Banach Space Valued Stochastic Differential Equations.

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Banach space valued stochastic differential equations

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STOCHASTIC DIFFERENTIAL EQUATIONS

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Let $W(t,\omega)$ be a Brownian motion on an abstract Wiener space $(i,H,B)$ corresponding to the canonical normal distribution on $H$. The well known theorem of Girsanov is proved for such a process with the perturbing term taking values in the Hilbert space $H$. Consider the stochastic integral equation $\xi(t) = x + \int_0^t A(s,\xi(s))dW(s) + \int_0^t \sigma(s,\xi(s))ds$ where $x$ is in $B$, $A(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are bounded continuous functions from $[0,\infty) \times B$ to $I + L_2(H)$ and $H$ respectively. Here $L_2(H)$ denotes the collection of Hilbert Schmidt operators on $H$. Furthermore, suppose for every $s \geq 0$, the restriction of $A(s,\cdot)$ to $H$ is invertible and $A(s,\cdot)$ and $\sigma(s,\cdot)$ are both Frechet differentiable in the directions of $H$ with bounded derivatives. Under suitable conditions, it is proved that for each $t \geq 0$, the measure generated by the solution $\xi(t)$ of the above stochastic integral equation is differentiable in the directions of $H$ in the sense of Fomin. By adding more conditions on $A$ and $\sigma$, it is shown that the transition probability associated with the solution of the stochastic differential equation $d\xi(t) = A(t,\xi(t))dW(t) + \sigma(t,\xi(t))dt$ satisfies the infinite dimensional Kolmogorov's forward equation in the distribution sense.
Introduction

Calculus of differentiable measures was introduced by Fomin in 1968 with the purpose of extending the theory of generalized functions to infinite dimensional spaces. The main motivation was to use this theory to advance in the field of partial differential equations on infinite dimensional vector spaces, despite the difficulties that arise partly from the fact that Lebesgue measure does not exist in such spaces. In this area, smooth measures (defined in section 2 of chapter 1) are used as elements of the space of test functions as well as solutions of the equations corresponding to infinite dimensional differential and pseudo-differential operators. In recent years this theory has found applications in other areas of analysis including functional analysis (geometry of Banach spaces) and the theory of stochastic processes. However, apart from its applications, this theory seems to be very interesting on its own.

In a finite dimensional space, every bounded $\sigma$-additive measure has a (possibly generalized) density with respect to Lebesgue measure. In such spaces, existence of this fundamental measure (Lebesgue measure) and abundance of smooth functions makes it possible, among other things, to express certain nice distributions as smooth functions. Hence, absence of Lebesgue measure and a large enough collection of smooth functions to work with are among the difficulties that one encounters trying to develop a distribution theory on infinite dimensional vector spaces. In other words, in contrast with the finite dimensional case, some distributions (set functions) in infinite dimensional spaces, can not be represented by point functions. Hence, in infinite dimensional spaces, set functions and point functions have to be studied in parallel rather than in connection with each other. But there is a way to get around this difficulty. Consider for example for a Borel measure $\mu$ on $\mathbb{R}$, the property of having an integrable density $\frac{d\mu}{dx}$ with respect to the Lebesgue
measure. By a theorem of Saks, this property is equivalent to the continuity of the function \( t \rightarrow \mu(A + t) \) for every Borel set \( A \). A property that can be generalized to infinite dimensional spaces. Similarly, it can be shown that \( \frac{d\mu}{dt} \) is a smooth function with integrable derivatives if and only if the map \( t \rightarrow \mu(A + t) \) is smooth for each Borel set \( A \). One can see that although Lebesgue measure does not exist in infinite dimensional spaces, a notion of differentiability can be defined directly on measures, thereby making it possible to extend many results of the theory of finite dimensional differential calculus to infinite dimensional spaces. This becomes important for example, in the development of distribution theory in Banach spaces. As is pointed out in [K5], constructing the theory of differentiable measures in Banach spaces, makes it possible to represent by smooth measures, certain nice distributions that can not be represented by smooth functions.

To illustrate the usefulness of the theory of differentiable measures, we consider another example. Let \((i, H, B)\) be an abstract Wiener space. Consider the stochastic differential equation

\[
d\xi(t) = A(t, \xi(t))dW(t) + \sigma(t, \xi(t))dt.
\]

It is well known that if \( B \) is finite dimensional and the coefficients \( A \) and \( \sigma \) satisfy the requirements of the existence and uniqueness theorem and are smooth in the second variable with bounded derivatives, then the transition probability (at each time \( t \)) associated with the solution of this equation has a smooth density with respect to the Lebesgue measure. Furthermore, this density satisfies the Kolmogorov forward equation. Our main purpose in this dissertation is to prove smoothness of measures induced by the solution \( \xi \) of the above stochastic differential equation, when \( B \) is an infinite dimensional Banach space and the coefficients \( A(t, \cdot) \) and \( \sigma(t, \cdot) \) are smooth in the directions of \( H \) and to show that these measures satisfy the Kolmogorov forward equation in distribution sense. For this purpose the necessary theorems are proved in chapters 1 and 2. In chapter 3, we prove differentiability of measures.
induced by certain types of stochastic differential equations. The main tool used in chapter 3 to prove differentiability of measures is the well known Cameron Martin Girsanov theorem whose proof is given in chapter 2.

This work is an extension (to infinite dimensional spaces) of the results obtained by Bichteler and Fonken [BF]. They used Girsanov theorem to prove absolute continuity (with respect to the Lebesgue measure) of measures induced by solutions of stochastic differential equations in finite dimensional spaces.
Chapter 1  Preliminaries

§1-Abstract Wiener space

Let $H$ be a real separable Hilbert space with norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$. Let the partially ordered collection of finite dimensional projections be denoted by $\mathcal{F}$. For each $t > 0$, we define the finitely additive Gauss measure $\mu_t$ with variance parameter $t$ on the field of cylinder subsets of $H$ by

$$\mu_t\{x \in H; Px \in E\} = \frac{1}{\sqrt{2\pi t}^\dim(PH)} \int_E e^{-|x|^2/2t}dx,$$

where $P \in \mathcal{F}$, $dx$ is the Lebesgue measure on $P(H)$ and $E$ is a Borel set in $P(H)$. If $H$ is infinite dimensional, the topology induced by the inner product of $H$ has so many open sets that prevents $\mu$ from being $\sigma$-additive on the algebra of cylinder subsets of $H$. Hence $\mu$ does not have a $\sigma$-additive extension to the Borel field of $H$. In other words $H$ is too small to support $\mu$. We get a $\sigma$-additive extension for $\mu$ by completing $H$ with respect to a norm strictly weaker than that of $H$, thereby enlarging the space $H$ (and reducing the number of open sets). This is done in the following way.

We call a norm $\| \cdot \|$ on $H$ a measurable norm, if for each $\epsilon > 0$, $\exists P_\epsilon \in \mathcal{F}$ such that $\mu_1\{x \in H; \| Px \| > \epsilon\} < \epsilon \forall P \perp P_\epsilon, P \in \mathcal{F}$. It can be shown that if $H$ is infinite dimensional, $|\cdot|$ is not a measurable norm. Now given a measurable norm $\| \cdot \|$, we can find a positive number $\epsilon$ such that $\| x \| < \epsilon |x|$ for all $x \in H$. $H$ is not complete with respect to $\| \cdot \|$ unless $H$ is finite dimensional (Completeness of $H$ with respect to $\| \cdot \|$ would imply the measurability of $|\cdot|$ which is not true if $H$ is infinite dimensional). Let $B$ be the separable Banach space obtained by completion of $H$ with respect to $\| \cdot \|$. Regarding each element in $B^*$ as an element of $H^* \equiv H$ by restriction, $B^*$ can be embedded in $H$ and this gives rise to the
following situation:

\[ B^* \subset H^* \equiv H \subset B. \]

For each \( x \in B^* \), \( |x| \leq c \|x\|_{B^*} \) and for each \( x \in H \), \( \|x\|_H \leq |x| \). \( B^* \) is dense in \( H \) and \( H \) is dense in \( B \). If \( x \in B^* \) and \( h \in H \), then \( (x, h) = (x, h) \) where \((\cdot, \cdot)\) is the natural pairing of elements of \( B^* \) and \( B \). The triple \((i, H, B)\), where \( i : H \to B \) is the continuous inclusion map is called an abstract Wiener space. A cylinder set in \( B \) is a set of the form \( \{x \in B; ((y_1, x), \ldots, (y_n, x)) \in E\} \), where \( y_1, \ldots, y_n \in B^* \) and \( E \in B \left(R^n\right) \). It can be shown that the \( \sigma \)-algebra generated by the cylinder sets of \( B \) is the Borel field of \( B \).

We now define the measure \( \tilde{\mu}_t \) on cylinder subsets of \( B \) in the following way.

\[ \tilde{\mu}_t\{x \in B; ((y_1, x), \ldots, (y_n, X)) \in E\} = \mu_t\{x \in H; ((y_1, x), \ldots, (y_n, x)) \in E\} \]

where \( \tilde{\mu}_t \) is the Gauss measure with variance parameter \( t \) on the field of cylinder subsets of \( H \). Obviously \( \tilde{\mu}_t(B) = 1 \).

**THEOREM 1.1.1 [Gross]**: \( \tilde{\mu}_t \) has a \( \sigma \)-additive extension to the Borel field of \( B \).

The \( \sigma \)-additive extension of \( \tilde{\mu}_t \) is denoted by \( \tilde{p}_t \) and is called the Wiener measure with variance \( t \). The measurability property of the norm \( \|\cdot\| \) is essential for a successful \( \sigma \)-additive extension of \( \mu_t \). In other words, if \( B \) is a Banach space obtained by completion of \( H \) with respect to a norm \( \|\cdot\| \) strictly weaker than \( |\cdot| \), the cylinder measure \( \mu_t \circ i^{-1} \) has a \( \sigma \)-additive extension to the Borel field of \( B \) if and only if \( \|\cdot\| \) is a measurable norm on \( H \).

**Example 1**: Let \( C[0, 1] \) denote the set of real valued continuous functions \( f \) in the unit interval \([0, 1]\) that vanish at 0, supplied with the sup norm \( \|f\| = \sup_{0 \leq x \leq 1} |f(x)| \). Let \( C'[0, 1] \) denote the subspace of \( C[0, 1] \) consisting of absolutely continuous functions \( g \) such that \( g' \) is square integrable with respect to the Lebesgue
measure. The space $C'[0,1]$ equipped with the inner product $\langle g, h \rangle = \int_0^1 g' h'$ is a Hilbert space. The triple $(i, C'[0,1], C[0,1])$ is an abstract Wiener space and the extension of the Gauss measure (defined on the cylinder sets of $C'[0,1]$) to the Borel field of $C[0,1]$ is the well known Wiener measure.

**Example 2:** Let $H$ be a separable Hilbert space and let $A$ be an injective Hilbert Schmidt operator on $H$. Define the norm $\| x \| = |Ax|$ for every $x \in H$. It can be shown that $\| \cdot \|$ is a measurable norm on $H$. If $B$ is the completion of $H$ with respect to the norm $\| \cdot \|$, then the triple $(i, H, B)$ is an abstract Wiener space.

In [K1] it is shown that if $B$ is a separable Banach space, then we can always find a dense Hilbert subspace $H$ of $B$ such that the norm of $B$ is a measurable norm on $H$ and $(i, H, B)$ is an abstract Wiener space. It is also proved that if $\mu$ is a nondegenerate Gaussian measure defined on the Borel field of $B$, then there is a dense Hilbert subspace $H$ of $B$ such that $(i, H, B)$ is an abstract Wiener space and the measure $\mu$ is the extension of the Gauss measure on $H$ defined above.

**Example 3:** This example is a generalization of example 1. Let $(i, H, B)$ be an abstract Wiener space. Denote by $C(B)$, the collection of continuous functions $f$ from $[0,1]$ into $B$ with $f(0) = 0$ supplied with the sup norm. Let $C'(H)$ denote the subspace of $C(B)$ consisting of the absolutely continuous functions taking values in $H$ with square integrable derivatives. We define in $C'(H)$ an inner product $\langle f, g \rangle = \int_0^1 \langle f', g' \rangle_H$. Supplied with this inner product, $C'(H)$ is a Hilbert space and the triple $(i, C'(H), C(B))$ is an abstract Wiener space. The collection $C(B)$ defines a Brownian motion process in $B$ whose law is the extension of the Gauss measure in the abstract Wiener space $(i, C'(H), C(B))$.

Below we mention some facts about abstract Wiener spaces which will be used later. All the results mentioned above and throughout this section are discussed and proved in [K1].
For each $x \in B^*$, the random variable $y \mapsto (x,y)$ from $(B,B(B),p_t)$ to $\mathbb{R}$ is normally distributed with mean 0 and variance $t|y|^2$. Now let $h \in H$ and $h_n$ be a sequence in $B^*$ that converges to $h$ in $|\cdot|$ norm. The sequence of random variables $(h_n, \cdot)$ is cauchy in $L^2(p_t)$ and hence converges to a random variable denoted by $(h, \cdot)$ which is normally distributed with mean 0 and variance $t|h|^2$. Obviously $(h, \cdot)$ is defined $p_t$-a.e. 

Let $p_t(x,dy) = p_t(dy-x)$. It can be shown that two measures $p_t(x,dy)$ and $p_s(z,dy)$ are either equivalent or singular. $p_t(x,dy)$ is equivalent to $p_s(z,dy)$ if and only if $s=t$ and $x-z \in H$. In this case

$$
\frac{dp_t(x,dy)}{dp_t(z,dy)} = \exp \left\{ \frac{1}{2t} \left( 2(x-z,y) - |x-z|_H^2 \right) \right\}.
$$

**THEOREM 1.1.2** (Fernique): There exists $\alpha > 0$ such that $\int_B e^{\alpha ||x||^2} p_1(dx) < \infty$.

**THEOREM 1.1.3** (Skorohod): There exists $\beta > 0$ such that $\int_B e^{\beta \|x\|} p_1(dx) < \infty$.

It follows from each of these two theorems that for each $q \geq 1$, the $B$ norm $\| \cdot \|$ is in $L^q(p_t)$ for every $t>0$.

**THEOREM 1.1.4** (Kuo): Let $(i,H,B)$ be an abstract Wiener space. Then there exists another abstract Wiener space $(i_0,H,B_0)$ and an increasing sequence $Q_n \subset F$ converging strongly to identity in $H$ such that (a) $B_0$-norm is stronger than $B$-norm (hence $B_0 \subset B$) (b) each $Q_n$ extends by continuity to a projection still denoted by $Q_n$ of $B_0$ , (c) $Q_n$ converges strongly to the identity in $B_0$ (with respect to $B_0$-norm) (d) $Q_n(B_0) \subset B_0^* \subset H^* \equiv H (e) \|Q_n \|_{B_0,B_0} \leq 1$ . and (f) if $p_t$ is the Borel measure with variance $t$ defined on the Borel field of $B$ , then $p_t(B_0) = 1$.

It follows from this theorem that the Banach space $B_0$ which contains the support of $p_t$, has a schauder basis $\{e_i\}_{i=1}^\infty$ in $B_0^*$, which is also an orthonormal
basis of $H$. Therefore for each $x \in B_0$, $\| \sum_{i=1}^{k} (e_i, x)e_i - x \|_{B_0} \to 0$ as $k \to \infty$.

Since $\| Q_n \|_{B_0, B_0} \leq 1$, it follows from Lebesgue's convergence theorem and the integrability of the norm $\| \cdot \|$ that $\sum_{i=1}^{k} (e_i, x)e_i$ converges to $x$ in $L^q(\hat{p}_t)$ for every $q \geq 1$. Here $\hat{p}_t$ is the Wiener measure on $B_0$.

In what follows, $L_1(H)$ and $L_2(H)$ denote the Banach space of trace class operators of $H$ (with norm $\| \cdot \|_1$) and the Hilbert space of Hilbert Schmidt operators of $H$ (with norm $\| \cdot \|_2$) respectively. If $X$ and $Y$ are Banach spaces, $L(X; Y)$ denotes the Banach space (with norm $\| \cdot \|_{X,Y}$) of bounded linear functions from $X$ to $Y$. If $X$ and $Y$ are Hilbert spaces, $L_2(X; Y)$ denotes the Hilbert space of Hilbert Schmidt operators from $X$ to $Y$.

**THEOREM 1.1.5** (Goodman): If $A \in L(B_0; B_0^*)$ and $\hat{A}$ is the restriction of $A$ to $H$, then $\hat{A} \in L_1(H)$, and

$$|\hat{A}|_1 \leq \left( \int_{B_0} \| x \|^2 \hat{p}_1(dx) \right) |A|_{B_0, B_0^*},$$

where $\hat{p}_1(dx)$ is the Wiener measure with variance 1 obtained by extending the Gauss measure $\mu_t$ to $B(B_0)$.

**THEOREM 1.1.6** (Kuo): Let $K$ be a Hilbert space. If $A \in L(B_0; K)$ then $\hat{A}$ belongs to the space $L_2(H, K)$. Furthermore,

$$|\hat{A}|_2 \leq \left( \int_{B_0} \| x \|^2 \hat{p}_1(dx) \right)^{1/2} |A|_{B_0, K}.$$

**THEOREM 1.1.7**: $L(B_0; K)$ is dense in $L_2(H, K)$. Furthermore if $A \in L(B_0; K)$, then $A^*(K) \subset B_0^*$.

Let $X$ be a Banach space. We say that a function $f : B \to X$ is differentiable in the directions of $H \ (H - C^1)$ if for each $x \in B$, there exists a bounded linear map
$Df(x) \in L(H; X)$ such that $|f(x+h) - f(x) - Df(x)(h)|_X = o(|h|_H)$. Note that this type of differentiability is weaker than Frechet differentiability. We define $n$-times differentiability in the direction of $H$ in the usual way. If $f$ is $n$-times differentiable in the directions of $H$, then for each $x \in B$ and $0 < i \leq n$, $D^i f(x) \in L^i(H; X)$. Although the collection of Frechet differentiable functions may be very restricted, the collection of bounded $H - C^\infty$ real valued functions with bounded derivatives is very large; in fact the intersection of this collection with the Banach space $C_b(B)$ of bounded uniformly continuous functions is a dense subspace of $C_b(B)$.

Abundance of bounded $H - C^\infty$ real valued functions with bounded derivatives leads one to look in this collection for a suitable choice of test functions. A study of distribution theory in Banach spaces was initiated in [K5] where three different classes of test functions were considered. For our purpose it is suitable to take the class $\mathcal{D}$ of test functions to be the collection of bounded $H - C^\infty$ functions such that $D^n f(x) \in L^n_2(H; R)$ $\forall x$, for each $n$, the map $D^n f$ from $B$ into $L^n_2(H; R)$ is bounded and Lip-1, $D^2 f(x) \in L_1(H)$ and the map $D^2 f$ from $B$ into $L_1(H)$ is bounded and Lip-1. We equip $\mathcal{D}$ with the topology according to which a net $\{f_\alpha\} \subset \mathcal{D}$ converges to $f \in \mathcal{D}$ if and only if $D^n f_\alpha$ converges to $D^n f$ pointwise and boundedly in $L^n_2(H; R)$ for all $n$ and $D^2 f_\alpha$ converges to $D^2 f$ pointwise and boundedly in $L_1(H)$. A continuous linear functional on $\mathcal{D}$ is called a distribution. We denote the collection of distributions by $\mathcal{D}'$ and supply this space with the weak topology, thus a net $\phi_\alpha \in \mathcal{D}'$ converges to $\phi \in \mathcal{D}'$ if $\phi_\alpha(f) \to \phi(f)$ for each $f \in \mathcal{D}$.

§2-Differentiable Measures

Throughout, $X$ will be a polish vector space and $\mathcal{M}$ will denote the collection of bounded $\sigma$-additive Borel measures on $X$. The total variation of $\nu \in \mathcal{M}$ will be denoted by $\|\nu\|$ and for each vector $h \in X$, $\nu_h(dx) = \nu(dx + h)$. $C_b(X)$ will denote
the Banach space of bounded uniformly continuous functions from $X$ to $\mathbb{R}$ supplied with the sup-norm $\| \cdot \|_\infty$.

**Theorem 1.2.1 [DS]**: If $\{ \nu_\alpha \} \in \mathcal{M}$ is a net that converges setwise, then the limit defines a set function that is $\sigma$-additive.

**Theorem 1.2.2 [DS]**: Let $\{ \nu_\alpha \} \in \mathcal{M}$ be a setwise bounded net, then $\nu_\alpha$ is uniformly bounded, i.e., $\sup_\alpha \| \nu_\alpha \| < \infty$.

On the set $\mathcal{M}$, we will be concerned with three topologies, $\tau_1$, $\tau_2$ and $\tau_3$. We denote the topology of weak convergence by $\tau_1$. A net $\{ \nu_\alpha \} \subset (\mathcal{M}, \tau_1)$ converges to $\nu \in \mathcal{M}$ if for each $f \in \mathcal{C}_b(X)$, $\int_X f(x) \nu_\alpha(dx) \xrightarrow{\alpha} \int_X f(x) \nu(dx)$. Thus the topology of weak convergence in $\mathcal{M}$ is the one induced by the weak* topology of the dual space of $\mathcal{C}_b(X)$. It is well known that $\{ \nu_\alpha \}$ converges to $\nu$ weakly if and only if for every $A \in \mathcal{B}(X)$ with $\nu(\partial A) = 0$, $\nu_\alpha(A) \xrightarrow{\alpha} \nu(A)$. Here $\partial A$ denotes the boundary of $A$. Topologies of setwise convergence and convergence in variation will be denoted by $\tau_2$ and $\tau_3$ respectively. $(\mathcal{M}, \tau_1)$ is a complete separable metric space (a suitable metric that generates $\tau_1$, is the well known Prohorov metric). It is also known that $(\mathcal{M}, \tau_3)$ with norm $\| \cdot \|$ is a Banach space and that convergence in $(\mathcal{M}, \tau_2)$ is equivalent to weak convergence in $(\mathcal{M}, \tau_3)$.

**Definition.** A measure $\nu \in \mathcal{M}$ is said to be continuous in the direction of $h \in X$ if the function $\phi: t \mapsto \nu_{th}$ from $\mathbb{R}$ to $(\mathcal{M}, \tau_2)$ is continuous.

It can be shown that continuity of $\phi$ from $\mathbb{R}$ to the topological space $(\mathcal{M}, \tau_2)$ is equivalent to continuity of $\phi$ from $\mathbb{R}$ to $(\mathcal{M}, \tau_3)$.

**Theorem 1.2.3 [Bl]**: Suppose $\nu \in \mathcal{M}$ is continuous in the direction of $h$, and $\mu \in \mathcal{M}$ is absolutely continuous with respect to $\nu$. Then $\mu$ is continuous in the direction of $h$. 
Definition. A measure \( \mu \in \mathcal{M} \) is said to be S-differentiable (or differentiable in the sense of skorokhod) in the direction of a nonzero vector \( h \in X \) if for each \( f \in C_b(X) \), \( \lim_{t \to 0} \int_X \frac{f(x-th)-f(x)}{t} \mu(dx) \) exists.

The existence of this limit for all \( f \in C_b(X) \) implies that if \( \{t_n\} \) is a sequence of real numbers converging to 0, then the sequence \( \{ \frac{\mu_h\cdot t_n - \mu}{t_n} \} \) is cauchy in \( (\mathcal{M}, \tau_2) \).

As mentioned above, the topological space \( (\mathcal{M}, \tau_2) \) is a complete separable metric space. Therefore there exists a measure \( d_{S:h}\mu \) such that \( \frac{\mu_h\cdot t_n - \mu}{t_n} \to d_{S:h}\mu \) weakly. \( d_{S:h}\mu \) is called the S-derivative of \( \mu \) in the direction of \( h \).

From the definition of weak convergence of measures, one can see that a measure \( \mu \in \mathcal{M} \) is S-differentiable in the direction of \( h \) if and only if there exists a measure \( d_{S:h}\mu \in \mathcal{M} \) such that \( \frac{\mu_h\cdot t_n - \mu}{t_n} \to d_{S:h}\mu(A) \) as \( t \to 0 \), for every Borel set \( A \) with \( d_{S:h}\mu(\partial A) = 0 \). A stronger notion of differentiability of measures is that of F-differentiability (introduced by Fomin) which is defined below.

Definition. A measure \( \nu \in \mathcal{M} \) is F-differentiable in the direction of \( h \) if the map \( \phi : t \mapsto \nu_{th} \) from \( R \) to \( (\mathcal{M}, \tau_2) \) is differentiable at 0.

It is shown in [ASF] that differentiability of the map \( \phi \) from \( R \) to \( (\mathcal{M}, \tau_2) \) is equivalent to differentiability of \( \phi \) from \( R \) to \( (\mathcal{M}, \tau_3) \). By theorems 1.2.1 and 1.2.2, there exists a measure in \( \mathcal{M} \) denoted by \( d_{F:h}\nu \) (and called F-derivative of \( \nu \) in the direction of \( h \)) such that \( \frac{\nu_h\cdot t_n - \nu}{t_n} \to d_{F:h}\nu \) both in \( \tau_2 \) and \( \tau_3 \) as \( t \) approaches 0. If \( \phi \) is n-times differentiable, then \( \nu \) is said to be n-times F-differentiable along \( h \) and it's n-th derivative is denoted by \( d_{F:h}^n\nu \). Since \( B(X) \) is translation invariant, differentiability of \( \phi \) at 0 implies that \( \phi \) is differentiable everywhere. Now suppose \( \nu \) is a positive measure and \( A \in B(X) \) is such that \( \nu(A) = 0 \). Then the function \( t \mapsto \nu_{th}(A) \) has a minimum at 0. This implies that the derivative of this function at 0, i.e., \( d_{F:h}\nu(A) \) is equal to 0. Therefore \( d_{F:h}\nu \) is absolutely continuous with
respect to $\nu$. Let $\rho_h = \frac{dF_{h,\nu}(dx)}{\nu(dx)}$. $\rho_h$ is called the logarithmic derivative of $\nu$ in the direction of $h$. We can prove the same result for a sign measure by considering its Jordan decomposition. It is shown in [ASF] that the map $h \mapsto d_h\nu$ is linear.

**Example 1:** Here we consider the simple example of a bounded Borel measure $\mu$ on $\mathbb{R}$. Let $h$ be a nonzero number. Then,

a) $\mu$ is continuous in the direction of $h$ if and only if it is absolutely continuous with respect to the Lebesgue measure.

b) $\mu$ is $s$-differentiable in the direction of $h$ if and only if it is absolutely continuous with respect to the Lebesgue measure and its Radon Nikodym derivative with respect to the Lebesgue measure is of bounded variation.

c) $\mu$ is $f$-differentiable in the direction of $h$ if and only if a version of its Radon Nikodym derivative $\frac{d\mu}{dx}$ with respect to the Lebesgue measure is absolutely continuous and $(\frac{d\mu}{dx})'$ is integrable with respect to the Lebesgue measure.

d) $\mu$ is infinitely many times $f$-differentiable if and only if a version of $\frac{d\mu}{dx}$ is infinitely differentiable with integrable derivatives of all orders.

**Example 2:** Let $(i, H, B)$ be an abstract Wiener space. Let $p_t$ be the Wiener measure on $B$ with the variance parameter $t > 0$. Using the fact that the measure $p_t$ is quasi-invariant in the directions of $H$ and that for each $h \in H$, the Radon Nikodym derivative $\frac{dp_t(dx+h)}{dp_t} = \exp \left\{ \frac{1}{2t} (2(h, x) - |h|^2) \right\}$ is a nice function that is $H - C^\infty$, one can show very easily that the measure $p_t$ is infinitely $f$-differentiable in the directions of $H$. In fact in this case, one can prove that the map $h \mapsto p_t(dx + h)$ from $H$ to $(\mathcal{M}, \tau_2)$ is Frechet differentiable. As mentioned above, both measures $d_{F;\nu}p_t$ and $d_{F;h,k}^2p_t$ are absolutely continuous with respect to $p_t$. The first and second derivatives of $p_t$ are given below. Notice that the Radon Nikodyms of these measures with respect to $p_t$ are smooth in the directions of $H$. 
\[ d_{F,h}p_t(dy) = \frac{1}{t}(h,y)p_t(dy) \]

and

\[ d_{F,h,k}p_t(dy) = \{ \frac{1}{t^2}(h,y)(k,y) - \frac{1}{t^2}(h,k) \} p_t(dy). \]

We define the subspace of differentiability of a measure \( \mu \in \mathcal{M} \) to be the linear space consisting of vectors along whose directions, \( \mu \) is F-differentiable. Although this subspace can not be the space \( X \) itself [ASF], it is a Banach space with certain nice properties [B3]. If \( X \) is a Banach space and \( \mu \) is a Gaussian measure on \( X \), then it is inferred from our discussion in section 1.1 that the subspace of differentiability of \( \mu \) is a Hilbert space \( H \) such that \( (i, H, X) \) is an abstract Wiener space.

One can see from example 1 that the regularity properties of a measure \( \mu \) on \( R \) (or on any n-dimensional Euclidean space) can all be stated equivalently in terms of its relation with respect to the Lebesgue measure. Existence of Lebesgue measure in finite dimensional Euclidean spaces reduces the theory of differentiable measures in such spaces to the theory of functions. Thus the theory of differentiable measures becomes of importance where there is no Lebesgue measure as is the case for infinite dimensional spaces.

**THEOREM 1.2.4 [B1,B2] :**

a) If \( \nu \in \mathcal{M} \) is S-differentiable along \( h \), then it is continuous along \( h \).

b) If \( \nu \in \mathcal{M} \) is F-differentiable along \( h \), then it is S-differentiable along \( h \).

c) If \( \nu \in \mathcal{M} \) is S-differentiable along \( h \) and \( d_{S,h} \nu \) is continuous along \( h \), then \( \nu \) is F-differentiable along \( h \).

**THEOREM 1.2.5 [ASF] :** If the measure \( \mu \) is differentiable along \( h \in X \), then for every real number \( t \), \( \| \mu_{th} - \mu \| \leq |t| \| d_h \mu \| \).
Proof: For each \( A \in B(X) \), apply the mean value theorem to the differentiable function \( s \mapsto \mu_{sh}(A) \) on the interval \([0, t]\). This gives us,

\[
\mu_{th}(A) - \mu(A) = t \left( d_h \mu(A) \right)_{\tau_0 h} \quad \text{for some } \tau_0 \in (0, t),
\]

\[
\mu_{th}(A) - \mu(A) \leq t \| (d_h \mu)_{\tau_0 h} \| = t \| d_h \mu \| .
\]

Therefore \( \| \mu_{th} - \mu \| \leq t \| d_h \mu \| . \) Q.E.D.

**Theorem 1.2.6 [ASF]**: If the measure \( \mu \) is differentiable in the direction \( h \in X \), then for every real number \( t \), \( \| \mu_{th} - \mu - td_h \mu \| \leq |t| \sup_{0 < r < t} \| (d_h \mu)_{\tau r h} - d_h \mu \| . \)

*Proof*: for each \( A \in B(X) \), an application of the mean value theorem to the differentiable map \( s \mapsto \mu_{sh}(A) - sd_h \mu(A) \) on the interval \([0, t]\) gives us,

\[
\mu_{ih}(A) - \mu(A) - td_h \mu(A) = t \left\{ (d_h \mu(A))_{\tau_0 h} - d_h \mu(A) \right\} \quad \text{for some } \tau_0 \in (0, t),
\]

\[
\leq |t| \sup_{0 < r < t} \left\{ (d_h \mu(A))_{\tau r h} - d_h \mu(A) \right\}
\]

\[
\leq |t| \sup_{0 < r < t} \| (d_h \mu)_{\tau r h} - d_h \mu \| .
\]

This inequality holds for every \( A \in B(X) \), hence

\[
\| \mu_{th} - \mu - td_h \mu \| \leq |t| \sup_{0 < r < t} \| (d_h \mu)_{\tau r h} - d_h \mu \| . \quad Q.E.D.
\]

**Theorem 1.2.7 [B2]**: Suppose that the measure \( \mu \) is \( F \)-differentiable in the direction of \( h \), \( f \) is a Borel measurable function differentiable in the direction of \( h \), 
\( d_h f(x) \in L^1(\mu) \) and \( f \in L^1(d_h \mu) \). Then \( \int d_h f(x) \mu(dx) = - \int f(x) d_h \mu(dx) \).

Let \( \mu \) and \( \nu \in \mathcal{M} \). The convolution \( \mu * \nu \) of the measures \( \mu \) and \( \nu \) is defined by, \( \mu * \nu(A) = \mu \times \nu \{ (x, y); x + y \in A \} \) for every \( A \in B(X) \). It is easy to check that \( \mu * \nu(A) = \int_X \mu(A - x) \nu(dx) \). It is also easy to check that if \( \mu \) is \( F \)-differentiable in the direction of \( h \), then so is \( \mu * \nu \) and \( d_h (\mu * \nu) = d_h \mu * \nu \).
**THEOREM 1.2.8**: Let \( \mu \) be a finite borel measure on \( X \). Let \( h \) be a fixed nonzero vector. If \( \mu \) has the property that for each bounded measurable function \( f \) that is differentiable along \( h \) and whose derivative along \( h \) is bounded, \( |\int_X d_h f(x) \nu(dx)| < C \| f \|_\infty \), for some positive constant \( C \), then \( \mu \) is \( S \)-differentiable (hence continuous) along \( h \).

**Proof**: Let \( g \) be a bounded uniformly continuous function. Let \( g_n = g * \lambda_n \) where \( \lambda_n \) is the Gaussian measure on \( \mathbb{R}^1 h \) with variance \( \frac{1}{n} \) and mean 0. Given \( \epsilon > 0 \), \( \exists \delta > 0 \) such that \( |t| < \delta \) implies that \( \sup_x |g(x - th) - g(x)| < \epsilon \).

\[
\sup_x |g_n(x) - g(x)| = \sup_x \left| \int_R [g(x - th) - g(x)] \frac{e^{-nt^2/2}}{\sqrt{2\pi/n}} dt \right|
\leq \int_{|t|<\delta} \frac{\epsilon}{\sqrt{2\pi/n}} dt + 2 \| g \|_\infty \int_{|t|>\delta} \frac{e^{-nt^2/2}}{\sqrt{2\pi/n}} dt
< \epsilon + 2 \| g \|_\infty \int_{t \geq \sqrt{n}\delta} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \to \epsilon \quad \text{as} \quad n \to \infty.
\]

Since we took \( \epsilon > 0 \) as an arbitrary positive number, we have \( \| g_n - g \|_\infty \to 0 \) as \( n \to \infty \).

Now, \( \lambda_n \) is differentiable along \( h \), so \( g_n = g * \lambda_n \) is also differentiable along \( h \) and \( d_h g_n = g * d_h \lambda_n \) and it is easy to see that \( d_h g_n \) is a bounded function.

Consider the functions \( \phi : t \mapsto \int g(x - th) \mu(dx) \) and \( \phi_n : t \mapsto \int g_n(x - th) \mu(dx) \) defined on some open interval \( I \) containing 0. Obviously \( \phi_n \to \phi \) as \( n \to \infty \) uniformly on \( I \). For each \( n \), \( \phi_n \) is differentiable on \( I \) and \( \phi'_n(t) = \int d_h g_n(x - th) \mu(dx) \). By assumption,

\[
\| \phi'_n - \phi'_m \| = \sup_{t \in I} \left| \int d_h [g_n(x - th) - g_m(x - th)] \mu(dx) \right|
\leq C \| g_n - g_m \|_\infty \to 0 \quad \text{as} \quad m, n \to \infty.
\]

Therefore \( \{\phi'_n\}_{n=1}^\infty \) converges uniformly to some continuous function. This proves that the map \( \phi : t \mapsto \int g(x - th) \mu(dx) \) is differentiable. But we took \( g \) as
an arbitrary bounded uniformly continuous function. Therefore \( \mu \) is S-differentiable and there exists a measure \( d_{S,h}\mu \) such that \( \phi'(0) = \int g(x)d_{S,h}\mu(dx) \). \( Q.E.D. \)

**Theorem 1.2.9:** Let \( \mu \) be a bounded Borel measure on \( X \). Let \( h \neq 0 \) be a fixed vector in \( X \). If \( \exists \xi_h \in L^1(\mu) \) such that \( \int d_h\phi(x)\mu(dx) = \int \phi(x)\xi_h(x)\mu(dx) \) for every bounded measurable function \( \phi \) that is differentiable along \( h \) with bounded \( d_h\phi \), then \( \mu \) is F-differentiable along \( h \) and \( d_{F,h}\mu(dx) = -\xi_h(x)\mu(dx) \).

**Proof:** Let \( g, \phi, g_n \) and \( \phi_n \) be as in the proof of the above theorem. It is obvious that \( \phi_n'(t) = \int g_n(x-th)\xi_h(x)\mu(dx) \). Therefore differentiability of \( \phi \) at 0 gives us,

\[
\lim_{t \to 0} \frac{\int g(x-th) - g(x)\mu(dx)}{t} = \int g(x)\xi_h(x)\mu(dx).
\]

It therefore follows that \( \mu \) is S-differentiable along \( h \) and \( d_{S,h}\mu(dx) = \xi_h(x)\mu(dx) \). But the measure \( \xi_h(x)\mu(dx) \) is absolutely continuous with respect to \( \mu \) and \( \mu \) being S-differentiable in the direction of \( h \) is continuous along \( h \). It therefore follows from theorems 1.2.3 and 1.2.4(c) that the measure \( \xi_h(x)\mu(dx) \) is continuous in the direction of \( h \) and that \( \mu \) is F-differentiable in the direction of \( h \). \( Q.E.D. \)

**Theorem 1.2.10:** Let \((i,H,B)\) be an abstract Wiener space. Let \( \mu \) be a bounded Borel measure on \( B \) satisfying the conditions of the previous theorem along the directions of \( H \). Furthermore suppose that the linear map \( h \mapsto \xi_h \) from \( H \) to \( L^1(\mu) \) is continuous, then the map \( h \mapsto \mu_h \) from \( H \) to \( (M,\tau_2) \) is Frechet differentiable.

**Proof:** It follows from the proof of the theorem 1.2.6 that for each \( A \in \mathcal{B}(B) \),

\[
|\mu_h(A) - \mu(A) - d_h\mu(A)| \leq \sup_{0<\tau<1} \left( (d_h\mu(A))_\tau - d_h\mu(A) \right)
\]

\[
= \sup_{0<\tau<1} \int \{1_{A+\tau h}(x) - 1_A(x)\}\xi_h(x)\mu(dx)
\]

\[
\leq 2 \int |\xi_h(x)|\mu(dx)
\]

\[
= o(|h|) \quad \text{by assumption.} \quad Q.E.D.
\]
THEOREM 1.2.11: Suppose the measure $\mu$ has the property that for every $h, k \in H$, there exist functions $\xi_1(h, x)$ and $\xi_2(h, k, x)$ in $L^1(\mu)$ such that for every function $f$ that is $H - C^2$ with bounded derivatives,

$$\int Df(x)(h)\mu(dx) = -\int f(x)\xi_1(h, x)\mu(dx)$$

and

$$\int D^2f(x)(h, k)\mu(dx) = \int f(x)\xi_2(h, k, x)\mu(dx),$$

then the measure $\mu$ is twice $F$-differentiable in the directions of $H$. Furthermore, if the map $(h, k) \mapsto \int |\xi_2(h, k, x)|\mu(dx)$ is continuous, then the map $h \mapsto \mu(A + h)$ is twice Frechét differentiable.

Proof: It follows from the theorem 1.2.9 that $\mu$ is once differentiable and $d_h\mu(dx) = \xi_1(h, x)\mu(dx)$. Now

$$-\int Df(x)(k)\xi_1(h, x)\mu(dx) = \int f(x)\xi_2(h, k, x)\mu(dx),$$

$$= \int f(x)\xi_2(h, k, x)\mu(dx).$$

This implies that the measure $\mu$ is twice differentiable. Q.E.D.

THEOREM 1.2.12: Suppose $\mu$ is a measure on $B$ that is $H - C^1$ and $d_h\mu(dx) = \langle \xi(x), h \rangle \mu(dx)$ where $\xi : B \rightarrow H$ is Bochner integrable with respect to $\mu$. Furthermore suppose that $f$ is a test function, $\sigma : B \rightarrow H$ is a Bochner integrable $H - C^1$ function such that $D\sigma(x) \in L_1(H)$ for every $x \in B$ and the function $x \mapsto |D\sigma(x)|_1$ from $B$ to $R$ is bounded and continuous. Then,

$$\int \langle \sigma(x), Df(x) \rangle \mu(dx) = -\int f(x)\left\{\langle \sigma(x), \xi(x) \rangle + \text{trace} D\sigma(x)\right\}\mu(dx).$$
\textbf{Proof :} By the above assumptions we have

\[ \int \left( \sigma(x), Df(x) \right) \mu(dx) = \int \left( \sigma(x), \sum_{i=1}^{\infty} \langle Df(x), e_i \rangle e_i \right) \mu(dx) \]

\[ = \sum_{i=1}^{\infty} \int \langle \sigma(x), e_i \rangle \langle Df(x), e_i \rangle \mu(dx) \]

\[ = \sum_{i=1}^{\infty} - \int f(x) \left\{ D\sigma(x)(e_i, e_i) + \langle \sigma(x), e_i \rangle \langle \xi(x), e_i \rangle \right\} \mu(dx) \]

\[ = - \int f(x) \left\{ \text{trace} D\sigma(x) + \langle \xi(x), \sigma(x) \rangle \right\} \mu(dx). \quad Q.E.D. \]

The case of Wiener measure is slightly different, since \( d_h \mu_1(dx) = -(h, x) \mu_1(dx) \) where \( x \in B \) and \( (h, x) \) is defined a.e. with respect to the measure \( \mu_1 \). But if \( \sigma : B \to B^* \) belongs to \( L^p(\mu_1) \) for some \( p > 1 \) and \( \{e_i\} \) is the orthonormal basis mentioned in theorem 1.1.4 and \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[ \left| \int \sum_{i=n}^{m} \langle \sigma(x), e_i \rangle (e_i, x) \mu_1(dx) \right| \leq \int \left| \langle \sigma(x), \sum_{i=n}^{m} (e_i, x) e_i \rangle \right| \mu_1(dx) \]

\[ \leq \left( \int |\sigma(x)|_B^p \mu_1(dx) \right)^{\frac{1}{p}} \left( \int \| \sum_{i=n}^{m} (e_i, x) e_i \|^q \mu_1(dx) \right)^{\frac{1}{q}} \]

\[ \to 0 \quad \text{By the remark following theorem 1.1.4.} \]

So we have the following theorem.

\textbf{Corollary 1.2.13 :} Suppose \( f \) is as in theorem 1.2.12 and \( \sigma : B \to B^* \) belongs to \( L^p(\mu_1) \) for some \( p > 1 \), then

\[ \int \left( \sigma(x), Df(x) \right) \mu_1(dx) = \int f(x) \left\{ (\sigma(x), x) - \text{trace} \ D\sigma(x) \right\} \mu_1(dx). \]

\textbf{Remark :} For the function \( \sigma \) given above, define the \( H \)-valued measure \( \nu \) on \( \mathcal{B}(B) \) by \( \nu(A) = \int_A \sigma(x) \mu(dx) \). It is very easy to check that if \( \sigma \) satisfies the conditions mentioned in theorem 1.2.12, then \( \nu \) is a countably additive measure that is \( H - C^1 \).
i.e., the map \( h \mapsto \mu(A + h) \) is Fréchet differentiable for each \( A \in B(B) \). Theorem 1.2.12 simply gives us an integration by parts formula for the measure \( \nu \). Obviously from the proof of the above theorem we see that integration by parts takes the form

\[
\int \langle Df(x), \sigma(x)\mu(dx) \rangle = -\int f(x) \sum_{i=1}^{\infty} D\left(\sigma(x)\mu(dx)\right)(e_i, e_i)
= -\int f(x) \text{trace} D\left(\sigma(x)\mu(dx)\right).
\]

**Theorem 1.2.14**: Let \( A : B \rightarrow L_1(H) \) be a continuous bounded map with range in the collection of positive definite trace class operators. Furthermore assume that \( A \) is \( H - C^2 \) and the maps \( DA \) and \( D^2A \) are bounded and continuous from \( B \) into \( L_1(H; L_2(H)) \) and \( L_1(L_2(H); L_2(H)) \) respectively; \( DA(x)(\cdot, h) \) and \( DA(\cdot, h, \cdot) \) are both in \( L_1(H) \) for all \( h \in H \) with bounded range in \( L_1(H) \). Let \( \mu \) be an \( H - C^2 \) measure with \( \mu'(dx)(h) = \langle \rho_1(x), h \rangle \mu(dx) \) and \( \mu''(dx)(h, k) = \langle \rho_2(x)h, k \rangle \mu(dx) \). Furthermore assume that \( \rho_1 \) and \( \rho_2 \) are Bochner integrable with respect to the measure \( \mu \) and take values in \( H \) and \( L(H) \) respectively. If \( f \) is a bounded real valued measurable function that is \( H - C^2 \) with bounded derivatives, then

\[
\int \text{trace} D^2f(x)\left(\sqrt{A(x)(\cdot)}, \sqrt{A(x)(\cdot)}\right)\mu(dx)
= \int_B f(x) \sum_{i,j} D^2\left\{ A(x)(e_i, e_j)\mu(dx)\right\}(e_i, e_j).
\]

**Proof**: Note that for any orthonormal basis \( \{e_i\}_{i=1}^{\infty} \) of \( H \),

\[
\text{trace} D^2f(x)\left(\sqrt{A(x)(\cdot)}, \sqrt{A(x)(\cdot)}\right) = \sum_{i=1}^{\infty} D^2f(x)\left(\sqrt{A(x)(e_i)}, \sqrt{A(x)(e_i)}\right)
\leq M_1 \sum_{i=1}^{\infty} |\sqrt{A(x)}(e_i)|_H^2
= M_1 |\sqrt{A(x)}|_{L_2(H)}^2
= M_1 |A(x)|_{L_1(H)} < M_2 < \infty.
\]
So by the Lebesgue's dominated convergence theorem,

\[ \int_B \text{trace } D^2 f(x) \left( \sqrt{A(x)}(\cdot), \sqrt{A(x)}(\cdot) \right) \]

\[ = \sum_{j,k=1}^{\infty} \int_B D^2 f(x)(e_j, e_k)(A(x)e_j, e_k)\mu(dx). \]

and the order of summation does not matter. But for each \( j \) and \( k \),

\[ \int D^2 f(x)(e_j, e_k)(A(x)e_j, e_k)\mu(dx) = \int f(x) \left\{ D^2 A(x)(e_j, e_k) + DA(x)(e_j, e_k)(\rho_1(x), e_k) + DA(x)(e_k, e_j)(\rho_1(x), e_j) + A(x)(e_j, e_k)(\rho_2(x), e_j, e_k) \right\} \mu(dx). \]

So now we have to show the convergence in \( L^1(\mu) \) of the sum of the integrand in the right hand side of the above equation.

\[ \int \sum_{j,k} D^2 A(x)(e_j, e_k, e_j, e_k)\mu(dx) \]

converges by our assumption on \( D^2 A \).

\[ \sum_{j,k} \int DA(x)(e_j, e_j, e_k)(\rho_1(x), e_k)\mu(dx) = \sum_j \int DA(x)(e_j, \rho_1(x))\mu(dx) \]

\[ = \int |\rho_1(x)|_{L^1(H)} |DA(x)(\cdot, \cdot)|_{L(H, L^1(H))}\mu(dx) < \infty. \]

The same argument shows that

\[ \sum_{j,k} \int DA(x)(e_k, e_j, e_k)(\rho_1(x), e_j)\mu(dx) < \infty. \]

Now

\[ \sum_{j,k} \int A(x)(e_j, e_k)(\rho_2(x)e_j, e_k)\mu(dx) = \sum_j \int (\rho_2^*(x)A(x)e_j, e_j)\mu(dx) < \infty. \]
Since $\sum_j (\rho_j^2(x)A(x)e_j,e_j) \leq |\rho_j^2(x)|_{L(H)}|A(x)|_{L^1(H)}$. Q.E.D.

Note that if $A(x) \in L_2(H)$ for all $x \in B$ and $C \in L_2(H)$. $A(x)$ and $C$ are both positive definite and the map $A$ from $B$ into $L_2(H)$ is continuous and bounded and $H - C^2$ with bounded and continuous functions $DA$ and $D^2A$ taking values in $L_H^2(H;R)$ and $L_H^4(H;R)$ respectively, then the function $A(x) \circ C$ satisfies the conditions stated in the above theorem.
§1-Stochastic integration in abstract Wiener Space

Throughout $(i,H,B)$ is considered to be an abstract Wiener space and the Banach space $B_0$, the orthonormal basis $\{e_i\}_{i=1}^\infty$, and the collection of orthogonal projections $\{Q_n\}_{n=1}^\infty$ are as in theorem 1.1.4.

**Definition**: Let $(\Omega,\mathcal{F},\mathcal{P})$ be a complete probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ be an increasing family of sub $\sigma$-fields of $\mathcal{F}$. Let $\{W(t,\cdot)\}_{t\geq 0}$ be a collection of random elements with state space $B$ satisfying the following conditions,

a) $W(0) = 0$ $\mathcal{P}$-a.s.,

b) for almost all $\omega \in \Omega$, the sample paths are continuous,

c) for each $t > 0$, $\sigma\{W_s; 0 \leq s \leq t\} \subset \mathcal{F}_t$,

d) for any $0 < s < t < \infty$, the random variable $W(t) - W(s)$ is independent of the $\sigma$-algebra $\mathcal{F}_s$,

e) $W(t) - W(s)$ is distributed according to the Wiener measure $\mu_{t-s}$.

Then the process $\{W(t)\}_{t\geq 0}$ is called a standard Brownian motion with state space $B$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

**Remark 1**: It is obvious that $\mathcal{P}\{\omega; W(t,\omega) \in B_0\} = \mu_t(B_0) = 1$.

**Remark 2**: It follows from the definition that if $Q : B \rightarrow H$ is a finite rank projection whose restriction to $H$ is an orthogonal projection, then the process $QW_t$ is also a Brownian motion in $Q(B)$.

**Theorem 2.1.1**: Let $(\Omega,\mathcal{F},\mathcal{P})$ and $\mathcal{F}_t$ be as above. A continuous $B$-valued random process $\{W(t,\omega)\}_{t\geq 0}$ adapted to the filtration $\mathcal{F}_t$ is a standard Brownian motion if and only if for each finite rank projection $Q : B \rightarrow H$ whose restriction to $H$ is orthogonal, the process $\{Q(t,\omega)\}_{t\geq 0}$ is a Brownian motion in $Q(B)$ with respect to the filtration $\mathcal{F}_t$. 

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Proof: "⇒" We only need to show (c) and (d) of the definition. We prove (d) first. Let $f$ be a bounded continuous function from $B$ to $R$. Then

$$\int_B f\{Q_n(W_t - W_s)\}d\mathcal{P} = \int_{Q_nH} f(x) \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{|x|^2}{2(t-s)}\right) dx$$

$$= \int_B f(Q_nx)d\mathcal{P}_{t-s}(x)$$

$$\rightarrow \int_B f(x)d\mathcal{P}_{t-s}(x) \text{ since } f(Q_n(x)) \rightarrow f(x) \text{ p}_{t-s} a.s.$$  

On the other hand

$$\int_B f\{Q_n(W_t - W_s)\}d\mathcal{P} \rightarrow \int_B f(W_t - W_s)d\mathcal{P} \text{ as } n \rightarrow \infty.$$  

Therefore $\mathcal{P}(W_t - W_s \in dx) = p_{t-s}(dx).$

To prove (c) we need to show that if $A \in B(B)$ and $C \in \mathcal{F}_s$, then

$$\mathcal{P}\{(W_t - W_s)^{-1}(A) \cap C\} = \mathcal{P}\{W_t - W_s \in A\}\mathcal{P}(C). \quad (*)$$

But our assumption implies that this equality holds if $A$ is a cylinder set. The same assumption implies that the collection of sets $A$ that satisfy $(*)$ for a fixed $C \in \mathcal{F}_s$ is an algebra. Furthermore this collection forms a monotone class (because $\mathcal{P}$ is a probability measure). Therefore this collection must be the Borel field of $B$. Q.E.D.

Existence: For each $0 = t_0 < t_1 < t_2 < \ldots < t_n$, consider the distribution $Q_{t_0,t_1,\ldots,t_n}$ on the measure space $(B \times B \times \ldots \times B, B(B) \times \ldots \times B(B))$ defined by

$$Q_{t_0,\ldots,t_n}(A_0 \times \ldots \times A_n) = \int_{A_n} \ldots \int_{A_1} \int_{A_0} p_{t_n-t_{n-1}}(y_{n-1},dy_{n-1}) \ldots p_{t_1}(y_0,dy_1) \delta_0(dy_0).$$

where $A_1, \ldots, A_n \in B(B)$ and $\delta_0$ is the dirac measure concentrated at 0. The collection $\{Q_{t_0\ldots t_n}\}_{0 < t_0 < t_1 < \ldots < t_n}$ is a consistent family of probability measures. (This follows from the fact that $p_{t-s}(x,A) = \int_B p_{u-s}(x,dy)p_{t-u}(y,A)$ for each $u; 0 \leq s < u < t$ and $A \in B(B)$). By Kolmogorov's extension theorem, there
exist a probability space \((\Omega, \mathcal{F}, \mathcal{P})\) and a collection \(\{X_t\}_{t \geq 0}\) of \(B\) valued random elements satisfying properties (a), (c) and (d) of the definition. It is obvious that 
\[
E\|X_t - X_s\|_4 = \int_B \|x\|_4 p_{t-s}(dx) = |t-s|^2 \int_B \|x\|_4 p_1(dx) < \infty.
\]
The same argument as in the finite dimensional case can be used to show that almost all sample paths of the process \(\{X_t\}_{t \geq 0}\) are continuous. For each \(t \geq 0\), we define \(\mathcal{F}_t\) to be the \(\sigma\)-algebra generated by the random elements \(\{X_s\}_{0 \leq s \leq t}\), then it can be shown that the process \(\{X_t\}_{t \geq 0}\) is a Brownian motion with respect to the filtration \(\{\mathcal{F}_t\}_{0 \leq t < \infty}\).

Remark 3: A Brownian motion \(\{W_t\}\) with state space \(B\) is a martingale since for \(s < t\),
\[
E(W_t|\mathcal{F}_s) = E(W_t - W_s|\mathcal{F}_s) + E(W_s|\mathcal{F}_s)
= E(W_t - W_s) + W_s
= \int_B x \ p_{t-s}(dx) + W_s
= W_s
\]
since the measure \(p_{t-s}\) is symmetric.

Let \(G\) be a Banach space. A process \(\xi : [0, \infty) \times \Omega \to G\) that is \((t, \omega)\)-jointly measurable is called nonanticipating if for each \(t \geq 0\), \(\xi(t)\) is \(\mathcal{F}_t\) measurable. For each \(p \geq 1\), \(0 \leq \alpha < \beta\), we introduce two classes of processes:

\(\mathcal{M}_{\alpha, \beta}[G]\) denotes the Banach space of nonanticipating processes with state space \(G\) satisfying 
\[
E \int_0^\beta \|\xi(t)\|^p_G dt < \infty.
\]
\(S_{\alpha, \beta}[G]\) denotes the Banach space of nonanticipating processes with state space \(G\) satisfying 
\[
E \sup_{\alpha \leq t \leq \beta} \|\xi(t)\|^p_G < \infty.
\]

Definition: A stochastic process \(\xi \in \mathcal{M}_{\alpha, \beta}[G]\) is called simple if there is a finite set of numbers \(\alpha = t_0 < t_1 < \ldots < t_n < \beta\) such that \(\xi(t) = \xi(t_j)\) for \(t_j \leq t < t_{j+1}\), \(j = 0, 1, 2, \ldots, n - 1\) and \(\xi(t) = \xi(t_n)\) when \(t_n \leq t \leq \beta\).
THEOREM 2.1.2: Let $K$ be a separable Hilbert space. If $\xi \in M_{a,\beta}^p[L_2(H;K)]$, then there exist a sequence $\{\xi_n\}$ of simple processes in $S_{a,\beta}^p[L(B_0;K)]$ such that $E \int_0^\beta |\xi(t) - \xi_n(t)|_2^2 dt \to 0$ as $n \to \infty$, where $\xi_n(t)$ is the restriction of $\xi_n(t)$ to $H$.

Proof: The same argument as in the finite dimensional case can be used to show that given any $\epsilon > 0$, $\xi$ can be approximated to within $\epsilon/2$ by a step process $\xi'$ in $S_{a,\beta}^p[L_2(H;K)]$ and then by taking composition of $\xi'$ with $Q_n$ with large enough $n$, $\xi'$ can be approximated to within $\frac{\epsilon}{2}$ by a simple process in $S_{a,\beta}^p[L(B_0;K)]$. Q.E.D.

Let $\xi(t,\omega)$ be a simple process in $M_{a,\beta}^p[L(B_0;K)]$ with jumps at $\alpha \leq t_1 < \ldots < t_n = \beta$. When $t_j \leq t < t_{j+1}$, $0 < j \leq n$, the $K$ valued random variable $\sum_{i=0}^n \xi(t_i)(W(t_{i+1}) - W(t_i)) + \xi(t_j)(W(t) - W(t_j))$, where $t_0 = \alpha$ and $t_{n+1} = \beta$ is denoted by $\int_\alpha^\beta \xi(s)dW(s)$. It is easy to check that if $a$ and $b$ are two real numbers and $\xi_1$ and $\xi_2$ are two simple processes in $M_{a,\beta}^p[L(B_0;K)]$, then for each $t$, $\alpha \leq t \leq \beta$, $\int_\alpha^t (a\xi_1(s) + b\xi_2(s))dW(s) = a\int_\alpha^t \xi_1(s)dW(s) + b\int_\alpha^t \xi_2(s)dW(s)$. In fact it is obvious that if $C \in L_2(K;K')$ where $K'$ is another separable Hilbert space, then $C\xi$ is a step process in $M_{a,\beta}^p[L(B_0;K')]$ and $\int_\alpha^t C\xi(s)dW(s) = C\int_\alpha^t \xi(s)dW(s)$.

Given $\xi \in M_{a,\beta}^2[L_2(H;K)]$, let $\{\xi_n\}$ be a sequence mentioned in theorem 2.1.2. It is proved in [K1] that for each $t$, $\alpha < t \leq \beta$, the sequence $\int_\alpha^t \xi_n(s)dW(s)$ is cauchy in $L^2(dP)$ and hence converges to a random element denoted by $\int_\alpha^t dW(s)$.

THEOREM 2.1.3 [K1]: If $\xi \in M_{a,\beta}^2[L_2(H;K)]$, the process $\{J_\xi(t)\}_{\alpha \leq t \leq \beta}$ where $J_\xi(t) = \int_\alpha^t \xi(s)dW(s)$ enjoys the following properties:

(a) $J_\xi$ has continuous sample paths a.s.,

(b) $J_\xi$ is a martingale,

(c) $\mathbb{P}\{\sup_0 \leq t \leq s |J_\xi(t)|_K > \delta\} \leq \delta^{-2}E|J_\xi(s)|_K^2$,

(d) $EJ_\xi(t)=0$ and $E|J_\xi(t)|_K^2 = E \int_\alpha^t |\xi(s)|_2^2 ds$. 
Remark 4: If $\xi$ belongs to the space $M^2_{a,\beta}[H]$, then the process $J_{\xi}(t)$ will be denoted by $\int_0^t \langle \xi(s), dW(s) \rangle$.

**Definition**: Let $K$ be a Hilbert subspace of a Banach space $G$ such that the norm of $G$ is strictly weaker than that of $K$. Then the couple $(K, G)$ is called a conditional Banach space.

**Definition**: Let $K_1$ and $K_2$ be two Hilbert spaces. A continuous bilinear map $S$ from $K_1 \times K_2$ is said to be of trace class type if (i) for each $x \in K_2$, $S_x \in L_1(K_1)$, where $\langle S_x y, z \rangle = \langle S(y, z), x \rangle$, and (ii) the linear functional $x \mapsto \text{trace } S_x$ is continuous.

It follows from this definition that there exists a unique vector, denoted by $\text{TRACE } S$, in $K_2$ such that $\langle \text{TRACE } S, x \rangle = \text{trace } S_x$ for all $x$ in $K_2$. It can also be shown that if $\{l_i\}$ is an orthonormal basis of $K_1$, then $\sum_{i=1}^{\infty} S(l_i, l_i)$ converges in $K_2$ to $\text{TRACE } S$.

**Lemma 2.1.4 [K1]**: If $T \in L_2(H; K_1)$ and $S$ is a continuous bilinear map from $K_1 \times K_2$ into $K_2$, then the bilinear map $S \circ [T \times T]$ from $H \times H$ into $K_2$ is of trace class type.

**Theorem 2.1.5 [K1] (Ito's lemma 1)**: Let $(K_1, G_1)$ and $(K_2, G_2)$ be two conditional Banach spaces. Let $\rho$ be a function from $[0, \infty) \times G_1$ into $G_2$ satisfying the following conditions,

(a) For each $x \in G_1$, $\rho(\cdot, x)$ is continuously differentiable and $\frac{\partial \rho}{\partial t}$ is continuous from $[0, \infty) \times G_1$ into $G_2$,

(b) for each $t \geq 0$, $\rho(t, \cdot) : G_1 \to G_2$ is twice Frechet differentiable such that $\rho'$ and $\rho''$ are $(t, x)$-jointly continuous. $\rho'(t, x)(K_1) \subset K_2$ and $\rho''(t, x)(K_1 \times K_1) \subset K_2$.

If $X(t) = x + \int_0^t \xi(s) dW(s) + \int_0^t \sigma(s) ds$, where $x \in G_1$, $\xi \in M^2_{a,\beta}[L_2(H; K_1)]$.
and $\sigma$ restricted to the interval $[\alpha, \beta]$ is integrable almost surely, then

$$
\rho(t, X(t)) = \rho(0, x) + \int_{\alpha}^{\beta} [\rho'(s, X(s)) \circ \xi(s)]dW(s) + \int_{\alpha}^{\beta} \left\{ \frac{\partial \rho}{\partial s}(s, X(s)) + 
+ \rho'(s, X(s))(\sigma(s)) + \frac{1}{2} \text{TRACE}(\rho''(s, X(s))) \circ [\xi(s) \times \xi(s)] \right\}ds.
$$

If $\xi(s) = J + \zeta(s)$, where $J \in L(B_0; G_1)$ and $\zeta \in \mathcal{M}_{\alpha, \beta}^2[L_2(H; K_1)]$, then we define $\int_{\alpha}^{\beta} \xi(s)dW(s) = J(W(\beta)) - J(W(\alpha)) + \int_{\alpha}^{\beta} \zeta(s)dW(s)$.

**THEOREM 2.1.6 [K1](Ito's lemma 2):** Let $(K_1, G_1)$ and $(K_2, G_2)$ be two conditional Banach spaces. Assume that $G_1$ has a Schauder basis. Let $\rho$ be a function from $[0, \infty) \times G_1$ into $G_2$ satisfying (a) of the above theorem and (b') for each $t \geq 0$, $\rho(t, \cdot)$ is twice $K_1$-differentiable such that $D\rho(t, x) \in L(K_1, K_2)$ and $D^2\rho(t, x)$ is a trace class type bilinear map from $K_1 \times K_1$ into $K_2$. Also the maps $D\rho : [0, \infty) \times G_1 \to L(K_1, K_2)$ and $\text{TRACE} \ D^2\rho : [0, \infty) \times G_1 \to K_2$ are continuous.

Let

$$
X(t) = x + \int_{\alpha}^{\beta} \xi(s)dW(s) + \int_{\alpha}^{\beta} \sigma(s)ds,
$$

where $\xi = J + \zeta$, $J \in L(B_0; G_1)$, $\zeta \in \mathcal{M}_{\alpha, \beta}^2[L_2(H; K_1)]$ and let $\sigma$ restricted to the interval $[\alpha, \beta]$ be integrable almost surely. Suppose $D\rho(t, x)(J(B_0)) \in K_2$ for all $t$ and $x$. Then

$$
\rho(t, X(t)) = \rho(0, x) + \int_{\alpha}^{\beta} [D\rho(s, X(s)) \circ \xi(s)]dW(s) + \int_{\alpha}^{\beta} \left\{ \frac{\partial \rho}{\partial s}(s, X(s)) + 
+ D\rho(s, X(s))(\sigma(s)) + \frac{1}{2} \text{TRACE}(D^2\rho(s, X(s))) \circ [\xi(s) \times \xi(s)] \right\}ds.
$$

As applications of Ito's lemma, we prove the following two theorems.

**THEOREM 2.1.7:** Let $K$ be a separable Hilbert space and $\xi \in \mathcal{M}_{\alpha, \beta}^{2m}[L_2(H; K)]$ for some integer $m \geq 1$; then the process $\{\int_{\alpha}^{\beta} \xi(s)dW(s)\}_{\alpha \leq \beta}$ belongs to the space $\mathcal{S}_{\alpha, \beta}^{2m}[K]$ and furthermore,

$$
E\left[ \sup_{\alpha \leq u \leq \beta} \left| \int_{\alpha}^{u} \xi(u)dW(u) \right|^{2m}_K \right] \leq \left( \frac{4m^3}{2m - 1} \right)^m \beta - \alpha \int_{\alpha}^{\beta} E|\xi(u)|^{2m}_K du.
$$
Proof: The result for $m=1$ follows from the fact that the process $\int_\alpha \xi(s)dW(s)$ is a martingale and $E|\int_\alpha^\beta \xi(s)dW(s)|_2^2 = E \int_\alpha^\beta |\xi(s)|^2 ds$ (use the submartingale inequality). So we assume that $m > 1$. First we show that for any $\xi \in \mathcal{M}^{2m}_{\alpha, \beta}[L_2(H; K)]$ we have

\[
(1) \quad E|\int_\alpha^\beta \xi(s)dW(s)| \leq m(2m-1)^m(\beta - \alpha)^{m-1} E \int_\alpha^\beta |\xi(s)|^2 ds.
\]

Let us assume for the moment that this inequality is true for every bounded simple process. Let $\xi_n \in S^{2m}_{\alpha, \beta}[L_2(B; K)]$ be such that $E \int_\alpha^\beta |\xi_n(s) - \xi(s)|^2 ds \to 0$ as $n \to \infty$. This of course implies that $|\int_\alpha^\beta \xi_n(s)dW(s)|_K \to |\int_\alpha^\beta \xi(s)dW(s)|_K$ in $L^2(dP)$ as $n \to \infty$. We may assume (by taking a subsequence if necessary) that this last convergence is almost surely. Now we have

\[
E|\int_\alpha^\beta \xi(s)dW(s)|^2 = E \lim_{n \to \infty} E|\int_\alpha^\beta \xi_n(s)dW(s)|^2_K
\]

\[
\leq \liminf_{n \to \infty} E|\int_\alpha^\beta \xi_n(s)dW(s)|^2_K \quad \text{Fatou's lemma.}
\]

\[
\leq \liminf_{n \to \infty} m(2m-1)^m(\beta - \alpha)^{m-1} E \int_\alpha^\beta |\xi_n(s)|^2 ds
\]

\[
= m(2m-1)^m(\beta - \alpha)^{m-1} E \int_\alpha^\beta |\xi(s)|^2 ds.
\]

So it suffices to prove (1) for bounded step functions $\xi$. To do this we apply Itô's formula to the function $f : K \to \mathbb{R}$ defined by $f(x) = |x|^{2m}$ and to the process $\int_\alpha \xi(s)dW(s)$. Note that

\[
f'(x)(\cdot) = 2m|x|^{2m-2} \langle x, \cdot \rangle_K
\]

and

\[
f''(x)(\cdot, \cdot) = 2m|x|^{2m-2} \langle \cdot, \cdot \rangle_K + 4m(m-1)|x|^{2m-4} \langle x, \cdot \rangle_K \langle x, \cdot \rangle_K.
\]
This gives us

\[ | \int_\alpha^\beta \xi(s)dW(s)|^{2m}_K = \int_\alpha^\beta (2m|\int_\alpha^s \xi(u)dW(u)|^{2m-2}_K \int_\alpha^\beta \xi(u)dW(u), \xi(s)dW(s))_K \]

\[ + \frac{1}{2} \int_\alpha^\beta \text{TRACE}[2m|\int_\alpha^s \xi(u)dW(u)|^{2m-2}_K \xi(s)(\cdot)|^2_K + \]

\[ + 4m(m-1)|\int_\alpha^s \xi(u)dW(u)|^{2m-4}_K \times \]

\[ \times (\int_\alpha^s \xi(u)dW(u), \xi(s)(\cdot))^2_K ds. \]

Now we need to show that

\[ E \int_\alpha^\beta \left| \xi(s)_* \left\{ 2m|\int_\alpha^s \xi(u)dW(u)|^{2m-2}_K \int_\alpha^s \xi(u)dW(u) \right\}_H^2 \right| ds < \infty. \]

We know that \( \xi(s) : B_o \to K \) and \( \xi(s)_*(K) \subset B_o^* \).

Let \( \alpha = t_0 < t_1 < \ldots < t_n = \beta \) be the partition corresponding to \( \xi \). Then,

\[ E \int_\alpha^\beta \left| \xi(u)dW(u)|^{4m-4}_K |\xi(s)_* \left( \int_\alpha^s \xi(u)dW(u) \right)|^2_H \right| ds \]

\[ \leq c_1 E \int_\alpha^\beta \left| \xi(u)dW(u)|^{4m-4}_K |\xi(s)_* \left( \int_\alpha^s \xi(u)dW(u) \right)|^2_{B_o} \right| ds \quad \text{c1's are constants}, \]

\[ \leq c_1 E \int_\alpha^\beta |\xi(\alpha)(W(s) - W(\alpha))|^{4m-4}_K |\xi(\alpha)_* \xi(\alpha)(W(s) - W(\alpha))|^2_{B_o^*} ds \]

\[ \leq c_2 E \int_\alpha^\beta |W(s) - W(\alpha)|^{4m-2} ds \]

\[ = c_2 \int_\alpha^\beta E|W(s) - W(\alpha)|^{4m-2} ds \]

\[ = c_2 \int_\alpha^\beta \int_B \|y\|^{4m-2} p_{\alpha-\alpha}(dy) ds \]

\[ = c_2 \int_\alpha^\beta \int_B (s - \alpha)^{2m-1} \|y\|^{4m-2} p_1(dy) ds < \infty. \]

Similarly one can show that

\[ E \int_{t_n}^{t_{n+1}} \left| \int_\alpha^s \xi(u)dW(u)|^{4m-4}_K |\xi(s)_* \left( \int_\alpha^s \xi(u)dW(u) \right)|^2_H \right| ds < \infty. \]
Therefore,
\[
E \int_{\alpha}^{\beta} (2m) \int_{\alpha}^{\beta} \xi(u) dW(u) \int_{k}^{2m-2} \int_{\alpha}^{\beta} \xi(u) dW(u), \xi(s) dW(s) K = 0
\]
and
\[
E| \int_{\alpha}^{\beta} \xi(s) dW(s) |^{2m}_{K} = \frac{1}{2} \int_{\alpha}^{\beta} E \sum_{j=1}^{\infty} (2m) \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m-2} | \xi(s)(e_j) |^{2}_{K} +
+ 4m(m-1) \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m-4}( \int_{\alpha}^{\beta} \xi(u) dW(u), \xi(s)(e_j) )^{2}_{K} ds.
\]
So
\[
E| \int_{\alpha}^{\beta} \xi(s) dW(s) |^{2m}_{K} \leq m(2m-1) \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m-2} | \xi(s) |^{2}_{K} ds.
\]
Now applying Hölder’s inequality with \( p = \frac{m}{m-1} \) and \( q = m \), we get
\[
E| \int_{\alpha}^{\beta} \xi(s) dW(s) |^{2m}_{K}
\leq \left( \int_{\alpha}^{\beta} E(\int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m-2}_{K} ds \right)^{\frac{m-1}{m}} \left( \int_{\alpha}^{\beta} E| \xi(s) |^{2m}_{K} ds \right) \frac{1}{m}.
\]
From (2) it is clear that the function \( t \rightarrow E| \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m}_{K} \) is monotone increasing. Therefore
\[
\int_{\alpha}^{\beta} E| \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m}_{K} ds \leq \int_{\alpha}^{\beta} E| \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m}_{K} ds
= (\beta - \alpha) E| \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m}_{K} ds.
\]
So from (3) it follows that
\[
E| \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m}_{K}
\leq m(2m-1) \{ (\beta - \alpha) E| \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m}_{K} \}^{\frac{m-1}{m}} \{ \int_{\alpha}^{\beta} E| \xi(s) |^{2m}_{K} \} \frac{1}{2}.
\]
Simplifying the inequality gives
\[
E| \int_{\alpha}^{\beta} \xi(u) dW(u) |^{2m}_{K} \leq m^m(2m-1)^m(\beta - \alpha)^{m-1} \int_{\alpha}^{\beta} E| \xi(s) |^{2m}_{K} ds.
\]
By the argument at the beginning of the proof, this inequality holds for every 
\( \xi \in \mathcal{M}_{a,\beta}^{2m}[L_2(H;K)] \). Now, the process \( \{ \int_0^t \xi(u)dW(u) \}_{\alpha \leq t \leq \beta} \) is a martingale; therefore

\[
E \sup_{\alpha \leq t \leq \beta} | \int_0^t \xi(u)dW(u) |_{K}^{2m} \leq \left( \frac{2m}{2m-1} \right) 2m E | \int_0^\beta \xi(u)dW(u) |_{K}^{2m}.
\]

This inequality together with (1) gives us the desired estimate. \( Q.E.D. \)

**THEOREM 2.1.8**: Consider the processes \( X_1 \) and \( X_2 \) where

\[
X_1(t) = \zeta_1 + \int_0^t \xi_1(s)dW(s) + \int_0^t \sigma_1(s)ds
\]

and

\[
X_2(t) = \zeta_2 + \int_0^t \xi_2(s)dW(s) + \int_0^t \sigma_2(s)ds.
\]

where \( \xi_1, \xi_2 \in \mathcal{M}_{0,\infty}^2[L_2(H;L_2(K))] \) and \( \sigma_1, \sigma_2 \in \mathcal{M}_{0,\infty}^1[L(K)] \) and \( \zeta_1 \) and \( \zeta_2 \) are \( L(K) \) valued square summable random variables that are \( \mathcal{F}_0 \) measurable. Then

\[
\begin{align*}
X_1(t)X_2(t) &= \xi_1\xi_2 + \int_0^t X_1(s)\xi_2(s)(dW(s)) + \int_0^t \xi_1(s)(dW(s))X_2(s) \\
X_1(t)X_2(t) &= \zeta_1\zeta_2 + \int_0^t X_1(s)\xi_2(s)(dW(s)) + \int_0^t \xi_1(s)(dW(s))X_2(s) \\
&+ \int_0^t \left( \sigma_1(s)X_2(s) + X_1(s)\sigma_2(s) + \sum_{i=1}^\infty \xi_1(s)(e_i) \circ \xi_2(s)(e_i) \right)ds.
\end{align*}
\]

**Proof**: We can combine both of these equations and write them in the following way,

\[
X(t) = \zeta + \int_0^t \xi(s)dW(s) + \int_0^t \sigma(s)ds,
\]

where \( \xi \in \mathcal{M}_{0,\infty}^2[L_2(H;L_2(K) \times L_2(K))] \) is defined by \( \xi(s)(h) = (\xi_1(s)(h), \xi_2(s)(h)) \) and \( \sigma \in \mathcal{M}_{0,\infty}^1[L(K) \times L(K)] \) is defined by \( \sigma(s)(h) = (\sigma_1(s)(h), \sigma_2(s)(h)) \) and \( \zeta = (\zeta_1, \zeta_2) \).

Let \( G_1 = L(K) \times L(K) \), \( G_2 = L(K) \), \( N_1 = L_2(K) \times L_2(K) \) and \( N_2 = L_2(K) \). Then \( (N_1, G_1) \) and \( (N_2, G_2) \) are both conditional Banach spaces. Now we apply Ito's formula to the function \( \rho : G_1 \rightarrow G_2 \) defined by \( \rho(A, B) = AB \) and to the process
$X(t)$. $\rho$ is twice Frechet differentiable; $\rho' : G_1 \rightarrow L(G_1 ; G_2)$ where $\rho'(A, B)(C, D) = AD + CB$ and $\rho'' : G_1 \rightarrow L(G_1 \times G_1 ; G_2)$ where $\rho''(A, B)[(C, D), (C', D')] = C'D + CD'$.

So

$$\rho'(X(t)) (\xi(t)) = \rho'(X_1(t), X_2(t)) (\xi_1(t), \xi_2(t))$$

$$= \xi_1(t) X_2(t) + X_1(t) \xi_2(t),$$

and

$$\rho''(X(t)) (\xi(t) \times \xi(t)) = \rho''(X_1(t), X_2(t)) [(\xi_1(t), \xi_2(t)), (\xi_1(t), \xi_2(t))]$$

$$= 2 \xi_1(t) \xi_2(t).$$

So

$$\text{TRACE } \rho''(X(t)) [\xi(t) \times \xi(t)] = 2 \sum \xi_1(t)(e_i) \xi_2(t)(e_i).$$

The conclusion of the theorem follows from Ito's formula. Q.E.D.

§2-Infinite dimensional stochastic differential equations

The following existence and uniqueness theorem was stated and proved in [K1, K2].

THEOREM 2.2.1 : Let $(K, G)$ be a conditional Banach space. Suppose that $f$ and $g$ are two functions satisfying the following conditions.

(a) $f$ is of the form $f(t,x,\omega) = C + h(t,x,\omega)$, where $C \in L(B; G)$ and $h : [\alpha, \beta] \times G \times \Omega \rightarrow L_2(H; K)$ is such that for each $\omega \in \Omega$, the map $(t, x) \mapsto h(t, x, \omega)$ is continuous. Let $g : [\alpha, \beta] \times G \times \Omega \rightarrow G$ be such that for each $\omega \in G$, the map $(t, x) \mapsto g(t, x, \omega)$ is continuous.

(b) For each $x \in G$, $f(\cdot, x, \cdot)$ and $g(\cdot, x, \cdot)$ are nonanticipating.

(c) There exists a positive number $\gamma$ such that with probability 1,

$$|h(t, x) - h(t, y)|_2 + |g(t, x) - g(t, y)|_G \leq \gamma |x - y|_G,$$

and

$$|h(t, x)|_2 + |g(t, x)|_G \leq \gamma (1 + |x|_G)$$

for all $x, y \in G$. 

Let $\zeta \in \mathcal{M}^{2,\alpha}_\beta[G]$ have continuous sample paths. Then the $G$-valued stochastic integral equation

$$Y(t) = \zeta(t) + \int_\alpha^t f(s, Y(s))dW(s) + \int_\alpha^t g(s, Y(s))ds$$

has a unique continuous solution $Y \in \mathcal{M}^{2,\alpha}_\beta[G]$. Moreover, $Y(t)$ is a Markov process if $\zeta(t)$ is so.

An easy application of Hölder's inequality and theorem 2.1.7 gives us the following estimate.

**THEOREM 2.2.2**: Suppose $\zeta$ in the above theorem belongs to $\mathcal{S}^{2m}_{\alpha,\beta}[G]$ for some positive integer $m$. If in addition to the assumptions stated above, $Y \in \mathcal{S}^{2m}_{\alpha,\beta}[G]$ and

$$|h(t, x)|^2 + |g(t, x)|^2 \leq m(1 + |x|^2),$$

then we have

$$E \sup_{\alpha \leq t \leq \beta} |Y(t)|^2 \leq D(1 + E \sup_{\alpha \leq t \leq \beta} |\zeta(t)|^2),$$

where $D$ is some positive constant.

Now consider the following two stochastic integral equations

$$Y_1(t) = \zeta_1(t) + \int_\alpha^t f(s, Y_1(s))dW(s) + \int_\alpha^t g(s, Y_1(s))ds$$

and

$$Y_2(t) = \zeta_2(t) + \int_\alpha^t f(s, Y_2(s))dW(s) + \int_\alpha^t g(s, Y_2(s))ds,$$

where $f$ and $g$ satisfy

$$|h(t, x) - h(t, y)|^2 + |g(t, x) - g(t, y)|^2 \leq \gamma|x - y|^2$$

in addition to the conditions stated in the theorem 2.2.1. Furthermore assume that $\zeta_1$ and $\zeta_2$ belong to $\mathcal{S}^{2m}_{\alpha,\beta}[G]$. Then using theorem 2.1.7 and Hölder's inequality we
arrive at the following estimate

\[ E \sup_{\alpha \leq s \leq t} |Y_1(s) - Y_2(s)|^{2m}_G \leq D_1 \left( E \sup_{\alpha \leq s \leq t} |\zeta_1(s) - \zeta_2(s)|^{2m}_G \right) + \\
+ \int_{\alpha}^{t} E \sup_{\alpha \leq s \leq u} |Y_1(s) - Y_2(s)|^{2m}_G du + \int_{\alpha}^{t} E \sup_{\alpha \leq s \leq u} |Y_1(s) - Y_2(s)|^{2m}_G du. \]

Invoking Gronwall's inequality we get

\[ E \sup_{\alpha \leq s \leq t} |Y_1(s) - Y_2(s)|^{2m}_G \leq D_1 E \sup_{\alpha \leq s \leq t} |\zeta_1(s) - \zeta_2(s)|^{2m}_G + \\
+ 2D_1 \int_{\alpha}^{t} e^{2D_1} E \sup_{\alpha \leq s \leq u} |\zeta_1(s) - \zeta_2(s)|^{2m}_G du. \quad (**) \]

Next we give another application of Ito's lemma.

**THEOREM 2.2.3** : Consider the following linear stochastic integral equation,

\[ \xi(t) = I + \int_{0}^{t} B(s)(dW(s))\xi(s) + \int_{0}^{t} \sigma(s)\xi(s)ds, \]

where \( B \) is in the affine space \( J + M_{0,\infty}^2[L_2(H; L_2(K))] \), \( J \in L(B_0; L(K)) \) and \( \sigma \in M_{0,\infty}^2[L_2(K)] \). This equation has a unique solution that is invertible almost surely for all \( t \).

**Proof** : Apply Ito's formula to the process \( \xi(t) \) and the function \( \rho : L(K) \to L(K) \) defined by \( \rho(A) = A^{-1} \). If \( \rho \) is defined at an element in \( L(K) \), then it is defined on a neighborhood of this element and it is infinitely Frechet differentiable in this neighborhood.

\( \rho' : L(K) \to L(L(K); L(K)) \) and \( \rho'' : L(K) \to L(L(K) \times L(K); L(K)) \).

\( \rho'(A)(D) = -A^{-1}DA^{-1} \) and \( \rho''(A)(D, E) = A^{-1}DA^{-1}EA^{-1} + A^{-1}EA^{-1}DA^{-1} \).

Therefore for those \( \omega \)'s where \( \xi(t)^{-1} \) exists,

\[ \rho'(\xi(t))B(t)\cdot\xi(t) = -\xi(t)^{-1}B(t)\cdot. \]
and

\[ \rho'(\xi(t))(\sigma(t)\xi(t)) = -\xi(t)^{-1}\sigma(t) \]

and

\[ \rho''(\xi(t))(B(t)(\cdot)\xi(t), B(t)(\cdot)\xi(t)) = 2\xi(t)^{-1}\{B(t)(\cdot)\}^2 \]

So

\[ \text{TRACE } \rho''(\xi(t))(B(t)(\cdot)\xi(t), B(t)(\cdot)\xi(t)) = 2\sum_{i=1}^{\infty} \xi(t)^{-1}\{B(t)(e_i)\}^2 \]

\[ = 2\xi(t)^{-1}\sum_{i=1}^{\infty}|B(t)(e_i)|^2. \]

The sum \( \sum_{i=1}^{\infty}|B(t)(e_i)|^2 \) converges in \( L_2(K) \) since

\[ |\sum_{i=m}^{\infty}|B(t)(e_i)|^2|_{L_2(K)} \leq \sum_{i=m}^{\infty}|B(t)(e_i)|^2_{L_2(K)} \]

\[ = \sum_{i=m}^{\infty}\sum_{j,k=1}^{\infty}|B(t)(e_i, l_j, l_k)|^2. \]

where \( \{l_j\} \) is an orthonormal basis of \( K \) and this last sum converges to zero as \( m \to \infty. \)

So we have shown that if the inverse of the solution \( \xi \) exists, it must satisfy the following linear stochastic integral equation

\[ \xi(t) = I - \int_{0}^{t} \xi(s)B(s)(dW(s)) + \int_{0}^{t} \xi(s)\{-\sigma(s) + \sum_{i=1}^{\infty}|B(t)(e_i)|^2\}ds. \]

This equation also satisfies the requirements of the existence and uniqueness theorem. It follows from theorem 2.1.8 that

\[ \xi(t)\xi(t) = I - \int_{0}^{t} \xi(s)\xi(s)B(s)(dW(s)) + \int_{0}^{t} B(s)(dW(s))\xi(s)\xi(s) + \]

\[ + \int_{0}^{t} \left(\sigma(s)\xi(s)\xi(s) + \xi(s)\xi(s)\{-\sigma(s) + \sum_{i=1}^{\infty}|B(s)(e_i)|^2\} - \right. \]

\[ - \sum_{i=1}^{\infty} B(s)(e_i)\xi(s)\xi(s)B(s)(e_i) \right)ds. \]
Obviously $\xi(s)\xi(s) = I$ is the unique solution of this equation. The same type of computation shows that $\xi(s)\xi(s) = I$, $\mathcal{P}$-a.s. Therefore for each $t$, $\xi(t)$ is invertible $\mathcal{P}$-a.s.. Since the process $\xi(t)$ is continuous almost surely, there exists an impossible event outside of which, $\xi(t)$ is invertible for all $t$.

Remark: Now consider the linear stochastic integral equation

$$\xi(t) = I + \int_0^t \xi(s)B(s)(dW(s)) + \int_0^t \xi(s)\sigma(s)ds$$

with $B$ and $\sigma$ satisfying the conditions stated in the above theorem. The same argument as in the proof of theorem 2.2.3 shows that $\xi(t)$ is invertible $\mathcal{P}$-a.s. for every $t$. It is shown in [K4] that if for each $x \in H$, $\{B(s)(x), \sigma(s)\}_{s \geq 0}$ forms a commutative family of operators in $L(K)$, then

$$\xi(t) = \exp \left( \int_0^t B(s)(dW(s)) + \int_0^t \left( \sigma(s) - \frac{1}{2} \sum_{i=1}^\infty (B(s)(e_i))^2 \right) ds \right)$$

It therefore follows that if $K = \mathbb{R}$, then the solution of the stochastic integral equation

$$\xi(t) = 1 + \int_0^t \xi(s)\langle B(s), dW(s) \rangle + \int_0^t \xi(s)\sigma(s)ds$$

is

$$\xi(t) = \exp \left( \int_0^t \langle B(s), dW(s) \rangle + \int_0^t (\sigma(s) - \frac{1}{2} ||B(s)||_H^2) ds \right).$$

§3-Equations depending on a parameter

It is shown in [K4] that if the coefficients $A$ and $\sigma$ of the $B$-valued stochastic integral equation $X^z(t) = x + \int_0^t A(s, X^z(s))dW(s) + \int_0^t \sigma(s, X^z(s))ds$ are differentiable in the directions of $H$ (in the second variable), then under some suitable conditions, the map $x \mapsto X^z(t)$ from $B$ to $\mathcal{M}^2_{0, t}[B]$ is $H - C^1$ and that the derivative is a process that satisfies a linear operator valued stochastic differential equation.
that is obtained by differentiating the above equation formally with respect to the starting point. Now we generalize this theorem to the case where the initial condition and the coefficients $A$ and $\sigma$ are smoothly dependent upon some variable.

**THEOREM 2.3.1**: Assume that $A$, $f$ and $\sigma$ satisfy the following conditions.

(a)- The map $f : B \to B$ is $H - C^1$ such that $Df(x) \in L(H;H)$ $\forall x$ and the function $Df : B \to L(H;H)$ is bounded and Lip-1.

(b)- $A$ is of the form $A(t,x,y) = J + K(t,x,y)$ where $J \in L(B,B)$, $K$ is a bounded continuous map from $[0, \infty) \times B \times B$ into $L_2(H)$ and $\sigma$ is a bounded continuous map from $[0, \infty) \times B \times B$ into $B$. Assume that for each $x$, $A(\cdot, x, \cdot)$ and $\sigma(\cdot, x, \cdot)$ both satisfy the conditions of the existence and uniqueness theorem.

(c)- For each $t$, $K(t,x,y)$ is $H - C^1$ in each variable $x$ and $y$, $D_1K(t,\cdot,x)$ and $D_2K(t,\cdot,x)$ are both bounded and $H$ Lip-1 and have values in $L_2(H;L_2(H))$. $D_1K(\cdot,\cdot,x)$ and $D_2K(\cdot,\cdot,x)$ are bounded continuous maps from $[0,s) \times B$ into $L_2(H;L_2(H))$ for each $s > 0$ and each $x \in B$.

(d)- $\sigma(t,x,y)$ is $H - C^1$ in each variable $x$ and $y$, $D_1\sigma(t,\cdot,x)$ and $D_2\sigma(t,\cdot,x)$ are both bounded and Lip-1 in the directions of $H$ and for each $s > 0$, $x \in B$, $D_1\sigma(t,\cdot,x)$ and $D_2\sigma(\cdot,\cdot,x)$ are both bounded and continuous maps from $[0,s) \times B$ into $L_2(H)$.

For each $x \in B$, let $\xi^x$ denote the solution of the stochastic integral equation

$$\xi^x(t) = f(x) + \int_0^t A(s,x,\xi^x(s))dW(s) + \int_0^t \sigma(s,x,\xi^x(s))ds,$$

then the map $B \to S^p_{0,s}(B)$ defined by $x \mapsto \xi^x$ is $H - C^1$ for every $p \geq 1$. Furthermore, for each $h \in H$, $D_x\xi^x(h)$ is the solution $\eta$ of the stochastic integral equation

$$\eta(t) = Df(x)h + \int_0^t DK(s,x,\xi^x(s))(h,\eta(s))dW(s) + \int_0^t D\sigma(s,x,\xi^x(s))(h,\eta(s))ds.$$
Proof: Throughout the proof, the norms in $S^p_{0,\theta}[B]$ and $S^p_{0,\theta}[L_2(H;K)]$ (where $K$ is a Hilbert space) will be denoted by $\|\cdot\|_{p,\theta}$ and $|\cdot|_{p,\theta}$ respectively. By Hölder’s inequality, we only need to consider the case where $p$ is an even integer. Theorem 2.1.7 implies that $\xi^z \in S^p_{0,z}[B]$ for all even $P$. From the hypothesis it follows that for each $h \in H, \xi_t^{z+h} - \xi_t^z \in H \forall x \in B, t > 0$. Now we define the processes

$$\Theta^h_t = \xi_t^{z+h} - \xi_t^z,$$

$$\Gamma^h_s = \int_0^1 DK(s, x + uh, \xi^z(s) + u\Theta^h_s)du$$

and

$$\Phi^h_s = \int_0^1 D\sigma(s, x + uh, \xi^z(s) + u\Theta^h_s)du.$$

So

$$\Theta^h_u = \{\int_0^1 Df(x + uh)du\}(h) + \int_0^u \Gamma^h_s(h, \Theta^h_s)dW(s) + \int_0^u \Phi^h_s(h, \Theta^h_s)ds.$$

Next we show that the map $x \mapsto \xi^z$ from $B$ to $S^p_{0,z}[B]$ is Lip-1. Throughout the proof $c_i$’s will be constants.

$$|\xi^{z+h} - \xi^z|_{p,u'}^p \leq c_1|h|^p + c_1\int_0^1 \Gamma^h_s(h, \Theta^h_s)dW(s)|_{p,u'}^p + c_1\int_0^1 \Phi^h_s(h, \Theta^h_s)ds|_{p,u'}^p.$$

Now,

$$\left|\int_0^1 \Gamma^h_s(h, \Theta^h_s)dW(s)\right|_{p,u'}^p \leq c_2 \int_0^u E\left|\Gamma^h_s(h, \Theta^h_s)\right|_{2}^p ds \quad \text{by theorem 2.1.7},$$

$$\leq c_3|h|^p + c_3 \int_0^u E|\Theta^h_s|_{H}^p ds.$$

Also

$$\left|\int_0^1 \Phi^h_s(h, \Theta^h_s)ds\right|_{p,u'}^p = E\left(\sup_{0 \leq s \leq u'} \left|\int_0^s \Phi^h_s(h, \Theta^h_s)ds\right|_{H}^p\right)$$

$$\leq E \int_0^u \left|\Phi^h_s(h, \Theta^h_s)\right|_{H}^p ds$$

$$\leq c_4|h|^p + c_4 \int_0^u E|\Theta^h_s|_{H}^p ds.$$
Therefore,

\[ |\xi^{z+h} - \xi^z|_{p,w'} \leq c_5 |h|^p + c_5 \int_0^{u'} E|\Theta^h_s|_H^p ds, \]

Since

\[ E|\Theta^h_s|_H^p \leq E \sup_{0 \leq t \leq s} |\Theta^h_t|_H^p = |\Theta^h_{(\cdot)}|_{p,s}^p, \]

we have

\[ |\Theta^h_{(\cdot)}|_{p,w'} \leq c_5 |h|^p + c_5 \int_0^{u'} |\Theta^h_{(\cdot)}|_{p,s}^p ds. \]

Invoking Gronwall’s lemma, we infer that

\[ |\xi^{z+h} - \xi^z|_{p,w'} \leq M|h|, \]

for some positive number M.

Now we have

\[ \xi^{z+h}(t) - \xi^z(t) - \eta(t) = \int_0^1 \{ Df(x + vh) - Df(x) \} dv(h) + \int_t^1 \{ \Gamma^h_s(h, \Theta^h_s) - DK(s, x, \xi^z_s)(h, \eta_s) \} + \int_t^1 \{ \Phi^h_s(h, \Theta^h_s) - D\sigma(s, x, \xi^z_s)(h, \eta_s) \} ds, \]

and

\[ \Gamma^h_s(h, \Theta^h_s) - DK(s, x, \xi^z_s)(h, \eta_s) = \Gamma^h_s(h, \Theta^h_s) - DK(s, x, \xi^z_s)(h, \Theta^h_s) + \]

\[ + DK(s, x, \xi^z_s)(0, \Theta^h_s) - DK(s, x, \xi^z_s)(h, \eta_s), \]

\[ = \int_0^1 \left( DK(s, x + vh, \xi^z_s + v\Theta^h_s) - DK(s, x, \xi^z_s) \right) dv(h, \Theta^h_s) + \]

\[ + DK(s, x, \xi^z_s)(0, \Theta^h_s - \eta_s). \]
So for \( 0 < u' < u \),

\[
\left| \int_0^{u'} \left( \Gamma_s^h(h, \Theta_s^h) - DK(s, x, \xi_s^x)(h, \eta_s) \right) dW(s) \right|^p_{p,u'} \leq c_6 \left\| \int_0^{u'} \left\{ DK(s, x + vh, \xi_s^x + v\Theta_s^h) - DK(s, x, \xi_s^x) \right\} dv(h, \Theta_s^h) dW(s) \right\|^p_{p,u'} + \\
c_6 \left\| DK(s, x, \xi_s^x)(0, \Theta_s^h - \eta_s) dW(s) \right\|^p_{p,u'}
\]

\[
\leq c_7 \int_0^{u'} E \left\{ DK(s, x + vh, \xi_s^x + v\Theta_s^h - DK(s, x, \xi_s^x)) dv(h, \Theta_s^h) \right\}^p_{L(H \times H; L_2(H))} ds + \\
c_7 \int_0^{u'} E \left\| DK(s, x, \xi_s^x)(0, \Theta_s^h - \eta_s) \right\|^p_{H} ds
\]

by Theorem 2.1.7.

\[
\leq c_8 \int_0^{u'} E \left( \int_0^{u'} \left| DK(s, x + vh, \xi_s^x + v\Theta_s^h) - DK(s, x, \xi_s^x) \right| dv \right)^p_{L(H \times H; L_2(H))} ds \times \sqrt{|h|_H^2 + |\Theta_s^h|_H^2} + c_8 \int_0^{u'} E|\Theta_s^h - \eta_s|_{H}^p ds
\]

\[
\leq c_9 \int_0^{u'} E \left( |h|_H^2 + |\Theta_s^h|_H^2 \right)^p ds + c_9 \int_0^{u'} E|\Theta_s^h - \eta_s|_{H}^p ds \quad DK \text{ is Lip} - 1.
\]

\[
\leq c_{10} |h|_H^{2p} + c_{10} \int_0^{u'} E \sup_{0 \leq t \leq s} |\Theta_t^h|_H^{2p} ds + c_{10} \int_0^{u'} E \sup_{0 \leq t \leq s} |\Theta_t^h - \eta_t|_{H}^p ds
\]

\[
= c_{10} |h|_H^{2p} + c_{10} \int_0^{u'} |\Theta_s^h|_{2p,s}^{2p} ds + c_{10} \int_0^{u'} |\Theta_s^h - \eta_s|_{p,s}^p ds.
\]

Note that \( |\Theta_s^h|_{2p,s}^{2p} \leq M^{2p}|h|_H^{2p} \forall s, 0 \leq s \leq u \). Therefore,

\[
\left\| \int_0^{u'} \left( \Gamma_s^h(h, \Theta_s^h) - DK(s, x, \xi_s^x)(h, \eta_s) \right) dW(s) \right\|^p_{p,u'} \leq c_{11} |h|_H^{2p} + c_{11} \int_0^{u'} |\Theta_s^h - \eta|_{p,s}^p ds.
\]
Similarly it can be shown that
\[ \left\| f - u'(h, \Theta - s^h) - D\sigma(s, x, \xi^x(h, \eta_s)) \right\|_{p, u'} \leq c_{12}\|h\|^2_H + c_{12} \int_0^{u'} \left\| \Theta^h_{(s)} - \eta \right\|_{p, s}^p \, ds. \]

Since by assumption, \( Df \) is Lip-1, we get from above estimates
\[ \left| \xi^{x+h} - \xi^x - \eta \right|_{p, u'}^p \leq c_{13}\|h\|^2_H + c_{13} \int_0^{u'} \left| \xi^{x+h} - \xi^x - \eta \right|_{p, s}^p \, ds. \]

Invoking Gronwall's lemma we infer that \( \left| \xi^{x+h} - \xi^x - \eta \right|_{p, u'}^p \leq c_{14}\|h\|^2_H \)

which implies that
\[ \left\| \xi^{x+h} - \xi^x - \eta \right\|_{p, u'} \leq c_{15}\left| \xi^{x+h} - \xi^x - \eta \right|_{p, u'}. \]

Therefore the map \( x \mapsto \xi^x \) from \( B \) to \( S^p_{0,t}[B] \) is Fréchet differentiable in the directions of \( H \). \( Q.E.D. \)

Using the same argument as in the proof of the above theorem, we arrive at the following result.

**THEOREM 2.3.2**: Let the functions \( f : B \mapsto S^p_{0,t}[H], \ g : B \mapsto S^p_{0,t}[L_2(H; L_2(H))] \) and \( z : B \mapsto S^p_{0,t}[L(H)] \) be Fréchet differentiable in the directions of \( H \) for every \( p \geq 1 \). Suppose \( z \) and \( g \) are bounded. If for each \( x \in B \), \( \xi^x \) is the solution of the linear stochastic integral equation
\[ \xi^x_t = \{ f(x) \}_t + \int_0^t \{ g(x) \}_s \xi^x(s) \, dW(s) + \int_0^t \{ z(x) \}_s \xi^x(s) \, ds, \]
then the map \( x \mapsto \xi^x \) from \( B \) to \( S^p_{0,t}[H] \) is Fréchet differentiable in the directions of \( H \) and the derivative \( \eta = \partial_x \xi^x(h) \) of this map in the direction of \( h \in H \) satisfies the stochastic integral equation
\[ \eta_t = \{ Df(x)(h) \}_t + \int_0^t \{ Dg(x)(h) \}_s \xi^x(s) \, dW(s) + \int_0^t \{ Dz(x)(h) \}_s \xi^x(s) \, ds + \int_0^t \{ g(x) \}_s \eta(s) \, dW(s) + \int_0^t \{ z(x) \}_s \eta(s) \, ds. \]
Note that the derivative of the solution \( \xi^x \) of the stochastic integral equation in theorem 2.3.1 is the solution of a linear stochastic integral equation which in turn is a differentiable function of \( x \) (by theorem 2.3.2), if one assumes higher order differentiability for the coefficients of the equation. Therefore combining the above two theorems, we arrive at the following result.

**THEOREM 2.3.3** : Suppose \( f, \sigma, A \) in theorem are bounded with bounded \( H \)-derivatives up to order \( m \). If the \( m \)-th derivative of each of these functions is \( \text{Lip-1} \) in the directions of \( H \), then the map \( x \mapsto \xi^x \) from \( B \) to \( S^q_{0,t}[B] \) is \( m \)-times Frechet differentiable in the directions of \( H \).

§4-Cameron Martin Girsanov Theorem

**LEMMA 2.4.1** : Let \( f \in \mathcal{M}^2_{0,T}[H] \) and for some \( \delta > 0 \),

\[
E \exp \left\{ (1 + \delta) \int_0^T |f(s)|^2_H \, ds \right\} < \infty.
\]

then

\[
E \exp \left\{ \int_{t_1}^{t_2} \langle f(s), dW(s) \rangle - \frac{1}{2} \int_{t_1}^{t_2} |f(s)|^2_H \, ds \right\} = 1
\]

for any \( 0 < t_1 < t_2 < T \).

**Proof** : Here we use the fact that the conclusion of the theorem holds when \( H \) is finite dimensional [F]. Therefore for each \( Q_n \) and \( 0 < t_1 < t_2 < T \) we have,

\[
E \exp \left\{ \int_{t_1}^{t_2} \langle Q_n f(s), dQ_n W(s) \rangle - \frac{1}{2} \int_{t_1}^{t_2} |Q_n f(s)|^2_H \, ds \right\} = 1.
\]

The equality \( \int_{t_1}^{t_2} \langle Q_n f(s), dQ_n W(s) \rangle = \int_{t_1}^{t_2} \langle Q_n f(s), dW(s) \rangle \), \( \forall n \), is obvious when \( f \) is a simple process in \( \mathcal{M}^2_{0,T}[H] \) and hence is true (by a passage to limit) for all \( f \in \mathcal{M}^2_{0,T}[H] \).
For each \( g \in \mathcal{M}^{2}_{0,T}[H] \), let

\[
\zeta(t_1, t_2, g) = \exp \left\{ \int_{t_1}^{t_2} \langle g(s), dW(s) \rangle - \frac{1}{2} \int_{t_1}^{t_2} |g(s)|^2_H ds \right\}.
\]

Now we show that the collection of random variables \( \left\{ \zeta(t_1, t_2, Q_n f) \right\}_{n=1}^{\infty} \) is uniformly integrable.

Let \( \epsilon > 0 \), then

\[
E \zeta(t_1, t_2, Q_n f)^{1+\epsilon} = E \exp \left\{ (1 + \epsilon) \int_{t_1}^{t_2} \langle Q_n f(s), dW(s) \rangle - \frac{(1 + \epsilon)^3}{2} \times \right.
\]

\[
\left. \times \int_{t_1}^{t_2} \langle Q_n f(s), dW(s) \rangle \exp \left\{ (1 + \epsilon)(2\epsilon + \epsilon^2) \int_{t_1}^{t_2} |Q_n f(s)|^2_H ds \right\} ds \right\}
\]

\[
\leq \left[ E \zeta(t_1, t_2, (1 + \epsilon)^2 Q_n f) \right]^{1+\epsilon} \left[ E \exp \left\{ \frac{(1 + \epsilon)^2(2 + \epsilon)}{2} \int_{t_1}^{t_2} |Q_n f(s)|^2_H ds \right\} \right]^{1+\epsilon}
\]

\[
= \left[ E \exp \left\{ \frac{(1 + \epsilon)^2(2 + \epsilon)}{2} \int_{t_1}^{t_2} |Q_n f(s)|^2_H ds \right\} \right]^{1+\epsilon}
\]

\[
\leq \left[ E \exp \left\{ \frac{(1 + \epsilon)^2(2 + \epsilon)}{2} \int_{t_1}^{t_2} |f(s)|^2_H ds \right\} \right]^{1+\epsilon} \text{ for all } n,
\]

\[
< \infty \text{ when } \epsilon \text{ is small enough.}
\]

Therefore the sequence \( \left\{ \zeta(t_1, t_2, Q_n f) \right\}_{n=1}^{\infty} \) is uniformly integrable.

We may assume (by taking a subsequence if necessary) that the sequence of random variables \( \int_{t_1}^{t_2} \langle Q_n f(s), dW(s) \rangle \) converges almost surely to \( \int_{t_1}^{t_2} \langle f(s), dW(s) \rangle \).

It therefore follows that \( \zeta(t_1, t_2, Q_n f) \to \zeta(t_1, t_2, f) \) \( \mathbb{P} \)-a.s.

Therefore from above we have

\[
1 = E \exp \left( \zeta(t_1, t_2, Q_n f) \right) \to E \exp \left( \zeta(t_1, t_2, f) \right) \text{ as } n \to \infty. \quad Q.E.D.
\]

Remark 1: It follows from the remark following theorem 2.2.3 that \( \zeta(t_1, t_2, f) \) satisfies the stochastic integral equation

\[
\xi(t) = 1 + \int_{t_1}^{t_2} \xi(s) \langle f(s), dW(s) \rangle.
\]
THEOREM 2.4.2 (Cameron Martin Girsanov): Let \( \{W(t)\}_{0 \leq t \leq T} \) be a \( B \)-valued Brownian motion adapted to the filtration \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) as defined in section 1. If \( f \in \mathcal{M}_{0,\infty}[H] \) and \( \zeta(t_1, t_2, f) \) is as in the proof of theorem 2.4.1, Then the process \( \tilde{W}(t) = W(t) - \int_0^t f(s)ds \) is a Brownian motion adapted to \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) with respect to the probability measure \( \tilde{\mathbb{P}}(d\omega) = \exp\zeta(0, T, f)\mathbb{P}(d\omega) \).

Proof: By theorem 2.4.1, \( \tilde{\mathbb{P}}(d\omega) \) is a probability measure. Let \( \tilde{E} \) denote the expectation with respect to measure \( \tilde{\mathbb{P}} \). If \( h \in B^* \), then

\[
\tilde{E}e^{i(h, \tilde{W}_t - \tilde{W}_s)} = \int_{\Omega} e^{i(h, \tilde{W}_t - \tilde{W}_s)} \zeta(0, T, f)\mathbb{P}(d\omega)
\]

\[
= \int \lim_{n \to \infty} e^{i(h, \zeta_n(\tilde{W}_t - \tilde{W}_s))} \zeta(0, T, Q_n f)\mathbb{P}(d\omega)
\]

\[
= \lim_{n \to \infty} \int e^{i(h, \zeta_n(\tilde{W}_t - \tilde{W}_s))} \zeta(0, T, Q_n f)\mathbb{P}(d\omega)
\]

since the collection \( \{ \zeta(0, T, Q_n f) \}_{n=1}^{\infty} \) is uniformly integrable,

\[
= \lim_{n \to \infty} e^{-\frac{1}{2}Q_n h^2(t-s)}
= e^{-\frac{1}{2}h^2(t-s)}
\]

So the \( B \)-valued random variable \( \tilde{W}_t - \tilde{W}_s \) is distributed according to \( p_{t-s} \).

Now let \( A \in \mathcal{F}_s \), then for each real number \( \lambda \).

\[
\tilde{E}e^{i\lambda(h, \tilde{W}_t - \tilde{W}_s)}e^{i\lambda A} = \int e^{i\lambda(h, \tilde{W}_t - \tilde{W}_s) + i\lambda A} \zeta(0, T, f)\mathbb{P}(d\omega)
\]

\[
= \lim_{n \to \infty} \int e^{i\lambda(h, \zeta_n(\tilde{W}_t - \tilde{W}_s)) + i\lambda A} \zeta(0, T, Q_n f)\mathbb{P}(d\omega)
\]

\[
= \lim_{n \to \infty} \int e^{i\lambda(Q_n h, \zeta_n(\tilde{W}_t - \tilde{W}_s))} \zeta(0, T, Q_n f)\mathbb{P}(d\omega) \times \int_A e^{i\lambda(0, T, Q_n f)}\mathbb{P}(d\omega)
\]

\[
= \int e^{i\lambda(h, \tilde{W}_t - \tilde{W}_s)} \zeta(0, T, f)\mathbb{P}(d\omega) \int_A e^{i\lambda(0, T, f)}\mathbb{P}(d\omega)
\]

\[
= \tilde{E}e^{i\lambda(h, \tilde{W}_t - \tilde{W}_s)} \tilde{E}e^{i\lambda A}.
\]

Hence the random variable \( \langle h, \tilde{W}_t - \tilde{W}_s \rangle \) is independent of the \( \sigma \)-algebra \( \mathcal{F}_s \). This implies that the \( B \)-valued random variable \( \tilde{W}_t - \tilde{W}_s \) is independent of \( \mathcal{F}_s \). The other conditions are obviously satisfied. The theorem is proved.
Let $f$ be a bounded nonanticipating process and $\tilde{W}(t)$ be as in theorem 2.4.2 and let $\mathcal{M}_{\alpha,\beta}^p[L_2(H;K)]$ denote the Banach space of nonanticipating processes with state space $L_2(H;K)$ that are $p$–integrable with respect to the measure $ds \times \mathcal{P}(d\omega)$. Assume that $g$ belongs to the space $\mathcal{M}_{\alpha,\beta}^2[L_2(H;K)]$. Then

$$\mathcal{E} \int_\alpha^\beta |g(s)|^2 ds = \int_\Omega \int_\alpha^\beta |g(s)|^2 \zeta(\alpha,\beta,f) \mathcal{P}(d\omega)$$

$$\leq C \int_\Omega \int_\alpha^\beta |g(s)|^2 \mathcal{P}(d\omega)$$

$$< \infty.$$ Therefore $g \in \mathcal{M}_{\alpha,\beta}^2[L_2(H;K)]$.

Now let $\{g_k\}$ be a sequence of simple processes in $\mathcal{M}_{\alpha,\beta}^2[L(B_0;K)]$ converging to $g$ in $\mathcal{M}_{\alpha,\beta}^2[L_2(H;K)]$, then $g_k$ converges to $g$ in $\mathcal{M}_{\alpha,\beta}^2[L_2(H;K)]$. Without loss of generality we may assume the following,

$$\int_\alpha^\beta g_k(s)dW(s) \rightarrow \int_\alpha^\beta g(s)dW(s) \text{ a.s.}$$

and

$$\int_\alpha^\beta g_k(s)d\tilde{W}(s) \rightarrow \int_\alpha^\beta g(s)d\tilde{W}(s) \text{ a.s.}$$

But it is readily seen that

$$\int_\alpha^\beta g_k(s)d\tilde{W}(s) = \int_\alpha^\beta g_k(s)dW(s) - \int_\alpha^\beta g_k(s)(f(s))ds.$$ So a passage to limit gives us

$$\int_\alpha^\beta g(s)d\tilde{W}(s) = \int_\alpha^\beta g(s)dW(s) - \int_\alpha^\beta g(s)(f(s))ds.$$ Now we use Girsanov theorem to obtain some results concerning the measures induced by a stochastic differential equation of the type

$$d\xi(t) = dW(t) + \sigma(t,\xi(t))dt \quad (*)$$
where the bounded $H$ valued function $\sigma(\cdot, \cdot)$ satisfies the conditions of the existence and uniqueness theorem. The solution of the equation (*) is a Markov process with transition probability $p(s, x, t, dy) = \mathcal{P}\{\xi_{s,x}(t) \in dy\}$ where $\xi_{s,x}$ is the solution that satisfies the initial condition $\xi(s) = x$ $\mathcal{P}$-a.s..

Consider the stochastic processes

$$\xi'_{s,x}(t) = x + W(t) - W(s)$$

and

$$\hat{W}(t) = W(t) - \int_s^t \sigma(u, \xi'_{s,x}(u))du \quad t \geq s.$$ 

By theorem 2.4.2, $\hat{W}(t)$ is a Brownian motion with respect to the probability measure

$$\hat{\mathcal{P}}(d\omega) = \exp\{\int_s^t \langle \sigma(u, \xi'_{s,x}(u)), dW(u) \rangle - \frac{1}{2} \int_s^t |\sigma(u, \xi'_{s,x}(u))|^2_H du\} \mathcal{P}(d\omega).$$

Obviously the measure induced by $\xi'_{s,x}(t)$, is the Wiener measure $p_{t-s}(x, dy)$ defined on $B$ that is smooth and quasi-invariant along the directions of $H$. Therefore,

$$p(s, x, t, dy) = \mathcal{P}(\xi_{s,x}(t) \in dy)$$

$$= \hat{\mathcal{P}}(\xi'_{s,x}(t) \in dy) \quad \text{by Girsanov theorem},$$

$$= 1_{\xi'_{s,x}(t) \in dy} \hat{\mathcal{P}}(d\omega)$$

$$= 1_{\xi'_{s,x}(t) \in dy} \frac{d\hat{\mathcal{P}}}{d\mathcal{P}}(\omega) \mathcal{P}(d\omega)$$

$$= \mathcal{E}\left(\frac{d\hat{\mathcal{P}}}{d\mathcal{P}}(\omega)\Big| \xi'_{s,x}(t) = y\right) \mathcal{P}_{\xi'_{s,x}(t)}(dy)$$

$$= \mathcal{E}\left(\frac{d\hat{\mathcal{P}}}{d\mathcal{P}}(\omega)\Big| \xi'_{s,x}(t) = y\right) p_{t-s}(x, dy)$$

where $\mathcal{E}\left(\frac{d\hat{\mathcal{P}}}{d\mathcal{P}}(\omega)\Big| \xi'_{s,x}(t) = y\right) : B \rightarrow \mathbb{R}$ is a Borel measurable function that belongs to $L^1(p_{t-s}(x, dy))$. Hence the measure $p(s, x, t, dy)$ is continuous in the directions
of $H$. Furthermore for each $h \in H$,

$$p(s, x, t, dy + h) = E\left( \frac{dP}{dP}(\omega) \bigg| \xi'_{s,x} = y + h \right) p_{t-s}(x + h, dy)$$
$$\ll p_{t-s}(x, dy)$$
$$= E\left( \frac{dP}{dP}(\omega) \bigg| \xi'_{s,x}(t) = y \right) p(s, x, t, dy)$$
$$\ll p(s, x, t, dy).$$

Therefore for each stochastic differential equation of the form (*), we obtain a large collection $\{ p(s, x, t, dy) \}_{0 < s < t, x \in B}$ of (in general non-Gaussian) measures that are quasi-invariant in the directions of $H$. 
\section{Smooth measures induced by solutions of stochastic differential equations}

In this section we use the Girsanov theorem to obtain some results on smoothness of measures induced by solutions of stochastic integral equations. We start by considering first a stochastic integral equation of the type

\[ \xi_t(t) = x + \int_0^t A(s)dW(s) + \int_0^t \sigma(s, \xi_t(s))ds \quad (\ast) \]

Where \( A(t) = I + K(t) \), \( K(\cdot) : [0, \infty) \rightarrow L_2(H) \) is continuous, \( K \in M_{0, \infty}[L_2(H)] \). \( A(t) \) restricted to \( H \) is bounded and separated from zero for all \( t \geq 0 \) and the \( H \) valued function \( \sigma(\cdot, \cdot) \) is bounded and satisfies the requirements of the existence and uniqueness theorem. In section 4 of chapter 2 we proved that in the special case where \( A(t) = I \), the measure induced by \( \xi_t(t) \) is continuous and quasi-invariant along the directions of \( H \). It was also mentioned there that the distribution of the random variable \( \xi_t(t) \) with respect to measure \( \mathcal{P} \) is exactly the same as the distribution of the random variable \( \xi'_t(t) = x + \int_0^t A(s)dW(s) \) with respect to the measure

\[ \hat{\mathcal{P}}(d\omega) = \Gamma(\omega)\mathcal{P}(d\omega) \]

where

\[ \Gamma = \exp \left\{ \int_0^t \langle A(s)^{-1}\sigma(s, \xi'_t(s)), dW(s) \rangle - \frac{1}{2} \int_0^t |A(s)^{-1}\sigma(s, \xi'_t(s))|^2_H ds \right\}. \]

Now consider the perturbed process \( W^h(t) = W(t) - \int_0^t A(s)^{-1}h ds \) and the corresponding Girsanov density

\[ G_h = \exp \left\{ \int_0^t \langle A(s)^{-1}h, dW(s) \rangle - \frac{1}{2} \int_0^t \|A(s)^{-1}h\|^2_H ds \right\}. \]
We use the perturbed equation
\[ \xi_x^{th}(t) = x + \int_0^t A(s) dW^h(s) \]
\[ = x + \int_0^t A(s) dW(s) - th \]
and the perturbed random variable
\[ \Gamma_h = \exp \left\{ \int_0^t \langle A(s)^{-1} \sigma(s, \xi_x^{th}(s)), dW^h(s) \rangle - \frac{1}{2} \int_0^t |A(s)^{-1} \sigma(s, \xi_x^{th}(s))|^2_H \right\} \mathcal{P}(d\omega) \]
to prove the following theorem.

**Theorem 3.1.1:** Suppose the coefficients \( A \) and \( \sigma \) of the equation (*) satisfy the conditions stated above. In addition suppose that the coefficient \( \sigma(s, x) \) is bounded and \( H - C^1 \) in \( x \) variable with bounded derivative for every \( s \geq 0 \). Then the measure \( \mathcal{P}_{\xi_x(t)} \) induced by the solution \( \xi_x \) at time \( t > 0 \) is \( H - C^1 \) in the sense of Fomin. Furthermore if \( \sigma(s, x) \) is \( H - C^n \) in the \( x \) variable with bounded derivatives, then the measure \( \mathcal{P}_{\xi_x(t)} \) is \( H - C^n \) in the sense of Fomin.

**Proof:** Let \( f \) be an \( H - C^1 \) real valued measurable function defined on \( B \) with bounded derivative. By the Cameron Martin Girsanov theorem, for each \( h \in H \), the distribution of the random variable \( f(\xi_x^{th}(t)) \Gamma_h \) with respect to the measure \( G_h(\omega) \mathcal{P}(d\omega) \) is exactly the same as the distribution of \( f(\xi_x(t)) \Gamma(\omega) \) with respect to \( \mathcal{P}(\omega) \), hence the function defined by
\[ h \to \int f(\xi_x^{th}(t)) \Gamma_h(\omega) G_h(\omega) \mathcal{P}(d\omega) \]
\[ = \int f(\xi_x^{th}(t)) \exp \left\{ \int_0^t \langle A(s)^{-1} \sigma(s, \xi_x^{th}(s)), dW(s) \rangle - \right\} \]
\[ \left\{ \int_0^t \langle A(s)^{-1} \sigma(s, \xi_x^{th}(s)), A(s)^{-1} h \rangle ds - \frac{1}{2} \int_0^t |A(s)^{-1} \sigma(s, \xi_x^{th}(s))|^2_H ds \right\} G_h(\omega) d\mathcal{P} \]
is constant. The differentiability of each factor of the integrand in the sense that would enable us to differentiate inside the integral sign has been established before.
So we differentiate with respect to \( h \) in the direction of a vector \( k \in H \) and then let \( h = 0 \). As the derivative is equal to 0, integration by part gives us

\[
\int D f(\xi_x(t))(k) \Gamma(\omega) \mathcal{P}(d\omega) = -\int f(\xi_x(t)) \left\{ \int_0^t \langle A(s)^{-1} D \sigma(s, \xi_x(s))(k), dW(s) \rangle + \right.
\]

\[
+ \frac{1}{t} \int_0^t \langle A(s)^{-1} \sigma(s, \xi_x(s)), A(s)^{-1} k \rangle ds
\]

\[- \int_0^t \langle A(s)^{-1} D \sigma(s, \xi_x'(s))(k), A(s)^{-1} \sigma(s, \xi_x'(s)) \rangle ds - \frac{1}{t} \int_0^t \langle A(s)^{-1} k, dW(s) \rangle \right\} \Gamma d\mathcal{P}.
\]

So by Girsanov theorem, we have

\[
\int D f(y) \mathcal{P}_{\xi_x(t)}(dy) = \int D f(\xi_x(t)) \mathcal{P}(d\omega)
\]

\[
= -\int f(\xi_x'(t)) \left\{ \int_0^t \langle A(s)^{-1} D \sigma(s, \xi_x'(s))(k), dW(s) \rangle + \right.
\]

\[
+ \frac{1}{t} \int_0^t \langle A(s)^{-1} \sigma(s, \xi_x'(s)), A(s)^{-1} k \rangle ds
\]

\[- \int_0^t \langle A(s)^{-1} D \sigma(s, \xi_x'(s))(k), A(s)^{-1} \sigma(s, \xi_x'(s)) \rangle
\]

\[- \frac{1}{t} \int_0^t \langle A(s)^{-1} k, dW(s) \rangle \right\} \Gamma(\omega) d\mathcal{P}
\]

\[
= \int f(\xi_x'(t))(J_1(\omega), k) \Gamma(\omega) d\mathcal{P}(d\omega)
\]

\[
= \int f(y) \left( E(J_1(\omega) | \xi_x'(t) = y), k \right) \mathcal{P}_{\xi_x(t)}(dy)
\]

where \( J_1 : \Omega \rightarrow B \) is a \( \mathcal{F}_t/B(B) \) random element. It follows from theorem 1.2.10 of chapter 1 that the measure \( \mathcal{P}_{\xi_x(t)}(dy) \) is \( H - C^1 \) in the sense of Fomin and

\[
d_{F,k} \mathcal{P}_{\xi_x(t)}(dy) = \left( E(J_1(\omega) | \xi_x'(t) = y), k \right) \mathcal{P}_{\xi_x(t)}(dy).
\]
Now we assume that the function $\sigma(s, x)$ is twice differentiable in the $x$ variable with bounded derivatives. For each element $k_2 \in H$, consider the map

$$ h \mapsto \int f(\xi_x^h(t)) \left\{ \int_0^t \langle A(s)^{-1} D\sigma(s, \xi_x^h(s))(k_2), dW^h(s) \rangle + \frac{1}{t} \int_0^t \langle A(s)^{-1} \sigma(s, \xi_x^h(s)), A(s)^{-1} k_2 \rangle ds - \int_0^t \langle A(s)^{-1} D\sigma(s, \xi_x^h(s))(k_2), A(s)^{-1} \sigma(s, \xi_x^h(s)) \rangle ds - \frac{1}{t} \int_0^t \langle A(s)^{-1} k_2, dW^h(s) \rangle \right\} \Gamma_h(\omega) G_h(\omega) \mathcal{P}(d\omega) $$

$$ = \int f(\xi_x^h(t)) \left\{ \int_0^t \langle A(s)^{-1} D\sigma(s, \xi_x^h(s))(k_2), dW(s) \rangle - \frac{1}{t} \int_0^t \langle A(s)^{-1} D\sigma(s, \xi_x^h(s))(k_2), A(s)^{-1} h ds \rangle + \frac{1}{t} \int_0^t \langle A(s)^{-1} \sigma(s, \xi_x^h(s)), A(s)^{-1} k_2 \rangle ds - \int_0^t \langle A(s)^{-1} D\sigma(s, \xi_x^h(s))(k_2), A(s)^{-1} \sigma(s, \xi_x^h(s)) \rangle ds - \frac{1}{t} \int_0^t \langle A(s)^{-1} k_2, dW(s) \rangle + \frac{1}{t} \int_0^t \langle A(s)^{-1} k_2, A(s)^{-1} h \rangle ds \right\} \Gamma_h(\omega) G_h(\omega) \mathcal{P}(d\omega). $$

Clearly the distribution of the random variable

$$ f(\xi_x^h(t)) \left\{ \int_0^t \langle A(s)^{-1} D\sigma(s, \xi_x^h(s))(k_2), dW^h(s) \rangle + \frac{1}{t} \int_0^t \langle A(s)^{-1} \sigma(s, \xi_x^h(s)), A(s)^{-1} k_2 \rangle ds - \frac{1}{t} \int_0^t \langle A(s)^{-1} k_2, dW^h(s) \rangle - \int_0^t \langle A(s)^{-1} D\sigma(s, \xi_x^h(s))(k_2), A(s)^{-1} \sigma(s, \xi_x^h(s)) \rangle ds \right\} \Gamma_h $$

with respect to the measure $G_h(\omega) \mathcal{P}(d\omega)$ is independent of $h$ and is the same as the distribution of the random variable $f(\xi_x^h(t))(J_1(\omega), k_2)$ with respect to the measure.
\( \mathcal{P}(d\omega) \). Therefore the above map is a constant map by Girsanov theorem. Differentiating this map with respect to \( h \) in the direction of \( k_1 \in H \) and then setting \( h = 0 \) gives us

\[
\int D^2 f(y)(k_1, k_2) \mathcal{P}_{\xi(t)}(dy) = -\int Df(y)(k_1) \left( E \{ J_1(\omega) \mid \xi(t) = y \} , k_2 \right)
\]

\[
= \int f(\xi'(t)) \left\{ \left( \int_0^t \langle A(s)^{-1} D^2 \sigma(s, \xi'(s))(k_1, k_2), dW(s) \rangle \right) + \frac{1}{t} \int_0^t \langle A(s)^{-1} D\sigma(s, \xi'(s))(k_2), A(s)^{-1}(k_1)ds \rangle 
\right.
\]

\[
+ \frac{1}{t} \int_0^t \langle A(s)^{-1} D\sigma(s, \xi'(s))k_1, A(s)^{-1}(k_2)ds \rangle 
\right.
\]

\[
- \int_0^t \langle A(s)^{-1} D\sigma(s, \xi'(s))(k_2), A(s)^{-1} D\sigma(s, \xi'(s))(k_1)ds \rangle 
\right.
\]

\[
- \int_0^t \langle A(s)^{-1} D^2 \sigma(s, \xi'(s))(k_1, k_2), A(s)^{-1}(s, \xi'(s))(ds) 
\right.
\]

\[
- \frac{1}{t^2} \int_0^t \langle A(s)^{-1} k_2, A(s)^{-1}(k_1)ds \rangle 
\right.
\]

\[
+ \left( \int_0^t \langle A(s)^{-1} D\sigma(s, \xi'(s))(k_2), dW(s) \rangle + \right.
\]

\[
+ \frac{1}{t} \int_0^t \langle A(s)^{-1} \sigma(s, \xi'(s)), A(s)^{-1}(k_2)ds \rangle 
\right.
\]

\[
- \int_0^t \langle A(s)^{-1} D\sigma(s, \xi'(s))(k_2), A(s)^{-1} \sigma(s, \xi'(s))(ds) 
\right.
\]

\[
- \frac{1}{t} \int_0^t \langle A(s)^{-1} k_2, dW(s) \rangle \right) \times 
\]

\[
\times \left( \int_0^t \langle A(s)^{-1} D\sigma(s, \xi'(s))(k_1), dW(s) \rangle + \right.
\]

\[
+ \frac{1}{t} \int_0^t \langle A(s)^{-1} \sigma(s, \xi'(s)), A(s)^{-1}(k_1)ds \rangle 
\right.
\]

\[
- \int_0^t \langle A(s)^{-1} D\sigma(s, \xi'(s))(k_1), A(s)^{-1} \sigma(s, \xi'(s))(ds) 
\right.
\]

\[
- \frac{1}{t} \int_0^t \langle A(s)^{-1} k_1, dW(s) \rangle \right) \right\} \Gamma(\omega) \mathcal{P}(d\omega). 
\]
Therefore,

\[
\int D^2 f(y)(k_1, k_2) \mathcal{P}_{\xi_x(t)}(dy) = \int D^2 f(\xi_x(t))(k_1, k_2) \mathcal{P}(d\omega)
\]

\[
= \int D^2 f(\xi_x(t))(k_1, k_2) \Gamma(\omega) \mathcal{P}(d\omega)
\]

\[
= \int f(\xi_x(t))J_2(\omega, k_1, k_2) \Gamma(\omega) \mathcal{P}(d\omega)
\]

\[
= \int f(y) E(J_2(\omega, k_1, k_2)|\xi_x(t) = y) \mathcal{P}_{\xi_x(t)}(dy),
\]

where \( J_2 \) is linear in \( k_1 \) and \( k_2 \) and defines a \( \mathcal{F}_t \) measurable map. Also it is clear from above that the map \((k_1, k_2) \mapsto E|J_2(\omega, k_1, k_2)|\) is continuous. So by the Girsanov theorem we have.

\[
\int D^2 f(y)(k_1, k_2) \mathcal{P}_{\xi_x(t)}(dy) = - \int Df(y)(k_1) \left( E(J_1(\omega)|\xi_x' = y), k_2 \right) \mathcal{P}_{\xi_x(t)}(dy)
\]

\[
= \int f(y) E(J_2(\omega, k_1, k_2)|\xi_x(t) = y) \mathcal{P}_{\xi_x(t)}(dy).
\]

Therefore the measure \( \mathcal{P}_{\xi_x(t)}(dy + x) \) is \( H - C^2 \) in the sense of Fomin. It is also obvious that if \( \sigma(s, \cdot) \) has higher order derivatives in the directions of \( H \) with bounded derivatives, then one can continue as above to prove higher order differentiability of the measure \( \mathcal{P}_{\xi_x(t)} \) in the sense of Fomin. \( Q.E.D. \)

**Example 1**: To check our results we consider the simplest case; \( \xi_x(t) = x + W(t) \).

Obviously \( \mathcal{P}_{\xi_x(t)}(dy) = p_t(dy - x) \). From above it follows that

\[
d_{F,k} \mathcal{P}_{\xi_x(t)}(dy - x) = -\frac{1}{t} (k, y) \mathcal{P}_{\xi_x(t)}(dy - x)
\]

and

\[
d_{F:k_1,k_2}^2 \mathcal{P}_{\xi_x(t)}(dy - x) = \left\{ -\frac{1}{t} (k_1, k_2) + \frac{1}{t^2} (k_1, y)(k_2, y) \right\} \mathcal{P}_{\xi_x(t)}(dy - x).
\]

**Example 2**: In this example we consider the stochastic integral equation

\[
\xi_x(t) = x + \int_0^t dW(s) + \int_0^t f(\xi(s))ds
\]
where \( f : B_0 \rightarrow H \) is \( H - C^2 \) such that \( Df(x) \in L_2(H) \) and \( D^2f(x) \) is a trace class type bilinear map from \( H \times H \) into \( H \). Also assume that the maps \( Df : B_0 \rightarrow L_2(H) \) and \( \text{TRACE } D^2f : B_0 \rightarrow H \) are continuous and bounded and \( Df(x) \) is symmetric for all \( x \) in \( B_0 \). By the above theorem, the measure \( \mathcal{P}_{\xi(t)} \) is \( H - C^2 \) in the sense of Fomin and the first derivative of its translation along \( x \), \( \mathcal{P}_{\xi(t)}(x;dy) \), is given by

\[
d_{F,H}\mathcal{P}_{\xi(t)}(x;dy) = E\left( \int_0^t \langle Df(W(s) + x)(h), dW(s) \rangle + \frac{1}{t} \int_0^t \langle f(W(s) + x), k \rangle \right. \\
\left. - \int_0^t \langle Df(W(s) + x)(k), f(W(s) + x) \rangle ds \right) \\
- \frac{1}{t} (k, W(t)) \bigg| W(t) = y \bigg) \mathcal{P}_{\xi(t)}(x;dy).
\]

Using Ito's formula, we get the following formula:

\[
d_{F,H}\mathcal{P}_{\xi(t)}(x;dy) = \left( f(y + x) - f(x) + E\left( - \int_0^t \sum_{i=1}^\infty D^2f(W(s) + x)(e_i, e_i) ds \right) \\
+ \frac{1}{t} \int_0^t f(W(s) + x) ds - \int_0^t Df(W(s) + x)f(W(s) + x) ds \bigg| W(t) = y \right) \\
- \frac{1}{t} y . h \bigg) \mathcal{P}_{\xi(t)}(x;dy).
\]

Now we Consider the stochastic integral equation

\[
\xi_s(t) = x + \int_0^t A(s, \xi(s)) dW(s) + \int_0^t \sigma(s, \xi(s)) ds.
\]

where \( A \) and \( \sigma \) satisfy the following conditions:

(1) : \( A(t, x) = 1 + K(t, x) \) where \( K : [0, \infty) \times B \rightarrow L_2(H) \) is a continuous map that is \( H - C^2 \) in \( x \) variable, \( DK(t, x) \in L_2(H; L_2(H)) \) and \( D^2K(t, x) \in L_2(H; L_2(H; L_2(H))) \) and \( DK(\cdot, \cdot) \) and \( D^2K(\cdot, \cdot) \) are bounded continuous maps from \([0, \infty) \times B \) into \( L_2(H; L_2(H)) \) and \( L_2(H; L_2(H; L_2(H))) \) respectively. \( \sigma(t, x) \)
is $H - C^2$ in $x$ variable with $D\sigma(t, x) \in L_2(H)$ and $D^2\sigma(t, x) \in L_2(H; L_2(H))$. $D\sigma(\cdot, \cdot)$ and $D^2\sigma(\cdot, \cdot)$ are bounded and continuous maps from $[0, \infty) \times B$ into $L_2(H)$ and $L_2(H; L_2(H))$ respectively.

It follows from the results obtained in chapter 2 that the process $\{\xi_x(t)\}_{0 \leq t < \infty}$ belongs to $S^p_{0, \infty}[B]$ for every $p \geq 1$ and the map $x \mapsto \xi_x$ from $B$ to $S^p_{0, \infty}[B]$ is $H - C^2$. Let $Y = D_x \xi_x$. Then the process $Y \in S^p_{0, \infty}[L(H)]$ satisfies the linear operator valued stochastic integral equation.

$$Y(t) = I + \int_0^t DK(s, \xi_x(s))(Y(s))(dW(s)) + \int_0^t D\sigma(s, \xi_x(s))(Y(s))ds$$

By theorem 2.2.3, the process $Y$ is invertible almost surely for all $t \geq 0$ and the inverse process $\hat{Y}$ satisfies the operator valued linear stochastic integral equation,

$$\hat{Y}(t) = I - \int_0^t \hat{Y}(s)DK(s, \xi_x(s))^*dW(s) + \int_0^t \hat{Y}(s)\{ - D\sigma(s, \xi_x(s)) + \sum_{i=1}^\infty \{DK(s, \xi_x(s))^*(e_i)\}^2\}ds.$$  

Here $F^*$ is the adjoint of $F$, i.e., $F^*(h)(k) = F(k)(h)$ for every $h, k \in H$.

(2) : Assume that there exists $\epsilon > 0$ such that for every $h \in H$, $s \in [0, \infty)$ and $x \in B$, $|A(s, x)h|_H \geq \epsilon|h|_H$. This condition implies that for each $s$ and $x$, the operator $A(s, x)$ restricted to $H$ is an invertible operator from $H$ to $H$ and it's inverse is bounded.

Let $\phi^m(s) = A(s, \xi_x(s))^{-1}Y(s)$ if $|A(s, \xi_x(s))^{-1}Y(s)|_{L(H)} \leq m$ and 0 otherwise. For each $h \in H$, consider the perturbed process $W^{m,h}(t) = W(t) - \int_0^t \phi^m(s)(h)ds$. By Girsanov theorem, $\{W^{m,h}(t)\}_{0 \leq t < \infty}$ is a Brownian motion (adapted to $F_t$) with respect to the probability measure $P_h(d\omega) = G(h)P(d\omega)$ where $G(h)$ is the Girsanov density.
Consider the perturbed stochastic integral equation

\[ \xi_z^{m,h}(t) = x + \int_0^t A(s, \xi_z^{m,h}(s))dW^m,\xi_z^{m,h}(s) + \int_0^t \sigma(s, \xi_z^{m,h}(s))ds \]

By theorem 2.3.2, for each \( T > 0 \), the map \( h \mapsto \xi_z^{m,h} \) from \( H \) to \( S_{0,T}^p[B] \) is Frechet differentiable and the process \( \eta^m = D_h \xi_z^{m,h} \bigg|_{h=0} \) satisfies the following stochastic integral equation

\[ \eta^m(t) = \int_0^t DK(s, \xi_z(s))(\eta^m(s))dW(s) + \int_0^t \left\{ D\sigma(s, \xi_z(s))\eta^m(s) - A(s, \xi_z(s))\phi^m(s) \right\}ds. \]

Similarity of this equation with the one satisfied by \( Y(t) \) leads us to look for a process \( \Theta(t) = \Theta_1(t)dW(t) + \Theta_2(t)dt \) such that \( d\eta^m(t) = d(Y(t)\Theta(t)) \). If such a process exists, then by theorem 2.1.8,

\[
d\eta^m(t) = Y(t)\Theta_1(t)dW(t) + DK(t, \xi_z(t))^\ast(dW(t))(Y(t)\Theta(t)) \\
+ \left\{ D\sigma(t, \xi_z(t))Y(t)\Theta(t) + Y(t)\Theta_2(t) \right\} \\
+ \sum_{i=1}^\infty DK(t, \xi_z(t))(Y(t))(c_i)\Theta_1(t)(c_i)dt \\
= Y(t)\Theta_1(t)dW(t) + DK(t, \xi_z(t))(\eta^m(t))(dW(t)) \\
+ \left\{ D\sigma(t, \xi_z(t))\eta^m(t) + Y(t)\Theta_2(t) \right\} \\
+ \sum_{i=1}^\infty DK(t, \xi_z(t))(Y(t))(c_i)\Theta_1(t)(c_i)dt.
\]

Obviously \( \Theta_1(t) = 0 \) for all \( t \) and \( \omega \) and therefore \( Y(t)\Theta_2(t) = -A(t, \xi_z^{m,h}(t))\phi^m(t) \) or \( d\Theta_2(t) = -\dot{Y}(t)A(t, \xi_z^{m,h}(t))\phi^m(t) \). Hence

\[ \Theta(t) = -\int_0^t \dot{Y}(s)A(s, \xi_z^{m,h}(s))\phi^m(s)ds \]
and

$$\eta^m(t) = -Y(t)\int_0^t \bar{Y}(s)A(s, \xi_{m,h}^x(s))\phi^m(s)ds \quad \mathcal{P} - a.s..$$

Note that almost all sample paths of the processes $\bar{Y}(t)$ and $Y(t)$ are bounded since both of them belong to the affine space $I + S_{0,\infty}^p[L_2(H)]$ for every $p \geq 1$. Lebesgue dominated convergence theorem implies that for almost every $\omega \in \Omega$, $\eta^m(t)$ converges to $-tY(t)$ as $m \to \infty$.

Next for each $h \in H$, consider the perturbed stochastic integral equation

$$Y_{m,h}(t) = I + \int_0^t DK(s, \xi_{m,h}^x(s))(Y_{m,h}(s))dW_{m,h}(s)$$

$$+ \int_0^t D\sigma(s, \xi_{m,h}^x(s))Y_{m,h}(s)ds$$

$$= I + \int_0^t DK(s, \xi_{m,h}^x(s))(Y_{m,h}(s))dW(s) + \int_0^t \left\{ D\sigma(s, \xi_{m,h}^x(s))Y_{m,h}(s) - DK(s, \xi_{m,h}^x(s))(Y_{m,h}(s))\phi^m(s)h \right\}ds.$$

It follows from theorem 2.3.2 that the map $h \to Y_{m,h}$ from $H$ to $S_{0,\infty}^p[L(H)]$ is Frechet differentiable. The process $\zeta^m = D_hY_{m,h}\Big|_{h=0}$ satisfies the following stochastic integral equation.

$$\zeta^m(t)(\cdot) = \int_0^t D^2K(s, \xi_x(s))(\eta^m(s)(\cdot))(Y(s), dW(s))$$

$$+ \int_0^t DK(s, \xi_x(s))(\zeta^m(s)(\cdot))(dW(s))$$

$$+ \int_0^t \left\{ D^2\sigma(s, \xi_x(s))(\eta^m(s)(\cdot))Y(s) + D\sigma(s, \xi_x(s))\zeta^m(s)(\cdot) - DK(s, \xi_x(s))(Y(s))\phi^m(s) \right\}ds.$$

Since the process

$$\left\{ \int_0^u D^2K(s, \xi_x(s))(\eta^m(s)(\cdot))(Y(s), dW(s)) + \int_0^u (D^2\sigma(s, \xi_x(s))(\eta^m(s)(\cdot))Y(s)$$

$$- DK(s, \xi_x(s))(Y(s))\phi^m(s)(\cdot))ds \right\}_{0 \leq u \leq \infty}$$
belongs to $S_{0,\infty}^p[L_2(H; L_2(H))]$, it therefore follows from section 2 of chapter 2 that the above stochastic integral equation has a unique solution in $S_{0,\infty}^p[L_2(H; L_2(H))]$. The process $\phi^m$ converges in $S_{0,\infty}^p[L(H)]$ to the process $A(\cdot, \xi_\tau(\cdot))Y(\cdot)$. By theorem 2.2.2, the solution $\zeta^m$ of the above stochastic integral equation converges in $S_{0,\infty}^p[L_2(H; L_2(H))]$ to a process $\zeta$ that satisfies the same equation with $\zeta^m(s)$ and $\phi^m(s)$ replaced by $\zeta(s)$ and $A(s, \xi_\tau(s))^{-1}Y(s)$.

It easily follows from our discussion above that for each $t \geq 0$ the map $h \mapsto Y^{m,h}(t)$ from $H$ to $L^p(d\mathcal{P})$ is Frechet differentiable for all $p \geq 1$. Therefore the map $h \mapsto \tilde{Y}^{m,h}(t)$ from $H$ to $L^p(d\mathcal{P})$ is also differentiable for each $t \geq 0$ and all $p \geq 1$ and

$$D_h \tilde{Y}^{m,h}(t)\bigg|_{h=0}(\cdot) = -\tilde{Y}(t)D_h Y^{m,h}(t)\bigg|_{h=0}(\cdot)\tilde{Y}(t) = -\tilde{Y}(t)\zeta^m(t)(\cdot)\tilde{Y}(t).$$

Let $f$ be an $H - C^1$ function on $B$ with bounded derivative. For each $k \in H$, consider the map $\theta$ from $H$ to $R$ defined by

$$\theta(h) = \int_{B} f(\xi^{m,h}_x)\langle \tilde{Y}^{m,h}(t)k, e_i \rangle G_h(\omega)d\mathcal{P}(d\omega).$$

The fact that the Girsanov density $G_h$ satisfies the stochastic equation

$$\lambda(t) = 1 + \int_{0}^{t} \lambda(s)(\phi^m(s)(h), dW(s))$$

implies that the map $h \mapsto G_h$ from $H$ to $L^p(d\mathcal{P})$ is Frechet differentiable for every $p \geq 1$. This together with differentiability of the maps $h \mapsto \xi^{m,h}_x(t)$ and $h \mapsto \tilde{Y}^{m,h}(t)$ imply that the map $h \mapsto \theta(h)$ is Frechet differentiable. Now, by Girsanov theorem, the distribution of the random variable $f(\xi^{m,h}_x(t))\langle \tilde{Y}^{m,h}(t)k, e_i \rangle$ with respect to the measure $G_h(\omega)d\mathcal{P}(d\omega)$ is exactly the same as the distribution of the random variable $f(\xi_{\tau}(t))\langle \tilde{Y}(t)k, e_i \rangle$ with respect to the measure $\mathcal{P}(d\omega)$. Therefore, the function $\theta$ must be a constant function. Differentiating $\theta$ with respect to
and then setting $h = 0$ gives us,

$$
\int Df(\xi_x(t))\eta^m(t)(e_i)(\tilde{Y}(t)k, e_i)d\mathcal{P} = -\int f(\xi_x(s))\left\{ -\langle \tilde{Y}(t)(\zeta^m(t)(e_i))\tilde{Y}(t)k, e_i \rangle \\
+ \langle \tilde{Y}(t)k, e_i \rangle \int_0^t \langle \phi^m(s)(e_i), dW(s) \rangle \right\}d\mathcal{P}.
$$

Taking limit as $m \to \infty$, we get

$$
\int Df(\xi_x(t))Y(t)((Y(t)k, e_i)e_i)d\mathcal{P} = -\int f(\xi_x(t))\left\{ \frac{1}{t}\langle \tilde{Y}(t)(\zeta(t)(e_i))\tilde{Y}(t)k, e_i \rangle \\
- \frac{1}{t}\langle \tilde{Y}(t)k, e_i \rangle \int_0^t \langle A(s, \xi_x(s))^{-1}(e_i), dW(s) \rangle \right\}d\mathcal{P}.
$$

Now,

$$
E\left| \sum_{i=n}^{m} Df(\xi_x(t))Y(t)((\tilde{Y}(t)k, e_i)e_i) \right|^2 \leq C_1 E\left| Y(t) \right|_H^2 \left| \sum_{i=n}^{m} (\tilde{Y}(t)k, e_i)e_i \right|_H^2 \\
\leq C_1 E\left\{ \left| Y(t) \right|_{L^1(H)}^2 \left| \sum_{i=n}^{m} (\tilde{Y}(t)k, e_i)e_i \right|_H^2 \right\} \\
= C_1 E\left\{ \left| Y(t) \right|_{L^1(H)}^2 \left( \sum_{i=n}^{m} (\tilde{Y}(t)k, e_i)^2 \right) \right\} \\
\to 0 \quad \text{as } n, m \to \infty,
$$

since $Y(t) \in L^p(d\mathcal{P})$ for all $p \geq 1$.

So taking summation with respect to $i$ of the left hand side of $(\#)$ we get

$$
\sum_{i=1}^{\infty} \int Df(\xi_x(t))Y(t)((\tilde{Y}(t)k, e_i)e_i)d\mathcal{P} = \int Df(\xi_x(t))Y(t)(\tilde{Y}(t)k)d\mathcal{P} \\
= \int Df(\xi_x(t))(k)d\mathcal{P}.
$$

Unfortunately the sum (with respect to $i$) of the integrand in the right hand side of the equation $(\#)$ does not always converge in $L^1(d\mathcal{P})$. To ensure convergence, we need to impose further assumptions on $K$ and $\sigma$. 

THEOREM 3.1.2: Suppose the coefficients $K$ and $\sigma$ of the equation (**) satisfy conditions (1) and (2) above. In addition suppose $K(s, x) = CK'(s, x)$ where $C \in L(H; B^*) \cup L_1(H)$ and $K'$ satisfies the same conditions mentioned in (1) and (2). Furthermore suppose $D\sigma(s, x) \in L(H, B^*)$ and $(D^2\sigma(s, x))(h) \in L_1(H)$ for all $h \in H$, $s \in [0, \infty)$ and $x \in B$ and $D\sigma$ and $D^2\sigma(\cdot, \cdot)(h)$ have bounded ranges in $L(H, B^*)$ and $L_1(H)$ respectively for every $h$ in $H$. Then the measure generated by $\xi_x(t)$ is $H - C^1$ in the sense of Fomin.

Proof: We only prove the case where $C \in L(H; B^*)$. The proof for the case $C \in L_1(H)$ is similar. Under the above assumptions the process $Y$ which solves the equation

$$Y(t) = I + C \int_0^t DK'(s, \xi_x(s))(Y(s))(dW(s)) + \int_0^t D\sigma(s, \xi_x(s))Y(s)ds$$

belongs to the affine space $I + S_{0, \infty}^p[L(H; B^*)]$. Its inverse process which satisfies another linear stochastic integral equation, also belongs to the same space. Now we prove the convergence in $L^2(d\mathcal{P})$ of the sum

$$\sum_{i=1}^\infty \langle \bar{Y}(t)k, e_i \rangle \int_0^t \langle A(s, \xi_x(s))^{-1}Y(s)e_i, dW(s) \rangle.$$

Now $Y(t) = I + Z_1(t)$ where $Z_1 \in S_{0, \infty}^p[L(H; B^*)] \forall p \geq 1$ and $A(t, \xi_x(t))^{-1}Y(t) = I + Z_2(t)$ where $Z_2 \in S_{0, \infty}^p[L(H)]$ for all $p \geq 1$.

So

$$\sum_{i=1}^\infty \langle \bar{Y}(t)k, e_i \rangle \int_0^t \langle A(s, \xi_x(s))^{-1}Y(s)e_i, dW(s) \rangle$$

$$= \sum_{i=1}^\infty \left\{ \langle k, e_i \rangle \langle e_i, W(t) \rangle + \langle Z_1(t)k, e_i \rangle \langle e_i, W(t) \rangle + \langle k, e_i \rangle \int_0^t \langle Z_2(s)e_i, dW(s) \rangle + \langle Z_1(t)k, e_i \rangle \int_0^t \langle Z_2(s)e_i, dW(s) \rangle \right\}.$$  

The sum $\sum_{i=1}^\infty \langle k, e_i \rangle \langle e_i, W(t) \rangle$ converges to the random variable $\langle k, W(t) \rangle$ almost surely and in $L^2(d\mathcal{P})$ by Kolmogorov's two series theorem.
Next we have
\[ \sum_{i=1}^{n} \langle Z_1(t)k, e_i \rangle (c_i, W(t)) = \langle Z_1(t)k, \left( \sum_{i=1}^{n} (c_i, W(t)) e_i \right) \rangle \]
\[ \rightarrow \langle Z_1(t)k, W(t) \rangle \quad \mathcal{P} \text{ a.s.} \]
Note that \( Z_1(t)k \in B^* \)
\[ = \langle k, Z_1(t)^*(W(t)) \rangle. \]

Convergence of this series in \( L^2(d\mathcal{P}) \) is obvious.

Now for each \( \omega \) outside a \( \mathcal{P} \)-null set,
\[ \sum_{i=1}^{\infty} \langle k, e_i \rangle \langle \int_{0}^{t} Z_2(s)^* dW(s), e_i \rangle \leq \sqrt{\sum_{i=1}^{\infty} \langle k, e_i \rangle^2} \sqrt{\sum_{i=1}^{\infty} \langle \int_{0}^{t} Z_2(s)^* dW(s), e_i \rangle^2} \]
\[ = |k|_H \left| \int_{0}^{t} Z_2(s)^* dW(s) \right|_H \]
\[ \leq |k|_H \left| \int_{0}^{t} Z_2(s)^* dW(s) \right|_H \]
\[ < \infty. \]

And from these estimates it follows that the third series converges in \( L^2(d\mathcal{P}) \).

Outside a \( \mathcal{P} \)-null set, we consider the fourth sum,
\[ \sum_{i=1}^{\infty} \langle Z_1(t), e_i \rangle \int_{0}^{t} \langle Z_2(s)e_i, dW(s) \rangle \]
\[ \leq \sqrt{\sum_{i=1}^{\infty} \langle Z_1(t)k, e_i \rangle^2} \sqrt{\sum_{i=1}^{\infty} \langle \int_{0}^{t} Z_2(s)^* dW(s), e_i \rangle^2} \]
\[ = \left| Z_1(t)k \right|_H \left| \int_{0}^{t} Z_2(s)^* dW(s) \right|_H. \]

Therefore,
\[ E \left| \sum_{i=1}^{\infty} \langle Z_1(t)k, e_i \rangle \int_{0}^{t} \langle Z_2(s)e_i, dW(s) \rangle \right| \leq E \left| Z_1(t)k \right|_H^2 \sqrt{E \left| \int_{0}^{t} Z_2(s)^* dW(s) \right|_H^2} \]
\[ = \sqrt{E \left| Z_1(t)k \right|_H^2} \int_{0}^{t} E \left| Z_2(s)^* \right|_{L^2(H)}^2 ds \]
\[ < \infty. \]
We have also shown that for each \( \omega \in \Omega \) outside a \( \mathcal{P} \)-null set, there exists an element \( \phi_1(\omega) \in H \) such that

\[
\sum_{i=1}^{\infty} \langle \tilde{Y}(t)k, e_i \rangle \int_0^t (A(s, \xi_x(s))^{-1}Y(s)e_i, dW(s)) = \langle k, \phi_1 \rangle + (k, W(t)) \cdot
\]

It is seen readily that the map \( \omega \to \phi_1(\omega) \) from \( \Omega \) to \( H \) is an \( \mathcal{F}_t/B(H) \) random element.

Now we prove the convergence of the following series:

\[
\sum_{i=1}^{\infty} \langle \tilde{Y}(t)(\zeta(t)(e_i)) \tilde{Y}(t)k, e_i \rangle.
\]

where

\[
\zeta(t)(e_i) = C \int_0^t D^2K(s, \xi_x(s))(Y(s)(e_i))(Y(s), dW(s))
\]

\[
+ C \int_0^t DK(s, \xi_x(s))(\zeta(s)(e_i))dW(s)
\]

\[
+ \int_0^t \left\{ D^2\sigma(s, \xi_x(s))(Y(s)(e_i))Y(s) + D\sigma(s, \xi_x(s))\zeta(s)(e_i) -
\right.
\]

\[
- CDK(s, \xi_x(s))(Y(s))A(s, \xi_x(s))^{-1}(e_i) \right\} ds
\]

\[
= CJ(t)(e_i) + \int_0^t \left\{ D^2\sigma(s, \xi_x(s))(Y(s)e_i)Y(s) + D\sigma(s, \xi_x(s))\zeta(s)e_i \right\} ds
\]

and \( J(t) \) has values in \( L_2(H; L_2(H)) \); \( E|J(t)|_2^p < \infty \) for all \( p \geq 1 \) and \( \zeta \) belongs to \( S_{0, \infty}^p[L_2(H; L_2(H))] \).

So

\[
\sum_{i=1}^{\infty} \langle \tilde{Y}(t)(\zeta(t)(e_i)) \tilde{Y}(t)k, e_i \rangle = \sum_{i=1}^{\infty} \langle J(t)(e_i) \tilde{Y}(t)(k), C^*\tilde{Y}(t)^*(e_i) \rangle
\]

\[
+ \sum_{i=1}^{\infty} \left\langle \int_0^t D^2\sigma(s, \xi_x(s))(Y(s)(e_i))Y(s)ds \tilde{Y}(t)(k), \tilde{Y}(t)^*e_i \right\rangle
\]

\[
+ \sum_{i=1}^{\infty} \left\langle \int_0^t D\sigma(s, \xi_x(s))\zeta(s)(e_i)ds \tilde{Y}(t)(k), \tilde{Y}(t)^*e_i \right\rangle.
\]
Now for each $\omega \in \Omega$ outside a $\mathcal{P}$-null set, we have

$$\sum_{i=1}^{\infty} \langle J(t)(e_i), \tilde{Y}(t)(k) \rangle \leq \sum_{i=1}^{\infty} |J(t)(e_i)|_{L_2(H)} |\tilde{C} \tilde{Y}(t)^*(e_i)|_H$$

$$\leq |\tilde{Y}(t)(k)|_H \sum_{i=1}^{\infty} |J(t)(e_i)|_{L_2(H)} |\tilde{C} \tilde{Y}(t)^*(e_i)|_H$$

$$\leq |\tilde{Y}(t)(k)|_H \sqrt{\sum_{i=1}^{\infty} |J(t)(e_i)|_{L_2(H)}^2} \sqrt{\sum_{i=1}^{\infty} |\tilde{C} \tilde{Y}(t)^*(e_i)|^2}$$

$$\leq |\tilde{Y}(t)|_{L(H)}^2 |k|_H |J(t)|_{L_2(H)} |\tilde{C} \tilde{Y}(t)^*|_{L_2(H)}$$

$$\leq |\tilde{Y}(t)|_{L(H)} |J(t)|_{L_2(H)} |\tilde{C} |_{L_2(H)} |k|_H$$

$$< \infty$$

The above inequalities also show that the series converges in $L^2(d\mathcal{P})$ and that for each $\omega \in \Omega$ outside a $\mathcal{P}$ measure 0, there exists an element $\phi_2(\omega) \in H$ such that

$$\sum_{i=1}^{\infty} \langle J(t)(e_i), \tilde{Y}(t)(k) \rangle \tilde{C} \tilde{Y}(t)^*(e_i) = \langle k, \phi_2(\omega) \rangle.$$ 

Obviously the map $\omega \mapsto \phi_2(\omega)$ from $\Omega$ to $H$ is $\mathcal{F}_t/B(H)$ measurable.

For each $\omega \in \Omega$ outside a $\mathcal{P}$-null set we have

$$\sum_{i=1}^{\infty} \int_0^t \langle \tilde{Y}(t)D^2\sigma(s, \xi_x(s))(Y(s)(e_i)) \tilde{Y}(t)(k), e_i \rangle ds$$

$$= \sum_{i=1}^{\infty} \int_0^t \langle \tilde{Y}(t)D^2\sigma(s, \xi_x(s))(Y(s)\tilde{Y}(t)(k))Y(s)(e_i), e_i \rangle ds$$

$$= \sum_{i=1}^{\infty} \int_0^t \langle \tilde{Y}(t)D^2\sigma(s, \xi_x(s))(Y(s)\tilde{Y}(t)(k))Y(s) ds e_i, e_i \rangle.$$ 

Here we used the fact that $D^2\sigma(s, x)(h, k) = D^2\sigma(s, x)(k, h)$, $\forall k, h \in H$. Noting
that the last integral converges in $L_1(H)$, we continue

$$= \text{trace} \int_0^t \tilde{Y}(t)D^2\sigma(s, \xi_z(s))(Y(s)\tilde{Y}(t)(k))Y(s)ds$$

$$= \int_0^t \text{trace}\{\tilde{Y}(t)D^2\sigma(s, \xi_z(s))(Y(s)\tilde{Y}(t)(k))Y(s)\}ds$$

$$\leq \int_0^t |\tilde{Y}(t)|_{L(H)}|D^2\sigma(s, \xi_z(s))|_{L(H, L_1(H))}|Y(s)\tilde{Y}(t)(k)|_H |Y(s)|_{L(H)}ds$$

$$\leq M_1(\omega)|\tilde{Y}(t)|_{L(H)}^2 \int_0^t |Y(s)|_{L(H)}^2 ds|k|_H$$

where $M_1$ is a constant.

So

$$\sum_{i=1}^\infty \int_0^t \langle \tilde{Y}(t)D^2\sigma(s, \xi_z(s))(Y(s)(e_i))Y(s)\tilde{Y}(t)(k), e_i \rangle_H = \langle \phi_3, k \rangle_H.$$ 

It is obvious that this sum converges in $L^2(d\mathcal{P})$ and that the map $\omega \mapsto \phi_3(\omega)$ is a $\mathcal{F}_t/\mathcal{B}(H)$ random element.

Finally we consider the last series. For each $\omega \in \Omega$ outside a set of $\mathcal{P}$ measure zero, we have

$$\sum_{i=1}^\infty \int_0^t \langle \tilde{Y}(t)D\sigma(s, \xi_z(s))\zeta(s)(e_i)\tilde{Y}(t)(k), e_i \rangle ds$$

$$= \sum_{i=1}^\infty \int_0^t \langle \tilde{Y}(t)D\sigma(s, \xi_z(s))\zeta(s)^*(\tilde{Y}(t)(k))(e_i), e_i \rangle ds$$

$$= \sum_{i=1}^\infty \langle \int_0^t \tilde{Y}(t)D\sigma(s, \xi_z(s))\zeta(s)^*(\tilde{Y}(t)(k))dsc, e_i \rangle$$

the integral converges in $L_1(H)$.

$$= \text{trace} \int_0^t \tilde{Y}(t)D\sigma(s, \xi_z(s))\zeta(s)^*(\tilde{Y}(t)(k))ds$$

$$= \int_0^t \text{trace} \tilde{Y}(t)D\sigma(s, \xi_z(s))\zeta(s)^*(\tilde{Y}(t)(k))ds$$

$$\leq M_2(\omega)|\tilde{Y}(t)|_{L(H)}^2 \int_0^t |\zeta(s)^*|_{L_2(H, L_2(H))} ds|k|_H.$$ 

So outside a set of measure zero, the above sum converges $\mathcal{P}$-a.s. and in $L^2(d\mathcal{P})$. 
Also for each $\omega \in \Omega$, outside a $\mathcal{P}$-null set, there exists an element $\phi_4(\omega)$ such that

$$
\sum_{i=1}^{\infty} \int_{0}^{t} \langle \bar{Y}(t) D\sigma(s, \xi_x(s)) \zeta(s)(e_i) \bar{Y}(t)(k), e_i \rangle = \langle \phi_4, k \rangle.
$$

and $\phi_4$ defines a $\mathcal{F}_t/\mathcal{B}(H)$ random element.

Let

$$
\phi(\omega) = -\frac{1}{\bar{t}} \phi_1(\omega) + \frac{1}{\bar{t}} \left\{ \phi_2(\omega) + \phi_3(\omega) + \phi_4(\omega) \right\}.
$$

Then,

$$
\sum_{i=1}^{\infty} \left\{ \frac{1}{\bar{t}} \langle D_h Y_t \rangle_{h=0} (e_i)(k), e_i \rangle - \frac{1}{\bar{t}} \langle \bar{Y}(t)(k), e_i \rangle \int_{0}^{t} \langle A(s, \xi_x(s))^{-1} Y(s)(e_i), dW(s) \rangle \right\}
$$

$$
= \langle \phi, k \rangle - \frac{1}{\bar{t}} \left( k, W(t) \right).
$$

The convergence is almost surely and in $L^2(d\mathcal{P})$.

Therefore for every $H - C^1$ function $f : B \rightarrow R$ with bounded derivative, we have

$$
\int_{\Omega} Df(\xi_x(t))(k) d\mathcal{P} = - \int_{\Omega} f(\xi_x(t)) \left\{ \langle \phi, k \rangle - \frac{1}{\bar{t}} \left( k, W(t) \right) \right\} d\mathcal{P}
$$

Or

$$
\int_{B} Df(y)(k) \mathcal{P}_{\xi_x(t)}(dy) = - \int_{B} f(y) \left\{ \langle E(\phi(\omega)|\xi_x(t) = y), k \rangle 
- \frac{1}{\bar{t}} \left( k, E(W(t)|\xi_x(t) = y) \right) \right\} \mathcal{P}_{\xi_x(t)}(dy).
$$

By theorem 1.2.9, the measure $\mathcal{P}_{\xi_x(t)}$ is differentiable in the directions of $H$ and

$$
d_{F, k} \mathcal{P}_{\xi_x(t)}(dy) = \left( E(\phi(\omega) - \frac{1}{\bar{t}} W(t)(\omega)|\xi_x(t) = y), k \right) \mathcal{P}_{\xi_x(t)}(dy)
$$

$$
= (g(y), k) \mathcal{P}_{\xi_x(t)}(dy)
$$

where $g : B \rightarrow B$ is $\mathcal{B}(B)/\mathcal{B}(B)$ measurable.
Furthermore,
\[
\sqrt{\int_B \langle g(y), k \rangle^2 \mathcal{P}_{\xi, t}(dy)} \leq \sqrt{\int_B \langle E(\phi(y)|\xi(t) = y), k \rangle^2 \mathcal{P}_{\xi, t}(dy) + \frac{1}{t} \sqrt{\int_B \langle E(W(t)|\xi(t) = y), k \rangle^2 \mathcal{P}_{\xi, t}(dy)}}
\]
\[
= \sqrt{\int_B \langle \phi(y), k \rangle^2 \mathcal{P}(dy) + \frac{1}{t} \int_B \langle k, W(t) \rangle^2 \mathcal{P}(dy)}
\]
\[
\leq M_3 |k|_H.
\]

So the map \( H \to L^2(d\mathcal{P}_{\xi, t}) \) defined by
\[
k \mapsto dF_k \mathcal{P}_{\xi, t}(dy)
\]
is bounded and linear. By theorem 1.2.10, the measure \( \mathcal{P}_{\xi, t}(dy) \) is Frechet differentiable in the directions of \( H \) in the sense of Fomin, i.e., for each \( A \in B(B) \), the map \( h \mapsto \mathcal{P}_{\xi, t}(A + h) \) from \( H \) to \( R \) is Frechet differentiable. \( \quad Q.E.D. \)

\section{Kolmogorov's equations}

Consider the stochastic differential equation
\[
d\xi(t) = A(t, \xi(t))dW(t) + \sigma(t, \xi(t))dt.
\]
where the coefficients \( A \) and \( \sigma \) satisfy the conditions of the existence and uniqueness theorem. There exists a Markov process whose transition probability coincides with \( p(t, x, s, dy) = \mathcal{P}(\xi_x(s) \in dy) \), where the process \( \{\xi_{x,t}(u)\}_{u \geq t} \) is the solution of the above stochastic differential equation that satisfies the initial condition \( \xi(t) = x \), almost surely. In addition to the assumptions stated in the existence and uniqueness theorem, assume that the coefficients \( A \) and \( \sigma \) satisfy the assumptions of theorem 2.3.1.
Fix $s > 0$ and $f \in \mathcal{D}$ and let $u(x,t) = Ef(\xi_x, t(s))$ where $0 \leq t \leq s$. An application of Ito's lemma gives us

\[
\frac{1}{h} E \left\{ f(\xi_{x, t-h}(t)) - f(x) \right\} = \frac{1}{h} \int_{t-h}^{t} Df(\xi_{x, t-h}(u))A(u, \xi_{x, t-h}(u))dW(u) + \frac{1}{h} \int_{t-h}^{t} \left\{ Df(\xi_{x, t-h}(u))\sigma(u, \xi_{t-h}(u)) + \frac{1}{2} \text{TRACE} D^2f(\xi_{x, t-h}(u)) \left[ \sigma(u, \xi_{x, t-h}(u)) \times \sigma(u, \xi_{x, t-h}(u)) \right] \right\} du.
\]

As the expectation of the stochastic integral is equal to zero; upon taking limit as $h \rightarrow 0$, we find that

\[
\lim_{h \rightarrow 0} \left\{ Ef(\xi_{x, t-h}(t)) - f(x) \right\} = Df(x)\sigma(t,x) + \frac{1}{2} \text{TRACE} D^2f(x) \left[ \sigma(t,x) \times \sigma(t,x) \right].
\]

Now

\[
u_{x.t-t-h) = Ef(\xi_{x, t-h}(s))
\]

\[= E u(\xi_{x, t-h}(t), t) \quad \text{By the Markov property of the solution } \xi.\]

So

\[
\lim_{h \rightarrow 0} \frac{1}{h} E \left[ u(\xi_{x, t-h}(t), t) - u(x,t) \right] = Df(x)(\sigma(t,x)) + \frac{1}{2} \text{TRACE} D^2f(x) (\sigma(t,x), \sigma(t,x)).
\]

But from above we have

\[
\frac{u(x,t) - u(x,t - h)}{h} = -E \frac{u(\xi_{x, t-h}(t), t) - u(x,t)}{h}.
\]

By the results obtained in chapter 2, the right hand side of this equation is differentiable with respect to $x$. Hence taking the limit as $h \rightarrow 0$, we find that

\[
\frac{\partial}{\partial t} u(x,t) \bigg|_t = -D u(x)(\sigma(t,x)) - \frac{1}{2} \sum_{i=1}^{\infty} \langle D^2 u(x)(\sigma(t,x)e_i), \sigma(t,x)e_i \rangle.\]
Remark [K4]: If $A$ and $\sigma$ are time independent, then the differential operator

$$ L = \frac{\partial}{\partial t} - D(\cdot)(\sigma(t,x)) - \frac{1}{2} \sum_{i=1}^{\infty} \langle D^2(\cdot)(x)(\sigma(t,x)e_i), \sigma(t,x)e_i \rangle $$

generates a semigroup $G$ on the Banach space of bounded continuous functions vanishing at infinity.

In the case where the stochastic differential equation takes values in an $n$-dimensional Euclidean space, it is known that if the coefficients are smooth, then the transition probability associated with the solution has a smooth density with respect to the Lebesgue measure, i.e., for each $0 \leq s < t$ and $x \in B$, there exists a smooth function $y \mapsto p(s,x;t,y)$ such that $P(\xi_t \in dy) = p(s,x;t,dy) = p(s,x;t,y)dy$. Furthermore, the function $p(s,x;t,y)$ satisfies the Kolmogorov's forward equation:

$$ \frac{\partial}{\partial t} \{p(s,x;t,y)\} = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j} \langle A(y,t)p(s,x;t,y)e_i, e_j \rangle - \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \langle \sigma(y,t)p(s,x;t,y), e_i \rangle $$

Where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis.

**THEOREM 3.2.1**: Consider the stochastic differential equation

$$ d\xi(t) = A(t,\xi(t))dW(t) + \sigma(t,\xi(t)) $$

Where $A$ and $\sigma$ satisfy the requirements of the existence and uniqueness theorem. Suppose $\sigma(\cdot,\cdot)$ is a $B^*$ valued bounded function that is $H - C^1$ and $D\sigma(t,x) \in L_1(H) \forall t \geq 0$ and $x \in B$. Let $A(t,\cdot) = I + K(t,\cdot)$ be such that $K(t,\cdot)$ is a bounded $H - C^2$ function taking values in the collection of positive definite Hilbert Schmidt operators. Furthermore, suppose $DK(t,x) \in L(H;L_1(H))$ and $D^2K(t,x) \in L_1(L_2(H);L_2(H))$ are bounded functions that satisfy the conditions mentioned in theorem 1.2.14. If the transition probability $p(s,x;t,dy)$ that is associated with the solution of the above SDE is $H - C^2$ in the sense of Fomin and...
has logarithmic derivatives as in theorem 1.2.14 and if $\text{trace } D^2 p(s, x; t, dy)$ exists in $\mathcal{D}'$, then the following holds as distributions in $\mathcal{D}'$.

$$\frac{\partial}{\partial t} p(s, x; t, dy) = \text{trace } D \left( \sigma(t, y) p(s, x; t, dy) \right) +$$

$$+ \frac{1}{2} \sum_{i,j} D^2 \left\{ A(t, y)^2 (e_i, e_j) p(s, x; t, dy) \right\} (e_i, e_j).$$

**Proof:** For each test function $f$, we have

$$\frac{1}{h} \int_B f(y) \left\{ p(s, x; t + h, dy) - p(s, x; t, dy) \right\} = \frac{1}{h} E \left\{ f(\xi, x(t + h)) - f(\xi, x(t)) \right\}$$

$$= \frac{1}{h} \int_{\Omega} \left\{ \int_t^{t+h} \left( Df(\xi, x(u)) \sigma(u, \xi, x(u)) \right) +$$

$$+ \frac{1}{2} \text{trace } A(u, \xi, x(u))^2 D^2 f(u, \xi, x(u)) \right\} du \right\} d\mathcal{P}.$$

Noting that the expectation of the stochastic integral is zero, we continue

$$= \frac{1}{h} \int_{\Omega} \left\{ \int_t^{t+h} \left( Df(\xi, x(u)) \sigma(u, \xi, x(u)) \right) +$$

$$+ \frac{1}{2} \text{trace } A(u, \xi, x(u))^2 D^2 f(u, \xi, x(u)) A(u, \xi, x(u)) \right\} du \right\} d\mathcal{P}.$$

Letting $h$ go to zero and using Lebesgue’s dominated convergence theorem, we get

$$\int_B f(y) \frac{\partial}{\partial t} p(s, x; t, dy) = \int_{\Omega} \left\{ (Df(\xi, x(t))), \sigma(t, \xi, x(t)) \right\} +$$

$$+ \frac{1}{2} \text{trace } A(t, \xi, x(t))^* D^2 f(t, \xi, x(t)) A(t, \xi, x(t)) \right\} d\mathcal{P}$$

$$= \int_{\Omega} \left\{ (Df(y), \sigma(t, y)) + \frac{1}{2} \text{trace } A(t, y)^* D^2 f(t, y) A(t, y) \right\} p(s, x; t, dy).$$

By assumption, $p(s, x; t, dy)$ is $H - C^2$ in the sense of Fomin. It follows from theorem 1.2.12 that

$$\int_B (Df(y), \sigma(t, y)) p(s, x; t, dy) = - \int_B f(y) \left\{ \text{trace } D \sigma(t, y) +$$

$$+ (\rho_1(y), \sigma(t, y)) \right\} p(s, x; t, dy).$$
Now
\[
\int_B \text{trace } A(t,y)^* D^2 f(y) A(t,y) p(s,x;t,dy)
\]
\[
= \int \sum_{i=1}^{\infty} D^2 f(y) \left( A(t,y) e_i, A(t,y) e_i \right) p(s,x;t,dy)
\]
\[
= \int_B \left( \text{trace } D^2 f(y) + 2 \text{trace } D^2 f(y) (K(t,y)(), \cdot) +
\right.
\]
\[
+ \text{trace } D^2 f(y) (K(t,y)(\cdot), K(t,y)(\cdot)) \right) p(s,x;t,dy).
\]

It follows from theorem 1.2.14 that
\[
\int_B \text{trace } D^2 f(y)(K(t,y)(), \cdot) p(s,x;t,dy)
\]
\[
= \int_B f(y) \sum_{i,j} D^2 \left\{ K(t,y)(e_i, e_j) p(s,x;t,dy) \right\} (e_i, e_j)
\]
and
\[
\int_B \text{trace } D^2 f(y)(K(t,y)(\cdot), K(t,y)(\cdot)) p(s,x;x,dy)
\]
\[
= \int_B f(y) \sum_{i,j} D^2 \left\{ K(t,y)^2(e_i, e_j) p(s,x;t,dy) \right\} (e_i, e_j).
\]

By assumption, trace \( D^2 p(s,x;t,dy) \) exists in \( D' \). Therefore we have
\[
\int_B \text{trace } A(t,y)^* D^2 f(y) A(t,y) p(s,x;t,dy)
\]
\[
= \int_B f(y) \sum_{i,j} D^2 \left\{ A(t,y)(e_i, e_j) p(s,x;t,dy) \right\}. \quad Q.E.D.
\]
Bibliography


Vita

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