCEP) provided that for each pair \((a, b) \in S \times S\) and each subsemigroup \( T \) of \( S \), \( \alpha^S \cap (T \times T) = \alpha^T(a, b) \).

1.6 [Garcia, 1988] A semigroup \( S \) has the congruence extension property (CEP) if and only if \( S \) has the principal congruence extension property (PCEP).

A homomorphism of a semigroup \( S \) is a map \( \phi : S \to T \) from \( S \) to a semigroup \( T \) such that \( \phi(xy) = \phi(x)\phi(y) \) for each \( x, y \in S \). The study of congruences in semigroups is closely related to the study of homomorphisms, since the kernel of a homomorphism \( \phi : S \to X \) defined by

\[
\ker \phi = \{(a, b) \in S \times S : \phi(a) = \phi(b)\}
\]
is a congruence on $S$, and each congruence $\sigma$ on a semigroup $S$ may be regarded as the kernel of the natural homomorphism $\pi: S \rightarrow S/\sigma$.

It remains an unsolved problem to determine whether the homomorphic image of a semigroup with the congruence extension property (CEP) also has CEP. We obtain some partial results concerning this question. As mentioned in [Garcia, 1988], it appears that associativity, or some consequence thereof, has an important role in a resolution of this problem. That the homomorphic image of a groupoid with CEP need not have CEP was demonstrated in [Biró, Kiss, and Pálfy, 1977].

The following lemmas are basic results concerning the congruence extension property and will be used in subsequent chapters. Lemmas 1.7, 1.8, and 1.9 are from [Garcia, 1988].

Let $\phi: S \rightarrow X$ be a homomorphism from a semigroup $S$ onto a semigroup $X$. Let $\alpha$ be a congruence on $X$ and $\rho$ a congruence on $S$. Then

\begin{align*}
\text{pullback of } \alpha := \{(a, b) \in S \times S: (\phi(a), \phi(b)) \in \alpha\}
\end{align*}

and

\begin{align*}
\text{pushout of } \rho := \{(\phi(a), \phi(b)) \in X \times X: (a, b) \in \rho\}.
\end{align*}

1.7 Lemma. Let $\phi: S \rightarrow X$ be a homomorphism from a semigroup $S$ onto a semigroup $X$. Let $\alpha$ be a congruence on $X$. Let $\rho$ be the pullback of $\alpha$ to $S$. Then $\rho$ is a congruence on $S$.

1.8 Lemma. Let $\phi: S \rightarrow X$ be a homomorphism from a semigroup $S$ onto a semigroup $X$. Let $\rho$ be a congruence on $S$. Let $\alpha$ be the pushout of $\rho$. Then $\alpha$ is a reflexive, symmetric, compatible relation on $X$. 
1.9 Lemma. Let $\phi: S \to X$ be a homomorphism from a semigroup $S$ onto a semigroup $X$. Let $\rho$ be a congruence on $S$ such that $\ker \phi \subseteq \rho$. Let $\alpha$ be the pushout of $\rho$ to $X$. Then $\alpha$ is a congruence on $X$. Moreover, the pullback of $\alpha$ is $\rho$.

1.10 Lemma. A semigroup $S$ has the congruence extension property (CEP) if and only if every isomorphic image of $S$ has the congruence extension property.

Proof. Suppose $S$ is a semigroup with CEP. Let $\phi: S \to X$ be an isomorphism. We claim that $X$ has CEP. Let $Y$ be a subsemigroup of $X$, and let $\alpha$ be a congruence on $Y$. Define $T = \phi^{-1}[Y]$. Define $\rho$ to be the pullback of $\alpha$ to $S$, that is,

$$\rho = \{(a, b) \in S \times S: (\phi(a), \phi(b)) \in \alpha\}.$$

Then by Lemma 1.7, $\rho$ is a congruence on $T$. Since $S$ has CEP, let $\bar{\rho}$ be an extension of $\rho$ to $S$. Now, since $\phi$ is an isomorphism, $\ker \phi = \Delta_S \subseteq \bar{\rho}$. Applying Lemma 1.9, we see that the pushout of $\bar{\rho}$ is a congruence on $X$. Let $\bar{\alpha}$ be the pushout of $\bar{\rho}$, i.e.,

$$\bar{\alpha} = \{(\phi(x), \phi(y)) \in X \times X: (x, y) \in \bar{\rho}\}.$$

We claim that $\bar{\alpha} \cap (Y \times Y) = \alpha$. Since $(\phi(a), \phi(b)) \in \alpha$ implies by definition of $\rho$ that $(a, b) \in \rho \subseteq \bar{\rho}$, it is clear that $\alpha \subseteq \bar{\alpha}$. Let $(\phi(a), \phi(b)) \in \bar{\alpha} \cap (Y \times Y)$. Then we have $(a, b) \in \bar{\rho} \cap (\phi^{-1}[Y] \times \phi^{-1}[Y]) = \bar{\rho} \cap (T \times T) = \rho$. Thus, $(\phi(a), \phi(b)) \in \alpha$, and $\bar{\alpha}$ is an extension of $\alpha$ to $X$. Hence, $X$ has CEP. The converse is immediate. 

If $S$ is a semigroup and $T$ is a subsemigroup of $S$, then $S$ is said to have the congruence extension property relative to $T$ provided that every congruence on $T$ has an extension to $S$. If $\mathcal{K}$ is a class of subsemigroups of a semigroup $S$ such that every congruence $\sigma$ on every member $T \in \mathcal{K}$ has an extension to $S$, then we say that $S$ has the congruence extension property relative to the class $\mathcal{K}$. 
Items 1.11 through 1.17 are known results concerning the relative congruence extension property.

1.11 [Garcia, 1988] Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$ such that $S \setminus T$ (the complement of $T$ in $S$) is an ideal of $S$. Then $S$ has the congruence extension property relative to $T$.

1.12 Corollary. Let $S$ be a monoid. Then $S$ has the congruence extension property relative to the group of units $H(1)$ of $S$.

1.13 [Stralka, 1972] Let $S$ be a band of groups such that $E(S)$ is a subsemigroup of $S$. Then $S$ has the congruence extension property relative to $E(S)$.

If $M$ is a subsemigroup of a semigroup $S$ and $\phi: S \to M$ is a homomorphism of $S$ onto $M$ such that $\phi|_M = 1_M$ (the identity function on $M$), then $\phi$ is called a homomorphic retraction of $S$ onto $M$ and $M$ is called a homomorphic retract of $S$.

1.14 Proposition. Let $S$ be a semigroup. Then $S$ has the congruence extension property relative to homomorphic retractions of $S$.

Proof. Let $S$ be a semigroup, let $\phi: S \to M$ be a homomorphic retraction, and let $\sigma$ be a congruence on $M$. Define $\sigma^* = \{(a, b) \in S \times S: (\phi(a), \phi(b)) \in \sigma\}$. Then $\sigma^*$ is an extension of $\sigma$ to $S$.

1.15 Corollary. Let $S$ be a commutative semigroup with a group minimal ideal $M(S)$. Then $S$ has the congruence extension property relative to $M(S)$.

Proof. Let $e$ denote the identity element of $M(S)$. Then $x \mapsto ex$ is a homomorphic retraction of $S$ onto $M(S)$.

1.16 Note. Proposition 1.14 and Corollary 1.15 have topological analogues; this will be discussed in Chapter 7. However, we remark that since every compact
commutative semigroup is known to have a compact (and hence closed) group minimal ideal, it has the congruence extension property relative to its minimal ideal. In particular, a finite commutative semigroup has CEP relative to its minimal ideal.

A semigroup $S$ is called medial provided that for each $a, b, c, d \in S$, we have that $abcd = acbd$. It has been shown in [Anderson and Hunter, 1962] that this is equivalent to the condition that for $a, b, c \in S$, $abca = acba$.

An element $r$ of a semigroup $S$ is called a regular element provided there exists $t \in S$ such that $rtr = r$. The element $t$ is called an inverse for $r$. A semigroup $S$ is said to be a regular semigroup if every element of $S$ is a regular element.

1.17 [Stralka, 1972] Let $S$ be a medial semigroup, and let $A$ be a subsemigroup of the regular elements of $S$ such that $A$ is a band of groups. Then $S$ has the congruence extension property relative to $A$.

In the following section we concern ourselves with congruences on groups.

A group $G$ is said to have the group congruence extension property (GCEP) provided that for each subgroup $H$ of $G$ and each congruence $\sigma$ on $H$, there exists an extension of $\sigma$ to $G$, i.e., there exists a congruence $\bar{\sigma}$ on $G$ such that $\bar{\sigma} \cap (H \times H) = \sigma$.

It is immediate that a group with the congruence extension property will also have the group congruence extension property. That the converse is not true is shown in [Garcia, 1988].

The following is a sequence of facts concerning congruences in groups.

1.18 [Clifford and Preston, 1961] Let $G$ be a group and $\sigma$ a congruence on $G$. Then there exists a normal subgroup $N$ of $G$ such that $(a, b) \in \sigma$ if and only if $ab^{-1} \in N$. 
1.19 Note. In 1.18, if $e$ denotes the identity of the group $G$, then we have that $N = \{ g \in G : (g, e) \in \sigma \}$.

1.20 [Biró, Kiss, and Pálfy, 1977] Let $G$ be a group. Then $G$ has the group congruence extension property (GCEP) if and only if whenever $H$ is a subgroup of $G$ and $K$ is a normal subgroup of $H$, there exists a normal subgroup $N$ of $G$ such that $N \cap H = K$.

1.21 Corollary. A group $G$ such that every subgroup $H$ of $G$ is normal in $G$ has the group congruence extension property (GCEP).

1.22 Corollary. Abelian and hamiltonian groups have GCEP.

We recall that a hamiltonian group is a non-abelian group in which every subgroup is normal. Thus, Corollaries 1.21 and 1.22 are equivalent statements.

1.23 Corollary. Each subgroup of a group with the group congruence extension property (GCEP) has the group congruence extension property.

1.24 [Biró, Kiss, and Pálfy, 1977] A group $G$ has the group congruence extension property (GCEP) if and only if each homomorphic image of $G$ has the group congruence extension property.

A group $G$ is called a $t$-group if the relation "is a normal subgroup of" is transitive among the subgroups of $G$, that is, if $L, M,$ and $N$ are subgroups of $G$ such that $L \triangleleft M \triangleleft N$, then $L \triangleleft N$ (where $\triangleleft$ indicates normal subgroup).

1.25 [Biró, Kiss, and Pálfy, 1977] A finite group $G$ has the group congruence extension property (GCEP) if and only if $G$ is a solvable $t$-group.

1.26 [Howie, 1976] Suppose $G$ is a group and $M$ and $N$ are normal subgroups of $G$. Let $\sigma_M$ and $\sigma_N$ be the congruences on $G$ defined by $M$ and $N$, respectively. Then $\sigma_M \cap \sigma_N = \sigma_{MN}$ and $\sigma_M \circ \sigma_N = \sigma_{MN}$. 
1.27 [Garcia, 1988] The union of an ascending family of groups with the group congruence extension property has the group congruence extension property.

1.28 [Garcia, 1988] Let $G$ be a group. Then $G$ has the congruence extension property (CEP) if and only if $G$ is a torsion group with the group congruence extension property (GCEP).

1.29 Note. The previous result is not true topologically, for the circle group $C = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x^2 + y^2 = 1\}$ has CEP topologically but is not torsion. This will be discussed in Chapter 7.

1.30 Examples. (1) It is well-known that the finite group $S_5$ is not solvable and hence does not have GCEP, by 1.25.

(2) By 1.22 the group $\mathbb{R}$ of real numbers has GCEP. However, it does not have CEP, as it contains $\mathbb{N}$ as a subsemigroup.

(3) The group $Q \times Q$ of the quaternions cross themselves does not have GCEP, even though $Q$ has CEP, and hence GCEP. Thus, GCEP and consequently CEP is not productive. [Garcia, 1988].

1.31 [Garcia, 1988] The homomorphic image of a group with the congruence extension property (CEP) has the congruence extension property.

We remark that ideals in semigroups are in some sense the analog of normal subgroups in groups. We have seen in 1.16 that congruence extension from a sub-group $H$ of a group $G$ is equivalent to the extension of normal subgroups of $H$ to normal subgroups of $G$. This motivates us to consider extending ideals of subsemigroups of a semigroup $S$ to ideals of $S$.

A semigroup $S$ is said to have the ideal extension property (IEP) if for each subsemigroup $T$ of $S$ and each ideal $I$ of $T$ there is an ideal $J$ of $S$ with $J \cap T = I$. 
The following list of items 1.32 through 1.40 is a sequence of known facts about the ideal extension property. Where no proof is given, see [Garcia, 1988].

1.32 A semigroup $S$ has the ideal extension property (IEP) if and only if each subsemigroup of $S$ has the ideal extension property.

Let $T$ be a subsemigroup of a semigroup $S$. The ideal of $S$ generated by $T$ is given by $S^1TS^1 = T \cup ST \cup TS \cup STS$. This is the smallest ideal of $S$ containing the subsemigroup $T$.

1.33 Let $T$ be a subsemigroup of a semigroup $S$, and let $I$ be an ideal of $T$. Then $I$ extends to an ideal of $S$ if and only if $S^1IS^1$ is an extension of $I$ to $S$.

Proof. Let $T$ be a subsemigroup of a semigroup $S$, and let $I$ be an ideal of $T$. Suppose that there is an ideal $J$ of $S$ with $J \cap T = I$. Since $S^1IS^1$ is the smallest ideal of $S$ containing $I$, we have $S^1IS^1 \subseteq J$. Thus, $S^1IS^1 \cap T \subseteq J \cap T \subseteq I$. Certainly, $I \subseteq S^1IS^1 \cap T$. Thus, $S^1IS^1$ is an extension of $I$ to $S$. The converse is immediate.

1.34 Example. It is known that the semigroup $(\mathbb{N}, +)$ does not have the ideal extension property. Consider the subsemigroup $T = \{2,3,4,5,\ldots\}$ of $\mathbb{N}$ and the ideal $I = \{2,4,5,6,\ldots\}$ of $T$.

1.35 A homomorphic image of a semigroup with the ideal extension property (IEP) has the ideal extension property.

Let $S$ be a semigroup and let $a \in S$. We let $J_S(a)$ denote the ideal of $S$ generated by the element $a$, that is, $J_S(a) = S^1aS^1 = \{a\} \cup aS \cup Sa \cup SaS$. Similarly, for a subsemigroup $T$ of $S$, and $a \in T$, we let $J_T(a)$ denote the ideal of $T$ generated by $a$, that is, $J_T(a) = T^1aT^1 = \{a\} \cup aT \cup Ta \cup TaT$.

A semigroup $S$ is said to have the principal ideal extension property (PIEP) if for each subsemigroup $T$ of $S$ and each element $a \in T$, $J_T(a) = J_S(a) \cap T$. 
An element $a$ of a semigroup $S$ is said to be a disruptive element provided there exists a subsemigroup $T$ of $S$ such that $a \in T$ and $J_T(a) \subseteq J_S(a) \cap (T \times T)$, where $\subseteq$ denotes proper containment. For a disruptive element $a \in S$ and the subsemigroup $T$ for which $J_T(a) \subseteq J_S(a) \cap (T \times T)$, we say $a$ is disruptive in $T$.

1.36 For semigroup $S$, the following are equivalent:

(1) $S$ has the ideal extension property (IEP);

(2) $S$ has the principal ideal extension property (PIEP); and

(3) $S$ contains no disruptive elements.

1.37 Let $S$ be a commutative semigroup, and let $T$ be a subsemigroup of $S$. Then no regular element of $T$ is disruptive in $T$. In particular, no idempotent element of a commutative semigroup is disruptive.

1.38 Each semilattice has the ideal extension property (IEP).

1.39 A commutative semigroup $S$ which has the congruence extension property (CEP) has the ideal extension property (IEP).

1.40 Note. It can be demonstrated by counterexample that the ideal extension property (IEP) is not productive. See [Garcia, 1988].

There is a natural partial ordering of the set $E$ of idempotents of a semigroup $S$. For $e, f \in E$, define $e \leq f$ to mean $ef = fe = e$. Then $\leq$ is a partial order on $E$, and if $e \leq f$ we say that $e$ is below $f$ or that $f$ is above $e$. If a semigroup $S$ has a zero element 0, then $0 \leq e$ for every $e \in E$. Also, if $S$ is a semigroup such that $S = E_S$ (i.e., $S$ is a band), then this forms a partial ordering of $S$.

An idempotent $e \in E$ is said to be primitive if the only idempotents of $S$ that are below $e$ are $e$ itself and 0 (if $S$ has a 0), and $e \neq 0$; that is, $e \in E$ is primitive provided $e \neq 0$ and $f \leq e$ implies that $f = e$ or $f = 0$. 
For a semigroup $S$ and $a \in S$, we let $\theta(a)$ denote the subsemigroup generated by the element $a$, i.e., $\theta(a) = \{a^n : n \in \mathbb{N}\}$. If $S$ is a semigroup such that $S = \theta(a)$ for some $a \in S$, then $S$ is called a cyclic semigroup with generator $a$. A semigroup $S$ is called $\theta$-finite provided that each cyclic subsemigroup of $S$ is finite.

For a semigroup $S$ and $a \in S$, the order of $a$ is defined to be the order of the subsemigroup $\theta(a)$. If $a$ is an element of finite order, then it is well-known that $\theta(a)$ contains exactly one idempotent.

Let $S$ be a semigroup, and let $a \in S$. If $a^m = a^n$ for some $m > n$, then the index of $a$ is defined to be the least such $n \in \mathbb{N}$. If $a^m \neq a^n$ for all $m \neq n$, we say that $a$ has infinite index. The index of $a$ is denoted by index$(a)$.

For an element $a$ of a semigroup $S$, index$(a)$ may be equivalently defined to be the least $n \in \mathbb{N}$ such that $a^n \in M(\theta(a))$ whenever $M(\theta(a)) \neq \emptyset$ and index$(a) = \infty$ otherwise.

We define $\text{index}(S)$ to be the maximum over $a \in S$ of index$(a)$, if this maximum exists. Otherwise, we say that $S$ has infinite index, or index$(S) = \infty$.

A semigroup $S$ is said to be periodic provided each element has finite index. In particular, if index$(S) < \infty$, then $S$ is periodic. However, by our definitions, it is possible that $S$ may have infinite index and be periodic.

The above discussion and the following result may be derived from material found in [Clifford and Preston, 1961].

**1.41** The following are equivalent for a semigroup $S$:

1. $S$ is periodic;
2. Each element of $S$ has finite order; and
3. Some power of each element of $S$ is idempotent.
1.42 [Garcia, 1988] Let $S$ be a cyclic semigroup. Then $S$ has the congruence extension property (CEP) if and only if $\text{index}(S) \leq 3$. In particular, an infinite cyclic semigroup does not have the congruence extension property (CEP).

1.43 [Garcia, 1988] Let $S$ be a semigroup with the congruence extension property (CEP) [or the ideal extension property (IEP)]. Then $\text{index}(S) \leq 3$. In particular, $S$ is periodic.

The converse of 1.43 is not true in general. However, we have the following:

1.44 [Aucoin, 1993] Let $S$ be a cyclic semigroup. Then $S$ has the ideal extension property (IEP) if and only if $\text{index}(S) \leq 3$.

1.45 [Stralka, 1972] Semilattices have the congruence extension property.

1.46 Note. Result 1.45 is not true topologically. In fact, in [Stralka, 1977] an example of a compact semilattice without CEP is given.

Let $S$ be a semigroup. For $a, b \in S$, we define $a \mathcal{L} b$ to mean that $a$ and $b$ generate the same principal left ideal of $S$. In other words, $\mathcal{L} = \{(a, b) \in S \times S : S^1a = S^1b\}$. Clearly, $\mathcal{L}$ is a right congruence on $S$, i.e., $\mathcal{L}$ is an equivalence relation on $S$ such that if $a \mathcal{L} b$ then $ac \mathcal{L} bc$ for all $c \in S$. If $a \mathcal{L} b$, then we say that $a$ and $b$ are $\mathcal{L}$-equivalent. For $a \in S$, we denote the set of all elements of $S$ which are $\mathcal{L}$-equivalent to $a$ by $L_a$. We call $L_a$ the $\mathcal{L}$-class of $a$.

Dually, for $a, b \in S$, we define $a \mathcal{R} b$ to mean that $a$ and $b$ generate the same principal right ideal of $S$, that is, $\mathcal{R} = \{(a, b) \in S \times S : aS^1 = bS^1\}$. Then $\mathcal{R}$ is a left congruence on $S$, and we denote by $R_a$ the $\mathcal{R}$-class containing the element $a$.

It is well-known that the relations $\mathcal{L}$ and $\mathcal{R}$ commute and that the relation $\mathcal{D}$ defined by $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ is the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$. The $\mathcal{D}$-class of an element $a \in S$ is denoted $D_a$. 
Analogously, we define \( J = \{(a, b) \in S \times S : S^1 a S^1 = S^1 b S^1\} \). Thus, \( a \) and \( b \) are \( J \)-equivalent if they generate the same principal (two-sided) ideal. It is well-known that \( D \subseteq J \), but in general \( D \neq J \).

Finally, we define \( H = \mathcal{L} \cap \mathcal{R} \). Then \( H \) is an equivalence relation, and we denote the \( H \)-class of \( a \in S \) by \( H_a \). Clearly, we have that \( H_a = L_a \cap R_a \). If \( S \) is a commutative semigroup, then we have that \( H = \mathcal{L} = \mathcal{R} = J = D \) and that \( H \) is a congruence on \( S \). This will be of particular importance in Chapter 5.

The above relations are frequently referred to as Green’s relations because they were first introduced and studied by [Green, 1951]. These relations play an intricate role in the discussion of completely simple semigroups in Chapter 6. For a complete discussion of the above, see [Clifford and Preston, 1961].

A commutative semigroup \( S \) is said to be archimedean provided that for any two elements of \( S \), each divides some power of the other. We will use ‘\( \mid \)’ to denote “divides”. Thus, \( a \mid b \) means that \( b = xa \) for some \( x \in S^1 \).

Define a relation \( \eta \) on a commutative semigroup \( S \) as follows:

\[
(a, b) \in \eta \iff a \mid b^n \text{ and } b \mid a^m \text{ for some } n, m \in \mathbb{N}
\]

The following two results are well-known and may be found in [Clifford and Preston, 1961].

1.47 The relation \( \eta \) on any commutative semigroup \( S \) is a congruence on \( S \), and \( S/\eta \) is the maximal semilattice homomorphic image of \( S \).

1.48 Every commutative semigroup \( S \) is uniquely expressible as a semilattice \( Y \) of archimedean semigroups \( C_\alpha (\alpha \in Y) \). The semilattice \( Y \) is isomorphic with the maximal semilattice homomorphic image \( S/\eta \) of \( S \), and the \( C_\alpha (\alpha \in Y) \) are the equivalence classes of \( S \) mod \( \eta \).
We call the archimedean semigroups $C_\alpha$ the archimedean components of $S$. Obviously, a commutative semigroup is archimedean provided that it has only one archimedean component (or $\eta$-class).

Archimedean semigroups are themselves of interest, as they are the "building blocks" of commutative semigroups. Note that a periodic semigroup is archimedean if and only if it is a commutative semigroup with exactly one idempotent. It is well-known that an archimedean semigroup (and hence, any archimedean component of a commutative semigroup) has at most one idempotent. Additionally, if an archimedean semigroup $S$ has an idempotent $e$, then every element $a \in S$ has an inverse with respect to $e$, that is, there is $a' \in S$ with $aa' = a'a = e$. See [Clifford and Preston, 1961].

If a commutative semigroup is periodic, then each of its archimedean components must contain an idempotent, as some power of each element of $S$ is idempotent and each component is a subsemigroup. Thus, $S = \bigcup_{e \in E} C(e)$, where $C(e)$ denotes the archimedean component of $S$ containing the idempotent $e$.

Archimedean semigroups which have a zero element will be of special interest in Chapter 4. Note that a commutative semigroup $S$ with zero is archimedean if and only if for each $x \in S$ there exists $n \in \mathbb{N}$ such that $x^n = 0$. Indeed, in an archimedean semigroup with zero, zero must divide some power of each element by definition of an archimedean semigroup. However, the only element that zero divides is zero itself and thus, we see that some power of each element of $S$ is zero. In particular, this shows that an archimedean semigroup $S$ with zero is periodic and that any subsemigroup $T$ of $S$ must contain zero.
CHAPTER 2

THE LATTICE OF CONGRUENCES OF A SEMIGROUP

In this chapter we consider the lattice of congruences of a semigroup $S$. As mentioned in Chapter 1, $\mathcal{C}(S) = \{\sigma: \sigma \text{ is a congruence on } S\}$ forms a lattice under the operations meet (denoted $\wedge$) and join (denoted $\vee$) defined by:

$$\sigma \wedge \rho = \sigma \cap \rho \quad \text{for } \sigma, \rho \in \mathcal{C}(S)$$

and

$$\sigma \vee \rho = (\sigma \cup \rho)_S \quad \text{for } \sigma, \rho \in \mathcal{C}(S).$$

In connection with open question (4) of [Garcia, 1988], we obtain a characterization for semigroups with CEP in terms of a condition on the lattice of congruences of the semigroup. Similarly, since the set of ideals of a semigroup $S$ forms a lattice under the operations intersection and union, we characterize semigroups with IEP in terms of a condition on the lattice of ideals of the semigroup. Examples are provided for illustrative purposes.

Let $S$ be a semigroup. Let $\mathcal{L}_S$ denote the lattice of congruences on $S$. Similarly, for each subsemigroup $T$ of $S$, let $\mathcal{L}_T$ denote the lattice of congruences on $T$, and let $X_T = \{\langle \sigma \rangle_S: \sigma \in \mathcal{L}_T\}$. Recall from Chapter 1 that $\langle \sigma \rangle_S$ is the congruence on $S$ generated by $\sigma$.

2.1 Proposition. Let $S$ be a semigroup. Then $S$ has the congruence extension property (CEP) if and only if for every subsemigroup $T$ of $S$, the map $\phi_T: \mathcal{L}_T \rightarrow X_T$ defined by $\sigma \mapsto \langle \sigma \rangle_S$ is a bijective correspondence.
Proof. For simplicity, denote the congruence on $S$ generated by $\sigma$ by $\langle \sigma \rangle$.
Suppose that $S$ has CEP. Let $T$ be a subsemigroup of $S$. The map $\phi_T$ is clearly onto.
Let $\sigma_1, \sigma_2 \in \mathcal{L}_T$. Then by 1.3, $\langle \sigma_1 \rangle, \langle \sigma_2 \rangle$ are extensions of $\sigma_1, \sigma_2$ to $S$, respectively.
Thus, $\sigma_1 = \langle \sigma_1 \rangle \cap (T \times T)$ and $\sigma_2 = \langle \sigma_2 \rangle \cap (T \times T)$. Therefore, if $\langle \sigma_1 \rangle = \langle \sigma_2 \rangle$, then we have that $\sigma_1 = \sigma_2$. Hence, the map $\phi_T$ is one-to-one and thus bijective, as desired.

Suppose, on the other hand, that $S$ does not have CEP. Then there exists a subsemigroup $T$ of $S$ and $\sigma \in \mathcal{L}_T$ such that $\sigma$ has no extension to $S$; in particular, we have that $\langle \sigma \rangle \cap (T \times T) \neq \sigma$. Define $\rho = \langle \sigma \rangle \cap (T \times T)$. Then $\rho \in \mathcal{L}_T$, so that $\langle \rho \rangle \in X_T$. To see that $\phi_T$ is not one-to-one, we will show that $\langle \rho \rangle = \langle \sigma \rangle$. Now, $\sigma \subseteq \rho \subseteq \langle \rho \rangle$, a congruence on $S$. Since $\langle \sigma \rangle$ is the smallest congruence on $S$ containing $\sigma$, we have that $\langle \sigma \rangle \subseteq \langle \rho \rangle$. Also, we have that $\langle \sigma \rangle$ is a congruence on $S$ containing $\rho$, so that $\langle \rho \rangle$ must be contained in $\langle \sigma \rangle$, since $\langle \rho \rangle$ is the smallest congruence on $S$ containing $\rho$. Hence, $\langle \sigma \rangle = \langle \rho \rangle$. We therefore conclude that $\phi_T$ is not bijective, as we wished to show. The proof is complete. \[\]

2.2 Note. The map $\phi_T$ in Proposition 2.1 is join preserving.

Proof. Let $S$ be a semigroup, let $T$ be a subsemigroup of $S$, and let $\sigma_1, \sigma_2 \in \mathcal{L}_T$.
We claim that $\langle \sigma_1 \lor \sigma_2 \rangle = \langle \sigma_1 \rangle \lor \langle \sigma_2 \rangle$. We first note that $\sigma_1 \lor \sigma_2 \subseteq \langle \sigma_1 \rangle \lor \langle \sigma_2 \rangle$. Thus, $\sigma_1 \lor \sigma_2 \subseteq \langle \sigma_1 \rangle \lor \langle \sigma_2 \rangle$, and hence we have that $\langle \sigma_1 \lor \sigma_2 \rangle \subseteq \langle \sigma_1 \rangle \lor \langle \sigma_2 \rangle$. Conversely, we have that $\sigma_1 \subseteq \sigma_1 \lor \sigma_2$, and $\sigma_2 \subseteq \sigma_1 \lor \sigma_2$. Hence, $\langle \sigma_1 \rangle \subseteq \sigma_1 \lor \sigma_2$ and $\langle \sigma_2 \rangle \subseteq \sigma_1 \lor \sigma_2$. Therefore, the union and hence the join of $\langle \sigma_1 \rangle$ and $\langle \sigma_2 \rangle$ is contained in $\langle \sigma_1 \lor \sigma_2 \rangle$.
Combining the two inclusions yields that $\langle \sigma_1 \lor \sigma_2 \rangle = \langle \sigma_1 \rangle \lor \langle \sigma_2 \rangle$, and thus $\phi_T$ is join preserving.
2.3 Example. This is an example to illustrate that the map $\phi_T$ in Proposition 2.1 need not be meet preserving. That $\langle \sigma_1 \cap \sigma_2 \rangle_S \subseteq \langle \sigma_1 \rangle_S \cap \langle \sigma_2 \rangle_S$ follows from the fact that $\langle \sigma_1 \cap \sigma_2 \rangle_S$ is the smallest congruence on $S$ containing $\sigma_1 \cap \sigma_2$. However, the reverse inclusion need not hold. To see this, let $S = \{1, 2, 3, 4, 5\}$ be the semigroup with multiplication given by the Cayley table:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 3 & 3 & 3 \\
1 & 1 & 3 & 4 & 5 \\
1 & 1 & 5 & 5 & 5 \\
\end{array}
\]

Then $S$ has CEP, as may be verified through an exhaustive computational check. (Note: Computer programs that check for various semigroup properties are extremely useful when conducting such exhaustive searches.) One may also verify that there are 10 congruence relations on $S$. We list them by exhibiting their congruence classes as follows:

\[
\begin{align*}
\sigma_1 & \quad \{1\}, \{2\}, \{3\}, \{4\}, \{5\} \\
\sigma_2 & \quad \{3, 5\}, \{1\}, \{2\}, \{4\} \\
\sigma_3 & \quad \{3, 4, 5\}, \{1\}, \{2\} \\
\sigma_4 & \quad \{1, 3, 5\}, \{2\}, \{4\} \\
\sigma_5 & \quad \{1, 3, 4, 5\}, \{2\} \\
\sigma_6 & \quad \{1, 2\}, \{3\}, \{4\}, \{5\} \\
\sigma_7 & \quad \{1, 2\}, \{3, 5\}, \{4\} \\
\sigma_8 & \quad \{1, 2\}, \{3, 4, 5\} \\
\sigma_9 & \quad \{1, 2, 3, 5\}, \{4\} \\
\sigma_{10} & \quad \{1, 2, 3, 4, 5\}
\end{align*}
\]
The lattice of congruences of \( S \) is depicted by the following diagram where each \( \sigma_i \), for \( 1 \leq i \leq 10 \), is labeled with only the subscript \( i \):

\[
\begin{array}{c}
\text{9} \quad \text{10} \\
\text{7} \quad \text{8} \\
\text{6} \quad \text{5} \\
\text{4} \quad \text{3} \\
\text{1} \\
\end{array}
\]

Let \( T = \{1, 2, 4, 5\} \). Then \( T \) is a subsemigroup of \( S \), and the congruences on \( T \) are given by the following:

\[
\begin{array}{l}
\rho_1 = \{1\}, \{2\}, \{4\}, \{5\} \\
\rho_2 = \{4, 5\}, \{1\}, \{2\} \\
\rho_3 = \{1, 5\}, \{2\}, \{4\} \\
\rho_4 = \{1, 4, 5\}, \{2\} \\
\rho_5 = \{1, 2\}, \{4\}, \{5\} \\
\rho_6 = \{1, 2\}, \{4, 5\} \\
\rho_7 = \{1, 2, 5\}, \{4\} \\
\rho_8 = \{1, 2, 4, 5\}
\end{array}
\]

The lattice of congruences of \( T \) is depicted by the following diagram:

\[
\begin{array}{c}
\text{9} \quad \text{6} \\
\text{5} \quad \text{4} \\
\text{2} \\
\text{1} \\
\end{array}
\]

We have that \( \langle \rho_1 \rangle_S = \sigma_1 \), \( \langle \rho_2 \rangle_S = \sigma_3 \), \( \langle \rho_3 \rangle_S = \sigma_4 \), \( \langle \rho_4 \rangle_S = \sigma_5 \), \( \langle \rho_5 \rangle_S = \sigma_6 \), \( \langle \rho_6 \rangle_S = \sigma_8 \), \( \langle \rho_7 \rangle_S = \sigma_9 \), and \( \langle \rho_8 \rangle_S = \sigma_{10} \).

Thus, it is clear that \( \phi_T \) is a bijection. One checks that \( \rho_2 \wedge \rho_3 = \rho_1 \) and \( \sigma_3 \wedge \sigma_4 = \sigma_2 \). However, since \( \langle \rho_1 \rangle_S \neq \sigma_2 \), we see that \( \phi_T \) is not meet preserving.
A $\Delta$-semigroup is a semigroup $S$ whose lattice of congruences $\mathcal{L}_S$ forms a chain, i.e., $\mathcal{L}_S$ is totally ordered under inclusion.

2.4 Corollary. Let $S$ be a $\Delta$-semigroup. Then $S$ has the congruence extension property (CEP) if and only if for each subsemigroup $T$ of $S$, the map $\phi_T : \mathcal{L}_T \to X_T$ defined by $\sigma \mapsto \langle \sigma \rangle_S$ is a lattice isomorphism.

Proof. Let $S$ be a $\Delta$-semigroup, and let $T$ be a subsemigroup of $S$. For simplicity, denote the congruence on $S$ generated by $\sigma$ by $\langle \sigma \rangle$.

Let $X_T = \{ \langle \sigma \rangle : \sigma \in \mathcal{L}_T \}$. We claim that $X_T$ is a sublattice of $\mathcal{L}_S$. We need to show that $X$ is closed under meets and joins. That $X_T$ is closed under joins follows from the argument given in the proof of Note 2.2. To see that $X_T$ is closed under meets, let $\langle \sigma_1 \rangle, \langle \sigma_2 \rangle \in X$. Then $\sigma_1, \sigma_2 \in \mathcal{L}_T$. Since $\mathcal{L}_S$ is a chain, we have that either $\langle \sigma_1 \rangle \subseteq \langle \sigma_2 \rangle$ or $\langle \sigma_2 \rangle \subseteq \langle \sigma_1 \rangle$. Without loss of generality, we assume the former. It follows then that $\sigma_1 = \langle \sigma_1 \rangle \cap (T \times T) \subseteq \langle \sigma_2 \rangle \cap (T \times T) = \sigma_2$. Thus, $\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \langle \sigma_1 \rangle$ and $\sigma_1 \cap \sigma_2 = \sigma_1$. Whence, $\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \langle \sigma_1 \cap \sigma_2 \rangle$. Therefore, we have that $X_T$ is a sublattice of $\mathcal{L}_S$.

Suppose that $S$ has CEP. Then by 1.3, for each congruence $\sigma$ on $T$, $\langle \sigma \rangle$ extends $\sigma$ to $S$. Then by Proposition 2.1 and Note 2.2, $\phi_T : \mathcal{L}_T \to X_T$ is a join-preserving bijective correspondence. That it is also meet preserving follows from the argument given above that $X_T$ is closed under meets. Therefore, we have that $\phi_T : \mathcal{L}_T \to X_T$ is a lattice isomorphism.

The converse is immediate from Proposition 2.1.

We note here that [Aucoin, 1993] characterizes commutative $\Delta$-semigroups with the congruence extension property (CEP).
Let $S$ be a semigroup. Let $I_S$ denote the lattice of ideals of $S$. Likewise, for each subsemigroup $T$ of $S$, let $I_T$ denote the lattice of ideals of $T$, and let $Y_T = \{ S^1IS^1 : I$ is an ideal of $T \}$.

2.5 Proposition. Let $S$ be a semigroup. Then $S$ has the ideal extension property if and only if for each subsemigroup $T$ of $S$, the map $\psi_T : I_T \rightarrow Y_T$ defined by $I \mapsto S^1IS^1$ is a bijective correspondence.

Proof. Suppose that $S$ has IEP. Let $T$ be a subsemigroup of $S$. The map $\psi_T : I_T \rightarrow Y_T$ defined by $I \mapsto S^1IS^1$ is clearly onto. We wish to see that it is also one-to-one. Let $I_1, I_2 \in I_T$. Suppose $S^1I_1S^1 = S^1I_2S^1$. Then by 1.33, we have that $I_1 = S^1I_1S^1 \cap T = S^1I_2S^1 \cap T = I_2$. Hence, $\psi_T$ is one-to-one, as desired.

Conversely, suppose that $S$ does not have IEP. Then there exist a subsemigroup $T$ of $S$ and an ideal $I$ of $T$ such that $S^1IS^1 \cap T \neq I$. Let $K = S^1IS^1 \cap T \in I_T$. We claim that $S^1KS^1 = S^1IS^1$. To see this, we first note that $K \subseteq S^1IS^1$ and thus $S^1KS^1 \subseteq S^1IS^1$. On the other hand, $I \subseteq K$, whereby $S^1IS^1 \subseteq S^1KS^1$. These containments yield that $S^1KS^1 = S^1IS^1$, but $K \neq I$. Therefore, $\psi_T$ is not bijective, as we wished to show. The proof is complete. 

2.6 Note. The map $\psi_T$ in Proposition 2.5 is join preserving.

Proof. Let $S$ be a semigroup, let $T$ be a subsemigroup of $S$, and let $I_1, I_2 \in I_T$. We claim that $S^1(I_1 \cup I_2)S^1 = S^1I_1S^1 \cup S^1I_2S^1$. To see this, let $xay \in S^1(I_1 \cup I_2)S^1$. Then $x, y \in S^1$ and $a \in I_1 \cup I_2$, so that $xay \in S^1I_1S^1 \cup S^1I_2S^1$. Conversely, suppose $xby \in S^1I_1S^1 \cup S^1I_2S^1$. Then $x, y \in S^1$ and $b \in I_1 \cup I_2$. Therefore, we have $xby \in S^1(I_1 \cup I_2)S^1 = S^1I_1S^1 \cup S^1I_2S^1$. Hence, the claim is proven.
2.7 Example. This is an example to illustrate that the map $\psi_T$ in Proposition 2.5 need not be meet preserving. That $S^1(I_1 \cap I_2)S^1 \subseteq S^1I_1S^1 \cap S^1I_2S^1$ follows from the fact that $S^1(I_1 \cap I_2)S^1$ is the smallest ideal of $S$ containing $I_1 \cap I_2$. However, the reverse inclusion need not hold. To illustrate this, we consider the semigroup $S = \{1, 2, 3, 4, 5\}$ with multiplication given by:

\[
\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 1 & 3 \\
1 & 2 & 3 & 4 & 1 \\
1 & 1 & 1 & 1 & 5 \\
\end{array}
\]

Then $S$ has IEP as may be verified by an exhaustive computational check. That $S$ has seven ideals may also be computationally confirmed. We list the ideal of $S$ as follows:

- $J_1 = \{1\}$
- $J_2 = \{1, 2\}$
- $J_3 = \{1, 3\}$
- $J_4 = \{1, 2, 3\}$
- $J_5 = \{1, 2, 3, 4\}$
- $J_6 = \{1, 2, 3, 5\}$
- $J_7 = \{1, 2, 3, 4, 5\}$

The lattice of ideals of $S$ is depicted by the following diagram where each $J_i$, for $1 \leq i \leq 7$ is labeled with only the subscript $i$: 

![Diagram of lattice of ideals](image)
Let $T = \{1, 3, 4, 5\}$. Then $T$ is a subsemigroup of $S$, and the ideals of $T$ are given by:

$I_1 = \{1\}$

$I_2 = \{1, 3\}$

$I_3 = \{1, 3, 4\}$

$I_4 = \{1, 3, 5\}$

$I_5 = \{1, 3, 4, 5\}$

The lattice of ideals of $T$ is depicted by the following diagram:

![Lattice Diagram](image)

Now, we have that $S^1I_1S^1 = J_1$, $S^1I_2S^1 = J_3$, $S^1I_3S^1 = J_5$, $S^1I_4S^1 = J_6$, and $S^1I_5S^1 = J_7$.

Thus, it is clear the $\psi_T$ is a bijection. One checks that $I_3 \cap I_4 = I_2$ and $J_5 \cap J_6 = J_4$. However, since $S^1I_2S^1 \neq J_4$, we see that $\psi_T$ is not meet preserving.

We have stated Propositions 2.1 and 2.5 for semigroups. Actually, each holds more generally for groupoids as the proofs demonstrate. Thus, the characterization in Proposition 2.1 does not appear to have the potential for aiding in a resolution of the homomorphism problem as hoped, that is, the problem of determining whether CEP is preserved by homomorphisms of semigroups.
CHAPTER 3

t-SEMIGROUPS

In this chapter, we discuss the notion of a t-semigroup (i.e., a semigroup $S$ in which the relation "is an ideal of" is transitive among the ideals of $S$). In connection with CEP and IEP, we determine that each of these properties for a semigroup $S$ implies that $S$ is a t-semigroup. For a cyclic semigroup $S$, we establish the equivalence of $S$ being a t-semigroup, $S$ having CEP, and $S$ having IEP. Examples are provided for illustrative purposes.

A semigroup $S$ is called a t-semigroup if the relation "is an ideal of" is transitive among the ideals of $S$. That is to say, $S$ is a t-semigroup provided that if $J$ is an ideal of $S$ and $I$ is an ideal of $J$, then $I$ is an ideal of $S$.

As remarked earlier, ideals of semigroups are in some sense analogous to normal subgroups of groups. Hence, t-semigroups are analogous to t-groups defined in Chapter 1.

3.1 Proposition. The homomorphic image of a t-semigroup is a t-semigroup.

Proof. Let $S$ be a t-semigroup, and let $\phi: S \to S^*$ be a homomorphism of $S$ onto $S^*$. We claim that $S^*$ is a t-semigroup. Let $J^*$ be an ideal of $S^*$, and let $I^*$ be an ideal of $J^*$. We will show that $I^*$ is an ideal of $S^*$.

Let $J = \phi^{-1}[J^*]$ and $I = \phi^{-1}[I^*]$. Then $J$ is an ideal of $S$, and $I$ is an ideal of $J$. Since $S$ is a t-semigroup, $I$ is an ideal of $S$. Let $x^* \in I^*$ and $y^* \in S^*$. Then there exist $x \in I$ and $y \in S$ such that $\phi(x) = x^*$ and $\phi(y) = y^*$. Since $I$ is an ideal of $S$,
xy \in I. Hence, x^* y^* = \phi(x)\phi(y) = \phi(xy) \in I^*. Likewise, y^* x^* \in I^*. Therefore, I^*
is an ideal of S^*, as we asserted. ■

3.2 Proposition. Suppose \{S_\alpha : \alpha \in A\} is a family of semigroups. Let

\[ S = \prod_{\alpha \in A} S_\alpha. \]

Then S is a t-semigroup if and only if S_\alpha is a t-semigroup for each \alpha \in A.

Proof. We first note that it is well-known that I = \prod_{\alpha \in A} I_\alpha is an ideal of

S = \prod_{\alpha \in A} S_\alpha if and only if I_\alpha is an ideal of S_\alpha for each \alpha \in A.

Suppose that S = \prod_{\alpha \in A} S_\alpha is a t-semigroup. Fix \beta \in A. We claim that S_\beta
is a t-semigroup. Let I_\beta be an ideal of S_\beta, and let K_\beta be an ideal of I_\beta. Consider
the ideal I = \prod_{\alpha \in A} J_\alpha of S, where J_\alpha = I_\alpha if \alpha \neq \beta and J_\beta = I_\beta. Consider also
the ideal K = \prod_{\alpha \in A} L_\alpha, where L_\alpha = S_\alpha if \alpha \neq \beta and L_\beta = K_\beta. Since S is a
t-semigroup, we have that K is an ideal of S. Thus, K_\beta is an ideal of S_\beta.

Conversely, suppose S_\alpha is a t-semigroup for each \alpha \in A. Let I be an ideal of
S, and let K be an ideal of I. Then I = \prod_{\alpha \in A} I_\alpha, where I_\alpha is an ideal of S_\alpha for
each \alpha \in A and K = \prod_{\alpha \in A} K_\alpha, where K_\alpha is an ideal of I_\alpha for each \alpha \in A. Since
each S_\alpha is a t-semigroup, we have that K_\alpha is an ideal of S_\alpha for each \alpha \in A. Hence,
K is an ideal of S. Therefore, S is a t-semigroup. ■

3.3 Example. This is an example to demonstrate that a subsemigroup of a
t-semigroup need not be a t-semigroup. Consider the semigroup S = \{1,2,3,4,5\}
with multiplication given by

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 1 \\
1 & 1 & 1 & 4 & 5 \\
1 & 4 & 5 & 1 & 1
\end{array}
\]

Then S is a t-semigroup, as the only ideals of S are \{1\} and S itself. Consider
$T = \{1, 2, 3, 4\}$. Then $T$ is a subsemigroup of $S$. To see that $T$ is not a t-semigroup, consider the ideal $I = \{1, 3, 4\}$ of $T$ and the ideal $K = \{1, 3\}$ of $I$. One sees that $K$ is not an ideal of $T$, as $3 \cdot 2 = 2 \not\in K$.

**3.4 Example.** An infinite cyclic semigroup is not a t-semigroup. We show that the additive semigroup $\mathbb{N}$ of natural numbers is not a t-semigroup. To see this, consider the ideal $J = \{3, 4, 5, \ldots\}$ of $\mathbb{N}$ and the ideal $I = \{3, 6, 7, \ldots\}$ of $J$. One checks that $I$ is not an ideal of $\mathbb{N}$, as $3 + 1 = 4 \notin I$. Hence, $\mathbb{N}$ is not a t-semigroup.

**3.5 Proposition.** Each semigroup with the ideal extension property (IEP) is a t-semigroup.

**Proof.** Let $S$ be a semigroup with IEP. Let $I$ be an ideal of $S$, and let $K$ be an ideal of $I$. Since $S$ has IEP, there exists an ideal $J$ of $S$ such that $K = J \cap I$. Since the (nonempty) intersection of ideals of $S$ is again an ideal of $S$, we have that $K$ is an ideal of $S$. \[\]

**3.6 Example.** This is an example to show that the converse of Proposition 3.5 does not hold in general. Let $S$ be the order 5 semigroup in Example 3.3, that is, $S$ has multiplication given by the Cayley table:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 1 \\
1 & 1 & 1 & 4 & 5 \\
1 & 4 & 5 & 1 & 1 \\
\end{array}
\]

Then $S$ is a t-semigroup, as the only ideals of $S$ are $\{1\}$ and $S$ itself. However, $S$ does not have IEP. To see this, consider the subsemigroup $T = \{1, 2, 3, 4\}$ of $S$ and the ideal $I = \{1, 2, 3\}$ of $T$. Clearly, no ideal of $S$ extends $I$ to $S$.\[\]
3.7 Lemma. Let $S$ be a cyclic semigroup. If $S$ is a $t$-semigroup, then $\text{index}(S) \leq 3$.

**Proof.** Let $S$ be a cyclic $t$-semigroup. Suppose, for the purpose of deriving a contradiction, that $\text{index}(S) \geq 4$. We write $S = \{a, a^2, a^3, \ldots, a^n\} \cup M(S)$. Let $I = \{a^2, a^3, \ldots, a^n\} \cup M(S)$, and $K = \{a^2, a^4, a^5, \ldots, a^n\} \cup M(S)$. Then $I$ is an ideal of $S$, and $K$ is an ideal of $I$. However, $K$ is not an ideal of $S$, since $a^3 = a \cdot a^2 \not\in SK$ and $a \not\in K$. This contradicts $S$ being a $t$-semigroup. Therefore, $\text{index}(S) \leq 3$. \[ \]

3.8 Theorem. Let $S$ be a cyclic semigroup. Then these are equivalent:

1. $S$ is a $t$-semigroup;
2. $S$ has index less than or equal to 3;
3. $S$ has the congruence extension property (CEP); and
4. $S$ has the ideal extension property (IEP).

**Proof.** Let $S$ be a cyclic semigroup. This is immediate from 3.7, 1.42, 1.39, and 3.5. Applying these results yields $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$. \[ \]

Theorem 3.8 shows that for a cyclic semigroup $S$, being a $t$-semigroup is equivalent to $S$ having IEP. However, Example 3.6 shows that this is not true in the non-commutative case. We conjecture the equivalence of these two conditions for commutative semigroups, although the proof remains elusive. However, an extensive computer search of commutative semigroups of orders 4, 5, 6, and 7 for a counterexample was conducted and none was found.

3.9 Example. This example shows that a semigroup $S$ having IEP does not imply that ideals of subsemigroups of $S$ are ideals of $S$. Let $S = \{1, 2, 3, 4\}$ with
multiplication given by the Cayley table:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 3 & 4 \\
1 & 1 & 4 & 3 \\
\end{array}
\]

Then \( S \) is a semigroup with IEP. Let \( T = \{1, 2, 3\} \), and let \( I = \{1, 3\} \). Then \( T \) is a subsemigroup of \( S \), and \( I \) is an ideal of \( T \). Clearly, \( I \) is not an ideal of \( S \), as \( 4 = 3 \cdot 4 \notin I \).

Let \( S \) be a semigroup. An element \( a \in S \) is said to be a disruptive element if there exists a subsemigroup \( T \) of \( S \) such that \( a \in T \) and \( J_T(a) \subset J_S(a) \cap (T \times T) \), where \( \subset \) denotes proper containment. For a disruptive element \( a \in S \) and the particular subsemigroup \( T \) for which \( J_T(a) \subset J_S(a) \cap (T \times T) \), we say that \( a \) is disruptive in \( T \).

An element \( r \) in a semigroup \( S \) is called a regular element provided there exists \( t \in S \) such that \( rtr = r \). The element \( t \) is called an inverse for \( r \). A semigroup \( S \) is said to be a regular semigroup provided every element of \( S \) is a regular element.

**3.10 Lemma.** Let \( S \) be a semigroup, and let \( r \) be a regular element of \( S \). Let \( I \) be an ideal of \( S \) such that \( r \in I \). Then \( r \) is not disruptive in \( I \).

**Proof.** Let \( r \) be a regular element of \( I \), and let \( t \) be an inverse for \( r \). Then \( rtr = r \) and \( trt = t \). Therefore, \( t \in I \), as \( I \) is an ideal of \( S \). Let \( p \in J_S(r) \cap I \). Then we have that \( p = urv \), for some \( u, v \in S^1 \). We now obtain that

\[
p = urv = u(rtr)v = ur(trt)v = (urt)r(trv) \in I^1rI^1 = J_I(r).
\]

Hence, \( J_I(r) = J_S(r) \cap I \), and \( r \) is not disruptive in \( I \).
3.11 Corollary. Let $S$ be a regular semigroup. Then $S$ is a $t$-semigroup.

Proof. Suppose $S$ is a regular semigroup. Let $I$ be an ideal of $S$, and let $K$ be an ideal of $I$. Then

$$K = \bigcup_{x \in K} J_I(x) = \bigcup_{x \in K} [J_S(x) \cap I] = \bigcup_{x \in K} J_S(x),$$

since $x \in K \subseteq I$ and $x$ regular implies that $x$ is not disruptive in $I$. Thus, $K$ is a union of ideals of $S$ and is hence an ideal of $S$. Therefore, $S$ is a $t$-semigroup. ~

3.12 Corollary. Each band is a $t$-semigroup.

Proof. This is immediate from Corollary 3.11. ~

3.13 Note. Let $S$ be a regular semigroup. Then $S = ES$ and $S = SE$.

Proof. Let $S$ be a regular semigroup. Let $a \in S$. Then there exists $x \in S$ such that $axa = a$ and $xax = x$. We claim that $xa, ax \in E$. Now,

$$(xa)(xa) = (xax)a = xa \quad \text{and} \quad (ax)(ax) = (axa)x = ax.$$ 

Thus, $xa, ax \in E$, and $a = axa \in ES \cap SE$. Hence, we have $S = ES$ and $S = SE$. ~

3.14 Example. This is an example to illustrate that the condition that $S$ be regular cannot be weakened to $ES = S = SE$ in Corollary 3.10. Let $S$ be the semigroup defined by the multiplication table:

$$
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 3 \\
1 & 1 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 3 & 2 & 5 \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$
Then $S$ is a commutative monoid and thus satisfies the condition $ES = S = SE$.

We see that $\{1, 2, 4\}$ is an ideal of $\{1, 2, 3, 4\}$ which is an ideal of $S$. However, $\{1, 2, 4\}$ is not an ideal of $S$, as $4 \cdot 5 = 3$.

3.15 Theorem. Let $S$ be a semigroup with the congruence extension property (CEP). Then $S$ is a t-semigroup.

Proof. Let $S$ be a semigroup with CEP. Suppose that $S$ is not a t-semigroup. Then there exists an ideal $I$ of $S$ and an ideal $K$ of $I$ such that $K$ is not an ideal of $S$. Since $K$ is not an ideal of $S$, there exists $a \in K$ such that $J_S(a) \not\subseteq K$. Let $s, t \in S^1$ such that $sat \notin K$. Then $sat \in I$, and $sat \neq uav$, for all $u, v \in I^1$.

Now, since $K$ is an ideal of $I$, we have that $J_I(a) \subseteq K$. Also, since $I$ is an ideal of $S$, $J_S(a) \subseteq I$. Thus, since $J_S(a) \not\subseteq K$, $J_I(a) \subseteq K$, and $J_S(a) \subseteq I$, we have that $J_S(a) \neq J_I(a)$. Therefore, $a$ is disruptive in $I$. By Lemma 3.9, $a$ is not a regular element of $I$. In particular, $a \neq a^3$.

Now, $(a, a^3) \in I \times I$, as $I$ is a subsemigroup of $S$. Also, $sat \in I$ and $sa^3t \in I$, as $I$ is an ideal of $S$. Thus, $(sat, sa^3t) \in \alpha^S(a, a^3) \cap (I \times I)$. In view of the facts that

(i) $sat \neq uav$, for all $u, v \in I^1$,

(ii) $sat \neq ua^3v$, for all $u, v \in I^1$, and

(iii) $sat \neq sa^3t$, as $sat \notin K$ and $sa^3t = (sa)a(at) \in IKI \subseteq K$,

we see that there is no finite transition in $I$ linking $sat$ to $sa^3t$; that is, there is no finite transition of the form $sat = x_1, x_2, \ldots, x_n = sa^3t$ with

$$(x_i, x_{i+1}) \in \{(uav, ua^3v), (ua^3v, uav): u, v \in I^1\}, \quad \text{for } 1 \leq i \leq n - 1.$$  

Thus, by the characterization of $\alpha^I(a, a^3)$ given in Chapter 1, we have that $(sat, sa^3t) \notin \alpha^I(a, a^3)$. Therefore, $\alpha^S(a, a^3) \cap (I \times I) \neq \alpha^I(a, a^3)$, contrary to $S$ having CEP. Hence, $S$ is a t-semigroup. \qed
3.16 Example. This is an example to show that the converse of Theorem 3.15 does not hold. For this purpose, consider $S = \{1, 2, 3, 4\}$ with multiplication given by the Cayley table:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 3 \\
1 & 1 & 3 & 4 \\
\end{array}
\]

Then $S$ is a commutative t-semigroup without CEP. To see that $S$ is a t-semigroup, we exhibit the ideals of $T$:

- $I_1 = \{1\}$
- $I_2 = \{1, 2\}$
- $I_3 = \{1, 3\}$
- $I_4 = \{1, 2, 3\}$
- $I_5 = \{1, 3, 4\}$
- $I_6 = \{1, 2, 3, 4\}$

One easily checks that each ideal of $I_i$ (for $i = 1, 2, ..., 6$) is an ideal of $S$. To see that $S$ does not have CEP, consider the subsemigroup $T = \{1, 2, 3\}$ of $S$ and the congruence $\sigma = \{(2, 3), (3, 2)\} \cup \Delta_T$. Then $\langle \sigma \rangle_S = \{(2, 3), (3, 2), (1, 3), (3, 1), (1, 2), (2, 1)\} \cup \Delta_S$ is clearly not an extension of $\sigma$. Thus, no extension of $\sigma$ to $S$ exists, and hence $S$ does not have CEP.

From our discussion of t-semigroups, two interesting questions arise about a semigroup $S$.

1. When is an ideal of a subsemigroup of $S$ an ideal of $S$?
2. When is a subsemigroup of $S$ an ideal of some ideal of $S$?

Question (1) arises from Example 3.8 and remains an open question. Question (2) is addressed in the following chapter.
In this chapter, we discuss semigroups $S$ with the property that every subsemigroup is an ideal of some ideal of $S$, or m-semigroups. We obtain that m-semigroups are periodic semigroups with zero and have index less than or equal to 5. It follows that commutative m-semigroups are archimedean semigroups with zero. Those commutative m-semigroups whose index is less than or equal to 3 are characterized.

4.1 Lemma. Let $S$ be a semigroup and let $T$ be a subsemigroup of $S$. Then there exists an ideal $J$ of $S$ such that $T$ is an ideal of $J$ if and only if $T$ is an ideal of $S^1TS^1$.

Proof. Let $S$ be a semigroup. Let $T$ be a subsemigroup of $S$. Suppose there exists an ideal $J$ of $S$ such that $T$ is an ideal of $J$. Then $J^1TJ^1 \subseteq T$. Since $S^1TS^1$ is the smallest ideal of $S$ containing $T$, we have that $S^1TS^1 \subseteq J$. Therefore, we have $(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq J^1TJ^1 \subseteq T$. Hence, $T$ is an ideal of $S^1TS^1$. The converse is immediate. \[\square\]

We say that a semigroup $S$ is an m-semigroup provided that for every subsemigroup $T$ of $S$, there exists an ideal $J$ of $S$ such that $T$ is an ideal of $J$, or equivalently, $T$ is an ideal of $S^1TS^1$.

4.2 Lemma. If $S$ is a m-semigroup, then every subsemigroup of $S$ is an m-semigroup.
Proof. Let $S$ be an $m$-semigroup. Let $R$ be a subsemigroup of $S$, and let $T$ be a subsemigroup of $R$. We claim that $T$ is an ideal of $R^1TR^1$. To see this, we first notice that $T$ is also a subsemigroup of $S$. Therefore, since $S$ is an $m$-semigroup, $T$ is an ideal of $S^1TS^1$. Thus,

$$(R^1TR^1)^1 \cdot T \cdot (R^1TR^1)^1 \subseteq (S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T.$$ 

Hence, $T$ is an $m$-semigroup.  

4.3 Lemma. Let $S$ be an $m$-semigroup. Let $\phi: S \to \hat{S}$ be a homomorphism from $S$ onto a semigroup $\hat{S}$. Then $\hat{S}$ is an $m$-semigroup.

Proof. Let $S$ be an $m$-semigroup. Let $\phi: S \to \hat{S}$ be a homomorphism from $S$ onto a semigroup $\hat{S}$. We claim that $\hat{S}$ is an $m$-semigroup. Let $\hat{T}$ be a subsemigroup of $\hat{S}$. Put $T := \phi^{-1}[\hat{T}]$. Then $T$ is a subsemigroup of $S$. Thus, $T$ is an ideal of $S^1TS^1$, as $S$ is an $m$-semigroup. Hence,

$$(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T.$$ 

Since $\phi$ is a homomorphism onto $\hat{S}$, we have that

$$(\hat{S}^1\hat{T}\hat{S}^1)^1 \cdot \hat{T} \cdot (\hat{S}^1\hat{T}\hat{S}^1)^1 = (\phi[S]^1\phi[T]\phi[S]^1)^1 \cdot \phi[T] \cdot (\phi[S]^1\phi[T]\phi[S]^1)^1$$

$$= \phi[S^1TS^1]^1 \cdot \phi[T] \cdot \phi[S^1TS^1]^1$$

$$= \phi[(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1] \subseteq \phi[T] = \hat{T}.$$ 

Hence, we have the desired result.  

4.4 Note. Example 4.27 demonstrates that the product of $m$-semigroups is not, in general, an $m$-semigroup. Proposition 4.28 shows that the product $S$ of
commutative semigroups \( S_\alpha \) with \( \text{index}(S_\alpha) \leq 3 \) is an m-semigroup if and only if each \( S_\alpha \) is an m-semigroup.

**4.5 Theorem.** Let \( S \) be an m-semigroup. Then \( \text{index}(S) \leq 5 \) and \( E(S) = \{0\} \).

**Proof.** Let \( S \) be an m-semigroup, and let \( a \in S \). We first claim that \( \theta(a) \) is finite. Suppose, for purposes of deriving a contradiction, that \( \theta(a) \) is not finite. Then \( \theta(a) = \{a^n : n \in \mathbb{N}, a^{n_1} \neq a^{n_2} \text{ for } n_1 \neq n_2 \} \) is a subsemigroup of \( S \). Now, \( \theta(a^2) = \{a^{2k} : k \in \mathbb{N}\} \) is a subsemigroup of \( \theta(a) \). By Lemma 4.2, we have that \( \theta(a) \) is an m-semigroup. Thus, \( [\theta(a)^1 \theta(a^2)^1 \theta(a)^1] \cdot [\theta(a)^1 \theta(a^2)^1 \theta(a)^1] \subseteq \theta(a^2) \). Hence, we have that \( a^5 = aa^2a^2 \in [\theta(a)^1 \theta(a^2)^1 \theta(a)^1] \cdot \theta(a^2) \cdot [\theta(a)^1 \theta(a^2)^1 \theta(a)^1] \subseteq \theta(a^2) \), a contradiction. Therefore, \( \theta(a) \) is finite and thus contains an idempotent.

We now claim that \( E(S) = \{0\} \). Let \( e \in E \). Then \( T = \{e\} \) is a subsemigroup of \( S \). Since \( S \) is an m-semigroup, \( (S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T \). Hence, \( (S^1eS^1)e(S^1eS^1)^1 = e \). Therefore, for all \( x \in S \),

\[
xe = xe^2 \in (S^1eS^1)e(S^1eS^1)^1 = e,
\]

\[
ex = e^2x \in (S^1eS^1)e(S^1eS^1)^1 = e,
\]

and \( e \) is a zero for \( S \). Thus, \( E(S) = \{0\} \).

Let \( a \in S \). Then we know that \( \theta(a) \) is finite and contains the idempotent 0. We claim that \( \text{index}(a) \leq 5 \). Let \( p \) be the smallest positive integer such that \( a^p = 0 \in E(S) \), and suppose that \( p \geq 6 \). Then \( \theta(a) = \{a, a^2, a^3, \ldots, a^{p-1}, a^p = 0\} \). Let \( T = \{a^2, a^4, a^6, a^7, \ldots, a^p = 0\} \). Then \( T \) is a subsemigroup of \( \theta(a) \), and

\[
a^5 = aa^2a^2 \in [\theta(a)^1 T \theta(a)^1] \cdot T \cdot [\theta(a)^1 T \theta(a)^1] \subseteq T,
\]

as \( \theta(a) \) is an m-semigroup. This is clearly a contradiction as \( a^5 \notin T \). Thus, \( p \leq 5 \), as desired. Therefore, \( \text{index}(a) \leq 5 \), for all \( a \in S \). Whence, \( \text{index}(S) \leq 5 \). \( \blacksquare \)
4.6 Example. This is an example to illustrate that the bound $\text{index}(S) \leq 5$ in Theorem 4.5 is the lowest possible upper bound. Let $S = \{1,2,3,4,5,6\}$ with multiplication given by the Cayley table:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 & 2 & 3 \\
1 & 1 & 1 & 2 & 2 & 3 \\
1 & 1 & 2 & 3 & 3 & 4 \\
\end{array}
\]

Then $S$ is a commutative m-semigroup whose index is 5. To see $\text{index}(S) = 5$, check the index of each element of $S$: $\text{index}(1)=1$, $\text{index}(2)=2$, $\text{index}(3)=2$, $\text{index}(4)=3$, $\text{index}(5)=3$, and $\text{index}(6)=5$. We exhibit the subsemigroups $T_i$ of $S$ and $S^1T_i$ for $i = 1, \ldots, 12$:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$T_i$</th>
<th>$S^1T_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{1}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{1,3}</td>
<td>{1,2,3}</td>
</tr>
<tr>
<td>3</td>
<td>{1,2}</td>
<td>{1,2}</td>
</tr>
<tr>
<td>4</td>
<td>{1,2,5}</td>
<td>{1,2,3,5}</td>
</tr>
<tr>
<td>5</td>
<td>{1,2,4}</td>
<td>{1,2,3,4}</td>
</tr>
<tr>
<td>6</td>
<td>{1,2,4,5}</td>
<td>{1,2,3,4,5}</td>
</tr>
<tr>
<td>7</td>
<td>{1,2,3}</td>
<td>{1,2,3}</td>
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<td>8</td>
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<td>9</td>
<td>{1,2,3,4}</td>
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<td>10</td>
<td>{1,2,3,4,6}</td>
<td>{1,2,3,4,6}</td>
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<td>11</td>
<td>{1,2,3,4,5}</td>
<td>{1,2,3,4,5}</td>
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<tr>
<td>12</td>
<td>{1,2,3,4,5,6}</td>
<td>{1,2,3,4,5,6}</td>
</tr>
</tbody>
</table>

One may check by inspection that $S$ is an m-semigroup.
4.7 Corollary. Let $S$ be an $m$-semigroup. Then $S$ is periodic and $E(S) = 0$.

4.8 Lemma. Let $S$ be a periodic semigroup with $E(S) = \{0\}$. For $a, b \in S$, $ab = b$ (dually, $ba = b$) if and only if $b = 0$.

**Proof.** Let $S$ be a periodic semigroup with $E(S) = \{0\}$. Let $a, b \in S$. Obviously, if $b = 0$, then $ab = 0 = b$. Let $b \in S$, and suppose that there exists $a \in S$ such that $ab = b$. Then $X = \{a : ab = b\} \neq \emptyset$. However, $X$ is a subsemigroup of $S$. Indeed, $a_1, a_2 \in X$ implies that $(a_1a_2)b = a_1(a_2b) = a_1b = b$; hence $a_1a_2 \in X$. Since $S$ is periodic, $0 \in X$. Therefore, $0 = 0 \cdot b = b$. \[\square\]

Let $S$ be an $m$-semigroup. Note that for all subsemigroups $T$ of $S$, we have that $S^1T^2 \subseteq T$ and $T^2S^1 \subseteq T$. For a commutative semigroup $S$, $S$ is an $m$-semigroup if and only if $S^1T^2 \subseteq T$ for all subsemigroups $T$ of $S$.

Let $S$ be a semigroup containing a zero element. The annihilator $S$ is defined to be $A(S) = \{x \in S : xS = Sx = \{0\}\}$. We frequently denote the annihilator of a semigroup with zero by simply $A$.

4.9 Proposition. Let $S$ be an $m$-semigroup. Then for each $x \in S$ such that $\text{index}(x) > 2$, $x^{\text{index}(x)-1} \in A$.

**Proof.** Let $S$ be an $m$-semigroup. By Proposition 4.5, $\text{index}(S) \leq 5$.

Let $x \in S$ such that $\text{index}(x) > 2$. Then $3 \leq \text{index}(x) \leq 5$. Consider the subsemigroup $T = \theta(x)$ of $S$. Since $S$ is an $m$-semigroup, $S^1\theta(x)^2 \subseteq \theta(x)$ and $\theta(x)^2S^1 \subseteq \theta(x)$.

Let $s \in S$. We wish to show that $sx^{\text{index}(x)-1} = 0$ and $x^{\text{index}(x)-1}s = 0$. We will show $sx^{\text{index}(x)-1} = 0$ for the case when $\text{index}(x) = 5$, and all other cases will
follow analogously. Suppose, then, that index(x)=5. We claim that $sx^4 = 0$. Now, 
$T = \theta(x) = \{0, x, x^2, x^3, x^4\}$ and $sx^2 \in S^1\theta(x)^2 \subseteq \theta(x)$. We consider cases for $sx^2$ equaling each element of $\theta(x)$.

Case 1. $sx^2 = 0$

If $sx^2 = 0$, then $sx^4 = (sx^2)x^2 = 0$, as desired.

Case 2. $sx^2 = x$

If $sx^2 = (sx)x = x$, then $x = 0$ by Lemma 4.8. Hence, $sx^4 = 0$.

Case 3. $sx^2 = x^2$

If $sx^2 = x^2$, then by Lemma 4.8 $x^2 = 0$. Hence, $sx^4 = 0$.

Case 4. $sx^2 = x^3$

If $sx^2 = x^3$, then $sx^4 = (sx^2)x^2 = x^3x^2 = x^5 = 0$.

Case 5. $sx^2 = x^4$

If $sx^2 = x^4$, then $sx^4 = (sx^2)x^2 = x^4x^2 = x^6 = 0$.

In each case, we have established that $sx^4 = 0$, as desired.

If index(x)=4, then $T = \theta(x) = \{0, x, x^2, x^3\}$. We claim that $sx^3 = 0$. Four cases analogous to Cases 1–4 above will establish this.

If index(x)=3, then $T = \{0, x, x^2\}$. Three cases analogous to Cases 1–3 will establish that $sx^2 = 0$.

Thus, for $3 \leq \text{index}(x) \leq 5$, we have shown that $sx^{\text{index}(x)-1} = 0$. Dually, we obtain that $x^{\text{index}(x)-1}s = 0$. The proof is complete. \]

**4.10 Corollary.** Let $S$ be an $m$-semigroup. Let $n$ denote index($S$), and suppose that $n > 2$. Then $x^{n-1} \in A$ for all $x \in S$.

**Proof.** Let $S$ be an $m$-semigroup with $2 < $ index($S$). Let $x \in S$. By Proposition 4.9, $x^{\text{index}(x)-1} \in A$. Certainly, index($x$) $\leq n$. We may assume that
index(x) < n for otherwise the result is clear. Then \( n\text{-index}(x) > 0 \). Hence,

\[
x^{n-1} = x^{\text{index}(x)-1} \cdot x^{n-\text{index}(x)} \in A \cdot S = 0.
\]

Therefore, \( x^{n-1} \in A \).

**4.11 Example.** This is an example to illustrate that Proposition 4.9, and hence Corollary 4.10, does not hold if \( \text{index}(S) = 2 \). Let \( S = \{1,2,3,4,5\} \) with multiplication given by:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 \\
1 & 1 & 2 & 2 \\
\end{array}
\]

Then \( S \) is a commutative semigroup with zero such that \( \text{index}(S) = 2 \). Note that the element designated "1" is the zero element of \( S \). We have that \( \text{index}(3)=2 \), but \( 3 \notin A \) as \( 3 \cdot 5 = 2 \neq 1 \). The semigroup \( S \) is an m-semigroup by Proposition 4.13.

**4.12 Note.** A semigroup \( S \) with zero satisfying the condition that \( S^2 \subseteq A \) has index less than or equal to 3. Let \( S \) be such a semigroup, and let \( x \in S \). Then \( x^3 = x(x^2) \in xA = \{0\} \). Hence, \( x^3 = 0 \), for all \( x \in S \), and \( \text{index}(S) \leq 3 \).

**4.13 Proposition.** If \( S \) is a semigroup with zero such that \( S^2 \subseteq A \), then \( S \) is an m-semigroup.

**Proof.** Let \( S \) be a semigroup with zero such that \( S^2 \subseteq A \). Suppose \( T \) is a subsemigroup of \( S \). Then \( 0 \in T \), by Note 4.12. Let \( x, y, z \in S \). Then since \( xyz = 0 \), we have that \( (S^1T^1)^1 \cdot T \cdot (S^1T^1)^1 \subseteq T \) and \( S \) is an m-semigroup.

**4.14 Remark.** Let \( S \) be a semigroup with zero. Then \( S^2 \subseteq A \) if and only if \( S^3 = 0 \). To see this, suppose that \( S^2 \subseteq A \). Let \( x, y, z \in S \). Then we have that
\[xyz = x(yz) \in xA = \{0\}.\] Hence, \(S^3 = 0\). Conversely, suppose that \(S^3 = 0\). Let \(a, b \in S\). We claim that \(ab \in A\). Indeed, let \(c \in S\). Then \(abc = 0\), since \(S^3 = 0\). Therefore, \(ab \in A\).

**4.15 Lemma.** Let \(S\) be a semigroup with zero. If \(S^3 = 0\), then \(\text{index}(S) \leq 3\) and \(S\) is an \(m\)-semigroup.

**Proof.** Immediate from Remark 4.14, Note 4.12, and Proposition 4.13.

We recall from Chapter 1 that a commutative semigroup \(S\) is said to be **archimedean** provided that for any two elements of \(S\), each divides some power of the other. We use "\(|\)" to denote "divides". If a relation \(\eta\) is defined on a commutative semigroup \(S\) by

\[
(a, b) \in \eta \equiv a|b^n \text{ and } b|a^m \text{ for some } n, m \in \mathbb{N},
\]

then we have the following two well-known results:

1. [Clifford and Preston, 1961] The relation \(\eta\) on any commutative semigroup \(S\) is a congruence on \(S\), and \(S/\eta\) is the maximal semilattice homomorphic image of \(S\).

2. [Clifford and Preston, 1961] Every commutative semigroup \(S\) is uniquely expressible as a semilattice \(Y\) of archimedean semigroups \(C_\alpha\) (\(\alpha \in Y\)). The semilattice \(Y\) is isomorphic with the maximal semilattice homomorphic image \(S/\eta\) of \(S\), and the \(C_\alpha\) (\(\alpha \in Y\)) are the equivalence classes of \(S \mod \eta\).

The next three results concern archimedean semigroups with zero.

**4.16 Lemma.** [Tamura, 1958] Let \(S\) be an archimedean semigroup with zero. Then for \(a, b \in S\), \(ab = b\) if and only if \(b = 0\).
4.17 Lemma. [Yamada, 1964] Let $S$ be a nontrivial, finite, archimedean semigroup with zero. Then the annihilator of $S$ contains a nonzero element.

Let $K$ be a semigroup. Let $L$ be a semigroup with a zero element $0$ having no element in common with $K$. Let $M = K \cup (L \setminus \{0\})$. If an associative binary operation $\circ$ is defined on $M$ satisfying:

$$
x \circ y = \begin{cases} xy, & \text{if } x, y \in K \text{ or if } x, y \in L \text{ and } xy \neq 0 \\ K, & \text{otherwise}, \end{cases}
$$

then $M$ is a semigroup with respect to $\circ$, and $M$ is called an extension of $K$ by $L$. If $K$ and $L$ are commutative, then $M$ is a commutative semigroup and is called a commutative extension of $K$ by $L$.

4.18 Lemma. [Yamada, 1964] A commutative extension of a null semigroup of order 2 by an archimedean semigroup with zero of order $n$ is an archimedean semigroup with zero of order $n + 1$, and every archimedean semigroup with zero of order $n + 1$ is a commutative extension of a null semigroup of order 2 by an archimedean semigroup with zero of order $n$.

4.19 Corollary. If $S$ is a commutative m-semigroup, then $S$ is an archimedean semigroup with zero such that $\text{index}(S) \leq 5$.

**Proof.** Let $S$ be a commutative m-semigroup. Then by Corollary 4.7, $S$ is periodic and $E(S) = \{0\}$. Thus, $S$ is an archimedean semigroup with zero. That $\text{index}(S) \leq 5$ was established in Theorem 4.5. \[\]

4.20 Example. This is an example to show that the converse of Corollary 4.19 does not hold. In order to see this, we take $S = \{1, 2, 3, 4, 5, 6, 7\}$ with multiplication
given by the following Cayley table:

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 2 & 3 \\
1 & 1 & 1 & 2 & 2 & 3 & 1 \\
1 & 1 & 2 & 3 & 3 & 1 & 6 \\
\end{array}
\]

Then \( S \) is an archimedean semigroup with zero such that \( \text{index}(S) = 3 \), but \( S \) is not an \( m \)-semigroup. To see that \( S \) is not an \( m \)-semigroup, consider the subsemigroup \( T = \{1, 6, 7\} \) of \( S \). We see that \( 2 = 4 \cdot 7 \cdot 7 \in S^1T^2 \), but \( 2 \notin T \).

Let \( S \) be a semigroup. Recall that \( \mathcal{H} = \{(a,b) \in S \times S : a S^1 = b S^1 \text{ and } S^1 a = S^1 b\} \).

If \( S \) is a commutative semigroup, then \( \mathcal{H} \) is a congruence on \( S \).

**4.21 Proposition.** Suppose \( S \) is an archimedean semigroup containing an idempotent. Then \( S \) is \( \mathcal{H} \)-trivial if and only if \( E(S) = \{0\} \).

**Proof.** Let \( S \) be an archimedean semigroup with an idempotent \( e \). Then \( E(S) = \{e\} \). Suppose first that \( S \) is \( \mathcal{H} \)-trivial, i.e., \( \mathcal{H} = \Delta_S \). Then \( a S^1 = b S^1 \) implies that \( a = b \) for \( a, b \in S \). Let \( a \in S \). We claim that \( ae = e \). Now, \( aeS^1 = eaS^1 \subseteq eS^1 \).

Since \( S \) is archimedean with idempotent \( e \), there is \( a' \in S \) with \( aa' = a'a = e \). Thus, for \( x \in S^1 \), \( ex = eex = eaa'x \). Therefore, \( eS^1 \subseteq eaS^1 \). Hence, \( aeS^1 = eS^1 \) which implies that \( ae = e \). Thus, \( e \) is a zero for \( S \).

Conversely, let \( E(S) = \{0\} \). Suppose that \( S \) is not \( \mathcal{H} \)-trivial. Then there are distinct \( a, b \in S \) such that \( (a,b) \in \mathcal{H} \). Then there exist \( x, y \in S \) such that \( a = bx \) and \( b = ay \). Now, \( (bx,b) = (a,b) \in \mathcal{H} \). Compatibility of \( \mathcal{H} \) yields that
(bx^2, bx) = (bx, b) \cdot x \in H. Consequently, \((bx^{n-1}, bx^n) \in H\) for all \(n \in \mathbb{N}\). By transitivity of \(H\), we have that \((b, bx^n) \in H\) for all \(n \in \mathbb{N}\). Since, \(S\) is archimedean with zero, there exists \(m \in \mathbb{N}\) such that \(x^m = 0\). Hence, \((b, 0) = (b, bx^m) \in H\). Thus, \(aS^1 = bS^1 = 0S^1 = \{0\}\). Therefore, \(a = bx = 0 = ay = b\), contrary to \(a \neq b\). Thus, \(H\) is trivial. 

4.22 Lemma. Let \(S\) be a finite archimedean semigroup with zero such that \(\text{index}(S) \leq 3\). If \(S^3 \neq 0\), then there exists \(w \in S\) such that \(w^2 \notin A\).

Proof. Let \(S\) be a finite archimedean semigroup with zero and suppose that \(\text{index}(S) \leq 3\). Suppose that \(S^3 \neq 0\). Then there exists \(x, y, z \in S\) such that \(xyz \neq 0\). We may assume that \(x, y,\) and \(z\) are distinct. Indeed, if not, by renaming elements we obtain \(w, u \in S\) with \(w^2u \neq 0\) or \(w^2 \notin A\). We will show that there is \(w \in \{x, y, z\}\) such that \(w^2 \notin A\). We let \(n\) denote the order of \(S\) and use mathematical induction.

Case 1. \(n = 3\)

Suppose that the order of \(S\) is 3. We have distinct \(x, y, z \in S\) such that \(xyz \neq 0\). Therefore, \(x, y, z \in S \setminus \{0\}\), contrary to \(0 \in S\) and \(|S| = 3\). Thus, \(S^3 = 0\). This case is complete.

Case 2. \(n = 4\)

Suppose that the order of \(S\) is 4. We have distinct \(x, y, z \in S\) such that \(xyz \neq 0\). Now, \(|S| = 4\) implies that \(S = \{x, y, z, 0\}\). Therefore, we have \(xyz \in \{0, x, y, z\}\). In any case, Lemma 4.16 yields that \(xyz = 0\), a contradiction. Thus, \(S^3 = 0\). This case is complete.

Case 3. \(n = 5\)

Suppose that the order of \(S\) is 5. We have distinct \(x, y, z \in S\) such that \(xyz \neq 0\). Then \(x, y, z \in S \setminus A\). Since \(|S| = 5\), we obtain that
$S = \{0, x, y, z, xyz\}$. By Lemma 4.17, $xyz \in A$. Now, by Lemma 4.16 we have that $xy \notin \{x, y\}$ and by assumption we have that $xy \notin \{0, xyz\} \subseteq A$. Hence, $xy = z$. Likewise, $xz = y$ and $yz = x$. Thus, $x, y, z \in H_x$. However, $H = \Delta_S$ by Proposition 4.21. Therefore, we have a contradiction. Hence, for any semigroup of order 5 with index($S$) ≤ 3, $S^3 = 0$. This case is complete.

Case 4. $n = 6$

Suppose that the order of $S$ is 6. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Then $x, y, z \in S \setminus A$. By Lemma 4.17, there exists a nonzero annihilator $u \in S$. By Lemma 4.18, $S$ is an ideal extension of $Z \setminus \{0_Z\}$ by $N = \{0_Z, u\}$ where $Z$ is an archimedean semigroup with zero of order 5 and $N$ is a null (or zero) semigroup. Now, $|S| = 6$ implies that $S = \{0_S, x, y, x, u, v\}$. Thus, $Z = \{0_Z, x, y, z, v\}$. We consider the product $xyz \in Z$. By the preceding case, $xyz = 0_Z \in Z$. Thus, $xyz = 0_S \in S$, a contradiction. Hence, $x, y$, and $z$ cannot be distinct. Whence, by renaming elements, we obtain $w, u \in S$ with $w^2 u \neq 0$, that is, $w^2 \notin A$. This case is complete.

Case 5. $n = k$

Suppose that the order of $S$ is $k$. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Assume that there exists $w \in \{x, y, z\}$ such that $w^2 \notin A$. This is our inductive hypothesis.

Case 6. $n = k + 1$

Suppose that the order of $S$ is $k + 1$. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Then $x, y, z \in S \setminus A$. By Lemma 4.17, there exists a nonzero
annihilator $u \in S$. By Lemma 4.18, $S$ is an ideal extension of $Z \setminus \{0_Z\}$ by $N = \{0_Z, u\}$ where $Z$ is an archimedean semigroup with zero of order $k$ and $N$ is a null (or zero) semigroup. Then $x, y, z \notin A(S)$ implies that $x, y, z \in Z$. Now, $xyz \neq 0_S$ implies that $xyz \neq 0_Z$ as a product in $Z$. By inductive hypothesis, there exists $w \in \{x, y, z\}$ such that $w^2 \notin A(Z)$. Therefore, $w^2 \notin A(S)$. Hence, the general case is complete.

Therefore, the lemma is established for all finite archimedean semigroups.

4.23 Corollary. Let $S$ be a finite archimedean semigroup with zero such that $\text{index}(S) \leq 3$. If $S^3 \neq 0$, then $S$ is not an m-semigroup.

Proof. Apply Lemma 4.22 and Corollary 4.10.

4.24 Corollary. Let $S$ be a finite archimedean semigroup with zero such that $\text{index}(S) \leq 3$. If $S$ is an m-semigroup, then $S^3 = 0$.

4.25 Proposition. Let $S$ be a finite archimedean semigroup with zero. Then $S^3 = 0$ if and only if $S$ is an m-semigroup and $\text{index}(S) \leq 3$.

Proof. This is immediate from Lemma 4.15 and Corollary 4.24.

4.26 Theorem. Let $S$ be an archimedean semigroup with zero. Then $S^3 = 0$ if and only if $S$ is an m-semigroup and $\text{index}(S) \leq 3$.

Proof. Let $S$ be an archimedean semigroup with zero. Suppose that $S$ is an m-semigroup and $\text{index}(S) \leq 3$. Suppose, for the purpose of deriving a contradiction, that $S^3 \neq 0$. Then there exists $x, y, z \in S$ such that $xyz \neq 0$. We have that $x, y, z$ are distinct by Corollary 4.10. Consider the subsemigroup
\( T = \langle x, y, z \rangle = \{ x, x^2, xy, xz, y, y^2, z, z^2, yz, 0 \} \) of \( S \). Then \( T \) is a finite archimedean semigroup with zero, \( \text{index}(T) \leq 3 \), and \( T \) is an m-semigroup. By Corollary 4.24, \( T^3 = 0 \). Then \( xyz \in T^3 \) implies that \( xyz = 0 \), a contradiction. Hence, \( S^3 = 0 \), as desired. The converse is immediate from Lemma 4.15. \( \blacksquare \)

4.27 Example. This is an example to show that the product of m-semigroups is not an m-semigroup in general. Let \( S \) be the semigroup from Example 4.6, i.e., \( S = \{1, 2, 3, 4, 5, 6\} \) with multiplication given by the following Cayley table:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 & 2 & 3 \\
1 & 1 & 1 & 2 & 2 & 3 \\
1 & 1 & 2 & 3 & 3 & 4 \\
\end{array}
\]

Then \( S \) is an archimedean m-semigroup. We consider the archimedean semigroup with zero \( S \times S \). To see that \( S \times S \) is not an m-semigroup, we consider \( T = \Delta_{S \times S} \). Then \( T \) is a subsemigroup of \( S \times S \). Now,

\[
(4, 6) \cdot (5, 5) \cdot (6, 6) = (4, 6) \cdot (3, 3) = (1, 2) \notin T.
\]

Hence, \((S \times S)^1 T^2 \not\subset T \) and \( S \times S \) is not an m-semigroup.

4.28 Proposition. Let \( \{S_\alpha : \alpha \in A\} \) be a family of archimedean semigroups with zero such that \( \text{index}(S_\alpha) \leq 3 \) for all \( \alpha \in A \). Let \( S = \prod\{S_\alpha : \alpha \in A\} \) with coordinate-wise multiplication. Then \( \text{index}(S) \leq 3 \), and \( S \) is an m-semigroup if and only if \( S_\alpha \) is an m-semigroup for each \( \alpha \in A \).

Proof. Let \( \{S_\alpha : \alpha \in A\} \) be a family of archimedean semigroups with zero such that \( \text{index}(S_\alpha) \leq 3 \) for all \( \alpha \in A \). Let \( S = \prod\{S_\alpha : \alpha \in A\} \). Then for each \( x \in S \), \( x^3 = 0 \) since \( x_\alpha^3 = 0_\alpha \) for each \( \alpha \in A \). Hence, \( \text{index}(S) \leq 3 \). Suppose that \( S \) is an m-semigroup. Then by Lemma 4.3, \( S_\alpha = \pi_\alpha[S] \) is an m-semigroup for each \( \alpha \in A \).
Conversely, suppose each $S_\alpha$ is an m-semigroup. Therefore, for each $\alpha \in A$, $S_\alpha^3 = 0_\alpha$. Let $T$ be a subsemigroup of $S$. Let $T_\alpha = \pi_\alpha[T]$ for each $\alpha \in A$. Then $T_\alpha$ is a subsemigroup of $S_\alpha$ for each $\alpha \in A$. Since each $S_\alpha$ is an m-semigroup, we have that $S_\alpha^1T_\alpha^2 \subseteq T_\alpha$, for each $\alpha \in A$.

Let $x \in S^1$ and let $y, z \in T$. Then $x = (x_\alpha), y = (y_\alpha)$, and $z = (z_\alpha)$, where $x_\alpha \in S_\alpha$ and $y_\alpha, z_\alpha \in T_\alpha$ for each $\alpha \in A$. Now, $xyz = (x_\alpha y_\alpha z_\alpha) = (0_\alpha) = 0 \in T$. Thus, $S^1T^2 \subseteq T$, and $S$ is an m-semigroup. \[\square\]

In [Aucoin, 1993], archimedean semigroups with zero that have the ideal extension property are characterized. The characterization is as follow:

4.29 Lemma. [Aucoin, 1993] Let $S$ be an archimedean semigroup with zero. Then the following are equivalent:

(1) If $x, y \in S$, $xy \neq 0$, then $xy = x^2 = y^2$.

(2) Each subsemigroup of $S$ is an ideal.

(3) $S$ has IEP.

4.30 Note. Let $S$ be archimedean semigroup with zero. If $S$ has the congruence extension property (CEP), then $S$ is an m-semigroup.

Proof. Let $S$ be an archimedean semigroup with zero. If $S$ has CEP, then by 1.39 $S$ has IEP. By Lemma 4.29, each subsemigroup of $S$ is an ideal. Therefore, $S$ is an m-semigroup. \[\square\]
In this chapter, we are concerned with semigroups which are bands of groups. Specifically, we consider the question: When does a band of groups have CEP?

We obtain that a semilattice of abelian groups has CEP if and only if each group is torsion, that is, if and only if each of the abelian groups has CEP. Since bands do not in general have CEP, an analogous statement for bands of abelian groups fails to hold. However, medial bands do have CEP, by [Stralka, 1972]. Therefore, we obtain that medial bands of abelian groups have CEP if and only if each of the abelian groups is torsion. The statement for a semilattice of abelian groups then follows.

A characterization for bands of (not necessarily abelian) groups with CEP is given in [Jones, 1992]. However, this characterization does not appear to give information about the structure of an extension for a given congruence. The characterization in the special case when $S$ is a completely simple semigroup leads us to conjecture that an arbitrary band of groups $S = \bigcup_{e \in E(S)} G_e$ has CEP if and only if $E(S)$ is a subsemigroup of $S$, $E(S)$ has CEP, and $G_e$ has CEP for each $e \in E(S)$. It is not immediately apparent that this is equivalent to the characterization given in [Jones, 1992].

Note that a semilattice of abelian groups $S$ is commutative; hence $\mathcal{H}$ is a congruence on $S$. We observe that $S/\mathcal{H}$ is isomorphic to the semilattice $E(S)$, and hence $S/\mathcal{H}$ has CEP. Consequently, for a commutative semigroup $S$, we consider whether
\( S/\mathcal{H} \) has CEP whenever \( S \) has CEP. Although this remains an open question, we reduce the question to a special case of the following: When does a semilattice of archimedean semigroups with zero each of which has CEP have CEP?

A semigroup \( S \) is called\textbf{ medial} provided that for each \( a, b, c, d \in S \), we have that \( abcd = acbd \). It has been shown in [Anderson and Hunter, 1962] that this is equivalent to the condition that for \( a, b, c \in S \), \( abca = acba \).

An element \( r \) of a semigroup \( S \) is called\textbf{ a regular element} provided there exists \( t \in S \) such that \( rtr = r \). The element \( t \) is called an\textbf{ inverse} for \( r \). A semigroup \( S \) is said to be a\textbf{ regular semigroup} if every element of \( S \) is a regular element.

5.1\textbf{ Note.} A medial group is abelian. To see this, let \( G \) be a medial group with identity \( e \). Let \( a, b \in G \). Then we have that

\[
ab = eabe = b(b^{-1}aba^{-1})a = b(ab^{-1}ba^{-1})a = b(ea^{-1})a = ba.
\]

5.2\textbf{ Remark.} A band of groups is regular. To see this, let \( S \) be a band of groups. Then \( S = \bigcup_{\alpha \in I} G_{\alpha} \), where \( I \) is a band and \( G_{\alpha} \) is a group for each \( \alpha \in I \). Let \( x \in S \). Then there exists \( \beta \in I \) such that \( x \in G_{\beta} \). Since \( G_{\beta} \) is a group, there exists \( x^{-1} \in G_{\beta} \) such that \( xx^{-1} = x^{-1}x = e_{\beta} \), where \( e_{\beta} \) is the identity of \( G_{\beta} \). Clearly, \( x^{-1} \in S \) and satisfies \( xx^{-1}x = x \), as desired.

5.3\textbf{ Theorem.} Let \( S \) be a medial semigroup which is a band of groups. Then \( S \) has the congruence extension property (CEP) if and only if each group is torsion.

\textbf{Proof.} Let \( S \) be a medial semigroup which is a band of groups. Then we have \( S = \bigcup_{\alpha \in I} G_{\alpha} \), where \( I \) is a medial band and \( G_{\alpha} \) is an abelian group for each \( \alpha \in I \). By 1.22, \( G_{\alpha} \) has GCEP for each \( \alpha \in I \).
Suppose there exists $\beta \in I$ such that $G_\beta$ is not torsion. Then by 1.28, $G_\beta$ does not have CEP. Hence, since CEP is hereditary by 1.1, $S$ does not have CEP.

Conversely, suppose that for each $\alpha \in I$, $G_\alpha$ is torsion. Then we remark that by 1.28, $G_\alpha$ has CEP for each $\alpha \in I$. Let $T$ be a subsemigroup of $S$. Let $J = \{\alpha \in I : T \cap G_\alpha \neq \emptyset\}$. Then $J$ is a band. Put $T_\alpha := T \cap G_\alpha$ for each $\alpha \in J$. Then since each $G_\alpha$ is torsion, $T_\alpha$ is a subgroup of $G_\alpha$ for each $\alpha \in J$. Thus, $T = \bigcup_{\alpha \in J} T_\alpha$ is a band of groups. Since $S$ is regular and medial and $T$ is a band of groups, we have that $S$ has CEP relative to $T$ by 1.17. However, $T$ was an arbitrary subsemigroup of $S$. Therefore, $S$ has CEP.

5.4 Corollary. Let $S$ be a semilattice of abelian groups. Then $S$ has the congruence extension property (CEP) if and only if each group is torsion.

5.5 Example. The cylinder $I_{\min} \times \mathbb{Z}$ does not have CEP. We recall that $I_{\min} = [0, 1]$ with multiplication $*$ given by $x * y = \min\{x, y\}$ for $x, y \in I_{\min}$. To see that the cylinder does not have CEP, we need only note that it contains a copy of $\mathbb{N} \subseteq \mathbb{Z}$ as a subsemigroup. This actually shows that for any semilattice $A$, $A \times \mathbb{Z}$ does not have CEP.

5.6 Example. This is an example to illustrate that in general bands do not have CEP. Let $S = \{1, 2, 3, 4\}$ be the semigroup with multiplication given by:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 4 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 \\
\end{array}
\]

Then $S$ is a band without CEP. To see this, consider the subsemigroup $T = \{1, 3, 4\}$ of $S$ and the congruence $\alpha^T(3, 4)$ on $T$. Then $(1, 4) = (2, 2) \cdot (3, 4) \in \alpha^S(3, 4) \cap T \times T$, but $(1, 4) \notin \alpha^T(3, 4)$. 
As noted before, if $S$ is a semilattice of abelian groups, then $S$ is commutative and $\mathcal{H}$ is a congruence on $S$. Also, we have that $S/\mathcal{H} \cong E(S)$ is a semilattice and therefore has CEP.

Let $S$ be a commutative semigroup. Recall from Chapter 1 that 

$$\mathcal{H} = \{(a, b) \in S \times S: aS^1 = bS^1 \text{ and } S^1a = S^1b\}$$

is a congruence on $S$. In the remainder of this chapter, we are concerned with determining whether $S$ having CEP is sufficient for $S/\mathcal{H}$ to have CEP. We obtain a reduction of the problem.

A commutative semigroup $S$ is said to be archimedean provided that for any two elements of $S$, each divides some power of the other. We use "$|$" to denote "divides".

If a relation $\eta$ is defined on a commutative semigroup $S$ by 

$$(a, b) \in \eta \equiv a|b^n \text{ and } b|a^m \text{ for some } n, m \in \mathbb{N},$$

then it is known that

(1) The relation $\eta$ is a congruence on $S$, and $S/\eta$ is the maximal semilattice homomorphic image of $S$; and

(2) The commutative semigroup $S$ is uniquely expressible as a semilattice $Y$ of archimedean semigroups $C_\alpha (\alpha \in Y)$. The semilattice $Y$ is isomorphic with the maximal semilattice homomorphic image $S/\eta$ of $S$, and the $C_\alpha (\alpha \in Y)$ are the equivalence classes of $S$ mod $\eta$.

We call the archimedean semigroups $C_\alpha$ the archimedean components of $S$. Obviously, a commutative semigroup is archimedean provided that it has only one archimedean component (or $\eta$-class). If a commutative semigroup is periodic, then each of its archimedean components must contain exactly one idempotent, as some
power of each element of $S$ is idempotent and each component is an archimedean subsemigroup. Thus, $S = \bigcup_{e \in E} C(e)$, where $C(e)$ denotes the archimedean component of $S$ containing the idempotent $e$.

5.7 Lemma. Let $S$ be an archimedean periodic semigroup. Then for the unique idempotent element $e$ of $S$, $\mathcal{H} = (H(e) \times H(e)) \cup \Delta_S$.

Proof. Let $S$ be an archimedean periodic semigroup. Then $S$ contains a unique idempotent $e$. That $(H(e) \times H(e)) \cup \Delta_S \subseteq \mathcal{H}$ is clear. To see the reverse inclusion, let $(a, b) \in \mathcal{H}$. We may assume that $a \neq b$. Then we have that $aS = bS$. Thus, there exist $x, y \in S$ such that $a = bx$ and $b = ay$.

Now, $(bx, b) = (a, b) \in \mathcal{H}$. We then have that $(bx^2, bx) \in \mathcal{H}$, by compatibility of $\mathcal{H}$. Hence, we obtain that $(bx^{n-1}, bx^n) \in \mathcal{H}$ for all $n \in \mathbb{N}$. Then by transitivity of $\mathcal{H}$, we have $(b, bx^n) \in \mathcal{H}$ for all $n \in \mathbb{N}$. Since $S$ is periodic, there exists $m \in \mathbb{N}$ such that $x^m = e$. Thus, $(b, be) \in \mathcal{H}$.

Since $S$ is archimedean with idempotent $e$, there is $b' \in S$ with $bb' = e = b'b$. Thus, $e = e^2 = b'be \in beS$. Certainly, $be \in eS$. Hence, $beS = eS$, and $(be, e) \in \mathcal{H}$.

Transitivity of $\mathcal{H}$ then yields that $(b, e) \in \mathcal{H}$. Hence, $b \in H(e)$. Since $a \in H_b$, we have also that $a \in H_e$. Therefore, $(a, b) \in H(e) \times H(e)$.

5.8 Lemma. Let $S$ be a commutative periodic semigroup with $E(S) = \{e\}$. Then $H(e)$ is the minimal ideal of $S$.

Proof. Let $S$ be a commutative periodic semigroup with $E(S) = \{e\}$. To see that $H(e)$ is an ideal of $S$, let $a \in H(e)$ and let $b \in S$. Then $aS^1 = eS^1$, so that $abS^1 \subseteq aS^1 = eS^1$. Now, there exists $n \in \mathbb{N}$ with $b^n = e$. Thus, $eS^1 = b^nS^1 \subseteq bS^1$. Therefore, $eS^1 = aS^1 = aeS^1 \subseteq abS^1$. Hence, $abS^1 = eS^1$, and $ab \in H(e)$. 

To see that $H(e)$ is the minimal, let $K$ be any ideal of $S$. We claim that $H(e) \subseteq K$. For $x \in K$, there is $m \in \mathbb{N}$ such that $x^m = e$. Thus, $e \in K$, and hence $eS^1 \subseteq K$. Let $a \in H(e)$. Then $aS^1 = eS^1 \subseteq K$. Hence, $a \in K$. The proof is complete. |

5.9 Corollary. Let $S$ be an archimedean semigroup containing an idempotent. Then $S$ is $H$-trivial if and only if $E(S) = \{0\}$.

Proof. Suppose first that $S$ is an archimedean semigroup with zero. Then $S$ is periodic, since for each $a \in S$ there exists $n \in \mathbb{N}$ with $a^n = 0$. Thus, by Lemma 5.7, $\mathcal{H} = (H(0) \times H(0)) \cup \Delta_S = \Delta_S$. Therefore, $S$ is $H$-trivial.

Conversely, suppose that $S$ is an $H$-trivial archimedean semigroup with idempotent $e$. We claim that $S$ is periodic. Indeed, since $S$ is archimedean, for $a \in S$ there is $n \in \mathbb{N}$ such that $a^n | e$ and $e | a$. Thus, there exist $x, y \in S$ with $a^n x = e$ and $e y = a$. Hence, $e y^n = a^n$, and $(e, a^n) \in \mathcal{H} = \Delta_S$. Therefore, $a^n = e$ and $S$ is periodic. Thus, $\Delta_S = \mathcal{H} = (H(e) \times H(e)) \cup \Delta_S$, by Lemma 5.7. Then since $H(e) = \{e\}$ is an ideal of $S$ by Lemma 5.8, we see that $S e = SH(e) \subseteq H(e) = \{e\}$. Therefore, $e$ is a zero for $S$, and $E(S) = \{0\}$. |

5.10 Proposition. If $S$ is an archimedean semigroup with the congruence extension property (CEP), then $S/\mathcal{H}$ has the congruence extension property.

Proof. Let $S$ be an archimedean semigroup with CEP. Then $S$ is periodic and $E(S) \neq \emptyset$. Since $S$ is archimedean with $E(S) \neq \emptyset$, $E(S) = \{e\}$. Then by Lemma 5.7, $\mathcal{H} = (H(e) \times H(e)) \cup \Delta_S$. By Lemma 5.8, $H(e)$ is an ideal of $S$. Thus, $S/\mathcal{H} = S/H(e)$ has CEP by Theorem 4.5 of [Aucoin, 1993]. |
We remark here that Theorem 4.5 of [Aucoin, 1993] states that for any semigroup \( S \) and any ideal \( I \) of \( S \), \( S/I \) has CEP provided \( S \) has CEP. In fact, it is shown in Corollary 4.15 of [Aucoin, 1993] that any homomorphic image of an archimedean semigroup with CEP retains CEP.

5.11 **Proposition.** Let \( S \) be a commutative semigroup with the ideal extension property (IEP). For any subsemigroup \( T \) of \( S \), \( H \cap (T \times T) = H_T \).

**Proof.** Let \( S \) be a commutative semigroup with IEP, and let \( T \) be a subsemigroup of \( S \). Let \((a, b) \in H \cap (T \times T)\). Then \( aS^1 = bS^1 \). Since \( S \) has IEP, \( aT^1 = aS^1 \cap T \) and \( bT^1 = bS^1 \cap T \). Therefore, \( aT^1 = bT^1 \), and \((a, b) \in H_T \). Conversely, let \((a, b) \in H_T \). Then \( aT^1 = bT^1 \), and there exist \( x, y \in T^1 \subseteq S^1 \) with \( a = bx \) and \( b = ay \). Thus, \((a, b) \in H \cap (T \times T)\). Hence, \( H \cap (T \times T) = H_T \).

Let \( S \) be a commutative semigroup with IEP. By 1.43, \( S \) is periodic. Thus, \( S = \bigcup_{e \in E} C(e) \), where \( C(e) \) denotes the archimedean component of \( S \) containing the idempotent \( e \). We have that \( H \subseteq \eta \). Indeed, for \((a, b) \in H \) there exist \( x, y \in S \) with \( a = bx \) and \( b = ay \). Hence, \( a|b \) and \( b|a \), and \((a, b) \in \eta \). Consequently, for \( e, f \in E(S) \) with \( e \neq f \), \( H \) cannot relate \( a \in C(e) \) to \( b \in C(f) \). Thus, \( H = \bigcup_{e \in E} H_{C(e)} \).

5.12 **Corollary.** Let \( S \) be a commutative semigroup with the ideal extension property (IEP). Then \( S \) is \( H \)-trivial if and only if \( S \) has trivial subgroups.

**Proof.** Let \( S \) be a commutative semigroup with IEP. Then \( S = \bigcup_{e \in E} C(e) \).

By Proposition 5.11, \( H \cap (C(e) \times C(e)) = H_{C(e)} \), for each \( e \in E \). Since \( H \subseteq \eta \),

\[ H = \bigcup_{e \in E} H_{C(e)} = \bigcup_{e \in E} (H(e) \times H(e)) \cup \Delta_{C(e)}, \]

by Lemma 5.7. One sees that \( H = \Delta_S \) if and only if \( H(e) = \{e\} \) for each \( e \in E \).
5.13 Note. This is the reduction of the problem of determining whether $S/\mathcal{H}$ has CEP for a commutative semigroup $S$ with CEP. Let $S$ be a commutative semigroup with CEP. Thus, $S = \bigcup_{e \in E} C(e)$, where $C(e)$ denotes the Archimedean component of $S$ containing the idempotent $e$. We have that $\mathcal{H} \subseteq \eta$. Consequently, $\mathcal{H} = \bigcup_{e \in E} \mathcal{H}_{C(e)}$. Therefore,

$$S/\mathcal{H} = \left( \bigcup_{e \in E} C(e) \right) / \mathcal{H} = \bigcup_{e \in E} (C(e)/\mathcal{H}_{C(e)}).$$

Now, for each $e \in S$, $C(e)/\mathcal{H}_{C(e)}$ is an Archimedean semigroup with zero having CEP. Thus, the question of when $S/\mathcal{H}$ retains CEP is related to the question of when a semilattice of Archimedean semigroups with zero each having CEP have CEP.
The primary objective of this chapter is to characterize completely simple semigroups with the congruence extension property (CEP). A construction is given which yields an alternative proof for the known result in the algebraic case and is amenable to direct extension to the topological result. A special case result was obtained in [Garcia, 1988] for completely simple semigroups in which the sandwich function maps the entire domain onto the identity of the group. In the algebraic case, a completely simple semigroup $S$ has CEP if and only if the Schützenberger group $G$ of $S$ has CEP and that $E(S)$ forms a subsemigroup of $S$. See [Jones, 1992]. In the topological case, we have a direct analog for compact completely simple semigroups. See Corollary 7.61.

Additionally, subsemigroups of a completely simple semigroup with torsion subgroups are characterized. We also demonstrate that the homomorphic image of a completely simple semigroup with CEP retains CEP.

The results of this chapter have topological analogues for compact completely simple topological semigroups. These are discussed in detail in Chapter 7.

We begin by recalling definitions and basic notions that will be needed.

There is a natural partial ordering of the set $E$ of idempotents of a semigroup $S$. Define $e \leq f$ (for $e, f \in E$) to mean $ef = fe = e$. Then $\leq$ is a partial order on $E$, and if $e \leq f$ we say that $e$ is below $f$ or that $f$ is above $e$. Note that if a semigroup $S$ has a zero element 0, then $0 \leq e$ for every $e \in E$. An idempotent $e \in E$ is said
to be **primitive** if the only idempotents of $S$ that are below $e$ are $e$ itself and 0 (if $S$ has a 0), and $e \neq 0$; that is, $e \in E$ is primitive if $e \neq 0$ and $f \leq e$ implies that $f = e$ or $f = 0$.

A semigroup $S$ is called **simple** if it contains no proper two-sided ideal. A semigroup $S$ is called **completely simple** if it is simple and contains a primitive idempotent, or equivalently, if it is simple and contains both a minimal left ideal and a minimal right ideal.

An element $a$ of a semigroup $S$ is called **regular** if $a \in aSa$. If every element of a semigroup $S$ is regular, then $S$ is called a **regular semigroup**.

The following are well-known and may be found in [Clifford and Preston, 1961].

1. Every idempotent in a completely simple semigroup is primitive.
2. A semigroup $S$ is completely simple if and only if it is a rectangular band of groups.
3. A completely simple semigroup is regular.
4. The homomorphic image of a completely simple semigroup is completely simple.

The following construction is well-known. If $X$ and $Y$ are nonempty sets, $S$ is a semigroup, and $\rho: Y \times X \to S$ is a function, then $[X, S, Y]_\rho$ is a semigroup on $X \times S \times Y$ with multiplication given by

$$(x, s, y) \cdot (x', s', y') = (x, s\rho(y, x')s', y').$$

The semigroup $[X, S, Y]_\rho$ is called the **Rees product of $S$ over $X$ and $Y$ with sandwich function $\rho$**. If $G$ is a group, then the semigroup $[X, G, Y]_\rho$ is called a **paragroup**.
We note that we may consider multiplication in the first and third coordinates as coordinate multiplication where \( X \) is a left trivial semigroup and \( Y \) is a right trivial semigroup.

The first three theorems are known. For proofs of Theorems 6.1 and 6.3, see [Clifford and Preston, 1961]. For a proof of Theorem 6.2, see [Carruth, Hildebrant, and Koch, 1983].

6.1 Theorem. [Rees, 1940] A semigroup \( S \) is completely simple if and only if it can be represented as \([L, G, R]_p\), where \( L \) is a left trivial semigroup, \( R \) is a right trivial semigroup, \( G \) is a group, and \( p: R \times L \rightarrow G \) is the sandwich function determining the multiplication in \( S \).

6.2 Theorem. [Sushkewitsch, 1937] Let \( S \) be a completely simple semigroup, and let \( e \in E \). Then \( eSe \) is a group. Define

\[
\rho: (eS \cap E) \times (Se \cap E) \rightarrow eSe \quad \text{by}
\]

\[
\rho(h, f) = hf,
\]

\[
\phi: [Se \cap E, eSe, eS \cap E]_p \rightarrow S \quad \text{by}
\]

\[
\phi(f, g, h) = fgh, \quad \text{and}
\]

\[
\hat{\phi}: S \rightarrow [Se \cap E, eSe, eS \cap E]_p \quad \text{by}
\]

\[
\hat{\phi}(s) = (s(es)e)^{-1}, ese, (ese)^{-1}s).
\]

Then \( \phi \) and \( \hat{\phi} \) are mutually inverse isomorphisms.

Conversely, if \( X \) and \( Y \) are nonempty sets, \( G \) is a group, and \( \rho: Y \times X \rightarrow G \) is a function, then \([X, G, Y]_p\) is a completely simple semigroup.
6.3 Theorem. [Clifford and Preston, 1961] Suppose that $S = [L, G, R]_p$ and $S^* = [L^*, G^*, R^*]_{p^*}$ are completely simple semigroups. Suppose there exist mappings $\gamma: L \to G^*$, $\delta: R \to G^*$, $\phi: L \to L^*$, $\psi: R \to R^*$, and a homomorphism $\omega: G \to G^*$ such that

$$\omega \circ p(r, l) = \delta(r)p^*(\psi(r), \phi(l))\gamma(l)$$

for every $r \in R$ and every $l \in L$. Define $\theta: S \to S^*$ by

$$\theta(l, g, r) = [\phi(l), \gamma(l)\omega(g)\delta(r), \psi(r)]$$

Then $\theta$ is a non-trivial homomorphism of $S$ into $S^*$, and conversely (since $S$ is regular) every non-trivial homomorphism of $S$ into $S^*$ is obtained in this way.

Note. If $\theta$ is an isomorphism from $S \to S^*$ in Theorem 6.3, then $\phi$, $\psi$, and $\omega$ are one-to-one and onto, for $\theta$ clearly induces a one-to-one mapping of the $\mathcal{R}[L]$-classes of $S$ onto those of $S^*$.

6.4 Lemma. [Clifford and Preston, 1961] Suppose $S = L \times G \times R$ with sandwich function $p: R \times L \to G$, and $S' = L \times G \times R$ with sandwich function $p': R \times L \to G$. If there exist mappings $\gamma: L \to G$ and $\delta: R \to G$ such that

$$p'(r, l) = \delta(r)p(r, l)\gamma(l) \quad \text{for all } (r, l) \in R \times L,$$

then $S$ is isomorphic to $S'$.

Proof. Denote the elements of $S$ by $(l, g, r)$ and the elements of $S'$ by $[l, g, r]$. Define $\phi: S' \to S$ by $\phi([l, g, r]) = (l, \gamma(l)g\delta(r), r)$. Then $\phi$ is a well-defined, bijective homomorphism. \qed
6.5 Remark. According to Lemma 6.4, given a completely simple semigroup $S = L \times G \times R$ with sandwich function $p: R \times L \to G$, we may "normalize" $p$, so that there is $\lambda \in L$ and $\mu \in R$ such that for every $l \in L$ and every $r \in R$ we have $p(r, \lambda) = e_G = p(\mu, l)$, where $e_G$ is the identity element of the group $G$. For this purpose, fix $\lambda \in L$ and $\mu \in R$. Define $\delta: R \to G$ by $\delta(r) = p(r, \lambda)^{-1}$, and define $\gamma: L \to G$ by $\gamma(l) = p(r, l)^{-1}p(\mu, \lambda)$. Then if we define $p': R \times L \to G$ by $p'(r, l) = \delta(r)p(r, l)\gamma(l)$, we see that $p'(r, \lambda) = p(r, \lambda)^{-1}p(r, \lambda)p(\mu, \lambda)^{-1}p(\mu, \lambda) = e_G$ for $r \in R$, and $p'(\mu, l) = p(\mu, \gamma(l)^{-1}p(\mu, \gamma(l)^{-1}p(\mu, \lambda) = e_G$ for $l \in L$.

We first determine the structure of subsemigroups of a completely simple semigroup $S$ with torsion subgroups before considering the extension of congruences on those subsemigroups to $S$. We are interested in the case when $S$ has torsion subgroups because if a semigroup $S$ has the congruence extension property (CEP), then $S$ has torsion subgroups. Indeed, if $S$ has CEP, then any subgroup of $S$ must inherit CEP. By 1.28, groups with CEP are torsion.

6.6 Lemma. Let $S$ be a completely simple semigroup with torsion subgroups. Let $T$ be a subsemigroup of $S$. Then $T$ is completely simple.

Proof. Let $S$ be a completely simple semigroup with torsion subgroups. Then $S = \bigcup_{I \times \Lambda} H_{i\lambda}$, where $I \times \Lambda$ is a rectangular band and each $H_{i\lambda}$ is a torsion group. Let $T$ be a subsemigroup of $S$. Then $T = T \cap S = T \cap \bigcup_{I \times \Lambda} H_{i\lambda} = \bigcup_{I \times \Lambda} (T \cap H_{i\lambda})$.

Now, each $H_{i\lambda}$ being torsion implies that $T \cap H_{i\lambda}$ is a subgroup of $H_{i\lambda}$ whenever $T \cap H_{i\lambda} \neq \emptyset$. 


Let

\[ I' = \{ i \in I : T \cap H_{i\lambda} \neq \emptyset \text{ for some } \lambda \in \Lambda \}, \quad \text{and} \]

let \( \Lambda' = \{ \lambda \in \Lambda : T \cap H_{i\lambda} \neq \emptyset \text{ for some } i \in I \} \).

Then \( T = \bigcup_{i' \times \Lambda'} H_{i\lambda} \) is a union of groups.

It remains to show that \( I' \times \Lambda' \) is a rectangular band. To see this, first let \( i \in I' \) and let \( \lambda \in \Lambda' \). Then there is \( j \in I \) and \( \mu \in \Lambda \) such that \( T \cap H_{i\mu} \neq \emptyset \) and \( T \cap H_{j\lambda} \neq \emptyset \). Thus, since \( H_{j\mu} \) contains an idempotent (as it is a group), and since \( T \) is a subsemigroup of \( S \), we see that \( T \cap H_{i\lambda} \neq \emptyset \). Then for \( (i_1, \lambda_1), (i_2, \lambda_2) \in I' \times \Lambda' \) we have that \( T \cap H_{i_1\lambda_1} \neq \emptyset \) and \( T \cap H_{i_2\lambda_2} \neq \emptyset \). Whence, \( T \cap H_{i_1\lambda_2} \neq \emptyset \). Thus, \( (i_1, \lambda_2) \in I' \times \Lambda' \), and \( I' \times \Lambda' \) is a rectangular band. Therefore, \( T \) is a rectangular band of groups and is hence completely simple. \( \blacksquare \)

6.7 Theorem. Let \( S = [L, G, R]_p \) be a completely simple semigroup, and suppose that \( p \) has been normalized so that there exist \( \lambda \in L \) and \( \mu \in R \) with \( p(r, \lambda) = e_G = p(\mu, l) \), for every \( r \in R \) and every \( l \in L \). Then every subsemigroup \( T \) of \( S \) is completely simple if and only if \( G \) is torsion.

Proof. Let \( S = [L, G, R]_p \) be a completely simple semigroup. Suppose that \( p \) has been normalized so that there exist \( \lambda \in L \) and \( \mu \in R \) with \( p(r, \lambda) = e_G = p(\mu, l) \) for every \( r \in R \) and every \( l \in L \).

Suppose that \( G \) is torsion. Let \( T \) be a subsemigroup of \( S \). Then \( S \) has torsion subgroups and \( T \) is completely simple by Lemma 6.6. Suppose, on the other hand, that \( G \) is not torsion. Then there exists \( h \in G \) such that \( h^m \neq e_G \) for all \( m \in \mathbb{N} \). Let \( H = \langle h \rangle \). Then \( H \) is an infinite cyclic semigroup with no idempotent. Consider the subsemigroup \( T = \{ \lambda \} \times H \times \{ \mu \} \) of \( S \). We claim that \( T \) does not contain an
idempotent and is hence not completely simple. To see this, let \( t = (\lambda, x, \mu) \in T \). Then \( t^2 = (\lambda, x, \mu)(\lambda, x, \mu) = (\lambda, x^2, \mu) \). Since \( x \neq x^2 \in H \), we have that \( t \neq t^2 \). Thus, \( T \) does not contain an idempotent and is therefore not completely simple. This completes the proof.

6.8 Corollary. Let \( S \) be a completely simple semigroup with torsion groups. Let \( T \) be a subsemigroup of \( S \), and let \( e \in E_S \cap T \). Then

\[
S \cong [Se \cap E_S, eSe, eS \cap E_S]_{\rho_S} \quad \text{and} \quad T \cong [Te \cap E_T, eTe, eT \cap E_T]_{\rho_T}.
\]

Proof. This is immediate from Theorem 6.2 and Lemma 6.6.

Note. Let \( S \) be a completely simple semigroup with torsion groups. Let \( T \) be a subsemigroup of \( S \), and let \( e \in E_S \cap T \). Then \( S \cong [Se \cap E_S, eSe, eS \cap E_S]_{\rho_S} \) and \( T \cong [Te \cap E_T, eTe, eT \cap E_T]_{\rho_T} \), by Corollary 6.8. Additionally, we make the following four observations.

1. We have the following containments:

\[
Te \cap E_T \subseteq Se \cap E_S
\]
\[
eT \cap E_T \subseteq eS \cap E_S
\]
\[
eTe \subseteq eSe
\]

Also, since \( eSe \) is torsion, we have that \( eTe \) is a subgroup of \( eSe \).

2. We have the following maps:

\[
\rho_S: (eS \cap E_S) \times (Se \cap E_S) \rightarrow eSe \quad \text{defined by}
\]
\[
\rho_S(h, f) = hf,
\]

and

\[
\rho_T: (eT \cap E_T) \times (Te \cap E_T) \rightarrow eTe \quad \text{defined by}
\]
\[
\rho_T(h, f) = hf.
\]

Thus, we see that \( \rho_T \) is the restriction of \( \rho_S \) to the set \((eT \cap E_T) \times (Te \cap E_T)\).
(3) Since we have

\[ \phi_S : [Se \cap ES, eSe, eS \cap ES]_{\rho_S} \to S \] defined by

\[ \phi_S(f, g, h) = fgh \quad \text{and} \quad \phi_T : [Te \cap ET, eTe, eT \cap ET]_{\phi_T} \to T \] defined by

\[ \phi_T(f, g, h) = fgh, \]

we see that \( \phi_S \) restricted to the semigroup \( [Te \cap ET, eTe, eT \cap ET]_{\phi_T} \) is \( \phi_T \).

(4) We analogously have that

\[ \hat{\phi}_S : S \to [Se \cap ES, eSe, eS \cap ES]_{\rho_S} \] defined by

\[ \hat{\phi}_S(s) = (s(ese)^{-1}, ese, (ese)^{-1}s) \quad \text{and} \quad \hat{\phi}_T : T \to [Te \cap ET, eTe, eT \cap ET]_{\rho_T} \] defined by

\[ \hat{\phi}_T(t) = (t(ete)^{-1}, ete, (ete)^{-1}t) \]

yields that \( \hat{\phi}_T \) is \( \hat{\phi}_S \) restricted to the semigroup \( T \).

6.9 Theorem. Let \( S = [L, G, R]_p \) be a completely simple semigroup with torsion subgroups. Let \( T \) be a subsemigroup of \( S \). Then \( T \) is completely simple, and there exist \( A \subseteq L, H < G, \) and \( B \subseteq R \) such that \( T \cong [A, H, B]_{p|B \times A} \).

Proof. We have that \( T \) is completely simple by Lemma 6.4. Let \( e \in E \cap T \). By Corollary 6.8, \( S \cong [Se \cap ES, eSe, eS \cap ES]_{\rho_S} \) and \( T \cong [Te \cap ET, eTe, eT \cap ET]_{\rho_T} \). Using Remark 6.5, we normalize \( \rho_S \) so that \( \rho_S(e, l) = e = \rho_S(r, e) \) for every \( l \in Se \cap E \) and every \( r \in eS \cap E \). Since \((e, e) \in (eT \cap E) \times (Te \cap E)\), we also have that \( \rho_T(b, e) = e = \rho_T(e, a) \), for every \( b \in Te \cap E \) and every \( a \in Te \cap E \).

Let \( \theta \) denote the isomorphism from \( [Se \cap E, eSe, eS \cap E]_{\rho_S} \) to \( S \). Then applying Theorem 6.3 and the subsequent note, there exist mappings \( \gamma : Se \cap E \to G \) and \( \delta : eS \cap E \to G \), one-to-one and onto mappings \( \phi : Se \cap E \to L \) and \( \psi : eS \cap E \to R \), and a one-to-one, onto homomorphism \( \omega : eSe \to G \). We define
A = \phi[T \cap E] \subseteq L,
B = \psi[eT \cap E] \subseteq R, \quad \text{and}
H = \omega[eTe] < G.

We claim \theta([T \cap T, eTe, eT \cap E]_{\rho_T}) = [A, H, B]_{\mu B \times A}. The map \gamma: Se \cap E \to G is defined by \theta(l, e, e) = [\phi(l), \gamma(l), \psi(e)] \in [L, G, R]_{\mu} = S. We wish to see that 
\gamma|_{T \cap E}: T \cap E \to H. Now, \theta(l, e, e) = (l, e, e)(l, e, e) \in [S \cap E_S, eS, eS \cap E_S]_{\rho_S}. Thus, \theta(l, e, e) = [\phi(l), \gamma(l), \psi(e)] \in E_S. Hence,

[\phi(l), \gamma(l), \psi(e)] = [\phi(l), \gamma(l)p(\psi(e), \phi(l))\gamma(l), \psi(e)].

Thus, \gamma(l) = p(\psi(e), \phi(l))^{-1}. If \l \in T \cap E, then \gamma(l) \in H as desired.

We wish to see also that \delta|_{eT \cap E}: eT \cap E \to H. Let \r \in eT \cap E, and let \l \in T \cap E. Then \omega \circ \rho_S(\r, \l) = \delta(\l)p(\psi(\r), \phi(\l))\gamma(\l) \in H. Using algebraic manipulations and the facts that p(\psi(\r), \phi(\l)) \in H and \gamma(\l) \in H, we see easily that \delta(\r) \in H.

To prove the claim, let \f \in T \cap E, g \in eTe, and k \in eT \cap E. Then

\theta(f, g, h) = [\phi(f), \gamma(f)\omega(g)\delta(k), \psi(k)] \in [A, H, B]_{\mu B \times A}.

Thus, we have that \theta([T \cap T, eTe, eT \cap E]_{\rho_T}) \subseteq [A, H, B]_{\mu B \times A}.

Conversely, let (a, h, b) \in [A, H, B]_{\mu B \times A}. Then

a = \phi(f) \in \phi[T \cap E],

b = \psi(k) \in \psi[eT \cap E], \quad \text{and}

h = \omega(g) \in \omega[eTe].

Now, \gamma(f) \in H = \omega[eTe] implies there is \tilde{g} \in eTe such that \gamma(f) = \omega(\tilde{g}). Likewise, there is \tilde{g} \in eTe such that \delta(k) = \omega(\tilde{g}) since \delta(k) \in H. Then we obtain that

\theta(f, \tilde{g}^{-1}g\tilde{g}^{-1}, k) = [\phi(f), \gamma(f)\omega(\tilde{g}^{-1}g\tilde{g}^{-1}), \psi(k)]

= [\phi(f), \gamma(f)\omega(\tilde{g})^{-1}\omega(g)\omega(\tilde{g})^{-1}, \psi(k)]

= [a, h, b].
Thus, \([A, H, B]_{p|B \times A} \subseteq \theta([Te \cap T, eTe, \epsilon T \cap E]_{\rho_T})\), and the claim is established.

Hence, \(T \cong [Te \cap T, eTe, \epsilon T \cap E]_{\rho_T} \cong [A, H, B]_{p|B \times A}\). The proof is complete.  

The results in the following discussion are due to [Tamura, 1960]. Proofs are indicated here for the sake of completeness.

Congruences on a completely simple semigroup \(S = L \times G \times R\) with sandwich function \(p: R \times L \to G\) may be characterized as triples of congruences on \(L, G,\) and \(R\) satisfying a “matching” condition. That is, a congruence \(\sigma\) on \(S\) is characterized as a triple of congruences \((\sigma_L, \sigma_G, \sigma_R)\) where \(\sigma_L\) is a congruence on \(L, \sigma_G\) is a congruence on \(G,\) and \(\sigma_R\) is a congruence on \(R\) such that

\[
\text{if } (a, x) \in \sigma_L \text{ and } (b, y) \in \sigma_R, \text{ then } (p(b, a), p(y, x)) \in \sigma_G. \tag{*}
\]

Indeed, for a congruence \(\sigma\) on \(S,\) define

\[
\sigma_L = \{(a, x) \in L \times L: \text{there is } b, y \in R \text{ and } u, v \in G \text{ with } ((a, u, b), (x, b, y)) \in \sigma\},
\]

\[
\sigma_R = \{(b, y) \in R \times R: \text{there is } a, x \in L \text{ and } u, v \in G \text{ with } ((a, u, b), (x, v, y)) \in \sigma\},
\]

and

\[
\sigma_G = \{(u, v) \in G \times G: \text{there is } a, x \in L \text{ and } b, y \in R \text{ with } ((a, u, b), (x, v, y)) \in \sigma\}.
\]

Then these are congruences on \(L, R,\) and \(G,\) respectively and \((*)\) is satisfied. Conversely, given two independently chosen congruences \(\sigma_L\) on \(L\) and \(\sigma_R\) on \(R,\) there is at least one congruence \(\sigma_G\) on \(G\) satisfying \((*)\). Then \((\sigma_L, \sigma_G, \sigma_R)\) determines a congruence \(\sigma\) as follows:

\[
((a, u, b), (x, v, y)) \in \sigma \text{ if and only if } (a, x) \in \sigma_L, \ (u, v) \in \sigma_G, \text{ and } (b, y) \in \sigma_R.\]

By Remark 6.5, we assume that there exists \(\lambda \in L\) and \(\mu \in R\) such that for every \(r \in R\) and every \(l \in L\) we have \(p(r, \lambda) = e = p(\mu, r),\) where \(e\) denotes the identity element of the group \(G.\)
To see that $\sigma_L$ (and analogously $\sigma_R$) are congruences, we first note that reflexivity and symmetry are clear. Now to see transitivity, let $(a, x), (x, t) \in \sigma_L$. Then there exist $u, v, w, h \in G$ and $b, y, c, k \in R$ such that

$$((a, u, b), (x, v, y)) \in \sigma \text{ and } ((x, w, c), (t, h, k)) \in \sigma.$$  

Then since $\sigma$ is compatible with multiplication in $S$, we have that

$$((a, uv^{-1}w, c), (x, w, c)) = ((a, u, b), (x, v, y)) \cdot (\lambda, v^{-1}w, c) \in \sigma.$$  

Since $\sigma$ is transitive, we now have $((a, uv^{-1}w, c), (t, h, k)) \in \sigma$. By definition of $\sigma_L$, we thus obtain $(a, x) \in \sigma_L$. Therefore, $\sigma_L$ is transitive and hence is an equivalence relation on $L$. Since $L$ is left trivial, $\sigma_L$ is a congruence on $L$. Similarly, $\sigma_R$ is a congruence on $R$.

We now wish to show that $\sigma_G$ is a congruence on $G$. First, $\sigma_G$ is clearly both reflexive and symmetric. To see that $\sigma_G$ is transitive, let $(u, v), (v, w) \in \sigma_G$. Then there exist $a, x, r, s \in L$ and $b, y, t, q \in R$ such that

$$((a, u, b), (x, v, y)) \in \sigma \text{ and } ((r, v, t), (s, w, q)) \in \sigma.$$  

Then since $\sigma$ is a congruence on $S$, we obtain

$$((r, u, t), (r, v, t)) = (r, e, \mu) \cdot ((a, u, b), (x, v, y)) \cdot (\lambda, e, t) \in \sigma.$$  

Now the transitivity of $\sigma$ yields that $((r, u, t), (s, w, q)) \in \sigma$. Therefore, $(u, v) \in \sigma_G$, and $\sigma_G$ is transitive, as desired. To see that $\sigma_G$ is compatible with multiplication in $G$, let $(u, v) \in \sigma_G$ and let $w \in G$. Then there exist $a, x \in L$ and $b, y \in R$ such that $((a, u, b), (x, v, y)) \in \sigma$. Since $\sigma$ is compatible with multiplication in $S$,

$$((a, uw, b), (x, vw, b)) = ((a, u, b), (x, v, y)) \cdot (\lambda, w, b) \in \sigma.$$  

Therefore, $(uw, vw) \in \sigma_G$. Likewise, $(wu, vw) \in \sigma_G$, and $\sigma_G$ is compatible.

We finally wish to see that (*) is satisfied. For this purpose, let $(a, x) \in \sigma_L$ and $(b, y) \in \sigma_R$. Then there are $u, v \in G$ such that $((a, u, b), (x, v, y)) \in \sigma$. [Indeed,
there are $s,t,w,q \in G$, $c,k \in R$, and $d,p \in L$ such that $((a,s,c),(x,t,k)) \in \sigma$ and $((d,w,b),(p,q,y)) \in \sigma$. Multiplying the former pair on the right by $(\lambda,e,b)$ and the latter pair on the left by $(x,tw^{-1},\mu)$ and using the transitivity of $\sigma$ yields $((a,s,b),(x,tw^{-1}q,y)) \in \sigma$. Put $u = s$ and $v = tw^{-1}q.$] Since $\sigma$ is a congruence on $S$, we have that

$$(a,e,b),(x,vu^{-1},b)) = ((a,u,b),(x,v,y)) \cdot (\lambda,u^{-1},b) \in \sigma.$$ 

Thus, $(e,vu^{-1}) \in \sigma_G$. Therefore, since $p(b,x) \in G$ and $\sigma_G$ is a congruence on $G$, $(p(b,x),p(b,x)vu^{-1}) \in \sigma_G$. Now

$$(a,p(b,a),b),(a,p(b,x)vu^{-1},b)) = (a,e,b) \cdot ((a,e,b),(x,vu^{-1},b)) \in \sigma.$$ 

Thus, $(p(b,a),p(b,x)uv^{-1}) \in \sigma_G$. By transitivity of $\sigma_G$, $(p(b,a),p(b,x)) \in \sigma_G$. Similarly, $(p(b,x),p(y,x)) \in \sigma_G$. Finally using again that $\sigma_G$ is transitive, we obtain $(p(b,a),p(y,x)) \in \sigma_G$, and (*) is satisfied.

Conversely, given independently chosen congruences $\sigma_L$ on $L$ and $\sigma_R$ on $R$, there exists at least one congruence $\sigma_G$ on $G$ satisfying (*), namely the universal congruence, $G \times G$. We show that $\sigma$ determined by $(\sigma_L,\sigma_G,\sigma_R)$ is a congruence on $S$. That $\sigma$ is an equivalence relation on $S$ follows easily from the fact that each of $\sigma_L$, $\sigma_G$, and $\sigma_R$ are equivalence relations. To see that $\sigma$ is compatible, let $((a,u,b),(x,v,y)) \in \sigma$, and let $(l,g,r) \in S$. Then

$$((l,g,r) \cdot ((a,u,b),(x,v,y)) = ((l, gp(r,a)u,b),(l, gp(r,x)v,y)).$$

Since $(a,x) \in \sigma_L$, $(r,r) \in \sigma_R$, and (*) is satisfied, we have $(p(r,a),p(r,x)) \in \sigma_G$. Compatibility of $\sigma_G$ yields $(gp(r,a)u,gp(r,x)v) \in \sigma_G$. Thus, we obtain that

$$((l,g,r) \cdot ((a,u,b),(x,v,y)) = ((l, gp(r,a)u,b),(l, gp(r,x)v,y)) \in \sigma.$$ 

Therefore, $\sigma$ is left compatible, and similarly $\sigma$ is right compatible. Hence, $\sigma$ is a congruence on $S$. 
6.10 Lemma. Let $S = [L, G, R]_p$ be a completely simple semigroup with torsion groups, let $T$ be a subsemigroup of $S$, and let $\sigma$ be a congruence on $T$. Thus, $T = A \times H \times B$, where $A \subseteq L$, $B \subseteq R$, and $H < G$, with sandwich function $p|_{B \times A}$; and $\sigma$ is determined by a triple $(\sigma_A, \sigma_H, \sigma_B)$ of congruences on $A$, $H$, and $B$ respectively such that if $(a, x) \in \sigma_A$ and $(b, y) \in \sigma_B$, then $(p(b, a), p(y, x)) \in \sigma_H$.

Then $\sigma$ has an extension $\bar{\sigma}$ to $S$ if and only if there exist

- $\sigma_L$ an extension of $\sigma_A$ to $L$,
- $\sigma_G$ an extension of $\sigma_H$ to $G$, and
- $\sigma_R$ an extension of $\sigma_B$ to $R$

such that $\bar{\sigma}$ is determined by $(\sigma_L, \sigma_G, \sigma_R)$.

Proof. By Remark 6.5 we may assume that $p$ has been normalized so that there exist $\lambda \in L$ and $\mu \in R$ such that $p(r, \lambda) = e = p(\mu, l)$ for every $r \in R$ and every $l \in L$, and that $p|_{B \times A}$ has been normalized so that there exist $\alpha \in A$ and $\beta \in B$ such that $p(b, \alpha) = e = p(\beta, a)$ for every $b \in B$ and every $a \in A$.

Suppose that $\bar{\sigma}$ is an extension of $\sigma$ to $S$, i.e., $\bar{\sigma} \cap (T \times T) = \sigma$. Then since $\bar{\sigma}$ is a congruence on $S$, there exist congruences $\sigma_L$ on $L$, $\sigma_G$ on $G$, and $\sigma_R$ on $R$ such that $\bar{\sigma}$ is determined by the "matching" triple $(\sigma_L, \sigma_G, \sigma_R)$.

Claims. (i) $\sigma_L \cap (A \times A) = \sigma_A$.

(ii) $\sigma_R \cap (B \times B) = \sigma_B$.

(iii) $\sigma_G \cap (H \times H) = \sigma_H$.

Proofs. (i): We first show that $\sigma_A \subseteq \sigma_L \cap (A \times A)$. Let $(a, x) \in \sigma_A$. Then there exist $b, y \in B$ and $u, v \in H$ such that $((a, u, b), (x, v, y)) \in \sigma \subseteq \bar{\sigma}$. Therefore, $(a, x) \in \sigma_L$. To see the reverse inclusion, let $(l_1, l_2) \in \sigma_L \cap (A \times A)$. Then there exist $r_1, r_2 \in R$ and $g_1, g_2 \in G$ such that $((l_1, g_1, r_1), (l_2, g_2, r_2)) \in \bar{\sigma}$. 

Multiplying on the right by \((\lambda, e, \beta) \in S\) yields\(\((l_1, g_1, \beta), (l_2, g_2, \beta)\) \in \tilde{\sigma}\).

Then by compatibility of \(\tilde{\sigma}\) we obtain\(\((l_1, g_1^n, \beta), (l_2, g_2^n, \beta)\) \in \tilde{\sigma}\) for every \(n \in \mathbb{N}\). Now, \(G\) is torsion implies that for each \(g \in G\) there is an \(m \in \mathbb{N}\) such that \(g^m = e\). Thus, \(\((l_1, e, \beta), (l_2, e, \beta)\) \in \tilde{\sigma} \cap (T \times T) = \sigma\). Therefore, \((l_1, l_2) \in \sigma_A\), as desired.

(ii): The proof of (ii) is analogous to the proof of (i).

(iii): First we show that \(\sigma_H \subseteq \sigma_G \cap (H \times H)\). Let \((u, v) \in \sigma_H\). Then there are \(a, x \in A\) and \(b, y \in B\) so that \(\((a, u, b), (x, v, y)\) \in \sigma \subseteq \tilde{\sigma}\). Hence, \((u, v) \in \sigma_G\). To prove the reverse inclusion, let \((u, v) \in \sigma_G \cap (H \times H)\). Then there exist \(l_1, l_2 \in L\) and \(r_1, r_2 \in R\) such that \(\((l_1, u, r_1), (l_2, v, r_2)\) \in \tilde{\sigma}\).

Since \(\tilde{\sigma}\) is a congruence on \(S\),

\[
(\alpha, e, \mu) \cdot ((l_1, u, r_1), (l_2, v, r_2)) \cdot (\lambda, e, \beta) \in \tilde{\sigma},
\]

i.e., \(\((\alpha, u, \beta), (\alpha, v, \beta)\) \in \tilde{\sigma} \cap (T \times T)\). Thus, \((u, v) \in \sigma_H\), as desired.

The established claims yield that \(\sigma_L\) is an extension of \(\sigma_A\) to \(L\), \(\sigma_G\) is an extension of \(\sigma_H\) to \(G\), and \(\sigma_R\) is an extension of \(\sigma_B\) to \(R\).

Conversely, suppose that there exists extensions \(\sigma_L\) of \(\sigma_A\) to \(L\), \(\sigma_G\) of \(\sigma_H\) to \(G\), and \(\sigma_R\) of \(\sigma_B\) to \(R\), so that if \((a, x) \in \sigma_L\) and \((b, y) \in \sigma_R\), then \((p(b, a), p(y, x)) \in \sigma_G\). Define \(\bar{\sigma} = \{(a, u, b), (x, v, y)\} \in S \times S: (a, x) \in \sigma_L, (u, v) \in \sigma_G, \text{ and } (b, y) \sigma_R\}\).

One verifies that \(\bar{\sigma}\) is a congruence on \(S\) and \(\sigma \subseteq \bar{\sigma}\). It remains to show that \(\bar{\sigma} \cap (T \times T) \subseteq \sigma\). For this purpose, let \((a, u, b), (x, v, y)\) \in \(\bar{\sigma} \cap (T \times T)\). Then

\[
(a, x) \in \sigma_L \cap (A \times A),
\]

\[
(u, v) \in \sigma_G \cap (H \times H), \text{ and}
\]

\[
(b, y) \in \sigma_R \cap (B \times B).
\]
Now since \((a, x) \in \sigma_A\), there exist \(h, g \in H\) and \(s, t \in B\) such that
\[
((a, h, s), (x, g, t)) \in \sigma.
\]
Since \((b, y) \in \sigma_B\), there exist \(z, w \in H\) and \(q, m \in A\) such that
\[
((q, z, b), (m, w, y)) \in \sigma.
\]
Since \((u, v) \in \sigma_H\), there exist \(r, k \in A\) and \(c, j \in B\) such that
\[
((r, u, c), (k, v, j)) \in \sigma.
\]
By compatibility of \(\sigma\) with multiplication on \(T\),
\[
((a, h, \beta), (x, g, \beta)) = ((a, h, s), (x, g, t)) \cdot (\alpha, e, \beta) \in \sigma.
\]
Thus, by compatibility again,
\[
((a, h^n, \beta), (x, g^n, \beta)) = ((a, h, \beta), (x, g, \beta))^n \in \sigma \quad \text{for every } n \in \mathbb{N}.
\]
Then since \(G\) is torsion we obtain that \(((a, e, \beta), (x, e, \beta)) \in \sigma\). Similarly,
\[
((\alpha, z, b), (\alpha, w, y)) = (\alpha, e, \beta) \cdot ((q, z, b), (m, w, y)) \in \sigma.
\]
Again using the facts that \(\sigma\) is compatible and \(G\) is torsion, we obtain that
\(((\alpha, e, b), (\alpha, e, y)) \in \sigma\). We put
\[
P_1 = ((a, e, \beta), (x, e, \beta)),
\]
\[
P_2 = ((r, u, c), (k, v, j)), \quad \text{and}
\]
\[
P_3 = ((\alpha, e, b), (\alpha, e, y)).
\]
Then since \(P_1, P_2, \text{ and } P_3 \in \sigma\) and \(\sigma\) is a congruence on \(T\), we have that
\[
P_1 P_2 P_3 = ((a, u, b), (x, v, y)) \in \sigma.
\]
Therefore, \(\sigma \cap (T \times T) \subseteq \sigma\), as desired. Thus, the proof is complete. \(\blacksquare\)

We note that there is a group, the Schützenberger group, associated to any \(\mathcal{H}\)-class \(H\). The Schützenberger group of \(H\), denoted \(\Gamma(H)\), is defined as follows:

\[\text{let } T(H) = \{t \in S^1 : Ht \subseteq H\}\]
\[\text{and } \Gamma(H) = \{\rho t|_H : t \in T(H)\}.\]
Thus, \( \Gamma(H) \) is the group of transformations of \( H \) induced by the inner right translations of \( S^1 \). If \( H \) and \( H' \) are two \( \mathcal{H} \)-classes in the same \( \mathcal{D} \)-class, the \( \Gamma(H) \cong \Gamma(H') \).

Also, if \( H \) is itself a group, then \( \Gamma(H) \cong H \).

In the case that \( S \) is a completely simple semigroup, we know that \( S \) is \( \mathcal{D} \)-simple and the \( \mathcal{H} \)-classes of \( S \) are the isomorphic maximal subgroups of \( S \). Thus, up to isomorphism, we may speak of the Schützenberger group of \( S \). If the Schützenberger group of \( S \) is torsion, then so is every subgroup of \( S \). Of course, if every subgroup of \( S \) is torsion, then the Schützenberger group being isomorphic to the maximal subgroups of \( S \) is torsion. Also, if \( S \) is a completely simple semigroup, then the group in Theorem 6.1 may be taken to be the Schützenberger group of \( S \). For an in depth discussion of the above, see [Clifford and Preston, 1961].

6.11 Theorem. Let \( S = [L,G,R]_p \) be a completely simple semigroup. Then \( S \) has the congruence extension property (CEP) if and only if the Schützenberger group of \( S \) is torsion and for every subsemigroup \( T = [A,H,B]_p \) of \( S \), where \( A \subseteq L \), \( B \subseteq R \), and \( H < G \) and every congruence \( \sigma = (\sigma_A,\sigma_H,\sigma_B) \) on \( T \), there exist

\[
\begin{align*}
\sigma_L & \text{ an extension of } \sigma_A \text{ to } L, \\
\sigma_G & \text{ an extension of } \sigma_H \text{ to } G, \text{ and} \\
\sigma_R & \text{ an extension of } \sigma_B \text{ to } R
\end{align*}
\]

such that \( (a,x) \in \sigma_L \) and \( (b,y) \in \sigma_R \) implies that \( (p(b,a),p(y,x)) \in \sigma_G \).

In particular, \( G \) has the congruence extension property.

Proof. Suppose that \( S = [L,G,R]_p \) has CEP. Without loss, we assume that \( p \) has been normalized so that there is \( \lambda \in L \) and \( \mu \in R \) with \( p(r,\lambda) = e = p(\mu,l) \)
for every \( r \in R \) and every \( l \in L \). Then \( G \cong \{ \lambda \} \times G \times \{ \mu \} \) has CEP as it is a subsemigroup of \( S \). Thus, \( G \) is torsion, and hence the Schützenberger group of \( S \) is torsion. By Lemma 6.10, the required extensions exist.

Conversely, let \( T \) be a subsemigroup of \( S \), and let \( \sigma \) be a congruence on \( T \). Then by Theorem 6.9, there exist \( A \subseteq L \), \( B \subseteq R \), and \( H < G \) such that \( T \cong [A, H, B]_{p|_{B \times A}} \). Again assume each of \( p \) and \( p|_{B \times A} \) have been normalized. We know \( \sigma = (\sigma_A, \sigma_H, \sigma_B) \), where \( \sigma_A \), \( \sigma_H \), and \( \sigma_B \) are congruences on \( A \), \( H \), and \( B \), respectively. Suppose that there exist extensions \( \sigma_L \) of \( \sigma_A \) to \( L \), \( \sigma_G \) of \( \sigma_H \) to \( G \), and \( \sigma_R \) of \( \sigma_B \) to \( R \), such that if \( (a, x) \in \sigma_L \) and \( (b, y) \in \sigma_R \), then \( (p(b, a), p(y, x)) \in \sigma_G \). Then by Lemma 6.10, there exists an extension \( \bar{\sigma} \) of \( \sigma \) to \( S \). Thus, \( S \) has CEP, as we wished to show.

We remark here that left trivial semigroups have the congruence extension property (CEP). Indeed, if \( L \) is a left trivial semigroup, then it is easily verified that every equivalence relation on \( L \) is a congruence on \( L \). Thus, if \( \sigma_A \) is any congruence on any subsemigroup \( A \) of \( L \), then \( \sigma_A \cup \Delta_L \) is a congruence on \( L \) extending \( \sigma_A \). Analogously, right trivial semigroups also have the congruence extension property (CEP).

6.12 Proposition. Let \( S = [L, G, R]_p \) be a completely simple semigroup with torsion subgroups. Let \( T \) be a subsemigroup of \( S \), and let \( \sigma \) be a congruence on \( T \). Thus, \( T \cong [A, H, B]_{p|_{B \times A}} \), where \( A \subseteq L \), \( B \subseteq R \), and \( H < G \); and \( \sigma = (\sigma_A, \sigma_H, \sigma_B) \), where \( \sigma_A \), \( \sigma_H \), and \( \sigma_B \) are congruences on \( A \), \( H \), and \( B \), respectively. Suppose \( \bar{\sigma} = (\sigma_L, \sigma_G, \sigma_R) \) is the smallest congruence on \( S \) containing \( \sigma \). Then \( \sigma_L = \sigma_A \cup \Delta_L \) and \( \sigma_R = \sigma_B \cup \Delta_R \).
Proof. Let \( \bar{\sigma} = (\sigma_L, \sigma_G, \sigma_R) \) be the smallest congruence on \( S \) containing \( \sigma \). Let 
\[ \rho = (\sigma_A \cup \Delta_L, \sigma_G, \sigma_B \cup \Delta_R) \]
Clearly we have that \( \sigma_A \cup \Delta_A \subseteq \sigma_L \) and \( \sigma_B \cup \Delta_R \subseteq \sigma_B \).
Now, if either containment is proper, we would have that \( \sigma \subseteq \rho \subset \bar{\sigma} \), contrary to the fact that \( \bar{\sigma} \) is the smallest congruence on \( S \) containing \( \sigma \). Thus, \( \sigma_A \cup \Delta_A = \sigma_L \) and \( \sigma_B \cup \Delta_R = \sigma_R \), as desired. \( \qed \)

6.13 Corollary. Let \( S = [L, G, R]_p \) be a completely simple semigroup with torsion subgroups. Let \( T \) be a subsemigroup of \( S \), and let \( \sigma \) be a congruence on \( T \).
Thus, \( T \cong [A, H, B]_{p|B \times A} \), where \( A \subseteq L \), \( B \subseteq R \), and \( H < G \); and \( \sigma = (\sigma_A, \sigma_H, \sigma_B) \), where \( \sigma_A \), \( \sigma_H \), and \( \sigma_B \) are congruences on \( A \), \( H \), and \( B \), respectively. Then \( \sigma \) has an extension of \( S \) if and only if there exists an extension \( \sigma_G \) of \( \sigma_H \) to \( G \) such that 
\[ \bar{\sigma} = (\sigma_A \cup \Delta_L, \sigma_G, \sigma_B \cup \Delta_R) \]
is an extension of \( \sigma \) to \( S \).

Proof. This is immediate from Proposition 6.12. \( \qed \)

6.14 Corollary. Let \( S = [L, G, R]_p \) be a completely simple semigroup. Let \( T = H_e \) for some \( e \in E \) be a subsemigroup of \( S \). Then every congruence \( \sigma \) on \( T \) can be extended to a congruence \( \bar{\sigma} \) on \( S \). Hence, \( S \) has the congruence extension property (CEP) relative to the \( H \)-classes of \( S \).

Proof. As usual we assume that \( p \) has been normalized so that there exist \( \lambda \in L \) and \( \mu \in R \) with \( p(r, \lambda) = \epsilon_G = p(\mu, l) \), for every \( r \in R \) and every \( l \in L \). We have that \( T = H_e \cong G \cong \{ \lambda \} \times G \times \{ \mu \} \), and that \( \sigma \) corresponds via the composed isomorphisms to a congruence \( (\sigma_{\{\lambda\}}, \sigma_G, \sigma_{\{\mu\}}) \), where \( \sigma_G \) is a congruence on \( G \), and \( \sigma_{\{\lambda\}} \) and \( \sigma_{\{\mu\}} \) are obviously the respective diagonal congruences.

Define \( \bar{\sigma} = (\Delta_L, \sigma_G, \Delta_R) \). Then \( \bar{\sigma} \) extends \( (\sigma_{\{\lambda\}}, \sigma_G, \sigma_{\{\mu\}}) \), and via the isomorphism between \( T \) and \( \{ \lambda \} \times G \times \{ \mu \} \), \( \sigma \) has an extension to \( S \). \( \qed \)
**6.15 Theorem.** Let \( S = [L, G, R]_p \) be a completely simple semigroup, and suppose that \( p \) has been normalized so that there exist \( \lambda \in L \) and \( \mu \in R \) with \( p(r, \lambda) = e_G = p(\mu, l) \), for every \( r \in R \) and every \( l \in L \). Then \( S \) has the congruence extension property (CEP) if and only if \( G \) has the congruence extension property and \( p(r, l) = e_G \) for every \( r \in R \) and every \( l \in L \).

**Proof.** Suppose first that \( G \) has CEP and that \( p(r, l) = e_G \) for every \( r \in R \) and every \( l \in L \). Let \( T \) be a subsemigroup of \( S \), and let \( \sigma \) be a congruence on \( T \). Then there exist \( A \subseteq L \), \( H < G \), and \( B \subseteq R \) and congruences \( \sigma_A \) on \( A \), \( \sigma_H \) on \( H \), and \( \sigma_B \) on \( B \) such that \( T \cong [A, H, B]_{p|_{B \times A}} \) and \( \sigma = (\sigma_A, \sigma_H, \sigma_B) \). Then since \( p \) maps the entire domain onto \( e_G \), we have that the equivalence relation \( \bar{\sigma} = (\sigma_A \cup \Delta_L, < \sigma_H \sigma_B \cup \Delta_R) \) is a congruence on \( S \). Since \( G \) has CEP, \( \bar{\sigma} \) clearly extends \( \sigma \). Therefore, \( S \) has CEP.

Conversely suppose that \( S \) has CEP. Then \( G \cong \{ \lambda \} \times G \times \{ \mu \} \) has CEP. Suppose that there exists \( r \in R \) and \( l \in L \) such that \( p(r, l) \neq e_G \). Let \( B = \{ r, \mu \} \), let \( A = \{ \lambda \} \), and let \( H = G \). Let \( T = A \times G \times B \). Then \( T \) is a subsemigroup of \( S \). Consider \( \sigma = (\Delta_A, \Delta_G, B \times B) \). Since \( p|_{B \times A} \) maps \( B \times A \) onto \( e_G \), we have that \( \sigma \) is a congruence on \( T \). Note that if \( \sigma_G \cap (G \times G) = \Delta_G \), then \( \sigma_G = \Delta_G \). Then by Corollary 6.13, \( \sigma \) has an extension to \( S \) if and only if \( (\Delta_L, \Delta_G, (B \times B) \cup \Delta_R) \) is an extension of \( \sigma \). We will show that \( (\Delta_L, \Delta_G, (B \times B) \cup \Delta_R) \) is not a congruence on \( S \). We have \( (r, \mu) \in (B \times B) \cup \Delta_R \) and \( (l, l) \in \Delta_L \). However, \( (p(r, l), p(\mu, \lambda)) \notin \Delta_G \), as \( p(r, l) \neq e_G \). Therefore, \( S \) does not have CEP, a contradiction.

**6.16 Corollary.** Let \( S = [L, G, R]_p \) be a completely simple semigroup. If \( G \) has the congruence extension property (CEP) and there is \( g \in G \) with \( p(r, l) = g \) for every \( r \in R \) and every \( l \in L \), then \( S \) has the congruence extension property.
Proof. By Remark 6.5, we normalize \( p \) so that there exist \( \lambda \in L \) and \( \mu \in R \) with \( p(r, \lambda) = e_G = p(\mu, l) \), for every \( r \in R \) and every \( l \in L \). Then \( p(r, l) = e_G \) for every \( r \in R \) and every \( l \in L \). We apply Theorem 6.15 to see that \( S \) has CEP.

6.17 Example. This is an example which illustrates the concepts presented in this chapter. Let \( L = \{ l_1, l_2, l_3 \} \), \( R = \{ r_1, r_2 \} \), and \( G = C_6 \), the cyclic group of order 6. Let \( S = [L, G, R]_p \), where \( p: R \times L \to G \) is defined by the matrix

\[
p = \begin{pmatrix} 1 & 3 & 6 \\ 3 & 2 & 3 \end{pmatrix}.
\]

We normalize \( p \) by first defining \( \gamma: L \to G \) by

\[
\gamma \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}
\]

and \( \delta: R \to G \) by

\[
\delta \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.
\]

Then by defining \( p': R \times L \to G \) by \( p'(r, l) = \delta(r)p(r, l)\gamma(l) \), we obtain an isomorphic copy \( S' \) of \( S \). We have that \( S' = [L, G, R]_{p'} \) where \( p' \) is given by the matrix

\[
p' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.
\]

By Theorem 6.15, we know that \( S \) cannot have the congruence extension property. We illustrate this. Let \( A = \{ l_1 \} \), let \( B = R \), and let \( H = G \). Let \( T = [A, G, B]_{p'|B \times A} \). Define \( \sigma = (\Delta_A, \Delta_G, B \times B) \). Since \( p'|B \times A: B \times A \to \{1\} \), we see that \( \sigma \) is a congruence on \( T \). Note that if \( \sigma_G \cap (G \times G) = \Delta_G \), then \( \sigma_G = \Delta_G \). According to Corollary 6.13, it is enough to show that \( (\Delta_L, \Delta_G, R \times R) \) is not a congruence on \( S \), as \( (B \times B) \cup \Delta_R = R \times R \). Indeed, \((l_2, l_2) \in \Delta_L \) and \((r_1, r_2) \in (R \times R) \), but \((p'(r_1, l_2), p'(r_2, l_1)) = (1, 2) \notin \Delta_G \). Thus, \( (\Delta_L, \Delta_G, R \times R) \) does not satisfy condition \( (*) \) and therefore is not a congruence on \( S \).
6.18 Example. This is the smallest example of a completely simple semigroup without the congruence extension property (CEP). Let \( L = \{l_1, l_2\} \), \( R = \{r_1, r_2\} \), and \( G = C_2 \), the cyclic group of order 2. Let \( S = [L, G, R]_p \), where the sandwich function \( p: R \times L \to G \) is defined by the matrix
\[
p = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.
\]
Then \( p \) is already normalized. We observe that if any of \( L \), \( G \), or \( R \) is made any smaller, then the normalized sandwich function \( p \) will be given by \( p(r, l) = 1 \) for every \( r \in R \) and every \( l \in L \).

By Theorem 6.15, \( S \) cannot have CEP. To illustrate this, let \( A = \{l_1\} \), let \( B = R \), and let \( H = G \). Let \( T = [A, H, B]_p |_{B \times A} \). Since \( p |_{B \times A}: B \times A \to \{1\} \), \( \sigma = (\Delta_A, \Delta_H, B \times B) \) is a congruence on \( T \). It is enough to show that \((\Delta_L, \Delta_G, R \times R)\) is not a congruence on \( S \), as \((B \times B) \cup \Delta_R = R \times R \), and the only congruence on \( G \) extending \( \Delta_G \) is \( \Delta_G \). Indeed, \((l_2, l_2) \in \Delta_L \) and \((r_1, r_2) \in (R \times R) \), but \((p(r_1, l_2), p(r_2, l_1)) = (1, 2) \notin \Delta_G \).

6.19 Proposition. Let \( S = [L, G, R]_p \) be a completely simple semigroup. Suppose that \( p \) has been normalized so that there exist \( \lambda \in L \) and \( \mu \in R \) with \( p(r, \lambda) = e_G = p(\mu, l) \) for every \( r \in R \) and every \( l \in L \). Then \( p(r, l) = e_G \) for every \( r \in R \) and every \( l \in L \) if and only if the set \( E \) consisting of idempotents of \( S \) is a subsemigroup of \( S \).

Proof. Suppose first that \( p(r, l) = e_G \) for every \( r \in R \) and every \( l \in L \). We claim that \( E = \{(l, e, r): l \in L, r \in R\} \). To see this, let \((a, u, b) \in E\). Then \((a, u, b) = (a, u, b)^2 = (a, u^2, b)\). Therefore, \( u = u^2 \in G \) which implies that \( u = e \). Conversely, it is clear that for \( l \in L \) and \( r \in R \), we have \((l, e, r)^2 = (l, e, r)\). Thus,
the claim is established. Now, \((a, e, b)(x, e, y) = (a, ep(b, x)e, y) = (a, e, y)\) yields that \(E\) is a subsemigroup of \(S\).

Suppose now that \(E\) is a subsemigroup of \(S\). Let \(r \in R\) and \(l \in L\). We have that \((\lambda, e, r)(\lambda, e, r) = (\lambda, e, r) \in E\) and \((l, e, \mu)(l, e, \mu) = (l, e, \mu) \in E\). Since \(E\) is a subsemigroup of \(S\), we have that \((\lambda, e, r)(l, e, \mu) = (\lambda, p(r, l), \mu) \in E\). Hence, we have that \((\lambda, p(r, l), \mu)^2 = (\lambda, p(r, l)^2, \mu) = (\lambda, p(r, l), \mu)\), which implies that \(p(r, l)^2 = p(r, l) \in G\). Thus, \(p(r, l) = e\), and the proof is complete.

6.20 Corollary. Let \(S\) be a completely simple semigroup with Schützenberger group \(G\). Then \(S\) has the congruence extension property (CEP) if and only if \(G\) has the congruence extension property and \(E\) is a subsemigroup of \(S\).

Proof. Realize \(S = [L, G, R]_p\), and normalize \(p\) as in Remark 6.5. Combining Theorem 6.15 and Proposition 6.19 yields the needed result.

As mentioned in Chapter 1, it remains an unsolved problem to determine whether the homomorphic image of a semigroup with the congruence extension property (CEP) retains CEP. However, in some cases it is possible to exploit a characterization of a class of semigroups with CEP to answer the homomorphic question affirmatively. The following corollary does this for completely simple semigroups.

6.21 Corollary. The homomorphic image of a completely simple semigroup with the congruence extension property has the congruence extension property.

Proof. Let \(S = [L, G, R]_p\) be a completely simple semigroup with CEP. Normalize \(p\) as in Remark 6.5. Then by Theorem 6.15, \(p(r, l) = e_G\) for every \(r \in R\) and every \(l \in L\), and by Proposition 6.19, \(E\) is a subsemigroup of \(S\).
Let \( \theta : S \to S^* \) be a homomorphism of \( S \) onto a semigroup \( S^* \). Then \( S^* \) is completely simple. Let \( S^* = [L^*, G^*, R^*] \). By Theorem 6.3, there exist mappings \( \gamma : L \to G^* \), \( \delta : R \to G^* \), onto mappings \( \phi : L \to L^* \), \( \psi : R \to R^* \), and an onto homomorphism \( \omega : G \to G^* \) such that \( \theta(l, g, r) = [\phi(l), \gamma(l)\omega(g)\delta(r), \psi(r)] \) and also \( \omega \circ p(r, l) = \delta(r)p^*(\psi(r), \phi(l))\gamma(l) \), for \( l \in L \) and \( r \in R \). Then we obtain

\[
e_{G^*} = \omega(e_G) = \omega \circ p(r, l) = \delta(r)p^*(\psi(r), \phi(l))\gamma(l), \quad \text{for } l \in L \text{ and } r \in R.
\]

To show that \( S^* \) has CEP, it suffices to show that \( G^* \) has CEP and that \( E_{G^*} \) is a subsemigroup of \( S^* \). That \( G^* \) has CEP follows from the fact that \( G^* \) is the homomorphic image of the group \( G \), which has CEP. We will show that \( E_{G^*} \) is a subsemigroup of \( S^* \). We know that \( E_S = \{(l, e, r): l \in L, r \in R\} \). One easily checks that \( \theta(E) = \{[\phi(l), \gamma(l)\delta(r), \psi(r)]: l \in L, r \in R\} \). Also, \( \theta(E) \) is a subsemigroup of \( S^* \), since \( E \) is a subsemigroup of \( S \) and \( \theta \) is a homomorphism.

We claim \( E_{G^*} = \{[\phi(l), \gamma(l)\delta(r), \psi(r)]: l \in L, r \in R\} \). In order to see this, let \( f = [\phi(l), \gamma(l)\omega(g)\delta(r), \psi(r)] \in E_{G^*} \). Then

\[
f^2 = [\phi(l), \gamma(l)\omega(g)\delta(r)p^*(\psi(r), \phi(l))\gamma(l)\omega(g)\delta(r), \psi(r)]
\]

\[
= [\phi(l), \gamma(l)\omega(g)^2\delta(r), \psi(r)].
\]

Since \( f^2 = f \), we have that \( \gamma(l)\omega(g)\delta(r) = \gamma(l)\omega(g)^2\delta(r) \). Since \( G^* \) is a group, we have that \( \omega(g) = \omega(g)^2 \) which implies that \( \omega(g) = e_{G^*} \). Therefore, we have that \( f = [\phi(l), \gamma(l)\delta(r), \psi(r)] \), as desired. To see the reverse inclusion, observe that

\[
[\phi(l), \gamma(l)\delta(r), \psi(r)]^2 = [\phi(l), \gamma(l)\delta(r)p^*(\psi(r), \phi(l))\gamma(l)\delta(r), \psi(r)]
\]

\[
= [\phi(l), \gamma(l)\delta(r), \psi(r)].
\]

The claim is established. Hence, \( E_{G^*} \) is a subsemigroup of \( S^* \), and \( S^* \) has CEP.
6.22 Proposition. Let \( \{S_\alpha : \alpha \in A \} \) be a family of completely simple semigroups. Let \( S = \prod \{S_\alpha : \alpha \in A \} \). Then \( S \) is a completely simple semigroup.

Proof. We will show that \( S \) contains a primitive idempotent and that \( S \) is simple. For each \( \alpha \in A \), let \( e_\alpha \) denote a primitive idempotent of \( S_\alpha \). Consider the element \( e \in S \) such that \( \pi_\alpha(e) = e_\alpha \). Then \( e \) is an idempotent of \( S \), as multiplication in \( S \) is coordinate-wise multiplication. Suppose there exists \( f \in E(S) \) with \( f \leq e \). Then for each \( \alpha \in A \), \( f_\alpha \leq e_\alpha \). Thus, for each \( \alpha \in A \), we have that \( f_\alpha = e_\alpha \). Hence, \( e = f \) and \( e \) is primitive. To see that \( S \) is simple, let \( I \) be an ideal of \( S \). Then \( I = \prod_{\alpha \in A} I_\alpha \), where \( I_\alpha \) is an ideal of \( S_\alpha \) for each \( \alpha \in A \). Since each \( S_\alpha \) is simple, we have that \( I_\alpha = S_\alpha \) for each \( \alpha \in A \). Therefore, \( I = S \) and \( S \) is simple.

Due to the above characterization and the fact that for an arbitrary semilattice of groups \( S \) we always have that \( E(S) \) is a subsemigroup of \( S \), we conjecture the following: A semilattice of groups has CEP if and only if each group has CEP. More generally, since every completely simple semigroup is a special-case band of groups, we conjecture that an arbitrary band of groups \( S \) has CEP if and only if \( E(S) \) is a subsemigroup of \( S \), \( E(S) \) has CEP, and each of the groups has CEP.
CHAPTER 7

TOPOLOGICAL RESULTS

In this chapter, the concepts of the preceding chapters are examined from a topological perspective. Topological analogues of the previous results are provided where appropriate. Other related topics are also discussed. For an introductory treatment of topological semigroups, see [Carruth, Hildebrant, and Koch, 1983].

A topological semigroup is a semigroup \( S \) with a Hausdorff topology such that the associative multiplication in \( S \) is continuous in the product topology of \( S \times S \). The condition that multiplication on \( S \) is continuous is equivalent to the condition that for each \( x, y \in S \) and each open set \( W \) with \( xy \in W \), there exists open set \( U \) and \( V \) such that \( x \in U \), \( y \in V \), and \( UV \subseteq W \). Note that any semigroup can be made into a topological semigroup by giving it the discrete topology.

A topological group is a group \( G \) with a Hausdorff topology such that multiplication and inversion are each continuous. The condition that inversion is continuous is equivalent to the condition that for each \( g \in G \) and each open set \( U \) with \( G \subseteq U \), there is an open set \( V \subseteq U \) such that \( g^{-1} \in V^{-1} \subseteq U^{-1} \).

We will use the convention that if the word “semigroup” [or “group”] is used with any topological modifier, then “topological semigroup” [or “topological group”] is understood.

A topological semigroup \( S \) is said to have the congruence extension property (CEP) provided that for each closed subsemigroup \( T \) of \( S \) and each closed congruence \( \sigma \) on \( T \), \( \sigma \) has a closed extension to \( S \).
The following topological results are well-known and will be used without further reference:

1. Suppose $X$ is a compact space, $Y$ is a Hausdorff space, and $f: X \to Y$ is a one-to-one, continuous, onto function. Then $f$ is a homeomorphism.

2. A closed subspace of a compact space is compact.

3. The continuous image of a compact space is compact.

4. A compact subspace of a Hausdorff space is closed.

5. The Tychonoff Theorem. Let $\{X_\alpha: \alpha \in A\}$ be a collection of non-empty topological spaces. Then $\prod\{X_\alpha: \alpha \in A\}$ is compact if and only if $X_\alpha$ is compact for each $\alpha \in A$.


7. Each compact semigroup contains a compact minimal [left,right] ideal.

8. Every compact simple semigroup is completely simple.

9. Closed subsemigroups of compact groups are subgroups.

The results in this chapter employ techniques using nets. For this reason, a short discussion of nets is included here.

A directed set $(D, \leq)$ is a set $D$ together with a relation $\leq$ on $D$ such that:

(i) $\alpha \leq \alpha$ for each $\alpha \in D$ (reflexive);

(ii) If $\alpha \leq \beta$ and $\beta \leq \gamma$ in $D$, then $\alpha \leq \gamma$ (transitive); and

(iii) If $\alpha, \beta \in D$, then there is $\gamma \in D$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$ (directed property).

A net in a topological space $X$ is a function $x: D \to X$ from a directed set $D$ into $X$. We use $x_\alpha$ to denote the image of $\alpha$ under $x$ and $\{x_\alpha\}_{\alpha \in D}$ to denote the net. When no confusion is likely, the directed set $D$ is not mentioned; that is, we use $\{x_\alpha\}$ (or simply $x_\alpha$) to denote the net.
If \( \{x_\alpha\}_{\alpha \in D} \) is a net in \( X \) and \( A \subseteq X \), then \( x_\alpha \) is eventually in \( A \) provided there exists \( \alpha \in D \) such that \( x_\beta \in A \) whenever \( \alpha \leq \beta \). This is denoted \( x_\alpha \in^e A \).

If \( \{x_\alpha\}_{\alpha \in D} \) is a net in \( X \) and \( A \subseteq X \), then \( x_\alpha \) is frequently in \( A \) provided for each \( \beta \in D \), there exists \( \gamma \in D \) such that \( \beta \leq \gamma \) and \( x_\gamma \in A \). This is denoted \( x_\alpha \in^f A \).

If \( \{x_\alpha\}_{\alpha \in D} \) is a net in \( X \) and \( p \in X \), then \( x_\alpha \) converges [clusters] to \( p \) provided \( x_\alpha \) is eventually [frequently] in each neighborhood of \( p \). This is denoted \( x_\alpha \xrightarrow{e} p \) or \( x_\alpha \xrightarrow{f} p \).

If \( E \) and \( D \) are directed sets, then a function \( \lambda : E \to D \) is cofinal in \( D \) if for each \( \delta \in D \), there exists \( \epsilon \in E \) such that \( \delta \leq \lambda(\epsilon) \) in \( D \) whenever \( \epsilon \leq \alpha \) in \( E \).

If \( \{x_\alpha\}_{\alpha \in D} \) is a net in \( X \), a net \( \{y_\beta\}_{\beta \in E} \) is a subnet of \( \{x_\alpha\}_{\alpha \in D} \) provided that there is a cofinal function \( \lambda : E \to D \) such that \( y_\beta = x_{\lambda(\beta)} \) for each \( \beta \in E \).

The following are well-known theorems about nets:

1. A space \( X \) is Hausdorff if and only if each net in \( X \) has at most one point of convergence.
2. Let \( E \) be a subset of a space \( X \). These are equivalent:
   a. \( E \) is closed;
   b. If \( x_\alpha \) is a net in \( E \) and \( x_\alpha \xrightarrow{e} p \), then \( p \in E \); and
   c. If \( x_\alpha \) is a net in \( E \) and \( x_\alpha \xrightarrow{f} p \), then \( p \in E \).
3. Let \( x_\alpha \) be a net in a space \( X \) with \( p \in X \). Then \( x_\alpha \xrightarrow{f} p \) if and only if there exists a subnet \( y_\beta \) of \( x_\alpha \) such that \( y_\beta \xrightarrow{f} p \).
4. Let \( x_\alpha \) be a net in a space \( X \) with \( p \in X \) such that \( x_\alpha \xrightarrow{e} p \). If \( y_\beta \) is a subnet of \( x_\alpha \), then \( y_\beta \xrightarrow{e} p \).
5. A space \( X \) is compact if and only if each net in \( X \) clusters to a point of \( X \).
(6) Let \( f : X \to Y \) be a function and let \( p \in X \). These are equivalent:

(a) \( f \) is continuous at \( p \);

(b) If \( x_\alpha \xrightarrow{\epsilon} p \), then \( f(x_\alpha) \xrightarrow{\epsilon} f(p) \); and

(c) If \( x_\alpha \xrightarrow{\delta} p \), then \( f(x_\alpha) \xrightarrow{\delta} f(p) \).

The closed congruence on a topological semigroup \( S \) generated by a given relation on \( S \) is the smallest closed congruence on \( S \) which contains the relation. Equivalently, it is the intersection of all closed congruences which contain the relation. If \( \sigma \) is a relation on \( S \), we let \( \sigma^* \) denote the closed congruence generated by \( \sigma \). Clearly, the smallest closed congruence on \( S \) generated by \( \sigma \) contains the smallest congruence on \( S \) generated by \( \sigma \), i.e., \( \langle \sigma \rangle_S \subseteq \sigma^* \). If \( \langle \sigma \rangle_S \) is closed, then \( \langle \sigma \rangle_S = \sigma^* \).

The following are easily verified topological analogues of Chapter 1 results unless otherwise indicated.

7.1 A topological semigroup \( S \) has the congruence extension property if and only if each closed subsemigroup of \( S \) has the congruence extension property.

7.2 Let \( T \) be a closed subsemigroup of a topological semigroup \( S \) and let \( \sigma \) be a closed congruence on \( T \). Then \( \sigma \) extends to a closed congruence on \( S \) if and only if \( \sigma^* \) is an extension of \( \sigma \) to \( S \).

7.3 Let \( S \) be a topological semigroup, \( I \) a closed ideal of \( S \), and \( \sigma \) a closed congruence on \( I \). Then \( \sigma \) extends to a closed congruence on \( S \) if and only if \( \sigma \cup \Delta_S \) is a closed congruence extending \( \sigma \) to \( S \).

Let \( S \) be a topological semigroup and let \( a \in S \). Then \( \Gamma(a) := \overline{\theta(a)} \) where \( \theta(a) = \{ a^n : n \in \mathbb{N} \} \) is called the monotethic subsemigroup of \( S \) generated
by $a$. If $S = \Gamma(a)$ for some $a \in S$, then $S$ is called a monothetic semigroup. If $\Gamma(a)$ is a compact monothetic semigroup, then its minimal ideal $M(\Gamma(a))$ is a compact abelian group and $\Gamma(a) = \theta(a) \cup M(\Gamma(a))$. Furthermore, $M(\Gamma(a))$ consists of the cluster points of $\Gamma(a)$. We define the monothetic index of the element $a$ as follows: $\text{mi}(a) = \min\{n \in \mathbb{N}: a^n \in M(\Gamma(a))\}$ if $\theta(a) \cap M(\Gamma(a)) \neq \emptyset$, and $\text{mi}(a) = \infty$ otherwise. The monothetic index of a semigroup $S$ is defined to be $\text{mi}(S) = \max\{\text{mi}(a) : a \in S\}$ if this maximum exists. Otherwise, $\text{mi}(S) = \infty$.

Recall from Chapter 1 that in the algebraic setting, $\text{index}(a)$ is the least $n \in \mathbb{N}$ such that $a^n \in M(\theta(a))$ if $M(\theta(a)) \neq \emptyset$ and otherwise, $\text{index}(a) = \infty$. Note that in a topological semigroup, we may have $\text{mi}(a) < \infty$ while $\text{index}(a) = \infty$. Also, for an algebraic semigroup, we defined $\text{index}(S) = \max\{\text{index}(a) : a \in S\}$ when this maximum exists and $\text{index}(S) = \infty$, otherwise. We noted that if $\text{index}(S') < \infty$, then $S$ is periodic. However, $\text{mi}(S) < \infty$ does not imply that $S$ is periodic. Hence, in spite of the fact that [Aucoin, 1993] shows that both the congruence extension property (CEP) and the ideal extension property (IEP) imply $\text{mi}(S) \leq 3$ for compact semigroups, we are not able to use certain techniques from previous chapters that employed periodicity. Compactness will often be used to justify statements that were previously justified by periodicity.

7.4 [Aucoin, 1993] Each compact semigroup $S$ with the congruence extension property (CEP) has monothetic index less than 4.

If $S$ is a topological semigroup and $T$ is a closed subsemigroup of $S$, then $S$ is said to have the congruence extension property relative to $T$ provided that every closed congruence on $T$ has an extension to $S$. If $\mathcal{K}$ is a class of closed subsemigroups of a topological semigroup $S$ such that every closed congruence $\sigma$ on
every member \( T \in \mathcal{K} \) has an extension to \( S \), then we say that \( S \) has the congruence extension property relative to the class \( \mathcal{K} \).

If \( M \) is a closed subsemigroup of a topological semigroup \( S \) and \( \phi: S \to M \) is a continuous homomorphism of \( S \) onto \( M \) such that \( \phi|_M = 1_M \) (the identity function on \( M \)), then \( \phi \) is called a homomorphic retraction of \( S \) onto \( M \) and \( M \) is called a homomorphic retract of \( S \).

7.5 Let \( S \) be a topological semigroup. Then \( S \) has the congruence extension property relative to closed homomorphic retractions of \( S \).

7.6 Let \( S \) be a commutative topological semigroup with a closed group minimal ideal \( M(S) \). Then \( S \) has the congruence extension property relative to \( M(S) \).

7.7 Corollary. Let \( S \) be a commutative compact semigroup. Then \( S \) has the congruence extension property (CEP) relative to \( M(S) \).

A topological semigroup \( S \) is said to have the ideal extension property (IEP) provided that for each closed subsemigroup \( T \) of \( S \) and each closed ideal \( I \) of \( T \), there exists a closed ideal \( J \) of \( S \) such that \( J \cap T = I \).

7.8 A topological semigroup \( S \) has the ideal extension property (IEP) if and only if each closed subsemigroup of \( S \) has the ideal extension property.

7.9 Let \( S \) be a compact semigroup, \( T \) a closed subsemigroup of \( S \) and \( I \) a closed ideal of the subsemigroup \( T \). Then there exists a closed ideal \( J \) of \( S \) such that \( J \cap T = I \) if and only if \( S^1 IS^1 \cap T = I \). Moreover, for \( x \in T \), there is a closed ideal of \( S \) extending \( T^1 x T^1 \) to \( S \) if and only if \( S^1 x S^1 \) extends \( T^1 x T^1 \) to \( S \).
7.10 A continuous homomorphic image of a compact semigroup with the ideal extension property (IEP) has IEP.

7.11 [Aucoin, 1993] Each compact semigroup S with the ideal extension property (IEP) has monothetic index less than 4.

Let S be a topological semigroup and let $a \in S$. We let $J_S(a)$ denote the ideal of S generated by the element a, that is, $J_S(a) = S^1aS^1 = \{a\} \cup aS \cup Sa \cup SaS$. We let $J^*_S(a)$ denote the closed ideal on S generated by a. If S is compact, we clearly have that $J^*_S(a) = J_S(a)$, as $S^1aS^1$ is compact and therefore closed.

Similarly, for a subsemigroup $T$ of $S$, and $a \in T$, we let $J_T(a)$ denote the ideal of T generated by a, that is, $J_T(a) = T^1aT^1 = \{a\} \cup aT \cup Ta \cup TaT$. We note that if S is compact and $T$ is closed, then $J_T(a) = J^*_T(a)$.

A topological semigroup S is said to have the principal ideal extension property (PIEP) if for each closed subsemigroup $T$ of $S$ and each element $a \in T$, $J^*_T(a) = J^*_S(a) \cap T$.

An element $a$ of a topological semigroup $S$ is said to be a disruptive element provided that there exists a closed subsemigroup $T$ of $S$ such that $a \in T$ and $J^*_T(a) \subset J^*_S(a) \cap (T \times T)$, where $\subset$ denotes proper containment. For a disruptive element $a \in S$ and the closed subsemigroup $T$ for which $J^*_T(a) \subset J^*_S(a) \cap (T \times T)$, we say $a$ is disruptive in $T$.

7.12 For compact semigroup $S$, the following are equivalent:

(1) $S$ has the ideal extension property (IEP);

(2) $S$ has the principal ideal extension property (PIEP); and

(3) $S$ contains no disruptive elements.
7.13 Let $S$ be a commutative topological semigroup, and let $T$ be a closed subsemigroup of $S$. Then no regular element of $T$ is disruptive in $T$.

7.14 In a commutative topological semigroup, no idempotent is disruptive.

7.15 A topological semilattice has the ideal extension property (IEP).

7.16 Note. In [Stralka, 1977] an example of a compact semilattice without CEP is given.

Let $(S, \leq)$ be a partially ordered topological semigroup. We say a congruence $\sigma$ on $S$ is monotone provided that $a \sigma b$ and $a \leq c \leq b$ imply that $a \sigma c$.

7.17 Proposition. Let $(S, \leq)$ be a partially ordered topological semigroup such that every closed subsemigroup of $S$ has a least upper bound and a greatest lower bound. Then closed monotone congruences on closed subsemigroups have closed extensions to $S$.

Proof. Let $(S, \leq)$ be a partially ordered topological semigroup such that every closed subsemigroup of $S$ has a least upper bound and a greatest lower bound. Suppose that $T$ is a closed subsemigroup of $S$ and that $\sigma$ is a closed monotone congruence on $T$. Let $T/\sigma$ denote the set of congruence classes of $\sigma$. Then for each $\bar{t} \in T/\sigma$,

\[
\text{let } a_{\bar{t}} = \text{glb}\{x \in T : x \sigma t\},
\]

and $b_{\bar{t}} = \text{lub}\{y \in T : t \sigma y\}$.

Since $T$ and $\sigma$ are closed, $a_{\bar{t}}, b_{\bar{t}} \in T$ and $a_{\bar{t}} \sigma t$ and $b_{\bar{t}} \sigma t$ for each $\bar{t} \in T/\sigma$. Then by transitivity of $\sigma$, $a_{\bar{t}} \sigma b_{\bar{t}}$. Since $\sigma$ is monotone, $[a_{\bar{t}}, b_{\bar{t}}] \cap T = \bar{t}$. 
Define
\[ \bar{\sigma} = \bigcup_{\bar{t} \in T/\sigma} ([a_{\bar{t}}, b_{\bar{t}}] \times [a_{\bar{t}}, b_{\bar{t}}]) \cup \Delta_S. \]

Then clearly \( \bar{\sigma} \) is a congruence on \( S \), and \( \bar{\sigma} \cap (T \times T) = \sigma \). We claim that \( \bar{\sigma} \) is closed. To see this, let \((x_\lambda, y_\lambda)\) be a net in \( \bar{\sigma} \) such that \((x_\lambda, y_\lambda) \xrightarrow{\sigma} (x, y)\). Thus, \( x_\lambda \xrightarrow{\sigma} x \) and \( y_\lambda \xrightarrow{\sigma} y \). Now, either there exist \( \bar{t} \in T/\sigma \) with \( x \in [a_{\bar{t}}, b_{\bar{t}}] \) or \( x \not\in [a_{\bar{t}}, b_{\bar{t}}] \) for every \( \bar{t} \in T/\sigma \). We consider these two cases.

Case 1. Suppose \( x \not\in [a_{\bar{t}}, b_{\bar{t}}] \) for every \( \bar{t} \in T/\sigma \).

Since \( x_\lambda \xrightarrow{\sigma} x \), we have that \( x_\lambda \in S \setminus [a_{\bar{t}}, b_{\bar{t}}] \). However, \( x_\lambda \sigma y_\lambda \) for all \( \lambda \); thus, \( x_\lambda = y_\lambda \). Hence, \( x = y \), and \( x\bar{\sigma}y \).

Case 2. Suppose there exist \( \bar{t} \in T/\sigma \) with \( x \in [a_{\bar{t}}, b_{\bar{t}}] \).

Subcase a. Suppose \( a_t \leq x \leq b_t \).

Since \( x_\lambda \xrightarrow{\sigma} x \), we have \( x_\lambda \in (a_t, b_t) \). Then \( x_\lambda \bar{\sigma}y_\lambda \) yields that \( y_\lambda \in (a_t, b_t) \).

Subcase b. Suppose \( x = a_t \).

Then either \( a_t <^f x_\lambda <^f b_t \) or \( x_\lambda <^f a_t \).

If \( a_t <^f x_\lambda <^f b_t \), then \( a_t <^f y_\lambda <^f b_t \) since \( x_\lambda \bar{\sigma}y_\lambda \). Thus, \( y_\lambda \xrightarrow{\sigma} y \) implies that \( a_t \leq y \leq b_t \). Hence, \( x\bar{\sigma}y \). If \( x_\lambda <^f a_t \), then \( x_\lambda \bar{\sigma}y_\lambda \) implies that \( x_\lambda =^f y_\lambda \). Hence, \( x = y \) and \( x\bar{\sigma}y \).

Subcase c. Suppose \( x = b_t \).

Then either \( a_t <^f x_\lambda <^f b_t \) or \( x_\lambda >^f b_t \).

If \( a_t <^f x_\lambda <^f b_t \), we conclude \( x\bar{\sigma}y \) as above. If \( x_\lambda >^f b_t \), then \( x_\lambda =^f y_\lambda \) and \( x = y \). Hence, \( x\bar{\sigma}y \).

Thus, \( \bar{\sigma} \) is closed, and the proof is complete. \( \square \)
Let \( I_m = [0,1] \) with multiplication given by \( xy = \min\{x,y\} \), for \( x, y \in I_m \). Then \( I_m \) is a commutative, partially ordered, compact, topological semigroup with the usual order and topology of the unit interval.

**7.18 Corollary.** The min interval \( I_m \) has the congruence extension property.

**Proof.** We first note that \( I_m \) is a partially ordered, compact semigroup. Thus, each closed subsemigroup of \( I_m \) has a least upper bound and a greatest lower bound. By Proposition 7.17, every closed monotone congruence on every closed subsemigroup of \( I_m \) has a closed extension to \( I_m \).

We claim that every closed congruence on every closed subsemigroup of \( I_m \) is a monotone congruence. To see this, let \( T \) be a closed subsemigroup of \( I_m \) and let \( \sigma \) be a closed congruence on \( T \). Suppose that \( a \sigma b \) and that \( a \leq c \leq b \). Then \( ac = a \) and \( bc = c \). Thus, by the compatibility of \( \sigma \), \( a \sigma c \). Therefore, \( \sigma \) is monotone, and \( I_m \) has CEP. \( \blacksquare \)

A topological group \( G \) is said to have the group congruence extension property (GCEP) provided that for each closed subgroup \( H \) of \( G \) and each closed congruence \( \sigma \) on \( H \), there exists a closed extension of \( \sigma \) to \( G \), i.e., there exists a closed congruence \( \bar{\sigma} \) on \( G \) such that \( \bar{\sigma} \cap (H \times H) = \sigma \).

It is immediate that a topological group with the congruence extension property will also have the group congruence extension property. The converse, however, is not true.

**7.19** Let \( G \) be a topological group and \( \sigma \) a closed congruence on \( G \). Then there exists a closed normal subgroup \( N \) of \( G \) such that \( (a, b) \in \sigma \) if and only if \( ab^{-1} \in N \).
7.20 Proposition. Let $G$ be a compact group. Then $G$ has the congruence extension property (CEP) if and only if for each closed subgroup $H$ of $G$ and each closed normal subgroup $N$ of $H$, there exists a closed normal subgroup $M$ of $G$ such that $N = M \cap H$.

Proof. Let $G$ be a compact group with identity $e$. Suppose that $G$ has CEP. Let $H$ be a closed subgroup of $G$, and let $N$ be a closed subgroup of $H$. Define $\sigma = \{(x,y) \in H \times H : xy^{-1} \in N\}$. Then $\sigma$ is a closed congruence on $H$. Since $G$ has CEP, there exists a closed extension $\bar{\sigma}$ of $\sigma$ to $G$. By 7.19, there exists a closed normal subgroup $M$ of $G$ such that $(x,y) \in G$ if and only if $xy^{-1} \in M$. We claim $N = M \cap H$. To see this, let $x \in N$. Now, $e \in H$; thus, $x = xe = xe^{-1} \in N$. Hence, $(x,e) \in \sigma \subseteq \bar{\sigma}$. Therefore, $xe^{-1} = x \in M$. Since $N \subseteq H$, we have $x \in M \cap H$. Conversely, let $x \in M \cap H$. Then $x = xe^{-1} \in M$, and $(x,e) \in \bar{\sigma}$. Since $x,e \in H$, $(x,e) \in \bar{\sigma} \cap (H \times H) = \sigma$. Thus, $x = xe^{-1} \in N$.

To see the converse, let $H$ be a closed subsemigroup of $G$ and let $\sigma$ be a closed congruence on $H$. Then since $G$ is compact, $H$ is a closed (and therefore compact) subgroup of $G$. Let $N$ be the closed subgroup of $H$ corresponding to $\sigma$, as guaranteed by 7.19. Therefore, $(x,y) \in \sigma$ if and only if $xy^{-1} \in N$. By assumption, there exists a closed normal subgroup $M$ of $G$ such that $N = M \cap H$. Define $\bar{\sigma} = \{(a,b) \in G \times G : ab^{-1} \in M\}$. Then one easily verifies that $\bar{\sigma}$ is a closed congruence on $G$ and that $\sigma = \bar{\sigma} \cap (H \times H)$.

7.21 Corollary. Every compact abelian group $G$ has the congruence extension property (CEP).

Proof. This is immediate from Proposition 7.20.
7.22 Remark. Topological CEP is different from algebraic CEP. Consider the compact abelian circle group \( C \). This group has CEP topologically by 7.21. However, it does not have CEP algebraically by 1.28, since it is not torsion.

The following are direct analogues of the results found in Chapter 2.

Recall that if \( \sigma \) is any relation on \( S \), we denote by \( \sigma^* \) the closed congruence generated by \( \sigma \). Clearly, \( (\sigma)_S \subseteq \sigma^* \).

We have that \( \mathcal{C}(S) = \{ \sigma : \sigma \text{ is a closed congruence on } S \} \) forms a lattice under the operations meet (denoted \( \wedge \)) and join (denoted \( \vee \)) defined by:

\[
\sigma \wedge \rho = (\sigma \cap \rho)^* \quad \text{and} \quad \sigma \vee \rho = (\sigma \cup \rho)_S^* \quad \text{for } \sigma, \rho \in \mathcal{C}(S).
\]

Let \( S \) be a topological semigroup. Let \( \mathcal{L}_S \) denote the lattice of closed congruences on \( S \). Similarly, for each closed subsemigroup \( T \) of \( S \), let \( \mathcal{L}_T \) denote the lattice of closed congruences on \( T \), and let \( X_T = \{ (\sigma)_S^* : \sigma \in \mathcal{L}_T \} \).

7.23 Proposition. Let \( S \) be a topological semigroup. Then \( S \) has the congruence extension property (CEP) if and only if for every closed subsemigroup \( T \) of \( S \), the map \( \phi_T : \mathcal{L}_T \to X_T \) defined by \( \sigma \mapsto (\sigma)_S^* \) is a bijective correspondence.

Proof. This is a direct analog of Proposition 2.1. \( \blacksquare \)

7.24 Note. The map \( \phi_T \) in Proposition 7.23 is join preserving although not necessarily meet preserving.

Proof. This is a direct analog of Note 2.2 and Example 2.3. \( \blacksquare \)

Let \( S \) be a topological semigroup. Let \( \mathcal{I}_S \) denote the lattice of closed ideals of \( S \). Likewise, for each closed subsemigroup \( T \) of \( S \), let \( \mathcal{I}_T \) denote the lattice of closed ideals of \( T \), and let \( Y_T = \{ S^1 I S^1 : I \text{ is an ideal of } T \} \).
7.25 Proposition. Let $S$ be a compact semigroup. Then $S$ has the ideal extension property if and only if for each closed subsemigroup $T$ of $S$, the map $\psi_T: \mathcal{I}_T \to \mathcal{Y}_T$ defined by $I \mapsto S^1 IS^1$ is a bijective correspondence.

Proof. This is a direct analog of Proposition 2.5.

7.26 Note. The map $\psi_T$ in Proposition 7.25 is join preserving but not necessarily meet preserving.

Proof. This is a direct analog of Note 2.6 and Example 2.7.

The following results are topological analogues of results found in Chapter 3.

A topological semigroup $S$ is called a t-semigroup if the relation “is a closed ideal of” is transitive among the closed ideals of $S$.

7.27 Proposition. Every continuous homomorphic image of a compact t-semigroup is a compact t-semigroup.

Proof. This is a direct analog of Proposition 3.1.

7.28 Proposition. Suppose $\{S_\alpha : \alpha \in A\}$ is a family of compact semigroups. Let $S = \prod_{\alpha \in A} S_\alpha$. Then $S$ is compact, and $S$ is a t-semigroup if and only if $S_\alpha$ is a t-semigroup for each $\alpha \in A$.

Proof. This is a direct analog of Proposition 3.2.

7.29 Note. A closed subsemigroup of a compact t-semigroup need not be a t-semigroup.

Proof. This is a direct analog of Example 3.3.
7.30 Proposition. Each compact semigroup with the ideal extension property (IEP) is a compact $t$-semigroup.

Proof. This is a direct analog of Proposition 3.5. 

Let $S$ be a topological semigroup. Recall that $a \in S$ is called a disruptive element provided there exists a closed subsemigroup $T$ of $S$ such that $a \in T$ and $J^*_{T}(a) \subset J^*_{S}(a) \cap (T \times T)$, where $\subset$ denotes proper containment. For a disruptive element $a \in S$ and the particular closed subsemigroup $T$ for which $J^*_{T}(a) \subset J^*_{S}(a) \cap (T \times T)$, we say $a$ is disruptive in $T$.

Recall also that an element $r$ in a semigroup $S$ is called a regular element provided there exists $t \in S$ such that $rtr = r$. The element $t$ is called an inverse for $r$. A semigroup $S$ is said to be a regular semigroup provided every element of $S$ is a regular element.

7.31 Lemma. Let $S$ be a topological semigroup, and let $r$ be a regular element of $S$. Let $I$ be a closed ideal of $S$ such that $r \in I$. Then $r$ is not disruptive in $I$.

Proof. This is a direct analog of Lemma 3.10.

7.32 Corollary. Each regular compact semigroup is a $t$-semigroup.

Proof. Let $S$ be a regular compact semigroup. Let $I$ be a closed ideal of $S$, and let $K$ be a closed ideal of $I$. Then $I$ is compact and $K$ is compact. Hence, $K$ is closed in $S$. Now,

$$K = \bigcup_{x \in K} J_I(x) = \bigcup_{x \in K} [J_S(x) \cap I] = \bigcup_{x \in K} J_S(x),$$

since $x \in K \subseteq I$ and $x$ regular implies that $x$ is not disruptive in $I$. Thus, $K$ is a
union of ideals of $S$ and is hence an ideal of $S$. Since $K$ is a closed ideal of $S$, $S$ is a t-semigroup.  

**7.33 Corollary.** Each compact band is a t-semigroup.

**Proof.** This is immediate from Corollary 7.32.

**7.34 Note.** Theorem 3.15 stating that a semigroup $S$ having CEP implies that $S$ is a t-semigroup does not topologize directly. This is due to the fact that the characterization of the congruence generated by a pair does not directly topologize, even in the compact case.

The following results are topological analogues of results found in Chapter 4.

We say that a topological semigroup $S$ is an m-semigroup provided that for every closed subsemigroup $T$ of $S$, there exists a closed ideal $J$ of $S$ such that $T$ is a closed ideal of $J$, or equivalently (for a compact semigroup $S$), $T$ is a closed ideal of $S^{1}TS^{1}$.

**7.35 Lemma.** Let $S$ be a compact semigroup and let $T$ be a closed subsemigroup of $S$. Then there exists a closed ideal $J$ of $S$ such that $T$ is a closed ideal of $J$ if and only if $T$ is a closed ideal of $S^{1}TS^{1}$.

**Proof.** This is a direct analog of Lemma 4.1.

**7.36 Lemma.** If $S$ is a compact m-semigroup, then every closed subsemigroup of $S$ is an m-semigroup.

**Proof.** This is a direct analog of Lemma 4.2.
7.37 Lemma. Let $S$ be a compact $m$-semigroup. Let $\phi: S \to \hat{S}$ be a continuous homomorphism from $S$ onto a semigroup $\hat{S}$. Then $\hat{S}$ is a compact $m$-semigroup.

Proof. This is a direct analog of Lemma 4.3. 

7.38 Note. The product of compact $m$-semigroups is not, in general, an $m$-semigroup. See Example 4.27 for a finite counterexample. Proposition 7.44 shows that the product $S$ of commutative topological semigroups $S_\alpha$ with $m(S_\alpha) \leq 3$ is a compact $m$-semigroup if and only if each $S_\alpha$ is a compact $m$-semigroup.

7.39 Theorem. Let $S$ be a compact $m$-semigroup. Then $m(S) \leq 5$ and $E(S) = \{0\}$.

Proof. Let $S$ be a compact $m$-semigroup. Then $S$ has a compact group minimal ideal $M(S)$. We claim that $M(S) = \{0\}$. We first show $E(S) = \{0\}$. Let $e \in E$. Then $T = \{e\}$ is a closed subsemigroup of $S$. Since $S$ is a compact $m$-semigroup, $(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T$. Hence, $(S^1eS^1)e(S^1eS^1)^1 = e$. Thus,

$$xe = xe^2 \in (S^1eS^1)e(S^1eS^1)^1 = e,$$

$$ex = e^2x \in (S^1eS^1)e(S^1eS^1)^1 = e,$$

for all $x \in S$, and $e$ is a zero for $S$. Thus, $E(S) = \{0\}$. Now, we have that $M(S)$ is a compact group containing a zero. Hence, $M(S) = \{0\}$.

Let $a \in S$. We now claim that $mS(a) \leq 5$. Now, $\theta(a) = \{a^n : n \in \mathbb{N}\}$ is a subsemigroup of $S$, and $\Gamma(a) = \overline{\theta(a)}$ is a closed and therefore compact subsemigroup of $S$. Now, $\theta(a^2) = \{a^{2k} : k \in \mathbb{N}\}$ is a subsemigroup of $\theta(a)$. Thus, $\Gamma(a^2)$ is a closed subsemigroup of $\Gamma(a)$. By Lemma 7.36, we have that $\Gamma(a)$ is a compact $m$-semigroup.
Thus, $[\Gamma(a) \Gamma(a^2) \Gamma(a)^1] \cdot \Gamma(a^2) \cdot [\Gamma(a) \Gamma(a^2) \Gamma(a)^1] \subseteq \Gamma(a^2)$. Hence,

$$a^5 = a a^2 a^2 \in [\Gamma(a) \Gamma(a^2) \Gamma(a)^1] \cdot \Gamma(a^2) \cdot [\Gamma(a) \Gamma(a^2) \Gamma(a)^1] \subseteq \Gamma(a^2) = \theta(a^2) \cup M(\Gamma(a)).$$

Since $a^5 \not\in \theta(a^2)$, we conclude that $a^5 \in M(\Gamma(a)) = \{0\}$. Therefore, we have $\theta(a) = \{a, a^2, a^3, a^4, a^5 = 0\}$ and $\theta(a) \cap M(\Gamma(a)) \neq \emptyset$. Hence, we obtain that $\text{mi}(a) = \min\{n \in \mathbb{N}: a^n \in M(\Gamma(a))\} = \min\{n \in \mathbb{N}: a^n \in \{0\}\} \leq 5$.

Since $\text{mi}(a) \leq 5$ for all $a \in S$, we have that $\text{mi}(S) \leq 5$.

7.40 Note. For a compact $m$-semigroup $S$, the concepts of index and monothetic index are equivalent. Indeed, let $S$ be a compact $m$-semigroup. Then by Theorem 7.39, $E(S) = 0$ and $\text{mi}(S) \leq 5$. Hence, we have that $M(S) = 0$, Thus for $a \in S$, $\text{mi}(a) = \min\{n \in \mathbb{N}: a^n \in M(\Gamma(a))\} = \min\{n \in \mathbb{N}: a^n = 0\} = \text{index}(a)$.

7.41 Corollary. Suppose $S$ is a compact $m$-semigroup. Then $S$ is periodic and $E(S) = 0$.

Proof. This is a direct corollary of Theorem 7.39 and Note 7.40.

7.42 Proposition. Let $S$ be a finite archimedean semigroup with zero. Then $S^3 = 0$ if and only if $S$ is an $m$-semigroup and $\text{index}(S) \leq 3$.

Proof. This is precisely Theorem 4.25.

7.43 Theorem. Let $S$ be a compact archimedean semigroup with zero. Then $S^3 = 0$ if and only if $S$ is an $m$-semigroup and $\text{mi}(S) \leq 3$.

Proof. This is a direct analog of Theorem 4.26.
7.44 Proposition. Let \( \{S_\alpha : \alpha \in A\} \) be a family of compact archimedean semigroups with zero such that \( \text{mi}(S_\alpha) \leq 3 \) for all \( \alpha \in A \). Let \( S = \prod \{S_\alpha : \alpha \in A\} \) with coordinate-wise multiplication. Then \( S \) is compact and \( \text{mi}(S) \leq 3 \). Moreover, \( S \) is an \( m \)-semigroup if and only if \( S_\alpha \) is an \( m \)-semigroup for each \( \alpha \in A \).

Proof. This is a direct analog of Proposition 4.28. 

7.45 Note. Results directly analogous to those in [Stralka, 1972] for compact semigroups are not possible. In fact, an example is given in [Stralka, 1972] of a compact semilattice without CEP. Therefore, results directly analogous to those in Chapter 5 are also not possible.

The results in the following section are topological versions of those in Chapter 6. Instead of topologizing each result in Chapter 6, we topologize only the most serviceable results. We first determine the structure of closed subsemigroups of compact, completely simple semigroups. Then we characterize congruences on such semigroups. Finally, we characterize compact, completely simple semigroups with the congruence extension property (CEP). As a corollary we obtain that the continuous homomorphic image of a compact, completely simple semigroup with CEP retains CEP.

7.46 Theorem. A compact semigroup \( S \) is completely simple if and only if it can be represented as \([L, G, R]_p\), where \( L \) is a compact left trivial semigroup, \( R \) is a compact right trivial semigroup, \( G \) is compact a group, and \( p : R \times L \to G \) is a continuous sandwich function determining the multiplication in \( S \).

Proof. This is the well-known topological version of Theorem 6.1.
7.47 Theorem. [Wallace, 1966] Let $S$ be a compact semigroup, and let $e \in E$. Then $eSe$ is a compact group. Define

$$\rho: (eS \cap E) \times (Se \cap E) \rightarrow eSe \quad \text{by}$$

$$\rho(h,f) = hf,$$

$$\phi: [Se \cap E, eSe, eS \cap E]_\rho \rightarrow S \quad \text{by}$$

$$\phi(f,g,h) = fgh, \quad \text{and}$$

$$\hat{\phi}: S \rightarrow [Se \cap E, eSe, eS \cap E]_\rho \quad \text{by}$$

$$\hat{\phi}(s) = (s(ese)^{-1}, ese, (ese)^{-1}s).$$

Then $\phi$ and $\hat{\phi}$ are mutually inverse topological isomorphisms.

Conversely, if $X$ and $Y$ are nonempty compact Hausdorff spaces, $G$ is a compact group, and $\rho: Y \times X \rightarrow G$ is a continuous function, then $[X,G,Y]_\rho$ is a compact completely simple semigroup.

Proof. A complete proof is given in [Carruth, Hildebrant, and Koch, 1983].

7.48 Theorem. Let $S = [L,G,R]_\rho$ and $S^* = [L^*,G^*,R^*]_{\rho^*}$ be compact completely simple semigroups. Suppose there exist continuous mappings $\gamma: L \rightarrow G^*$, $\delta: R \rightarrow G^*$, $\phi: L \rightarrow L^*$, $\psi: R \rightarrow R^*$, and a continuous homomorphism $\omega: G \rightarrow G^*$ such that $\omega \circ p(r,l) = \delta(r)p^*(\psi(r),\phi(l))\gamma(l)$ for every $r \in R$ and every $l \in L$. Define $\theta: S \rightarrow S^*$ by $\theta(l,g,r) = [\phi(l),\gamma(l)\omega(g)\delta(r),\psi(r)]$. Then $\theta$ is a non-trivial continuous homomorphism of $S$ into $S^*$, and conversely every non-trivial continuous homomorphism of $S$ into $S^*$ is obtained in this way.

Proof. This is a direct topological analog of Theorem 6.3.
7.49 Note. If \( \theta \) is a topological isomorphism from \( S \to S^* \) in Theorem 7.48, then \( \phi, \psi, \) and \( \omega \) are one-to-one and onto, for \( \theta \) clearly induces a one-to-one mapping of the \( \mathcal{R}[\mathcal{L}] \)-classes of \( S \) onto those of \( S^* \).

7.50 Proposition. Let \( S \) be a compact completely simple semigroup. Let \( T \) be a closed subsemigroup of \( S \). Then \( T \) is compact and completely simple.

Proof. Let \( S \) be a compact completely simple semigroup. Then \( S = \bigcup_{I \times \Lambda} H_{i\lambda} \), where \( I \times \Lambda \) is a rectangular band and each \( H_{i\lambda} \) is a compact group. Let \( T \) be a closed (and therefore compact) subsemigroup of \( S \). Then

\[
T = T \cap S = T \cap \bigcup_{I \times \Lambda} H_{i\lambda} = \bigcup_{I \times \Lambda} (T \cap H_{i\lambda}).
\]

Now, each \( H_{i\lambda} \) being compact implies that \( T \cap H_{i\lambda} \) is a closed subgroup of \( H_{i\lambda} \), whenever \( T \cap H_{i\lambda} \neq \emptyset \).

Let

\[
I' = \{i \in I : T \cap H_{i\lambda} \neq \emptyset \text{ for some } \lambda \in \Lambda\}, \quad \text{and}
\]

let \( \Lambda' = \{\lambda \in \Lambda : T \cap H_{i\lambda} \neq \emptyset \text{ for some } i \in I\}. \)

Then \( T = \bigcup_{I' \times \Lambda'} H_{i\lambda} \) is a union of compact groups.

That \( I' \times \Lambda' \) is a rectangular band is the same as in the proof of Lemma 6.6.

Therefore, \( T \) is a rectangular band of groups. \( \blacksquare \)

7.51 Corollary. Let \( S \) be a compact completely simple semigroup. Let \( T \) be a closed subsemigroup of \( S \), and let \( e \in E_S \cap T \). Then

\[
S \cong [Se \cap E_S, eSe, eS \cap E_S]_{\rho_S} \quad \text{and} \quad T \cong [Te \cap E_T, eTe, eT \cap E_T]_{\rho_T}.
\]

Proof. This is immediate from Proposition 7.50 and Theorem 7.48. \( \blacksquare \)
7.52 Lemma. Suppose that $S = L \times G \times R$ with continuous sandwich function $p: R \times L \rightarrow G$, and that $S' = L \times G \times R$ with continuous sandwich function $p': R \times L \rightarrow G$ are compact completely simple semigroups. If there exist continuous mappings $\gamma: L \rightarrow G$ and $\delta: R \rightarrow G$ such that

$$p'(r, l) = \delta(r)p(r, l)\gamma(l)$$

for all $(r, l) \in R \times L,$ then $S$ is topologically isomorphic to $S'$.

Proof. Denote the elements of $S$ by $(l, g, r)$ and the elements of $S'$ by $[l, g, r]$. Define $\phi: S' \rightarrow S$ by $\phi([l, g, r]) = (l, \gamma(l)g\delta(r), r)$. Then $\phi$ is a well-defined, continuous, bijective homomorphism. Since $S$ is compact and $S'$ is Hausdorff, we know that $\phi$ is a homeomorphism, and hence a topological isomorphism.

7.53 Remark. According to Lemma 7.52, given a compact completely simple semigroup $S = L \times G \times R$ with continuous sandwich function $p: R \times L \rightarrow G$, we may “normalize” $p$, so that there is $\lambda \in L$ and $\mu \in R$ such that for every $l \in L$ and every $r \in R$ we have $p(r, \lambda) = e_G = p(\mu, l)$, where $e_G$ is the identity element of the group $G$. For this purpose, fix $\lambda \in L$ and $\mu \in R$. Define $\delta: R \rightarrow G$ by $\delta(r) = p(r, \lambda)^{-1}$, and define $\gamma: L \rightarrow G$ by $\gamma(l) = p(r, l)^{-1}p(\mu, \lambda)$. Then $\gamma$ and $\delta$ are continuous. If we define $p': R \times L \rightarrow G$ by $p'(r, l) = \delta(r)p(r, l)\gamma(l)$, we see that

$$p'(r, \lambda) = p(r, \lambda)^{-1}p(r, \lambda)p(\mu, \lambda)^{-1}p(\mu, \lambda) = e_G$$

for $r \in R$, and

$$p'(\mu, l) = p(\mu, \lambda)^{-1}p(\mu, l)p(\mu, \lambda)^{-1}p(\mu, \lambda) = e_G$$

for $l \in L$.

The results in the following are topological analogues of those in [Tamura, 1960]. Closed congruences on a completely simple semigroup $S = L \times G \times R$ with continuous sandwich function $p: R \times L \rightarrow G$ may be characterized as triples of closed congruences on $L$, $G$, and $R$ satisfying a “matching” condition. That is to
say, a closed congruence \( \sigma \) on \( S \) is characterized as a triple of closed congruences \((\sigma_L, \sigma_G, \sigma_R)\) where \( \sigma_L \) is a closed congruence on \( L \), \( \sigma_G \) is a closed congruence on \( G \), and \( \sigma_R \) is a closed congruence on \( R \) such that

\[
\text{if } (a, x) \in \sigma_L \text{ and } (b, y) \in \sigma_R, \text{ then } (p(b, a), p(y, x)) \in \sigma_G. \quad (*)
\]

Indeed, for a closed congruence \( \sigma \) on \( S \), define

\[
\sigma_L = \{(a, x) \in L \times L: \text{there is } b, y \in R \text{ and } u, v \in G \text{ with } ((a, u, b), (x, b, y)) \in \sigma\},
\]

\[
\sigma_R = \{(b, y) \in R \times R: \text{there is } a, x \in L \text{ and } u, v \in G \text{ with } ((a, u, b), (x, v, y)) \in \sigma\},
\]

and

\[
\sigma_G = \{(u, v) \in G \times G: \text{there is } a, x \in L \text{ and } b, y \in R \text{ with } ((a, u, b), (x, v, y)) \in \sigma\}.
\]

Then these are closed congruences on \( L \), \( R \), and \( G \), respectively and (*) is satisfied.

Conversely, given two independently chosen closed congruences \( \sigma_L \) and \( \sigma_R \) on \( L \) and \( R \) respectively, there is at least one closed congruence \( \sigma_G \) on \( G \) satisfying (*).

Then \((\sigma_L, \sigma_G, \sigma_R)\) determines a closed congruence \( \sigma \) as follows:

\[
((a, u, b), (x, v, y)) \in \sigma \text{ if and only if } (a, x) \in \sigma_L, \quad (u, v) \in \sigma_G, \quad \text{and } (b, y) \in \sigma_R.
\]

Justification of the above characterization follows. By Remark 7.53, we assume that there exists \( \lambda \in L \) and \( \mu \in R \) such that for every \( r \in R \) and every \( l \in L \) we have \( p(r, \lambda) = e = p(\mu, r) \), where \( e \) denotes the identity element of the compact group \( G \).

To see that \( \sigma_L \) (and analogously \( \sigma_R \)) are closed congruences, we first note that \( \sigma_L \) is a congruence from Chapter 6. We only show that \( \sigma_L \) is closed. Let \((a_\alpha, x_\alpha)\) be a net in \( \sigma_L \) such that \((a_\alpha, x_\alpha) \xrightarrow{\alpha} (a, x)\). Then there exist \( b_\alpha, y_\alpha \in R \) and \( u_\alpha, v_\alpha \in G \) such that \((a_\alpha, u_\alpha, b_\alpha), (x_\alpha, v_\alpha, y_\alpha) \in \sigma\). Now, since \( R \) and \( G \) are compact, we have that \( b_\alpha \xrightarrow{f} b \) for some \( b \in R \), \( y_\alpha \xrightarrow{f} y \) for some \( y \in R \), \( u_\alpha \xrightarrow{f} u \) for some \( u \in G \), and \( v_\alpha \xrightarrow{f} v \) for some \( v \in G \). Therefore, \(((a_\alpha, u_\alpha, b_\alpha), (x_\alpha, v_\alpha, y_\alpha)) \xrightarrow{f} ((a, u, b), (x, v, y))\).
Then since \( \sigma \) is closed, by passing to subnets we see that \(((a, u, b), (x, v, y)) \in \sigma \). Thus, \( \sigma_L \) is a closed congruence on \( L \). Similarly, \( \sigma_R \) is a closed congruence on \( R \) and \( \sigma_G \) is a closed congruence on \( G \). We note that (*) is satisfied as in Chapter 6.

Conversely, given independently chosen closed congruences \( \sigma_L \) on \( L \) and \( \sigma_R \) on \( R \), there exists at least one closed congruence \( \sigma_G \) on \( G \) satisfying (*), namely the universal closed congruence, \( G \times G \). We show that \( \sigma \) determined by \((\sigma_L, \sigma_G, \sigma_R)\) is a closed congruence on \( S \). That \( \sigma \) is a congruence on \( S \) satisfying (*) follows easily as in Chapter 6. We wish to see that \( \sigma \) is closed. Let \(((a_\alpha, u_\alpha, b_\alpha), (x_\alpha, v_\alpha, y_\alpha)) \) be a net in \( \sigma \) converging to \((a, u, b), (x, v, y)) \). Then for every \( \alpha \), \((a_\alpha, x_\alpha) \in \sigma_L, (u_\alpha, v_\alpha) \in \sigma_G \), and \((b_\alpha, y_\alpha) \in \sigma_R \). Since each of \( \sigma_L \), \( \sigma_G \), and \( \sigma_R \) are closed, we have that \((a_\alpha, x_\alpha) \xrightarrow{\sigma} (a, x) \in \sigma_L, (u_\alpha, v_\alpha) \xrightarrow{\sigma} (u, v) \in \sigma_G \), and \((b_\alpha, y_\alpha) \xrightarrow{\sigma} (a, x) \in \sigma_R \). Thus, we have \(((a, u, b), (x, v, y)) \in \sigma \). Hence, \( \sigma \) is a closed congruence on \( S \).

We now determine when a compact completely simple semigroup \( S \) has the congruence extension property (CEP).

We remark that left trivial topological semigroups have CEP. Indeed, if \( L \) is a left trivial topological semigroup, then every closed equivalence relation on \( L \) is a closed congruence on \( L \). If \( \sigma_A \) is a closed congruence on a closed subsemigroup \( A \) of \( L \), then \( \sigma_A \cup \Delta_L \) is a closed congruence on \( L \) extending \( \sigma_A \). Analogously, right trivial topological semigroups also have CEP.

7.54 Proposition. Let \( S = [L, G, R]_p \) be a compact completely simple semigroup. Let \( T = H_e \) for some \( e \in E \) be a subsemigroup of \( S \). Then \( T \) is a closed subsemigroup of \( S \) and every closed congruence \( \sigma \) on \( T \) can be extended to a closed congruence \( \tilde{\sigma} \) on \( S \). Hence, \( S \) has the congruence extension property (CEP) relative to the \( \mathcal{H} \)-classes of \( S \).
Proof. We first wish to see that $T$ is a closed subsemigroup of $S$. Let $x_\alpha$ be a net in $H_e$ converging to some $x \in S$. Then for each $\alpha$, $x_\alpha \in eS \cap Se$. Thus, there exist nets $s_\alpha$ and $t_\alpha$ in $S$ such that $x_\alpha = es_\alpha$ and $x_\alpha = t_\alpha e$. Then continuity of multiplication in $S$ yields that $es_\alpha \rightarrow x$ and $t_\alpha e \rightarrow x$. However, since $S$ is compact we also have that the nets $s_\alpha \rightarrow s$ for some $s \in S$, and $t_\alpha \rightarrow t$ for some $t \in S$. Thus, by passing to subnets, we obtain that $x = es$ and $s = te$. Hence, $x \in H_e = T$, and $T$ is closed.

We assume that $p$ has been normalized so that there is $\lambda \in L$ and $\mu \in R$ with $p(r, \lambda) = e_G = p(\mu, l)$, for every $r \in R$ and every $l \in L$. Then we have the following: $T = H_e \cong_r G \cong_r \{\lambda\} \times G \times \{\mu\}$ and $\sigma$ corresponds via the composed isomorphisms to a closed congruence $(\sigma_{\{\lambda\}}, \sigma_G, \sigma_{\{\mu\}})$, where $\sigma_G$ is a closed congruence on $G$, and $\sigma_{\{\lambda\}}$ and $\sigma_{\{\mu\}}$ are obviously the respective closed diagonal congruences.

Define $\bar{\sigma} = (\Delta_L, \sigma_G, \Delta_R)$. Then $\bar{\sigma}$ is a closed congruence on $S$ and extends $(\sigma_{\{\lambda\}}, \sigma_G, \sigma_{\{\mu\}})$. Then via the topological isomorphism between $T$ and $\{\lambda\} \times G \times \{\mu\}$, $\sigma$ has an extension to $S$. 

The next two lemmas are topological analogues of results from [Garcia, 1988].

7.55 Lemma. Let $G$ be a compact group with identity $e$, let $L$ be a compact left trivial semigroup, and let $R$ be a compact right trivial semigroup. Suppose $S = L \times G \times R$, and let $T$ be a closed subsemigroup of $S$. If $(a, g, b) \in T$, then $(a, e, b) \in T$.

Proof. By the Tychonoff Theorem, $S$ is compact. Let $(a, g, b) \in T$. Then $(a, g^n, b) = (a, g, b)^n \in T$, for every $n \in \mathbb{N}$. Since $T$ is closed and $(a, g^n, b) \rightarrow (a, e, b)$, we have that $(a, e, b) \in T$, as desired.
7.56 Lemma. Let $G$ be a compact group with identity $e$, let $L$ be a compact left trivial semigroup, and let $R$ be a compact right trivial semigroup. Suppose $S = L \times G \times R$, and suppose $T$ is a closed subsemigroup of $S$. Then there exists $A \subseteq L$, $H < G$, and $B \subseteq R$ such that $T = A \times H \times B$.

Proof. Let 

$$H = \{g \in G: (a, g, b) \in T \text{ for some } a \in L \text{ and } r \in R\},$$

$$A = \{a \in L: (a, g, b) \in T \text{ for some } g \in G \text{ and } r \in R\},$$

and

$$B = \{r \in R: (a, g, b) \in T \text{ for some } a \in L \text{ and } g \in G\}.$$ 

Then $H$ is a subsemigroup of the compact group $G$. Thus, $H$ is a subgroup of $G$. Also, $A \subseteq L$, $B \subseteq R$, and $T \subseteq A \times H \times B$. To see that $A \times H \times B \subseteq T$, let $(a, h, b) \in A \times H \times B$. Since $a \in A$, there exist $g \in G$ and $y \in R$ with $(a, g, y) \in T$. Since $h \in H$, there exist $l \in L$ and $r \in R$ such that $(l, h, r) \in T$. Because $b \in B$, there exist $x \in L$ and $u \in G$ such that $(x, u, b) \in T$. By Lemma 7.55, we have that $(a, e, y), (x, e, b) \in T$. Thus, $(a, h, b) = (a, e, y) \cdot (l, h, r) \cdot (x, e, b) \in T$.

7.57 Theorem. Let $S = [L, G, R]_p$ be a compact completely simple semigroup, and suppose that $p$ has been normalized so that there exist $\lambda \in L$ and $\mu \in R$ with $p(r, \lambda) = e_G = p(\mu, l)$, for every $r \in R$ and every $l \in L$. Then $S$ has the congruence extension property (CEP) if and only if $G$ has the congruence extension property and $p(r, l) = e_G$ for every $r \in R$ and every $l \in L$.

Proof. Suppose first that $G$ has CEP and that $p(r, l) = e_G$ for every $r \in R$ and every $l \in L$. Let $T$ be a closed subsemigroup of $S$, and let $\sigma$ be a closed congruence on $T$. Then there exist $A \subseteq L$, $H < G$, and $B \subseteq R$ and closed congruences $\sigma_A$ on $A$, $\sigma_H$ on $H$, and $\sigma_B$ on $B$ such that $T = A \times H \times B$ and $\sigma = (\sigma_A, \sigma_H, \sigma_B)$. 
Let \( \sigma_L \) be an extension of \( \sigma_A \) to \( L \), let \( \sigma_G \) be an extension of \( \sigma_H \) to \( G \), and let \( \sigma_R \) be an extension of \( \sigma_B \) to \( R \). Define

\[
\bar{\sigma} = \{ ((a, u, b), (x, v, y)) \in S \times S : (a, x) \in \sigma_L, (u, v) \in \sigma_G, \text{ and } (b, y) \in \sigma_R \}.
\]

One verifies that \( \bar{\sigma} \) is a closed congruence on \( S \) and \( \sigma \subseteq \bar{\sigma} \). It remains to show that \( \bar{\sigma} \cap (T \times T) \subseteq \sigma \). For this purpose, let \( ((a, u, b), (x, v, y)) \in \bar{\sigma} \cap (T \times T) \). Then

\[
(a, x) \in \sigma_L \cap (A \times A),
\]

\[
(u, v) \in \sigma_G \cap (H \times H), \text{ and}
\]

\[
(b, y) \in \sigma_R \cap (B \times B).
\]

Now, since \( (a, x) \in \sigma_A \), there are \( h, g \in H \) and \( s, t \in B \) with \( ((a, h, s), (x, g, t)) \in \sigma \). Since \( (b, y) \in \sigma_B \), there are \( z, w \in H \) and \( q, m \in A \) with \( ((q, z, b), (m, w, y)) \in \sigma \). Since \( (u, v) \in \sigma_H \), there exist \( r, k \in A \) and \( c, j \in B \) with \( ((r, u, c), (k, v, j)) \in \sigma \). Note that since \( (a, u, b) \in T \), we have that \( (a, e, b) \in T \). Then by compatibility of \( \sigma \),

\[
((a, h, b), (x, g, b)) = ((a, h, s), (x, g, t)) \cdot (a, e, b) \in \sigma.
\]

Thus, by compatibility again,

\[
((a, h^n, b), (x, g^n, b)) = ((a, h, b), (x, g, b))^n \in \sigma \quad \text{for every } n \in \mathbb{N}.
\]

Then since \( G \) is compact, we obtain that the cluster points of the nets \( g^n \) and \( h^n \) are idempotent. The only idempotent in the group \( G \) is the identity \( e \). Thus, we have \( ((a, h^n, b), (x, g^n, b)) \xrightarrow{\sigma} ((a, e, b), (x, e, b)) \). Hence, since \( \sigma \) is closed, we have that \( ((a, e, b), (x, e, b)) \in \sigma \).

Likewise, \( ((a, z, b), (a, w, y)) = (a, e, b) \cdot ((q, z, b), (m, w, y)) \in \sigma \). By the identical technique, we obtain that \( ((a, e, b), (a, e, y)) \in \sigma \).

We put

\[
P_1 = ((a, e, b), (x, e, b)),
\]

\[
P_2 = ((r, u, c), (k, v, j)), \text{ and}
\]

\[
P_3 = ((a, e, b), (a, e, y)).
\]
Then since $P_1, P_2,$ and $P_3 \in \sigma$ and $\sigma$ is a congruence on $T$, we have that

$$P_1 P_2 P_3 = ((a, u, b), (x, v, y)) \in \sigma.$$ 

Therefore, $\bar{\sigma} \cap (T \times T) \subseteq \sigma$, as desired. The converse is a direct analog of the converse of Theorem 6.15.

7.58 Corollary. Let $S = [L, G, R]_p$ be a compact completely simple semigroup. If $G$ has the congruence extension property (CEP) and there exists $g \in G$ such that $p(r, l) = g$ for every $r \in R$ and every $l \in L$, then $S$ has the congruence extension property.

Proof. By Remark 7.53, we normalize $p$ so that there exist $\lambda \in L$ and $\mu \in R$ with $p(r, \lambda) = e_G = p(\mu, l)$, for every $r \in R$ and every $l \in L$. Then $p(r, l) = e_G$ for every $r \in R$ and every $l \in L$. We apply Theorem 7.57 to see that $S$ has CEP.

7.59 Note. The set $E$ of idempotents of a topological semigroup $S$ is closed.

Proof. Suppose that $S$ is a topological semigroup, and let $E$ denote the set of idempotents of $S$. Let $e_\alpha$ be a net in $E$, and suppose that $e_\alpha \xrightarrow{\xi} x$. Then by continuity of multiplication, we have that $e_\alpha = e_\alpha e_\alpha \xrightarrow{\xi} x^2$. Uniqueness of limits then implies that $x^2 = x$, and hence $x \in E$. Therefore, $E$ is closed.

7.60 Proposition. Let $S = [L, G, R]_p$ be a completely simple semigroup, and suppose that $p$ has been normalized so that there exists $\lambda \in L$ and $\mu \in R$ with $p(r, \lambda) = e_G = p(\mu, l)$, for every $r \in R$ and every $l \in L$. Then $p(r, l) = e_G$ for every $r \in R$ and every $l \in L$ if and only if $E$ is a closed subsemigroup of $S$.

Proof. This is a direct analog of Corollary 6.19.
7.61 Corollary. Suppose $S$ is a compact completely simple semigroup. Let $G$ be the Schützenberger group of $S$. Then $S$ has the congruence extension property (CEP) if and only if $G$ has the congruence extension property and $E$ is a closed subsemigroup of $S$.

**Proof.** Realize $S = [L, G, R]_p$, and normalize $p$ as in Remark 7.53. Combining Theorem 7.57 and Proposition 7.60 yields the needed result. 

7.62 Proposition. The continuous homomorphic image of a compact completely simple semigroup is a compact completely simple semigroup.

**Proof.** This follows from the fact that the homomorphic image of a completely simple semigroup is completely simple (see [Clifford and Preston, 1961]) and the fact that continuous images of compact spaces are compact.

7.63 Corollary. The continuous homomorphic image of a compact completely simple semigroup with the congruence extension property (CEP) has the congruence extension property.

**Proof.** This is a direct analog of Corollary 6.21.

7.64 Proposition. Let $\{S_\alpha : \alpha \in A\}$ be a family of compact completely simple semigroups. Let $S = \prod\{S_\alpha : \alpha \in A\}$. Then $S$ is a compact completely simple semigroup.

**Proof.** This follows from Proposition 6.22 and the Tychonoff Theorem.
The purpose of this chapter is to summarize the results of the preceding chapters and to present and discuss open questions for future research. We have studied the congruence extension property (CEP), the ideal extension property (IEP), \(t\)-semigroups, \(m\)-semigroups and related concepts in semigroups. After considering these topics in the purely algebraic setting in Chapters 1–6, we have examined each topologically in Chapter 7.

We summarize the attributes of \(t\)-semigroups, \(m\)-semigroups, and completely simple semigroups in the following table. The term “hereditary” refers to whether the property is preserved by subsemigroups; the term “productive” refers to whether the property is productive; and the term “homomorphic” refers to whether the property is preserved by homomorphisms.

<table>
<thead>
<tr>
<th></th>
<th>Hereditary</th>
<th>Productive</th>
<th>Homomorphic</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)-semigroup</td>
<td>no (3.3)</td>
<td>yes (3.2)</td>
<td>yes (3.1)</td>
</tr>
<tr>
<td>(m)-semigroup</td>
<td>yes (4.2)</td>
<td>no (4.27)</td>
<td>yes (4.3)</td>
</tr>
<tr>
<td>completely simple</td>
<td>no (6.7)</td>
<td>yes, if index (\leq 3) (4.28)</td>
<td>yes, see [Clifford and Preston, 1961]</td>
</tr>
<tr>
<td>semigroup</td>
<td>yes, iff torsion subgroups (6.8)</td>
<td>yes (6.22)</td>
<td></td>
</tr>
</tbody>
</table>

From the above table, we see that completely simple semigroups with torsion subgroups form a semigroup variety as do \(m\)-semigroups with index less than 4.
From a topological perspective, “hereditary” refers to whether the property is inherited by closed subsemigroups and “homomorphic” refers to whether the property is preserved by continuous homomorphisms. The corresponding table for compact semigroups is below.

<table>
<thead>
<tr>
<th></th>
<th>Hereditary</th>
<th>Productive</th>
<th>Homomorphic</th>
</tr>
</thead>
<tbody>
<tr>
<td>compact t-semigroup</td>
<td>no (7.29)</td>
<td>yes (7.28)</td>
<td>yes (7.27)</td>
</tr>
<tr>
<td>compact m-semigroup</td>
<td>yes (7.36)</td>
<td>no (7.38)</td>
<td>yes (7.37)</td>
</tr>
<tr>
<td></td>
<td>yes, if mi ≤ 3 (7.44)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>compact completely simple semigroup</td>
<td>yes (7.50)</td>
<td>yes (7.64)</td>
<td>yes (7.62)</td>
</tr>
</tbody>
</table>

Again, we see that compact completely simple semigroups form a semigroup variety as do m-semigroups with monothetic index less than or equal to 3.

In Chapter 2, a characterization is given for semigroups with CEP in terms of a condition on the lattice of congruences. Specifically, it is obtained that a semigroup $S$ has the congruence extension property (CEP) if and only if for every subsemigroup $T$ of $S$, the map $\phi_T: \mathcal{L}_T \rightarrow X_T$ defined by $\sigma \mapsto (\sigma)_S$ is a bijective correspondence. A similar result is given for semigroups with the ideal extension property (IEP). Each of these holds more generally for groupoids.

With the following diagrams, we illustrate the implication relationships among t-semigroups, the ideal extension property (IEP), the congruence extension property (CEP), various index conditions, and m-semigroups.
IMPLICATION DIAGRAMS.

\[
\begin{align*}
\text{IEP} & \quad 3.5 \\
\text{t-semigroup} & \quad 3.6 \\
\implies & \\
\text{index}(S) & \leq 3
\end{align*}
\]

For a cyclic semigroup \(S\),

\[
\begin{align*}
\text{t-semigroup} & \quad \iff \\
\text{index}(S) & \leq 3 \\
\implies & \\
\text{IEP} & \quad \iff \\
\text{CEP} & \quad (3.8)
\end{align*}
\]

\[
\begin{align*}
\text{band} & \quad 3.12 \\
\text{regular semigroup} & \quad 3.11 \\
\implies & \\
\text{t-semigroup} & \quad 3.15 \\
\implies & \\
\text{CEP} & \quad 3.16 \\
\iff & \\
\text{m-semigroup} & \quad 4.5 \\
\iff & \\
\text{index}(S) & \leq 5 \quad \text{and} \quad E(S) = \{0\}
\end{align*}
\]

For an archimedean semigroup \(S\) with zero,

\[
\begin{align*}
\text{m-semigroup} & \quad 4.25 \\
\iff & \\
S^3 & = 0
\end{align*}
\]

These implication diagrams carry over topologically in the compact case with monothetic index replacing index except for 3.15. Theorem 3.15 does not topologize directly due to the fact that the characterization of the congruence generated by a pair does not directly topologize, even in the compact case.
Chapter 6 deals with completely simple semigroups with CEP. A construction is given which yields an alternative proof for the known result in the algebraic case and is amenable to direct extension to the topological result. In the process of obtaining the construction, subsemigroups of completely simple semigroups with torsion subgroups are studied. It is obtained that all subsemigroups of a completely simple semigroup $S$ are themselves completely simple if and only if $S$ has torsion subgroups. Additionally, it is demonstrated that homomorphic images of completely simple semigroups with CEP retain CEP.

In Chapter 5, we considered bands of groups. By employing a result in [Stralka, 1972], we obtained that a semilattice of abelian groups has the congruence extension property (CEP) if and only if each of the groups is torsion, that is, if and only if each of the abelian groups has CEP. We recall that completely simple semigroups form a special-case band of groups. Thus, we combine our knowledge of the characterization of completely simple semigroups with CEP with our knowledge of techniques used in [Stralka, 1972] to arrive at the conjectures in Open Question 4.

The following is a list of open questions concerning the congruence extension property (CEP), the ideal extension property (IEP), $t$-semigroups, $m$-semigroups and related concepts. Although each question is phrased for the case of algebraic semigroups, each may also be considered as a topological question.

OPEN QUESTIONS.

1. It is not known whether the congruence extension property (CEP) implies the ideal extension property (IEP). We conjecture this is true. The commutative case was shown by [Garcia, 1988]. See 1.39. The proof for the commutative case
does not seem to generalize to the non-commutative case. An extensive computer search was conducted of non-commutative semigroups of orders 4, 5, and 6 and no counterexample was found.

2. We conjecture that a commutative semigroup $S$ has the ideal extension property (IEP) if and only if $S$ is a $t$-semigroup. That any semigroup with IEP is a $t$-semigroup was shown in Proposition 3.5. Theorem 3.8 illustrates this equivalence for cyclic semigroups, but the proof does not appear to generalize to the commutative case. However, an extensive computer search of commutative semigroups of orders 4, 5, 6, and 7 was conducted and no counterexample was found. One potential avenue of exploration may be to exploit the characterization of commutative semigroups with IEP given in [Aucoin, 1993]. As shown by Example 3.6, a non-commutative $t$-semigroup need not have IEP.

3. We discuss three related questions.

a. In a semigroup $S$, a necessary and sufficient condition for an ideal of a subsemigroup of $S$ to be an ideal of $S$ is not known. This question arises somewhat naturally in Chapter 3. We have not addressed it here. However, if we consider this as a global property, that is, every ideal of every subsemigroup must be an ideal of $S$, then we have that every subsemigroup of $S$ must be an ideal of $S$. Semigroups of this type are characterized in [Kimura, Tamura, and Merkel, 1965]. Such semigroups were called $\sigma$-semigroups in that work.

b. It has yet to be determined whether a $t$-semigroup with zero is an $m$-semigroup. If this is true, then every subsemigroup of a $t$-semigroup $S$ with zero is an ideal of $S$. Again, these have been characterized.

c. It is unknown whether a commutative $t$-semigroup is a $\sigma$-semigroup.
4. We conjecture that an arbitrary band of groups $S$ has CEP if and only if $E(S)$ is a subsemigroup of $S$, $E(S)$ has CEP, and each of the groups has CEP. If this is true, then it will follow that a semilattice of groups has CEP if and only if each group has CEP. This conjecture arises from our studies in Chapters 5 and 6 and it is hoped that the techniques in [Stralka, 1972] may aid in a resolution of this question.

5. When does a semilattice of archimedean semigroups with zero each of which having the congruence extension property (CEP) have CEP? This question a special case of a question considered in [Aucoin, 1993]. It is known that the answer is not always, but it is hoped that the answer to this question will aid in determining if $S/\mathcal{H}$ retains CEP for a commutative semigroup $S$ with CEP.

6. It remains to be determined whether the homomorphic image of a semigroup with the congruence extension property (CEP) retains CEP. This is generally considered to be the major open question concerning CEP. Some partial results have been obtained. The question is answered affirmatively for ideal semigroups in [Garcia, 1988], for homomorphisms determined by an ideal in [Aucoin, 1993], for archimedean semigroups in [Aucoin, 1993], and for regular semigroups in [Jones, 1992]. It does not appear that the characterization of semigroups with CEP given in Chapter 2 in terms of a condition on the lattice of congruences will aid in a resolution of this problem, as hoped. As mentioned in [Garcia, 1988], it appears that associativity, or some consequence thereof, has an important role in the solution of the problem. That the homomorphic image of a groupoid with CEP need not have CEP was demonstrated in [Biró, Kiss, and Pálfy, 1977].

In the topological case, it isn’t necessarily true that the continuous homomor-
A homomorphic image of a compact semigroup with CEP has CEP. A example was given in [Stralka, 1977] showing that the continuous homomorphic image of a compact semilattice with CEP need not have CEP. That every continuous homomorphic image of a compact completely simple semigroup with CEP retains CEP is illustrated in Chapter 7.

7. A characterization for (compact) semigroups with the congruence extension property (CEP) is still unknown. It is hoped that such a characterization may lead to a resolution of the previous question. Some useful special case results have been found. Specifically, $\Delta$-semigroups with CEP, archimedean semigroups with CEP, and commutative ideal semigroups with CEP are each characterized in [Aucoin, 1993], regular semigroups with CEP are characterized in [Jones, 1992], and compact completely simple semigroups with CEP are characterized in Chapter 7. In each of these cases, the characterization lead to an affirmative answer to the previous question for that case.
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WALLACE, A. D.

YAMADA, M.

YAMADA, M. AND TAMURA, T.
Jill Ann Dumesnil was born in Houston, Texas on January 31, 1966. She attended Port Neches - Groves High School in Port Neches, Texas where she graduated co-valedictorian in May, 1984. Four years later she received a Bachelor of Science in Mathematical Sciences from Lamar University in Beaumont, Texas. In August, 1988 she began her graduate studies at Louisiana State University in Baton Rouge as a Board of Regents Fellow. She received her Master of Science degree in mathematics in August, 1990. She served half-time as an instructor in the Department of Mathematics at LSU while completing her doctoral dissertation.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Jill Ann Dumesnil

Major Field: Mathematics

Title of Dissertation: The Congruence Extension Property and Related Topics in Semigroups

Approved:

[Signature]
Major Professor and Chairman

[Signature]
Dean of the Graduate School

EXAMINING COMMITTEE:

J. Hurrelbrink

E. Ku

O. McGehee

R. L. E. Mason

Date of Examination:

02/10/93