The Congruence Extension Property, the Ideal Extension Property, and Ideal Semigroups.

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The congruence extension property, the ideal extension property, and ideal semigroups

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THE CONGRUENCE EXTENSION PROPERTY,
THE IDEAL EXTENSION PROPERTY,
AND IDEAL SEMIGROUPS

A Dissertation

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in

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by
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Preface

Primary motivation for this work was provided by the 1988 doctoral dissertation of Josefa Garcia concerning the congruence extension property and related topics in algebraic semigroups. Other major influences on this work include selected papers by A.R. Stralka, B. Biró, E. Kiss and P. Pálfy, M. Petrich, T. Tamura, and M. Yamada.

The objectives of this research are:

(1) To characterize commutative semigroups with the congruence extension property (CEP) and the ideal extension property (IEP), respectively.

(2) To characterize commutative semigroups in which each congruence is determined by an ideal.

(3) To examine the question of whether CEP is preserved by homomorphisms.

(4) To explore these concepts in a topological setting.

It is well-known that a commutative semigroup is a semilattice of archimedean semigroups called archimedean components. A characterization of commutative semigroups with IEP is given in terms of multiplicative conditions within and between the archimedean components of the semigroup in Chapter 2. The work in this chapter was motivated by an attempt to characterize commutative semigroups with CEP as it is known from the work of Garcia that CEP implies IEP in commutative semigroups. Accordingly, a similar characterization of commutative semigroups with CEP is sought in Chapter 3. Utilizing the ideal extension structure of periodic archimedean semigroups given in the work of Tamura and the characterization of congruences on certain extensions of semigroups given in the 1967 paper
of Petrich, archimedean semigroups with CEP are characterized. The remainder of Chapter 3 is devoted to establishing necessary conditions on multiplication between the archimedean components of a commutative semigroup with CEP. Using these results, one can determine the possibilities for multiplication among any three finite components of a commutative semigroup with CEP based on the ordering of idempotents lying in these components.

Chapter 4 explores the question of whether CEP is preserved by homomorphisms. It is proved that Rees quotients of semigroups with CEP have CEP and that the homomorphic image of an archimedean semigroup with CEP has CEP. In addition, it is established that for a semigroup \( S \) with CEP, the question of whether a homomorphic image \( \phi(S) \) has CEP is equivalent to the question of whether joining \( \ker \phi \) to extensions of certain congruences preserves the extension. Employing this equivalence, a computer search was conducted which showed that the homomorphic images of semigroups of order six or less and commutative semigroups of order seven with CEP must retain CEP. Examples are given to illustrate that techniques used in treating special cases cannot be applied in general.

In Chapter 5, commutative semigroups in which each congruence is determined by an ideal are characterized. These semigroups, called "ideal semigroups", arose in the work of Garcia in connection with the study of the relationship between CEP and IEP. However, the inclination to study such semigroups arises naturally when one considers the correspondence between congruences and normal subgroups in groups and between congruences and ideals in rings. There is not a one-to-one correspondence between ideals and congruences in semigroups. Thus, it seems natural to consider the structure of semigroups in which there is such a correspondence.
The characterization of commutative ideal semigroups given provides a description of the $H$-order graph (or divisibility ordering) of such a semigroup. Using this characterization, the $H$-order graphs of commutative ideal semigroups with IEP and CEP respectively are completely described. Commutative semigroups with CEP whose congruences form a chain are characterized as an unexpected corollary to this work.

The concepts of IEP, CEP and ideal semigroups are examined in a topological setting in Chapter 6. In particular, a number of analogues of results discussed above are proven for compact unipotent semigroups. An example is cited from the 1972 paper of Stralka which shows that CEP is not always preserved by continuous homomorphisms in compact semigroups. In light of this, it is interesting that the special case result that Rees quotients of semigroups with CEP retain CEP has a direct topological analog for compact semigroups.

In Chapter 7, a summary of results established along with a list of open questions arising from this work are given.
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Abstract

A semigroup has the congruence extension property (CEP) provided that each congruence on each subsemigroup of $S$ extends to a congruence on $S$. The ideal extension property (IEP) for semigroups is defined analogously. A characterization of commutative semigroups with IEP is given in terms of multiplicative conditions within and between the archimedean components of the semigroup. A similar characterization of commutative semigroups with CEP is sought. Toward this end, archimedean semigroups with CEP are characterized in terms of multiplicative structure and a number of necessary conditions on multiplication between the archimedean components of a commutative semigroup with CEP are established. In the topological setting, compact commutative unipotent semigroups with IEP and CEP respectively are characterized.

Relative to the question of whether CEP is preserved by homomorphisms, necessary and sufficient conditions for a given [continuous] homomorphic image to retain CEP are given. It is proved that the homomorphic image of an archimedean semigroup with CEP has CEP and that Rees quotients of compact semigroups with CEP have CEP.

An ideal semigroup is a semigroup in which each congruence is determined by an ideal. A characterization of commutative ideal semigroups is given which provides a description of the $\mathcal{H}$-order graph (or divisibility ordering) of such a semigroup. Consequently, the $\mathcal{H}$-order graphs of commutative ideal semigroups with IEP and CEP respectively are completely described. Commutative semigroups with CEP whose congruences form a chain are characterized as an unexpected corollary.
The primary purpose of this chapter is to provide definitions and results that will be used in the chapters that follow. In establishing this background material, known results are listed along with related results which are easily obtained. Congruences and the congruence extension property (CEP) are treated first in this chapter, followed by the ideal extension property (IEP) and ideal semigroups. A discussion of commutative semigroup theory follows these topics and the chapter concludes with a short discussion of \( \Delta \)-semigroups.

A congruence \( \sigma \) on a semigroup \( S \) is an equivalence relation on \( S \) which is compatible with multiplication in \( S \). That is, a relation \( \sigma \) is a congruence on \( S \) provided:

1. \( \sigma \) is reflexive (\( \Delta_S \subseteq \sigma \) where \( \Delta_S = \{(x, x) : x \in S\} \));
2. \( \sigma \) is symmetric (\( \sigma^{-1} = \sigma \) where \( \sigma^{-1} = \{(b, a) : (a, b) \in \sigma\} \));
3. \( \sigma \) is transitive (\( \sigma \circ \sigma \subseteq \sigma \)); and
4. \( \sigma \) is compatible with multiplication (If \( (a, b) \in \sigma \) and \( s \in S \), then \( (sa, sb), (as, bs) \in \sigma \)).

In the work that follows, we will often consider the congruence on \( S \) generated by a given relation on \( S \). This is most simply defined as the smallest congruence on \( S \) which contains the given relation. In particular, we will consider congruences generated by a single pair. We have the following definition. A congruence \( \alpha \) on a semigroup \( S \) is called a principal congruence provided \( \alpha \) is generated by a single
pair \((a, b) \in S \times S\). We denote the congruence on \(S\) generated by the pair \((a, b)\) by \(\alpha^S(a, b)\).

If \(T\) is a subsemigroup of a semigroup \(S\) and \(\sigma\) is a congruence on \(T\), then a congruence \(\overline{\sigma}\) on \(S\) is called an extension of \(\sigma\) provided \(\overline{\sigma} \cap (T \times T) = \sigma\).

A semigroup \(S\) is said to have the congruence extension property (CEP) provided that for each subsemigroup \(T\) of \(S\) and each congruence \(\sigma\) on \(T\), \(\sigma\) has an extension to \(S\).

A semigroup \(S\) is said to have the principal congruence extension property (PCEP) provided that for each subsemigroup \(T\) of \(S\) and each \((a, b) \in T \times T\), \(\alpha^S(a, b) \cap (T \times T) = \alpha^T(a, b)\). As stated below, it is known that CEP is equivalent to PCEP.

In [Biró, Kiss, and Pálfy, 1977], congruence extension properties are explored for a number of algebraic structures. Of interest to us is the case of groups. A group \(G\) is said to have the group congruence extension property (GCEP) provided that for each subgroup \(H\) of \(G\) and each congruence \(\sigma\) on \(H\), there exists an extension of \(\sigma\) to \(G\). For a group \(G\), the relationship between CEP (considering the group \(G\) as a semigroup) and GCEP will be given in the list of known results which follows. Considering this relationship, we recall the well known fact that a subsemigroup of a torsion group is a subgroup.

Before listing known results concerning congruences and congruence extension, we review a number of definitions of terms that appear in the statements of these results.

Let \(\sigma\) and \(\delta\) be congruences on a semigroup \(S\). The meet (denoted \(\wedge\)) of \(\sigma\) and \(\delta\) is the congruence on \(S\) defined by \(\sigma \wedge \delta = \sigma \cap \delta\). The join (denoted \(\vee\)) of \(\sigma\) and
$\delta$ is defined by $\sigma \vee \delta = (\sigma \cup \delta)_S$, the congruence on $S$ generated by $\sigma \cup \rho$. The set of congruences on a semigroup $S$ forms a lattice with respect to the operations of meet and join.

Let $\phi$ be a homomorphism from a semigroup $S$ to a semigroup $X$. A subset $T$ of a semigroup $S$ is said to be $\phi$-saturated provided that $\phi^{-1}(\phi(T)) = T$. A subset $T$ of a semigroup $S$ is said to be saturated with respect to a congruence $\sigma$ on $S$ provided that if $(x, y) \in \sigma$ and $x \in T$, then $y \in T$. Equivalently, a subset $T$ is saturated with respect to $\sigma$ if and only if $\sigma \subseteq (T \times T) \cup (S \setminus T \times S \setminus T)$. Furthermore, it is straightforward to show that $T$ is $\phi$-saturated if and only if $T$ is saturated with respect to $\ker \phi$.

A semigroup $S$ is said to be medial if for each $x, y, z, w \in S$, $xyzw = xzyw$. A semigroup $S$ in which each element is idempotent is called a band. A semilattice is a commutative band. (Thus, any semilattice is a medial band.) An element $r$ of a semigroup $S$ is a regular element provided there exists $t \in S$ such that $rtr = r$. A semigroup in which each element is regular is a regular semigroup. Observe that any idempotent element is regular in any subsemigroup which contains it.

For $S$ a semigroup, $S^1$ denotes the semigroup with an identity adjoined and $S_0$ denotes the semigroup with a zero adjoined. We use $E_S$ to denote the set of idempotents of $S$ or simply $E$ when no confusion is likely. Finally, we denote the minimal ideal of a semigroup $S$ by $M(S)$ when it exists.

There is a natural ordering of the set $E_S$ of idempotents of a semigroup $S$ given by the following. Define $e \leq f$, $(e, f \in E_S)$ to mean $e = ef = fe$. Then it is well-known that $\leq$ is a partial ordering (i.e. a reflexive, anti-symmetric, and transitive relation) of the set of idempotents $E_S$. In particular, if $S$ is a band, then
this gives a partial ordering on $S$ and we establish the following notation. For $A \subseteq S$, we define $\uparrow A := \{x \in S : a \leq x \text{ for some } a \in A\}$. Likewise, we set $\downarrow A := \{x \in S : x \leq a \text{ for some } a \in A\}$.

For $S$ a semigroup and $a \in S$, we denote the subsemigroup of $S$ consisting of all powers of $a$ by $\theta(a)$. Also, $\text{index}(a)$ is the least $n \in \mathbb{N}$ such that $a^n = a^m$ for some $m > n$. If $a^n \neq a^m$ for all $n \neq m$, then $\text{index}(a) = \infty$. Equivalently, $\text{index}(a)$ is the least $n \in \mathbb{N}$ such that $a^n \in M(\theta(a))$ if $M(\theta(a)) \neq \emptyset$ and otherwise, $\text{index}(a) = \infty$. We define $\text{index}(S)$ to be $\max\{\text{index}(a) : a \in S\}$ if this maximum exists. Otherwise, $\text{index}(S) = \infty$. Clearly, $\text{index}(\theta(a)) = \text{index}(a)$ according to these definitions. A semigroup $S$ is said to be periodic provided each element has finite index. In particular, if $\text{index}(S) < \infty$, then $S$ is periodic. (Note that it is possible that $S$ be periodic and have $\text{index}(S) = \infty$.) Equivalently, a semigroup $S$ is periodic provided that some power of each element is idempotent.

If $M$ is a subsemigroup of a semigroup $S$ and $\phi : S \to M$ is a homomorphism of $S$ onto $M$ such that $\phi|_M = 1_M$, then $\phi$ is called a homomorphic retraction of $S$ onto $M$ and $M$ is called a homomorphic retract of $S$.

The following is a list of known results concerning the congruence extension property (CEP) along with several related propositions and corollaries which are easily drawn from known results. Short arguments are given to justify these propositions and corollaries. Also included is an alternate proof of one of the results. When neither a reference nor a proof is given, the fact is well-known and may be found in [Garcia, 1988]

1.1 A semigroup $S$ has the congruence extension property (CEP) if and only if each subsemigroup of $S$ has the congruence extension property.
1.2 Let $T$ be a subsemigroup of a semigroup $S$ and let $\sigma$ be a congruence on $T$. Then the congruence on $S$ generated by $\sigma$ is given by

$$\langle \sigma \rangle_S = \bigcup_{n \in \mathbb{N}} \rho^n$$

where $\rho = \{(xay, xby) : (a, b) \in \sigma \cup \Delta_S, \ x, y \in S^1\}$.

1.3 Let $T$ be a subsemigroup of a semigroup $S$ and let $\sigma$ be a congruence on $T$. Then $\sigma$ has an extension to $S$ if and only if $\langle \sigma \rangle_S$ is an extension of $\sigma$ to $S$.

(Note that $\sigma \subseteq \langle \sigma \rangle_S \cap (T \times T)$ always holds. Thus, to see that $\langle \sigma \rangle_S$ is an extension, one need only check the reverse inclusion.)

1.4 Let $S$ be a semigroup and let $(a, b) \in S \times S$. Then $\alpha^S(a, b)$ is given by

$$\alpha^S(a, b) = \bigcup_{n \in \mathbb{N}} \beta^n$$

where $\beta = \{(xcy, xdy) : (c, d) \in \{(a, b), (b, a)\} \cup \Delta_S, \ x, y \in S^1\}$.

1.5 Corollary. Let $S$ be a semigroup and let $u, v, a, b \in S$, $u \neq v$. Then $(u, v) \in \alpha^S(a, b)$ provided there is a finite transition $u = x_0, x_1, ..., x_n = v$ such that $(x_i, x_{i+1}) \in \{(xay, xby), (xby, xay) : x, y \in S^1\}$.

Proof. This is immediate from 1.4 and the fact that we may assume that $x_i \neq x_{i+1}$ for $0 \leq i \leq n - 1$ for a transition $u = x_0, ..., x_n = v$ where $u \neq v$ as this assumption does not eliminate any pairs from $\alpha^S(a, b) \setminus \Delta^S$.

1.6 Let $S$ be a semigroup. Let $\sigma$ and $\delta$ be congruences on $S$. Then $\sigma \cup \delta$ is a reflexive, symmetric, compatible relation on $S$ and

$$\sigma \vee \delta = \langle \sigma \cup \delta \rangle_S = \bigcup_{n \in \mathbb{N}} (\sigma \cup \delta)^n$$
1.7 Proposition. A subset $T$ of a semigroup $S$ is saturated with respect to a relation $\delta$ on $S$ if and only if $T$ is saturated with respect to

$$\bigcup_{n \in \mathbb{N}} \delta^n$$

In particular, a subset $T$ is saturated with respect to $\alpha^S(a, b)$ if and only if $T$ is saturated with respect to $\{(xay, xby), (xby, xay) : x, y \in S^1\}$.

Proof. If a subset $T$ of a semigroup $S$ is saturated with respect to a relation $\delta$ on $S$ and $x_0, x_1, \ldots, x_n$ is a transition such that $(x_i, x_{i+1}) \in \delta$ for $0 \leq i \leq n - 1$, then clearly $x_0 \in T$ implies $x_i \in T$ for $0 \leq i \leq n$ and $x_0 \notin T$ implies $x_i \notin T$ for $0 \leq i \leq n$. Hence, the forward implication of the first statement holds and the converse is clear. The final statement is an easy consequence in light of the remarks concerning $\alpha^S(a, b)$ that appear above. \[\square\]

1.8 [Garcia, 1988] A semigroup $S$ has the congruence extension property (CEP) if and only if $S$ has the principal congruence extension property (PCEP).

1.9 [Garcia, 1988] The following are equivalent in a semigroup $S$.

(1) $S$ has CEP.

(2) $S^1$ has CEP.

(3) $S_0$ has CEP.

1.10 [Stralka, 1972] A medial band has the congruence extension property (CEP).

1.11 Corollary. A semilattice has the congruence extension property (CEP).

This is an immediate corollary of 1.10. However, the construction of the extensions in the proof 1.10 is quite involved, and it is possible in the case
of a semilattice to simplify the construction. Since we will be particularly interested in extending congruences on semilattices in the chapters that follow, we would like to obtain a simple description of extensions in this case. For this reason, we provide the following alternate proof of the Corollary which clearly exhibits the extension Stralka constructed in the special case of a semilattice.

**Alternate Proof.** Let $T$ be a subsemilattice of a semilattice $S$ and let $\sigma$ be a congruence on $T$. We produce a congruence $\overline{\sigma}$ which extends $\sigma$. Set

$$\overline{\sigma} = \bigcap_{e \in T} ((\uparrow [e]_\sigma \times \uparrow [e]_\sigma) \cup ((S \setminus \uparrow [e]_\sigma) \times (S \setminus \uparrow [e]_\sigma)))$$

Now the intersection of a collection of congruences is again a congruence, so we need only show that for $e \in T$,

$$((\uparrow [e]_\sigma \times \uparrow [e]_\sigma) \cup ((S \setminus \uparrow [e]_\sigma) \times (S \setminus \uparrow [e]_\sigma)))$$

is a congruence on $S$. Toward this end, we show that $S \setminus \uparrow [e]_\sigma$ is a prime ideal of $S$. Let $f \in S$ and let $g \in S \setminus \uparrow [e]_\sigma$. If $gf \notin S \setminus \uparrow [e]_\sigma$, then there is some $h \in [e]_\sigma$ such that $gfh = h$. Thus, $hg = (gfh)g = g^2fh = gfh = h$ and this implies that $x \in \uparrow [e]_\sigma$ which is a contradiction. Hence, we conclude that $xS \setminus \uparrow [e]_\sigma$ and $S \setminus \uparrow [e]_\sigma$ is an ideal. We show that it is prime. Let $f, g \in \uparrow [e]_\sigma$. Then there exist $h, h' \in [e]_\sigma$ such that $hf = h$ and $h'g = h'$. Now $(hh')(fg) = (hf)(h'g) = hh'$. Also, $hh' \in [e]_\sigma$ since $[e]_\sigma$ is a subsemigroup as it is the homomorphic preimage of an idempotent. Hence, $fg \in \uparrow [e]_\sigma$ and $S \setminus \uparrow [e]_\sigma$ is a prime ideal of $S$. Now certainly for $e \in T$, $(\uparrow [e]_\sigma \times \uparrow [e]_\sigma) \cup ((S \setminus \uparrow [e]_\sigma) \times (S \setminus \uparrow [e]_\sigma)))$ is an equivalence on $S$ and compatibility of this relation follows easily from the fact that $S \setminus \uparrow [e]_\sigma$ is a
prime ideal of $S$. Hence, for $e \in T$, $(\uparrow [e]_\sigma \times \uparrow [e]_\sigma) \cup ((S \setminus \uparrow [e]_\sigma) \times (S \setminus \uparrow [e]_\sigma))$ is a congruence on $S$. Therefore, by a previous remark, $\sigma$ is a congruence on $S$. We must show that $\sigma \cap (T \times T) = \sigma$. To see the reverse inclusion, let $(e, f) \in \sigma$ and let $h \in T$. If $e \in \uparrow [h]_\sigma$, then there exists $g \in [h]_\sigma$ such that $eg = g$. Thus, $(g, fg) = (eg, fg) \in \sigma$ so that $fg \in [g]_\sigma = [h]_\sigma$. Also, $f(fg) = f^2g = fg$ so we conclude that $f \in [h]_\sigma$. Dually, assuming that $f \in [h]_\sigma$ yields that $e \in [h]_\sigma$. According to these observations, we have $(e, f) \in (\uparrow [h]_\sigma \times \uparrow [h]_\sigma) \cup ((S \setminus \uparrow [h]_\sigma) \times (S \setminus \uparrow [h]_\sigma))$ for arbitrarily chosen $h \in T$. This shows that $(e, f) \in \sigma$ and so the reverse inclusion holds. To see that the forward inclusion holds, let $(e, f) \in \sigma \cap (T \times T)$. Certainly, we have $e \in [e]_\sigma \subseteq \uparrow [e]_\sigma$ and $f \in [f]_\sigma \subseteq \uparrow [f]_\sigma$. Thus, since $e \in \uparrow [e]_\sigma$ and $(e, f) \in (\uparrow [e]_\sigma \times \uparrow [e]_\sigma) \cup ((S \setminus \uparrow [e]_\sigma) \times (S \setminus \uparrow [e]_\sigma))$ as $(e, f) \in \sigma$, we obtain $f \in \uparrow [e]_\sigma$. Likewise, $e \in \uparrow [f]_\sigma$. Thus, there exist $g, h \in T$ such that $(g, e) \in \sigma$, $(h, f) \in \sigma$, $gf = g$ and $he = h$. Now $(hg, he) \in \sigma$ and $(hg, fg) \in \sigma$ by compatibility. Hence, by transitivity, $(h, g) = (he, fg) \in \sigma$ and again by transitivity, this yields $(e, f) \in \sigma$ as desired. Thus, $\sigma$ extends $\sigma$ to $S$ and $S$ has the congruence extension property (CEP). \[\square\]

1.12 [Stralka, 1972] Let $S$ be a band of groups such that $E_S$ is a subsemigroup of $S$. Then each congruence on $E_S$ can be extended to $S$.

1.13 [Stralka, 1972] Let $S$ be a medial semigroup and let $A$ be a subsemigroup of the regular elements of $S$ such that $A$ is a band of groups. Then each congruence on $A$ can be extended to $S$.

1.14 [Garcia, 1988] A cyclic semigroup $S$ has the congruence extension property (CEP) if and only if $\text{index}(S) \leq 3$. 
1.15 [Garcia, 1988] Let $S$ be a semigroup with the congruence extension property (CEP). Then $\text{index}(S) \leq 3$. (In particular, a semigroup with CEP is periodic.)

1.16 [Clifford and Preston, 1961] Let $G$ be a group and $\sigma$ a congruence on $G$. Then there exists a normal subgroup $N$ of $G$ such that $(a, b) \in \sigma$ if and only if $ab^{-1} \in N$.

1.17 [Biró, Kiss, and Pálfy, 1977] A group $G$ has the group congruence extension property (GCEP) if and only if whenever $H$ is a subgroup of $G$ and $K$ is a normal subgroup of $H$, there exists a normal subgroup $N$ of $G$ such that $N \cap H = K$.

1.18 Corollary. A group $G$ such that each subgroup is normal has the group congruence extension property (GCEP). In particular, an abelian group has GCEP.

1.19 [Garcia, 1988] A group $G$ has the congruence extension property (CEP) if and only if $G$ is a torsion group with the group congruence extension property (GCEP).

1.20 Corollary. An abelian group $G$ has the congruence extension property (CEP) if and only if $G$ is a torsion group.

(Remarks on a direct proof of one direction of this result. Let $T$ be a subsemigroup of a torsion abelian group $G$ and $\sigma$ a congruence on $T$. Now $T$ is a subgroup of $G$ since $G$ is torsion so there exists a normal subgroup $N$ of $T$ (by 1.16) such that $(a, b) \in \sigma$ provided $ab^{-1} \in N$ and $a, b \in T$. But $N$ is also a normal subgroup of $G$ as $G$ is abelian and it is easily seen that the congruence $\overline{\sigma} := \{(a, b) \in S \times S : ab^{-1} \in N\}$ extends $\sigma$ to $G$. )
1.21 [Garcia, 1988] The homomorphic image of a group with the congruence extension property (CEP) has the congruence extension property.

1.22 Proposition. Let $S$ be a semigroup, $M$ a homomorphic retract of $S$, and $\sigma$ a congruence on $M$. Then there exists an extension of $\sigma$ to $S$.

Proof. If $\phi : S \to M$ is a homomorphic retraction and $\sigma$ is a congruence on $M$, then $\bar{\sigma} = \{(a, b) \in S \times S : (\phi(a), \phi(b)) \in \sigma\}$ is an extension of $\sigma$ to $S$. $\blacksquare$

1.23 Corollary. Let $S$ be a commutative semigroup having a group minimal ideal $M$. Then each congruence on $M$ can be extended to $S$.

Proof. The map defined by $x \mapsto ex$ where $e$ is the identity of $M$ is a homomorphic retraction. $\blacksquare$

1.24 Proposition. Let $S$ be a semigroup, $Q$ a subsemigroup of $S$ and $T$ a subsemigroup of $Q$. Let $\sigma$ be a congruence on $T$. Let $\delta$ be an extension of $\sigma$ to $Q$ and let $\gamma$ be an extension of $\delta$ to $S$. Then $\gamma$ is an extension of $\sigma$ to $S$.

Proof. Simply observe that
\[
\gamma \cap (T \times T) = (\gamma \cap (Q \times Q)) \cap (T \times T) = \delta \cap (T \times T) = \sigma. \blacksquare
\]

1.25 Corollary. Let $S$ be a commutative semigroup having a torsion group minimal ideal $M$ and let $T$ be a subsemigroup of $M$. Then each congruence on $T$ can be extended to $S$.

Proof. Let $\sigma$ be a congruence on $T$. By 1.20, there exists an extension $\delta$ of $\sigma$ to $M$. By 1.23, there exists an extension $\gamma$ of $\delta$ to $S$. Thus, by Proposition 1.24, $\gamma$ extends $\sigma$ to $S$. $\blacksquare$
1.26 [Garcia, 1988] Let $S$ be a semigroup and $T$ a subsemigroup of $S$ such that $S \setminus T$ is an ideal of $S$. Then every congruence on $T$ can be extended to $S$.

1.27 Proposition. Let $S$ be a semigroup, $I$ an ideal of $S$, and $\sigma$ a congruence on $I$. Then $\sigma$ has an extension to $S$ if and only if $\sigma \cup \Delta_S$ is a congruence extending $\sigma$ to $S$.

Proof. Only the forward implication must be proven. Observe that $\langle \sigma \rangle_S \subseteq (I \times I) \cup \Delta_S$ as $(I \times I) \cup \Delta_S$ is a congruence on $S$ containing $\sigma$. Now if $\sigma$ has an extension to $S$, then by 1.3, $\langle \sigma \rangle_S$ is an extension. Thus, we have $\langle \sigma \rangle_S = \langle \sigma \rangle_S \cap ((I \times I) \cup \Delta_S) = ((\sigma)_{S \cap (I \times I)}) \cup ((\sigma)_{S \cap \Delta_S}) = \sigma \cup \Delta_S$. Hence, $\sigma \cup \Delta_S$ is a congruence extending $\sigma$ to $S$. $

From the results above we see, in particular, that semilattices and torsion abelian groups always have the congruence extension property (CEP). Other simple examples of semigroups with CEP include zero semigroups and semigroups of order less than four. That a zero semigroup has CEP follows easily from the fact that any equivalence on a zero semigroup is a congruence. That a semigroup of order less than four has CEP follows from the fact that the only congruences on a proper subsemigroup $T$ of such a semigroup are the trivial ones ($\Delta_T$ and $T \times T$) and these always extend. In semigroups which do not generally have CEP, we are often interested in determining whether there are certain subsemigroups relative to which congruences can be extended. Along these lines, we see from above that congruences can always be extended from subsemigroups whose complements are ideals and from homomorphic retracts (in particular, from group minimal ideals of commutative semigroups). Stralka's results on extending from idempotent subsemigroups in bands of groups and from subsemigroups of the regular elements which are
bands of groups in a medial semigroup are also results of the "relative" type. In addition to the results listed above which give sufficient conditions to allow congruence extension, we see several results which list properties that semigroups with CEP share. Namely, any element in a semigroup with CEP must have index less than or equal to three and hence, semigroups with CEP are periodic. Also, adjoining a zero or an identity is a process that retains CEP. Finally, with regard to the results listed above, we note that the first seven results of the list which deal with characterizations of congruences generated by relations and with principal congruences will provide useful techniques and reduction tools for proofs involving congruence extensions in the chapters that follow.

We now consider the related topic of ideal extension. A semigroup $S$ is said to have the **ideal extension property** (IEP) provided that for each subsemigroup $T$ of $S$ and each ideal $I$ of $T$, there exists an ideal $J$ of $S$ such that $J \cap T = I$.

A semigroup $S$ is said to have the **principal ideal extension property** (PIEP) provided that for each subsemigroup $T$ of $S$ and each $a \in T$,

$$T^1 a T^1 = S^1 a S^1 \cap T.$$ 

The following is a list of facts concerning the ideal extension property. Unless otherwise indicated, results for which a proof is not provided can be found in [Garcia, 1988].

1.28 A semigroup $S$ has the ideal extension property (IEP) if and only if each subsemigroup of $S$ has the ideal extension property.

1.29 Let $S$ be a semigroup, $T$ a subsemigroup of $S$ and $I$ an ideal of the subsemigroup $T$. Then there exists an ideal $J$ of $S$ such that $J \cap T = I$ if and only if $S^1 I S^1 \cap T = I$. 
Proof. Note that if $J \cap T = I$, then $J$ is an ideal of $S$ containing $I$ so $S^1IS^1 \subseteq J$. Thus, $S^1IS^1 \cap T \subseteq J \cap T = I$. Hence, $S^1IS^1 \cap T = I$ as the other inclusion always holds. Conversely, just let $J = S^1IS^1$. 

1.30 A homomorphic image of a semigroup with the ideal extension property (IEP) has IEP.

1.31 A semigroup $S$ has the ideal extension property (IEP) if and only if $S$ has the principal ideal extension property (PIEP).

1.32 A commutative semigroup with the congruence extension property (CEP) has the ideal extension property (IEP).

1.33 Corollary. Each semilattice has the ideal extension property (IEP).

Proof. Apply 1.32 and 1.11. 

1.34 Proposition. A cyclic semigroup $S$ has the ideal extension property (IEP) if and only if $\text{index}(S) \leq 3$. (For a proof of this proposition, see the section on monothetic semigroups in Chapter 6. The algebraic analogue of 6.6 for cyclic semigroups proves this proposition and the corollary which follows. We present the algebraic statements of these results here for use in the following chapter.)

1.35 Corollary. Let $S$ be a semigroup with the ideal extension property (IEP). Then $\text{index}(S) \leq 3$. (Hence, a semigroup with IEP is periodic.)

1.36 A group $G$ has the ideal extension property (IEP) if and only if $G$ is torsion.

Proof. If a group $G$ has IEP, then it follows from 1.35 that $G$ is torsion. Conversely, given a torsion group $G$ and a subsemigroup $T$ of $G$, $T$ is a subgroup of $G$ as remarked above. Thus, the only ideals of $T$ are the trivial
ones as \( T \) is a group and these certainly extend to the corresponding trivial ideals of \( G \). Hence, \( G \) has IEP. \( \blacksquare \)

Noting from above that the congruence extension property (CEP) implies the ideal extension property (IEP) in the case of commutative semigroups, it is reasonable to expect that IEP shares some of the known properties of CEP. Indeed, we see that IEP is hereditary, is equivalent to PIEP, is shared by semilattices, and implies that each element has index less than or equal to three. Also, any zero semigroup has IEP since any subset containing zero is an ideal and any semigroup of order less than four has IEP since a proper subsemigroup of such a semigroup has only the trivial ideals and these always extend. However, we also see that IEP is retained by homomorphic images, and whether this holds for CEP remains an open question.

We now consider semigroups in which each congruence is determined by an ideal. A congruence \( \sigma \) on a semigroup \( S \) is called an ideal congruence provided that there exists an ideal \( I \) of \( S \) such that \( \sigma = (I \times I) \cup \Delta_S \). A semigroup \( S \) is called an ideal semigroup if each congruence on \( S \) is an ideal congruence.

The following is a list of results concerning ideal semigroups which can be found in [Garcia, 1988].

1.37 An ideal semigroup has a zero element.

1.38 The homomorphic image of an ideal semigroup is an ideal semigroup.

1.39 An ideal semigroup \( S \) has the congruence extension property (CEP) if and only if \( S \) has the ideal extension property (IEP) and each subsemigroup of \( S \) is an ideal semigroup.

1.40 An ideal semigroup \( S \) is congruence free if and only if \( S \) is 0-simple.

In the chapters that follow we will be particularly interested in determining the
structure of commutative semigroups which have the properties discussed above. For this reason, we need to consider known results concerning the structure of a general commutative semigroup.

A commutative semigroup $S$ is said to be archimedean provided that for any two elements of $S$, each divides some power of the other. We will use "$|$" to denote "divides". Thus, $a|b$ means that $b = xa$ for some $x \in S^1$.

Define a relation $\eta$ on a commutative semigroup $S$ as follows:

$$(a, b) \in \eta \equiv a|b^n \text{ and } b|a^m \text{ for some } n, m \in \mathbb{N}$$

1.41 [Clifford and Preston, 1961] The relation $\eta$ on any commutative semigroup $S$ is a congruence on $S$, and $S/\eta$ is the maximal semilattice homomorphic image of $S$.

1.42 [Clifford and Preston, 1961] Every commutative semigroup $S$ is uniquely expressible as a semilattice $Y$ of archimedean semigroups $C_\alpha (\alpha \in Y)$. The semilattice $Y$ is isomorphic with the maximal semilattice homomorphic image $S/\eta$ of $S$, and the $C_\alpha (\alpha \in Y)$ are the equivalence classes of $S$ mod $\eta$.

We call the archimedean semigroups $C_\alpha$ the archimedean components of $S$. Obviously, a commutative semigroup is archimedean provided that it has only one archimedean component (or $\eta$-class). For convenience, we make the following notational adjustment. Instead of indexing the archimedean components as above, let $\mathcal{A}$ be a system of distinct representatives of the archimedean components. Then a commutative semigroup $S$ is a semilattice of archimedean semigroups $C(x) (x \in \mathcal{A})$. That is,

$$S = \bigcup_{x \in \mathcal{A}} C(x) \text{ where } C(x)C(y) \subseteq C(xy)$$
Note that the "where" statement above is simply the condition that $S$ is a semilattice of its components.

Since a semigroup which has either the congruence extension property (CEP) or the ideal extension property (IEP) is periodic according to 1.15 and 1.35, we are particularly interested in the structure of periodic commutative semigroups. Generally, it is well-known that an archimedean semigroup (and hence, any archimedean component of a commutative semigroup) has at most one idempotent. If a commutative semigroup is periodic, then each of its archimedean components must contain an idempotent as some power of each element of $S$ is idempotent and each component is a subsemigroup. Therefore, in the periodic case we may choose the unique idempotent in each archimedean component to form the system of distinct representatives $A$ and we have

$$S = \bigcup_{e \in E_S} C(e) \text{ where } C(e)C(f) \subseteq C(ef)$$

Furthermore, in the periodic case we have $C(e) = \{x \in S : x^n = e \text{ for some } n \in \mathbb{N}\}$. (Note that from the previous comments it is easy to see that a periodic semigroup is archimedean if and only if it is a commutative semigroup with exactly one idempotent.)

In order to obtain a more complete description of the structure of periodic commutative semigroups, we must consider the structure of the archimedean components themselves. Of special interest to us will be archimedean semigroups which have a zero element. These types of semigroups are sometimes referred to as "nil" semigroups. Hence, we will often use the letter $N$ to denote them. However, we will refer to such semigroups as archimedean semigroups with zero. The following is a list of known results concerning archimedean semigroups with zero.
1.43 Let $S$ be a commutative semigroup with zero. Then $S$ is archimedean if and only if for each $x \in S$, $x^n = 0$ for some $n \in \mathbb{N}$. In particular, if $S$ is an archimedean semigroup with zero, then $S$ is periodic, any subsemigroup of $S$ must contain zero, and any finitely generated subsemigroup of $S$ is finite.

(To see that the first statement holds, note that in an archimedean semigroup $S$ with zero, zero must divide some power of each element by definition of an archimedean semigroup. However, the only element that zero divides is zero itself and thus, we see that some power of each element of $S$ is zero. Conversely, if for each $x \in S$, $x^n = 0$ for some $n \in \mathbb{N}$, then $S$ is a commutative periodic semigroup so by the note above, $C(0) = \{x \in S : x^n = 0 \text{ for some } n \in \mathbb{N}\} = S$. Hence, $S$ is archimedean. To see that the final statement holds, let $T$ be a subsemigroup of an archimedean semigroup $S$ with zero. For $x \in T$, $0 = x^n \in T$ for some $n \in \mathbb{N}$. Suppose $T$ is finitely generated. It is clear that the cyclic semigroup generated by each generator is finite. Taking all products between elements of these cyclic semigroups yields $T$. There are a finite number of products as there are a finite number of generators. Hence, $T$ is finite.)

1.44 [Yamada, 1969] Let $S$ be an archimedean semigroup with zero. Then for $a, b \in S$, $ab = b$ if and only if $b = 0$.

(This is actually stated for finite semigroups in [Yamada, 1969], but it is true in general. To see this let $ab = b$. As above, $a^n = 0$ for some $n \in \mathbb{N}$. Thus, $0 = 0 \cdot b = a^n b = a^{n-1} (ab) = a^{n-1} b = \ldots = ab = b$ as desired. The converse is obvious.)
1.45 [Yamada, 1964] The annihilator of a nontrivial, finite, archimedean semigroup with zero has a nonzero element. (The annihilator of a semigroup $S$ is given by $A(S) = \{ x \in S : xS = Sx = \{0\} \}$.)

In the results that follow, we will see that archimedean semigroups with zero and commutative groups form the "building blocks" of general periodic archimedean semigroups. To see how these "building blocks" are connected we recall the definition of an ideal extension of a semigroup by a semigroup with zero.

A semigroup $V$ is an ideal extension of a semigroup $S$ by a semigroup with zero $T$ if it contains $S$ as an ideal, and if $V/S$ is isomorphic with $T$.

By a partial homomorphism of a partial groupoid $S$ into a partial groupoid $T$, we mean a mapping $\phi$ of $S$ into $T$ such that if $a, b \in S$ and $ab$ is defined in $S$, then the product $\phi(a)\phi(b)$ is defined in $T$ and is equal to $\phi(ab)$. By a partial isomorphism of $S$ onto $T$, we mean a one-to-one partial homomorphism $\phi$ of $S$ onto $T$.

Now for $V$ an ideal extension of $S$ by $T$, let $\alpha : V \rightarrow V/S$ be the natural homomorphism and let $\beta$ be an isomorphism from $V/S$ to $T$. Then since $\alpha$ is one-to-one on $V\setminus S$, it is easy to see that $\beta \circ \alpha|_{V\setminus S}$ is a partial isomorphism from $V\setminus S$ onto $T\setminus \{0\}$. It is customary to identify $V\setminus S$ with $T\setminus \{0\}$. We write $V = S \cup T\setminus \{0\}$.

According to the definition above, a mapping $\phi : T\setminus \{0\} \rightarrow S$ is a partial homomorphism if and only if $\phi(a)\phi(b) = \phi(ab)$ whenever $ab \neq 0$. Concerning ideal extensions, we have the following theorem.
1.46 [Clifford and Preston, 1961] A partial homomorphism $\phi : T \setminus \{0\} \to S$ determines an ideal extension $V$ of $S$ by $T$ as follows: ($\ast$ denotes multiplication in $S$ and $\odot$ denotes multiplication in $T$)

$$
ab = \begin{cases} 
a \ast b & \text{if } a, b \in S; 
a \odot b & \text{if } a, b \in T \setminus \{0\} \text{ and } a \odot b \neq 0; 
\phi(a) \ast \phi(b) & \text{if } a, b \in T \setminus \{0\} \text{ and } ab = 0; 
\phi(a) \ast b & \text{if } a \in T \setminus \{0\} \text{ and } b \in S; 
a \ast \phi(b) & \text{if } a \in S \text{ and } b \in T \setminus \{0\}. 
\end{cases}
$$

If $S$ has an identity element, then every extension of $S$ by $T$ is found in this fashion.

We now return to the task of describing periodic archimedean semigroups. According to [Tamura, 1969a], any archimedean semigroup is of one of the following forms:

(1) An archimedean semigroup with zero.

(2) An ideal extension of an abelian group $G$, $|G| > 1$, by an archimedean semigroup with zero.

(3) An archimedean torsion-free semigroup.

Note that if we allow $|G| = 1$ in (2), we obtain a semigroup of type (1). That is, an ideal extension of a trivial group by an archimedean semigroup with zero is an archimedean semigroup with zero. (In general, the zero of the semigroup that one is “extending by” is replaced with the semigroup that one is “extending” making that semigroup an ideal of the extension. In this case, the trivial group replaces the zero of the original archimedean semigroup.) At the other extreme, an ideal extension of a group by a trivial archimedean semigroup with zero is the original group. Thus, ideal extensions of abelian groups by archimedean semigroups with zero are precisely those semigroups listed in (1) and (2) above and this collection certainly includes all abelian groups and archimedean semigroups with zero. Now by definition of
periodicity, it is obvious that any periodic archimedean semigroup cannot be of form (3). Thus, by these observations along with the fact that ideal extensions of monoids (hence, also of groups) are determined by partial homomorphisms by 1.46, we have the following result.

1.47 Corollary. A periodic archimedean semigroup $S$ is an ideal extension of an abelian group $G$ by an archimedean semigroup with zero $N$ which is determined by a partial homomorphism $\phi : N\setminus\{0\} \to G$.

It is well-known that a subgroup of a semigroup $S$ which is an ideal must be the minimal ideal $M(S)$ of $S$. Thus, in 1.47, $M(S) = G$. Now it is clear that a subsemigroup of a periodic semigroup is again periodic. Hence, from the discussion above we obtain that a periodic commutative semigroup $S$ is a semilattice of periodic archimedean semigroups $C(e)$ ($e \in E_S$), and for each $e \in E_S$, $C(e)$ is an ideal extension of an abelian group, which we will denote by $G_e$, by an archimedean semigroup with zero $N_e$ which is determined by a partial homomorphism $\phi_e : N_e\setminus\{0\} \to G_e$. Furthermore, $G_e = M(C(e))$ for each $e \in E_S$.

We now present a summary theorem regarding the structure of periodic commutative semigroups. (Recall that in the set $E_S$ of idempotents of a semigroup $S$, $e \leq f$ means $e = ef = fe$.)

1.48 Periodic Commutative Semigroup Structure Theorem. Let $S$ be a periodic commutative semigroup. Then

$$S = \bigcup_{e \in E_S} C(e)$$

where each $C(e)$, $(e \in E_S)$ is an archimedean component subsemigroup of $S$ and the following hold:

(1) $C(e)C(f) \subseteq C(ef)$ for $e, f \in E_S$
(2) For \( x \in C(e), e \in E_S \), there exists \( n \in \mathbb{N} \) such that \( x^n = e \). Consequently, if \( T \) is a subsemigroup of \( S \) and \( T \cap C(e) \neq \emptyset \), then \( e \in T \).

(3) For each \( e \in E_S \), \( C(e) \) is an ideal extension of an abelian group \( G_e = M(C(e)) \) by an archimedean semigroup with zero \( N_e \) which is determined by a partial homomorphism \( \phi_e : N_e \setminus \{0_e\} \rightarrow G_e \).

(4) If \( e \leq f \), \( (e, f \in E_S) \), then \( C(f)M(C(e)) \subseteq M(C(e)) \).

(5) For \( T \) a subsemigroup of \( S \), \( T = \bigcup_{e \in E_T} C_T(e) \) where \( C_T(e) = C(e) \cap T \).

As noted earlier, a semigroup \( S \) which has the congruence extension property (CEP) or the ideal extension property (IEP) is periodic. Hence, the structure described above holds in commutative semigroups with these properties. Furthermore, using the index condition guaranteed by 1.15 and 1.35, we obtain the following proposition.

**1.49 Proposition.** Let \( S \) be a commutative semigroup with the ideal extension property (IEP) or the congruence extension property (CEP). Then \( S \) is periodic and 1.48 holds in \( S \). Moreover, if \( x \in C(e), e \in E_S \), then we have \( x^n \in M(C(e)) = G_e \) for all \( n \geq 3 \).
(We need only see that the last statement holds. According to [Tamura, 1954b], $G_e$ is the greatest group of $C(e)$. That is, any subgroup of $C(e)$ is contained in $G_e$. Now for $x \in C(e)$, $M(\theta(x))$ is a subgroup of $C(e)$. Hence, $M(\theta(x)) \subseteq G_e$. By 1.15 and 1.35, $x^n \in M(\theta(x))$ for all $n \geq 3$. Therefore, $x^n \in M(\theta(x)) \subseteq G_e = M(C(e))$ for all $n \geq 3$ as desired.)

1.50 Corollary. Let $S$ be an archimedean semigroup with zero with the ideal extension property (IEP) or the congruence extension property (CEP). Then $x^n = 0$ for all $n \geq 3$ and $x \in S$.

(This is immediate from 1.49.)

A semigroup is called a $\Delta$-semigroup if its congruences form a chain under inclusion. In [Tamura, 1969b], commutative $\Delta$-semigroups are characterized. We recall several definitions before presenting this characterization.

If $S$ is a semigroup and $a, b \in S$, define $a \leq_{\mathcal{H}} b$ provided $a \in S^1 b \cap b S^1$. It is well-known that $\leq_{\mathcal{H}}$ is a quasi-order (i.e. a reflexive and transitive relation). We say that a semigroup $S$ is an $\mathcal{H}$-chain if $\leq_{\mathcal{H}}$ is a total order (i.e. a partial order in which each pair of elements is comparable) on $S$. Define $\mathcal{H} = \leq_{\mathcal{H}} \cap \leq_{\mathcal{H}}^{-1}$. It is well-known that $\mathcal{H}$ is an equivalence on any semigroup $S$ and if $S$ is commutative, then $\mathcal{H}$ is a congruence on $S$. Furthermore, for $S$ a commutative semigroup, $a \leq_{\mathcal{H}} b$ provided $a \in b S^1$ and $(a, b) \in \mathcal{H}$ provided $a S^1 = b S^1$. Note that for a commutative semigroup $S$, $\leq_{\mathcal{H}}$ is a divisibility ordering on $S$. That is, $a \leq_{\mathcal{H}} b$ provided $b | a$.

It is easy to see that $\mathcal{H} = \Delta_S$ if and only if $\leq_{\mathcal{H}}$ is a partial order on $S$. That is, anti-symmetry of $\leq_{\mathcal{H}}$ implies and is implied by $\mathcal{H} = \Delta_S$. Also, if $S$ is an archimedean semigroup with zero, then anti-symmetry of $\leq_{\mathcal{H}}$ follows from 1.44. Hence, $\leq_{\mathcal{H}}$ is a partial order and $\mathcal{H} = \Delta_S$ for any archimedean semigroup with zero.
A group $G$ is said to be a quasicyclic group provided that for some prime $p$, $G$ is the union of an ascending chain of cyclic groups $C_n$ of order $p^n$. That is,

$$G = \bigcup_{n=1}^{\infty} C_n \text{ where } C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n \subseteq \ldots \text{ and } |C_n| = p^n$$

Note that a quasicyclic group is torsion as each of its elements lies in an subgroup of order $p^n$, for some $n$.

1.51 [Tamura, 1969b] A commutative semigroup $S$ is a $\Delta$-semigroup if and only if $S$ is of one of the following types:

(1) A quasicyclic group;

(2) A quasicyclic group with zero adjoined;

(3) An archimedean semigroup with zero which is an $\mathcal{H}$-chain;

(4) A semigroup of type (3) with identity adjoined.
In this chapter, we will characterize commutative semigroups which have the ideal extension property (IEP). This characterization describes the multiplicative structure of commutative semigroups with IEP. Establishing this characterization was motivated not only by an interest in IEP itself, but also by the fact that in the category of commutative semigroups, the congruence extension property (CEP) implies IEP. The characterization obtained here will be used in the following chapter in which we seek to characterize commutative semigroups with CEP.

Based on the facts that IEP is hereditary, that any semigroup with IEP is periodic, and that any periodic commutative semigroup is a semilattice of its periodic archimedean components, we first characterize those periodic archimedean semigroups which have IEP. Once this is done, one might hope that a commutative semigroup has IEP provided that its archimedean components have IEP. However, as we will see in Example 2.6, this is not the case. Instead, a necessary and sufficient condition on the multiplication between components is given which completes the characterization of commutative semigroups with IEP.

As indicated above, we begin by characterizing periodic archimedean semigroups which have IEP. We recall from Chapter 1 that a periodic archimedean semigroup is an ideal extension of an abelian group $G$ by an archimedean semigroup $N$ with zero. (The term “ideal extension” as defined and discussed in Chapter 1 is
not to be confused with the ideal extension property.) It is natural to conjecture, and we will indeed show, that such a semigroup has IEP provided that $G$ and $N$ have IEP. According to Chapter 1, $G$ has IEP provided that it is torsion. Thus, we need only determine conditions under which an archimedean semigroup with zero has IEP. This is accomplished in the following lemma.

2.1 Lemma. Let $S$ be an archimedean semigroup with zero. Then the following are equivalent:

(1) If $x, y \in S$ and $xy \neq 0$, then $xy = x^2 = y^2$.

(2) Each subsemigroup of $S$ is an ideal.

(3) $S$ has IEP.

Proof. To see that (1) implies (2), assume (1) holds and let $T$ be a subsemigroup of $S$. By the remarks in Chapter 1 on archimedean semigroups with zero, $0 \in T$. Let $x \in T$ and $y \in S$. If $xy = 0$, then $xy \in T$. Otherwise, by (1), $xy = x^2 \in T$. Thus, $T$ is an ideal and (2) holds.

Now assume (2) holds. Let $T$ be a subsemigroup of $S$ and let $I$ be an ideal of $T$. Then $I$ is a subsemigroup of $S$ and hence an ideal of $S$ by (2). Thus, $I \cap T = I$ is its own extension to $S$ and $S$ has IEP. Thus, (2) implies (3). (Note that this holds for any semigroup.)

To see that (3) implies (1), suppose $S$ has IEP and let $x, y \in S$ with $xy \neq 0$. Suppose that it is not the case that $xy = x^2 = y^2$. Then we must have either $xy \neq x^2$ or $xy \neq y^2$. We may assume without loss of generality that $xy \neq x^2$. Let $T = (0, xy, x)$. According to 1.50, $z^n = 0$ for all $z \in S$, $n \geq 3$. Using this fact, $T = \{0, x, xy, x^2, x^2y, x^2y^2\}$ gives a complete listing of the elements of $T$ where these elements are not necessarily distinct. Note that $xy \in xS^4 \cap T$. 
Claim: $xy \notin xT^1$

Now $x \cdot 0 = x \cdot x^2 = x \cdot x^2 y = x \cdot x^2 y^2 = 0 \neq xy$. Also $xy \notin x^2$ by assumption and $x \neq xy \neq x \cdot xy$ by 1.44 since $x \neq 0 \neq xy$. Thus, $xy \notin xT^1$.

Thus, $xy \in xS^1 \cap T$, but $xy \notin xT^1$ so $xS^1 \cap T \neq xT^1$. By 1.29, this contradicts that $S$ has IEP. We conclude that (3) implies (1).

2.2 Examples. This pair of simple examples illustrates how property (1) of Lemma 2.1 can be used, in particular, to easily determine whether a given finite archimedean semigroup with zero has IEP. Semigroup (1) does not have IEP while semigroup (2) has IEP.

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \\
1 & 1 & 2 & 2 & 1 & 1 & 1 & 2 \\
1 & 1 & 2 & 1 & 2 & 1 & 1 & 2 \\
\end{array}
\]

(1) \hspace{1cm} (2)

Note that in semigroup (1), $3 \cdot 4 = 2 \neq 1$, (1 is the zero of these semigroups), but $3^2 \neq 2$ in violation of property (1). Translating property (1), one sees that if a nonzero entry appears in the $(n, m)$ position of the table, then that same nonzero entry must appear in the $n^{th}$ and $m^{th}$ diagonal positions, "completing the nonzero square" in a sense. Certainly, this holds in semigroup (2) which has IEP.

Furthermore, note that in semigroup (1), $\{1,3\}$ is a subsemigroup which is not an ideal of $S$, violating condition (2), but $\{1,3\}$ is an ideal of the subsemigroup $\{1,2,3\}$ which has no extension to $S$ as any ideal of $S$ containing $\{1,3\}$ also contains 2.

As promised, we now show that a periodic archimedean semigroup $S$ has IEP provided that $G$ and $N$ have IEP, where $S$ is an ideal extension of $G$, an abelian group, by $N$, an archimedean semigroup with zero. We write $S = G \cup N \setminus \{0\}$. For a
definition and discussion of the structure of semigroups which are ideal extensions, the reader should refer to Chapter 1.

2.3 Lemma. Let $S$ be a periodic archimedean semigroup. Hence, $S$ is an ideal extension of an abelian group $G$ by an archimedean semigroup $N$ with zero. Then $S$ has IEP if and only if $G$ and $N$ have IEP.

Proof. Suppose $S$ has IEP. Since IEP is hereditary and is retained by homomorphic images (1.28 and 1.30), $G$ and $N$ have IEP as $G$ is an ideal (hence, a subsemigroup) of $S$ and $N \cong S/G$.

Now suppose that $G$ and $N$ have IEP. Let $T$ be a subsemigroup of $S$ and let $I$ be an ideal of $T$. We have $G \cap T$ is a subsemigroup of $G$ and $G$ is a torsion group as it has IEP(1.36). Hence, $G \cap T$ is a subgroup of $G$. Also, since $G$ is an ideal of $S$, $G \cap T$ is an ideal of $T$. Thus, $G \cap T$ is a group which is an ideal of $T$. Hence, $M(T) = G \cap T$, as it is well-known that any subgroup which is an ideal must be the minimal ideal. We claim that $S^1 I \cap T = I$. We need only show the forward inclusion. Let $s \in S^1$, $x \in I$ such that $sx \in T$. Now if $sx \in G$, then $sx \in G \cap T = M(T) \subseteq I$. If $sx \notin G$, then $s, x \notin G$ as $G$ is an ideal. Furthermore, $s, x \in N \setminus \{0\}$ and $sx \neq 0$. Now $N$ has IEP, so (1) of Lemma 2.1 holds. Hence, $sx = x^2$. Thus, $sx = x^2 \in I$ as $x \in I$ and our claim is proven. Thus, $S^1 I$ extends $I$ and $S$ has IEP.

2.4 Corollary. Let $S$ be a periodic archimedean semigroup. Then $S$ has IEP if and only if $G$ is torsion and $xy = x^2 = y^2$ for all $x, y \in N$ with $xy \neq 0$.

Proof. Apply Lemmas 2.1 and 2.3 and 1.36.
2.5 Examples. This pair of examples is provided to illustrate the results above and to give the reader a better understanding of ideal extensions. Semigroup (1) is an archimedean semigroup which has IEP while semigroup (2) is an archimedean semigroup which does not have IEP.

\[
\begin{align*}
(1) & & 1 & 2 & 2 & 2 & 2 & 6 & 6 & 1 & 1 & 1 & 5 & 5 & 5 \\
& & 2 & 6 & 6 & 6 & 1 & 1 & 1 & 1 & 1 & 5 & 5 & 5 \\
& & 2 & 6 & 7 & 6 & 7 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 \\
& & 2 & 6 & 6 & 7 & 7 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 \\
& & 2 & 6 & 7 & 7 & 7 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 \\
& & 1 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 & 1 & 1 & 1 \\
& & 1 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 & 1 & 1 & 1 \\
& & 6 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 & 6 & 1 & 1 & 2 \\
(2) & & 1 & 2 & 2 & 2 & 2 & 6 & 6 & 1 & 1 & 5 & 5 & 5 \\
& & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
& & 1 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 & 1 & 1 & 1 \\
& & 2 & 6 & 6 & 7 & 7 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 \\
& & 1 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 & 1 & 1 & 1 \\
& & 6 & 1 & 1 & 1 & 2 & 2 & 5 & 5 & 5 & 6 & 1 & 1 & 2 \\
\end{align*}
\]

Semigroup (1) is an ideal extension of the cyclic group \{1, 2, 6\} of order three by the archimedean semigroup with zero \{0, 3, 4, 5, 7\}. Notice here that the only “nonzero” product among the elements \{3, 4, 5, 7\} (i.e. products that do not fall in the group) is 7 and one easily sees that property (1) of 2.1 holds in \{0, 3, 4, 5, 7\}. Thus, the (obviously torsion) group and archimedean semigroup with zero which compose semigroup (1) each have IEP and the semigroup itself has IEP. Semigroup (2) is an ideal extension of the cyclic group of order two \{1, 5\} by the archimedean semigroup with zero \{0, 2, 3, 4, 6, 7\}. Notice that \(7 \cdot 4 = 6\) is a “nonzero” product, but \(7 \cdot 4 = 6 \neq 2 = 4^2\) in violation of property (1) of 2.1. Thus, \{0, 2, 3, 4, 6, 7\} does not have IEP. Hence, semigroup (2) does not have IEP. One notes that \{1, 2, 4, 5\} is an ideal of the subsemigroup \{1, 2, 4, 5, 6\} which does not extend as any ideal of semigroup (2) which contains \{1, 2, 4, 5\} also contains 6.

Having completed the characterization of periodic archimedean semigroups with IEP, we turn our attention to general commutative semigroups. As indicated in the introduction to this chapter, we know that a commutative semigroup
with IEP is a semilattice of periodic archimedean semigroups (called archimedean components and denoted $C(e)$ where $e$ is the unique idempotent of the component) each of which has IEP as IEP is a hereditary property. However, the fact that a given commutative semigroup's components each have IEP is not enough to insure IEP in the entire semigroup. Consider the following examples.

2.6 Examples. In the pair of examples below, each archimedean component has IEP, but semigroup (1) has IEP while semigroup (2) does not have IEP.

![Examples](image)

In semigroup (1), there are three archimedean components, $C(1) = \{1, 2\}$, $C(3) = \{3, 4, 5\}$, and $C(6) = \{6, 7\}$. That each of these components has IEP is clear for a number of reasons. One observes that each is a zero semigroup and has order less than or equal to three and as noted in Chapter 1, any semigroup with either of these properties has IEP. Also, each is an archimedean semigroup with zero which trivially satisfies property (1) of 2.1. Now $M(C(1)) = \{1\}$, $M(C(3)) = \{3\}$, and $M(C(6)) = \{6\}$. We observe that for $e, f \in E_S = \{1, 3, 6\}, x \in C(e)$, and $y \in C(f)$, we have that either $xy = x$, $xy = y$, or $xy \in M(C(ef))$. However, in semigroup (2) having three components, $C(1) = \{1, 2, 3\}$, $C(4) = \{4, 5, 6\}$, and $C(7) = \{7\}$ with $M(C(1)) = \{1\}$, $M(C(4)) = G_4 = \{4, 5\}$, and $M(C(7)) = \{7\}$, each component has IEP by the order condition above or by applying lemmas from this chapter, but $4 \cdot 3 = 2 \not\in M(C(1)) = \{1\}$ so the condition on multiplication
between components that was observed to hold in semigroup (1) which has IEP does not hold in semigroup (2) which lacks IEP. Furthermore, one can see that \{1, 3\} is an ideal of the subsemigroup \(C(1) = \{1, 2, 3\}\) which has no extension to \(S\) as any ideal of semigroup (2) containing \{1, 3\} also contains 2.

While the example above gives only an indication that multiplication between components of a commutative semigroup \(S\) with IEP must obey the observed condition, each possible case (based on the relationship of idempotents) can be resolved to yield the following characterization of commutative semigroups which have IEP.

2.7 Theorem. Let \(S\) be a commutative semigroup. Then \(S\) has IEP if and only if

\[
S = \bigcup_{e \in E_S} C(e)
\]

where (1) \(C(e)\) has IEP for all \(e \in E_S\) and;

(2) If \(e\) and \(f\) are distinct idempotents, \(x \in C(e)\), and \(y \in C(f)\), then either

\[xy \in M(C(ef)), \quad xy = x, \quad \text{or} \quad xy = y.\]

Proof. Suppose \(S\) has IEP. According to 1.35, semigroups with IEP are periodic. Hence, by 1.48,

\[
S = \bigcup_{e \in E_S} C(e)
\]

Also, IEP is hereditary by 1.28. Hence, (1) holds. To show that (2) holds, let \(e, f \in E_S\) be distinct, \(x \in C(e)\), and \(y \in C(f)\).

Claim 1: If \(e = ef\), then \(xy \in M(C(e))\) or \(xy = x\).

Let \(e = ef\) and suppose that \(xy \notin M(C(e))\) and \(xy \neq x\). We seek a contradiction in each case below.
**Case 1:** Assume $xy \neq x^2$.

We first show that $C(e)^1x = \{x, x^2\} \cup M(C(e))$. The reverse containment is clear as $C(e)^1x$ is an ideal of $C(e)$ and contains $x$. In order to prove the forward containment, let $z \in C(e)$. If $zx \notin M(C(e)) = G_e$, then $z, x \notin M(C(e)) = G_e$. Furthermore, $z, x \in N_e$ and $zx \neq 0_{N_e}$. Thus, $zx = x^2$ by Corollary 2.4 since $C(e)$ has IEP as (2) holds. Whence, the forward containment is proven. Now we have $xy \in S^1x \cap C(e)$, but $xy \notin \{x, x^2\} \cup M(C(e)) = C(e)^1x$ according to our assumptions. Thus, $S^1x \cap C(e) \neq C(e)^1x$ contrary to $S$ having IEP.

**Case 2:** Assume $xy = x^2$.

Let $T = M(C(e)) \cup \theta(y) \cup \{x^2\}$. We show that $T$ is a subsemigroup.

Now $M(C(e))^2 \subseteq M(C(e)) \subseteq T$ and $\theta(y)^2 \subseteq \theta(y) \subseteq T$. Also, we have $(x^2)^2 = x^4 \in M(C(e)) \subseteq T$ by 1.49. According to 1.48(4), we know that $(M(C(e)))^2 \subseteq (M(C(e)))(C(f)) \subseteq M(C(e)) \subseteq T$ and certainly, $(M(C(e)))x^2 \subseteq M(C(e)) \subseteq T$. Now $x^2y = x(xy) = xx^2 = x^3 \in M(C(e))$ by 1.49 and for $n \geq 2$,

$$x^2y^n = x(xy)y^{n-1} = x(x^2)y^{n-1} = x^3y^{n-1} \in (M(C(e)))(C(f)) \subseteq M(C(e))$$

by 1.49 and 1.48(4). Hence, $x^2(\theta(y)) \subseteq M(C(e)) \subseteq T$. This completes the proof that $T$ is a subsemigroup.

The next claim is that $x^2 = xy \notin T^1y$. Now $C(e) \cap C(f) = \emptyset$ and $x^2 \in C(e)$ so $x^2 \neq 1 \cdot y \in C(f)$ and $x^2 \notin (\theta(y))y \subseteq \theta(y) \subseteq C(f)$. Also,

$$x^2 \neq x^2y = x(xy) = x(x^2) = x^3 \in M(C(e))$$

by our assumptions. Finally, if it were true that $x^2 \in M(C(e)) \cdot y$, then
we would obtain that

\[ xy = x^2 \in (M(C(e)))y \subseteq (M(C(e)))(C(f)) \subseteq M(C(e)) \]

contrary to our assumptions. Thus, \( x^2 = xy \notin T^1y \). However, we have

\[ x^2 = xy \in S^1y \cap T \text{ so } S^1y \cap T \neq T^1y \text{ contrary to } S \text{ having IEP.} \]

Thus, Claim 1 is proven. Dually, if \( f = ef \), then either \( xy \in M(C(f)) \) or \( xy = y \).

Claim 2: If \( ef \neq e \) and \( ef \neq f \), then \( xy \in M(C(ef)) \).

Let \( ef \neq e \), \( ef \neq f \) and suppose \( xy \notin M(C(ef)) \). First note that if \( xv = xy = uy \) for some \( u \in M(C(e)) \), \( v \in M(C(f)) \), then using the fact that \( f \) is an identity for \( u \) and \( e \) is an identity for \( v \) and \( y^m = f \) for some \( m \), we obtain that

\[ xy = xv = xvf = xvy^m = (xy)vy^{m-1} = (uy)vy^{m-1} = uvy^m = uvf \]

\[ = (ue)vf = (ef)(uv) \in (M(C(ef))(C(ef)) \subseteq M(C(ef)) \]

which is a contradiction. Thus, either \( xy \neq xv \) for all \( v \in M(C(f)) \) or \( xy \neq uy \) for all \( u \in M(C(e)) \). We may assume without loss of generality that \( xy \neq xv \) for all \( v \in M(C(f)) \) as the proof in the other case is dual to that which follows.

Let \( T = M(C(ef)) \cup \theta(y) \cup \theta(y)\theta(x) \). We first show that \( T \) is a subsemigroup. Certainly,

\[ M(C(ef))^2 \subseteq M(C(ef)) \subseteq T, \ \theta(y)^2 \subseteq \theta(y) \subseteq T, \text{ and } (\theta(x)\theta(y))^2 \subseteq \theta(x)\theta(y) \subseteq T. \]

Also, \( (\theta(x)\theta(y))\theta(y) \subseteq \theta(x)\theta(y) \subseteq T \) and

\[ \theta(x)\theta(y)M(C(ef)) \subseteq C(ef)M(C(ef)) \subseteq M(C(ef)) \subseteq T. \]

Finally,

\[ \theta(y)M(C(ef)) \subseteq C(f)M(C(ef)) \subseteq M(C(ef)) \subseteq T \]
by 1.48(4) as \( ef \leq f \). Thus, \( T \) is a subsemigroup. We claim that \( xy \notin T^1y \). Now \( C(f) \cap C(ef) = \emptyset \) and \( xy \in C(ef) \) so \( xy \neq 1 \cdot y \in C(f) \) and \( xy \notin (\theta(y))y \subseteq C(f) \).

Furthermore, we know that

\[
xy \notin (M(C(ef)))y \subseteq (M(C(ef)))(C(f)) \subseteq M(C(ef))
\]

by assumption. Finally, suppose that \( xy \in (\theta(x)\theta(y))y \). In this case, \( xy = x^ny^m \) for some \( n \geq 1, m \geq 2 \). Now if \( xy = x^ny^m \) where \( n, m \geq 2 \), then

\[
xy = x^ny^m
\]

\[
= (xy)(x^{n-1}y^{m-1})
\]

\[
= (x^ny^m)(x^{n-1}y^{m-1})
\]

\[
= (xy)(x^{2n-2}y^{2m-2})
\]

\[
= (x^ny^m)(x^{2n-2}y^{2m-2})
\]

\[
= x^{3n-2}y^{3m-2}
\]

\[
= (xy)^3(x^{3n-5}y^{3m-5}).
\]

Now \( (xy)^3 \in M(C(ef)) \) and \( x^{3n-5}y^{3m-5} \in C(ef) \) since \( 3n - 5, 3m - 5 \geq 1 \). Thus, \( xy = (xy)^3(x^{3n-5}y^{3m-5}) \in (M(C(ef)))(C(ef)) \subseteq M(C(ef)) \) contrary to assumption. Thus, \( xy \neq x^ny^m \) where \( n, m \geq 2 \). Now if \( xy = x^ny^m \) for \( m \geq 2 \), then \( xy = xy^m = (xy)y^{m-1} = (x^ny^m)y^{m-1} = xy^{2m-1} \). Now \( 2m - 1 \geq 3 \) so by 1.49, \( y^{2m-1} \in M(C(f)) \). This contradicts that \( xy \neq xv \) for all \( v \in M(C(f)) \). Thus, \( xy \notin (\theta(x)\theta(y))y \). This completes the proof that \( xy \notin T^1y \).

We have \( xy \in S^1y \cap T \) and \( xy \notin T^1y \) so \( S^1y \cap T \neq T^1y \) which is contrary to \( S \) having IEP. This completes the proof of Claim 2. By Claims 1 and 2, the forward implication of the theorem is proven.
Conversely, suppose

\[ S = \bigcup_{e \in E_S} C(e) \]

where (1) and (2) hold and let \( T \) be a subsemigroup of \( S \). Let \( I \) be an ideal of \( T \).

Now \( C(e) \) is periodic for each \( e \in E_S \) by (2). Hence, as discussed in Chapter 1, if \( x \in C(e) \), then \( x^n = e \) for some \( n \in \mathbb{N} \). Thus, some power of each element of \( S \) is idempotent. Hence, \( S \) is periodic so 1.48 holds. By 1.48(5), \( T = \bigcup_{e \in E_T} C_T(e) \) where \( C_T(e) = C(e) \cap T \). We claim that \( S^1 I \cap T = I \). We need only show the forward inclusion so let \( s \in S^1, x \in I \) such that \( sx \in T \). We must show that \( sx \in I \). Now since (1) holds and semilattices have IEP (1.33), we have:

\[ C(e)^1(I \cap C_T(e)) \cap C_T(e) = I \cap C_T(e) \text{ for all } e \in E_S \]

(*)

and

\[ E_S^1(I \cap E_T) \cap E_T = I \cap E_T \]

(**)

since \( I \cap C_T(e) \) is an ideal of \( C_T(e) \) and \( I \cap E_T \) is an ideal of \( E_T \). We have \( s \in C(e) \) and \( x \in C_T(f) \cap I \) for some \( e, f \in E_S \). In each case below, we show \( sx \in I \).

**Case 1:** Assume \( e = f \).

Now we have \( s, x \in C(e) \) so \( sx \in C(e) \) and by assumption \( sx \in T \). Thus, we have \( sx \in C(e) \cap T = C_T(e) \). Hence, \( sx \in C(e)^1(I \cap C_T(e)) \cap C_T(e) = I \cap C_T(e) \) by (\( * \)). Therefore, \( sx \in I \).

**Case 2:** Assume \( e \neq f \).

We have the following three subcases as (2) holds:

**Subcase 1:** Assume \( sx \in M(C(ef)) \).

By 1.48(2), \( x^m = f \) for some \( m \). Thus, \( f = x^m \in I \cap E_T \) as \( x \in I \subseteq T \).

Hence, \( ef \in E^1_S(I \cap E_T) \). Also, by 1.48(2), \( (sx)^n = ef \) for some \( n \) as
Thus, \( ef = (sx)^m \in T \) as \( sx \in T \) so \( ef \in E_T \). Therefore, we conclude that \( ef \in E^1_S(\mathcal{I} \cap E_T) \cap E_T = \mathcal{I} \cap E_T \) by (**). Hence, \( ef \in I \) so \( ef \in (G_{ef} \cap T) \cap I \neq \emptyset \). Now \( G_{ef} \cap T \) is a subsemigroup of the torsion (as \( S \) has IEP) group \( G_{ef} \) so it is a subgroup of \( T \) which meets the ideal \( I \) of \( T \). It is well-known that any subgroup which meets an ideal must be contained in that ideal. Thus, \( G_{ef} \cap T \subseteq I \). Hence, \( sx \in M(C(ef)) \cap T = G_{ef} \cap T \subseteq I \).

**Subcase 2:** Assume \( sx = x \). Then \( sx = x \in I \).

**Subcase 3:** Assume \( sx = s \).

Then \( s = sx \in T \) so \( sx \in I \) as \( x \in I \) and \( I \) is an ideal of \( T \).

Therefore, in all cases, \( sx \in I \) and this proves that \( S^1 I \) extends \( I \). We conclude \( S \) has IEP. \( \square \)

### 2.8 Corollary. Let \( S \) be a commutative semigroup. Then \( S \) has IEP if and only if

\[
S = \bigcup_{e \in E_S} C(e)
\]

where (1) For each \( e \in E_S \), \( G_e \) is torsion and \( xy = x^2 = y^2 \) for all \( x, y \in N_e \setminus \{0_e\} \) with \( xy \neq 0_e \) and;

(2) If \( e \) and \( f \) are distinct idempotents, \( x \in C(e) \), and \( y \in C(f) \), then either \( xy \in M(C(ef)) \), \( xy = x \), or \( xy = y \).

**Proof.** Apply Theorem 2.7 and Corollary 2.4. \( \square \)
CHAPTER 3

THE CONGRUENCE EXTENSION PROPERTY
FOR COMMUTATIVE SEMIGROUPS

In this chapter, we obtain a partial characterization of commutative semigroups which have the congruence extension property (CEP). Toward this end, we will again employ the general structure of periodic commutative semigroups given in 1.48 as semigroups with CEP are periodic according to Chapter 1. Thus, any commutative semigroup which is a candidate for having CEP will be a semilattice of its archimedean components $C(e)$ ($e \in E_S$) where for each $e \in E_S$, $C(e)$ is an ideal extension of an abelian group $G_e = M(C(e))$ by an archimedean semigroup $N_e$ with zero. Following the same procedure used in treating IEP in the previous chapter, we first determine when a given component $C(e)$ has CEP. That is, we characterize those periodic archimedean semigroups which have CEP. Indeed, we will prove that an archimedean semigroup $S$ has CEP provided that the group and archimedean semigroup with zero that compose $S$ have CEP. Since we know that an abelian group has CEP provided that it is torsion, a complete characterization of archimedean semigroups with CEP will be obtained by characterizing archimedean semigroups with zero which have CEP. In considering a general commutative semigroup $S$, it is certainly true that if $S$ has CEP, then each of its components has CEP as CEP is hereditary. However, as in the case of IEP, one cannot expect that the converse holds. Thus, we seek necessary and sufficient conditions on the multiplication between components which have CEP to insure that the entire semigroup
has CEP. As indicated in the previous chapter, we will employ the characterization of commutative semigroups with IEP in the proofs that follow as CEP implies IEP in the category of commutative semigroups. We begin with the promised characterization of archimedean semigroups with zero which have CEP.

3.1 Lemma. Let $S$ be an archimedean semigroup with zero. Then $S$ has the congruence extension property (CEP) if and only if the following hold:

1. If $xy \neq 0$, then $xy = x^2 = y^2$.
2. If $xy = xz = z^2 = x^2 = y^2 \neq 0$, then $z^2 = yz$.

Proof. Assume $S$ has CEP. By 1.33, $S$ has IEP as $S$ is a commutative semigroup. Thus, by Lemma 2.1, (1) holds. To see that (2) holds, assume that we have elements $x, y,$ and $z$ such that $xy = xz = z^2 = x^2 = y^2 \neq 0$. For the purpose of contradiction, suppose $z^2 \neq yz$. Let $T = \langle z, x \rangle$. Now using our assumptions and the fact that $u^3 = 0$ for all $u \in S$ (1.50), we can give a complete listing of the elements of $T$. One checks that $T = \{0, z, x, z^2\}$ and $z^2 = x^2 = xz = y^2 = xy$. We claim that $\alpha^T(x, z) = (\{x, z\} \times \{x, z\}) \cup \Delta_T$. We show that $\{(tx, tz) : t \in T\} \subseteq \Delta_T$. Now $(0 \cdot x, 0 \cdot z) = (0, 0) \in \Delta_T$, $(zx, zz) \in \Delta_T$, $(xx, xz) \in \Delta_T$, and finally, $(z^2 x, z^2 z) = (x^2 x, z^2 z) = (0, 0) \in \Delta_T$. Thus, we have $\{(tx, tz) : t \in T\} \subseteq \Delta_T$ and from this and 1.5 it is clear that

$$\alpha^T(x, z) = (\{x, z\} \times \{x, z\}) \cup \Delta_T.$$

By assumption, $z^2 \neq zy$ so $(z^2, zy) \notin \Delta_T$. Also, $z^2 \neq z$ and $zy \neq z$ by 1.44 as $z \neq 0$. Hence, we conclude that $(z^2, zy) \notin (\{x, z\} \times \{x, z\}) \cup \Delta_T = \alpha^S(x, z)$. However, $(z^2, zy) = (xy, zy) \in \alpha^S(x, z)$. In addition, since (1) holds, $T$ is an ideal by Lemma 2.1. Thus, $zy \in T$ as $z \in T$ so we have $(z^2, zy) \in \alpha^S(x, z) \cap (T \times T)$, but $(z^2, zy) \notin \alpha^T(x, z)$ contrary to $S$ having CEP. Therefore, (2) holds.
Now assume that (1) and (2) hold. We show that $S$ has CEP. Let $T$ be a subsemigroup of $S$. Then by Lemma 2.1, $T$ is an ideal of $S$ as (1) holds. Let $\sigma$ be a congruence on $T$. We must prove that $\langle \sigma \rangle S \cap (T \times T) = \sigma$. Using 1.2, one can easily see that it suffices to show that $(sx, sy) \in \sigma$ for each $(x, y) \in \sigma$ and $s \in S$.

Let $(x, y) \in \sigma$ and $s \in S$. If $sx = 0 = sy$, then $(sx, sy) = (0, 0) \in \sigma$. If $sx \neq 0$ and $sy \neq 0$, then by (1), $sx = s^2 = sy \in T$ as $T$ is an ideal and $(sx, sy) \in \Delta_T \subseteq \sigma$.

Finally suppose $sx \neq 0$ and $sy = 0$. (The case where $sx = 0$ and $sy \neq 0$ is dual.) Then by (1), $sx = s^2 = x^2$. We claim that $xy = 0$. If not, then by (1), $xy = x^2 = y^2$ and we have $sx = xy = s^2 = x^2 = y^2 \neq 0$, but $s^2 \neq 0 = sy$ contrary to (2).

Thus, $xy = 0$. Hence, $(sx, sy) = (x^2, 0) = (x^2, xy) \in \sigma$. Thus, $(sx, sy) \in \sigma$ for each $(x, y) \in \sigma$ and $s \in S$ so $\langle \sigma \rangle S \cap (T \times T) = \sigma$. Therefore, $S$ has CEP. \[\square\]

3.2 Examples. In this pair of archimedean semigroups with zero which illustrate the necessity of property (2) of 3.1, semigroup (1) does not have CEP while semigroup (2) has CEP.

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 2 & 1 \\
1 & 1 & 1 & 2 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 \\
1 & 1 & 2 & 1 & 1 & 2 \\
1 & 1 & 2 & 1 & 1 & 2 \\
1 & 1 & 2 & 1 & 1 & 2 \\
\end{array}
\]

(1) (2)

Note that property (1) of 3.2 holds in semigroup (1), but property (2) fails as $3 \cdot 5 = 4 \cdot 5 = 3^2 = 4^2 = 5^2 = 2 \neq 1$, but $3 \cdot 4 = 1 \neq 2$. Thus, semigroup (1) provides an example of an archimedean semigroup with zero which has IEP and lacks CEP. We observe that $\alpha^{\{1,2,3,5\}}(3,5)$ is given by the classes $\{1\}\{2\}\{3,5\}$ and has no extension to $S$ as any attempt to extend relates 1 to 2. Semigroup (2) which has CEP certainly satisfies both conditions of 3.1.
As indicated in the introduction to this chapter, our next task is to prove that a periodic archimedean semigroup (i.e. an ideal extension of an abelian group $G$ by an archimedean semigroup $N$ with zero) has CEP provided that $G$ and $N$ have CEP.

In order to accomplish this, we would like to express congruences on an ideal extension $V$ of a semigroup $S$ by a semigroup $T$ in terms of congruences on $S$ and $T$ in some sense. We have the following definition from [Petrich, 1967].

Let $V$ be an ideal extension of a semigroup $S$ by a semigroup $T$. (Refer to Chapter 1 for a review of ideal extensions.) Let $\sigma$ be a congruence on $S$. Let $P$ be an ideal of $T$ and let $\tau$ be a congruence on $T/P$. We define a relation $\nu$ on $V$ as follows:

$$(a, b) \in \nu \equiv \begin{cases} 
    a, b \in T \setminus P \text{ and } (a, b) \in \tau \text{ or;}
    
    a, b \in S \text{ and } (a, b) \in \sigma \text{ or;}
    
    a, b \in P \setminus \{0\} \text{ and there exist } a', b' \in S \\
    \text{ such that } (a'x, ax), (b'x, bx) \in \sigma \text{ for all } x \in S
    
    \text{ and } (a', b') \in \sigma \text{ or;}
    
    a \in P \setminus \{0\}, b \in S \text{ and there exists } a' \in S \\
    \text{ such that } (a'x, ax) \in \sigma \text{ for all } x \in S \text{ and } (a', b) \in \sigma
\end{cases}$$

We write $\nu = (\sigma, P, \tau)$.

### 3.3 Lemma.

Let $V$ be an ideal extension of a monoid $S$ by a semigroup $T$. Hence, the extension $V$ is determined by a partial homomorphism $\phi : T \setminus \{0\} \to S$. Let $\sigma$ be a congruence on $S$ and let $P$ be an ideal of $T$. Then for $a, b \in P \setminus \{0\}$, there exist $a', b' \in S$ such that $(a'x, ax), (b'x, bx) \in \sigma$ for all $x \in S$ and $(a', b') \in \sigma$ if and only if $(\phi(a), \phi(b)) \in \sigma$ and for $a \in P \setminus \{0\}, b \in S$, there exists $a' \in S$ such that $(a'x, ax) \in \sigma$ for all $x \in S$ and $(a', b) \in \sigma$ if and only if $(\phi(a), b) \in \sigma$. 
Proof. Let $a, b \in P \setminus \{0\}$. Assume that there exist $a', b' \in S$ such that $(a'x, ax), (b'x, bx) \in \sigma$ for all $x \in S$ and $(a', b') \in \sigma$. Now in particular, we have $(a'e, ae), (b'e, be) \in \sigma$ where $e$ is the identity of $S$. That is, by definition of multiplication in $V$, $(a', \phi(a)) = (a'e, \phi(a)e) \in \sigma$ and $(b', \phi(b)) = (b'e, \phi(b)e) \in \sigma$. Thus, by transitivity, $(\phi(a), \phi(b)) \in \sigma$ as $(a', b') \in \sigma$. Conversely, assume that $(\phi(a), \phi(b)) \in \sigma$. Then setting $a' = \phi(a)$ and $b' = \phi(b)$, we have $a'x = ax$ and $b'x = bx$ for all $x \in S$ by definition of multiplication in $V$. Thus, $(a'x, ax), (b'x, bx) \in \Delta_S \subseteq \sigma$ for all $x \in S$.

Also, $(a', b') = (\phi(a), \phi(b)) \in \sigma$ by assumption so the converse is proven.

Now let $a \in P \setminus \{0\}$, $b \in S$. Assume there exist $a' \in S$ such that $(a'x, ax) \in \sigma$ for all $x \in S$ and $(a', b) \in \sigma$. In particular, $(a'e, ae) \in \sigma$ where $e$ is the identity of $S$. That is, by definition of multiplication in $V$, $(a', \phi(a)) = (a'e, \phi(a)e) \in \sigma$. Thus, by transitivity, $(\phi(a), b) \in \sigma$ as $(a', b) \in \sigma$. Conversely, assume that $(\phi(a), b) \in \sigma$. Then setting $a' = \phi(a)$, we have $a'x = ax$ for all $x \in S$ by definition of multiplication in $V$. Thus, $(a'x, ax) \in \Delta_S \subseteq \sigma$ for all $x \in S$. Also, $(a', b) = (\phi(a), b) \in \sigma$ by assumption so the proof is complete.

Combining this lemma with the definition of the relation $\nu = (\sigma, P, \tau)$ above we obtain the following. If $V$ is an ideal extension of a monoid $S$ by a semigroup $T$, $\sigma$ a congruence on $S$, $P$ an ideal of $S$, $\tau$ a congruence on $T/P$, and $\nu = (\sigma, P, \tau)$, then we have

$$
(a, b) \in \nu \equiv \begin{cases} 
(a, b) \in T \setminus P \text{ and } (a, b) \in \tau \text{ or; } \\
(a, b) \in S \text{ and } (a, b) \in \sigma \text{ or; } \\
(a, b) \in P \setminus \{0\} \text{ and } (\phi(a), \phi(b)) \in \sigma \text{ or; } \\
a \in P \setminus \{0\}, \ b \in S \text{ and } (\phi(a), b) \in \sigma
\end{cases}
$$

Combining Corollary 2 and Proposition 2 from [Petrich, 1967], we have the following theorem.
3.4 Theorem. Let $V$ be an ideal extension of $S$ by $T$ such that $S/\alpha$ is weakly reductive for each congruence $\alpha$ on $S$ and such that the extension $V$ is determined by a partial homomorphism $\phi : T \setminus \{0\} \to S$. Then $\nu$ is a congruence on $V$ if and only if $\nu = (\sigma, P, \tau)$ for some ideal $P$ of $S$ and congruences $\sigma$ on $S$ and $\tau$ on $T/P$ such that $[0]_\tau = \{0\}$ and such that $(a, b) \in \tau$ implies $(\phi(a), \phi(b)) \in \sigma$ for all $a, b \in T \setminus P$.

Now since the homomorphic image of a monoid is a monoid and monoids are weakly reductive, we conclude that $S/\sigma$ is weakly reductive for each congruence $\alpha$ on a monoid $S$. Furthermore, as noted above, any ideal extension of a monoid by a semigroup is determined by a partial homomorphism. Combining these facts with the theorem above we have the following corollary.

3.5 Corollary. Let $V$ be an ideal extension of a monoid $S$ by a semigroup $T$. (Hence, $S/\alpha$ is weakly reductive for each congruence $\alpha$ on $S$ and the extension $V$ is determined by a partial homomorphism $\phi : T \setminus \{0\} \to S$.) Then $\nu$ is a congruence on $V$ if and only if $\nu = (\sigma, P, \tau)$ for some ideal $P$ of $S$ and congruences $\sigma$ on $S$ and $\tau$ on $T/P$ such that $[0]_\tau = \{0\}$ and such that $(a, b) \in \tau$ implies $(\phi(a), \phi(b)) \in \sigma$ for all $a, b \in T \setminus P$.

3.6 Theorem. Let $S$ be a periodic archimedean semigroup. Hence, $S$ is an ideal extension of a commutative group $G$ by an archimedean semigroup with zero $N$ determined by a partial homomorphism $\phi : N \setminus \{0\} \to G$. Then $S$ has the congruence extension property (CEP) if and only if $G$ and $N$ have CEP.

Proof. Suppose $S$ has CEP. Then since CEP is hereditary, $G$ has CEP. Furthermore, by Proposition 4.5, $N \cong S/G$ has CEP as $G$ is an ideal of $S$ and $S$ has CEP.
To prove the converse, suppose that $G$ and $N$ have CEP. Let $T$ be a subsemigroup of $S$. Let $e$ denote the identity of $G$. By 1.48(2), $e \in T$. We have $T = T \cap S = T \cap (G \cup N \setminus \{0\}) = (T \cap G) \cup (T \cap N \setminus \{0\})$. Now $e \in T \cap G \neq \emptyset$. If $T \cap N \setminus \{0\} = \emptyset$, then $T \subseteq G = M(S)$. By 1.25, any congruence on a subsemigroup of a minimal ideal which is a group in a commutative semigroup $S$ can be extended to $S$. Hence, we may assume that $T \setminus (T \cap G) = T \cap N \setminus \{0\} \neq \emptyset$.

Now the fact that $G$ has CEP implies that $G$ is torsion. Thus, the subsemigroup $T \cap G$ is a subgroup of $G$. Also, $T \cap G$ is an ideal of $T$ as $G$ is an ideal of $S$.

We define $i^* : T/(T \cap G) \to S/G$ by $i^*([t]_{T/(T \cap G)}) = [t]_{S/G}$. It is straightforward to show that $i^*$ is a well-defined monomorphism. Let $N' = T/(T \cap G)$. Then $N'$ is embedded in $S/G \cong N$ via $i^*$ so we may consider $N'$ a subsemigroup of $N$. Following the usual conventions for ideal extensions as discussed above and in Chapter 1, we identify points of $N' \setminus \{0\} = (T/(T \cap G)) \setminus \{0\}$ with points of $T \setminus (T \cap G)$ as the restriction of the natural homomorphism from $T$ onto $T/(T \cap G)$ to $T \setminus (T \cap G)$ is a partial isomorphism. In light of this identification, we may write $T = (T \cap G) \cup (T \setminus (T \cap G)) \cup (T \cap G) \cup N' \setminus \{0\}$. Furthermore, if $a \in N' \setminus \{0\}$, then $\phi(a) = \phi(a)e = a \cdot e \in T$ as $a, e \in T$ by the comments above. Thus, $\phi(N' \setminus \{0\}) \subseteq T \cap G$ and this yields that $\phi|_{N' \setminus \{0\}} : N' \setminus \{0\} \to T \cap G$ is a partial homomorphism which determines $T$ as an ideal extension of $T \cap G$ by $N'$.

Now let $\nu$ be a congruence on $T$. Then by Corollary 3.5, $\nu = (\sigma, P, \tau)$ where $\sigma$ is a congruence on $T \cap G$, $P$ is an ideal of $N'$ and $\tau$ is a congruence on $N'/P$ such that $[0]_{\tau} = \{0\}$ and such that $(a, b) \in \tau$ implies $(\phi(a), \phi(b)) \in \sigma$ for all $a, b \in N'/P$. 


That is, we have

\[(a, b) \in \nu \equiv \begin{cases} 
    a, b \in N' \setminus P \text{ and } (a, b) \in \tau \text{ or; } \\
    a, b \in T \cap G \text{ and } (a, b) \in \sigma \text{ or; } \\
    a, b \in P \setminus \{0\} \text{ and } (\phi(a), \phi(b)) \in \sigma \text{ or; } \\
    a \in P \setminus \{0\}, b \in T \cap G \text{ and } (\phi(a), b) \in \sigma
\end{cases}\]

Now since \(G\) has CEP, \(\sigma\) extends to a congruence \(\bar{\sigma}\) on \(G\). Also, since \(N\) is an archimedean semigroup with zero which has CEP, (1) of Lemma 3.1 must hold and by Lemma 2.1 this implies that each subsemigroup of \(N\) must be an ideal. Now \(P\) is an ideal of \(N'\) which, in turn, is a subsemigroup of \(N\). In particular, \(P\) is a subsemigroup of \(N\). Thus, \(P\) is an ideal of \(N\). Furthermore, by the obvious embedding, we may consider \(N'/P\) to be a subsemigroup of \(N/P\). Now \(N/P\) is again an archimedean semigroup with zero which has CEP by Proposition 4.5. Thus, \(N'/P\) is an ideal of \(N/P\) as justified above. Now \(\tau\) extends from \(N'/P\) to a congruence \(\bar{\tau}\) on \(N/P\). From the observation that \(N'/P\) is an ideal of \(N/P\) and 1.27, we may assume that \(\bar{\tau} = \tau \cup \Delta_{N'/P}\). Thus, we have a triple \((\bar{\sigma}, P, \bar{\tau})\). It is clear that \([0]_{\bar{\tau}} = \{0\}\) as \(\bar{\tau} = \tau \cup \Delta_{N'/P}\) and \([0]_{\tau} = \{0\}\). Also, if \((a, b) \in \bar{\tau} = \tau \cup \Delta_{N'/P}\) and \(a, b \in N'\setminus P\), then either \((a, b) \in \tau\) or \(a = b\). Thus, \((\phi(a), \phi(b)) \in \sigma \cup \Delta_G \subseteq \bar{\sigma}\). Hence, this triple satisfies the conditions of Corollary 3.5 and putting \(\bar{\nu} = (\bar{\sigma}, P, \bar{\tau})\) we conclude that \(\bar{\nu}\) is a congruence on \(S\). That is,

\[(a, b) \in \bar{\nu} \equiv \begin{cases} 
    a, b \in N' \setminus P \text{ and } (a, b) \in \bar{\tau} \text{ or; } \\
    a, b \in G \text{ and } (a, b) \in \bar{\sigma} \text{ or; } \\
    a, b \in P \setminus \{0\} \text{ and } (\phi(a), \phi(b)) \in \bar{\sigma} \text{ or; } \\
    a \in P \setminus \{0\}, b \in G \text{ and } (\phi(a), b) \in \bar{\sigma}
\end{cases}\]

We need only show that \(\bar{\nu} \cap (T \times T) = \nu\). Let \((a, b) \in \bar{\nu} \cap (T \times T)\). We consider each of the four cases for \((a, b) \in \bar{\nu} \cap (T \times T)\) as determined by the definition of
In the first case, $a, b \in N \setminus P \cap T$ and $(a, b) \in \overline{\sigma}$. Now noting that $0 \in P$, we have $N \setminus P \cap T = T \cap N \setminus \{0\} \cap P^c = T \setminus (T \cap G) \cap P^c = N' \setminus \{0\} \cap P^c = N' \setminus P$. Thus, $a, b \in N' \setminus P$ and $(a, b) \in \overline{\sigma} \cap ((N' \setminus P) \times (N' \setminus P)) = \tau$ by following Petrich's identification conventions. Hence, $(a, b) \in \nu$ by definition of $\nu$. In the second case, $a, b \in G \cap T$ and $(a, b) \in \overline{\sigma}$. Thus, $(a, b) \in \overline{\sigma} \cap ((T \cap G) \times (T \cap G)) = \sigma$ and hence, $(a, b) \in \nu$. In the third case, $a, b \in P \setminus \{0\} \subseteq T$ and $(\phi(a), \phi(b)) \in \overline{\sigma}$. Now since $P \setminus \{0\} \subseteq N' \setminus \{0\}$ and $\phi(N' \setminus \{0\}) \subseteq T \cap G$, we have $(\phi(a), \phi(b)) \in T \cap G$. Thus, $(\phi(a), \phi(b)) \in \overline{\sigma} \cap ((G \cap T) \times (G \cap T)) = \sigma$. Hence, $(a, b) \in \nu$. In the last case, $a \in P \setminus \{0\} \subseteq T, b \in G \cap T$, and $(\phi(a), b) \in \overline{\sigma}$. Thus, as above $\phi(a) \in T \cap G$ so $(\phi(a), b) \in \overline{\sigma} \cap ((T \cap G) \times (T \cap G)) = \sigma$ and hence, $(a, b) \in \nu$. This completes the proof of the forward containment. The reverse containment is easily proven. Thus, $\overline{\nu}$ extends $\nu$ and $S$ has CEP.

3.7 Examples. This pair of examples is provided to illustrate the preceding theorem and to further familiarize the reader with ideal extensions.

Semigroup (1) is an ideal extension of the cyclic group $\{1, 2, 6\}$ of order three by the archimedean semigroup with zero $\{0, 3, 4, 5, 7\}$. Note that property (1) of 3.1 holds in $\{0, 3, 4, 5, 7\}$ and semigroup (1) has IEP. However, property (2) of 3.1 does not hold in $\{0, 3, 4, 5, 7\}$ and hence this archimedean semigroup with zero does not have CEP. Thus, by the theorem, semigroup (1) does not have CEP. Semigroup (2)
is an ideal extension of the commutative (obviously torsion) group \{1, 5, 6, 7\} which has CEP by the archimedean semigroup with zero \{0, 2, 3, 4\} which satisfies both conditions of 3.1 and hence, has CEP. Thus, semigroup (2) has CEP.

Examining extensions of congruences in archimedean semigroups with CEP such as the one given above leads one to believe that in such a semigroup, one can always choose an extension with the saturation property given by the following proposition. This certainly does not hold for general semigroups with CEP, but when it does hold in a semigroup \(S\) with CEP, it allows one to conclude that homomorphic images of \(S\) retain CEP as shown in the following chapter. Thus, we will be able to conclude that the homomorphic image of an archimedean semigroup with CEP has CEP. This gives us some hope that the homomorphic image of a commutative semigroup with CEP will retain CEP as commutative semigroups are composed of archimedean semigroups in the manner discussed at length previously. The following proposition is included here rather than in the next chapter because its proof relies heavily on the proof of Theorem 3.6.

**3.8 Proposition.** Let \(S\) be an archimedean semigroup with CEP. Hence, \(S\) is a periodic semigroup which is an ideal extension of a commutative group \(G\) by an archimedean semigroup with zero \(N\) determined by a partial homomorphism \(\phi : N\setminus\{0\} \to G\). Let \(T\) be a subsemigroup of \(S\) and let \(\nu\) be a congruence on \(T\). Then there exists an extension \(\overline{\nu}\) of \(\nu\) such that \(T\) is saturated with respect to \(\overline{\nu}\).

**Proof.** As shown in the proof of Theorem 3.6, \(T = (T \cap G) \cup N'\setminus\{0\}\) where \(N' \cong T/(T \cap G)\), \(N'\) is a subsemigroup of \(N\), and \(\phi|_{N'\setminus\{0\}} : N'\setminus\{0\} \to T \cap G\) is a partial homomorphism which determines \(T\) as an ideal extension of \(T \cap G\) by
$N'$. Also, as in the proof of Theorem 3.6, $\nu = (\sigma, P, \tau)$ where $\sigma$ is a congruence on $T \cap G$, $P$ is an ideal of $N'$ and $\tau$ is a congruence on $N'/P$ such that $[0]_\tau = \{0\}$ and such that $(a, b) \in \tau$ implies $(\phi(a), \phi(b)) \in \sigma$ for all $a, b \in N'/P$. That is, we have

$$(a, b) \in \nu \equiv \begin{cases} a, b \in N' \setminus P \text{ and } (a, b) \in \tau \text{ or}; \\ a, b \in T \cap G \text{ and } (a, b) \in \sigma \text{ or}; \\ a, b \in P \setminus \{0\} \text{ and } (\phi(a), \phi(b)) \in \sigma \text{ or}; \\ a \in P \setminus \{0\}, b \in T \cap G \text{ and } (\phi(a), b) \in \sigma \end{cases}$$

Again, by the argument in the proof of Theorem 3.6, $\nu$ extends to the congruence $\overline{\nu} = (\overline{\sigma}, P, \overline{\tau})$ where $\overline{\tau} = \tau \cup \Delta_{N/P}$ and $\overline{\sigma}$ is any extension of $\sigma$ to $G$. That is, $\overline{\nu}$ is defined by

$$(a, b) \in \overline{\nu} \equiv \begin{cases} a, b \in N \setminus P \text{ and } (a, b) \in \overline{\tau} \text{ or}; \\ a, b \in G \text{ and } (a, b) \in \overline{\sigma} \text{ or}; \\ a, b \in P \setminus \{0\} \text{ and } (\phi(a), \phi(b)) \in \overline{\sigma} \text{ or}; \\ a \in P \setminus \{0\}, b \in G \text{ and } (\phi(a), b) \in \overline{\sigma} \end{cases}$$

For the proof of this proposition we must consider a particular extension $\overline{\sigma}$. By 1.16, there exists a normal subgroup $M$ of the subgroup $T \cap G$ such that $(x, y) \in \sigma$ if and only if $xy^{-1} \in M$. Now since $G$ is commutative, $M$ is a normal subgroup of $G$ and by 1.20, we may extend $\sigma$ by $\overline{\sigma}$ where $(x, y) \in \overline{\sigma}$ if and only if $xy^{-1} \in M$. For this choice of extension $\overline{\sigma}$, we consider the extension $\overline{\nu} = (\overline{\sigma}, P, \overline{\tau})$ of $\nu$ and show that $T$ is saturated with respect to $\overline{\nu}$ as defined in Chapter 1. Let $(a, b) \in \overline{\nu}$ and suppose $a \in T$. We must show that $b \in T$. We have four cases to consider by definition of $\overline{\nu}$. In the first case, $a, b \in N \setminus P$ and $(a, b) \in \overline{\tau} = \tau \cup \Delta_{N/P}$. Thus, either $b = a \in T$ or $(a, b) \in \tau$ which implies $b \in T$. In the second case, $a, b \in G$ and $(a, b) \in \overline{\sigma}$. Thus, by definition of $\overline{\sigma}$, $ba^{-1} \in M \subseteq T \cap G$. Hence, $b = (ba^{-1})a \in T$ as $ba^{-1}, a \in T$. In
the third case, \( a, b \in P \setminus \{0\} \subseteq T \) so \( b \in T \). In the last case, \( a \in P \setminus \{0\}, b \in G \), and \((\phi(a), b) \in \sigma\). Thus, \( b\phi(a)^{-1} \in M \subseteq T \cap G \) and \( \phi(a) \in T \) since \( a \in T \) as shown in the proof of Theorem 3.6. Hence, \( b = b\phi(a)^{-1}\phi(a) \in T \). Therefore, \( b \in T \) in each case and the proof of the proposition is complete.

We now return to the task of determining the structure of commutative semigroups with CEP. Now that we have determined when a given component of a commutative semigroup \( S \) has CEP, we turn our attention to how the components interact in the presence on CEP. As illustrated in the examples which follow, there certainly exist commutative semigroups which lack CEP, but whose components all have CEP. One special case in which CEP within components implies CEP within the entire commutative semigroup is the case in which each component is an abelian group. This fact is proven in [Dumesnil, 1993] as an application of a theorem in [Stralka, 1972]. In our general case, we know that the condition on multiplication between components given in Theorem 2.7 must hold in a commutative semigroup with CEP as CEP implies IEP in the category of commutative semigroups. However, this condition can and must be strengthened in the case of CEP. Consider the following examples.

3.9 Examples. In this pair of commutative semigroups, each archimedean component of each semigroup has CEP. (This holds trivially as each component has order less than four and any such semigroup has CEP according to Chapter 1.) Also, each semigroup has IEP so in particular, condition (2) of Theorem 2.7 holds in each semigroup. However, semigroup (2) does not have CEP while semigroup (1) has CEP.
In semigroup (2), note that $\alpha^{C(3)}(4,5)$ is given by the class listing, $\{3\}{4,5}$, and any congruence on $S$ containing this congruence relates 3 to 5 so that this congruence does not have an extension. Also, note that in this semigroup 6 acts as an identity for $5 \in C(3)$ and multiplies $4 \in C(3)$ into $\{3\} = M(C(3))$. Now, semigroup (1) has CEP and one observes that there is a uniformity in the way that the elements of one component act on the elements of a given distinct component. For example, 7 acts as an identity for each element of $C(1) = \{1,2\}$ and multiplies each element of $C(4) = \{4,5,6\}$ into $\{4\} = M(C(4))$. Similar statements can be made for the other elements.

We formalize the notion of “uniformity” discussed in the example above in the following proposition.

**3.10 Proposition.** Let $S$ be a commutative semigroup with the congruence extension property (CEP). Let $e, f \in E, e \neq f$. Then the following hold:

1. If $fe \neq e$ and $fe \neq f$, then $C(f)C(e) \subseteq M(C(fe))$.

2. If $fe = e$, then either
   (i) $C(f)C(e) \subseteq M(C(e))$ or
   (ii) $C(f)x = x$ for all $x \in C(e) \setminus M(C(e))$ and $C(f)M(C(e)) \subseteq M(C(e))$.

3. If $fe = f$, then the dual to (2) holds.
Proof. Since $S$ is a commutative semigroup with CEP, $S$ has IEP by 1.33. Thus, by Theorem 2.7, we have the following property:

if $x \in C(e)$ and $y \in C(f)$, then either $yx = x$, $yx = y$, or $yx \in M(C(fe))$. (*)

In addition, $C(e)$ and $N_e$ have CEP for all $e \in E$ as CEP as noted above. Thus, in particular, condition (1) of Lemma 3.1 holds in $N_e$ for all $e \in E$.

To see that (1) holds, we note that we always have $C(f)C(e) \subseteq C(fe)$ and since $e, f, fe$ are distinct idempotents, we know that $C(e), C(f),$ and $C(fe)$ are disjoint. Thus, by (*), if $x \in C(e)$ and $y \in C(f)$, then it must be the case that $yx \in M(C(fe))$. That is, $C(f)C(e) \subseteq M(C(fe))$. (Note that this always holds in the presence of IEP. This will not be true of condition (2).)

In order to prove (2), let $fe = e$. In this case, if $x \in C(e)$ and $y \in C(f)$, then by (*), either $yx = x$ or $yx \in M(C(e))$ as $C(f)C(e) \subseteq C(e)$ and $C(e)$ and $C(f)$ are disjoint. We prove that (2) holds by means of a number of claims.

Claim 1: If $fx = x$ for some $x \in C(e) \setminus M(C(e))$, then $fx' = x'$ for all $x' \in C(e) \setminus M(C(e))$.

Suppose that $fx = x$ and $fx' \neq x'$ for some $x, x' \in C(e) \setminus M(C(e))$. Note that $x, x' \in C(e) \setminus M(C(e)) = C(e) \setminus G_e = N_e \setminus \{0_{N_e}\}$ and $x \neq x'$. By (*), $fx' \in M(C(e))$.

Now $(x, fx') = (fx, fx') \in \alpha^S(x, x') \cap (C(e) \times C(e))$. We obtain a contradiction by showing that $(x, fx') \notin \alpha^C(x, x')$. For this purpose consider any transition of the form $x = x_0, x_1, \ldots, x_n$ such that for $0 \leq i \leq n - 1$, either $(x_i, x_{i+1}) = (s_i x, s_i' x')$ or $(x_i, x_{i+1}) = (s_i' x', s_i x)$ for some $s_i \in C(e)^1$. We claim that for any such transition, $s_i = 1$ for $0 \leq i \leq n - 1$. The proof of this claim is by induction on $n$. In the case that $n = 0$, we have either $x = x_0 = s_0 x$ or $x = x_0 = s_0 x'$. For each of these possibilities, we know that $s_0 \notin M(C(e))$ since $x \notin M(C(e))$. If $x = s_0 x'$, then $s_0 \neq 1$ as $x \neq x'$.
so we show that this assumption leads to a contradiction. Assuming \( x = s_0 x' \), we have \( x, x', s_0 \in C(e) \setminus M(C(e)) = C(e) \setminus G_e = N_e \setminus \{0_{N_e}\} \) by the observations above. Thus, \( s_0 x' = x \neq 0_{N_e} \). Hence, \( x = s_0 x' = x'^2 \), since (1) of Lemma 3.1 holds in \( N_e \) as noted above. Therefore, \( f x = f x'^2 = (f x') x' \in M(C(e)) \subseteq M(C(e)) \) and this is a contradiction. Thus, we conclude that \( x = s_0 x \). Now 1.44 implies that \( s_0 \not\in C(e) \setminus M(C(e)) = N_e \setminus \{0_{N_e}\} \) since \( x \in N_e \setminus \{0_{N_e}\} \). Thus, \( s_0 \not\in C(e) \) as we noted above that \( s_0 \not\in M(C(e)) \). We conclude that \( s_0 = 1 \). Now assume that \( s_i = 1 \) for \( 0 \leq i \leq k - 1 \) for any transition of the above form of length \( k \). Consider a transition of length \( k + 1 \). Then by the induction hypothesis, \( s_i = 1 \) for \( 0 \leq i \leq k - 1 \). We must show that \( s_k = 1 \). Now by the definition of these transitions and the fact that \( s_{k-1} = 1 \), we know that either

\[
\begin{align*}
x = s_{k-1} x = x_k &= \begin{cases} s_k x & \text{or} \\ s_k x' \end{cases} \\
x' = s_{k-1} x' = x_k &= \begin{cases} s_k x & \text{or} \\ s_k x' \end{cases}
\end{align*}
\]

In the first case, we have either \( x = s_k x \) or \( x = s_k x' \) and the argument to show that \( s_k = 1 \) proceeds just as in the case \( n = 0 \) above. In the second case, either \( x' = s_k x \) or \( x' = s_k x' \). By a dual to the argument given in the proof of case \( n = 0 \), we obtain that \( x' = x^2 \) if \( x' = s_k x \). Thus, if \( x' = s_k x \), then \( f x' = f x^2 = (f x) x = x^2 = x' \) contrary to our assumption. Thus, \( x' = s_k x' \). Again, just as in the proof of case \( n = 0 \), we obtain that \( s_k \not\in C(0) \) by using 1.44 and we conclude that \( s_k = 1 \). Thus, the induction is complete. Now \( s_i = 1 \) for \( 0 \leq i \leq n - 1 \) certainly yields that \( x_i \in \{s_i x, s_i x'\} = \{x, x'\} \). Thus, we have shown that \( x \) is linked only to \( x' \) by pairs from \( \{(s x, s x'), (s x', s x) : s \in C(e)^1\} \). Hence, \( [x]_{\alpha C(e)(x,x')} = \{x, x'\} \).

Now \( f x' \in M(C(e)) \) and \( x, x' \not\in M(C(e)) \) so \( f x' \not\in \{x, x'\} = [x]_{\alpha C(e)(x,x')} \). Thus,
(x, fx') \not\in \alpha^{C(e)}(x, x')$, but $(x, fx') \in \alpha^S(x, x') \cap (C(e) \times C(e))$ as noted above. Thus, $\alpha^S(x, x') \cap (C(e) \times C(e)) \neq \alpha^{C(e)}(x, x')$ contrary to $S$ having CEP and Claim 1 is proven.

**Claim 2:** If $fx = x$ for all $x \in C(e) \setminus M(C(e))$, then

$$C(f)x = x \text{ for all } x \in C(e) \setminus M(C(e)).$$

Assume $fx = x$ for all $x \in C(e) \setminus M(C(e))$. Let $y \in C(f) \setminus \{f\}$, $x \in C(e) \setminus M(C(e))$. We must show that $yx = x$. Suppose $yx \neq x$. Then by $(\ast)$, $yx \in M(C(e))$. Now $S$ is periodic as it has CEP and $y \in C(f)$ so $y^m = f$ for some $m \in \mathbb{N}$. Thus,

$$x = fx = y^m x = (yx)y^{m-1} \in M(C(e)C(f) \subseteq M(C(e)) \text{ by } 1.48(4).$$

This is a contradiction. Hence, $yx = x$ and Claim 2 is proven.

**Claim 3:** If $fx \neq x$ for all $x \in C(e) \setminus M(C(e))$, then

$$C(f)(C(e) \setminus M(C(e))) \subseteq M(C(e)).$$

Assume that $fx \neq x$ for all $x \in C(e) \setminus M(C(e))$ and let $y \in C(f)$, $x \in C(e) \setminus M(C(e))$. If $yx \not\in M(C(e))$, then by $(\ast)$, $yx = x$. As above, $y^m = f$ for some $m \in \mathbb{N}$. Thus,

$$x = yx = y(yx) = y(y(yx)) = \ldots = y^m x = fx \text{ contrary to the hypothesis. Hence, } yx \in M(C(e)) \text{ and Claim 3 is proven.}$$

We combine these claims to prove that (2) holds. If the hypothesis of Claim 1 holds, then combining Claims 1 and 2 with the fact that $C(f)M(C(e)) \subseteq M(C(e))$, we have $C(f)x = x$ for all $x \in C(e) \setminus M(C(e))$ and $C(f)M(C(e)) \subseteq M(C(e))$. If the hypothesis of Claim 1 does not hold, then the hypothesis of Claim 3 holds and combining Claim 3 with the fact that $C(f)M(C(e)) \subseteq M(C(e))$ as above, we have $C(f)C(e) \subseteq M(C(e))$. This completes the proof of (2). Condition (3) is dual and the proposition is proven.
While the lemma above gives significant information on how multiplication behaves between components of a semigroup with CEP, these conditions are again not enough in general to insure that a periodic semigroup whose components have CEP has CEP. Before presenting examples which illustrate the insufficiency of these conditions in general, we present a very special case in which the conditions obtained thus far are sufficient to insure the presence of CEP. Motivation for considering this particular special case will become apparent in Chapter 5.

3.11 Proposition. Let $S$ be a commutative semigroup with zero such that $S = C(0) \cup E_S \setminus \{0\}$, $C(0)$ has the congruence extension property (CEP), and for each $e \in E_S \setminus \{0\}$, either $ex = x$ for all $x \in C(0)$ or $ex = 0$ for all $x \in C(0)$. Then $S$ has CEP.

Proof. If $C(0)$ is trivial, then $S$ is a semilattice and has CEP. Thus, we may assume that $C(0)$ is nontrivial. Let $T$ be a subsemigroup of $S$. $C(0)$ is periodic, as noted in Chapter 1. Thus, it is clear that $S$ is periodic. Hence, according to 1.48(5), $T = C_T(0) \cup E_T \setminus \{0\}$ where $C_T(0) = C(0) \cap T$. Let $\sigma$ be a congruence on $T$. We show that $\sigma$ has an extension to $S$ in each of the two cases below.

**Case 1:** Assume $0 \notin T$.

In this case, $C_T(0) = C(0) \cap T = \emptyset$ according to 1.48(2). Thus, $\sigma$ is a congruence on $T = E_T$ which is a subsemilattice of $E_S$. Thus, $\sigma$ extends to a congruence $\sigma_{E_S}$ on $E_S$ as semilattices have CEP. Let $\overline{\sigma} = \sigma_{E_S} \cup (\{0\}_{\sigma_{E_S}} \cup C(0)) \times (\{0\}_{\sigma_{E_S}} \cup C(0))$. We show that $\overline{\sigma}$ is a congruence on $S$ extending $\sigma$. Clearly, $\overline{\sigma}$ is reflexive and symmetric. To see that $\overline{\sigma}$ is transitive, it suffices to show that if $(x, y) \in \sigma_{E_S}$ and $(y, z) \in (\{0\}_{\sigma_{E_S}} \cup C(0)) \times (\{0\}_{\sigma_{E_S}} \cup C(0))$, then $(x, z) \in \overline{\sigma}$. For such $x, y,$ and $z,$
y ∈ [0]σ_{Es} as y ∈ σ_{Es} ∩ ([0]σ_{Es} ∪ C(0)). Thus, x ∈ [0]σ_{Es} as (x, y) ∈ σ_{Es}. Hence, (x, z) ∈ ([0]σ_{Es} ∪ C(0)) × ([0]σ_{Es} ∪ C(0)) ⊆ σ. To see that σ is compatible, let s ∈ S and (x, y) ∈ σ. As noted previously, the class of zero in any congruence on a semigroup with zero must be an ideal. In particular, C(0) and [0]σ_{Es} are ideals of S and E_S respectively. Thus, if s ∈ C(0), then (sx, sy) ∈ C(0) × C(0) ⊆ σ. Otherwise, s ∈ E_S \{0\}. In this case, if (x, y) ∈ σ_{Es}, then (sx, sy) ∈ σ_{Es} and if

\((x, y) ∈ ([0]σ_{Es} ∪ C(0)) × ([0]σ_{Es} ∪ C(0)),\)

then (sx, sy) ∈ ([0]σ_{Es} ∪ C(0)) × ([0]σ_{Es} ∪ C(0)). Therefore, σ is a congruence on S. Also, since C(0) ∩ T = ∅ and σ_{Es} extends σ from T = E_T to E_S, we have σ ∩ (T × T) = σ_{Es} ∩ (T × T) = σ. Hence, σ is a congruence on S which extends σ.

**Case 2:** Assume 0 ∈ T.

Now σ|_{E_T} is a congruence on the subsemilattice E_T of E_S. Hence, σ|_{E_T} extends to a congruence σ_{Es} on E_S. We may assume that σ_{Es} is the congruence on E_S generated by the relation σ|_{E_T} according to 1.3. Set

\[ E_1 = \{ e ∈ E_S : ex = x \text{ for all } x ∈ C(0) \} \]

and

\[ E_2 = \{ e ∈ E_S : ex = 0 \text{ for all } x ∈ C(0) \}. \]

By our assumptions, E_S = E_1 ∪ E_2.

**Claim 1:** E_2 is a prime ideal of E_S.

To see that E_2 is an ideal of E_S, let e ∈ E_2 and f ∈ E_S. Then for all x ∈ C(0), ef = (ex)f = 0 · f = 0 so ef ∈ E_2. To see that E_2 is prime, let e, f ∈ E_S \ E_2 = E_1. Then for all x ∈ C(0), ef = e(fx) = ex = x so ef ∈ E_1 and E_2 is a prime ideal of E_S.
Claim 2: If \( \sigma|_{E_T} \subseteq (E_1 \times E_1) \cup (E_2 \times E_2) \), then \( \sigma|_{E_S} \subseteq (E_1 \times E_1) \cup (E_2 \times E_2) \).

Suppose \( \sigma|_{E_T} \subseteq (E_1 \times E_1) \cup (E_2 \times E_2) \). Recall that \( \sigma|_{E_S} \) is the congruence on \( E_S \) generated by the relation \( \sigma|_{E_T} \) by assumption. Hence, according to 1.4 and 1.7, it suffices to show that

\[
(\{ (ex, ey) : (x, y) \in \sigma|_{E_T}, \ e \in E_S^1 \} \subseteq (E_1 \times E_1) \cup (E_2 \times E_2) )
\]

Let \( e \in E_S \) and \( (x, y) \in \sigma|_{E_T} \). Then \( (x, y) \in (E_1 \times E_1) \cup (E_2 \times E_2) \). If \( (x, y) \in E_2 \times E_2 \), then \( (ex, ey) \in E_2 \times E_2 \) as \( E_2 \) is an ideal of \( E_S \). Otherwise, \( (x, y) \in E_1 \times E_1 \). In this case, if \( e \in E_2 \), then \( (ex, ey) \in E_2 \times E_2 \) as \( E_2 \) is an ideal and if \( e \in E_S \setminus E_2 = E_1 \), then \( (ex, ey) \in E_1 \times E_1 \) as \( E_2 \) is prime.

This completes the proof of the claim.

Claim 3: If \( \sigma|_{E_T} \not\subseteq (E_1 \times E_1) \cup (E_2 \times E_2) \), then \( \sigma|_{C_T(0)} = C_T(0) \times C_T(0) \).

If \( \sigma|_{E_T} \not\subseteq (E_1 \times E_1) \cup (E_2 \times E_2) \), then there exists \( (x, y) \in \sigma|_{E_T} \cap (E_1 \times E_2) \).

Now for all \( z \in C_T(0) \), \( (z, 0) = (zx, zy) \in \sigma \). Thus, \( C_T(0) \subseteq [0]_\sigma \). Hence, \( \sigma|_{C_T(0)} = C_T(0) \times C_T(0) \).

Now \( C(0) \) has CEP so \( \sigma|_{C_T(0)} \) extends to a congruence \( \sigma|_{C(0)} \) on \( C(0) \). Set

\[
\bar{\sigma} = \sigma|_{C(0)} \cup \sigma|_{E_S} \cup (([0]_{\sigma|_{C(0)}} \cup [0]_{\sigma|_{E_S}}) \times ([0]_{\sigma|_{C(0)}} \cup [0]_{\sigma|_{E_S}})).
\]

We show that \( \bar{\sigma} \) is a congruence on \( S \) extending \( \sigma \). Clearly, \( \bar{\sigma} \) is reflexive and symmetric. Transitivity follows as \( ([0]_{\sigma|_{C(0)}} \cup [0]_{\sigma|_{E_S}}) \times ([0]_{\sigma|_{C(0)}} \cup [0]_{\sigma|_{E_S}}) \subseteq \bar{\sigma} \) and \( C(0) \cap E_S = \{0\} \). To see that \( \bar{\sigma} \) is compatible, let \( s \in S \) and \( (x, y) \in \bar{\sigma} \). First assume that \( s \in E_S \setminus \{0\} \). If \( (x, y) \in \sigma|_{E_S} \), then \( (sx, sy) \in \sigma|_{E_S} \subseteq \bar{\sigma} \). If \( (x, y) \in \sigma|_{C(0)} \), then by hypothesis, either \( (sx, sy) = (0, 0) \) or \( (sx, sy) = (x, y) \) so \( (sx, sy) \in \bar{\sigma} \). Assume \( s \in C(0) \). If \( (x, y) \in \sigma|_{C(0)} \), then \( (sx, sy) \in \sigma|_{C(0)} \subseteq \bar{\sigma} \). Suppose \( (x, y) \in \sigma|_{E_S} \). If \( (x, y) \in E_1 \times E_1 \), then \( (sx, sy) = (s, s) \in \bar{\sigma} \) and if \( (x, y) \in E_2 \times E_2 \), then we have
If \((x, y) \not\in (E_1 \times E_1) \cup (E_2 \times E_2)\), then certainly it is clear that \(\sigma_{E_S} \not\subseteq (E_1 \times E_1) \cup (E_2 \times E_2)\). Hence, \(\sigma_{|E_T} \not\subseteq (E_1 \times E_1) \cup (E_2 \times E_2)\) according to Claim 2, and by Claim 3, \(\sigma_{|C_T(0)} = C_T(0) \times C_T(0)\). Therefore, we may assume that \(\sigma_{C(0)} = C(0) \times C(0)\). Thus, \((sx, sy) \in C(0) \times C(0) = \sigma_{C(0)} \subseteq \bar{\sigma}\), since \(s \in C(0)\) and \(C(0)\) is an ideal. It remains to show that if \((x, y) \in [0]_{\sigma_{C(0)}} \times [0]_{\sigma_{E_S}}\) and \(s \in S\), then \((sx, sy) \in \bar{\sigma}\). Now \((x, y) \in [0]_{\sigma_{C(0)}} \times [0]_{\sigma_{E_S}}\) implies that \((x, 0) \in \sigma_{C(0)}\) and \((0, y) \in \sigma_{E_S}\). As argued above, \((sx, 0) = (sx, s \cdot 0) \in \bar{\sigma}\) and \((0, sy) = (s \cdot 0, sy) \in \bar{\sigma}\).

Thus, by transitivity, \((sx, sy) \in \bar{\sigma}\) as required and \(\bar{\sigma}\) is compatible. Whence, \(\bar{\sigma}\) is a congruence on \(S\).

We must show that \(\bar{\sigma} \cap (T \times T) = \sigma\). To see that the forward inclusion holds, let \((x, y) \in \bar{\sigma} \cap (T \times T)\). Then either \((x, y) \in \sigma_{C(0)} \cap (T \times T) \subseteq \sigma_{|C_T(0)} \subseteq \sigma\), or \((x, y) \in \sigma_{E_S} \cap (T \times T) \subseteq \sigma_{|E_T} \subseteq \sigma\), or \((x, y) \in ([0]_{\sigma_{C(0)}} \cup [0]_{\sigma_{E_S}}) \times ([0]_{\sigma_{C(0)}} \cup [0]_{\sigma_{E_S}})\). In the last case, without loss of generality we may assume that \((x, y) \in [0]_{\sigma_{C(0)}} \times [0]_{\sigma_{E_S}}\).

Thus, \((x, 0) \in \sigma_{C(0)} \cap (T \times T) \subseteq \sigma_{|C_T(0)} \subseteq \sigma\) and \((0, y) \in \sigma_{E_S} \cap (T \times T) \subseteq \sigma_{|E_T} \subseteq \sigma\).

Hence, \((x, y) \in \sigma\) by transitivity and the forward inclusion holds. To see that the reverse inclusion holds, first note that \(\sigma_{|C_T(0)} \subseteq \bar{\sigma}\) and \(\sigma_{|E_T} \subseteq \bar{\sigma}\). Thus, it suffices to show that if \((x, y) \in \sigma \cap (E_S \times C(0))\), then \((x, y) \in \bar{\sigma}\). For such \(x\) and \(y\), there exists \(n \in \mathbb{N}\) such that \(y^n = 0\). Thus, \((x, 0) = (x_n, y^n) \in \sigma\) so \(x \in [0]_{\sigma_{|E_T}} \subseteq [0]_{\sigma_{E_S}}\). By transitivity, \((y, 0) \in \sigma\) so \(y \in [0]_{\sigma_{|C_T(0)}} \subseteq [0]_{\sigma_{C(0)}}\). Hence, \((x, y) \in ([0]_{\sigma_{C(0)}} \cup [0]_{\sigma_{E_S}}) \times ([0]_{\sigma_{C(0)}} \cup [0]_{\sigma_{E_S}}) \subseteq \bar{\sigma}\) and the reverse containment holds. Whence, \(\bar{\sigma}\) is a congruence on \(S\) extending \(\sigma\).

By Cases 1 and 2, we conclude that \(S\) has CEP.

3.12 Corollary. Let \(S\) be a commutative semigroup with a zero element such that \(S = C(0) \cup E_S \setminus \{0\}\). Then \(S\) has CEP if and only if \(C(0)\) has the congruence
extension property (CEP) and for each \( e \in E_S \setminus \{0\} \), either \( ex = x \) for all \( x \in C(0) \) or \( ex = 0 \) for all \( x \in C(0) \).

**Proof.** If \( S \) has CEP then \( C(0) \) has CEP as CEP is hereditary and for each \( e \in E_S \setminus \{0\} \), either \( ex = x \) for all \( x \in C(0) \) or \( ex = 0 \) for all \( x \in C(0) \) by Proposition 3.10. According to Proposition 3.11, the converse holds. \( \blacksquare \)

Thus, in the special case that a commutative periodic semigroup \( S \) has a zero and each component not containing zero is trivial, we see that \( S \) has CEP if and only if the components have CEP and the conditions of Proposition 3.10 hold. As promised, we now present examples to illustrate the insufficiency of these conditions in general.

**3.13 Examples.** All of the components of the following commutative semigroups have CEP and the conditions of Lemma 3.10 hold in each semigroup, but none of these semigroups have CEP.

\[
\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 3 & 3 & 3 & 3 & 3 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 3 & 4 & 4 & 4 & 4 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 1 & 3 & 4 & 4 & 4 & 4 & 1 & 1 & 1 & 4 & 4 & 4 & 5 & 5 & 5 \\
1 & 1 & 3 & 4 & 4 & 4 & 4 & 1 & 1 & 1 & 4 & 4 & 4 & 6 & 6 & 6 \\
1 & 2 & 3 & 4 & 4 & 4 & 7 & 1 & 1 & 1 & 4 & 5 & 6 & 7 \\
\text{(1)} & \text{(2)} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 3 & 1 & 2 & 3 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 1 & 1 & 4 & 4 & 4 & 4 & 1 & 2 & 2 & 4 & 4 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 4 & 5 & 5 & 5 & 4 & 1 & 2 & 2 & 4 & 5 & 5 & 2 & 2 \\
1 & 1 & 1 & 4 & 5 & 5 & 5 & 4 & 1 & 2 & 2 & 4 & 5 & 6 & 6 & 2 \\
1 & 2 & 3 & 4 & 4 & 4 & 7 & 1 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\
\text{(3)} & \text{(4)} \\
\end{array}
\]
In semigroup (1), $\alpha^{1,2,4,5,6}(2,5)$ is given by the class listing \{1,4\}2,5\{6\} any congruence on (1) which contains this congruence relates 2 to 4. Thus, this congruence does not extend. Note that in the idempotent ordering $1 \leq 4 \leq 7$ and that 7 acts as an identity on $C(1)$ and multiplies $C(4)$ into $M(C(4)) = \{4\}$ while 4 multiplies $C(1)$ into $M(C(1)) = \{1\}$. One can check that if either 7 acted as an identity for $C(4)$ or 4 acted as an identity for $C(1)$, then $\alpha^{1,2,4,5,6}(2,5)$ would indeed have an extension.

In semigroup (2), $\alpha^{1,2,4,5}(2,5)$ is given by the class listing \{1,4\}2,5\{6\}. Since any congruence on (2) which contains this congruence relates 1 to 5 this congruence has no extension. In the idempotent ordering, $1 \leq 4 \leq 7$ and 7 acts as an identity on $C(4)$ and multiplies $C(1)$ into $M(C(1)) = \{1\}$ while 4 multiplies $C(1)$ into $M(C(1)) = \{1\}$. One can check that if either 7 multiplied $C(4)$ into $M(C(4)) = \{4\}$ or 4 acted as an identity for $C(1)$, then $\alpha^{1,2,4,5}(2,5)$ would extend to a congruence on (2).

In semigroup (3), $\alpha^{1,2,5,6}(2,6)$ is given by the class listing \{1,5\}2,6\{6\}. Since any congruence on (2) which contains this congruence relates 1 to 2 this congruence has no extension. In the idempotent ordering, $1 \leq 4 \leq 7$, $1 \leq 5$, and 5 and 7 are not comparable. Now 7 acts as an identity on $C(1)$ while 5 multiplies $C(1)$ into $M(C(1)) = \{1\}$. One can check that if 5 acted as an identity for $C(1)$, then $\alpha^{1,2,5,6}(2,6)$ would extend to a congruence on (3).

In semigroup (4), $\alpha^{2,3,4,5,7}(5,7)$ is given by the class listing \{2,3,4\}5,7\{7\}. Since any congruence on (2) which contains this congruence relates 2 to 5 this congruence has no extension. In the idempotent ordering, $2 \leq 4 \leq 6$, $2 \leq 3$, and 3 is comparable to neither 6 nor 4. Now 6 acts as an identity on $C(4)$. One can check
that if 6 multiplied \( C(4) \) into \( M(C(4)) = \{4\} \), then \( \alpha^{1,2,4,5}(2,5) \) would extend to a congruence on \( (4) \).

Thus, we see that more information concerning interaction between components is needed. We seek this information based on idempotent configurations. The following proposition lists the possibilities for multiplication between finite components of a commutative semigroup \( S \) with CEP which contain linearly ordered idempotents \( e_1, e_2, \) and \( e_3 \). The conditions of this proposition are motivated by examples 3.13(1) and 3.13(2).

**3.14 Proposition.** Let \( S \) be a commutative semigroup with the congruence extension property (CEP) such that for each \( e \in E_S \), \( C(e) \) is finite. Then for \( e_1, e_2, e_3 \in E_S \) such that \( e_1 < e_2 < e_3 \), one of the following holds:

(i) \( C(e_3)x = x \) for all \( x \in C(e_1)\setminus M(C(e_1)) \) and \( C(e_3)x = x \) for all \( x \in C(e_2)\setminus M(C(e_2)) \).

(ii) \( C(e_3)x = x \) for all \( x \in C(e_1)\setminus M(C(e_1)), C(e_3)C(e_2) \subseteq M(C(e_2)) \), and \( C(e_2)x = x \) for all \( x \in C(e_1)\setminus M(C(e_1)) \).

(iii) \( C(e_3)C(e_1) \subseteq M(C(e_1)) \) and \( C(e_3)C(e_2) \subseteq M(C(e_2)) \).

(iv) \( C(e_3)C(e_1) \subseteq M(C(e_1)), C(e_3)x = x \) for all \( x \in C(e_2)\setminus M(C(e_2)) \), and \( C(e_2)x = x \) for all \( x \in C(e_1)\setminus M(C(e_1)) \).

**Proof.** Let \( e_1, e_2, e_3 \in E_S \) such that \( e_1 < e_2 < e_3 \). Suppose that \( C(e_3)x = x \) for all \( x \in C(e_1)\setminus M(C(e_1)) \). We show that either (i) or (ii) holds. If (i) does not hold, then there exists \( x \in C(e_2)\setminus M(C(e_2)) \) such that \( C(e_3)x \neq x \). (In particular, this implies that \( C(e_2)\setminus M(C(e_2)) \neq \emptyset \)) Then according to Proposition 3.10(2), \( C(e_3)C(e_2) \subseteq M(C(e_2)) \). We show that (ii) holds by proving that \( C(e_2)x = x \) for
all $x \in C(e_1) \setminus M(C(e_1))$. Suppose this does not hold. That is, suppose that there exists $x \in C(e_1) \setminus M(C(e_1))$ such that $C(e_2)x \neq x$. (In particular, this implies that $C(e_1) \setminus M(C(e_1)) \neq \emptyset$.) Then by Proposition 3.10(2),

$$C(e_2)C(e_1) \subseteq M(C(e_1)). \quad (\star)$$

As noted above, $C(e_2) \setminus M(C(e_2)) \neq \emptyset$ and $C(e_1) \setminus M(C(e_1)) \neq \emptyset$. Thus, $|N_{e_1}|$, $|N_{e_2}| \geq 2$ and by assumption, these are finite. Hence, according to 1.45, we may choose $u \in A(N_{e_1}) \setminus \{0_{e_1}\}$ and $v \in A(N_{e_2}) \setminus \{0_{e_2}\}$. We have

$$ux \in M(C(e_1)) \text{ for all } x \in C(e_1) \text{ and } vx \in M(C(e_2)) \text{ for all } x \in C(e_2). \quad (\star\star)$$

If $s \in C(e_1)$, then $(su, sv) \in M(C(e_1)) \times M(C(e_1))$ by $(\star\star)$ and $(\star)$. If $s \in C(e_2)$, then $(su, sv) \in M(C(e_1)) \times M(C(e_2))$ by $(\star)$ and $(\star\star)$. Set $T = C(e_1) \cup C(e_2)$ and $Q = M(C(e_1)) \cup M(C(e_2))$. Then according to the observations listed above,

$$\{(su, sv) : s \in T^1\} \subseteq (\{u, v\} \times \{u, v\}) \hat{\cup} (Q \times Q).$$

We conclude that

$$\alpha^T(u, v) \subseteq (\{u, v\} \times \{u, v\}) \hat{\cup} (Q \times Q) \cup \Delta_T.$$ 

Thus, $(u, e_3v) \not\subseteq \alpha^T(u, v)$ as $e_3v \in C(e_3)C(e_2) \subseteq M(C(e_2)) \subseteq Q$. However, $u = e_3u$ by the initial assumption above. Thus, $(u, e_3v) = (e_3u, e_3v) \in \alpha^S(u, v) \cap (T \times T)$. Hence, $\alpha^S(u, v) \cap (T \times T) \neq \alpha^T(u, v)$ contrary to $S$ having CEP. This completes the proof that if $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$, then either (i) or (ii) must hold.

Suppose that it is not the case that $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$. That is, assume that there exists $x \in C(e_1) \setminus M(C(e_1))$ such that $C(e_3)x \neq x$. Then by Proposition 3.10(2), $C(e_3)C(e_1) \subseteq M(C(e_1))$. We will show that either (iii) or (iv) must hold. If (iii) does not hold, then $C(e_3)C(e_2) \not\subseteq M(C(e_2))$. (In particular,
according to 1.48(4), this implies that $C(e_2) \setminus M(C(e_2)) \neq \emptyset$. Then according to Proposition 3.10(2), $C(e_3)x = x$ for all $x \in C(e_2) \setminus M(C(e_2))$. We show that (iv) holds by proving that $C(e_2)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$. Suppose this does not hold. That is, suppose that there exists $x \in C(e_1) \setminus M(C(e_1))$ such that $C(e_2)x \neq x$. (In particular, this implies that $C(e_1) \setminus M(C(e_1)) \neq \emptyset$.) Then by Proposition 3.10(2),

$$C(e_2)C(e_1) \subseteq M(C(e_1)).$$  

(†)

As noted above, $C(e_2) \setminus M(C(e_2)) \neq \emptyset$ and $C(e_1) \setminus M(C(e_1)) \neq \emptyset$. Thus, $|N_{e_1}|, |N_{e_2}| \geq 2$ and by assumption, these are finite. Hence, according to 1.45, we may choose $u \in A(N_{e_1}) \setminus \{0_{e_1}\}$ and $v \in A(N_{e_2}) \setminus \{0_{e_2}\}$. We have

$$ux \in M(C(e_1)) \text{ for all } x \in C(e_1) \text{ and } vx \in M(C(e_2)) \text{ for all } x \in C(e_2).$$  

(‡)

Now if $s \in C(e_1)$, then $(su, sv) \in M(C(e_1)) \times M(C(e_1))$ by (‡) and (†). If $s \in C(e_2)$, then $(su, sv) \in M(C(e_1)) \times M(C(e_2))$ by (†) and (‡). Set $T = C(e_1) \cup C(e_2)$ and $Q = M(C(e_1)) \cup M(C(e_2))$. Then by the observations above,

$$\{(su, sv) : s \in T^1\} \subseteq (\{u, v\} \times \{u, v\}) \cup (Q \times Q).$$

We conclude that

$$\alpha^T(u, v) \subseteq (\{u, v\} \times \{u, v\}) \cup (Q \times Q) \cup \Delta_T.$$  

Thus, $(e_3u, v) \not\in \alpha^T(u, v)$ as $e_3u \in C(e_3)C(e_1) \subseteq M(C(e_1)) \subseteq Q$. However, $v = e_3v$ so $(e_3u, v) = (e_3u, e_3v) \in \alpha^S(u, v) \cap (T \times T)$. Hence, $\alpha^S(u, v) \cap (T \times T) \neq \alpha^T(u, v)$ contrary to $S$ having CEP. This completes the proof that if it is not the case that $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$, then either (iii) or (iv) must hold. Whence, one of (i)-(iv) must hold given $e_1, e_2, e_3 \in E_S$ such that $e_1 < e_2 < e_3$. 

In examining the results obtained thus far, we note that if $e \leq f$ in a commutative semigroup $S$ with CEP, then according to Proposition 3.10, there are two
possibilities for the action of $C(f)$ on $C(e)$. Namely, each element of $C(f)$ may act as an identity for each element of $N_e$ or otherwise, all products between $C(f)$ and $C(e)$ must be contained in $M(C(e))$. The proposition above indicates that given idempotents $e_1 \leq e_2 \leq e_3$ such that $C(e_i)$ is finite for each $i$, if $C(e_3)$ acts differently on $C(e_1)$ and $C(e_2)$ respectively (i.e. neither 3.14(1) nor 3.14(3) holds), then $C(e_2)$ must act as an identity on $N_{e_1}$. Hence, we have an initial idea of how components interact in the presence of CEP. Next, we consider the action of components whose idempotents are not related on components whose idempotents lie below one or both of the non-related idempotents. The conditions of this proposition are motivated by examples 3.13(3) and 3.13(4).

3.15 Proposition. Let $S$ be a commutative semigroup with the congruence extension property (CEP) such that for each $e \in E_S$, $C(e)$ is finite. Then for $e_1, e_2, e_3, e_4 \in E_S$ such that $C(e_2) \setminus M(C(e_2)) \neq \emptyset$, $C(e_3) \setminus M(C(e_3)) \neq \emptyset$, $C(e_2 e_3) \setminus M(C(e_2 e_3)) \neq \emptyset$, $e_2 \not\leq e_3$, $e_3 \not\leq e_2$, $e_1 \leq e_2 e_3$, $e_4 < e_2$, and $e_4 \not\leq e_3$, the following hold:

(i) Either $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$ and $C(e_2)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$

or $C(e_3)C(e_1) \subseteq M(C(e_1))$ and $C(e_2)C(e_1) \subseteq M(C(e_1))$.

(ii) $C(e_2)C(e_4) \subseteq M(C(e_4))$.

Proof. Let $e_1, e_2, e_3, e_4 \in E_S$ such that $e_2 \not\leq e_3$, $e_3 \not\leq e_2$, $e_1 \leq e_2 e_3$, $e_4 < e_2$, and $e_4 \not\leq e_3$. To see that (i) holds, first suppose that $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$. We show that $C(e_2)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$. Suppose this is not the case. Then there exists $x \in C(e_1) \setminus M(C(e_1))$ such that $C(e_2)x \neq x$. (In particular, this implies that $C(e_1) \setminus M(C(e_1)) \neq \emptyset$.) Then according
to Proposition 3.10(2),
\[ C(e_2)C(e_1) \subseteq M(C(e_1)). \] (*)

By assumption and the note above, \( C(e_2) \setminus M(C(e_2)) \neq \emptyset \) and \( C(e_1) \setminus M(C(e_1)) \neq \emptyset \). Thus, \(|N_{e_1}|, |N_{e_2}| \geq 2\) and by assumption, these are finite. Hence, according to 1.45, we may choose \( u \in A(N_{e_2}) \setminus \{0_{e_2}\} \) and \( v \in A(N_{e_1}) \setminus \{0_{e_1}\} \). We have

\[ vx \in M(C(e_1)) \text{ for all } x \in C(e_1) \text{ and } ux \in M(C(e_2)) \text{ for all } x \in C(e_2). \] (**) 

If \( s \in C(e_1) \), then \((su, sv) \in M(C(e_1)) \times M(C(e_1))\) by (*) and (**). If \( s \in C(e_2) \), then \((su, sv) \in M(C(e_2)) \times M(C(e_1))\) by (*) and (**). Set \( T = C(e_1) \cup C(e_2) \) and \( Q = M(C(e_1)) \cup M(C(e_2)) \). Then by the observations above,

\[ \{(su, sv) : s \in T\} \subseteq (\{u, v\} \times \{u, v\}) \cup (Q \times Q). \]

We conclude that

\[ \alpha^T(u, v) \subseteq (\{u, v\} \times \{u, v\}) \cup (Q \times Q) \cup \Delta_T. \]

Thus, \((v, e_2v) \not\in \alpha^T(u, v)\) as \( e_2v \in C(e_2)C(e_1) \subseteq M(C(e_1)) \subseteq Q \) by (*). We show that \((v, e_2v) \in \alpha^S(u, v)\). Now \((e_3u, e_3v) \in \alpha^S(u, v)\) and \((e_2e_3u, e_2e_3v) \in \alpha^S(u, v)\).

By Proposition 3.10(1), \( e_3u \in C(e_3)C(e_2) \subseteq M(C(e_2e_3)) = G_{e_2e_3} \). Thus, we have

\[ e_2e_3u = (e_2e_3)(e_3u) = e_3u \text{ as } e_2e_3 \text{ is the identity of } G_{e_2e_3}. \]

Hence, by transitivity, we have \((e_3v, e_2e_3v) \in \alpha^S(u, v)\). However, \( v = e_3v \) by one of the assumptions above. Thus, \( e_2e_3v = e_2v \) and \((v, e_2v) = (e_3v, e_2e_3v) \in \alpha^S(u, v) \cap (T \times T)\). Hence, \( \alpha^S(u, v) \cap (T \times T) \neq \alpha^T(u, v) \) contrary to \( S \) having CEP. Thus, if \( C(e_3)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \), then \( C(e_2)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \).

Now suppose that \( C(e_3)x \neq x \) for some \( x \in C(e_1) \setminus M(C(e_1)) \). Then by Proposition 3.10(2), \( C(e_3)C(e_1) \subseteq M(C(e_1)) \). If \( C(e_2)C(e_1) \not\subseteq M(C(e_1)) \), then by Proposition 3.10(2), \( C(e_2)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \) and, as argued in the previous paragraph, this implies that \( C(e_3)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \) contrary to the
assumption. Thus, we conclude that $C(e_2)C(e_1) \subseteq M(C(e_1))$. Hence, if it is not the case that $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$, then $C(e_3)C(e_1) \subseteq M(C(e_1))$ and $C(e_2)C(e_1) \subseteq M(C(e_1))$. This completes the proof that (i) holds.

Now we show that $C(e_2)C(e_4) \subseteq M(C(e_4))$. By assumption, we have $e_2 \not\leq e_3$, $e_3 \not\leq e_2$, $e_4 < e_2$, and $e_4 \not\leq e_3$. If $C(e_2)C(e_4) \not\subseteq M(C(e_4))$, then according to 1.48(4), $C(e_4) \setminus M(C(e_4)) \neq \emptyset$ and according to Proposition 3.10(2),

$$C(e_2)x = x \text{ for all } x \in C(e_4) \setminus M(C(e_4)).$$

Now $e_3e_4 = e_3(e_4e_2) = e_2e_3e_4 = (e_2e_3)(e_3e_4)$ so $e_3e_4 \leq e_2e_3$. We consider the cases in which $e_3e_4 = e_2e_3$ and $e_3e_4 < e_2e_3$ separately below.

**Case 1:** Assume $e_3e_4 = e_2e_3$.

We have the following configuration of idempotents:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
| & | & |
\end{array}
\]

By assumption and the note above, $C(e_3) \setminus M(C(e_3)) \neq \emptyset$ and $C(e_4) \setminus M(C(e_4)) \neq \emptyset$. Thus, $|N_{e_3}|, |N_{e_4}| \geq 2$ and by assumption, these are finite. Hence, according to 1.45, we may choose $u \in A(N_{e_3}) \setminus \{0_{e_3}\}$ and $v \in A(N_{e_4}) \setminus \{0_{e_4}\}$. We have

$$ux \in M(C(e_3)) \text{ for all } x \in C(e_3) \text{ and } vx \in M(C(e_4)) \text{ for all } x \in C(e_4).$$

Note that $T = C(e_3) \cup C(e_4) \cup C(e_3e_4)$ is a subsemigroup of $S$. Now if $s \in C(e_3)$, then $(su, sv) \in M(C(e_3)) \times M(C(e_3e_4))$ by (†) and by Proposition 3.10(1). For each $s \in C(e_4)$, $(su, sv) \in M(C(e_3e_4)) \times M(C(e_4))$ by Proposition 3.10(1) and (†). If $s \in C(e_3e_4)$, then either $(su, sv) \in M(C(e_3e_4)) \times M(C(e_3e_4))$ or $(su, sv) = (s, s)$ as
(i) holds. Set \( Q = M(C(e_3)) \cup M(C(e_4)) \cup M(C(e_3 e_4)) \). Then by the observations above, \( \{(su, sv) : s \in T^1\} \subseteq ((\{u, v\} \times \{u, v\}) \cup (M(C(e_3)) \cup (Q \times Q) \cup \Delta_T. \) We conclude that \( \alpha^T(u, v) \subseteq ((\{u, v\} \times \{u, v\}) \cup (Q \times Q) \cup \Delta_T. \) Thus, \( (v, e_2 u) \not\in \alpha^T(u, v) \) as \( e_2 u \in C(e_2)C(e_3) \subseteq M(C(e_2 e_3)) = M(C(e_3 e_4)) \subseteq Q \) by Proposition 3.10(1).

However, \( v = e_2 v \) by (†). Thus, \( (v, e_2 u) = (e_2 v, e_2 u) \in \alpha^S(u, v) \cap (T \times T). \) Hence, \( \alpha^S(u, v) \cap (T \times T) \neq \alpha^T(u, v) \) contrary to \( S \) having CEP. Therefore, we conclude that \( C(e_2)C(e_4) \subseteq M(C(e_4)) \) in Case 1.

**Case 2:** Assume \( e_3 e_4 < e_2 e_3 \).

In this case, \( e_2 e_3 \not\leq e_4 \) as otherwise, \( e_2 e_3 = e_2 e_3 e_4 = e_3 e_4 \), contrary to assumption. Also, \( e_4 \not\leq e_2 e_3 \) as \( e_4 \not\leq e_3 \). We have the following configuration of idempotents:

![Diagram](image)

Note that \( T = C(e_2 e_3) \cup C(e_4) \cup C(e_3 e_4) \) is a subsemigroup of \( S \). By assumption and the note above, \( C(e_2 e_3) \setminus M(C(e_2 e_3)) \neq \emptyset \) and \( C(e_4) \setminus M(C(e_4)) \neq \emptyset \). Thus, \( |N_{e_4}|, |N_{e_2 e_3}| \geq 2 \) and by assumption, these are finite. Hence, according to 1.45, we may choose \( u \in A(N_{e_2 e_3}) \setminus \{0 e_2 e_3\} \) and \( v \in A(N_{e_4}) \setminus \{0 e_4\} \). We have

\[
ux \in M(C(e_2 e_3)) \text{ for all } x \in C(e_2 e_3) \text{ and } vx \in M(C(e_4)) \text{ for all } x \in C(e_4). \quad (††).
\]

According to (††) and Proposition 3.10(1), if \( s \in C(e_2 e_3) \), then we may conclude that \( (su, sv) \in M(C(e_2 e_3)) \times M(C(e_2 e_3 e_4)) = M(C(e_2 e_3)) \times M(C(e_3 e_4)) \) and if \( s \in C(e_4) \), then \( (su, sv) \in M(C(e_2 e_3 e_4)) \times M(C(e_4)) = M(C(e_3 e_4)) \times M(C(e_4)). \) If \( s \in C(e_3 e_4) \), then either \( (su, sv) \in M(C(e_3 e_4)) \times M(C(e_3 e_4)) \) or \( (su, sv) = (s, s) \) as (i) holds. Let \( Q = M(C(e_2 e_3)) \cup M(C(e_4)) \cup M(C(e_3 e_4)) \). By previous observations,
\({\{(su, sv) : s \in T^1\} \subseteq (\{u, v\} \times \{u, v\}) \cup (Q \times Q) \cup \Delta_T.}\) Therefore, we conclude that \(\alpha_T(u, v) \subseteq (\{u, v\} \times \{u, v\}) \cup (Q \times Q) \cup \Delta_T.\) Thus, we have \((v, e_2u) \not\in \alpha_T(u, v)\) as \(e_2u \in C(e_2)C(e_2e_3) \subseteq M(C(e_2e_3)) \subseteq Q\) by 1.48(4). However, \(v = e_2v\) by \(\dagger\). Thus, \((v, e_2u) = (e_2v, e_2u) \in \alpha_S(u, v) \cap (T \times T).\) Hence, \(\alpha_S(u, v) \cap (T \times T) \neq \alpha_T(u, v)\) contrary to \(S\) having CEP. Therefore, we conclude that \(C(e_2)C(e_4) \subseteq M(C(e_4))\) in Case 2.

This completes the proof that (ii) holds and the proposition is proven. \(\blacksquare\)

Given any pair of idempotents in a commutative semigroup \(S\) with CEP which has finite components, we can now employ Propositions 3.14 and 3.15 to describe the possibilities for the action of these components on a third component whose idempotent is below one or both of the idempotents in the original pair. Consider the following diagram.

For any pair of comparable idempotents, we may consider a third idempotent which is below one or both of these idempotents. For example, consider \(e_2 \leq e_5.\) We have \(e_3 \leq e_5\) and \(e_3 \not\leq e_2\) choosing \(e_3\) as a third idempotent. Also, we have \(e_1 \leq e_2, e_5\) choosing \(e_1\) as a third idempotent. In these cases, Proposition 3.14 can be used to determine possibilities for multiplication between components \(C(e_2), C(e_3),\) and \(C(e_5)\) as \(e_2 \leq e_3 \leq e_5\) and between components \(C(e_1), C(e_2),\) and...
$C(e_5)$ as $e_1 \leq e_2 \leq e_5$. Now consider $e_4 \parallel e_5$. Choosing $e_3$ as a third idempotent, Proposition 3.10(1) and 3.15(ii) describe multiplication between $C(e_4)$, $C(e_5)$, and $C(e_3)$. Choosing $e_1$ as a third idempotent, Proposition 3.10(1) and 3.15(i) describe possibilities for multiplication between $C(e_4)$, $C(e_5)$, and $C(e_1)$.

Combining results obtained in this chapter and employing the hereditary property of CEP, we obtain the following theorem.

3.16 Summary Theorem. Let $S$ be a commutative semigroup with the congruence extension property (CEP). Then

$$S = \bigcup_{e \in E_S} C(e)$$

and the following hold:

(1) For each $e \in E_S$, $G_e$ is torsion, $xy = x^2 = y^2$ for all $x, y \in N_e \setminus \{0_e\}$ with $xy \neq 0_e$, and if $x, y, z \in N_e \setminus \{0_e\}$ such that $xy = xz = z^2 = x^2 = y^2 \neq 0_e$, then $z^2 = yz$. Moreover, this holds if and only if $C(e)$ has CEP for each $e \in E_S$.

(2) For $e, f \in E_S$ such that $e \parallel f$, $C(f)C(e) \subseteq M(C(fe))$. If $e \leq f$, then either

(i) $C(f)C(e) \subseteq M(C(e))$ or

(ii) $C(f)x = x$ for all $x \in C(e) \setminus M(C(e))$ and $C(f)M(C(e)) \subseteq M(C(e))$.

Furthermore, if $C(e)$ is finite for each $e \in E_S$, then the following hold:

(3) For $e_1, e_2, e_3 \in E_S$ such that $e_1 < e_2 < e_3$, one of the following holds:

(i) $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$ and $C(e_3)x = x$ for all $x \in C(e_2) \setminus M(C(e_2))$.

(ii) $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$, $C(e_3)C(e_2) \subseteq M(C(e_2))$, and $C(e_2)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$. 


(iii) $C(e_3)C(e_1) \subseteq M(C(e_1))$ and $C(e_3)C(e_2) \subseteq M(C(e_2))$.

(iv) $C(e_3)C(e_1) \subseteq M(C(e_1))$, $C(e_3)x = x$ for all $x \in C(e_2)\backslash M(C(e_2))$, and $C(e_2)x = x$ for all $x \in C(e_1)\backslash M(C(e_1))$.

(4) For $e_i \in E_S$ (1 ≤ $i$ ≤ 4) such that $e_2 \not\subseteq e_3$, $e_3 \not\subseteq e_2$, $e_1 \leq e_2 e_3$, $e_4 < e_2$, $e_4 \not\subseteq e_3$, $C(e_2)\backslash M(C(e_2)) \neq \emptyset$, $C(e_3)\backslash M(C(e_3)) \neq \emptyset$, and $C(e_2 e_3)\backslash M(C(e_2 e_3)) \neq \emptyset$, the following hold:

(i) Either $C(e_3)x = x$ for all $x \in C(e_1)\backslash M(C(e_1))$ and $C(e_2)x = x$ for all $x \in C(e_1)\backslash M(C(e_1))$ or $C(e_3)C(e_1) \subseteq M(C(e_1))$ and $C(e_2)C(e_1) \subseteq M(C(e_1))$.

(ii) $C(e_2)C(e_4) \subseteq M(C(e_4))$. 
In examining any class of semigroups, it is natural to ask whether that class forms a variety (i.e. a class which is closed under the operations of taking subsemigroups, homomorphic images, and products.) With regard to the properties treated in previous chapters, it is known that the ideal extension property (IEP) is hereditary and is preserved by homomorphisms, but it is not productive. Therefore, the class of semigroups having IEP does not form a variety. It is also known that the congruence extension property (CEP) is hereditary but not productive. Hence, the class of semigroups having CEP is not a variety. However, the question of whether CEP is preserved by homomorphisms remains open. The purpose of this chapter is to answer this question in several special cases and to further examine the general question.

It is well-known that each homomorphism on a semigroup gives rise to a congruence on that semigroup. Namely, for a homomorphism $\phi$ on $S$,

$$\ker \phi = \{ (x, y) \in S \times S : \phi(x) = \phi(y) \}$$

is a congruence on $S$. Furthermore, for a congruence $\sigma$ on a semigroup $S$, the natural map from $S$ to $S/\sigma$ is a homomorphism. Using this correspondence between homomorphisms and congruences, we can translate the question of whether homomorphic images of a semigroup $S$ have CEP into a question concerning congruences on the semigroup $S$ and certain subsemigroups of $S$. We make this precise in the course of this chapter.
Upon initial consideration of the question of whether CEP is preserved by homomorphisms, one is naturally inclined to begin with a congruence $\alpha$ on a sub-semigroup of an image $X$ of a semigroup $S$ with CEP, pull $\alpha$ back through the homomorphism, extend this relation as $S$ has CEP, and then push the extension through the homomorphism out to $X$ with the hope that this pushed-out relation will be a congruence on $X$ which extends $\alpha$. We formalize the notions of the pullback and pushout of a congruence with the following definition. Let $\phi: S \to X$ be a homomorphism from a semigroup $S$ onto a semigroup $X$. Let $\alpha$ be a congruence on $X$ and $\rho$ a congruence on $X$ and $\rho$ a congruence on $S$. Then

**pullback of $\alpha$** := \{(x, y) \in S \times S: (\phi(x), \phi(y)) \in \alpha\}

and

**pushout of $\rho$** := \{(\phi(x), \phi(y)) \in X \times X: (x, y) \in \rho\}.

There are a number of problems with following the initial impulse to perform the pullback-pushout process described above. As is indicated in the following lemmas, the pullback of a congruence is always a congruence, but the pushout of a congruence may fail to be transitive due to lack of injectivity of the given homomorphism. In light of this, one might expect that most of the potential problems with the pullback-pushout process are eliminated when dealing with congruences on $S$ which contain $\ker \phi$. Indeed, this is the case and as a result we will give special consideration to the join of $\ker \phi$ and the extension of the pullback of $\alpha$ in making the aforementioned translation of the main problem. The first three lemmas are from [Garcia, 1988].

**4.1 Lemma.** Let $\phi: S \to X$ be a homomorphism from a semigroup $S$ onto a semigroup $X$. Let $\alpha$ be a congruence on $X$. Let $\rho$ be the pullback of $\alpha$ to $S$. Then $\rho$ is a congruence on $S$. 
4.2 Lemma. Let \( \phi: S \to X \) be a homomorphism from a semigroup \( S \) onto a semigroup \( X \). Let \( \rho \) be a congruence on \( S \). Let \( \alpha \) be the pushout of \( \rho \). Then \( \alpha \) is a reflexive, symmetric, compatible relation on \( X \).

4.3 Lemma. Let \( \phi: S \to X \) be a homomorphism from a semigroup \( S \) onto a semigroup \( X \). Let \( \rho \) be a congruence on \( S \) such that \( \ker(\phi) \subseteq \rho \). Let \( \alpha \) be the pushout of \( \rho \) to \( X \). Then \( \alpha \) is a congruence on \( X \). Moreover, the pullback of \( \alpha \) is \( \rho \).

4.4 Lemma. Let \( \phi: S \to X \) be a homomorphism from a semigroup \( S \) onto a semigroup \( X \). Let \( Y \) be a subsemigroup of \( X \) and let \( T = \phi^{-1}[Y] \). Let \( \alpha \) be a congruence on \( Y \) and let \( \gamma \) be a congruence on \( X \). Let \( \rho \) be the pullback of \( \alpha \) and let \( \tau \) be the pullback of \( \gamma \). Then \( \gamma \) extends \( \alpha \) if and only if \( \tau \) extends \( \rho \).

Proof. We have
\[
\tau = \{(a, b) \in S \times S : (\phi(a), \phi(b)) \in \gamma\}
\]
and
\[
\rho = \{(a, b) \in S \times S : (\phi(a), \phi(b)) \in \alpha\}.
\]
Let \( \gamma \) extend \( \alpha \). Then \( \gamma \cap (Y \times Y) = \alpha \). We must see that \( \tau \cap (Y \times Y) = \rho \). Clearly, \( \rho \subseteq \tau \), since \( \alpha \subseteq \gamma \). Thus, \( \rho \subseteq \tau \cap (T \times T) \). Now let \( (a, b) \in \tau \cap (T \times T) \). Then \( (\phi(a), \phi(b)) \in \gamma \cap (Y \times Y) = \alpha \). Thus, \( (a, b) \in \rho \). Hence, \( \rho = \tau \cap (T \times T) \).

Conversely, let \( \tau \) extend \( \rho \). Then \( \tau \cap (T \times T) = \rho \). We show that \( \gamma \cap (Y \times Y) = \alpha \).

Let \( (x, y) \in \alpha \). Choose \( a, b \in T \) such that \( \phi(a) = x \) and \( \phi(b) = y \). Then since \( (\phi(a), \phi(b)) = (x, y) \in \alpha \), \( (a, b) \in \rho \subseteq \tau \). Thus, we have \( (x, y) = (\phi(a), \phi(b)) \in \gamma \) and \( \alpha \subseteq \gamma \cap (Y \times Y) \). Let \( (x, y) \in \gamma \cap (Y \times Y) \). \( \phi(a) = x \) and \( \phi(b) = y \) for some \( a, b \in T \) and \( (\phi(a), \phi(b)) = (x, y) \in \gamma \). Thus, \( (a, b) \in \tau \cap (T \times T) = \rho \). Hence, we have \( (x, y) = (\phi(a), \phi(b)) \in \alpha \) and \( \alpha = \gamma \cap (Y \times Y) \).
The following proposition affirmatively answers the general question posed in this chapter in the case that the homomorphic image under consideration is a Rees quotient. The proof presented here can be directly topologized for the class of compact semigroups as shown in Chapter 6. For more discussion of proof techniques used below, the reader is referred to the end of this chapter.

4.5 Proposition. Let $S$ be a semigroup with the congruence extension property (CEP) and let $I$ be an ideal of $S$. Then $S/I$ has CEP.

Proof. Let $\phi : S \to S/I$ be the natural surmorphism. Then $\ker \phi = (I \times I) \cup \Delta_S$ and $\phi(I) = 0 \in S/I$. Let $Y$ be a subsemigroup of $S/I$ and let $T = \phi^{-1}(Y)$. Let $\alpha$ be a congruence on $Y$ and let $\rho$ be the pullback of $\alpha$ to $T$. By Lemma 4.1, $\rho$ is a congruence on $T$.

Claim 1: $I \cup T$ is a subsemigroup of $S$.

Since $I$ is an ideal, $I \cdot I \cup I \cdot T \cup T \cdot I \subseteq I$ and $T \cdot T \subseteq T$ as $T$ is a subsemigroup. Thus, $(I \cup T) \cdot (I \cup T) = I \cdot I \cup I \cdot T \cup T \cdot I \cup T \cdot T \subseteq I \cup T$.

Claim 2: $\rho \cup (I \times I)$ is a congruence on $I \cup T$

Case 1: Assume $0 \in Y$.

Then $I \subseteq \phi^{-1}(Y) = T$ and thus $T \cup I = T$. Also $I \times I \subseteq \rho$ as $(0,0) \in \alpha$. Hence, $\rho \cup (I \times I) = \rho$, a congruence on $T = I \cup T$.

Case 2: Assume $0 \notin Y$.

Then $I \cap T = \emptyset$. Now $\rho \cup (I \times I)$ is reflexive as $\Delta_{I \cup T} = \Delta_I \cup \Delta_T \subseteq \rho \cup (I \times I)$.

Also, $\rho \cup (I \times I)$ is symmetric as $\rho$ and $I \times I$ are symmetric and since $I \cap T = \emptyset$, it is clear that $\rho \cup (I \times I)$ is transitive as $\rho$ and $I \times I$ are transitive. To show compatibility, let $(a, b) \in \rho \cup (I \times I)$ and let $s \in I \cup T$.

If $(a, b) \in I \times I$ or $s \in I$, then $(sa, sb) \in I \times I$ as $I$ is an ideal. Otherwise,
$(a, b) \in \rho, s \in T$ and $(sa, sb) \in \rho$ as $\rho$ is compatible. Thus, $\rho \cup (I \times I)$ is a congruence on $I \cup T$.

Claim 3: $(I \times I) \cap (T \times T) \subseteq \rho$

In the proof of Claim 2 we saw that if $0 \in Y$, then $I \subseteq T$, $I \times I \subseteq \rho$ and hence, $(I \times I) \cap (T \times T) \subseteq \rho$. Otherwise, $0 \notin Y$, $I \cap T = \emptyset$, and $(I \times I) \cap (T \times T) = \emptyset \subseteq \rho$.

Now by Claims 1 and 2, $\rho \cup (I \times I)$ is a congruence on the subsemigroup $I \cup T$.

Thus, since $S$ has CEP, there is a congruence $\sigma$ on $S$ such that

$$\sigma \cap ((I \cup T) \times (I \cup T)) = \rho \cup (I \times I).$$

Claim 4: $\sigma$ extends $\rho$ from $T$ to $S$.

$$\sigma \cap (T \times T) = (\sigma \cap ((T \cup I) \times (T \cup I))) \cap (T \times T)$$

$$= (\rho \cup (I \times I)) \cap (T \times T)$$

$$= (\rho \cap (T \times T)) \cup ((I \times I) \cap (T \times T))$$

$$\subseteq \rho \cup \rho = \rho.$$

Let $\gamma$ be the pushout of $\sigma$. Note that $\ker \phi = (I \times I) \cup \Delta S \subseteq \sigma$. Thus, by Lemma 4.3, $\gamma$ is a congruence on $S/I$ and the pullback of $\gamma$ is $\sigma$. We have that the pullback of $\gamma = \sigma$ extends the pullback of $\alpha = \rho$ by Claim 4. Thus, applying Lemma 4.4, $\gamma$ extends $\alpha$ and $S/I$ has CEP.

4.6 Corollary. Let $S$ be an ideal semigroup with the congruence extension property (CEP). Then any homomorphic image of $S$ has CEP.

Proof. This is immediate from Proposition 4.5.

We now turn our attention toward the general question of whether the homomorphic image of a semigroup with the congruence extension property (CEP) has
CEP. As indicated in the introduction, we can translate this question into a question concerning only congruences on the original semigroup. That is, we can state necessary and sufficient conditions on the interactions of congruences on certain subsemigroups of a semigroup to insure that its image retains CEP. As the first step in achieving this goal, we present the following result.

**4.7 Proposition.** Let \( \phi : S \rightarrow X \) be a homomorphism from a semigroup \( S \) onto a semigroup \( X \). Let \( Y \) be a subsemigroup of \( X \) and let \( \alpha \) be a congruence on \( Y \). Let \( \rho \) be the pullback of \( \alpha \) to \( T = \phi^{-1}(Y) \). Then \( \langle \alpha \rangle_X \) is the pushout of \( \ker \phi \vee \langle \rho \rangle_S \).

**Proof.** We must show that \( \langle \alpha \rangle_X = \{(\phi(a), \phi(b)) : (a, b) \in \ker \phi \vee \langle \rho \rangle_S \} \).

To see that the forward inclusion holds, note first that the pushout of \( \ker \phi \vee \langle \rho \rangle_S \) (i.e. the right hand side above) is a congruence on \( X \) by Lemma 4.3 as \( \ker \phi \) is contained in \( \ker \phi \vee \langle \rho \rangle_S \). Certainly, the pushout of \( \rho \) is contained in the pushout of \( \ker \phi \vee \langle \rho \rangle_S \) and the pushout of \( \rho \) is \( \alpha \). Thus, the pushout of \( \ker \phi \vee \langle \rho \rangle_S \) is a congruence on \( X \) which contains \( \alpha \). Hence, the pushout of \( \ker \phi \vee \langle \rho \rangle_S \) must contain \( \langle \alpha \rangle_X \) and the forward inclusion is proven.

To prove the reverse inclusion, let \( (x, y) \in \{(\phi(a), \phi(b)) : (a, b) \in \ker \phi \vee \langle \rho \rangle_S \} \). Then \( (x, y) = (\phi(a), \phi(b)) \) for some \( (a, b) \in \ker \phi \vee \langle \rho \rangle_S \). Thus, according to 1.6, there is a finite transition \( a = x_0, x_1, \ldots, x_n = b \) such that \( (x_i, x_{i+1}) \in \ker \phi \cup \langle \rho \rangle_S \). We claim that \( (\phi(x_i), \phi(x_{i+1})) \in \langle \alpha \rangle_X \) for \( 0 \leq i \leq n - 1 \). Now for fixed \( i \), if \( (x_i, x_{i+1}) \in \ker \phi \), then \( (\phi(x_i), \phi(x_{i+1})) \in \Delta_X \subseteq \langle \alpha \rangle_X \). Otherwise, \( (x_i, x_{i+1}) \in \langle \rho \rangle_S \). In this case, according to 1.2, there is a finite transition \( x_i = y_0, y_1, \ldots, y_m = x_{i+1} \) such that \( (y_j, y_{j+1}) \in \{(\text{sat}, \text{sbt}) : (a, b) \in \rho, s, t \in S^1 \} \). Thus, for a given \( j \),

\[
(\phi(y_j), \phi(y_{j+1})) = (\phi(s_j a_j t_j), \phi(s_j b_j t_j)) = (\phi(s_j)\phi(a_j)\phi(t_j), \phi(s_j)\phi(b_j)\phi(t_j)) \in \langle \alpha \rangle_X
\]

since \( \langle \alpha \rangle_X \) is compatible and \( (\phi(a_j), \phi(b_j)) \in \langle \alpha \rangle_X \) as \( \alpha \) is the pushout of \( \rho \) and
Thus, we obtain that \((\phi(x_i), \phi(x_{i+1})) = (\phi(y_0), \phi(y_m)) \in (\alpha)_X\) since 
\((\alpha)_X\) is transitive and our claim is proven. Thus, by transitivity of 
\((\alpha)_X\), we have 
\((x, y) = (\phi(a), \phi(b)) = (\phi(x_0), \phi(x_n)) \in (\alpha)_X\) as required and the proposition is proven.

4.8 Corollary. Let \(\phi : S \to X\) be a homomorphism from a semigroup \(S\) onto a semigroup \(X\). Let \(Y\) be a subsemigroup of \(X\) and let \(\alpha\) be a congruence on \(Y\). Let \(\rho\) be the pullback of \(\alpha\) to \(T = \phi^{-1}(Y)\). Then the following are equivalent:

1. \(\alpha\) extends to a congruence on \(X\).
2. The pushout of \(\ker \phi \lor (\rho)_S\) is an extension of \(\alpha\).
3. \(\ker \phi \lor (\rho)_S\) is an extension of \(\rho\).

Proof. The equivalence of (1) and (2) follows from 1.3 and Proposition 4.7. To see that (2) and (3) are equivalent, first note that according to Lemma 4.3, the pullback of the pushout of \(\ker \phi \lor (\rho)_S\) is \(\ker \phi \lor (\rho)_S\) as \(\ker \phi \subseteq \ker \phi \lor (\rho)_S\). Thus, according to Lemma 4.4, \(\ker \phi \lor (\rho)_S\) (the pullback of the pushout of \(\ker \phi \lor (\rho)_S\)) extends \(\rho\) (the pullback of \(\alpha\)) if and only if the pushout of \(\ker \phi \lor (\rho)_S\) extends \(\alpha\).

The following theorem gives the promised conditions on the interaction of certain congruences of a semigroup which insure that a given image retains the congruence extension property (CEP).

4.9 Theorem. Let \(\phi : S \to X\) be a homomorphism from a semigroup \(S\) onto a semigroup \(X\). Then \(X\) has the congruence extension property (CEP) if and only if for each subsemigroup \(T\) of \(S\) which is \(\phi\)-saturated and each congruence \(\rho\) on \(T\) which contains \(\ker \phi|_T\), \(\ker \phi \lor (\rho)_S\) extends \(\rho\).
Proof. Suppose $X$ has the congruence extension property (CEP). Let $T$ be a subsemigroup of $S$ which is $\phi$-saturated. Let $\rho$ be a congruence on $T$ which contains $\ker \phi|_T$. Let $Y = \phi(T)$. Then $\phi^{-1}(Y) = \phi^{-1}(\phi(T)) = T$ as $T$ is $\phi$-saturated. Let $\alpha$ be the pushout of $\rho$ to $Y$. Since $\ker \phi|_T \subseteq \rho$, $\alpha$ is a congruence on $Y$ and $\rho$ is the pullback of $\alpha$ to $T$ according to Lemma 4.3. Now since $X$ has CEP, $\alpha$ extends to a congruence on $X$. Thus, according to Corollary 4.8, $\ker \phi \vee \langle \rho \rangle_S$ extends $\rho$. This proves the forward implication.

Now suppose that for each subsemigroup $T$ of $S$ which is $\phi$-saturated and each congruence $\rho$ on $T$ which contains $\ker \phi|_T$, $\ker \phi \vee \langle \rho \rangle_S$ extends $\rho$. Let $Y$ be a subsemigroup of $X$ and let $\alpha$ be a congruence on $Y$. We must show that $\alpha$ extends to a congruence on $X$. Let $T = \phi^{-1}(Y)$. Then $T = \phi^{-1}(\phi(\phi^{-1}(Y))) = \phi^{-1}(\phi(T))$ so $T$ is $\phi$-saturated. Let $\rho$ be the pullback of $\alpha$ to $T$. Then since $\Delta_Y \subseteq \alpha$, $\ker \phi|_T \subseteq \rho$. Thus, by hypothesis, $\ker \phi \vee \langle \rho \rangle_S$ extends $\rho$. Hence, by Corollary 4.8, $\alpha$ extends to a congruence on $X$ and the reverse implication is proven.

It has been determined through a computer search that for each semigroup with CEP of order six or less and each commutative semigroup of order seven, the condition given in Theorem 4.9 holds. Thus, we conclude that the homomorphic images of semigroups of order six or less and commutative semigroups of order seven with CEP must retain CEP. In the statement of Theorem 4.9, note that if $S$ has the congruence extension property (CEP), then we know that $\langle \rho \rangle_S$ extends $\rho$ to $S$. Thus, the question of whether the image $X$ retains CEP is equivalent to the question of whether joining $\ker \phi$ to $\langle \rho \rangle_S$ destroys its extending property with respect to $\rho$. In considering this question, one might conjecture that if $S$ is any semigroup, $T$ is any $\phi$-saturated subsemigroup of $S$, and $\rho$ is a congruence on $T$.
containing \( \ker \phi \mid T \) such that \( \langle \rho \rangle_S \) extends \( \rho \), then \( \ker \phi \lor \langle \rho \rangle_S \) extends \( \rho \). The idea here is that perhaps the presence of CEP in a semigroup is needed only to insure that \( \langle \rho \rangle_S \) extends \( \rho \) in considering whether the condition given in Theorem 4.9 must always hold. However, this conjecture is false as shown in the example below. This indicates that if it is true that homomorphic images of semigroups with CEP retain CEP, then the presence of CEP in a semigroup plays a larger role than that which is noted above in insuring that a given image has CEP.

4.10 Example. This example shows that \( \ker \phi \lor \langle \rho \rangle_S \) is not necessarily an extension of \( \rho \) under the conditions of Theorem 4.9 even when \( \langle \rho \rangle_S \) extends \( \rho \) in a semigroup which lacks CEP. For the semigroup \( S \) listed below, let \( \phi \) be the homomorphism determined by the congruence \( \ker \phi = \{5, 6\} \times \{5, 6\} \cup \Delta_S \). Let \( T \) be the subsemigroup \( \{1, 3, 4\} \). Then \( T \) is \( \phi \)-saturated. Let \( \rho \) be the congruence on \( T \) given by the class listing \( \{1\}\{3, 4\} \). Note that \( \ker \phi \mid T = \Delta_T \subseteq \rho \). Now \( \langle \rho \rangle_S \) which is given by the class listing \( \{1, 2, 5\}\{3, 4, 6\} \) extends \( \rho \). However, \( \ker \phi \lor \langle \rho \rangle_S = S \times S \) which is not an extension of \( \rho \). We see that \( S \) does not have CEP as \( \{3, 5\}\{6\} \) gives a congruence on the subsemigroup \( \{3, 5, 6\} \) which has no extension to \( S \).

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 & 5 & 5 \\
3 & 3 & 3 & 3 & 3 & 3 \\
3 & 4 & 3 & 4 & 6 & 6 \\
5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
\]

Note that in this example, we have a transition of elements linking 1 to 3 given by 1, 5, 6, 3 where \( (1, 5) \in \langle \rho \rangle_S \setminus \ker \phi \), \( (5, 6) \in \ker \phi \setminus \langle \rho \rangle_S \), and \( (6, 3) \in \langle \rho \rangle_S \setminus \ker \phi \). However, \( (1, 3) \notin \rho \). The following proposition shows that for semigroups with CEP, one need only consider transitions that are of the type just noted.
4.11 Proposition. Let $\phi : S \to X$ be a homomorphism from a semigroup $S$ which has the congruence extension property (CEP) onto a semigroup $X$. Let $T$ be a subsemigroup of $S$ which is $\phi$-saturated and let $\rho$ be a congruence on $T$ which contains $\ker \phi|_T$. Then $\ker \phi \lor \langle \rho \rangle_S$ extends $\rho$ if and only if for each transition of the form $a = x_0, x_1, \ldots, x_n = b$ where $a, b \in T$, $n$ is odd and for $0 \leq i \leq n - 1$,

$$(x_i, x_{i+1}) \in \begin{cases} \langle \rho \rangle_S \ker \phi & \text{if } i \text{ is even} \\ \ker \phi \setminus \langle \rho \rangle_S & \text{if } i \text{ is odd} \end{cases}$$

we have $(a, b) \in \rho$.

Proof. Let the hypotheses given in the first two statements of the proposition hold. The forward implication of the third statement is immediate. We will prove the contrapositive of the reverse implication. Suppose $\ker \phi \lor \langle \rho \rangle_S$ does not extend $\rho$. Let $(a, b) \in (\ker \phi \lor \langle \rho \rangle_S) \cap (T \times T)$ such that $(a, b) \notin \rho$. We show that there is a transition of the forbidden form linking the pair $(a, b) \notin \rho$. Before proceeding with the proof, we make two observations. First, note that

$$(\ker \phi \cup \langle \rho \rangle_S) \cap (T \times T) = \ker \phi|_T \cup (\langle \rho \rangle_S \cap (T \times T)) \subseteq \rho \cup \rho = \rho. \quad (*)$$

by the assumption that $\ker \phi|_T \subseteq \rho$ and the fact that $\langle \rho \rangle_S$ extends $\rho$ as $S$ has CEP. Second, note that

$$\ker \phi \subseteq (T \times T) \cup (S \setminus T \times S \setminus T). \quad (**)$$

as $T$ is $\phi$-saturated.

Applying 1.6, we have $a = x_0, x_1, \ldots, x_n = b$ where $(x_i, x_{i+1}) \in \ker \phi \cup \langle \rho \rangle_S$. If $x_i \in T$ for all $i$, then $(x_i, x_{i+1}) \in (\ker \phi \cup \langle \rho \rangle_S) \cap (T \times T) \subseteq \rho$ for all $i$ by $(*)$ and hence $(a, b) \in \rho$ by transitivity, contrary to choice of $(a, b)$. Thus, we may choose the least $j$ such that $x_j \notin T$. Then $(x_i, x_{i+1}) \in (\ker \phi \cup \langle \rho \rangle_S) \cap (T \times T) \subseteq \rho$ for $0 \leq i \leq j - 2$. 
Hence, \((x_0, x_{j-1}) \in \rho \subseteq \langle \rho \rangle\). Also, \((x_{j-1}, x_j) \in (\ker \phi \cup \langle \rho \rangle_S) \cap (T \times S \setminus T)\). Thus, by \((**)\), \((x_{j-1}, x_j) \in \langle \rho \rangle_S \setminus \ker \phi\). Therefore, \((x_0, x_j) \in (\ker \phi \cup \langle \rho \rangle_S) \cap (T \times S \setminus T)\) and we conclude by \((**)\) that \((x_0, x_j) \in \langle \rho \rangle_S \setminus \ker \phi\). By reducing the transition in this manner, we may assume that \((x_0, x_1) \in \langle \rho \rangle_S \setminus \ker \phi\). Furthermore, for a given \(i\) such that \((x_i, x_{i+1}) \in \langle \rho \rangle_S \setminus \ker \phi\), we may assume that \((x_{i+1}, x_{i+2}) \in \ker \phi \langle \rho \rangle_S\). Otherwise, we could reduce the transition to obtain a transition with this property using transitivity. Likewise, for a given \(i\) such that \((x_i, x_{i+1}) \in \ker \phi \langle \rho \rangle_S\), we may assume that \((x_{i+1}, x_{i+2}) \in \langle \rho \rangle_S \setminus \ker \phi\). Combining these observations with the allowable assumption that \((x_0, x_1) \in \langle \rho \rangle_S \setminus \ker \phi\), we obtain that

\[(x_i, x_{i+1}) \in \begin{cases} \langle \rho \rangle_S \setminus \ker \phi & \text{if } i \text{ is even} \\ \ker \phi \langle \rho \rangle_S & \text{if } i \text{ is odd} \end{cases}\]

In particular, either \((x_{n-1}, x_n) \in \langle \rho \rangle_S \setminus \ker \phi\) or \((x_{n-1}, x_n) \in \ker \phi \langle \rho \rangle_S\). We have \(x_n = b \in T\) so we conclude that \((x_{n-1}, x_n) \notin \ker \phi \langle \rho \rangle_S\) by applying \((**)\). Thus, we have \((x_{n-1}, x_n) \in \langle \rho \rangle_S \setminus \ker \phi\) and consequently \(n - 1\) is even. Hence, \(n\) is odd and we have a transition \(a = x_0, x_1, ..., x_n = b\) of the form which is prohibited when \((a, b) \notin \rho\). This completes the proof of the contrapositive. \(\blacksquare\)

4.12 Corollary. Let \(\phi : S \to X\) be a homomorphism from a semigroup \(S\) which has the congruence extension property (CEP) onto a semigroup \(X\). Let \(T\) be a subsemigroup of \(S\) which is \(\phi\)-saturated and let \(\rho\) be a congruence on \(T\) which contains \(\ker \phi | T\). If there do not exist elements \(x, y, z \in S\) such that \((x, y) \in \langle \rho \rangle_S \cap (T \times S \setminus T)\) and \((y, z) \in \ker \phi \langle \rho \rangle_S\), then \(\ker \phi \cup \langle \rho \rangle_S\) extends \(\rho\). In particular, if \(T\) is saturated with respect to \(\langle \rho \rangle_S\), then \(\ker \phi \cup \langle \rho \rangle_S\) extends \(\rho\).

Proof. Let the hypotheses given in the first two statements of the proposition hold and suppose that elements \(x, y, z \in S\) such that \((x, y) \in \langle \rho \rangle_S \cap (T \times S \setminus T)\) and
(y, z) ∈ ker φ \(\langle\rho\rangle_S\) do not exist. We show that there is no transition of the form \(a = x_0, x_1, ..., x_n = b\) where \(a, b \in T\), \(n\) is odd and

\[(x_i, x_{i+1}) \in \begin{cases} \langle\rho\rangle_S \ker \phi & \text{if } i \text{ is even} \\ \ker \phi \langle\rho\rangle_S & \text{if } i \text{ is odd} \end{cases}\]

If there were such a transition, then \((x_0, x_1) \in \langle\rho\rangle_S \ker \phi\) and \((x_1, x_2) \in \ker \phi \langle\rho\rangle_S\).

Also, \(x_1 \in S \setminus T\) for otherwise we obtain \((x_1, x_2) \in \ker \phi \cap (T \times T) \subseteq \rho \subseteq \langle\rho\rangle_S\) as \(T\) is \(\phi\)-saturated and this is a contradiction. Thus, \((x_0, x_1) \in \langle\rho\rangle_S \cap (T \times S \setminus T)\) and \((x_1, x_2) \in \ker \phi \langle\rho\rangle_S\) contrary to our assumption. Therefore, there is no such transition and applying Proposition 4.11, \(\ker \phi \vee \langle\rho\rangle_S\) extends \(\rho\) as desired. To see the final statement, note that if \(T\) is saturated with respect to \(\langle\rho\rangle_S\), then elements \(x, y, z\) satisfying the conditions given in this Corollary certainly cannot exist. Hence, \(\ker \phi \vee \langle\rho\rangle_S\) extends \(\rho\). 

**4.13 Example.** In this example, there exist elements \(x, y, z \in S\) such that \((x, y) \in \langle\rho\rangle_S \cap (T \times S \setminus T)\) and \((y, z) \in \ker \phi \langle\rho\rangle_S\) for a certain choice of \(\phi, T,\) and \(\rho\) satisfying the initial hypotheses of Corollary 4.12 such that \(\ker \phi \vee \langle\rho\rangle_S\) extends \(\rho\). For the semigroup \(S\) with CEP given below, let \(\phi\) be the homomorphism determined by the congruence \(\ker \phi = \{3, 4\} \times \{3, 4\} \cup \Delta_S\). Let \(T\) be the subsemigroup \(\{1, 2, 5\}\). Then \(T\) is \(\phi\)-saturated. Let \(\rho\) be the congruence on \(T\) given by the class listing \(\{2\}\{1, 5\}\). Then \(\ker \phi \mid_T = \Delta_T \subseteq \rho\). Now \(\langle\rho\rangle_S\) which is given by the class listing \(\{1, 3, 5\}\{2\}\{4\}\) extends \(\rho\) and \(\ker \phi \vee \langle\rho\rangle_S\) which is given by the class listing \(\{1, 3, 4, 5\}\{2\}\) extends \(\rho\). Note that \((1, 3) \in \langle\rho\rangle_S \cap (T \times S \setminus T)\) and \((3, 4) \in \ker \phi \langle\rho\rangle_S\), but for each transition of the form given in Theorem 4.9 between elements \(a\) and \(b\), we have \((a, b) \in \rho\) as is required for \(\ker \phi \vee \langle\rho\rangle_S\) to extend \(\rho\).
4.14 Corollary. Let $\phi : S \to X$ be a homomorphism from a semigroup $S$ which has the congruence extension property (CEP) onto a semigroup $X$. If for each subsemigroup $T$ of $S$ which is $\phi$-saturated and each congruence $\rho$ on $T$ which contains $\ker \phi|_T$, $T$ is saturated with respect to $(\rho)_S$, then $X$ has CEP.

Proof. Apply Theorem 4.9 and Corollary 4.12.  

Although the corollary above seems to be very restrictive, it can be applied to produce the following interesting corollary. This result is somewhat significant given the fact that any commutative semigroup is a semilattice of archimedean semigroups.

4.15 Corollary. Let $S$ be an archimedean semigroup which has the congruence extension property (CEP). Then any homomorphic image of $S$ has CEP.

Proof. According to Proposition 3.8, if $\rho$ is a congruence on a subsemigroup $T$ of an archimedean semigroup $S$ which has the congruence extension property (CEP), then $\rho$ has an extension to $S$ with respect to which $T$ is saturated. It is clear that if $T$ is saturated with respect to a given extension $\overline{\rho}$ of $\rho$, then $T$ is saturated with respect to any extension of $\rho$ which is contained in $\overline{\rho}$. In particular, $T$ is saturated with respect to $(\rho)_S$. Thus, an archimedean semigroup $S$ with CEP satisfies the hypothesis of Corollary 4.14 for any choice of homomorphism $\phi$. Hence, the corollary is proven.  

As indicated above, this result gives useful information when considering the homomorphism problem for commutative semigroups. For any congruence \( \sigma \) which is contained in the congruence \( \eta \) which gives the greatest semilattice decomposition of a periodic commutative semigroup \( S \), it is easy to see that \( S/\sigma \) is a semilattice of archimedean semigroups \( C(e)/\sigma|_{C(e)} \) where \( C(e) \) \( (e \in E_S) \) are the archimedean components of \( S \). According to Corollary 4.15, \( C(e)/\sigma|_{C(e)} \) will have the congruence extension property (CEP) if \( C(e) \) has CEP. Hence, if \( S \) is a commutative semigroup with CEP and \( \sigma \) is a congruence on \( S \) contained in \( \eta \), then \( C(e) \) has CEP for each \( e \in E_S \) as CEP is hereditary and by the observations above, \( S/\sigma \) is a semilattice of archimedean semigroups each of which has CEP. That is, each of the archimedean components of \( S/\sigma \) has CEP. We know from Chapter 3 that a commutative semigroup whose archimedean components have CEP does not necessarily have CEP. Furthermore, a good portion of Chapter 3 is devoted to determining conditions under which such a semigroup has CEP. A number of necessary conditions have been found, but a sufficient list has not been determined. Once a complete list is established, one must show that such conditions are preserved by homomorphisms in order to conclude that \( S/\sigma \) retains CEP for any \( \sigma \subseteq \eta \). One such congruence \( \sigma \) on a commutative semigroup \( S \) is Green's Relation \( \mathcal{H} \). A complete discussion and reduction of the homomorphism problem for the congruence \( \mathcal{H} \) is included in [Dumesnil, 1993].

We now consider another condition under which \( \ker \phi \cup \langle \rho \rangle_S \) extends \( \rho \) in order to identify additional special cases in which homomorphic images retain CEP.

4.16 Proposition. Let \( \phi : S \to X \) be a homomorphism from a semigroup \( S \) which has the congruence extension property (CEP) onto a semigroup \( X \). Let
Let $T$ be a subsemigroup of $S$ and let $\rho$ be a congruence on $T$ containing $\ker \phi|_T$. If $\ker \phi \cup \langle \rho \rangle_S$ is a congruence on $S$, then $\ker \phi \vee \langle \rho \rangle_S = \ker \phi \cup \langle \rho \rangle_S$ is an extension of $\rho$.

**Proof.** By assumption, $\ker \phi \cap (T \times T) \subseteq \rho$ and since $S$ has the congruence extension property, $(\rho)_S \cap (T \times T) = \rho$. Thus, if $\ker \phi \cup \langle \rho \rangle_S$ is a congruence on $S$, then

\[(\ker \phi \vee \langle \rho \rangle_S) \cap (T \times T) = (\ker \phi \cup \langle \rho \rangle_S) \cap (T \times T)\]
\[= (\ker \phi \cap (T \times T)) \cup (\langle \rho \rangle_S \cap (T \times T))\]
\[\subseteq \rho \cup \rho\]
\[= \rho.\]

Thus, $\ker \phi \vee \langle \rho \rangle_S = \ker \phi \cup \langle \rho \rangle_S$ is an extension of $\rho$. \]

**4.17 Corollary.** Let $\phi : S \to X$ be a homomorphism from a semigroup $S$ which has the congruence extension property (CEP) onto a semigroup $X$. If for each subsemigroup $T$ of $S$ which is $\phi$-saturated and each congruence $\rho$ on $T$ which contains $\ker \phi|_T$, $\ker \phi \cup \langle \rho \rangle_S$ is a congruence on $S$, then $X$ has CEP.

**Proof.** Apply Theorem 4.9 and Proposition 4.16. \]

As we will see in Example 4.19, the hypothesis of Corollary 4.17 may certainly fail to hold. One very special case in which it does hold is presented in the following corollary.

**4.18 Corollary.** Let $S$ be a $\Delta$-semigroup with the congruence extension property (CEP). Then any homomorphic image of $S$ has CEP.
Proof. Let \( \phi : S \rightarrow X \) be a homomorphism from \( S \) onto \( X \). Let \( T \) be a subsemigroup of \( S \) and let \( \rho \) be a congruence on \( T \). Since \( S \) is a \( \Delta \)-semigroup, either \( \ker \phi \subseteq \langle \rho \rangle_S \) or \( \langle \rho \rangle_S \subseteq \ker \phi \). Thus, \( \ker \phi \cup \langle \rho \rangle_S \) is equal to either \( \ker \phi \) or \( \langle \rho \rangle_S \). In either case, \( \ker \phi \cup \langle \rho \rangle_S \) is a congruence. Hence, the hypothesis of Corollary 4.17 is certainly satisfied and \( X \) has CEP. This proves the corollary. \( \blacksquare \)

4.19 Example. The purpose of this example is to show that the hypothesis of Corollary 4.17 need not hold even when \( \ker \phi \) is an ideal congruence and to further illustrate the concepts presented in this chapter. For the semigroup \( S \) with CEP given below, let \( \phi \) be the homomorphism which is determined by the congruence \( \ker \phi = \{1, 4\} \times \{1, 4\} \cup \Delta_S \). Note that \( \{1, 4\} \) is an ideal of \( S \). Let \( T \) be the subsemigroup \( \{5, 6\} \). Then \( T \) is \( \phi \)-saturated. Let \( \rho = \{5, 6\} \times \{5, 6\} \). Then certainly \( \ker \phi |_T = \Delta_T \subseteq \rho \). Now \( \langle \rho \rangle_S \) is given by the class listing \( \{1, 3\} \{5, 6\} \{2\} \{4\} \) and \( \ker \phi \cup \langle \rho \rangle_S = \{(1, 4), (4, 1), (1, 3), (3, 1), (5, 6), (6, 5)\} \cup \Delta \) is not a congruence. We see that \( \ker \phi \cup \langle \rho \rangle_S \) is given by the class listing \( \{1, 3, 4\} \{5, 6\} \{2\} \) and extends \( \rho \).

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 3 & 1 & 1 & 3 \\
1 & 1 & 1 & 4 & 4 & 4 \\
1 & 2 & 1 & 4 & 5 & 5 \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]

Furthermore, \( \phi(S) \) has Cayley table given by

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 3 & 1 & 1 & 3 \\
1 & 2 & 1 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 5 & 5
\end{array}
\]

where \( \phi(1) = \phi(4) = 1, \phi(2) = 2, \phi(3) = 3, \phi(5) = 4, \) and \( \phi(6) = 5 \). One can check that this semigroup has CEP in agreement with Proposition 4.5. Note that the
congruence \( \alpha = \{4,5\} \times \{4,5\} \) on the subsemigroup \( \{4,5\} \) of \( \phi(S) \) pulls back to the congruence \( \rho = \{5,6\} \times \{5,6\} \) considered above. Moreover, \( \langle \alpha \rangle_{\phi(S)} \) is given by the class listing \( \{1,3\}\{4,5\}\{2\} \) in \( \phi(S) \) and as assured by Corollary 4.8, it is clear that this is the push-out of the extension \( \ker \phi \vee \langle \rho \rangle_S \) (i.e. \( \{1,3,4\}\{5,6\}\{2\} \)) of \( \rho \).

To illustrate the proof technique used in Proposition 4.5, note that \( \{1,4\}\{5,6\} \) gives the congruence \( \rho \cup (I \times I) \) where \( I = \{1,4\} \) on the subsemigroup \( T \cup I \) of \( S \) and this congruence extends to the congruence \( \langle \rho \cup (I \times I) \rangle_S \) which is given by the class listing \( \{1,3,4\}\{5,6\}\{2\} \). We observe that \( \langle \rho \cup (I \times I) \rangle_S = \ker \phi \vee \langle \rho \rangle_S \) where \( \ker \phi = I \times I \cup \Delta_S \) and as noted above, this congruence extends \( \rho \) as is required to obtain an extension in the image. It is easy to verify that the equality noted in the previous statement holds generally. So we see that in the proof of Proposition 4.5, we were able to take advantage of the fact that \( \ker \phi \vee \langle \rho \rangle_S \) could be written as the extension of a congruence \( \delta \) on a subsemigroup \( Q \) containing \( T \) where \( \delta \) was an extension of \( \rho \) to \( Q \). One might conjecture that this fortunate situation might occur in other special cases. However, it is difficult in general to find suitable subsemigroups \( Q \) and congruences \( \delta \) which allow the use of this double-extension technique.
In this chapter, we characterize commutative ideal semigroups. The first characterization obtained describes commutative ideal semigroups in terms of multiplicative structure. This characterization is then translated into a description of the $\mathcal{H}$-order graph of a commutative ideal semigroup. Thus, we obtain a rather clear picture of the structure of such semigroups. As a corollary to this characterization, we are able to give an explicit description of commutative ideal semigroups which have the ideal extension property (IEP) or the congruence extension property (CEP). In addition, an unexpected corollary to these results is a characterization of commutative $\Delta$-semigroups which have CEP.

According to Chapter 1, any ideal semigroup has a zero element. To see this, simply consider that since $\Delta S$ must be an ideal congruence, some point of $S$ must be an ideal of $S$ and hence, a zero for $S$. The following two basic lemmas give information on ideal congruences in semigroups with zero which will be used in proving a number of results in the remainder of this chapter.

5.1 Lemma. Let $S$ be a semigroup with zero. Let $\sigma$ be a congruence on $S$. Then the following are equivalent:

1. $\sigma$ is an ideal congruence.

2. $\sigma = ([0]_\sigma \times [0]_\sigma) \cup \Delta$

3. If $x \neq y$ and $(x, y) \in \sigma$, then $(x, 0) \in \sigma$. 

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Proof. Assume (1) holds and write \( \sigma = (I \times I) \cup \Delta \) where \( I \) is an ideal of \( S \). Now \( 0 \in I \) so clearly \( I = [0]_{\sigma} \). Thus, \( \sigma = ([0]_{\sigma} \times [0]_{\sigma}) \cup \Delta \) and (1) implies (2). Conversely, we note that \([0]_{\sigma}\) is an ideal since \((x, 0) \in \sigma\) and \(s, t \in S\) implies \((sxt, 0) = (sxt, s \cdot 0 \cdot t) \in \sigma\). Thus, (2) implies (1).

To prove that (2) implies (3), let \( x, y \in S, x \neq y, (x, y) \in \sigma = ([0]_{\sigma} \times [0]_{\sigma}) \cup \Delta \). Now \((x, y) \notin \Delta \) so \( x, y \in [0]_{\sigma} \). Hence, \((x, 0) \in \sigma\) and (3) holds. Finally, we show (3) implies (2). Let \((a, b) \in \sigma\). If \((a, b) \notin \Delta \), then assuming (3), \((a, 0) \in \sigma\). By transitivity, \((b, 0) \in \sigma\). Thus, \((a, b) \in [0]_{\sigma} \times [0]_{\sigma}\) and this shows \(\sigma \subseteq ([0]_{\sigma} \times [0]_{\sigma}) \cup \Delta\). The other containment is clear so (2) holds.

For future reference, we remark that any nontrivial class of an ideal congruence in a semigroup with zero must be the class of zero.

5.2 Lemma. Let \( S \) be a semigroup with zero and let \( a, b \in S, a \neq b \). Then \( \alpha^S(a, b) \) is an ideal congruence if and only if \((a, 0) \in \alpha^S(a, b)\).

Proof. Assume \( \alpha^S(a, b) \) is an ideal congruence. By assumption, \( a \neq b \) and certainly, \((a, b) \in \alpha^S(a, b)\). Thus, \([a]_{\alpha^S(a, b)} \) is a nontrivial class. Hence, by the remark preceding this lemma, \([a]_{\alpha^S(a, b)} = [0]_{\alpha^S(a, b)} \). Therefore, \((a, 0) \in \alpha^S(a, b)\).

Now assume \((a, 0) \in \alpha^S(a, b)\). By Lemma 5.1, it will suffice to show that if \( x \neq y \) and \((x, y) \in \alpha^S(a, b)\), then \((x, 0) \in \alpha^S(a, b)\). Let \( x, y \in S, x \neq y, \) and \((x, y) \in \alpha^S(a, b)\). Then by 1.5 there is a transition, \( x = x_1, x_2, ..., x_n = y \), where \((x_i, x_{i+1}) \in \{(sat, sbt), (sbt, sat) : s, t \in S^1}\). Either \( x = x_1 = sat \) or \( x = x_1 = sbt \) for some \(s, t \in S^1 \). Now \((a, 0) \in \alpha^S(a, b)\) and by transitivity, \((b, 0) \in \alpha^S(a, b)\). Thus, either \((x, 0) = (sat, s \cdot 0 \cdot t) \in \alpha^S(a, b)\) or \((x, 0) = (sbt, s \cdot 0 \cdot t) \in \alpha^S(a, b)\). Hence, applying Lemma 5.1, \( \alpha^S(a, b) \) is an ideal congruence.
The next result indicates that a semigroup is an ideal semigroup provided that it is "principally" an ideal semigroup. Furthermore, these equivalences will provide useful proof techniques in what follows.

5.3 Lemma. Let $S$ be a semigroup with zero. Then the following are equivalent:

(1) $S$ is an ideal semigroup.

(2) $\alpha^S(a, b)$ is an ideal congruence for all $a, b \in S$

(3) $(a, 0) \in \alpha^S(a, b)$ for all $a, b \in S$ with $a \neq b$.

Proof. That (1) implies (2) is immediate. To see prove that (2) implies (1), let $\sigma$ be a congruence on $S$. We show that $\sigma$ is an ideal congruence. By Lemma 5.1, it suffices to show that if $a \neq b$ and $(a, b) \in \sigma$, then $(a, 0) \in \sigma$. Now $(a, b) \in \sigma$ implies $\alpha^S(a, b) \subseteq \sigma$. Also, since $\alpha^S(a, b)$ is an ideal congruence, $a \neq b$, and $(a, b) \in \alpha^S(a, b)$, Lemma 5.2 implies $(a, 0) \in \alpha^S(a, b) \subseteq \sigma$. Thus, (2) implies (1).

To see that (2) implies (3), let $a \neq b$. Now $\alpha^S(a, b)$ is an ideal congruence containing $(a, b)$ so by Lemma 5.2, $(a, 0) \in \alpha^S(a, b)$ and (3) holds. To prove that (3) implies (2), let $a, b \in S$. If $a \neq b$, then assuming (3) in light of Lemma 5.2 yields that $\alpha^S(a, b)$ is an ideal congruence. If $a = b$, then $\alpha^S(a, b) = \Delta$, an ideal congruence as $S$ has a zero. Thus, (3) implies (2).

We define a semigroup $S$ to be weakly reductive away from zero if and only if given $a, b \in S \setminus \{0\}$ such that $ax = bx$ and $xa = xb$ for all $x \in S$, we have $a = b$. In the commutative case, we could replace the term weakly reductive with reductive and make the obvious adjustment in the definition.

Note that the previous results do not depend on commutativity. In addition, we have the following result which holds for any ideal semigroup.
5.4 Proposition. Let $S$ be an ideal semigroup. Then $S$ is weakly reductive away from zero.

Proof. Suppose not. Then there exist $a, b \in S \setminus \{0\}$, $a \neq b$ such that $ax = bx$ and $xa = xb$ for all $x \in S$. Now $\{(a, b) \times \{a, b\}\} \cup \Delta$ is clearly an equivalence and it is compatible since $(ax, bx), (xa, xb) \in \Delta$ for all $x \in S$. Hence, $\{(a, b) \times \{a, b\}\} \cup \Delta$ is a congruence. Now $\{a, b\}$ is a nontrivial class not containing zero. Thus, by a previous remark, $\{(a, b) \times \{a, b\}\} \cup \Delta$ is not an ideal congruence. This is a contradiction as $S$ is an ideal semigroup and the proposition is proven.

In preparation for the following lemma, we recall the following definitions from Chapter 1.

In a commutative semigroup $S$, $a \leq_H b$ provided $a \in bS^1$ and $(a, b) \in H$ provided $aS^1 = bS^1$. We write $a \parallel_H b$ if $a \not\leq_H b$ and $b \not\leq_H a$. Generally, $\leq_H$ is a quasi-order and for commutative semigroups, $H$ is a congruence. It is easy to see that $H = \Delta_S$ if and only if $\leq_H$ is a partial order on $S$. That is, anti-symmetry of $\leq_H$ implies and is implied by $H = \Delta_S$.

5.5 Lemma. Let $S$ be a commutative ideal semigroup. Then $H = \Delta$.

Proof. Since $S$ is commutative, $H$ is a congruence on $S$. Thus, $H$ is an ideal congruence as $S$ is an ideal semigroup. By Lemma 5.1, $H = ([0]_H \times [0]_H) \cup \Delta$. Now if $(a, 0) \in H$, then $aS^1 = 0 \cdot S^1 = \{0\}$. Thus, $a = 0$. Therefore, $[0]_H = \{0\}$ and $H = \Delta$. Thus, by this lemma and the remarks preceding it, $\leq_H$ is a partial order on any commutative ideal semigroup. Recall from Chapter 1 that any commutative
semigroup is a disjoint union of its archimedean components, which are the classes of a congruence $\eta$ on the semigroup. Thus, if $S$ is a commutative ideal semigroup, then by Lemma 5.1(2), $\eta = ([0]_\eta \times [0]_\eta) \cup \Delta$. Translating this into the notation established in Chapter 1, we see that $S$ is the disjoint union of $C(0) = [0]_\eta$ and any number of trivial components. Since these trivial components are semigroups, they must be idempotents. Thus, $S$ is the disjoint union of $C(0)$ and the nonzero idempotents of $S$ since $0$ is the only idempotent in $C(0)$, an archimedean semigroup.

Whence, we begin to see the structure of commutative ideal semigroups. However, one can not expect that any semigroup with this structure is an ideal semigroup. Consider the following example.

5.6 Example. The semigroup below is the disjoint union of its zero component $C(1) = \{1, 2\}$ and its nonzero idempotents $\{3, 4, 5, 6, 7\}$. However, the congruence given by the class listing $\{1, 2, 3, 4\}\{5, 6, 7\}$ is not an ideal congruence. Hence, this is not an ideal semigroup.

```
1 1 1 1 1 1 1 1
1 1 1 1 2 2 2 2
1 1 3 1 1 1 3 3
1 1 1 4 1 4 1 1
1 2 1 1 5 5 5 5
1 2 1 4 5 6 5 5
1 2 3 1 5 5 5 7
```

The next four results provide additional properties of commutative ideal semigroups as well as an alternative proof that the structure obtained above holds in ideal semigroups. The lemma below is key to the proof of the fact which follows.

5.7 Lemma. Let $S$ be a commutative semigroup. Let $I$ be an ideal of $S$ and $T$ a subsemigroup of $S$ such that $T \cap I = \emptyset$. Then there exists a prime ideal $I'$ of $S$ containing $I$ such that $I' \cap T = \emptyset$. 
Proof. Set \( I' = \{ x \in S : xS^1 \cap T = \emptyset \} \). Now if \( x \in I \), then \( xS^1 \subseteq I \) so \( xS^1 \cap T = \emptyset \) and \( x \in I' \). Hence, \( I \subseteq I' \). Also, \( I' \cap T = \emptyset \) since \( x \in T \) implies \( x \in xS^1 \cap T \) which implies \( x \notin I' \). We must show that \( I' \) is a prime ideal of \( S \). For this purpose, let \( x \in I' \) and \( y \in S \). Suppose that \( xy \notin I' \). Then \( (xy)s \in T \) for some \( s \in S^1 \). Thus, \( x(ys) \in T \cap xS^1 \) contrary to \( x \in I' \). Hence, \( xy \in I' \) and \( I' \) is an ideal of \( S \). To see that \( I' \) is prime, let \( x, y \in S \setminus I' \). Then there exist \( s, t \in S^1 \) such that \( xs \in T \) and \( yt \in T \). Thus, \((xy)(st) = (xs)(yt) \in (T)(T) \cap xyS^1 \subseteq T \cap xyS^1 \). Thus, \( xy \in S \setminus I' \) and \( I' \) is a prime ideal of \( S \). 

A note of interest concerning the lemma above is that while this result does not hold generally in noncommutative semigroups, it seems to hold in any semigroup which has CEP. The following result is very useful in obtaining additional information about the multiplicative structure of commutative ideal semigroups.

5.8 Lemma. Let \( S \) be a commutative ideal semigroup. Then each nontrivial subsemigroup \( T \) of \( S \) contains zero.

Proof. Let \( T \) be a subsemigroup of a commutative ideal semigroup \( S \). Suppose \( 0 \notin T \). We must show that \( T \) is trivial. Now \( \{0\} \) is an ideal of \( S \) and \( T \) is a subsemigroup of \( S \) such that \( \{0\} \cap T = \emptyset \). Thus, by Lemma 5.7, there is a prime ideal \( I' \) of \( S \) such that \( \{0\} \subseteq I' \) and \( I' \cap T = \emptyset \). Hence, \( T \subseteq S \setminus I' \). The relation \( \sigma = (I' \times I') \cup (S \setminus I' \times S \setminus I') \) is certainly an equivalence on \( S \) and compatibility follows from the fact that \( I' \) is a prime ideal. Thus \( \sigma \) is a congruence on \( S \) and hence, an ideal congruence. Now \([0]_\sigma = I' \) and by Lemma 5.1, \( \sigma = ([0]_\sigma \times [0]_\sigma) \cup \Delta \). Thus, \((I' \times I') \cup (S \setminus I' \times S \setminus I') = \sigma = (I' \times I') \cup \Delta \). Hence, \( S \setminus I' \) must be trivial. Therefore, \( T \) is trivial as \( T \subseteq S \setminus I' \).
5.9 Lemma. Let $S$ be a commutative semigroup with zero. Then each nontrivial subsemigroup contains zero if and only if $S = C(0) \cup E \setminus \{0\}$ and $ef = 0$ for all $e \neq f, e, f \in E \setminus \{0\}$.

Proof. Suppose each nontrivial subsemigroup of $S$ contains zero. Since $S$ is a commutative semigroup with zero we may write $S = C(0) \cup \bigcup_{i} C_i$ where $C(0), C_i$ are the archimedean components of $S$. Each $C_i$ is a subsemigroup not containing zero. Thus, by assumption, $C_i$ is trivial for each $i$. Now a trivial subsemigroup is an idempotent so each $C_i$ is just an idempotent. Hence, $\bigcup_{i} C_i = E \setminus \{0\}$ as zero is the only idempotent in $C(0)$. Therefore, $S = C(0) \cup E \setminus \{0\}$. Now let $e \neq f, e, f \in E \setminus \{0\}$. If $ef \neq 0$, then $\{e, f, ef\}$ forms a nontrivial subsemigroup of $S$ not containing zero contrary to the hypothesis. Thus, $ef = 0$.

Now suppose $S = C(0) \cup E \setminus \{0\}$ and $ef = 0$ for all $e \neq f, e, f \in E \setminus \{0\}$. Let $T$ be a nontrivial subsemigroup of $S$. If $C(0) \cap T \neq \emptyset$, then let $x \in C(0) \cap T$. Now as discussed in Chapter 1, $x^n = 0$ for some $n$ so $0 = x^n \in T$. Otherwise, $T \subseteq E \setminus \{0\}$. Since $T$ is nontrivial we may choose $e, f \in T \subseteq E \setminus \{0\}$. Then by assumption, $0 = ef \in T$ contrary to $T \subseteq E \setminus \{0\}$. Hence, $C(0) \cap T \neq \emptyset$ and so $0 \in T$.

As promised, we now have an alternative argument that the structure given above for commutative ideal semigroups holds. In addition, we have a description of multiplication in the subsemigroup of idempotents of a commutative ideal semigroup. This is stated in the following corollary.

5.10 Corollary. Let $S$ be a commutative ideal semigroup. Then $S = C(0) \cup E \setminus \{0\}$ and if $e \neq f, e, f \in E \setminus \{0\}$, then $ef = 0$.

Proof. Apply Lemmas 5.8 and 5.9.
In the following example, we see that the conditions given by the corollary above are not sufficient to imply that a commutative semigroup is an ideal semigroup.

5.11 Example. This commutative semigroup is the disjoint union of its zero component \{1,2,3,4\} and its nonzero idempotents \{5,6\} and \(5\cdot6 = 1\) (the zero of the semigroup), but it is not an ideal semigroup as the class listing \{1,2\}\{3,4\}\{5\}\{6\} gives a congruence which is not an ideal congruence.

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 3 \\
1 & 1 & 1 & 2 & 1 & 3 \\
1 & 1 & 1 & 1 & 5 & 1 \\
1 & 1 & 3 & 3 & 1 & 6 \\
\end{array}
\]

The lemma which follows gives a condition which sheds more light on why the example above fails to be an ideal semigroup.

5.12 Lemma. Let \(S\) be a commutative ideal semigroup. Let \(x \in C(0)\) and let \(e \in E \setminus \{0\}\). Then either \(xe = 0\) or \(xe = x\).

Proof. By Corollary 5.10, we have \(S = C(0) \cup E \setminus \{0\}\). Let \(x \in C(0), e \in E \setminus \{0\}\). Suppose \(xe \neq 0\) and \(xe \neq x\). We will show that \(\alpha^S(x, xe)\) is not an ideal congruence by means of the following claim.

Claim: For each \(s \in S\), \((sx, sxe) \in (\{x, xe\} \times \{x, xe\}) \cup (S \setminus \{x, xe\} \times S \setminus \{x, xe\})\).

Let \(s \in S\). It suffices to show that \(sx \in \{x, xe\}\) if and only if \(sxe \in \{x, xe\}\).

First assume \(sx \in \{x, xe\}\). Now if \(sx = x\), then \(sxe = xe\). Otherwise, \(sx = xe\) and \(sxe = (xe)e = xe^2 = xe\) as \(e \not\in C(0)\) implies \(e \in E\). Conversely, assume \(sxe \in \{x, xe\}\). If \(sxe = x\), then \((se)x = sxe = x\). Note here that \(x \neq 0\) as \(xe \neq 0\).

Thus, by 1.44, \(se \not\in C(0)\). Hence, \(s \not\in C(0)\) as \(C(0)\) is an ideal so \(s \in E \setminus \{0\}\). Therefore, we have \(sx = s(sxe) = s^2xe = sxe = x\). Otherwise, \(sxe = sx\). In this
case, \( s \not\in C(0) \) by 1.44 as \( xe \neq 0 \) and \( xe \in C(0) \) \((xe \in C(0) \text{ since } C(0) \text{ is an ideal and } x \in C(0))\). Thus, \( s \in E \setminus \{0\} \). If \( s \neq e \), then \( se = 0 \) by Corollary 5.10 as \( s, e \in E \setminus \{0\} \) and \( xe = sx \neq 0 \cdot x = 0 \), a contradiction. Thus, \( s = e \) and \( sx = ex = xe \). The claim is proven.

Now by the claim and notes 1.4 and 1.7,
\[
\alpha^S(x, xe) \subseteq (\{x, xe\} \times \{x, xe\}) \cup (S \setminus \{x, xe\} \times S \setminus \{x, xe\}).
\]
Hence, it is clear that \([x]_{\alpha^S(x, xe)} = \{x, xe\} \). Thus, \( \{x, xe\} \) forms a nontrivial class not containing zero as \( x \neq xe, x \neq 0 \), and \( xe \neq 0 \). Whence \( \alpha^S(x, xe) \) is not an ideal congruence. This is a contradiction and the proof is complete.

Thus far, we have obtained conditions on multiplication between the idempotents and between the zero component and nonzero idempotents of a commutative ideal semigroup. Once again, these conditions are not sufficient to imply that a commutative semigroup is an ideal semigroup. Consider the following example.

**5.13 Example.** This commutative semigroup satisfies the conditions given by Corollary 5.10 and Lemma 5.12, but is not an ideal semigroup as the congruence given by the listing \( \{1\}\{2\}\{3, 4\}\{5\} \) is not an ideal congruence.

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 4 \\
1 & 2 & 3 & 4 \\
\end{array}
\]

Note that \( 3 \cdot 3 = 3 \cdot 4 = 2 \neq 1 \) so that this semigroup does not have a cancellation property even when the products are not zero. We formalize this notion in the following definition.

A semigroup \( S \) has **nonzero cancellation** if and only if \( xy = xz \neq 0 \) implies \( y = z \). One should note that the properties of reductivity away from zero defined
previously and nonzero cancellation defined here are independent properties. For
example, the semigroup above is not nonzero cancellative, but it is reductive away
from zero. (In fact, it is reductive as 5 is an identity for the semigroup.) This is not
surprising as reductivity certainly does not imply cancellation in general. However,
one can also find examples of semigroups which are nonzero cancellative and are not
reductive away from zero. One trivial example is a zero semigroup of order greater
than two.

The following lemma will be very useful in proving the characterization theorem
for commutative ideal semigroups later in this chapter.

5.14 Lemma. Let $S$ be a commutative ideal semigroup. Then $S$ has nonzero
cancellation.

Proof. Suppose $xy = xz \neq 0$. We must show that $y = z$. Define

$$\sigma := \{(u, v) \in S \times S : xu = xv\}.$$ 

It is straightforward to show that $\sigma$ is a congruence on $S$. Now $(y, z) \in \sigma$. However,
if $(y, 0) \in \sigma$, then $xy = x \cdot 0 = 0$ contrary to assumption. Thus, $(y, 0) \not\in \sigma$. Hence,
if $y \neq z$, then $\sigma$ is not an ideal congruence according to Lemma 5.1. We conclude
that $y = z$. $lacksquare$

The next lemma is an interesting observation which results from the multiplica­
tive conditions previously determined to hold in commutative ideal semigroups, but
is not of great importance to us in completing the characterization of commutative
ideal semigroups.

5.15 Proposition. Let $S$ be a commutative ideal semigroup. If $x \in S$ is not
an identity for $S$, then $x$ is a zero divisor.
Proof. By Corollary 5.10, \( S = C(0) \cup E \setminus \{0\} \). Let \( x \in S \) such that \( x \) is not an identity for \( S \). If \( x \in C(0) \), then as discussed in Chapter 1, \( x^n = 0 \) for some \( n \in \mathbb{N} \). Hence, \( x \) is a zero divisor. Otherwise, \( x \in E \setminus \{0\} \). In this case, since \( x \) is not an identity for \( S \), there is some \( y \in S \) such that \( xy \neq y \). Clearly, \( y \neq 0 \). We claim that \( xy = 0 \). First suppose \( y \in E \setminus \{0\} \). If \( x = y \), we have \( xy = y^2 = y \) contrary to \( xy \neq y \). Thus, \( x \neq y \) and \( x, y \in E \setminus \{0\} \) so by Corollary 5.10, \( xy = 0 \). Now suppose \( y \not\in E \setminus \{0\} \). Then \( y \in C(0) \) and \( x \in E \setminus \{0\} \) so by Lemma 5.12, \( xy = y \) or \( xy = 0 \). Hence, \( xy = 0 \) as \( xy \neq y \). Thus, in any case, \( xy = 0 \) and \( x \) is a zero divisor.

By Corollary 5.10 and Lemma 5.12, we have conditions on multiplication between idempotents and between the zero component and the nonzero idempotents of a commutative ideal semigroup. However, we have little information on the behavior of elements inside of the zero component. This missing information will be given in condition (4) of the characterization theorem which follows. First consider the following examples.

5.16 Examples. These commutative semigroups satisfy the conditions given by Corollary 5.10 and Lemma 5.12 and have nonzero cancellation, but are not ideal semigroups.

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 & 1 \\
1 & 1 & 1 & 5 & 1 & 6 \\
1 & 1 & 1 & 1 & 1 & 6 \\
\end{array}
\quad
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 2 & 1 & 1 & 3 \\
1 & 1 & 1 & 2 & 1 & 4 \\
1 & 1 & 1 & 1 & 5 & 1 \\
1 & 2 & 3 & 4 & 1 & 6 \\
(1) & & & & & \\
(2) & & & & & \\
\end{array}
\]

In semigroup (1), the congruence given by the class listing \( \{1\} \{2, 3\} \{4\} \{5\} \{6\} \) is not an ideal congruence. Furthermore, note that 2 and 3 are not \( H \)-comparable and no idempotent acts as an identity for either 2 or 3. In semigroup (2), the
congruence given by the class listing \{1,2\}\{3,4\}\{5\}\{6\} is not an ideal congruence. Note that 3 and 4 are not \mathcal{H}\text{-comparable and 6 acts as an identity for both 3 and 4. The purpose of these observations will become clear in condition (4) of the following characterization theorem for commutative ideal semigroups.

5.17 Multiplicative Structure Characterization Theorem for Commutative Ideal Semigroups. Let \( S \) be a commutative semigroup. Then \( S \) is an ideal semigroup if and only if the following hold:

1. \( S \) has a zero and \( S = C(0) \cup E \setminus \{0\} \).
2. If \( e, f \in E \setminus \{0\}, e \neq f \), then \( ef = 0 \).
3. If \( a \in C(0), e \in E \setminus \{0\} \), then \( ae = 0 \) or \( ae = a \).
4. If \( a, b \in C(0) \), then one of the following holds:
   (i) \( a \leq_{\mathcal{H}} b \) or \( b \leq_{\mathcal{H}} a \)
   (ii) \( a \parallel_{\mathcal{H}} b \) and both of the following hold:
   (A) There is some \( e \in E \setminus \{0\} \) such that either \( ea = a \) or \( eb = b \).
   (B) If \( e \in E \setminus \{0\} \) such that \( ea = a \), then \( eb = 0 \).

Proof. Let \( S \) be a commutative ideal semigroup. Then by 1.37, Lemma 5.12 and Corollary 5.10, (1), (2), and (3) hold. It remains to show that (4) holds. Suppose (4) does not hold. Then there exist \( a, b \in C(0) \) such that neither (i) nor (ii) hold. That is, we have \( a, b \in C(0) \) such that

\[ a \neq bs, \; b \neq as \quad \text{for all} \; s \in S^1 \]  

\[ (* \) 

and either

\[ ea \neq a, \; eb \neq b \quad \text{for all} \; e \in E \setminus \{0\} \]  

\[ (†) \]
or
\[ e'a = a \text{ and } e'b = b \text{ for some } e' \in E \setminus \{0\} \] (†)

It is clear by (*) that \( a \neq 0 \neq b \) and \( a \neq b \). We will obtain a contradiction by showing that \( \alpha^S(a, b) \) is not an ideal congruence. Consider a transition of form \( a = x_0, x_1, \ldots, x_n \) where either \( (x_i, x_{i+1}) = (s_i a, s_i b) \) or \( (x_i, x_{i+1}) = (s_i b, s_i a) \) for some \( s_i \in S \) \((0 \leq i \leq n - 1)\). Describing the transition with this notation yields that for each \( i \), either \( s_{i-1} a = x_i = s_i a, s_{i-1} b = x_i = s_i b, \) or \( s_{i-1} b = x_i = s_i a \). Also, note that if \( x_i = s_i a \), then \( x_{i+1} = s_i b \) and if \( x_i = s_i b \), then \( x_{i+1} = s_i a \). We show by the induction that for any such transition, \( x_i \in \{a, b\} \) for \( 0 \leq i \leq n \). If \( n = 0 \), then \( x_0 = a \in \{a, b\} \). Assume \( x_i \in \{a, b\} \) for any such transition with \( n = k \). Let \( a = x_0, \ldots, x_{k+1} \) be a transition of the above form. Then by induction hypothesis, \( x_i \in \{a, b\} \) for \( 0 \leq i \leq k \). We show that \( x_{k+1} \in \{a, b\} \).

Now either \( x_k = a \) or \( x_k = b \). We may assume without loss of generality that \( x_k = a \) since the case where \( x_k = b \) is completely dual. By (*) and the remark on notation above we have \( a = x_k = s_{k-1} a = s_k a \) where \( s_{k-1}, s_k \in S^1 \). Thus, by the note above, since \( x_k = s_k a \), we have \( x_{k+1} = s_k b \). There are two cases to consider as either (†) or (‡) holds.

**Case 1:** Assume (†) holds.

In this case, \( a \neq e a \) for all \( e \in E \setminus \{0\} \). Also, \( a \neq s a \) for all \( s \in C(0) \) by 1.44 as \( a \neq 0 \). Thus, \( a \neq s a \) for all \( s \in S \). Now since \( a = s_k a \) where \( s_k \in S^1 \) we conclude \( s_k = 1 \). Thus, \( x_{k+1} = s_k b = 1 \cdot b = b \in \{a, b\} \).

**Case 2:** Assume (‡) holds.

Now 1.44, \( a = s_k a \) and \( a \neq 0 \) imply \( s_k \notin C(0) \). Thus, \( s_k \in (E \setminus \{0\})^1 \). Now is \( s_k = 1 \), then \( x_{k+1} = s_k b = b \) and we are done. Otherwise, \( s_k \in E \setminus \{0\} \).
and by (†) there is some $e' \in E \backslash \{0\}$ such that $e'a = a$ and $e'b = b$. Thus, $s_k a = e'a = a \neq 0$ so by Lemma 5.14, $s_k = e'$ and $x_{k+1} = s_k b = e'b = b$.

Thus, in all cases, $x_{k+1} \in \{a, b\}$ and the induction is complete. This shows that $a$ is linked only to $b$ by pairs from $\{(sa, sb), (sb, sa) : s \in S^1\}$. Hence, $[a]_{\alpha^S(a,b)} = \{a, b\}$.

Therefore, $\{a, b\}$ forms a nontrivial class of $\alpha^S(a,b)$ which does not contain zero as $a \neq b$ and $a \neq 0 \neq b$. Thus, $\alpha^S(a,b)$ is not an ideal congruence contrary to $S$ an ideal semigroup. We conclude that (4) holds.

Conversely, suppose (1)-(4) hold. By Lemma 5.3, it suffices to show that $(a, 0) \in \alpha^S(a,b)$ for all $a \neq b$, $a, b \in S$. For this purpose, let $a, b \in S$, $a \neq b$. If $a = 0$ or $b = 0$, then $(a, 0) \in \alpha^S(a,b)$ so assume $a \neq 0 \neq b$.

**Case 1: $a, b \notin C(0)$**

Then $ab = 0$ as $a \neq b$ and $a, b \in E \backslash \{0\}$. Thus, $(a, 0) = (a^2, ab) \in \alpha^S(a,b)$.

**Case 2: $a \notin C(0)$ and $b \in C(0)$**

Since $b^n = 0$ for some $n$, $(a, 0) = (a^n, b^n) \in \alpha^S(a,b)$.

**Case 3: $a \in C(0)$ and $b \notin C(0)$**

Since $a^n = 0$ for some $n$, $(0, b) = (a^n, b^n) \in \alpha^S(a,b)$. By transitivity, $(a, 0) \in \alpha^S(a,b)$.

**Case 4: $a, b \in C(0)$**

By (4), we have two subcases:

**Subcase 1: Assume $a \leq_H b$ or $b \leq_H a$**

We may assume without loss of generality that $b \leq_H a$, as dually we can show $(b, 0) \in \alpha^S(a,b)$ and then use transitivity. Thus, $b = as$ for some $s \in S^1$. Now $s \neq 1$ as $a \neq b$. If $s \notin C(0)$, then $b = as = 0$ or $b = as = a$.
as (3) holds and neither of these is the case. Hence, \( s \in C(0) \) so \( s^m = 0 \) for some \( m \). Now \( b = as \) implies \( (a, as) \in \alpha^S(a, b) \). But by compatibility, this implies \( (as^n, as^{n+1}) \in \alpha^S(a, b) \) for all \( n \in \mathbb{N} \). Thus, by transitivity, \( (a, as^n) \in \alpha^S(a, b) \) for all \( n \in \mathbb{N} \). Thus, \( (a, 0) = (a, as^m) \in \alpha^S(a, b) \).

Subcase 2: Assume \( a \parallel_H b \) and (A) and (B) hold.

By (A), there is some \( e \in E \setminus \{0\} \) such that \( ea = a \) or \( eb = b \). Suppose \( ea = a \). Then \( eb = 0 \) by (B) and \( (a, 0) = (ea, eb) \in \alpha^S(a, b) \). Likewise, in the case that \( eb = b \), we obtain \( ea = 0 \) and \( (b, 0) = (ea, eb) \in \alpha^S(a, b) \).

Hence, by transitivity, \( (a, 0) \in \alpha^S(a, b) \).

Thus, \( S \) is an ideal semigroup. 

We translate this characterization into a description of the \( \mathcal{H} \)-order graph of a commutative ideal semigroup. In doing so, we use the following notation:

\[
\downarrow_{\mathcal{H}} x = \{ y \in S : y \leq_{\mathcal{H}} x \}
\]

\[
\downarrow_{\mathcal{H}} x = (\downarrow_{\mathcal{H}} x) \setminus \{x\}.
\]

It is easily seen by definition of \( \leq_{\mathcal{H}} \) that \( \downarrow_{\mathcal{H}} x = S^1 x \). \( \uparrow_{\mathcal{H}} x \) is defined analogously.

We say that \( S \) is an \( \mathcal{H} \)-forest provided \( \leq_{\mathcal{H}} \) is a partial order and \( \downarrow_{\mathcal{H}} x \) is a chain for all \( x \in S \). \( S \) is an \( \mathcal{H} \)-tree provided \( S \) is an \( \mathcal{H} \)-forest with zero.

5.18 Lemma Let \( S \) be a semigroup with zero such that \( \leq_{\mathcal{H}} \) is a partial order.

Then the following are equivalent:

1. \( S \) is an \( \mathcal{H} \)-tree and \( \uparrow_{\mathcal{H}} x \) is a chain for all \( x \neq 0 \).
2. \( S \) is an \( \mathcal{H} \)-tree and for all \( x \parallel_{\mathcal{H}} y \), \( \downarrow_{\mathcal{H}} x \cap \downarrow_{\mathcal{H}} y = \{0\} \).
3. \( S \) is the 0-disjoint union of its maximal \( \mathcal{H} \)-chains.
Moreover, in a semigroup $S$ satisfying these equivalent conditions, $xy = 0$ if $x \parallel_H y$, and for each maximal $H$-chain $M$, $\leq_{H_M} = \leq_H \cap (M \times M)$.

**Proof.** To see that (1) implies (2), note that if $z \in \downarrow_H x \cap \downarrow_H y$, then $x, y \in \uparrow_H z$. Thus, if (1) holds, then for any such $z \neq 0$, we must have $x$ and $y$ $H$-comparable. Hence, (2) holds.

To see that (2) implies (3), suppose that $S$ is an $H$-tree and for all $x \parallel_H y$, $\downarrow_H x \cap \downarrow_H y = \{0\}$. Certainly, $S$ is the union of its maximal $H$-chains. We need only show that if $M_1$ and $M_2$ are distinct maximal $H$-chains, then $M_1 \cap M_2 = \{0\}$.

Let $x \in M_1 \cap M_2$. Now $\downarrow_H x$ is a chain as $S$ is an $H$-tree. By maximality of $M_1$ and $M_2$, we have $\downarrow_H x \subseteq M_1 \cap M_2$. Also, since $M_1$ and $M_2$ and distinct and maximal, there exist $y \in M_1 \setminus M_2$ and $z \in M_2 \setminus M_1$. By choice of $y$, $z$, and $x$ and the fact that $\downarrow_H x \subseteq M_1 \cap M_2$, we must have $y, z \geq x$ and $y \parallel_H z$. Thus, we conclude that $x \in \downarrow_H y \cap \downarrow_H z = \{0\}$. Hence $x = 0$ as required.

To see that (3) implies (1), suppose that $S$ is the 0-disjoint union of its maximal $H$-chains. We need only show that if $x \neq 0$, then $\downarrow_H x$ and $\uparrow_H x$ are chains. Suppose that $\downarrow_H x$ is not a chain for some $x \neq 0$. Then there exist $y, z \in \downarrow_H x$ such that $y \parallel_H z$. We may choose a maximal chain $M_1$ containing $y$ and $x$ and a maximal chain $M_2$ containing $z$ and $x$. Then $M_1$ and $M_2$ are distinct as $y \notin M_2$ and $z \notin M_1$. Thus, $M_1 \cap M_2 = \{0\}$ contrary to $x \in M_1 \cap M_2$. Thus, $\downarrow_H x$ is a chain as required. The argument that $\uparrow_H x$ is a chain is similar.

To see that the final statement holds, assume that the equivalent conditions (1)-(3) hold. Then if $x \parallel_H y$, we have $xy \in \downarrow_H x \cap \downarrow_H y = \{0\}$. Hence, $xy = 0$. Let $M$ be a maximal $H$-chain. We claim that $\leq_{H_M} = \leq_H \cap (M \times M)$. We need only show the reverse inclusion. Let $x, y \in M$ such that $x \leq_H y$. Then $x = sy$ for some $s \in S^1$. 
If \( s \in M^1 \), then \( x \leq_{\mathcal{H}_M} y \) as required. Otherwise, \( s \in S \setminus M \). In this case, since \( M \) is maximal and \( y \in M \), we have \( s \parallel_{\mathcal{H}} y \). Hence, as argued above, \( x = sy = 0 \). Certainly, \( x = 0 \leq_{\mathcal{H}_M} y \) as required. Thus, \( \leq_{\mathcal{H}_M} = \leq_{\mathcal{H}} \cap (M \times M) \). 

We say that \( S \) is an \( \mathcal{H} \)-fan provided that \( S \) satisfies the equivalent conditions given in the lemma above.

### 5.19 \( \mathcal{H} \)-Order Graph Characterization Theorem for Commutative Ideal Semigroups.

*Let* \( S \) *be a commutative semigroup. Then* \( S \) *is an ideal semigroup if and only if the following hold:

(a) \( S = C(0) \cup E \setminus \{0\} \).

(b) \( S \) is an \( \mathcal{H} \)-fan.

(c) Each \( e \in E \setminus \{0\} \) is an \( \mathcal{H} \)-maximal element and at most one maximal \( \mathcal{H} \)-chain is not bounded above by an idempotent.

**Proof.** Suppose \( S \) is an ideal semigroup. Then by conditions (1), (2), and (3) of Theorem 5.17, we have \( S = C(0) \cup E \setminus \{0\} \) (hence, (a) holds), \( xy = 0 \) for \( x \neq y, x, y \in E \setminus \{0\} \), and \( xy = 0 \) or \( xy = x \) for \( x \in C(0), y \in E \setminus \{0\} \). To show that (b) holds, we first observe that \( \downarrow_{\mathcal{H}} x \subseteq C(0) \) for all \( x \in S \) as follows. If \( x \in C(0) \), then \( \downarrow_{\mathcal{H}} x \subseteq \downarrow_{\mathcal{H}} x = S^1x \subseteq C(0) \) as \( C(0) \) is an ideal of \( S \). If \( x \in E \setminus \{0\} \), then by the notes on multiplication above we have \( \downarrow_{\mathcal{H}} x = S^1x \subseteq C(0) \cup \{x\} \) so \( \downarrow_{\mathcal{H}} x = (\downarrow_{\mathcal{H}} x) \setminus \{x\} \subseteq (C(0) \cup \{x\}) \setminus \{x\} \subseteq C(0) \).

Now let \( x \in S \). Let \( a, b \in \downarrow_{\mathcal{H}} x \). We claim that \( a \) and \( b \) are \( \mathcal{H} \)-comparable. We may assume that \( a \neq 0 \neq b \) and \( a, b \in \downarrow_{\mathcal{H}} x \) for otherwise our claim holds trivially. Thus, by the observation above, \( a, b \in C(0) \). Now \( a <_{\mathcal{H}} x \) and \( b <_{\mathcal{H}} x \) so \( a = xs \) and \( b = xt \) for some \( s, t \in S \).
Now if $a$ and $b$ are not $\mathcal{H}$-comparable, then (4)(i) of Theorem 5.17 does not hold so (4)(ii) must hold. Thus by (4)(ii)(A), there is some $e \in E \setminus \{0\}$ such that either $ea = a$ or $eb = b$. Suppose without loss of generality that $ea = a$ as the case where $eb = b$ is dual. Now if $ex = 0$, then $a = ea = exs = 0 \cdot s = 0$ contrary to our assumptions. Thus, $ex \neq 0$. If $x \in E \setminus \{0\}$, then $x = e$ and $ex = x^2 = x$ for otherwise $ex = 0$ by Theorem 5.17(2). If $x \in C(0)$, then by Theorem 5.17(3), $ex = x$. Hence, in any case, $ex = x$ and $eb = ext = xt = b$. Therefore, $ea = a \neq 0$ and $eb = b \neq 0$ contrary to (4)(ii)(B). We conclude $a$ and $b$ must be $\mathcal{H}$-comparable.

This shows that $\downarrow_{\mathcal{H}} x$ is a chain and thus $S$ is an $\mathcal{H}$-tree.

To complete the proof that $S$ is an $\mathcal{H}$-fan, we use the equivalent statement of the $\mathcal{H}$-fan definition cited before this theorem. Suppose $x \parallel_{\mathcal{H}} y$. Let $s \in \downarrow_{\mathcal{H}} x \cap \downarrow_{\mathcal{H}} y$. Then $s = xu = yv$ for some $u, v \in S^1$. We show that $s = 0$. We have $x \neq 1$, $y \neq 1$ as $x \not\preceq_{\mathcal{H}} y$ and $y \not\preceq_{\mathcal{H}} x$. If $u \in E \setminus \{0\}$, then by Theorem 5.17(3), either $s = xu = 0$ or $s = xu = x$. If $s = xu = 0$, then we are done. Otherwise, $yv = s = xu = x$ contrary to $x \not\preceq_{\mathcal{H}} y$. Thus, we may assume that $u \in C(0)$ and likewise, $v \in C(0)$. Now if $u \succeq_{\mathcal{H}} v$, then $u = vt$ for some $t \in S^1$ and $xtv = xvt = xu = yv$. If $yv = 0$, then we are done. Otherwise, $(xt)v = yv \neq 0$ so by Lemma 5.14, $xt = y$ contrary to $y \not\preceq_{\mathcal{H}} x$. Thus, we may assume $u \not\preceq_{\mathcal{H}} v$ and likewise, $v \not\preceq_{\mathcal{H}} u$. We conclude that (4)(ii) of Theorem 5.17 must hold for $u$ and $v$. Thus, by (4)(ii)(A), $eu = u$ or $ev = v$ for some $e \in E \setminus \{0\}$. Assume without loss of generality that $eu = u$. Then by (4)(ii)(B), $ev = 0$. Thus, $s = xu = xeu = exu = eyv = yev = y \cdot 0 = 0$ as required and the proof that (b) holds is complete.

To see that (c) holds, suppose $e \in E \setminus \{0\}$ and $y \succeq_{\mathcal{H}} e$. We must show that $e = y$. Now $e = yx$ for some $x \in S^1$. Also, since $e \not\in C(0)$ and $C(0)$ is an ideal,
we have $y, x \not\in C(0)$. Thus, $y, x \in E \setminus \{0\}$. If $y \neq x$, then by condition (2) of Theorem 5.17, $e = yx = 0$ and this is a contradiction. Thus, $y = x \in E$ and $e = yx = y^2 = y$. Hence, $e$ is $\mathcal{H}$-maximal. To complete the proof of (c), recall that $S$ is the 0-disjoint union of its maximal $\mathcal{H}$-chains as we have shown that $S$ is an $\mathcal{H}$-fan above. Suppose that there exist two distinct maximal $\mathcal{H}$-chains $M_1$ and $M_2$ which are not bounded above by an idempotent. Since elements of $E \setminus \{0\}$ are maximal, we conclude that $M_1 \cap E \setminus \{0\} = \emptyset = M_2 \cap E \setminus \{0\}$. Thus, $M_1, M_2 \subseteq C(0)$ and we may choose $x \in M_1 \subseteq C(0)$ and $y \in M_2 \subseteq C(0)$ such that $x\|_\mathcal{H} y$. Now by the remarks above, there is no idempotent above either $x$ or $y$. In particular, we see that 5.17(4)(ii)(A) does not hold. By this contradiction, we conclude that at most one maximal $\mathcal{H}$-chain is not bounded above by an idempotent.

Conversely, suppose that (a)-(c) hold. We show that (1)-(4) of Theorem 5.17 hold. (1) is given by (a). To show that (2) holds, let $e, f \in E \setminus \{0\}$, $e \neq f$. Then by (c), $e$ and $f$ are $\mathcal{H}$-maximal so $e\|_\mathcal{H} f$. Thus, by (b), $ef \subseteq \downarrow_\mathcal{H} e \cap \downarrow_\mathcal{H} f = \{0\}$ so $ef = 0$ and (2) holds.

To see that (3) holds, let $x \in C(0)$ and $y \in E \setminus \{0\}$. We must show that either $xy = 0$ or $xy = x$. If $x\|_\mathcal{H} y$, then $xy \subseteq \downarrow_\mathcal{H} x \cap \downarrow_\mathcal{H} y = \{0\}$. That is, $xy = 0$. Otherwise, either $x \leq_\mathcal{H} y$ or $y \leq_\mathcal{H} x$. Now by (c), $y$ is $\mathcal{H}$-maximal so $x \leq_\mathcal{H} y$. Thus, $x = ys$ for some $s \in S^1$. Hence, $xy = (ys)y = y^2s = ys = x$. Therefore, (3) holds.

To see that (4) holds, let $x, y \in C(0)$ such that $x \not\leq_\mathcal{H} y$ and $y \not\leq_\mathcal{H} x$ (i.e. $x\|_\mathcal{H} y$). We must show that (4)(ii)(A) and (B) hold. Since $x\|_\mathcal{H} y$, we have $x \in M_1$ and $y \in M_2$ where $M_1$ and $M_2$ are distinct maximal $\mathcal{H}$-chains. According to (c), we may assume without loss of generality that $M_1$ is bounded above by an idempotent $e'$. Thus, $x = e's$ for some $s \in S^1$. Now $xe' = e'se' = e's^2 = e's = x$ so (4)(ii)(A)
holds. To see that (B) holds, assume $xe = x$ for $e \in E \setminus \{0\}$. Then $x \leq e$ and $x \not\leq_H y$ and $y \not\leq_H x$ so $y \not\leq_H x$ as $S$ is a tree by (b). Thus, in particular, $ye \neq y$. Hence, $ye = 0$ as we have shown that (3) holds above. Therefore, (4)(ii)(B) holds. This completes the proof that (4) holds and we conclude that $S$ is an ideal semigroup by Theorem 5.17. 

The $\mathcal{H}$-order graph of a commutative ideal semigroup described in Theorem 5.19 suggests the following diagram:

Note that there may be an infinite number of maximal $\mathcal{H}$-chains or “ribs” in the fan and that any $\mathcal{H}$-chain may be infinite. We present Cayley tables and $\mathcal{H}$-graphs of two finite commutative ideal semigroups to further illustrate the characterization given in Theorem 5.19.

5.20 Example. In commutative ideal semigroup (1) below, each maximal $\mathcal{H}$-chain is bounded above by an idempotent. In commutative ideal semigroup (2), exactly one maximal $\mathcal{H}$-chain is not bounded above by an idempotent.
Restating the characterization given in Theorem 5.19, a commutative ideal semigroup is the 0-disjoint union of its maximal $\mathcal{H}$-chains where at most one maximal $\mathcal{H}$-chain is not bounded by an idempotent and each element which is not an $\mathcal{H}$-maximal idempotent must lie in the archimedean component of zero. Furthermore, according to Lemma 5.18, for each maximal $\mathcal{H}$-chain $M$, the $\mathcal{H}$-order on $S$ restricts to the $\mathcal{H}$-order on $M$. In addition, each maximal $\mathcal{H}$-chain $M$ satisfies the conditions of Theorem 5.19. Hence, each maximal $\mathcal{H}$-chain $M$ is an ideal subsemigroup of $S$. Combining these observations, we conclude that the problem of completely describing a commutative ideal semigroup reduces to that of describing its maximal ideal $\mathcal{H}$-chains. Toward this end, we have the following corollary concerning commutative ideal $\mathcal{H}$-chains.

5.21 Corollary. Let $S$ be a commutative ideal semigroup. Then the following are equivalent:

(1) $S$ is an $\mathcal{H}$-chain.
(2) The principal ideals of $S$ form a chain.

(3) The ideals of $S$ form a chain.

(4) $S$ is a $\Delta$-semigroup.

(5) $S$ is an archimedean semigroup or $S = T^1$ where $T$ is an archimedean semigroup.

(6) $S$ is an archimedean semigroup or $S$ is a monoid.

Proof. That (1) is equivalent to (2) is clear by definition of the $H$ order. To see that (2) implies (3), let $I$ and $J$ be ideals of $S$. Suppose $I \not\subseteq J$. We show that $J \subseteq I$ assuming (2). Let $x \in J$. Since $I \not\subseteq J$, there exists $a \in I$ such that $a \not\subseteq J$. Thus, $a \neq x$. By (2), either $aS^1 \subseteq xS^1$ or $xS^1 \subseteq aS^1$. Thus, either $a = xu$ for some $u \in S^1$ or $x = ay$ for some $Y \in S^1$. If $a = xu$, then $a \in J$ as $x \in J$ and $J$ is an ideal and this is a contradiction. Hence, $x = ay$ and since $a \in I$ we have $x = ay \in I$ as $I$ is an ideal. Therefore, $J \subseteq I$ and (2) implies (3). (Note that this implication does not depend on $S$ an ideal semigroup.) That (3) implies (2) is obvious. That (3) is equivalent to (4) is easy to see as every congruence is determined by an ideal. We have (1)-(4) are equivalent.

Now we show that (5) implies (1). Let $a, b \in S$. We must show that $a$ and $b$ are $H$-comparable. Now either $S = C(0)$ or $S = C(0)^1$ by (5). If $a = 0$, $a = 1$, $b = 0$, or $b = 1$, then we are done. Thus, we may assume $a, b \in C(0) \setminus \{0\}$. Hence, either (i) or (ii) of condition (4) of Theorem 5.17 must hold. Now if $S = C(0)$ then clearly (ii)(A) does not hold. If $S = C(0)^1$, then (ii)(B) does not hold as $1 \cdot a = a \neq 0$ and $1 \cdot b = b \neq 0$. Thus, (ii) does not hold so (i) must hold and $a$ and $b$ are $H$-comparable. To see that (1) implies (5), first note that by conditions (1) of Theorem 5.19, $S = C(0) \cup E \setminus \{0\}$ and any nonzero idempotent is $H$-maximal. Thus,
since \( S \) is an \( \mathcal{H} \)-chain, \( S \) has at most one nonzero idempotent so either \( S = C(0) \) in which case \( S \) is archimedean or \( S = C(0) \cup \{e\} \). In the second case, we show that \( e \) is an identity for \( S \) to complete the proof. Let \( x \in S \). Since \( e \) is \( \mathcal{H} \)-maximal and \( S \) is an \( \mathcal{H} \)-chain, \( x \leq_{\mathcal{H}} e \) so \( x = ey \) some \( y \in S^1 \). Hence, \( ex = eey = ey = x \). Thus, \( e \) is an identity for \( S \) and in this case \( S = C(0)^1 \). Let \( T = C(0) \) to obtain (5).

We need only show (5) is equivalent to (6). Certainly, (5) implies (6). To see that (6) implies (5), we have \( S = C(0) \cup \{1\} \cup E \setminus \{0,1\} \) as \( S \) is a commutative monoid. Now if \( E \setminus \{0,1\} \neq \emptyset \), then let \( e \in E \setminus \{0,1\} \). Then \( \{e, 1\} \) forms a nontrivial subsemigroup not containing zero contrary to Lemma 5.8. Thus, for \( T = C(0) \), \( S = C(0) \cup \{1\} = T^1 \) and the corollary is proven.

5.22 Corollary. Let \( S \) be an archimedean semigroup with zero. Then \( S \) is an ideal semigroup if and only if \( S \) is an \( \mathcal{H} \)-chain

Proof. If \( S \) is an archimedean ideal semigroup, then it is an \( \mathcal{H} \)-chain by Corollary 5.21((5) implies (1)). Conversely, if \( S \) is an archimedean \( \mathcal{H} \)-chain with zero we have \( S = C(0) \) and \( S \) clearly satisfies the conditions of Theorem 5.19. Thus, \( S \) is an ideal semigroup.

According to Corollary 5.21, a commutative ideal \( \mathcal{H} \)-chain is an archimedean \( \mathcal{H} \)-chain with zero or such a semigroup with an identity adjoined. Thus, the problem of describing commutative ideal \( \mathcal{H} \)-chains reduces to that of describing archimedean \( \mathcal{H} \)-chains with zero. We answer this question in a special case with the following proposition.

5.23 Proposition. Let \( T \) be an archimedean \( \mathcal{H} \)-chain with zero such that \( T \) has an \( \mathcal{H} \)-maximal element \( x \). Then \( T = \theta(x) = \{x, x^2, x^3, \ldots, x^{n-1}, 0\} \), the cyclic
semigroup of order $n$ with idempotent zero for some $n$. Moreover, if $S$ is a commutative ideal semigroup, then for each maximal $\mathcal{H}$-chain $M$ such that $M \cap C(0)$ has an $\mathcal{H}$-maximal element $x$, $M \cap C(0) = \theta(x)$.

Proof. Certainly, $T = \downarrow_{\mathcal{H}} x$. Thus, we show that $\downarrow_{\mathcal{H}} x = \theta(x)$. We need only show the forward inclusion. Let $y \in \downarrow_{\mathcal{H}} x$. Then $y = xt_1$ for some $t_1 \in T^1$. We must show that $xt_1 = x^n$ for some $n \in \mathbb{N}$. We have $x^m = 0$ for some $m \in \mathbb{N}$ as $T$ is an archimedean semigroup with zero. If $t_1 = 1$, then $xt_1 = x \in \theta(x)$. Thus, we may assume that $t_1 \in T = \downarrow_{\mathcal{H}} x$. Write $t_1 = xt_2$ for $t_2 \in T^1$. Now if $t_2 = 1$, then $t_1 = xt_2 = x$. Thus, $xt_1 = x^2 \in \theta(x)$. Hence, we may assume that $t_2 \in T = \downarrow_{\mathcal{H}} x$. Continuing this process, we may assume for each $0 \leq i \leq m$ that $t_i \in \downarrow_{\mathcal{H}} x$ and we obtain $xt_1 = x(xt_2) = x(x(xt_3)) = \ldots = x^m t_m = 0 \cdot t_m = 0 \in \theta(x)$. Hence, $y = xt_1 \in \theta(x)$ and $T = \downarrow_{\mathcal{H}} x = \theta(x)$ as required. Now if $S$ is a commutative ideal semigroup, then for each maximal $\mathcal{H}$-chain $M$ such that $M \cap C(0)$ has an $\mathcal{H}$-maximal element $x$, $M \cap C(0)$ is an archimedean $\mathcal{H}$-chain in itself as $\leq_{\mathcal{H},M} = \leq_{\mathcal{H}} \cap (M \times M)$. Therefore, applying the statement just proved, $M \cap C(0) = \theta(x)$. 

According to the proposition above and previous observations, a finite maximal $\mathcal{H}$-chain in a commutative ideal semigroup is either a finite cyclic semigroup with zero or such a semigroup with an identity adjoined. In order to examine infinite maximal $\mathcal{H}$-chains in commutative ideal semigroups, we must consider the structure of infinite archimedean $\mathcal{H}$-chains. We conclude from the proposition above that such an $\mathcal{H}$-chain has no maximal element. While a complete description of an arbitrary infinite $\mathcal{H}$-chain does not appear here, one such $\mathcal{H}$-chain is the half-open nilpotent interval $[1/2, 1)$. One begins to see some of the forms that an infinite archimedean $\mathcal{H}$-chain may take by examining infinite $\mathcal{H}$-chains which are embedded in the nilpo-
tent interval $[1/2, 1)$. For example, the rational powers of a given element $x \in [1/2, 1)$ form an $\mathcal{H}$-chain which may be infinite. Furthermore, $Q \cap [1/2, 1)$ is an infinite $\mathcal{H}$-chain contained in $[1/2, 1)$. The reader is referred to Chapter 6 for a discussion of intervals which are ideal semigroups in a topological setting.

We now consider the question of what conditions guarantee that a subsemigroup of a commutative ideal semigroup is again an ideal semigroup. In general, we know that the property of being an ideal semigroup is not hereditary even in the commutative case. Observe that in Example 19(1), $C(0)$ is a subsemigroup of an ideal semigroup which is not itself an ideal semigroup as it is not an $\mathcal{H}$-chain.

We noted previously that each maximal $\mathcal{H}$-chain in a commutative ideal semigroup is again an ideal semigroup. In addition, the following corollary provides one possible answer to the question of what conditions guarantee that a given subsemigroup inherits the property of being an ideal semigroup. We remark here that any semigroup of the form $S = C(0) \cup E \setminus \{0\}$ is periodic. Hence, the notes on periodic commutative semigroups from Chapter 1 apply to ideal semigroups.

5.24 Corollary. Let $S$ be a commutative ideal semigroup. Let $T$ be a non-trivial subsemigroup of $S$. Then $T$ is an ideal semigroup if and only if condition (4) of Theorem 5.17 holds in $T$.

Proof. If $T$ is an ideal semigroup, then certainly (4) holds by applying the theorem. Conversely, we need only see that (1)-(3) hold in $T$ to conclude that $T$ is an ideal semigroup. Since any nontrivial subsemigroup of $S$ must contain zero, $0 \in T$. As above, $S$ is periodic so 1.48(5) holds and we have $T = C_T(0) \cup E_T \setminus \{0\}$ where $C_T(0) = C(0) \cap T$. Thus, (1) holds in $T$ and it is then clear that (2) and (3) hold in $T$ as they hold in $S$. #
We now turn to the task of characterizing commutative ideal semigroups which have the ideal extension property (IEP) or the congruence extension property (CEP).

5.25 Proposition. Let $S$ be a commutative ideal semigroup. Then $S$ has IEP if and only if for each $x \in C(0)$, $\downarrow x$ is either the trivial semigroup, the zero semigroup of order two, or the cyclic semigroup of order three with idempotent zero.

Proof. Suppose that $S$ has IEP. Then $C(0)$ has IEP as IEP is hereditary. Let $x \in C(0)$. We claim that $\downarrow x = \{x, x^2, 0\}$. We need only prove the forward inclusion. Let $y \in \downarrow x$. Then we have $y = xs$ for some $s \in S^1$. If $s = 1$, then we have $y = x \in \{x, x^2, 0\}$. If $s \in E \setminus \{0\}$, then according to Theorem 5.17(3), $y \in \{x, 0\} \subseteq \{x, x^2, 0\}$. Otherwise, $s \in C(0)$. In this case, if $y = xs \neq 0$, then $y = xs = x^2$ by Lemma 2.1. Hence, we conclude that $y \in \{x, x^2, 0\}$ in all cases and the forward inclusion is proven. Therefore, $\downarrow x = \{x, x^2, 0\}$. Certainly, if $x = 0$, then $\downarrow x = \{0\}$, the trivial semigroup. Also, if $x \neq 0$ and $x^2 = 0$, then $\downarrow x = \{x, 0\}$, the zero semigroup of order two. In the remaining case, $x^2 \neq 0$. We have $\downarrow x = \{x, x^2, 0\}$ and $x, x^2$ and 0 are distinct since $x^2 \neq 0$ (hence, $x \neq 0$) and $x^2 \neq x$ as the only idempotent is zero. Therefore, $\downarrow x$ is the cyclic semigroup of order three with idempotent zero. Therefore, $\downarrow x$ must have one of the three forms listed.

Conversely, suppose that for each $x \in C(0)$, $\downarrow x$ is either the trivial semigroup, the zero semigroup of order two, or the cyclic semigroup of order three with idempotent zero. We will show that the conditions of Theorem 2.7 are satisfied in order to conclude that $S$ has IEP. As noted previously, any commutative ideal
semigroup is periodic so the initial condition of Theorem 2.7 is satisfied. Also, combining 5.17(2) and 5.17(3), we see that the condition on multiplication between components given by 2.7(2) is satisfied. We need only see that $C(0)$ has IEP (condition 2.7(1)). According to Lemma 2.1, it suffices to show that if $x, y \in C(0)$, then either $xy = 0$ or $xy = x^2 = y^2$. If $x$ and $y$ are not $\mathcal{H}$-comparable, then according to Theorem 5.19, $xy \in \downarrow_{\mathcal{H}} x \cap \downarrow_{\mathcal{H}} y = \{0\}$. Otherwise, we may assume that $y \in \downarrow_{\mathcal{H}} x$. By examining the assumed possibilities for $\downarrow_{\mathcal{H}} x$, we conclude that if $x \neq y$, then $xy = 0$. Certainly, if $x = y$, then $xy = x^2 = y^2$. Thus, by Lemma 2.1, $C(0)$ has IEP.

By Theorem 2.7, this completes the proof that $S$ has IEP.

5.26 Corollary. Let $S$ be a commutative ideal semigroup. Then the following are equivalent:

(1) $S$ has CEP.

(2) $S$ has IEP and $C(0)$ is an $\mathcal{H}$-chain.

(3) $C(0)$ is either the trivial semigroup, the zero semigroup of order two, or the cyclic semigroup of order three with idempotent zero.

Proof. Combining 1.33, 1.39, and Corollary 5.22, we obtain that (1) implies (2). Now assume (2). According to Proposition 5.25, for each $x \in C(0)$, $\downarrow_{\mathcal{H}} x$ is isomorphic to one of the three semigroups given in (3) above. Combining this with the fact that $C(0)$ is an $\mathcal{H}$-chain, we conclude that $C(0) = \downarrow_{\mathcal{H}} x$ for some $x \in C(0)$ and $C(0)$ must have one of these three forms. Therefore, (2) implies (3).

To prove that (3) implies (1), we first note that if $C(0)$ is trivial, then $S$ is a semilattice and therefore, has CEP. In the other two cases given by (3), we apply Proposition 3.12 to show that $S$ has CEP. Each of the two remaining possibilities for $C(0)$ has CEP as each has order less than or equal to three. Now since condition
(3) of Theorem 5.17 holds in $S$, it is clear that if $C(0)$ is the zero semigroup of order two, then $S$ satisfies the remaining condition of Proposition 3.12. Hence, in this case, $S$ has CEP. Finally, if $C(0)$ is the cyclic semigroup of order three with idempotent zero, we must see that the remaining condition of Proposition 3.12 holds. Let $e \in E\{0\}$. By condition (3) of Theorem 5.17, $ex = x$ or $ex = 0$ for each $x \in C(0) = \{y, y^2, 0\}$. Now if $e \cdot y = y$, then $e \cdot y^2 = (e \cdot y) \cdot y = y \cdot y = y^2$ and if $e \cdot y = 0$, then $e \cdot y^2 = (e \cdot y) \cdot y = 0 \cdot y = 0$. This shows that the remaining condition of Proposition 3.12 holds and $S$ has CEP in all cases. Thus, (3) implies (1).

5.27 Proposition. Let $S$ be a commutative ideal semigroup. Then $S$ has CEP if and only if each subsemigroup $T$ of $S$ is an ideal semigroup.

Proof. According to 1.39, the forward implication holds. To prove the converse, suppose that each subsemigroup $T$ of $S$ is an ideal semigroup. According to 1.39, we need only show that $S$ has IEP. By Proposition 5.25, it suffices to show that for each $x \in C(0)$, $\downarrow x$ is either the trivial semigroup, the zero semigroup of order two, or the cyclic semigroup of order three with idempotent zero. Let $x \in C(0)$. Then by assumption, $\downarrow x$ is an ideal semigroup. Hence, $\downarrow x$ is an $H$-chain in itself by Corollary 5.22 as it is an archimedean ideal semigroup with zero. Thus, $\downarrow x$ is an archimedean $H$-chain with zero having $H$-maximal element $x$. Hence, according to Proposition 5.23, $\downarrow x = \theta(x) = \{x, x^2, x^3, ..., x^{n-1}, x^n = 0\}$, the cyclic semigroup of order $n$ with idempotent zero for some $n$. One verifies easily that if $n \geq 4$, then all products among distinct elements $\{x^{n-2}, x^{n-1}, x^n = 0\}$ are zero. Hence, $\{x^{n-2}, x^{n-1}, x^n = 0\}$ is the zero semigroup of order three and this archimedean semigroup is not an $H$-chain in itself. Therefore, according to Corollary 5.22, $\{x^{n-2}, x^{n-1}, x^n = 0\}$ forms a subsemigroup which is not an ideal
semigroup contrary to hypothesis. Whence, we may assume that \( n \leq 3 \) and we conclude that \( \downarrow_{\mathcal{H}} x \) is a cyclic semigroup of order three or less having idempotent zero. Thus, \( \downarrow_{\mathcal{H}} x \) is either the trivial semigroup, the zero semigroup of order two, or the cyclic semigroup of order three with idempotent zero. This completes the proof of the converse.

Let \( S \) be a semigroup in which \( \leq_{\mathcal{H}} \) is a partial order. For any pair of \( \mathcal{H} \)-comparable elements \( a, b \in S \),

\[
[a, b] = \{ x \in S : a \leq_{\mathcal{H}} x \leq_{\mathcal{H}} b \}.
\]

If all chains from \( a \) to \( b \) are finite and if among the maximal chains there exists one of length \( n \), then the interval \( [a, b] \) has length \( n \) (denoted \( l[a, b] = n \)). If \( S \) has a zero element, then for any element \( x \in S \) such that \( l[0, x] = n < \infty \), we write \( \text{height}(x) = l[0, x] = n \). We use the notions of height and length to give explicit descriptions of ideal semigroups with the ideal extension property (IEP) and the congruence extension property (CEP) respectively in the following two results.

5.28 Proposition. Let \( S \) be a commutative semigroup. Then \( S \) is an ideal semigroup with IEP if and only if the following hold:

(a) \( S = C(0) \cup E \setminus \{0\} \).

(b) \( S \) is an \( \mathcal{H} \)-fan such that \( \text{height}(x) \leq 2 \) for all \( x \in C(0) \).

(c) Each \( e \in E \setminus \{0\} \) is an \( \mathcal{H} \)-maximal element and at most one \( \mathcal{H} \)-maximal element of \( S \) is not an idempotent.

Proof. Suppose that \( S \) is an ideal semigroup with IEP. We show that (a)-(c) hold. Applying Theorem 5.19, we need only show that \( \text{height}(x) \leq 2 \) for all
\(x \in C(0)\). According to Proposition 5.25, \(|\downarrow_{\mathcal{H}} x| \leq 3\) for all \(x \in C(0)\). Hence, it is clear that \(\text{height}(x) \leq 2\) for all \(x \in C(0)\).

Conversely, suppose that (a)-(c) hold. We show that the conditions of Theorem 5.19 hold in order to conclude that \(S\) is an ideal semigroup. We need only show that if \(x \parallel_{\mathcal{H}} y\), then either \(x \leq_{\mathcal{H}} e\) or \(y \leq_{\mathcal{H}} e\) for some \(e \in E \setminus \{0\}\). According to (b), \(\mathcal{H}\)-chains in \(C(0)\) (and in \(S\)) are finite. Thus, it is clear that we may choose distinct \(\mathcal{H}\)-maximal elements above the \(\mathcal{H}\)-parallel elements \(x\) and \(y\). By (c), at least one of these \(\mathcal{H}\)-maximal elements must be an idempotent. Hence, there exists an \(\mathcal{H}\)-maximal idempotent \(e\) such that either \(x \leq_{\mathcal{H}} e\) or \(y \leq_{\mathcal{H}} e\) as required. Thus, \(S\) is an ideal semigroup. To see that \(S\) has IEP, we show that \(\downarrow_{\mathcal{H}} x\) has one of the three forms given in Proposition 5.25 for each \(x \in C(0)\). Let \(x \in C(0)\).

By assumption, \(\text{height}(x) \leq 2\). If \(\text{height}(x) = 0\), then \(\downarrow_{\mathcal{H}} x = \{0\}\), the trivial semigroup. If \(\text{height}(x) = 1\), then \(\downarrow_{\mathcal{H}} x = \{0, x\}\), the zero semigroup of order two as \(x\) is not idempotent. Finally, suppose that \(\text{height}(x) = 2\). In this case, we write \(\downarrow_{\mathcal{H}} x = \{0, y, x\}\) where \(0 \prec_{\mathcal{H}} y \prec_{\mathcal{H}} x\). Applying 1.44, we obtain that \(yx = 0 = y^2\).

Thus, all products excluding \(x^2\) between elements of \(\downarrow_{\mathcal{H}} x\) are zero. Also, \(x^2 \neq x\) and if \(x^2 = 0\), then by the previous observation, \(S\) is the zero semigroup of order three which is not an ideal semigroup. Hence, we conclude that \(x^2 = y, 0, x, x^2\) are distinct and \(\downarrow_{\mathcal{H}} x\) is the cyclic three element semigroup with idempotent zero. This completes the proof that \(S\) has IEP. Therefore, \(S\) is an ideal semigroup with IEP.

According to Proposition 5.28, the \(\mathcal{H}\)-order graph of a typical commutative ideal semigroup with the ideal extension property (IEP) might be given by the following diagram.
5.29 Proposition. Let $S$ be a commutative semigroup. Then $S$ is an ideal semigroup with CEP if and only if the following hold:

(a) $S = C(0) \cup E \setminus \{0\}$.

(b) $S$ is an $\mathcal{H}$-fan such that $C(0)$ is an $\mathcal{H}$-chain with $\text{length}(C(0)) \leq 2$.

(c) Each $e \in E \setminus \{0\}$ is an $\mathcal{H}$-maximal element and at most one $\mathcal{H}$-maximal element of $S$ is not an idempotent.

Proof. Suppose that $S$ is an ideal semigroup with CEP. We show that (a)-(c) hold. Applying Theorem 5.19, we need only show that $C(0)$ is an $\mathcal{H}$-chain with $\text{length}(C(0)) \leq 2$. According to Corollary 5.26, $C(0)$ is an $\mathcal{H}$-chain and $|C(0)| \leq 3$. Hence, it is clear that $\text{length}(C(0)) \leq 2$ and (a)-(c) must hold.

Conversely, suppose that (a)-(c) hold in $S$. Since $C(0)$ is an $\mathcal{H}$-chain with $\text{length}(C(0)) \leq 2$, it is clear that $\text{height}(x) \leq 2$ for all $x \in C(0)$. Hence, conditions (a)-(c) of Proposition 5.28 hold and we conclude that $S$ is an ideal semigroup with IEP. Furthermore, since $C(0)$ is an $\mathcal{H}$-chain, we can apply Corollary 5.26 to conclude that $S$ has CEP. Therefore, $S$ is an ideal semigroup with CEP.

According to Proposition 5.29, the $\mathcal{H}$-order graph of a typical commutative ideal semigroup with the congruence extension property (CEP) might be given by the following diagram.
The final result of this chapter gives a characterization of commutative $\Delta$-semigroups which have the congruence extension property (CEP) as promised in the introduction.

5.30 Theorem. Let $S$ be a commutative semigroup. Then the following are equivalent.

(1) $S$ is a $\Delta$-semigroup with CEP.

(2) $S$ is a $\Delta$-semigroup with IEP.

(3) $S$ is of one of the following forms:

(a) A quasicyclic group.

(b) A quasicyclic group with zero adjoined.

(c) The trivial semigroup, the zero semigroup of order two, or the cyclic semigroup of order three with idempotent zero.

(d) One of the semigroups listed in (c) with an identity adjoined.

Proof. Since commutative semigroups with CEP have IEP, (1) implies (2). Suppose that $S$ is a $\Delta$-semigroup with IEP. By 1.51, the commutative $\Delta$-semigroups are precisely quasicyclic groups, quasicyclic groups with zero adjoined, archimedean semigroups with zero ($S = C(0)$) which are $\mathcal{H}$-chains, and semigroups of this form with an identity adjoined ($C(0)^1$). To see that (3) holds, we need only show that if
$S = C(0)$ is an $\mathcal{H}$-chain, then $S$ has one of the forms given in (c). Now by Theorem 5.19 and Corollary 5.22, archimedean semigroups with zero which are $\mathcal{H}$-chains are ideal semigroups. Thus, if $S$ has this form, then $S = C(0)$ is a commutative ideal semigroup with IEP in which $C(0)$ is an $\mathcal{H}$-chain and according to Corollary 5.26, $S = C(0)$ is either the trivial semigroup, the zero semigroup of order two, or the cyclic semigroup of order three with idempotent zero. Thus, $S$ has one of the forms listed in (c). Hence, (2) implies (3).

To see that (3) implies (1), note that quasicyclic groups have CEP since they are torsion commutative groups (1.20) and adjoining a zero retains CEP by 1.9. Thus, semigroups of forms (a) and (b) have CEP. Also, each semigroup given in (c) has CEP trivially as the order of each is less than or equal to three and adjoining an identity preserves CEP by 1.9. Thus, semigroups of forms (c) and (d) have CEP. Finally, semigroups of form (a)-(d) are $\Delta$-semigroups according to 1.51 and we conclude that (3) implies (1).
CHAPTER 6

TOPOLOGICAL RESULTS

The purpose of this chapter is to examine in a topological setting those properties of semigroups treated algebraically in previous chapters.

A semigroup $S$ is a topological semigroup provided that it has a Hausdorff topology in which multiplication on $S$ is continuous.

The results in this chapter employ techniques using nets. For this reason, a short discussion of nets is included here.

A directed set is a pair $(D, \leq)$, where $D$ is a non-empty set and $\leq$ is a reflexive and transitive relation on $D$ such that for $\alpha, \beta \in D$ there exists $\gamma \in D$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A net is a set $X$ is a function from a directed set $D$ into $X$. We denote the image of $\alpha \in D$ by $x_\alpha$ and the net itself by $\{x_\alpha\}_{\alpha \in D}$ or simply $\{x_\alpha\}$ when no confusion seems likely.

If $\{x_\alpha\}_{\alpha \in D}$ is a net in $X$ and $A \subseteq X$, then $x_\alpha$ is eventually in $A$ provided there exists $\alpha \in D$ such that $x_\beta \in A$ whenever $\alpha \leq \beta$. This is denoted $x_\alpha \in^e A$. We say that $x_\alpha$ is frequently in $A$ provided for each $\alpha \in D$, there exists $\beta \geq \alpha$ such that $x_\beta \in A$. This is denoted $x_\alpha \in^f A$.

A net $\{x_\alpha\}$ converges to $x$ (denoted $x_\alpha \xrightarrow{c} x$) provided that $\{x_\alpha\}$ is eventually in each neighborhood of $x$ and $\{x_\alpha\}$ clusters to $x$ (denoted $x_\alpha \xrightarrow{d} x$) provided that $\{x_\alpha\}$ is frequently in each neighborhood of $x$.

The following topological results pertaining to nets will be used throughout this chapter.
(i) A net in a Hausdorff space converges to at most one point.

(ii) A space \( X \) is compact if and only if each net in \( X \) clusters.

(iii) A subset \( C \) of a space \( X \) is closed if and only if for each net \( \{x_\alpha\} \) in \( C \) converging to a point \( x \in X \), we have \( x \in C \).

(iv) A function \( f : X \to Y \) is continuous if and only if \( x_\alpha \to x \) in \( X \) implies \( f(x_\alpha) \to f(x) \) in \( Y \). (Convergence can be replaced by clustering here.)

There are many well-known and useful facts concerning compact semigroups.

One such fact is that a compact semigroup must contain an idempotent. Also, any compact semigroup contains a unique minimal ideal which is compact. Indeed, such a minimal ideal is what is known as a completely simple semigroup. A characterization of completely simple semigroups with CEP is given in [Dumesnil, 1993]. Furthermore, since the minimal ideal of a commutative semigroup is a group, a compact commutative semigroup has a minimal ideal which is a compact abelian group. Also of use to us will be the fact that a closed subsemigroup of a compact group is a subgroup. Finally, note that since a topological semigroup is Hausdorff by definition, a subsemigroup \( T \) of a compact semigroup \( S \) is closed if and only if it is compact. Thus, in a compact semigroup, the product of closed (i.e. compact) subsets is closed (i.e. compact) by continuity of multiplication on \( S \). In particular, ideals generated by closed subsets (e.g. principal ideals) are closed.

Let \( S \) be a topological semigroup and let \( a \in S \). Then \( \Gamma(a) := \overline{\theta(a)} \) where \( \theta(a) = \{a^n : n \in \mathbb{N}\} \) is called the monothetic subsemigroup of \( S \) generated by \( a \). If \( S = \Gamma(a) \) for some \( a \in S \), then \( S \) is called a monothetic semigroup. If \( \Gamma(a) \) is a compact monothetic semigroup, then its minimal ideal \( M(\Gamma(a)) \) is a compact abelian group and \( \Gamma(a) = \theta(a) \cup M(\Gamma(a)) \). Furthermore, \( M(\Gamma(a)) \) consists of the
cluster points of \( \Gamma(a) \). Hence, any subset of \( \Gamma \) containing \( M(\Gamma(a)) \) is closed. We define the monothetic index of \( a \), denoted \( \text{mi}(a) \), to be the smallest \( n \in \mathbb{N} \) such that \( a^n \in M(\Gamma(a)) \) if \( \theta(a) \cap M(\Gamma(a)) \neq \emptyset \). Otherwise, \( \text{mi}(a) = \infty \). We define the monothetic index of a semigroup \( S \) (denoted \( \text{mi}(S) \)) to be \( \max\{\text{mi}(a) : a \in S\} \) if this maximum exists. Otherwise, \( \text{mi}(S) = \infty \).

Recall from Chapter 1 that in the purely algebraic setting, \( \text{index}(a) \) is the least \( n \in \mathbb{N} \) such that \( a^n \in M(\theta(a)) \) if \( M(\theta(a)) \neq \emptyset \) and otherwise, \( \text{index}(a) = \infty \). We note that in a topological semigroup, we may have \( \text{mi}(a) < \infty \) while \( \text{index}(a) = \infty \). Consider the circle group \( S^1 \).

In algebraic semigroups, we defined \( \text{index}(S) \) to be \( \max\{\text{index}(a) : a \in S\} \) when this maximum exists and otherwise, \( \text{index}(S) = \infty \). If \( \text{index}(S) < \infty \), then \( S \) is periodic. However, it is clear from the example above and from the definition of the analogously defined monothetic index of a semigroup \( S \) that \( \text{mi}(S) < \infty \) does not imply that \( S \) is periodic. Hence, in spite of the fact that we will show that both the congruence extension property (CEP) and the ideal extension property (IEP) (as defined for topological semigroups below) imply \( \text{mi}(S) < 3 \) in the category of compact semigroups, we will not be able to use certain techniques from previous chapters that employed periodicity in attempting to partially characterize compact semigroups with IEP and CEP. Compactness will often be used to justify statements that were previously justified by periodicity.

In Chapters 2 and 3, we characterized unipotent commutative semigroups having IEP and CEP respectively as part of an attempt to characterize commutative semigroups with these properties since a commutative semigroup with either of these properties is a semilattice of unipotent semigroups. We will present topologi-
cal versions of these characterizations in this chapter. From Chapter 1, a periodic unipotent commutative semigroup is archimedean and is an ideal extension of an abelian group by an archimedean semigroup with zero. As discussed above, compact semigroups with IEP or CEP are not periodic in general. Hence, we cannot assume that compact unipotent commutative semigroups with CEP or IEP share the structure described above which holds in the algebraic setting. Thus, we prove the following two lemmas on the structure of compact unipotent commutative semigroups in preparation for characterizing semigroups of this type which have IEP/CEP.

6.1 Lemma. A compact commutative unipotent semigroup $S$ is an ideal extension of a compact abelian group by a compact commutative semigroup with a unique idempotent which is a zero.

Proof. By previous remarks on compact semigroup theory, $S$ has a unique minimal ideal $M(S)$ which is a compact abelian group as $S$ is commutative. Certainly $S$ is an ideal extension of $M(S)$ by $S/M(S)$ and $S/M(S)$ is a compact commutative semigroup which has a zero. Since $S$ is compact, $\phi(E(S)) = E(S/M(S))$ where $\phi$ is the natural map from $S$ to $S/M(S)$. This yields that $S/M(S)$ has exactly one idempotent and the proof is complete. $\blacksquare$

6.2 Lemma. A compact commutative semigroup with $mi(S) < \infty$ which has a unique idempotent zero is an archimedean semigroup with zero.

Proof. According to 1.43, it suffices to show that some power of each element is zero. Since $mi(S) < \infty$, for each $a \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in M(\Gamma(a))$. Now $\Gamma(a)$ is a compact semigroup so it must contain an idempotent. Hence, $0 \in \Gamma(a)$
as zero is the unique idempotent of $S$. Thus, $M(\Gamma(a)) = \{0\}$. Therefore, for each $a \in S$, there exists $n \in \mathbb{N}$ such that $a^n = 0$ as required. 

A topological semigroup $S$ is said to have the ideal extension property (IEP) provided that for each closed subsemigroup $T$ of $S$ and each closed ideal $I$ of $T$, there exists a closed ideal $J$ of $S$ such that $J \cap T = I$.

6.3 A topological semigroup $S$ has the ideal extension property (IEP) if and only if each closed subsemigroup of $S$ has the ideal extension property.

6.4 Let $S$ be a compact semigroup, $T$ a closed subsemigroup of $S$ and $I$ a closed ideal of the subsemigroup $T$. Then there exists a closed ideal $J$ of $S$ such that $J \cap T = I$ if and only if $S^1 IS^1 \cap T = I$. Moreover, for $x \in T$, there is a closed ideal of $S$ extending $T^1 x T^1$ to $S$ if and only if $S^1 x S^1$ extends $T^1 x T^1$ to $S$.

6.5 A continuous homomorphi's image of a compact semigroup with the ideal extension property (IEP) has IEP.

6.6 Proposition. A compact monothetic semigroup $\Gamma(a)$ has the ideal extension property (IEP) if and only if $\text{mi}(a) \leq 3$.

Proof. Suppose that $4 \leq \text{mi}(a)$. Then $T = \{a^2, a^3, a^4, \ldots\} \cup M(\Gamma(a))$ is a closed subsemigroup of $\Gamma(a)$ and $I = \{a^2, a^4, a^5, \ldots\}$ is a closed ideal of $T$. Now $a^3 \in \Gamma(a)^1 \Gamma(a)^1 \cap T$, but $a^3 \notin I$. Thus, by 6.4, there is no closed ideal extending $I$ to $\Gamma(a)$ and $\Gamma(a)$ does not have IEP.

Conversely, suppose that $\text{mi}(a) \leq 3$. Then one of the following holds:

(1) $\Gamma(a) = M(\Gamma(a))$;

(2) $\Gamma(a) = \{a\} \cup M(\Gamma(a))$; or

(3) $\Gamma(a) = \{a, a^2\} \cup M(\Gamma(a))$. 
Let $T$ be a closed subsemigroup of $\Gamma(a)$. If $a \in T$, then $T = \Gamma(a)$ and there is nothing to prove. Thus, in any of the cases listed above, we may assume that $a \not\in T$. Also, if $T \subseteq M(\Gamma(a))$, then $T$ is a closed subsemigroup of the compact group $M(\Gamma(a))$. Hence, $T$ is a subgroup of $M(\Gamma(a))$. Thus, the only ideal of $T$ is $T$ and clearly this extends to $\Gamma(a)$. According to these remarks, we see that $\Gamma(a)$ has IEP if it is of form (1) or (2), and to see that form (3) has IEP, we need only show that if $I$ is a closed ideal of a closed subsemigroup $T$ such that $a \not\in T$ and $a^2 \in T$, then $I$ extends to a closed ideal of $\Gamma(a)$. For such $I$ and $T$, note that $M(\Gamma(a)) \cap T$ is a closed subsemigroup (hence, a subgroup) of $M(\Gamma(a))$ which is contained in $T$. Also, $M(\Gamma(a)) \cap T$ is an ideal of $T$. Hence, $M(\Gamma(a)) \cap T$ is the minimal ideal of $T$ ($M(T)$) since a group which is an ideal of a semigroup must be the minimal ideal of that semigroup. Now if $a^2 \not\in I$, then $I$ is a closed subgroup of $M(\Gamma(a))$ which is also an ideal of $T$. Thus, $M(T) = I$. Therefore, $M(\Gamma(a)) \cap T = M(T) = I$ and $I$ extends to the closed ideal $M(\Gamma(a))$ of $\Gamma(a)$. Otherwise, $a^2 \in I$ and we show that the closed ideal $\{a^2\} \cup M(\Gamma(a))$ of $\Gamma(a)$ extends $I$. Certainly, $I \subseteq (\{a^2\} \cup M(\Gamma(a))) \cap T$ as $a \not\in I$. As shown above, $M(\Gamma(a)) \cap T = M(T)$, so

$$(\{a^2\} \cup M(\Gamma(a))) \cap T = \{a^2\} \cup (M(\Gamma(a)) \cap T) = \{a^2\} \cup M(T) \subseteq I$$

as $I$ is an ideal of $T$ and $a^2 \in I$. Hence, $(\{a^2\} \cup M(\Gamma(a))) \cap T = I$. Therefore, a semigroup of form (3) has IEP and the proof is complete.

6.7 Corollary. Each compact semigroup $S$ with the congruence extension property (CEP) has monothetic index less than 4.

Proof. Apply Proposition 6.6 and 6.3.
We have the following analogue to Lemma 2.1 for compact archimedean semi-
groups. An argument similar to that used in the proof of 2.1 justifies this result
as compactness (sometimes finiteness) of subsemigroups and ideals used yields the
closed condition where needed.

**6.8 Lemma.** Let $S$ be a compact archimedean semigroup with zero. Then the
following are equivalent:

1. If $x, y \in S$, $xy \neq 0$, then $xy = x^2 = y^2$.
2. Each closed subsemigroup of $S$ is a closed ideal.
3. $S$ has IEP.

**6.9 Lemma.** Let $S$ be a compact commutative unipotent semigroup. Hence,
$S$ is an ideal extension of a compact abelian group $G$ by a compact commutative
semigroup $N$ with unique idempotent zero. Then $S$ has IEP if and only if $N$ is an
archimedean semigroup with zero having IEP.

**Proof.** Suppose $S$ has IEP. By 6.5, IEP is retained by homomorphic images so
$N \cong S/G$ has IEP. Thus, by 6.7, $\text{mi}(N) < \infty$. Hence, by 6.2, $N$ is an archimedean
semigroup with zero.

Now suppose that $N$ is an archimedean semigroup with zero having IEP. Let
$T$ be a closed subsemigroup of $S$ and let $I$ be a closed ideal of $T$. We have $G \cap T$ is
a closed subsemigroup of $G$ and $G$ is a compact group so $G \cap T$ is a closed subgroup
of $G$. Also, since $G$ is an ideal of $S$, $G \cap T$ is a closed ideal of $T$. Thus, $G \cap T$ is a
group which is an ideal of $T$. Hence, $M(T) = G \cap T$, as it is well-known that any
subgroup which is an ideal must be the minimal ideal. We claim that $S^1 I \cap T = I$.
We need only show the forward inclusion. Let $s \in S^1$, $x \in I$ such that $sx \in T$. Now
if \( sx \in G \), then \( sx \in G \cap T = M(T) \subseteq I \). If \( sx \notin G \), then \( s, x \notin G \) as \( G \) is an ideal. Furthermore, \( s, x \in N \setminus \{0\} \) and \( sx \neq 0 \). Now \( N \) has IEP, so (1) of Lemma 6.8 holds. Hence, \( sx = x^2 \). Thus, \( sx = x^2 \in I \) as \( x \in I \) and our claim is proven. Thus, \( S^1I \) is a closed ideal of \( S \) extending \( I \) and \( S \) has IEP.

We note here that the congruence \( \eta \) which gives the decomposition of a commutative semigroup into its archimedean components is not a closed congruence. For this reason, results concerning interaction between components of commutative semigroups with IEP do not seem to have direct topological analogues.

We turn to the examination of compact semigroups with the congruence extension property (CEP). A topological semigroup \( S \) is said to have the congruence extension property (CEP) provided that for each closed subsemigroup \( T \) of \( S \) and each closed congruence \( \sigma \) on \( T \), \( \sigma \) can be extended to a closed congruence on \( S \).

The closed congruence on a topological semigroup \( S \) generated by a given relation on \( S \) is the smallest closed congruence on \( S \) which contains the given relation. Equivalently, it is the intersection of all closed congruences which contain the given relation. If \( \sigma \) is a relation on \( S \), we denote the closed congruence generated by \( \sigma \) by \( \sigma^* \). It is clear that the smallest closed congruence on \( S \) contains the smallest congruence on \( S \). Notationally, we have \( \langle \sigma \rangle_S \subseteq \sigma^* \). Certainly, if \( \langle \sigma \rangle_S \) is closed, then \( \langle \sigma \rangle_S = \sigma^* \). In particular, we will consider the closed congruence on a topological semigroup \( S \) generated by a single pair. We denote the closed congruence on \( S \) generated by the pair \((a, b)\) by \( \alpha^{S*}(a, b) \). As a special case of the remark above we have \( \alpha^S(a, b) \subseteq \alpha^{S*}(a, b) \).

In [Hofmann and Mostert, 1966], a characterization of the closed congruence generated by a given relation is given. Unfortunately, this characterization does not
lend itself to proof methods as readily as the algebraic characterization of the congruence generated by a relation. Thus, many algebraic results whose proofs relied on the characterization of congruences generated by relations cannot be directly topologized. In some cases examined below, the congruence generated algebraically by a relation is closed topologically allowing direct topological analogues. In general, it can be difficult to compute the closed congruence generated by a relation.

The results which are listed below are topological versions of basic algebraic results given in Chapter 1. Proofs of these results in the topological setting are straightforward analogues of those in the algebraic setting.

6.10 A topological semigroup $S$ has the congruence extension property (CEP) if and only if each closed subsemigroup of $S$ has the congruence extension property.

6.11 Let $T$ be a closed subsemigroup of a topological semigroup $S$ and let $\sigma$ be a closed congruence on $T$. Then $\sigma$ extends to a closed congruence on $S$ if and only if $\sigma^*$ is an extension of $\sigma$ to $S$. Moreover, for $a, b \in T$, $\alpha^{T^*}$ extends to a closed congruence on $S$ if and only if $\alpha^{S^*}$ is an extension of $\alpha^{T^*}$.

6.12 Let $G$ be a topological group and $\sigma$ a closed congruence on $G$. Then there exists a closed normal subgroup $N$ of $G$ such that $(a, b) \in \sigma$ if and only if $ab^{-1} \in N$.

6.13 Let $S$ be a topological semigroup, $I$ a closed ideal of $S$, and $\sigma$ a closed congruence on $I$. Then $\sigma$ extends to a closed congruence on $S$ if and only if $\sigma \cup \Delta_S$ is a congruence extending $\sigma$ to $S$.

6.14 Let $S$ be a topological semigroup, $M$ a closed homomorphic retract of $S$, and $\sigma$ a closed congruence on $M$. Then there exists a closed extension of $\sigma$ to $S$.

6.15 Let $S$ be a commutative topological semigroup having a minimal ideal $M$. Then each closed congruence on $M$ can be extended to a closed congruence on $S$. 
6.16 Let $S$ be a topological semigroup, $Q$ a closed subsemigroup of $S$ and $T$ a closed subsemigroup of $Q$. Let $\sigma$ be a closed congruence on $T$. Let $\delta$ be a closed congruence extending $\sigma$ to $Q$ and let $\gamma$ be a closed congruence extending $\delta$ to $S$. Then $\gamma$ is a closed congruence extending $\sigma$ to $S$.

We also have the following useful result from [Dumesnil, 1993].

6.17 Compact abelian groups have the congruence extension property (CEP).

(For future reference, we give the following remarks on the proof of this result. If $T$ is a closed subsemigroup (hence a subgroup by previous remarks) of a compact abelian group $G$ and $\sigma$ a closed congruence on $T$, then by 6.12, there exists a closed normal subgroup $N$ of $T$ such that $(a, b) \in \sigma$ provided $ab^{-1} \in N$ and $a, b \in T$. Now $N$ is a normal subgroup of $G$ as $G$ is commutative and $\bar{\sigma} := \{(a, b) \in S \times S : ab^{-1} \in N\}$ is a closed congruence extending $\sigma$.)

6.18 Let $S$ be a compact commutative semigroup and let $T$ be a closed subsemigroup of the minimal ideal $M$. Then each congruence on $T$ can be extended to $S$. (This is a corollary of 6.17, 6.16, and 6.15.)

A detailed proof of the following result is given to illustrate useful techniques for working with monothetic semigroups.

6.19 Proposition. A compact monothetic semigroup $\Gamma(a)$ has the congruence extension property (CEP) if and only if $\text{mi}(a) \leq 3$.

Proof. Suppose that $4 \leq \text{mi}(a)$. Then $T = \{a^2, a^3, a^4, \ldots\} \cup M(\Gamma(a))$ is a closed subsemigroup of $\Gamma(a)$ and $I = \{a^2, a^4, a^5, \ldots\}$ is a closed ideal of $T$. Now $\sigma = (I \times I) \cup \Delta_T$ is a closed congruence on $T$, but $\sigma^*$ is not an extension of $\sigma$ as $(a^3, a^5) \in \sigma^* \cap (T \times T)$ and $(a^3, a^5) \notin \sigma$. Thus, $\sigma$ has no extension.
Conversely, suppose that $\text{mi}(a) \leq 3$. Then one of the following holds:

1. $\Gamma(a) = M(\Gamma(a))$;
2. $\Gamma(a) = \{a\} \cup M(\Gamma(a))$; or
3. $\Gamma(a) = \{a, a^2\} \cup M(\Gamma(a))$.

In case (1), $\Gamma(a)$ has CEP by 6.7, since $M(\Gamma(a))$ is a compact abelian group. Let (2) hold. Let $T$ be a proper closed subsemigroup of $\Gamma(a) = \{a\} \cup M(\Gamma(a))$. Then since $T$ is proper, $a \not\in T$ and $T$ is a closed subsemigroup of $M(\Gamma(a))$. According to 6.18, each congruence on $T$ can be extended to $\Gamma(a)$ and $\Gamma(a)$ has CEP. Now suppose that $\Gamma(a) = \{a, a^2\} \cup M(\Gamma(a))$. Let $T$ be a proper closed subsemigroup of $\Gamma(a)$. Again, since $T$ is proper, $a \not\in T$. If $a^2 \not\in T$, then $T$ is a closed subsemigroup of $M(\Gamma(a))$ and each congruence on $T$ can be extended to $\Gamma(a)$ by 6.18. Otherwise, $T = \{a^2\} \cup H$ where $H = T \cap M(\Gamma(a))$ is a closed subsemigroup (hence, a closed subgroup) of the compact group $M(\Gamma(a))$. We must show that any congruence on $T = \{a^2\} \cup H$ can be extended to a closed congruence on $\Gamma(a)$. To accomplish this, we first show that any closed congruence on $\{a^2\} \cup M(\Gamma(a))$ extends to a closed congruence on $\Gamma(a)$. Let $\sigma$ be a closed congruence on $\{a^2\} \cup M(\Gamma(a))$. Let $\overline{\sigma} = \sigma \cup \Delta_{\Gamma(a)}$. It is clear that $\overline{\sigma}$ is a closed equivalence which extends $\sigma$. We need only check compatibility. It suffices to show that $(xa, ya) \in \overline{\sigma}$ for $(x, y) \in \sigma$. Let $(x, y) \in \sigma$. Since $\sigma|_{M(\Gamma(a))}$ is a closed congruence on the group $M(\Gamma(a))$, there exists a closed normal subgroup $N$ of $M(\Gamma(a))$ such that

$$\sigma|_{M(\Gamma(a))} = \{(c, d) \in M(\Gamma(a)) \times M(\Gamma(a)) : cd-1 \in N\}$$

according to 6.12. If $(x, y) \in \sigma|_{M(\Gamma(a))}$, then $(xa)(ya)^{-1} = xy^{-1} \in N$ so we have $(xa, ya) \in \sigma|_{M(\Gamma(a))} \subseteq \sigma \subseteq \overline{\sigma}$. Otherwise, we may assume that $(x, y) = (a^2, y)$ for some $y \in M(\Gamma(a))$. We must show that $(a^3, ya) \in \overline{\sigma}$. First note that if $e$ is the
identity of $M(\Gamma(a))$, then since $a^3 \in M(\Gamma(a))$, $a^3 = a^3 e = a^3(yy^{-1}) = (a^2y^{-1})(ya)$. Thus, since $ya \in M(\Gamma(a))$, $a^3(ya)^{-1} = a^2y^{-1}$. We need only show that $a^2y^{-1} \in N$ so that $a^3(ya)^{-1} = a^2y^{-1} \in N$ and $(a^3, ya) \in \sigma|_{M(\Gamma(a))} \subseteq \sigma \subseteq \sigma$ as required.

Now since $a^4 \in M(\Gamma(a))$ and $(a^2, y) \in \sigma$, $(a^6, a^4y) = (a^4a^2, a^4y) \in \sigma$ as $\sigma$ is a congruence on $\{a^2\} \cup M(\Gamma(a))$. However, $(a^6, a^4y) \in M(\Gamma(a)) \times M(\Gamma(a))$ so we have $(a^6, a^4y) \in \sigma|_{M(\Gamma(a))}$. Thus, $a^6(a^4y)^{-1} \in N$. But

$$a^2y^{-1} = a^2(ey^1) = a^2(a^4(a^4)^{-1})y^{-1} = a^6(a^4)^{-1}y^{-1} = a^6(a^4y)^{-1}$$

so $a^2y^{-1} \in N$ and the proof that $\sigma$ extends from $\{a^2\} \cup M(\Gamma(a))$ to a closed congruence on $\Gamma(a)$ is complete.

We now return to the case in which $T = \{a^2\} \cup H$ where $H = T \cap M(\Gamma(a))$ is a closed subgroup of $M(\Gamma(a))$. Let $\sigma$ be a closed congruence on $T = \{a^2\} \cup H$. We first extend $\sigma$ to a closed congruence $\delta$ on $\{a^2\} \cup M(\Gamma(a))$ and then employ that which was proved above to extend to $\Gamma(a)$. Now $\sigma|_H$ is a closed congruence on the subgroup $H$ of the compact abelian group $M(\Gamma(a))$. Hence, according to 6.12 and the comments on the proof of 6.17, there exists a closed normal subgroup $N$ of $H$ such that $\sigma|_H = \{(c, d) \in H \times H : cd^{-1} \in N\}$ and

$$\sigma_M = \{(c, d) \in M(\Gamma(a)) \times M(\Gamma(a)) : cd^{-1} \in N\}$$

is a closed congruence on $M(\Gamma(a))$ which extends $\sigma|_H$. Let $\delta = \sigma \cup \sigma_M$. Then $\delta \cap (T \times T) = (\sigma \cup \sigma_M) \cap (T \times T) = \sigma \cup (\sigma_M \cap (H \times H)) = \sigma \cup \sigma|_H = \sigma$.

Also, it is clear that $\delta$ is a reflexive, symmetric, closed relation on $\{a^2\} \cup M(\Gamma(a))$. We check transitivity. Let $(x, y), (y, z) \in \delta$. We must show that $(x, y) \in \delta$. If $(x, y), (y, z) \in \sigma$ or $(x, y), (y, z) \in \sigma_M$, then we are done. Thus, by duality, it suffices to show that if $(x, y) \in \sigma \setminus \sigma_M$ and $(y, z) \in \sigma_M$, then $(y, z) \in \sigma$. Since $(y, z) \in \sigma_M$ implies $y \in M(\Gamma(a))$ and $(x, y) \in \sigma$ implies $y \in T = \{a^2\} \cup H$, we
have \( y \in T \cap M(\Gamma(a)) = H \). But \((y, z) \in \sigma_M\) implies that \( zy^{-1} \in N \subseteq H \) so \( z = (zy^{-1})y \in HH \subseteq H \). Thus, \((y, z) \in \sigma_M \cap (H \times H) = \sigma|_H \subseteq \sigma\) as required and \( \delta \) is transitive. We show that \( \delta \) is compatible. Let \((x, y) \in \delta \) and \( z \in \{a^2\} \cup M(\Gamma(a))\).

If \((x, y) \in \sigma_M \) and \( z \in M(\Gamma(a))\), then \((xz, yz) \in \sigma_M \subseteq \delta\). If \((x, y) \in \sigma_M \) and \( z = a^2\), then \((xa^2)(ya^2)^{-1} = xy^{-1} \in N\) so \((xa^2, ya^2) \in \sigma_M \subseteq \delta\). If \((x, y) \in \sigma\) and \( z = a^2\), then \((xz, yz) \in \sigma \subseteq \delta\) as \( \sigma \) is a congruence on \( \{a^2\} \cup H\). Finally, suppose that \((x, y) \in \sigma \setminus \sigma_M \) and \( z \in M(\Gamma(a))\). We may assume that \((x, y) = (a^2, y)\) for some \( y \in T\). If \( y = a^2\), then we have \((xz, yz) \in \Delta_M \subseteq \sigma_M \subseteq \delta\). Otherwise, \((x, y) = (a^2, y)\) for some \( y \in H\). We show that \((a^2z, yz) \in \sigma_M \subseteq \delta\). First note that \(a^2z(yz)^{-1} = a^2(y^{-1}y)z(yz)^{-1} = a^2y^{-1}(yz)(yz)^{-1} = a^2y^{-1}\). Thus, we need only show that \(a^2y^{-1} \in N\) to obtain that \((a^2z, yz) \in \sigma_M \subseteq \delta\). Since \( T = \{a^2\} \cup H\) is a subsemigroup, we have \(a^2n \in H\) for all \( n \geq 2\). Now \((a^6, a^4y) \in \sigma\) since \((a^2, y) \in \sigma\), \(a^4 \in H \subseteq T\), and \( \sigma \) is a congruence on \( T\). Also, \((a^6, a^4y) \in \sigma|_H\), so

\[
a^2y^{-1} = a^2(a^4(a^4)^{-1})y^{-1} = a^6(a^4)^{-1}y^{-1} = a^6(a^4y)^{-1} \in N
\]

as required and \( \delta \) is compatible. Thus, \( \delta \) is a closed congruence on \( \{a^2\} \cup M(\Gamma(a))\) which extends \( \sigma\). Now by the argument presented in the previous paragraph, there is a closed congruence \( \rho \) which extends \( \delta \) from \( \{a^2\} \cup M(\Gamma(a))\) to \( \Gamma(a)\). According to 6.15, since \( \rho \) extends \( \delta \) and \( \delta \) extends \( \sigma \), \( \rho \) extends \( \sigma \) to \( \Gamma(a)\).

6.20 Corollary. Each compact semigroup \( S \) with the congruence extension property (CEP) has monothetic index less than 4.


We now determine the structure of compact commutative unipotent semigroups with CEP. The details of the proof of the following analogue of Lemma 3.1 are provided to illustrate special cases in which \( \alpha^T(u, v) = \alpha^{T^*}(u, v)\).
6.21 Lemma. Let $S$ be a compact archimedean semigroup with zero. Then $S$ has the congruence extension property (CEP) if and only if the following hold:

(1) If $xy \neq 0$, then $xy = x^2 = y^2$.

(2) If $xy = xz = z^2 = x^2 = y^2 \neq 0$, then $z^2 = yz$.

Proof. Assume $S$ has CEP. To see that (1) holds, let $x, y \in S$ with $xy \neq 0$. Suppose that it is not the case that $xy = x^2 = y^2$. Then we must have either $xy \neq x^2$ or $xy \neq y^2$. Clearly, we may assume without loss of generality that $xy \neq x^2$. Now let $T = \langle 0, xy, x \rangle$. According to 1.50, $z^n = 0$ for all $z \in S$, $n \geq 3$. Using this fact, we give a complete listing of the elements of $T$. One checks that $T = \{0, x, xy, x^2, x^2y, x^2y^2\}$ where these elements are not necessarily distinct. Since $T$ is finite, $T$ is a closed subsemigroup. Consider $\alpha^T(0, x)$. By previous discussion in this chapter and 1.5, we have $\bigcup_{n \in \mathbb{N}} \{(t \cdot 0, t \cdot x), (t \cdot x, t \cdot 0) : t \in T^1\}^n \cup \Delta_T = \alpha^T(0, x) \subseteq \alpha^{T*}(0, x)$. However, $(t \cdot 0, t \cdot x) = (0, tx)$ for all $t \in T^1$. Thus,

$$\alpha^T(0, x) = \bigcup_{n \in \mathbb{N}} \{(t \cdot 0, t \cdot x), (t \cdot x, t \cdot 0) : t \in T^1\}^n \cup \Delta_T = (xT^1 \times xT^1) \cup \Delta_T.$$ 

Hence, $(xT^1 \times xT^1) \cup \Delta_T \subseteq \alpha^{T*}(0, x)$ and since $(xT^1 \times xT^1) \cup \Delta_T$ is a closed congruence containing $(0, x)$, $\alpha^{T*}(0, x) \subseteq (xT^1 \times xT^1) \cup \Delta_T$. Therefore,

$$\alpha^{T*}(0, x) = (xT^1 \times xT^1) \cup \Delta_T.$$ 

Likewise, we have $\alpha^{S*}(0, x) = (xS^1 \times xS^1) \cup \Delta_S$. Now $xy \in xS^1$ and we obtain that $(0, xy) \in \alpha^{S*}(0, x) \cap (T \times T) = ((xS^1 \times xS^1) \cup \Delta_S) \cap (T \times T)$. But one can show (by the argument given in Lemma 2.1) that $xy \not\in xT^1$ and hence, $(0, xy) \not\in \alpha^{T*}(0, x)$. By note 6.11, this is contrary to $S$ having CEP.

To see that (2) holds, let $xy = xz = z^2 = x^2 = y^2 \neq 0$. For the purpose of contradiction, suppose $z^2 \neq yz$. Let $T = \langle z, x \rangle$. Now using our assumptions and the fact that $u^3 = 0$ for all $u \in S$ (1.50), we can give a complete listing of the elements
of $T$. One checks that $T = \{0, z, x, z^2 = x^2 = xz = y^2 = xy\}$. Then $T$ is closed as it is finite. As shown in the proof of 3.1, $(\{x, z\} \times \{x, z\}) \cup \Delta_T = \alpha^T(x, z)$. Thus, $(\{x, z\} \times \{x, z\}) \cup \Delta_T = \alpha^T(x, z) \subseteq \alpha^{T^*}(x, z)$. But $(\{x, z\} \times \{x, z\}) \cup \Delta_T$ is a closed congruence on $T$ containing $(x, z)$ so $\alpha^{T^*}(x, z) \subseteq (\{x, z\} \times \{x, z\}) \cup \Delta_T$. Therefore, $\alpha^{T^*}(x, z) = (\{x, z\} \times \{x, z\}) \cup \Delta_T$. By assumption, $z^2 \neq zy$ so $(z^2, zy) \notin \Delta_T$. Also, $z^2 \neq z$ and $zy \neq z$ by 1.44 as $z \neq 0$. Hence,

$$(z^2, zy) \notin (\{x, z\} \times \{x, z\}) \cup \Delta_T = \alpha^{T^*}(x, z).$$

However, $(z^2, zy) = (xy, zy) \in \alpha^S(x, z) \subseteq \alpha^{S^*}(x, z)$. In addition, since (1) holds, $T$ is an ideal by Lemma 2.1. Thus, $zy \in T$ so $(z^2, zy) \in \alpha^{S^*}(x, z) \cap (T \times T)$, but $(z^2, zy) \notin \alpha^{T^*}(x, z)$ contrary to $S$ having CEP. Therefore, (2) holds.

Now assume that (1) and (2) hold. We show that $S$ has CEP. Let $T$ be a closed subsemigroup of $S$. Then by Lemma 6.8, $T$ is a closed ideal of $S$ as (1) holds. Let $\sigma$ be a closed congruence on $T$. We must prove that $\sigma^* \cap (T \times T) = \sigma$. It suffices to show that $(sx, sy) \in \sigma$ for each $(x, y) \in \sigma$ and $s \in S$. Let $(x, y) \in \sigma$ and $s \in S$. If $sx = 0 = sy$, then $(sx, sy) = (0, 0) \in \sigma$. If $sx \neq 0$ and $sy \neq 0$, then by (1), $sx = s^2 = sy \in T$ as $T$ is an ideal and $(sx, sy) \in \Delta_T \subseteq \sigma$. Finally suppose $sx \neq 0$ and $sy = 0$. (The case where $sx = 0$ and $sy \neq 0$ is dual.) Then by (1), $sx = s^2 = x^2$. We claim that $xy = 0$. If not, then by (1), $xy = x^2 = y^2$ and we have $sx = xy = s^2 = x^2 = y^2 \neq 0$, but $s^2 \neq 0 = sy$ contrary to (2). Thus, $xy = 0$. Hence, $(sx, sy) = (x^2, 0) = (x^2, xy) \in \sigma$. Thus, $(sx, sy) \in \sigma$ for each $(x, y) \in \sigma$ and $s \in S$ so $\sigma^* \cap (T \times T) = \sigma$. Therefore, $S$ has CEP and the proof of this lemma is complete. ☐

Our next task is to prove that a compact commutative unipotent semigroup (hence, an ideal extension of a compact abelian group $G$ by a compact commutat-
tive semigroup \( N \) with unique idempotent zero) has CEP provided that \( N \) is an archimedean semigroup with zero having CEP.

In order to accomplish this, we would like to express congruences on a compact ideal extension \( V \) of a compact semigroup \( S \) by a compact semigroup \( T \) in terms of congruences on \( S \) and \( T \) in some sense. Recall that we have the following definition for algebraic semigroups from [Petrich, 1967].

Let \( V \) be an ideal extension of a semigroup \( S \) by a semigroup \( T \). Let \( \sigma \) be a congruence on \( S \). Let \( P \) be an ideal of \( T \) and let \( \tau \) be a congruence on \( T/P \). We define a relation \( \nu \) on \( V \) as follows:

\[
(a, b) \in \nu \equiv \begin{cases} 
    a, b \in T\setminus P & \text{and } (a, b) \in \tau \text{ or;}
    a, b \in S & \text{and } (a, b) \in \sigma \text{ or;}
    a, b \in P\setminus\{0\} & \text{such that } (a'x, ax), (b'x, bx) \in \sigma \text{ for all } x \in S \\
    \text{and } (a', b') \in \sigma \text{ or;}
    a \in P\setminus\{0\}, b \in S & \text{and there exists } a' \in S \\
    \text{such that } (a'x, ax) \in \sigma \text{ for all } x \in S \text{ and } (a', b) \in \sigma
\end{cases}
\]

We write \( \nu = (\sigma, P, \tau) \).

According to [Petrich, 1967], each congruence on an ideal extension \( V \) is of the form given above where \( \sigma, P, \) and \( \tau \) satisfy certain properties. We topologize this result as follows.

6.22 Proposition. Each closed congruence \( \nu \) on a compact ideal extension \( V \) of a compact semigroup \( S \) by a compact semigroup \( T \) is of the form \((\sigma, P, \tau)\) where \( \sigma \) is a closed congruence on \( S \), \( P \) is a closed ideal of \( T \) such that

\((*)\) for each \( a \in P\setminus\{0\}, \) there exists \( b \in S \) such that \((bx, ax), (xb, xa) \in \sigma \) for all \( x \in S \)

and \( \tau \) is a closed congruence on \( T/P \) such that \([0]_\tau = \{0\}\) and
(**) for all $a, b \in T \setminus P$ and $x, y \in S$ such that $(a, b) \in \tau$ and $(x, y) \in \sigma$, $(ax, by), (xa, yb) \in \sigma$.

**Proof.** Let $\nu$ be a closed congruence on $V$. In [Petrich, 1967], it is shown that the congruence $\nu$ is given by $(\sigma, P, \tau)$ where $\sigma = \nu|_S$ is a congruence on $S$, 

$$P = \{ a \in T \setminus \{0\} : (a, b) \in \nu \text{ for some } a \in S \} \cup \{0\}$$

is an ideal of $T$ satisfying (*) and $\tau$ defined on $T/P$ by 

$$(a, b) \in \tau \text{ if and only if } (a, b) \in \nu \text{ for } a, b \in T \setminus P \text{ and } (0, 0) \in \tau$$

is a congruence on $T/P$ satisfying (**). We need only show that $\sigma$, $P$, and $\tau$ are closed. Now $\sigma = \nu \cap (S \times S)$ so $\sigma$ is closed as $\nu$ and $S$ are closed. To see that $P$ is a closed ideal of $T$, let $\{x_\alpha\}$ be a net in $P$ such that $x_\alpha \rightarrow x$. We must show that $x \in P$. If $x_\alpha =^f 0$, then $x_\alpha \rightarrow 0$ and we conclude $x = 0 \in P$. Otherwise, $x_\alpha \in^e P \setminus \{0\}$ and we may assume $x_\alpha \in P \setminus \{0\}$ for all $\alpha$. In this case, for each $\alpha$, there exists $y_\alpha \in S$ such that $(x_\alpha, y_\alpha) \in \nu$. Now $S$ is compact, so $y_\alpha \rightarrow y$ for some $y \in S$. Thus, $(x_\alpha, y_\alpha) \rightarrow (x, y) \in \nu$ as $\nu$ is closed. Hence, $x \in P$ and $P$ is a closed ideal of $T$. To see that $\tau$ is a closed congruence on $T/P$, let $\{(x_\alpha, y_\alpha)\}$ be a net in $\tau$ such that $(x_\alpha, y_\alpha) \rightarrow (x, y)$. We show that $(x, y) \in \tau$. If $(x_\alpha, y_\alpha) =^f (0, 0)$, then $(x_\alpha, y_\alpha) \rightarrow (0, 0)$ and we conclude $(x, y) = (0, 0) \in \tau$. Otherwise, $(x_\alpha, y_\alpha) \in^e \nu$ and $x_\alpha, y_\alpha \in T \setminus P$ so we may assume that this holds for all $\alpha$. Then since $\nu$ is closed, $(x, y) \in \nu$. Thus, if $x, y \in T \setminus P$, then $(x, y) \in \tau$. Also, if $x = y = 0 \in T/P$, then $(x, y) \in \tau$. Hence, we will show that either $x, y \in T \setminus P$ or $x = y = 0 \in T/P$ to complete the proof that $\tau$ is a closed congruence. It suffices to show that if $x \in P$ (i.e. $x = 0 \in T/P$), then $y \in P$. By definition of $P$, if $x \in P$, then there exists $z \in S$
such that \((x, z) \in \nu\). By transitivity, \((y, z) \in \nu\) so \(y \in P\) as required. This completes the proof of the proposition.  

As a corollary to the results in [Hildebrant, 1982], we have the following topological result concerning ideal extensions.

6.23 Proposition. A continuous partial homomorphism \(\phi : T \setminus \{0\} \to S\) determines a compact ideal extension \(V\) of a compact semigroup \(S\) by a compact semigroup \(T\) as follows: (* denotes multiplication in \(S\) and \(\odot\) denotes multiplication in \(T\))

\[
a \odot b = \begin{cases} 
  a \ast b & \text{if } a, b \in S; \\
  a \odot b & \text{if } a, b \in T \setminus \{0\} \text{ and } a \odot b \neq 0; \\
  \phi(a) \ast \phi(b) & \text{if } a, b \in T \setminus \{0\} \text{ and } ab = 0; \\
  a \ast \phi(b) & \text{if } a \in S \text{ and } b \in T \setminus \{0\}.
\end{cases}
\]

If \(S\) has an identity element, then every extension of \(S\) by \(T\) is found in this fashion.

6.24 Theorem. Let \(V\) be a compact ideal extension of a compact semigroup \(S\) by a compact semigroup \(T\) such that the extension \(V\) is determined by a partial homomorphism \(\phi : T \setminus \{0\} \to S\). Any closed congruence \(\nu\) on \(V\) is of the form \((\sigma, P, \tau)\) for some closed ideal \(P\) of \(S\) and closed congruences \(\sigma\) on \(S\) and \(\tau\) on \(T/P\) such that \([0]_{\tau} = \{0\}\) and such that \((a, b) \in \tau\) implies \((\phi(a), \phi(b)) \in \sigma\) for all \(a, b \in T \setminus P\).

Proof. By Proposition 6.22, \(\nu\) is of the form \((\sigma, P, \tau)\) where \(\sigma\) is a closed congruence on \(S\), \(P\) is a closed ideal of \(T\) such that

(*): for each \(a \in P \setminus \{0\}\), there exists \(b \in S\) such that \((bx, ax), (xb, xa) \in \sigma\) for all \(x \in S\)

and \(\tau\) is a closed congruence on \(T/P\) such that \([0]_{\tau} = \{0\}\) and

(**): for all \(a, b \in T \setminus P\) and \(x, y \in S\) such that \((a, b) \in \tau\) and \((x, y) \in \sigma, (ax, by), (xa, yb) \in \sigma\).
In [Petrich, 1967], it is shown that for an extension determined by a partial homomorphism \( \phi \), (*) holds and (**) is equivalent to the condition that \((a, b) \in \tau \) implies \((\phi(a), \phi(b)) \in \sigma \) for all \(a, b \in T \setminus P\). Hence, the theorem holds.

We have the following analogue of Lemma 3.3.

**6.25 Lemma.** Let \( V \) be a compact ideal extension of a compact monoid \( S \) by a compact semigroup \( T \). Hence, the extension \( V \) is determined by a continuous partial homomorphism \( \phi : T \setminus \{0\} \to S \). Let \( \sigma \) be a closed congruence on \( S \) and let \( P \) be a closed ideal of \( T \). Then for \( a, b \in P \setminus \{0\} \), there exist \( a', b' \in S \) such that \((a'x, ax), (b'x, bx) \in \sigma \) for all \( x \in S \) and \((a', b') \in \sigma \) if and only if \((\phi(a), \phi(b)) \in \sigma \) and for \( a \in P \setminus \{0\} \), \( b \in S \), there exists \( a' \in S \) such that \((a'x, ax) \in \sigma \) for all \( x \in S \) and \((a', b) \in \sigma \) if and only if \((\phi(a), b) \in \sigma \).

Combining this Lemma with the definition of the relation \( \nu = (\sigma, P, \tau) \) above we obtain the following. If \( V \) is an ideal extension of a monoid \( S \) by a semigroup \( T \), \( \sigma \) a congruence on \( S \), \( P \) an ideal of \( S \), \( \tau \) a congruence on \( T/P \), and \( \nu = (\sigma, P, \tau) \), then we have

\[
(a, b) \in \nu \equiv \begin{cases} 
  a, b \in T \setminus P \text{ and } (a, b) \in \tau \text{ or; } \\
  a, b \in S \text{ and } (a, b) \in \sigma \text{ or; } \\
  a, b \in P \setminus \{0\} \text{ and } (\phi(a), \phi(b)) \in \sigma \text{ or; } \\
  a \in P \setminus \{0\}, \ b \in S \text{ and } (\phi(a), b) \in \sigma 
\end{cases}
\]

**6.26 Theorem.** Let \( S \) be a compact commutative unipotent semigroup. Hence, \( S \) is a compact ideal extension of a compact commutative group \( G \) by a compact commutative semigroup \( N \) with unique idempotent zero which is determined by a continuous partial homomorphism \( \phi : N \setminus \{0\} \to G \). Then \( S \) has the congruence extension property (CEP) if and only if \( N \) is an archimedean semigroup with zero having CEP.
**Proof.** Suppose $S$ has CEP. By Theorem 6.31, $N \cong S/G$ has CEP as $G$ is a closed ideal of $S$ and $S$ has CEP. Thus, by 6.7, $\text{mi}(N) < \infty$. Hence, by 6.2, $N$ is an archimedean semigroup with zero.

To prove the converse, suppose that $N$ is an archimedean semigroup with zero having CEP. By 6.17, $G$ has CEP since it is a compact abelian group. Let $T$ be a closed subsemigroup of $S$. Let $e$ denote the identity of $G$. Since $T$ is a compact subsemigroup, $e \in T$ as it must contain an idempotent and $e$ is the only idempotent. We have $T = T \cap S = T \cap (G \cup N \setminus \{0\}) = (T \cap G) \cup (T \cap N \setminus \{0\})$. Now $e \in T \cap G \neq \emptyset$. If $T \cap N \setminus \{0\} = \emptyset$, then $T \subseteq G = M(S)$. By 6.18, any congruence on a subsemigroup of a minimal ideal which is a group in a commutative semigroup $S$ can be extended to $S$. Hence, we may assume that $T \setminus (T \cap G) = T \cap N \setminus \{0\} \neq \emptyset$. Since $G$ is compact, the subsemigroup $T \cap G$ is a subgroup of $G$. Also, $T \cap G$ is an ideal of $T$ as $G$ is an ideal of $S$. By the argument given in Theorem 3.6, we obtain that $\phi|_{N' \setminus \{0\}} : N' \setminus \{0\} \to T \cap G$ is a continuous partial homomorphism which determines $T$ as a compact ideal extension of $T \cap G$ by $N'$.

Now let $\nu$ be a closed congruence on $T$. Then by Theorem 6.22, $\nu = (\sigma, P, \tau)$ where $\sigma$ is a closed congruence on $T \cap G$, $P$ is a closed ideal of $N'$ and $\tau$ is a closed congruence on $N'/P$ such that $[0]_\tau = \{0\}$ and such that $(a, b) \in \tau$ implies $(\phi(a), \phi(b)) \in \sigma$ for all $a, b \in N'/P$. That is, we have

$$(a, b) \in \nu \equiv \begin{cases} a, b \in N' \setminus P \text{ and } (a, b) \in \tau \text{ or;} \\ a, b \in T \cap G \text{ and } (a, b) \in \sigma \text{ or;} \\ a, b \in P \setminus \{0\} \text{ and } (\phi(a), \phi(b)) \in \sigma \text{ or;} \\ a \in P \setminus \{0\}, b \in T \cap G \text{ and } (\phi(a), b) \in \sigma \end{cases}$$

Now since $G$ has CEP, $\sigma$ extends to a closed congruence $\overline{\sigma}$ on $G$. We consider a particular extension $\overline{\sigma}$. Now by 6.12, there exists a closed normal subgroup $M$ of
the closed subgroup $T \cap G$ such that $(x, y) \in \sigma$ if and only if $xy^{-1} \in M$. Now since $G$ is commutative, $M$ is a normal subgroup of $G$ and by the note on the proof of 6.17, we may extend $\sigma$ by a closed congruence $\overline{\sigma}$ where $(x, y) \in \overline{\sigma}$ if and only if $xy^{-1} \in M$. Also, as argued in the proof of 3.6, $P$ is a closed ideal of $N$ and we may assume that $\overline{\tau} = \tau \cup \Delta_{N/P}$ is a closed congruence extending $\tau$ as $N$ has CEP as 6.8, 6.13, and 6.21 hold. Thus, we have a triple $(\overline{\sigma}, P, \overline{\tau})$. It is clear that $[0]_{\overline{\tau}} = \{0\}$ as $\overline{\tau} = \tau \cup \Delta_{N/P}$ and $[0]_{\tau} = \{0\}$. Also, if $(a, b) \in \tau = \tau \cup \Delta_{N/P}$ and $a, b \in N \setminus P$, then either $(a, b) \in \tau$ or $a = b$. Thus, by our known condition on $\tau$, $(\phi(a), \phi(b)) \in \sigma \cup \Delta_G \subseteq \overline{\sigma}$. Hence, this triple satisfies the conditions of Corollary 3.5 and putting $\nu = (\overline{\sigma}, P, \overline{\tau})$ we conclude that $\nu$ is a congruence on $S$. That is,

$$(a, b) \in \nu \iff \begin{cases} 
  a, b \in N \setminus P \text{ and } (a, b) \in \tau \text{ or; } \\
  a, b \in G \text{ and } (a, b) \in \overline{\sigma} \text{ or; } \\
  a, b \in P \setminus \{0\} \text{ and } (\phi(a), \phi(b)) \in \overline{\sigma} \text{ or; } \\
  a \in P \setminus \{0\}, b \in G \text{ and } (\phi(a), b) \in \overline{\sigma}
\end{cases}$$

By the proof of 3.6, this congruence extends $\nu$. We need only show that $\nu$ is closed. For this purpose, let $\{(x_\alpha, y_\alpha)\}$ be a net in $\nu$ such that $(x_\alpha, y_\alpha) \xrightarrow{\nu} (x, y)$. We must show that $(x, y) \in \nu$. We know that for each $\alpha$, one of the following holds:

$$(x_\alpha, y_\alpha) \in \nu \iff \begin{cases} 
  x_\alpha, y_\alpha \in N \setminus P \text{ and } (x_\alpha, y_\alpha) \in \tau \text{ or; } \\
  x_\alpha, y_\alpha \in G \text{ and } (x_\alpha, y_\alpha) \in \overline{\sigma} \text{ or; } \\
  x_\alpha, y_\alpha \in P \setminus \{0\} \text{ and } (\phi(x_\alpha), \phi(y_\alpha)) \in \overline{\sigma} \text{ or; } \\
  x_\alpha \in P \setminus \{0\}, y_\alpha \in G \text{ and } (\phi(x_\alpha), y_\alpha) \in \overline{\sigma}
\end{cases}$$

Suppose first that $x_\alpha, y_\alpha \in^f G$ and $(x_\alpha, y_\alpha) \in^f \overline{\sigma}$. In this case, since $(x_\alpha, y_\alpha) \in^f \overline{\sigma}$ and $(x_\alpha, y_\alpha) \xrightarrow{\nu} (x, y)$, we conclude that some subnet of $\{(x_\alpha, y_\alpha)\}$ contained in $\overline{\sigma}$ converges to $(x, y)$. Thus, $(x, y) \in \overline{\sigma}$ as $\overline{\sigma}$ is closed. Hence, $(x, y) \in \nu$. If it is not true
that $x_\alpha, y_\alpha \in f^\ast G$ and $(x_\alpha, y_\alpha) \in f^\ast \overline{\sigma}$, then there exists $\beta$ such that for each $\alpha \geq \beta$, one of the following holds:

\[
\begin{cases}
  x_\alpha, y_\alpha \in N \setminus P \text{ and } (x_\alpha, y_\alpha) \in \overline{\tau} \text{ or;} \\
  x_\alpha, y_\alpha \in P \setminus \{0\} \text{ and } (\phi(x_\alpha), \phi(y_\alpha)) \in \sigma \text{ or;} \\
  x_\alpha \in P \setminus \{0\}, \ y_\alpha \in G \text{ and } (\phi(x_\alpha), y_\alpha) \in \overline{\tau}
\end{cases}
\]

Hence, we may assume that one of the above holds for each $\alpha$. Suppose now that $x_\alpha, y_\alpha \in f^\ast P \setminus \{0\}$ and $(\phi(x_\alpha), \phi(y_\alpha)) \in f^\ast \sigma$. Then $x_\alpha, y_\alpha \in f^\ast P \setminus \{0\} \subseteq T$. Thus, $(x_\alpha, y_\alpha) \in \overline{\nu} \cap (T \times T) = \nu$. Therefore, since $(x_\alpha, y_\alpha) \xrightarrow{\zeta} (x, y)$, some subnet of \{(x_\alpha, y_\alpha)\} contained in $\nu$ converges to $(x, y)$. Hence, $(x, y) \in \nu \subseteq \overline{\nu}$ as $\nu$ is closed.

If it is not the case that $x_\alpha, y_\alpha \in f^\ast P \setminus \{0\}$ and $(\phi(x_\alpha), \phi(y_\alpha)) \in f^\ast \overline{\sigma}$, then as above we may assume that one of the following holds for each $\alpha$:

\[
\begin{cases}
  x_\alpha, y_\alpha \in N \setminus P \text{ and } (x_\alpha, y_\alpha) \in \overline{\tau} \text{ or;} \\
  x_\alpha \in P \setminus \{0\}, \ y_\alpha \in G \text{ and } (\phi(x_\alpha), y_\alpha) \in \overline{\sigma}
\end{cases}
\]

Suppose that $x_\alpha \in f^\ast P \setminus \{0\}$, $y_\alpha \in f^\ast G$ and $(\phi(x_\alpha), y_\alpha) \in f^\ast \overline{\sigma}$. Now $x_\alpha \in f^\ast P \setminus \{0\} \subseteq T$ so $\phi(x_\alpha) \in f^\ast T \cap G$. Also, $(\phi(x_\alpha), y_\alpha) \in f^\ast \overline{\sigma}$ so $y_\alpha(\phi(x_\alpha))^{-1} \in f^\ast N \subseteq T \cap G$. Thus, $y_\alpha = (y_\alpha(\phi(x_\alpha))^{-1})(\phi(x_\alpha) \in f^\ast (T \cap G)(T \cap G) \subseteq T \cap G \subseteq T$. Hence, $(x_\alpha, y_\alpha) \in f^\ast T$ and $(x_\alpha, y_\alpha) \in \overline{\nu} \cap (T \times T) = \nu$. As above, we conclude $(x, y) \in \nu \subseteq \overline{\nu}$. Finally, if it is not the case that $(x, y) \in \nu \subseteq \overline{\nu}$, then we may assume that $x_\alpha, y_\alpha \in N \setminus P$ and $(x_\alpha, y_\alpha) \in \overline{\tau} = \tau \cup \Delta_{N/P}$ for each $\alpha$. If $(x_\alpha, y_\alpha) \in f^\ast \tau$, then $(x_\alpha, y_\alpha) \in f^\ast \nu$ and as reasoned above, $(x, y) \in \nu \subseteq \overline{\nu}$. Otherwise, we may assume that $(x_\alpha, y_\alpha) \in \Delta_{N/P}$ for each $\alpha$ and by our assumption that $x_\alpha, y_\alpha \in N \setminus P$ for each $\alpha$, we conclude that $(x_\alpha, y_\alpha) \in \Delta_{S}$ for each $\alpha$. Hence, $(x, y) \in \Delta_{S} \subseteq \overline{\nu}$. This completes the proof that $\overline{\nu}$ is closed. 

As in the case of IEP, the fact that the congruence which gives the decomposition of a commutative semigroup into components is not closed prevents direct
topologization of results concerning the interaction of the archimedean components of a compact commutative semigroup in the presence of CEP.

We now consider the continuous homomorphic image of a compact semigroup with CEP. In [Stralka, 1977], an example is given which shows that the homomorphic image of a compact semigroup with CEP need not have CEP. Stralka defines $S$ to be $(C \times I) \cup (I \times \{0\})$ where $C$ is the standard Cantor chain and $\ker(\Psi)$ to be the congruence generated by the equivalences:

\[(0,0) \sim (1,0), (0,1/2) \sim (1/3,1/2),\]
\[(2/3,1/2) \sim (1,1/2), (0,3/4) \sim (1/9,3/4), \text{ etc.}\]

Then $\Psi(S)$ is a Cantor fan with endpoints $E(\Psi(S))$. If this image space had CEP, then the standard "gap closer" congruence on $E(\Psi(S))$ would have to extend to $\Psi(S)$ which would force the resulting image space to be two-dimensional, but this is not possible according to [Lawson, 1972]. In light of this example, it is interesting to examine certain special cases and conditions under which the continuous homomorphic image of a compact semigroup with CEP retains CEP. We begin by recalling the following definitions from Chapter 4. Let $\phi: S \rightarrow X$ be a homomorphism from a semigroup $S$ onto a semigroup $X$. Let $\alpha$ be a congruence on $X$ and $\rho$ a congruence on $S$. Then

**pullback of** $\alpha := \{(x,y) \in S \times S: (\phi(x),\phi(y)) \in \alpha\}$

and

**pushout of** $\rho := \{((\phi(x),\phi(y)) \in X \times X: (x,y) \in \rho\}$.

We now present topological analogues of four basic lemmas from Chapter 4. Net arguments are provided when needed.
6.27 Lemma. Let \( \phi : S \to X \) be a continuous homomorphism from a compact semigroup \( S \) onto a semigroup \( X \). Let \( \alpha \) be a closed congruence on \( X \). Let \( \rho \) be the pullback of \( \alpha \) to \( S \). Then \( \rho \) is a closed congruence on \( S \).

Proof. Applying Lemma 4.1, we need only show that \( \rho \) is closed. For this purpose, let \( \{(x_\beta, y_\beta)\} \) be a net contained in \( \rho \) such that \( (x_\beta, y_\beta) \xrightarrow{\phi} (x, y) \). We must show that \( (x, y) \in \rho \). For each \( \beta \), \( (x_\beta, y_\beta) \in \rho \) so \( (\phi(x_\beta), \phi(y_\beta)) \in \alpha \). By continuity of \( \phi \), \( (\phi(x_\beta), \phi(y_\beta)) \xrightarrow{\phi} (\phi(x), \phi(y)) \). Thus, since \( \alpha \) is closed, \( (\phi(x), \phi(y)) \in \alpha \). Hence, \( (x, y) \in \rho \).

6.28 Lemma. Let \( \phi : S \to X \) be a continuous homomorphism from a compact semigroup \( S \) onto a semigroup \( X \). Let \( \rho \) be a closed congruence on \( S \). Let \( \alpha \) be the pushout of \( \rho \). Then \( \alpha \) is a reflexive, symmetric, compatible, closed relation on \( X \).

Proof. Applying Lemma 4.2, we need only show that \( \alpha \) is closed. For this purpose, let \( \{(x_\beta, y_\beta)\} \) be a net contained in \( \alpha \) such that \( (x_\beta, y_\beta) \xrightarrow{\phi} (x, y) \). We must show that \( (x, y) \in \alpha \). For each \( \beta \), \( (x_\beta, y_\beta) \in \alpha \) so there exists \( (a_\beta, b_\beta) \in \rho \) such that \( (\phi(a_\beta), \phi(b_\beta)) = (x_\beta, y_\beta) \). Now \( \rho \) is a closed relation contained in the compact set \( S \times S \) so \( \rho \) is compact. Hence, \( (a_\beta, b_\beta) \xrightarrow{\phi} (a, b) \) for some \( (a, b) \in \rho \). By continuity of \( \phi \), \( (\phi(a_\beta), \phi(b_\beta)) \xrightarrow{\phi} (\phi(a), \phi(b)) \). However, \( (\phi(a_\beta), \phi(b_\beta)) = (x_\beta, y_\beta) \) and \( (x_\beta, y_\beta) \xrightarrow{\phi} (x, y) \). Thus, \( (\phi(a), \phi(b)) = (x, y) \). Hence, \( (x, y) \in \alpha \) as \( (a, b) \in \rho \).

6.29 Lemma. Let \( \phi : S \to X \) be a continuous homomorphism from a compact semigroup \( S \) onto a semigroup \( X \). Let \( \rho \) be a closed congruence on \( S \) such that \( \ker \phi \subseteq \rho \). Let \( \alpha \) be the pushout of \( \rho \) to \( X \). Then \( \alpha \) is a closed congruence on \( X \). Moreover, the pullback of \( \alpha \) is \( \rho \).
6.30 Lemma. Let $\phi: S \to X$ be a continuous homomorphism from a topological semigroup $S$ onto a topological semigroup $X$. Let $Y$ be a closed subsemigroup of $X$ and let $T = \phi^{-1}[Y]$. Let $\alpha$ be a closed congruence on $Y$ and let $\gamma$ be a closed congruence on $X$. Let $\rho$ be the pullback of $\alpha$ and let $\tau$ be the pullback of $\gamma$. Then $\gamma$ extends $\alpha$ if and only if $\tau$ extends $\rho$.

The double-extension technique used to prove that the Rees quotients retain CEP in the purely algebraic setting lends itself well to topologization. Thus, we obtain the following theorem.

6.31 Theorem. Let $S$ be a compact semigroup with the congruence extension property (CEP) and let $I$ be and ideal of $S$. Then $S/I$ has CEP.

Proof. Let $\phi: S \to S/I$ be the natural map. Then $\ker \phi = (I \times I) \cup \Delta_S$ is a closed congruence on $S$ and $\phi(I) = 0 \in S/I$. Let $Y$ be a closed subsemigroup of $S/I$ and let $T = \phi^{-1}(Y)$. Let $\alpha$ be a closed congruence on $Y$ and let $\rho$ be the pullback of $\alpha$ to $T$. By Lemma 6.27, $\rho$ is a closed congruence on $T$.

As shown in the proof of Proposition 4.5, $I \cup T$ is a subsemigroup of $S$ and $\rho \cup (I \times I)$ is a congruence on $I \cup T$. Furthermore, since $I$, $T$, and $\rho$ are closed, $I \cup T$ is a closed subsemigroup of $S$ and $\rho \cup (I \times I)$ is a closed congruence on $I \cup T$.

There is a closed congruence $\sigma$ on $S$ such that $\sigma \cap ((I \cup T) \times (I \cup T)) = \rho \cup (I \times I)$ as $S$ has CEP and by the argument given in 4.5, $\sigma$ extends $\rho$ from $T$ to $S$.

Let $\gamma$ be the pushout of $\sigma$. Note that $\ker \phi = (I \times I) \cup \Delta_S \subseteq \sigma$. Thus, by Lemma 6.29, $\gamma$ is a closed congruence on $S/I$ and the pullback of $\gamma$ is $\sigma$. We have that the pullback of $\gamma = \sigma$ extends the pullback of $\alpha = \rho$ from above. Thus, applying Lemma 6.30, $\gamma$ extends $\alpha$ and $S/I$ has CEP. $\blacksquare$
6.32 Corollary. Let $S$ be an ideal compact semigroup with the congruence extension property (CEP). Then any continuous homomorphic image of $S$ has CEP.

Proof. This is immediate from Theorem 6.31. 

An attempt to directly topologize Lemma 4.7 fails as its proof is heavily dependent on the algebraic characterization of the congruence generated by a relation. However, using alternative techniques, we obtain the following analogue of 4.8.

6.33 Lemma. Let $\phi : S \to X$ be a continuous homomorphism from a compact semigroup $S$ onto a semigroup $X$. Let $Y$ be a closed subsemigroup of $X$ and let $\alpha$ be a closed congruence on $Y$. Let $\rho$ be the pullback of $\alpha$ to $T = \phi^{-1}(Y)$. Then $\alpha$ extends to a closed congruence on $X$ if and only if $\ker \phi \vee \rho^*$ is an extension of $\rho$.

Proof. Suppose that $\alpha$ extends to a closed congruence $\gamma$ on $X$. Let $\delta$ be the pullback of $\gamma$ to $S$. It is clear that $\delta$ must contain $\rho$. Also, according to Lemma 6.27, $\delta$ is a closed congruence on $S$. Thus, we obtain that $\delta$ must contain $\rho^*$. Also, since $\Delta_X \subseteq \gamma$, $\delta$ contains $\ker \phi$. Hence, $\delta$ must contain $\ker \phi \vee \rho^*$. By Lemma 6.30, $\delta$ (the pullback of $\gamma$) extends $\rho$ (the pullback of $\alpha$). Thus, $\delta \cap (T \times T) = \rho$. Hence, $(\ker \phi \vee \rho^*) \cap (T \times T) \subseteq \delta \cap (T \times T) = \rho$. Therefore, $(\ker \phi \vee \rho^*) \cap (T \times T) = \rho$ and $\ker \phi \vee \rho^*$ is an extension of $\rho$.

Conversely, suppose that $\ker \phi \vee \rho^*$ is an extension of $\rho$. Let $\gamma$ be the pushout of $\ker \phi \vee \rho^*$. By Lemma 6.29, $\gamma$ is a closed congruence on $X$ and the pullback of $\gamma$ is $\ker \phi \vee \rho^*$. Thus, the pullback of $\gamma$ $(\ker \phi \vee \rho^*)$ extends the pullback of $\alpha$ ($\rho$) so by Lemma 6.30, $\gamma$ is a closed congruence extending $\alpha$. 

Applying the lemma above and an argument similar to that given in the proof of 4.9, we obtain the following theorem.
6.34 Theorem. Let \( \phi : S \to X \) be a continuous homomorphism from a compact semigroup \( S \) onto a semigroup \( X \). Then \( X \) has the congruence extension property (CEP) if and only if for each closed subsemigroup \( T \) of \( S \) which is \( \phi \)-saturated and each closed congruence \( \rho \) on \( T \) which contains \( \ker \phi|_T \), \( \ker \phi \lor \rho^* \) extends \( \rho \).

Thus, we have conditions under which the continuous homomorphic image of a compact semigroup has CEP. In the example given by Stralka which was cited previously, \( \ker \Psi \lor \rho^* \) does not extend \( \rho \) where \( \rho \) is the pullback of the standard "gap closer" and \( \Psi(S) \) does not have CEP.

We have the following three direct analogues of propositions 4.16, 4.17, and 4.18. Proofs in the case of compact semigroups hold as compactness insures that the appropriate congruences are closed. Other results listed in Chapter 4 fail to topologize directly again due to the difference in the characterization of the closed congruence generated by a relation and the congruence generated algebraically by a relation.

6.35 Proposition. Let \( \phi : S \to X \) be a continuous homomorphism from a compact semigroup \( S \) which has the congruence extension property (CEP) onto a semigroup \( X \). Let \( T \) be a closed subsemigroup of \( S \) and let \( \rho \) be a closed congruence on \( T \) containing \( \ker \phi|_T \). If \( \ker \phi \lor \rho^* \) is a congruence on \( S \), then \( \ker \phi \lor \rho^* = \ker \phi \lor \rho^* \) is a closed congruence extending \( \rho \).

6.36 Corollary. Let \( \phi : S \to X \) be a continuous homomorphism from a compact semigroup \( S \) which has the congruence extension property (CEP) onto a semigroup \( X \). If for each closed subsemigroup \( T \) of \( S \) which is \( \phi \)-saturated and each
closed congruence $\rho$ on $T$ which contains $\ker \phi|_T$, $\ker \phi \cup \rho^*$ is a closed congruence on $S$, then $X$ has CEP.

6.37 Corollary. Let $S$ be a compact $\Delta$-semigroup with the congruence extension property (CEP). Then any continuous homomorphic image of $S$ has CEP.

Having considered IEP and CEP in a topological setting, we close this chapter with a short discussion of topological ideal semigroups. Topologically, an ideal semigroup is a semigroup in which each closed congruence has form $(I \times I) \cup \Delta$ where $I$ is a closed ideal of $S$. Very few of the results from Chapter 5 on the structure of commutative ideal semigroups have direct topological analogues. Primarily, this is due to the use of the algebraic characterization of the congruence generated by a relation in proofs of these results and the fact that the semilattice decomposition of a commutative semigroup does not necessarily decompose the semigroup into closed subsemigroups. Examples of this will be provided in the discussion of intervals below.

Some of the results from Chapter 5 which do topologize are as follows. A topological ideal semigroup does have a zero as the diagonal congruence is a closed congruence (recall that topological semigroups are assumed to be Hausdorff) and this forces some point to be an ideal (hence, a zero). Also, a compact ideal semigroup is weakly reductive away from zero and a compact commutative ideal semigroup has nonzero cancellation as the congruences used in proving these results in Chapter 5 are necessarily closed congruences. Finally, one can show that a compact commutative ideal semigroup is $\mathcal{H}$-trivial using the fact that $\mathcal{H}$ is a closed congruence on compact commutative semigroups.
To illustrate the differences between ideal semigroups in the algebraic and topological settings, we consider the usual, nilpotent, and min intervals. According to results on $I$-semigroups found in [Carruth, Hildebrant, and Koch, 1983], the usual and nilpotent intervals are ideal semigroups when considered topologically. The archimedean components of the nilpotent interval are $[0,1)$ and 1 and from the algebraic characterization of commutative ideal semigroups given in Chapter 5, one can see that the nilpotent interval is also an ideal semigroup when considered algebraically. However, the archimedean components of the usual interval are 0, (0, 1), and 1 and these give a class listing for a non-ideal congruence in the purely algebraic setting. Thus, the usual interval is an ideal semigroup when considered topologically, but not algebraically. The min interval is neither an algebraic nor a topological ideal semigroup. In particular, the congruence $([1/3, 2/3] \times [1/3, 2/3]) \cup \Delta$ on the min interval is not determined by an ideal.
CHAPTER 7

SUMMARY AND OPEN QUESTIONS

Two of the main results presented in the preceding chapters provide characterizations of commutative ideal semigroups and commutative semigroups with the ideal extension property (IEP). A third result partially describes the structure of commutative semigroups with the congruence extension property (CEP).

Commutative semigroups with IEP are characterized as commutative semigroups having the following structure:

\[ S = \bigcup_{e \in E_S} C(e) \]

where

1. For each \( e \in E_S \), \( G_e \) is torsion and \( xy = x^2 = y^2 \) for all \( x, y \in N_e \setminus \{0_e\} \) with \( xy \neq 0_e \) and;

2. If \( e \) and \( f \) are distinct idempotents, \( x \in C(e) \), and \( y \in C(f) \), then either \( xy \in M(C(ef)) \), \( xy = x \), or \( xy = y \).

Here \( C(e) \) denotes the archimedean component with idempotent \( e \) which is an ideal extension of the group \( G_e \) by the nilpotent semigroup \( N_e \). Condition (1) gives the conditions under which a given component \( C(e) \) has IEP and condition (2) describes how multiplication between components must behave in the presence of IEP.

The primary motivation for obtaining the characterization of commutative semigroups with IEP stated above was to begin to determine the structure of commutative semigroups with CEP as it is known that CEP implies IEP in commutative
semigroups. If $S$ is a commutative semigroups with CEP, then

$$S = \bigcup_{e \in E_S} C(e)$$

and the following hold:

1. For each $e \in E_S$, $G_e$ is torsion, $xy = x^2 = y^2$ for all $x, y \in N_e \setminus \{0_e\}$ with $xy \neq 0_e$, and if $x, y, z \in N_e \setminus \{0_e\}$ such that $xy = xz = z^2 = x^2 = y^2 \neq 0_e$, then $z^2 = yz$.

2. For $e, f \in E_S$ such that $e \parallel f$, $C(f)C(e) \subseteq M(C(ef))$. If $e \leq f$, then either
   (i) $C(f)C(e) \subseteq M(C(e))$ or
   (ii) $C(f)x = x$ for all $x \in C(e) \setminus M(C(e))$ and $C(f)M(C(e)) \subseteq M(C(e))$.

Condition (1) characterizes when a given component $C(e)$ has CEP. Note that the conditions given are stronger than those given in the case of IEP. Condition (2) provides information on interaction of components in the presence of CEP. In some sense, components can either act on each other as identities or all products must lie in the minimal ideal of the product component. This uniformity of action stated in condition (2) is again stronger than condition (2) of the structure theorem for IEP. However, there is more that can be said about interaction of components in the presence of CEP. We have the following additional conditions on multiplication between finite components of a commutative semigroup with CEP based on the ordering of idempotents.

If $C(e)$ is finite for each $e \in E_S$, then the following hold:

3. For $e_1, e_2, e_3 \in E_S$ such that $e_1 < e_2 < e_3$, one of the following holds:
   (i) $C(e_3)x = x$ for all $x \in C(e_1) \setminus M(C(e_1))$ and $C(e_3)x = x$ for all $x \in C(e_2) \setminus M(C(e_2))$. 
(ii) \( C(e_3)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \), \( C(e_3)C(e_2) \subseteq M(C(e_2)) \), and \( C(e_2)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \).

(iii) \( C(e_3)C(e_1) \subseteq M(C(e_1)) \) and \( C(e_3)C(e_2) \subseteq M(C(e_2)) \).

(iv) \( C(e_3)C(e_1) \subseteq M(C(e_1)) \), \( C(e_3)x = x \) for all \( x \in C(e_2) \setminus M(C(e_2)) \), and \( C(e_2)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \).

(4) For \( e_i \in E_S \) (1 \( \leq i \leq 4 \)) such that \( e_2 \not\leq e_3 \), \( e_3 \not\leq e_2 \), \( e_1 \leq e_2e_3 \), \( e_4 < e_2 \), \( e_4 \not\leq e_3 \), \( C(e_2) \setminus M(C(e_2)) \neq \emptyset \), \( C(e_3) \setminus M(C(e_3)) \neq \emptyset \), and \( C(e_2e_3) \setminus M(C(e_2e_3)) \neq \emptyset \), the following hold:

(i) Either \( C(e_3)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \) and \( C(e_2)x = x \) for all \( x \in C(e_1) \setminus M(C(e_1)) \) or \( C(e_3)C(e_1) \subseteq M(C(e_1)) \) and \( C(e_2)C(e_1) \subseteq M(C(e_1)) \).

(ii) \( C(e_2)C(e_4) \subseteq M(C(e_4)) \).

Condition (3) essentially states that for idempotents \( e_1 < e_2 < e_3 \), if \( C(e_3) \) does not act in the same way on \( C(e_2) \) and \( C(e_1) \), then \( C(e_2) \) must act as an identity on the non-group part of \( C(e_1) \). Condition (4) gives conditions on the interaction between components whose idempotents are not related and a third component lying below one or both of the non-related components. Using these conditions, one can determine the possibilities for multiplication among any three finite components of a commutative semigroup with CEP based on the ordering of idempotents lying in these components.

The third main structure theorem presented in this work characterizes commutative ideal semigroups. That is, commutative semigroups in which each congruence is determined by an ideal are characterized as commutative semigroups of form \( S = C(0) \cup E \setminus \{0\} \) which are \( \mathcal{H} \)-fans in which each \( e \in E \setminus \{0\} \) is an \( \mathcal{H} \)-maximal.
element and at most one maximal $\mathcal{H}$-chain is not bounded above by an idempotent. Pictorially, this is represented as:

Combining the structure theorems above, characterizations of commutative semigroups with any two of the three properties described above are obtained. In particular, commutative ideal semigroups with IEP are commutative semigroups of form $S = C(0) \cup E \setminus \{0\}$ which are $\mathcal{H}$-fans such that $\text{height}(x) \leq 2$ for all $x \in C(0)$, each $e \in E \setminus \{0\}$ is an $\mathcal{H}$-maximal element, and at most one $\mathcal{H}$-maximal element of $S$ is not an idempotent as illustrated in the following diagram:

Commutative ideal semigroups with CEP are commutative $\mathcal{H}$-fans with form $S = C(0) \cup E \setminus \{0\}$ such that $C(0)$ is an $\mathcal{H}$-chain with $\text{length}(C(0)) \leq 2$, each nonzero idempotent is an $\mathcal{H}$-maximal element, and at most one $\mathcal{H}$-maximal element of $S$ is
not an idempotent as illustrated in the following diagram:

An unexpected corollary to the results listed above is that commutative semi­
groups with CEP or IEP whose congruences form a chain are characterized as
semigroups with one of the following forms:

(a) A quasicyclic group.

(b) A quasicyclic group with zero adjoined.

(c) The trivial semigroup, the zero semigroup of order two, or the cyclic semi­
group of order three with idempotent zero.

(d) One of the semigroups listed in (c) with an identity adjoined.

In addition to the results concerning structure discussed above, this work also
explores the homomorphic image of semigroups with CEP. Two of the main results
obtained here concerning the question of whether CEP is preserved by homomor­
phisms are that CEP is preserved by Rees quotients and that the homomorphic
image of an archimedean semigroup with CEP has CEP. The second result has im­
lications in considering the homomorphism question for commutative semigroups
as it shows that the image of any component of a commutative semigroup which
has CEP retains CEP. It is shown that the homomorphic image \( X = \phi(S) \) has
CEP provided that for each subsemigroup \( T \) of \( S \) which is \( \phi \)-saturated and each
congruence \( \rho \) on \( T \) which contains \( \ker \phi|_T \), \( \ker \phi \vee \langle \rho \rangle_S \) extends \( \rho \). Using this result, an exhaustive computer search was conducted through semigroups of order seven which showed that CEP is preserved by homomorphisms in these semigroups with small orders.

Finally, the main results which are obtained in the topological setting are those concerning monothetic index and the structure of compact commutative unipotent semigroups with IEP and CEP respectively which arise as analogues to algebraic results which characterize archimedean semigroups with these properties. Results from [Petrich, 1967] concerning congruences on ideal extensions are topologized in the special case under consideration to obtain the characterization of compact commutative unipotent semigroups with CEP. Compact Rees quotients are shown to preserve CEP and an analogue of the result above concerning conditions under which homomorphisms preserve CEP is proven. However, an example due to Stralka is provided which shows that continuous homomorphisms do not preserve CEP in compact semigroups in general.

**OPEN QUESTIONS**

1. The main question which arises in this work is that of how one might complete the characterization of commutative semigroups with CEP. It is conjectured that a sufficient list of conditions on the interaction of archimedean components of a commutative semigroup may be obtained to insure the presence of CEP. Combining this with the characterization of archimedean semigroups
given here, a complete characterization of commutative semigroups with CEP would be established.

(2) The question of whether CEP is preserved by homomorphisms remains open. The conditions established in Chapter 4 which are necessary and sufficient for a given image to have CEP might be employed in answering this question. In addition, since it was shown that the homomorphic image of an archimedean semigroup with CEP retains CEP, it seems likely that the homomorphism question in the case of commutative semigroups could be answered once the characterization discussed in (1) is established by either proving or disproving that the conditions on interaction of components are preserved by homomorphisms.

(3) Complete characterizations compact ideal semigroups and of compact semigroups with CEP and IEP respectively have not been found. Characterizations of compact unipotent commutative semigroups with CEP and IEP given in this work may shed some light on seeking characterizations of compact commutative semigroups with CEP and IEP.

(4) Characterizations of noncommutative ideal semigroups and of noncommutative semigroups with CEP and IEP have not been found. In each of these cases one might consider the semilattice decomposition of general semigroups as this approach proved effective in the commutative case. There is greater difficulty in using this approach in the noncommutative case as the components are not as easy to determine and do not possess many of the useful properties which are present in archimedean semigroups.

(5) Finally, one might wish to consider the structure of semigroups in which each congruence (possibly) excluding the diagonal is determined by an ideal. We
use the term nearly ideal semigroup for such semigroups. In a nearly ideal semigroup, one might find a group replacing the zero which is always present in ideal semigroups due to the fact that the diagonal must be an ideal congruence. Allowing for nontrivial groups in this way would produce a class of semigroups which is larger than the class of ideal semigroups, warranting its own investigation while retaining certain characteristics found to hold in ideal semigroups.
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