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Combinatorial and Asymptotic Statistical Properties of Partitions and Unimodal Sequences

Walter McFarland Bridges
Louisiana State University and Agricultural and Mechanical College

Combinatorial and Asymptotic Statistical Properties of Partitions and Unimodal Sequences

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Walter McFarland Bridges
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Asymptotic Notation

We use the following standard asymptotic notation:

- We write $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.
- We write $f(x) = O(f(x))$, or equivalently $f(x) \ll g(x)$, if $\limsup_{x\to\infty} \frac{f(x)}{g(x)}$ is bounded from above. We write $f(x) = \omega(g(x))$ if $f(x) \gg g(x)$.
- We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
- We write $f(x) \asymp g(x)$ if $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

Abstract

Our main results are asymptotic zero-one laws satisfied by the diagrams of unimodal sequences of positive integers. These diagrams consist of columns of squares in the plane; the upper boundary is called the shape. For various types of unimodal sequences, we show that, as the number of squares tends to infinity, 100% of shapes are near a certain curve—that is, there is a single limit shape. Similar phenomena have been well-studied for integer partitions, but several technical difficulties arise in the extension of such asymptotic statistical laws to unimodal sequences. We develop a widely applicable method for obtaining these limit shapes, based in part on a method of Petrov. We also mention a few notable corollaries—for example, we obtain a limit shape for so-called "overpartitions" by a simple DeSalvo-Pak-type transfer.

To aid in the proof of these limit shapes, we prove an asymptotic formula for the number of partitions of the integer n into distinct parts where the largest part is at most $t\sqrt{n}$ for fixed t. Our method follows a probabilistic approach of Romik, who gave a simpler proof of Szekeres' asymptotic formula for distinct parts partitions when instead the number of parts is bounded by $t\sqrt{n}$. The probabilistic approach is equivalent to a circle method/saddle-point method calculation, but it makes some of the steps more intuitive and even predicts the shape of the asymptotic formula, to some degree.

Finally, motivated by certain problems concerning Rogers-Ramanujan-type identities, we give combinatorial proofs of three families of inequalities among certain types of integer partitions.

Chapter 1: Introduction

A *partition* of an integer n is a way of writing n as a sum of positive integers:

Definition 1.1. *A* partition λ *of an integer* n *is a multi-set of positive integers* $\{\lambda_1, \ldots, \lambda_\ell\}$*, whose* parts *satisfy*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1 \qquad \textit{and} \qquad \sum_{j=1}^{\ell} \lambda_j = n.$$

The size *of* λ *is denoted* $|\lambda| = n$*.*

(See [2], §1.1.) For example, the partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. The study of these simple combinatorial objects, a vast subject in its own right, has developed in tandem with many areas of mathematics and physics. In representation theory, each partition of n leads to an irreducible representation of the symmetric group S_n. In statistical mechanics, limit shapes for partitions describe the limiting distribution of energy levels in ideal gas. And in number theory, various generating functions related to partitions, old and new, have been important examples of automorphic forms.

Visually, a partition λ may be represented by its *Ferrer's diagram*, in which parts are displayed as rows of dots. For example, the Ferrer's diagram of the partition $5+3+2+2$ is the array below.

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \\ \bullet & \bullet & & & \end{array}$$

An important involution on partitions of n is *conjugation*, which is performed on the Ferrer's diagrams by reflecting about the 45 degree diagonal starting at the

top left dot. Equivalently, the columns of a Ferrer's diagram form the conjugate partition. Thus, the conjugate of $5+3+2+2$ is $4+4+2+1+1$.

Generating functions have long been used to obtain both combinatorial and asymptotic information about partitions. These are typically given as basic[1] hypergeometric series. For example, if $\mathcal{P}$ denotes the set of all (unrestricted) partitions, then the generating function for $p(n)$, the number of partitions of n, may be written in any of the following forms:

$$P(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} = \sum_{n \geq 0} p(n) q^n = \prod_{k \geq 1} \frac{1}{1-q^k}, \qquad \text{where } q \in \mathbb{C} \text{ and } |q| < 1.$$

For convenience, we have set $p(0) = 1$ which includes the "empty partition of 0" as a member of $\mathcal{P}$. The combinatorial proof of the above is to expand each factor in the product as a geometric series. Identities between these q-series may lead to identities between certain types of partitions, and it is often easier to manipulate the entire q-series rather than to show a direct combinatorial connection between types of partitions.

In this thesis, we study some combinatorial and asymptotic statistical properties of partitions as well as unimodal sequences, a related generalization.

1.1 Limit Shapes for Unimodal Sequences

Much is known about the asymptotic behavior of various statistics for partitions. For example, one can ask, what is the average size of the largest part in partitions of n? Erdős and Lehner showed that it is roughly $A\sqrt{n}\log(A\sqrt{n})$ for 100% of partitions of n in the asymptotic limit, where $A = \frac{\sqrt{6}}{\pi}$. In fact, they found that the largest part obeys the following extreme value distribution.

[1] Here, "basic" means to base-q, in contrast to (regular) hypergeometric series.

Theorem (Theorem 1.1 of [24], reformulated)**.** *Let $\mathcal{P}_n$ denote the set of partitions of n and let P_n be the uniform probability measure on $\mathcal{P}_n$. Then*

$$\lim_{n\to\infty} P_n\left(\lambda \in \mathcal{P}_n : \frac{\lambda_1 - A\sqrt{n}\log(A\sqrt{n})}{A\sqrt{n}} \leq x\right) = e^{-e^{-x}}.$$

Note that, due to conjugation of partitions, the above is also the distribution for the *number of parts* in partitions of n.

Comparatively less is known about the asymptotic behavior of statistics for *unimodal sequences*, objects closely related to partitions in that summands are allowed to increase and then decrease:

Definition 1.2. *A* unimodal sequence $\lambda = \{\lambda_j\}_{j=1}^{s}$ *of* size n *is a multi-set of positive integers that sum to n, whose* parts *satisfy:*

$$\lambda : \qquad 0 < \lambda_1 \leq \cdots \leq \lambda_{k-1} \leq \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_s > 0 \qquad \textit{and} \qquad \sum_{j=1}^{s} \lambda_j = n. \tag{1.1}$$

The peaks *of λ are λ_k and any other parts equal to λ_k.*

For example, the unimodal sequences of size 4 are 4, $3+1$, $1+3$, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Stanley's survey [38] collects a variety of unimodal sequences arising throughout mathematics—in combinatorics, geometry, Lie algebras, finite groups, and more.

Similar to Ferrer's diagrams of partitions, the *diagram* of a unimodal sequence λ is the set of adjacent columns of unit squares in the plane, where the j-th column has λ_j squares. To fix a centering of a diagram, we will always choose to place the left-most peak vertex on the y-axis[2]. In [11], the author studied the asymptotic behavior of the *shape*, $\varphi(\lambda)$, which is the top border of the diagram of λ.

To compare diagrams of size $n \to \infty$ to a fixed curve, it is convenient to rescale them to have area 1, so let us define the *renormalized shape* $\widetilde{\varphi}(\lambda)$ to be the shape

[2]Our results hold regardless of which peak vertex we fix as the center.

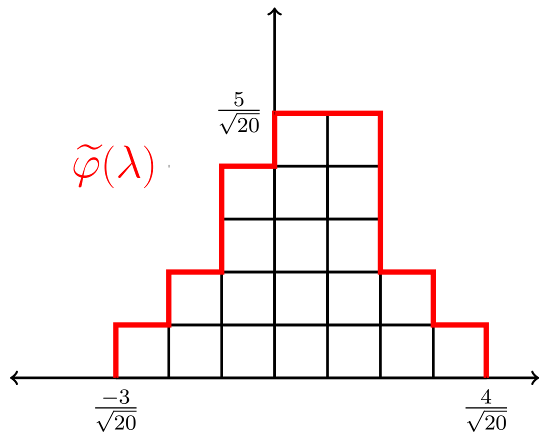

Figure 1.1. Diagram and renormalized shape for $\lambda = (1, 2, 4, 5, 5, 2, 1)$ of size 20.

obtained from $\varphi(\lambda)$ by rescaling both the x- and y-axes by $\frac{1}{\sqrt{n}}$ when the size of λ is n (see Figure 1.1).

Roughly, the question we want to answer is the following: What are the typical shapes of diagrams of size n, as $n \to \infty$? Here, "typical" will mean "under the uniform probability measure on diagrams of size n". It turns out that, for the types of unimodal sequences we consider, there is a single *limit shape*.

This type of striking 0-1 law has been well-studied for integer partitions. In [25], Fristedt introduced a probabilistic model to make a deep study of the limiting behavior of partitions. This machinery was subsequently used by Vershik in [41] to state many types of limit shapes that were finally proved in [22]. For more information on the history of limit shapes for partitions, see Sections 1 and 12 of [21]. We discuss and apply Fristedt's methodology to solve a different problem in Section 3.1.2.

For unrestricted partitions of size n, the limit shape is

$$y = f_p(x) = -\frac{\sqrt{6}}{\pi} \log\left(1 - e^{-\frac{\pi}{\sqrt{6}}x}\right). \tag{1.2}$$

Note that this can be symmetrized as $e^{\frac{\pi}{\sqrt{6}}x} + e^{\frac{\pi}{\sqrt{6}}y} = 1$, which respects conjugation. An "elementary" proof (one that does not require measure theory) that (1.2) is

the limit shape for partitions was given by Petrov in [35], and we will utilize his approach.

A different type of problem was recently solved by DeSalvo and Pak, who found conditions under which partition bijections allow for the transfer of limit shapes [21]. We will see one result of this type in Section 2.4.

Following the notation of Bringmann and Mahlburg in [16], let $\mathscr{S}(n)$ denote the set of (unrestricted) unimodal sequences of size n, and denote its cardinality by $s(n)$. Let $\mathscr{D}(n)$ denote the set of *strongly unimodal sequences* of size n, and denote its cardinality by $d(n)$; these have the added requirement that all of the inequalities in (1.1) are strict. Finally, let $\mathscr{D}_m(n)$ denote the set of *semi-strict unimodal sequences* of size n, and denote its cardinality by $dm(n)$; here, we require that there be a single peak and that the inequalities to the left of it in (1.1) are strict.

For a function $f(x)$, let $N_\epsilon(f)$ denote the set of points in the plane whose horizontal distance from $y = f(x)$ is at most ϵ, together with ϵ neighborhoods of the x- and y-axes. (The latter components of N_ϵ are necessary to account for vertical and horizontal asymptotes of functions.)

Theorem 1.3 (Strongly Unimodal Sequences)**.** *Let $\epsilon > 0$ be arbitrary and let*

$$f_d(x) := \begin{cases} -\frac{\sqrt{6}}{\pi}\log\left(e^{-\frac{\pi}{\sqrt{6}}x} - 1\right) & \text{if } x \in \left[-\frac{\sqrt{6}}{\pi}\log(2), 0\right), \\ -\frac{\sqrt{6}}{\pi}\log\left(e^{\frac{\pi}{\sqrt{6}}x} - 1\right) & \text{if } x \in \left(0, \frac{\sqrt{6}}{\pi}\log(2)\right]. \end{cases}$$

Then

$$\lim_{n\to\infty} \frac{1}{d(n)} \cdot \#\left\{\lambda \in \mathscr{D}(n) : \widetilde{\varphi}(\lambda) \subset N_\epsilon(f_d)\right\} = 1. \tag{1.3}$$

We can show that $\int_{\mathbb{R}} f_d(x)dx = 1$ using the dilogarithm function, $\mathrm{Li}_2(z)$, defined for $z \in \mathbb{C} \setminus (-\infty, -1)$ by the integral

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-w)}{w}dw,$$

taking the principal branch of the complex logarithm. We also have the Taylor expansion $\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$ for $|z| \leq 1$, and hence $\mathrm{Li}_2(1) = \frac{\pi^2}{6}$ and $\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$. (See [1], §27.7, where $f(x) = \mathrm{Li}_2(1-x)$.)

Thus,

$$\int_{\mathbb{R}} f_d(x)dx = 2\int_0^{\frac{\sqrt{6}}{2}\log 2} -\frac{\sqrt{6}}{\pi}\log\left(e^{\frac{\pi}{\sqrt{6}}x} - 1\right)dx = \frac{12}{\pi^2}\int_1^2 \frac{\log(t-1)}{t}dt$$

$$= \frac{12}{\pi^2}\left(-\mathrm{Li}_2(1-t) - \log(t-1)\log t\right)\Big|_{t=1}^{2} = -\frac{12}{\pi^2}\mathrm{Li}_2(-1) = 1.$$

Integrals of the other functions below are similarly evaluated in terms of $\mathrm{Li}_2(z)$.

Theorem 1.4 (Unrestricted Unimodal Sequences)**.** *Let $\epsilon > 0$ be arbitrary and let*

$$f_s(x) := \begin{cases} -\frac{\sqrt{3}}{\pi}\log\left(1 - e^{\frac{\pi}{\sqrt{3}}x}\right) & \text{if } x < 0, \\ -\frac{\sqrt{3}}{\pi}\log\left(1 - e^{-\frac{\pi}{\sqrt{3}}x}\right) & \text{if } x > 0. \end{cases}$$

Then

$$\lim_{n\to\infty} \frac{1}{s(n)} \cdot \#\left\{\lambda \in \mathscr{S}(n) : \widetilde{\varphi}(\lambda) \subset N_\epsilon(f_s)\right\} = 1.$$

Remark 1.5. *The limit shape of Theorem 1.4 also holds for "unimodal sequences with summits", which are distinguished from unrestricted unimodal sequences by designating one peak as the "summit". These were called "stacks with summits" in [16], and the number of unimodal sequences with summits was denoted by $ss(n)$. In particular, we have $ss(n) \sim s(n)$ (see [16]). It is straightforward to repeat our proof of Theorem 1.4 for unimodal sequences with summits with very little change.*

Remark 1.6. *It is not surprising that the limit shapes for unrestricted and strongly unimodal sequences are made of two halves of the limit shapes for unrestricted and distinct parts partitions (see [41], Th. 4.4 and Th. 4.5). In particular, each half of the limit shape for unrestricted unimodal sequences is the curve* (1.2) *scaled down so that the area beneath it is $\frac{1}{2}$.*

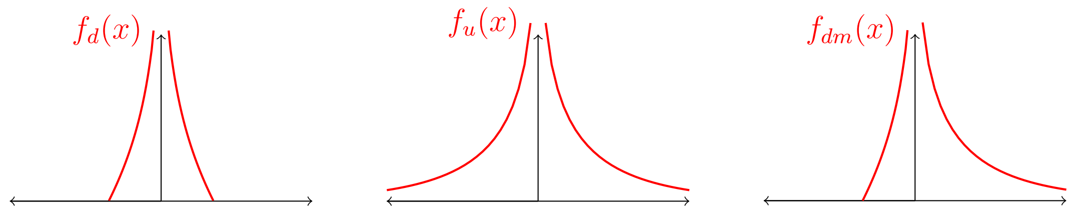

Figure 1.2. Respective limit shapes for strongly, unrestricted and semi-strict unimodal sequences

Theorem 1.7 (Semi-strict Unimodal Sequences)**.** *Let $\epsilon > 0$ be arbitrary and let*

$$f_{dm}(x) := \begin{cases} -\frac{2}{\pi}\log\left(e^{-\frac{\pi}{2}x} - 1\right) & \text{if } x \in \left[-\frac{2}{\pi}\log 2, 0\right), \\ -\frac{2}{\pi}\log\left(1 - e^{-\frac{\pi}{2}x}\right) & \text{if } x > 0. \end{cases}$$

Then

$$\lim_{n\to\infty} \frac{1}{dm(n)} \cdot \#\left\{\lambda \in \mathscr{D}_m(n) : \widetilde{\varphi}(\lambda) \subset N_\epsilon(f_{dm})\right\} = 1.$$

Remark 1.8. *We observe that the left-half of $f_{dm}(x)$ is the limit shape for distinct parts partitions scaled so that the area beneath it is $\frac{1}{3}$ ([41], Th. 4.5), while the right half is the limit shape for unrestricted partitions scaled so that the area beneath it is $\frac{2}{3}$. We discuss the appearance of these constants further in Section 2.4.*

Our proofs of the main results are structured as follows. Following a method of Petrov ([35], §6), we will obtain limit shapes for the left and right halves "in isolation", showing that as $n \to \infty$, 0% of left (resp. right) halves of shapes are *not* in an ϵ neighborhood of some left (resp. right) limit shape. For Theorem 1.3, this is enough to complete the proof. However, for Theorems 1.4 and 1.7, we will need to analyze peaks more closely; we will show that, on average, peaks are $\omega(\sqrt{n})$, so that a degenerate "completely flat" limit shape does not occur. To show this, we require the asymptotic formula stated in the next Section and proved in Chapter 3.

1.2 Partitions into Distinct Parts with Bounded Largest Part

To complete the proof of Theorem 1.7 in Chapter 3, we require an asymptotic formula for the number of partitions of n into distinct parts where the largest part is bounded by $t\sqrt{n}$ for fixed t. To the best of our knowledge, the author's work in [10] is the first occurrence of such an asymptotic formula.

Let $q(n)$ denote the number of distinct parts partitions of n. These numbers are easily seen to be generated by the following infinite product:

$$\sum_{k\geq 0} q(n)x^n = \prod_{k\geq 1}(1+x^k).$$

Pioneering work of Hardy and Ramanujan used the modular properties of the infinite product to obtain an asymptotic series for $q(n)$ (and similar enumerations) after representing these coefficients as contour integrals around the origin ([27], §7.1). The main term in Hardy and Ramanujan's asymptotic series is

$$q(n) \sim \frac{1}{4\sqrt[4]{3}n^{\frac{3}{4}}} e^{\frac{\pi}{\sqrt{3}}\sqrt{n}}. \tag{1.4}$$

The *circle method* is now often used as an umbrella term for the asymptotic analysis of contour integrals, including Hardy-Ramanjuan's method and its many variants, as well as certain cases of the *saddle-point method*. For an exposition of Hardy, Ramanujan, and Rademacher's original work, see [2] Ch. 5-6 and for the saddle-point method, see [26] Ch. VIII.

A more recent approach to these asymptotic statistics, begun by Fristedt in [25] and used by Romik in [36], is to reformulate the proof using probability theory. This can make some of the steps more intuitive. We explain these ideas further in Section 3.1.

Let $t > 0$ be fixed. We study a restriction of $q(n)$ defined as the coefficient of x^n in the following generating function:

$$q_t(n) := \text{Coeff } [x^n] \ \mathcal{Q}_{t,n}(x), \qquad \text{where} \qquad \mathcal{Q}_{t,n}(x) := \prod_{k \le t\sqrt{n}} (1 + x^k).$$

Thus, $q_t(n)$ is the number of distinct parts partitions of n with largest part is at most $t\sqrt{n}$. The smallest possible largest part in a distinct parts partition of n is k, where

$$1 + 2 + \cdots + (k-1) = \frac{k(k-1)}{2} < n \le \frac{k(k+1)}{2}.$$

Thus, we ignore the range $t \le \sqrt{2}$, where often $q_t(n) = 0$, and consider only $t > \sqrt{2}$. We prove the following asymptotic formula for $q_t(n)$. Here and throughout, $\lfloor \alpha \rfloor$ denotes the greatest integer less than or equal to α and $\{\alpha\} := \alpha - \lfloor \alpha \rfloor$.

Theorem 1.9. *Let $t > \sqrt{2}$. Define $\beta : (\sqrt{2}, \infty) \to \left(-\infty, \frac{\pi}{2\sqrt{3}}\right)$ implicitly as a function of t so that*

$$1 = \int_0^t \frac{ue^{-\beta u}}{1 + e^{-\beta u}} du. \tag{1.5}$$

Let

$$B(t) := 2\beta + t \log\left(1 + e^{-\beta t}\right) \qquad \text{and} \qquad A_n(t) := \frac{e^{\frac{\beta t}{2}} + e^{-\frac{\beta t}{2}}}{2\left(1 + e^{-\beta t}\right)^{\{t\sqrt{n}\}}} \sqrt{\frac{\beta'(t)}{\pi t}}. \tag{1.6}$$

Then

$$q_t(n) \sim \frac{A_n(t)}{n^{3/4}} e^{B(t)\sqrt{n}}.$$

The oscillatory factor $\left(1 + e^{-\beta t}\right)^{-\{t\sqrt{n}\}}$ is present because $t\sqrt{n}$ is not always an integer. Numerically, this oscillation is also reflected in $q_t(n)$, which appears to be non-increasing for t close to $\sqrt{2}$.

Remark 1.10. *We record properties of the functions $\beta(t), B(t)$ and $A(t) := A_n(t) \cdot \left(1 + e^{-\beta t}\right)^{\{t\sqrt{n}\}}$ in Section 3.2. In particular, we show that β and B are strictly*

increasing, and we show that $\beta(t)$*,* $B(t)$ *and* $A(t)$ *tend to* $\frac{\pi}{2\sqrt{3}}$*,* $\frac{\pi}{\sqrt{3}}$ *and* $\frac{1}{4\sqrt[4]{3}}$*, respectively, as* $t \to \infty$*. Thus, Theorem 1.9 is consistent with Hardy and Ramanujan's asymptotic formula, and* (1.4) *could be recovered if we were allowed to take* $t \to \infty$*.*

Remark 1.11. *It has been shown that the largest part of a typical distinct parts partition of* n *is* $c\sqrt{n}\log n$ *for some* c *([25], Thm. 9.4), so that our* $q_t(n)$ *counts (asymptotically)* 0% *of distinct parts partitions of* n*. In fact, since* $B(t)$ *is strictly increasing to* $\frac{\pi}{\sqrt{3}}$*, it follows that* $q_t(n) = o\,(q(n))$ *for any fixed* t*. Thus, Theorem 1.9 too implies that* 0% *of partitions of* n *have largest part at most* $t\sqrt{n}$*, as* $n \to \infty$*.*

Szekeres found an asymptotic formula for distinct parts partitions when instead the *number of parts* is at most $t\sqrt{n}$ ([39], [40]). When parts are allowed to repeat, bounding the number of parts and bounding the size of the largest part give the same enumeration function due conjugation. But here, when parts are distinct, these two notions are different.

Szekeres' proof in [40] is based on the saddle-point method, and later Romik [36] recast and simplified this proof using Fristedt's probabilistic machinery [25]. Although equivalent to a circle method calculation, Romik's proof motivates some of the more technical steps in the proof and even predicts the shape of the asymptotic formula, to some degree. Our proof here closely follows Romik.

1.3 Partition Inequalities

Our motivation for proving inequalities among types of partitions of n stems, ultimately, from the Rogers-Ramanujan identities. These have been intensely studied and generalized since their first appearance and feature surprising connections to affine Lie algebras. We refer the reader to [31], §1.1 for a brief summary of these connections. To state the Rogers-Ramanujan identities, we use the standard q-Pochhammer symbol,

$$(a;q)_n := \prod_{j=0}^{n-1}\left(1-aq^j\right), \quad (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \quad \text{and}$$

$$(a_1,\ldots,a_r;q)_n := (a_1;q)_n\cdots(a_r;q)_n.$$

By convention, an empty product equals 1. The Rogers-Ramanujan identities are as follows.

$$\mathcal{RR}_1: \quad \sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty},$$

$$\mathcal{RR}_2: \quad \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

(See [2], Ch. 7.) The identity $\mathcal{RR}_1$ may be interpreted as an equality of certain partition generating functions, giving that the number of partitions of n such that the gap between successive parts is at least 2 equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$. Similarly, $\mathcal{RR}_2$ gives that the number of partitions of n such that the gap between successive parts is at least 2 and 1 does not occur as a part equals the number of partitions of n into parts congruent to $\pm 2 \pmod 5$.

1.3.1 Ehrenpreis Problems

For two q-series $f(q) = \sum_{n\geq 0} a_n q^n$ and $g(q) = \sum_{n\geq 0} b_n q^n$, we write $f(q) \succeq g(q)$ if $a_n \geq b_n$ for all n. By the above combinatorial interpretation of the Rogers-Ramanujan identities (or by simple algebra) it follows that

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n} - \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} \succeq 0. \tag{1.7}$$

Thus, the Rogers-Ramanujan identities imply the following inequality, which is far from obvious.

$$\frac{1}{(q,q^4;q^5)_\infty} - \frac{1}{(q^2,q^3;q^5)_\infty} \succeq 0. \tag{1.8}$$

At the 1987 A.M.S. Institute on Theta Functions, the problem was posed by Leon Ehrenpreis to provide a proof of (1.8) that did not reference the (heavy-handed) Rogers-Ramanujan identities. (See [5], §1.)

Solutions to Ehrenpreis' Problem have been given in various ways. In the course of proving (1.8), Andrews and Baxter [5] were led to a new "motivated" proof of the Rogers-Ramanujan Identities themselves; "motivated proofs" were subsequently given for infinite families of Rogers-Ramanujan type identities in [20], [29] and [32]. Recalling the combinatorial interpretation of the products, (1.8) may be interpreted as a *partition inequality*—namely, that there are more partitions of n into parts congruent to $\pm 1 \pmod 5$ than into parts congruent to $\pm 2 \pmod 5$. Under this interpretation, a direct combinatorial proof of (1.8) was provided by Kadell [28], who constructed an injection between these types of partitions. Later, Andrews developed the Anti-telescoping Method for showing positivity in differences of products like (1.8) [4].

We make use of injections and Andrews' Method of Anti-telescoping to prove several partition inequalities. The motivation for our first partition inequality comes from an "Ehrenpreis Problem" for recently conjectured sum-product identities of Kanade-Russell to be described further below.

Our first result extends the following theorem of Berkovich and Garvan, who generalized (1.8) to an arbitrary modulus.

Theorem (Theorem 5.3 of [7])**.** *Suppose $L \geq 1$ and $1 \leq r < \frac{M}{2}$. Then*

$$\frac{1}{(q, q^{M-1}; q^M)_L} - \frac{1}{(q^r, q^{M-r}; q^M)_L} \succeq 0$$

if and only if $r \nmid (M-r)$.

Our extension is independent of the modulus.

Theorem 1.12. *Let a, b, c and M be integers satisfying $1 < a < b < c$ and $1 + c = a + b$. Then if $a \nmid b$,*

$$\frac{1}{(q, q^c; q^M)_L} - \frac{1}{(q^a, q^b; q^M)_L} \succeq 0 \qquad \textit{for any } L \geq 0.$$

Note that we do not necessarily assume $a, b, c \leq M$. Translated into a partition inequality, Theorem 1.1 says that there are more partitions of n into parts of the forms $Mj + 1$ and $Mj + c$ than there are partitions of n into parts of the forms $Mj + a$ and $Mj + b$, where $1 \leq j \leq L$.

1.3.2 McLaughlin's Two-Variable Inequalities

Partition inequalities with a fixed number of parts were considered by McLaughlin in [33]. Answering two of McLaughlin's questions, we give combinatorial proofs of finite analogues of Theorems 7 and 8 from [33].

Theorem 1.13. *Let a, b and M be integers satisfying $1 \leq a < b < \frac{M}{2}$ and $\gcd(b, M) = 1$. Define $c(m, n)$ by*

$$\frac{1}{(zq^a, zq^{M-a}; q^M)_L(1 - q^{LM+a})} - \frac{1}{(zq^b, zq^{M-b}; q^M)_L} =: \sum_{m,n \geq 0} c(m, n) z^m q^n.$$

Then for any $L, n \geq 0$, we have $c(m, nM) \geq 0$. If in addition M is even and a is odd, then we also have $c\left(m, nM + \frac{M}{2}\right) \geq 0$ for every $n \geq 0$.

Note that we do not necessarily make the assumption $\gcd(a, M) = 1$ that is in [33]. While these partition inequalities hold only for n in certain residue classes (mod M), Theorem 1.13 is a strengthening of Theorem 1.12 for these n. The following is a distinct parts analogue.

Theorem 1.14. *Let a, b and M be integers satisfying $1 \leq a < b < \frac{M}{2}$ and $\gcd(b, M) = 1$. Define $d(m, n)$ by*

$$(-zq^a, -zq^{M-a}; q^M)_L \left(1 + zq^{LM+a}\right) - (-zq^b, -zq^{M-b}; q^M)_L =: \sum_{m,n \geq 0} d(m, n) z^m q^n.$$

Then for any $L, n \geq 0$, we have $d(m, nM) \geq 0$. If in addition M is even and a is odd, then we also have $d\left(m, nM + \frac{M}{2}\right) \geq 0$ for every $n \geq 0$.

Remark 1.15. *Taking the limit as $L \to \infty$ in Theorems 1.13 and 1.14 recovers McLaughlin's original partition inequalities.*

1.4 Structure of the Thesis

In Chapter 2, we prove Theorems 1.3, 1.4 and 1.7. We then state several consequences of these limit shapes in Section 2.4.

Chapter 3 contains the proof of Theorem 1.9 which is required to complete the proof of Theorem 1.7 in Chapter 2. The proof is outlined in Section 3.1, where we motivate a Fristedt-type probabilistic model and state three propositions that together imply Theorem 1.9. In Section 3.2 we record some properties of the functions $\beta(t)$, $B(t)$ and $A(t)$, including those mentioned in Remark 1.10. Section 3.4 provides the proofs of two technical lemmas; these could be useful in similar asymptotic analysis and may be of independent interest.

Chapter 4 contains the proofs of Theorems 1.12, 1.13 and 1.14. In Section 4.4, we apply these theorems and Andrews' Method of Anti-telescoping to solve Ehrenpreis Problems for conjectures of Kanade-Russell.

In Chapter 5, we conclude by reviewing our results and by speculating on possible future work.

Chapter 2: Limit Shapes

For the proofs of Theorems 1.3, 1.4 and 1.7, we will need one more definition. A renormalized shape $\widetilde{\varphi}(\lambda)$ consists of line segments that meet at 90° corners, which we will call *vertices*, but we will exclude the top two corners at the peaks from this set, as well as the points on the x-axis (see Figure 2.1). Let $V_\ell(\lambda)$ be the set of left vertices of $\widetilde{\varphi}(\lambda)$ and let $V_r(\lambda)$ be the set of right vertices (see Figure 2.1). For fixed ϵ and large n, we clearly have $\widetilde{\varphi}(\lambda) \subset N_\epsilon(f)$ if and only if

$V_\ell(\lambda) \cup V_r(\lambda) \subset N_\epsilon(f)$.

2.1 Proof of Theorem 1.3

We will find it easier to analyze the left part of the shape after translating into the first quadrant. Let

$$\overline{f_d}(x) := f_d\left(x - \frac{\sqrt{6}}{\pi}\log(2)\right) = -\frac{\sqrt{6}}{\pi}\log\left(2e^{-\frac{\pi}{\sqrt{6}}x} - 1\right) \quad \text{for } x \in \left[0, \frac{\sqrt{6}}{\pi}\log(2)\right).$$

We will also make use of the inverse function for $\overline{f_d}$, namely

$$\overline{g_d}(y) := \frac{\sqrt{6}}{\pi}\left(\log(2) - \log\left(1 + e^{-\frac{\pi}{\sqrt{6}}y}\right)\right) \qquad \text{for } y \in [0, \infty).$$

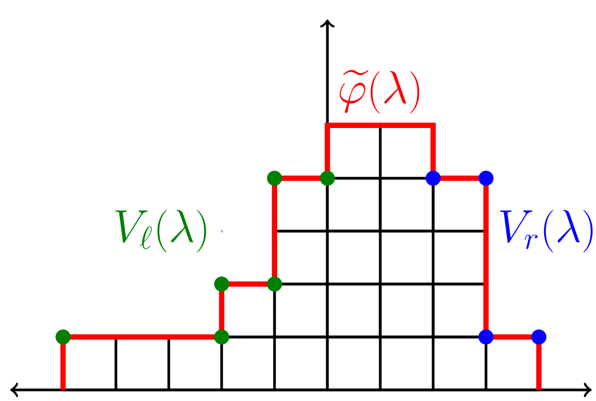

Figure 2.1. Renormalized shape, $\widetilde{\varphi}(\lambda)$; left vertices, $V_\ell(\lambda)$; and right vertices, $V_r(\lambda)$, for the unimodal sequence $\lambda = (1, 1, 1, 2, 4, 5, 5, 4, 1)$.

Let $\overline{V_\ell}(\lambda)$ be the left vertices after translating $\widetilde{\varphi}(\lambda)$ to the right by $\frac{\sqrt{6}}{\pi}\log(2)$. We want to show a left limit shape for the left half of diagrams, in the sense that

$$\lim_{n\to\infty} \frac{1}{d(n)} \cdot \#\left\{\lambda \in \mathscr{D}(n) : \overline{V_\ell}(\lambda) \not\subset N_\epsilon(\overline{f_d})\right\} = 0. \tag{2.1}$$

We estimate the above count using the following inequalities:

$$\begin{aligned}
&\#\left\{\lambda \in \mathscr{D}(n) : V_\ell(\lambda) \not\subset N_\epsilon(f_d)\right\} \\
&\leq \sum_{a<\sqrt{2n}} \#\left\{\lambda \in \mathscr{D}(n) : \frac{1}{\sqrt{n}}(a,b) \in \overline{V_\ell}(\lambda),\ \left|\overline{g_d}\left(\frac{b}{\sqrt{n}}\right) - \frac{a}{\sqrt{n}}\right| > \epsilon\right\} &(2.2)\\
&\leq 2\sum_{a<\sqrt{2n}} \#\left\{\lambda \in \mathscr{D}(n) : \lambda \text{ has exactly } a \text{ left parts} \leq b,\ \left|\overline{g_d}\left(\frac{b}{\sqrt{n}}\right) - \frac{a}{\sqrt{n}}\right| > \epsilon\right\}. &(2.3)
\end{aligned}$$

(2.2) follows from the definition of N_ϵ, and (2.3) is easy to see geometrically. After multiplying (2.3) by $\frac{1}{d(n)}$, we will show that each summand is $e^{-C\sqrt{n}+o(\sqrt{n})}$, where $C > 0$ is independent of a. It then follows that

$$\lim_{n\to\infty} \frac{1}{d(n)} \cdot \#\left\{\lambda \in \mathscr{D}(n) : \overline{V_\ell}(\lambda) \not\subset N_\epsilon(\overline{f_d})\right\} \leq \lim_{n\to\infty} \sqrt{2n} \cdot e^{-C\sqrt{n}+o(\sqrt{n})} = 0,$$

so (2.1) holds.

Now, $d(n)$ appears as the n-th coefficient in $\mathscr{D}(q)$, so clearly

$$d(n) \leq q^{-n}\mathscr{D}(q) \qquad \text{for } q \in (0,1). \tag{2.4}$$

The following Lemma shows that we can choose q depending on n that concentrates the mass of $\mathscr{D}(q)$ in the single term $d(n)q^n$, in the sense that after taking a logarithm, (2.4) becomes an asymptotic. We elaborate further on these ideas in Section 3.1.1.

Lemma 2.1. *There exists a unique $c > 0$ such that for $q = e^{-\frac{c}{\sqrt{n}}}$, we have*

$$\log\left(q^{-n}\mathscr{D}(q)\right) \sim \log d(n) = \frac{2\pi}{\sqrt{6}}\sqrt{n} + o(\sqrt{n}).$$

Proof. Recall that $\log d(n) \sim \frac{2\pi}{\sqrt{6}}\sqrt{n}$. (See [16], Table 1.) Theorem 4.3 of [15] states that $\mathscr{D}(e^{-t}) \sim \frac{1}{4}e^{\frac{\pi^2}{6t}}$ as $t \to 0^+$. Letting $t = \frac{c}{\sqrt{n}}$, we see that

$$\log\left(e^{c\sqrt{n}}\mathscr{D}\left(e^{-\frac{c}{\sqrt{n}}}\right)\right) \sim \left(c + \frac{\pi^2}{6c}\right)\sqrt{n}.$$

We take $c = \frac{\pi}{\sqrt{6}}$, and the lemma is proved. □

Throughout we let $c := \frac{\pi}{\sqrt{6}}$ and $q = e^{-\frac{c}{\sqrt{n}}}$. We will use Lemma 2.1 in the form $\frac{q^{-n}\mathscr{D}(q)}{d(n)} \sim e^{o(\sqrt{n})}$.

Using standard combinatorial techniques, the generating function $\mathscr{D}(q)$ is obtained by summing over peaks as

$$\mathscr{D}(q) := \sum_{n\geq 0} d(n)q^n = \sum_{n\geq 0} q^{n+1}\prod_{j=1}^{n}(1+q^j)^2.$$

Here, the two products generate the partitions to the left and right of the peak $n+1$. Similarly, the number of $\lambda \in \mathscr{D}(n)$ with exactly a left parts at most b is the coefficient of $z^a q^n$ in

$$\sum_{m>b} q^{m+1}\prod_{j=1}^{m}(1+q^j)^2 \prod_{j\leq b}\frac{1+zq^j}{1+q^j}. \tag{2.5}$$

The latter product has the effect of replacing the original factor $1+q^j$ generating a left part $j \leq b$ with $1+zq^j$. Written as above, we see that the latter product is independent of m, and therefore may be factored out. Hence, by the principle used in (2.4) and by Lemma 2.1, we how have

$$\begin{aligned}\{\lambda \in \mathscr{D}(n) : \lambda \text{ has exactly } a \text{ left parts} \leq b\} \leq \frac{q^{-n}\mathscr{D}(q)}{d(n)}z^{-a}\prod_{j\leq y\sqrt{n}}\frac{1+zq^j}{1+q^j} \\ =: e^{o(\sqrt{n})}\cdot e^{U(\tau)},\end{aligned} \tag{2.6}$$

where we have set $b = y\sqrt{n}$ and $z = e^{\tau}$ for $\tau \in \mathbb{R}$ and $y \geq 0$, and

$$U(\tau) := -\tau a + \sum_{1\leq j\leq y\sqrt{n}}\left(\log(1+e^{\tau - c\frac{j}{\sqrt{n}}}) - \log(1+e^{-c\frac{j}{\sqrt{n}}})\right).$$

Note that $U(0) = 0$, thus to get exponential decay in (2.6), we must have $\tau \neq 0$. Taking derivatives, we have

$$U'(\tau) = -a + \sum_{1 \leq j \leq y\sqrt{n}} \frac{e^{\tau - \frac{cj}{\sqrt{n}}}}{1 + e^{\tau - \frac{cj}{\sqrt{n}}}}; \qquad U''(\tau) = \sum_{1 \leq j \leq y\sqrt{n}} \frac{e^{\tau - \frac{cj}{\sqrt{n}}}}{\left(1 + e^{\tau - \frac{cj}{\sqrt{n}}}\right)^2}.$$

Let Σ'_y and Σ''_y denote the two sums directly above. Then multiplying by $\frac{1}{\sqrt{n}}$, we get Riemann sums for the following integrals:

$$\frac{1}{\sqrt{n}} \Sigma'_y \to \int_0^y \frac{e^{\tau - ct}}{1 + e^{\tau - ct}} dt = \frac{1}{c} \left(\log(1 + e^{\tau}) - \log(1 + e^{\tau - cy}) \right), \tag{2.7}$$

and

$$\frac{1}{\sqrt{n}} \Sigma''_y \to \int_0^y \frac{e^{\tau - ct}}{(1 + e^{\tau - ct})^2} dt = \frac{1}{c} \left(\frac{e^{\tau}}{1 + e^{\tau}} - \frac{e^{\tau - cy}}{1 + e^{\tau - cy}} \right). \tag{2.8}$$

Let $\delta > 0$. The integrands in (2.7) and (2.8) are monotonically decreasing functions of t; hence from integral comparison, we have, for $|\tau| < \delta$ and $y \in [0, \infty)$,

$$\left| \frac{1}{c} \left(\log(1 + e^{\tau}) - \log\left(1 + e^{\tau - cy}\right) \right) - \frac{1}{\sqrt{n}} \Sigma'_y \right| < \frac{1}{\sqrt{n}} \cdot \frac{e^{\delta}}{1 + e^{\delta}},$$

and

$$\left| \frac{1}{c} \left(\frac{e^{\tau}}{1 + e^{\tau}} - \frac{e^{\tau - cy}}{1 + e^{\tau - cy}} \right) - \frac{1}{\sqrt{n}} \Sigma''_y \right| < \frac{1}{\sqrt{n}} \cdot \frac{e^{\delta}}{(1 + e^{\delta})^2}.$$

Thus the convergence in (2.7) and (2.8) is uniform in $y \in [0, \infty)$ and $|\tau| < \delta$.

Using Taylor's Theorem, we now have

$$\begin{aligned} U(\tau) &\leq \tau U'(0) + \frac{\tau^2}{2} \sup_{|\sigma| < \delta} |U''(\sigma)| \\ &\sim \tau\sqrt{n} \left(-\frac{a}{\sqrt{n}} + \frac{1}{c} \left(\log(2) - \log(1 + e^{-cy}) \right) + O(\tau) \right) \\ &= \tau\sqrt{n} \left(-\frac{a}{\sqrt{n}} + \overline{g_d}(y) + O(\tau) \right), \end{aligned}$$

where, because of uniformity in (2.8), $O(\tau)$ does not depend on y. Thus, choosing τ small in absolute value and positive or negative as needed, we get that $U(\tau) \leq -C\sqrt{n}$ for some $C > 0$ that holds for all $\left(\frac{a}{\sqrt{n}}, y\right)$ with $\left| -\frac{a}{\sqrt{n}} + \overline{g_d}(y) \right| \geq \epsilon$. Using this in (2.6), we obtain (2.1).

Thus, fixing bottom left vertices at $-\frac{\sqrt{6}}{\pi}\log(2)$,

$$\lim_{n\to\infty}\frac{1}{d(n)}\cdot\#\left\{\lambda\in\mathscr{D}(n) : V_\ell(\lambda)\subset N_\epsilon(f_d)\right\}=1,$$

and by symmetry, we can (only) say

$$\lim_{n\to\infty}\frac{1}{d(n)}\cdot\#\left\{\lambda\in\mathscr{D}(n) : V_r(\lambda)\subset N_\epsilon(f_d^*)\right\}=1,$$

where f_d^* is f_d with its right-half translated by some amount x_0. Hence,

$$\lim_{n\to\infty}\frac{1}{d(n)}\cdot\#\left\{\lambda\in\mathscr{D}(n) : V_\ell(\lambda)\cup V_r(\lambda)\subset N_\epsilon(f_d^*)\right\}=1,$$

but since the diagrams have area 1 and ϵ may be made arbitrarily small, we must have $x_0=0$. Thus, finally,

$$\lim_{n\to\infty}\frac{1}{d(n)}\cdot\#\left\{\lambda\in\mathscr{D}(n) : V_\ell(\lambda)\cup V_r(\lambda)\subset N_\epsilon(f_d)\right\}=1,$$

and this implies Theorem 1.

2.2 Proof of Theorem 1.4

For the proof of Theorem 1.4, we need to estimate a slightly different product, and again we will want to do our manipulations in the first quadrant. Once we have left and right limit shapes, gluing them together will only be valid once we know that, for 100% of stacks as $n\to\infty$, peaks occur with multiplicity $o(\sqrt{n})$. We show that this follows from a well-known asymptotic for partitions with restricted largest part.

Throughout the proof let $c:=\frac{\pi}{\sqrt{3}}$ and $q=e^{-\frac{c}{\sqrt{n}}}$. This c is the constant needed in the following lemma, an analogue of Lemma 2.1.

Lemma 2.2. *There exists a unique $c>0$ such that for $q=e^{-\frac{c}{\sqrt{n}}}$, we have*

$$\log\left(q^{-n}\mathscr{S}(q)\right)\sim\log s(n)=\frac{2\pi}{\sqrt{3}}\sqrt{n}+o(\sqrt{n}).$$

Proof. Recall that $\mathscr{S}(q) = \prod_{m\geq 1} \frac{1}{(1-q^m)^2} \cdot L(q)$. From the well-known transformation of the Dedekind η-function ([6], Th. 3.1), one has

$$\log \prod_{m\geq 1} \frac{1}{1-e^{-mt}} \sim \frac{\pi^2}{6t}.$$

By Wright Lemma 2 [43] we have $\log L(e^{-t}) \sim \log \frac{1}{2}$. Letting $t = \frac{c}{\sqrt{n}}$, we have

$$\log\left(e^{-c\sqrt{n}} \mathscr{S}\left(e^{-\frac{c}{\sqrt{n}}}\right)\right) \sim \left(c + \frac{\pi^2}{3c}\right)\sqrt{n}.$$

We take $c = \frac{\pi}{\sqrt{3}}$, minimizing the term on the right, and the lemma is proved. □

Let $\overline{f_s}(t)$ be the left-half of f_s translated right into the first quadrant as follows

$$\overline{f_s} : \left[0, -\frac{1}{c}\log\left(1-e^{-c\epsilon}\right)\right) \mapsto [\epsilon, \infty), \qquad \overline{f_s}(x) = -\frac{1}{c}\log\left(1 - e^{cx}(1-e^{-c\epsilon})\right).$$

We will also make use of the inverse for $\overline{f_s}$ which is

$$\overline{g_s} : [\epsilon, \infty) \mapsto \left[0, -\frac{1}{c}\log\left(1-e^{-c\epsilon}\right)\right), \qquad \overline{g_s}(y) := \frac{1}{c}\log\left(\frac{1-e^{-cy}}{1-e^{-c\epsilon}}\right).$$

By Lemma 2.2, an upper bound for the proportion of the number of stacks of size n with a left parts that lie in $[\epsilon\sqrt{n}, y\sqrt{n}]$, is

$$\frac{q^{-n}\mathscr{S}(q)}{s(n)} z^{-a} \prod_{\epsilon\sqrt{n}\leq j\leq y\sqrt{n}} \frac{1-q^j}{1-zq^j} =: e^{o(\sqrt{n})} \cdot e^{U(\tau)}, \tag{2.9}$$

where $z = e^\tau$ for $\tau \in \mathbb{R}$ and

$$U(\tau) := -\tau a + \sum_{\epsilon\sqrt{n}\leq j\leq y\sqrt{n}} \left(\log(1-e^{-c\frac{j}{\sqrt{n}}}) - \log(1-e^{\tau - c\frac{j}{\sqrt{n}}})\right).$$

We find the derivatives

$$U'(\tau) = -a + \sum_{\epsilon\sqrt{n}\leq j\leq y\sqrt{n}} \frac{e^{\tau-\frac{cj}{\sqrt{n}}}}{1-e^{\tau-\frac{cj}{\sqrt{n}}}}; \qquad U''(\tau) = \sum_{\epsilon\sqrt{n}\leq j\leq y\sqrt{n}} \frac{e^{\tau-\frac{cj}{\sqrt{n}}}}{\left(1-e^{\tau-\frac{cj}{\sqrt{n}}}\right)^2}.$$

Let Σ'_y and Σ''_y denote the two sums directly above. Then multiplying by $\frac{1}{\sqrt{n}}$, we get Riemann sums for the following integrals:

$$\frac{1}{\sqrt{n}}\Sigma'_y \to \int_\epsilon^y \frac{e^{\tau-ct}}{1-e^{\tau-ct}}dt$$

$$
\begin{aligned}
&= \frac{1}{c}\left(-\log(1-e^{\tau-c\epsilon}) + \log(1-e^{\tau-cy})\right)\\
&= \frac{1}{c}\log\left(\frac{1-e^{\tau-cy}}{1-e^{\tau-c\epsilon}}\right), \qquad (2.10)
\end{aligned}
$$

and

$$
\frac{1}{\sqrt{n}}\Sigma_y'' \to \int_\epsilon^y \frac{e^{\tau-ct}}{(1-e^{\tau-ct})^2}dt = \frac{1}{c}\left(\frac{e^{\tau-c\epsilon}}{1-e^{\tau-c\epsilon}} - \frac{e^{\tau-cy}}{1-e^{\tau-cy}}\right). \qquad (2.11)
$$

Let $\delta > 0$. From integral comparison and monotonicity of the integrand, it is easy to see that, for $|\tau| < \delta$ and $y \in [\epsilon, \infty)$,

$$
\left|\frac{1}{c}\left(-\log(1-e^{\tau-c\epsilon}) + \log\left(1-e^{\tau-cy}\right)\right) - \frac{1}{\sqrt{n}}\Sigma_y'\right| < \frac{1}{\sqrt{n}}\cdot\frac{e^{\delta-c\epsilon}}{1-e^{\delta-c\epsilon}},
$$

and

$$
\left|\frac{1}{c}\left(\frac{e^{\tau}}{1-e^{\tau-c\epsilon}} - \frac{e^{\tau-cy}}{1-e^{\tau-cs}}\right) - \frac{1}{\sqrt{n}}\Sigma_y''\right| < \frac{1}{\sqrt{n}}\cdot\frac{e^{\delta-c\epsilon}}{(1-e^{\delta-c\epsilon})^2}.
$$

Thus the convergence in (2.10) and (2.11) is uniform in $y \in [\epsilon, \infty)$ and $|\tau| < \delta$. Using Taylor's Theorem, we now have

$$
\begin{aligned}
U(\tau) \le \tau U'(0) + \frac{\tau^2}{2}\sup_{|\sigma|<\delta}|U''(\sigma)| &\sim \tau\sqrt{n}\left(-\frac{a}{\sqrt{n}} + \frac{1}{c}\log\left(\frac{1-e^{-cy}}{1-e^{-c\epsilon}}\right) + O(\tau)\right)\\
&\sim \tau\sqrt{n}\left(-\frac{a}{\sqrt{n}} + \overline{g_s}(y) + O(\tau)\right)
\end{aligned}
$$

where, because of uniformity in (2.11), $O(\tau)$ does not depend on y. Thus, when $\left|-\frac{a}{\sqrt{n}} - \overline{g_s}(y)\right| \ge \epsilon$, we conclude, as in the proof of Theorem 1.3, that

$$
\lim_{n\to\infty}\frac{1}{s(n)}\cdot\#\{\lambda \in \mathscr{S}(n) : V_\ell(\lambda) \not\subset N_\epsilon(f_s)\} = 0.
$$

By symmetry, we can also say that

$$
\lim_{n\to\infty}\frac{1}{s(n)}\cdot\#\{\lambda \in \mathscr{S}(n) : V_r(\lambda) \not\subset N_\epsilon(f_s^*)\} = 0,
$$

where f_s^* is f_s with its right-half translated by some t_0. We cannot, however, use an area argument to immediately conclude that $t_0 = 0$.

For strongly unimodal sequences, the fact that parts on either side are distinct and the total area is one forced a limit shape from the *negative* result that on average 0% of left (resp. right) halves of diagrams are *not* near a left (resp. right) limit shape. But we do not have this forcing in this case because of peaks. For example, if peaks are at most $t\sqrt{n}$ on average where t is fixed, then the vertical asymptotes are not approached in the limit shape. Thus, we show that, indeed, "all of the limit shape is used," in the sense that peaks are $\omega(\sqrt{n})$ on average.

Lemma 2.3. *Let $t > 0$ be an arbitrary fixed constant. If $k = t\sqrt{n}$ and $s_{\leq k}(n)$ denotes the number of unimodal sequences of size n in which peaks are size at most k, then*

$$\lim_{n\to\infty} \frac{s_{\leq k}(n)}{s(n)} = 0.$$

From this we can conclude that peaks are $\omega(\sqrt{n})$ on average, and hence occur with multiplicity $o(\sqrt{n})$ on average. Thus, "gluing" left and right limit shapes together at the origin is valid.

Proof of Lemma 2.3. Let $\mathscr{S}_k(n)$ denote the set of stacks of size n in which peaks have size k. Let $\mathscr{P}_{\leq k}(n)$ denote the set of partitions of n into parts $\leq k$. Let $s_k(n)$ and $p_{\leq k}(n)$ be the cardinality of these sets, respectively. Then we have an injection

$$\mathscr{S}_k(n) \hookrightarrow \bigcup_{m=0}^{n} \mathscr{P}_{\leq k}(m) \times \mathscr{P}_{\leq k}(n-m), \tag{2.12}$$

given by cutting a stack $\lambda \in \mathscr{S}_k(n)$ in half directly right of the left-most peak.[3] Thus, we may write

[3]We can be more precise about the image in (2.12), but we will not need to be.

$$
\begin{aligned}
s_k(n) &\leq \sum_{m=0}^{n} p_{\leq k}(m) p_{\leq k}(n-m) \\
&\leq 2 \sum_{0 \leq m \leq \epsilon \cdot n} p_{\leq k}(m) p_{\leq k}(n-m) + \sum_{\epsilon \cdot n \leq m \leq (1-\epsilon) n} p_{\leq k}(m) p_{\leq k}(n-m) \qquad (2.13) \\
&=: 2\Sigma_1 + \Sigma_2
\end{aligned}
$$

for some $\epsilon = \epsilon(t) \in (0,1)$ to be specified later. Asymptotics for $p_{\leq k}(n)$ when $k = t\sqrt{n}$ were given first by Szekeres [40], reformulated and reproved by Canfield [17] and later by Romik [36]. From Romik's formulation,

$$
p_{\leq k}(n) \ll e^{H(t)\sqrt{n}},
$$

where

$$
\begin{aligned}
H(t) &= 2\alpha(t) - t \log\left(1 - e^{-t\alpha(t)}\right) \\
\alpha :&[0,\infty) \to \left[0, \frac{\pi}{\sqrt{6}}\right) \text{ defined by } \alpha(t)^2 = \mathrm{Li}_2\left(1 - e^{-t\alpha(t)}\right)
\end{aligned}
$$

We now show that $\alpha : [0,\infty) \to \left[0, \frac{\pi}{\sqrt{6}}\right)$ is strictly increasing; in particular, $\alpha(t)$ is well-defined as above. One finds

$$
\alpha'(t) = \frac{t\alpha(t)}{2(e^{t\alpha(t)} - 1) - t^2}.
$$

The numerator is positive for $t > 0$, so it remains to show that the denominator is positive for $t > 0$. We will actually show

$$
\frac{t^2}{e^{t\alpha(t)} - 1} < 1, \qquad \text{for } t > 0. \qquad (2.14)
$$

Following Canfield ([17], Comment 19), we have

$$
\begin{aligned}
\alpha(t)^2 &= \mathrm{Li}_2\left(1 - e^{-t\alpha(t)}\right) \\
&= -\int_0^{1-e^{-t\alpha(t)}} \frac{\log(1-z)}{z} dz
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{t\alpha(t)} \frac{u}{e^u - 1} du && \text{(substituting } z = 1 - e^{-u}\text{)} \\
&> t\alpha(t) \cdot \frac{t\alpha(t)}{e^{t\alpha(t)} - 1}. && \text{(since the integrand is increasing)}
\end{aligned}
$$

From this, (2.14) follows, so α is strictly increasing. Next, it may be checked that $H'(t) = -\log\left(1 - e^{-t\alpha(t)}\right) > 0$, so that H is strictly increasing. Furthermore, we have

$$
\lim_{t\to\infty} H(t) = \lim_{t\to\infty} 2\alpha(t) = \pi\sqrt{\frac{2}{3}}.
$$

Returning to (2.13), we may write any $m \in [\epsilon n, (1-\epsilon)n]$ as sn for some $s \in [\epsilon, 1-\epsilon]$. Thus,

$$
\Sigma_2 \ll n \exp\left(\sqrt{n} \sup_{s\in[\epsilon,1-\epsilon]} \left(\sqrt{s}H\left(\frac{t}{\sqrt{s}}\right) + \sqrt{1-s}H\left(\frac{t}{\sqrt{1-s}}\right)\right)\right).
$$

Since H is strictly increasing to $\pi\sqrt{\frac{2}{3}}$, for any fixed t, ϵ and all $s \in [\epsilon, 1-\epsilon]$, there is a $B = B_{\epsilon,t} < \pi\sqrt{\frac{2}{3}}$ such that $H\left(\frac{t}{\sqrt{s}}\right), H\left(\frac{t}{\sqrt{1-s}}\right) \leq B$. Hence, we may bound the above by

$$
n \exp\left(\sqrt{n} \sup_{s\in[\epsilon,1-\epsilon]} \left(\sqrt{s}\cdot B + \sqrt{1-s}\cdot B\right)\right) = n\exp\left(\sqrt{n}\cdot\sqrt{2}\cdot B\right).
$$

Now,

$$
2\Sigma_1 \ll np_{\leq k}(\lfloor \epsilon n \rfloor)p_{\leq k}(n) \ll n\exp\left(\sqrt{n}\left(\sqrt{\epsilon}H\left(\frac{t}{\sqrt{\epsilon}}\right) + \pi\sqrt{\frac{2}{3}}\right)\right).
$$

Since H is bounded, we may choose $\epsilon = \epsilon(t)$ so that

$$
C := \sqrt{\epsilon}H\left(\frac{t}{\sqrt{\epsilon}}\right) + \pi\sqrt{\frac{2}{3}} < \pi\frac{2}{\sqrt{3}}.
$$

Thus, altogether we have

$$
s_{\leq k}(n) \leq ns_k(n) \ll n^2\exp\left(\sqrt{n}\cdot C\right) + n^2\exp\left(\sqrt{n}\cdot B\sqrt{2}\right),
$$

where $C, B\sqrt{2} < \pi\frac{2}{\sqrt{3}}$. Recalling that $s(n) \sim \frac{1}{2^3\cdot 3^{3/4}\cdot n^{5/4}}e^{\pi\frac{2}{\sqrt{3}}\sqrt{n}}$ ([43], Th. 2), we have finished the proof of Lemma 2.3. □

By our earlier observations, the the proof of Theorem 1.4 is now complete.

2.3 Proof of Theorem 1.7

Here, the derivations of left and right limit shapes are similar, respectively, to those in Sections 2.1 and 2.2. Thus, in this section we content ourselves to proving an analogue of Lemmas 2.1 and 2.2 and to showing that peaks are $\omega(\sqrt{n})$ on average; as in Section 2.2, this is necessary to avoid the possibility of a degenerate limit shape.

As required by our technique, the next lemma shows that there is a choice of q so that

$$\frac{q^{-n}\mathscr{D}_m(q)}{dm(n)} = e^{o(\sqrt{n})}.$$

Lemma 2.4. *There exists $c > 0$ such that for $q = e^{-\frac{c}{\sqrt{n}}}$, we have*

$$\log\left(q^{-n}\mathscr{D}_m(q)\right) \sim \log dm(n) = \pi\sqrt{n} + o(\sqrt{n}).$$

Proof. By equation (3.3) and Theorem 1.3 of [16], we have $\log dm(n) \sim \pi\sqrt{n}$, and

$$\log\left(e^{c\sqrt{n}}\mathscr{D}_m\left(e^{-\frac{c}{\sqrt{n}}}\right)\right) \sim \left(c + \frac{\pi^2}{4c}\right)\sqrt{n}.$$

We take $c = \frac{\pi}{2}$, and the lemma is proved. □

With Lemma 2.4 in hand, we may derive left and right limit shapes as in Sections 2.1 and 2.2, respectively. Thus, the proof will be completed by the following lemma, which shows that peaks are $\omega(\sqrt{n})$ on average.

Lemma 2.5. *Let $t > 2$ be an arbitrary fixed constant. If $k = t\sqrt{n}$ and $dm_{\leq k}(n)$ denotes the number of stacks of size n in which the peak is at most k, then*

$$\lim_{n\to\infty} \frac{dm_{\leq k}(n)}{dm(n)} = 0.$$

Remark 2.6. *Since $t_1 \leq t_2$ implies $dm_{\leq t_1\sqrt{n}}(n) \leq dm_{\leq t_2\sqrt{n}}(n)$, the conclusion of Lemma 2.5 holds with any $t \geq 0$.*

Proof of Lemma 2.5. Let $\mathscr{Q}_{\leq k}(n)$ and $q_{\leq k}(n)$ be, respectively, the set and number of distinct-parts partitions of n whose largest part is at most k. As in Section 2.2, we define a map

$$\mathscr{D}_{m,k}(n) \hookrightarrow \bigcup_{m=0}^{n} \mathscr{P}_{\leq k}(m) \times \mathscr{Q}_{\leq k}(n-m), \tag{2.15}$$

by sending the peak and left parts to a distinct partition, and by sending the right parts to an unrestricted partition. Thus,

$$\begin{aligned} dm_k(n) \leq & \sum_{m \leq \epsilon n} \left(q_{\leq k}(m) p_{\leq k}(n-m) + q_{\leq k}(n-m) p_{\leq k}(m)\right) \\ & + \sum_{\epsilon n \leq m \leq (1-\epsilon)n} q_{\leq k}(m) p_{\leq k}(n-m) \\ =: & \Sigma_1 + \Sigma_2. \end{aligned} \tag{2.16}$$

We complete the proof by citing Theorem 1.9, proved in Chapter 3, which in a weak form states

$$q_{\leq k}(n) \ll e^{B(t)\sqrt{n}},$$

for $k = t\sqrt{n}$ for fixed $t > \sqrt{2}$. We also show in Section 3.2 that $B(t)$ is a strictly increasing function with $\lim_{t\to\infty} B(t) = \frac{\pi}{\sqrt{3}}$. For $m \in [\epsilon n, (1-\epsilon)n]$, we will write $m = sn$, for $s \in [\epsilon, 1-\epsilon]$. Thus, as in Section 2.2,

$$\begin{aligned} \Sigma_2 & \ll n \exp\left(\sqrt{n} \sup_{s\in[\epsilon,1-\epsilon]} \left(\sqrt{s} H\left(\frac{t}{\sqrt{s}}\right) + \sqrt{1-s} B\left(\frac{t}{\sqrt{1-s}}\right)\right)\right) \\ & \ll n \exp\left(\sqrt{n} \cdot C \sup_{s\in[\epsilon,(1-\epsilon)]} \left(\sqrt{2s} + \sqrt{1-s}\right)\right), \end{aligned}$$

where $C < \frac{\pi}{\sqrt{3}}$. Thus,

$$\Sigma_2 \ll n \exp\left(\sqrt{n} \cdot C\sqrt{3}\right) = o\left(\pi \exp(\sqrt{n})\right).$$

Now,

$$\Sigma_1 \ll n q_{\leq k}\left(\lfloor \epsilon n \rfloor\right) p_{\leq k}(n) + n q_{\leq k}(n) p_{\leq k}\left(\lfloor \epsilon n \rfloor\right))$$

$$\ll n\exp\left(\sqrt{n}\left(\sqrt{\epsilon}B\left(\frac{t}{\sqrt{\epsilon}}\right)+\pi\sqrt{\frac{2}{3}}\right)\right)+n\exp\left(\sqrt{n}\left(\frac{\pi}{\sqrt{6}}+\sqrt{\epsilon}H\left(\frac{t}{\sqrt{\epsilon}}\right)\right)\right),$$

where we have used the asymptotic formulas for $p(n)$ and $q(n)$ ([2], Th. 6.2). Since H is bounded, we may choose $\epsilon = \epsilon(t)$ so that $\Sigma_1 \ll o\left(\exp(\pi\sqrt{n})\right)$.

Finally, since $dm(n) \sim \frac{1}{16n}e^{\pi\sqrt{n}}$ ([16], Th. 1.3), we have $dm_{\leq k}(n) = o(dm(n))$ as required. □

This concludes the proof of Theorem 1.7.

2.4 Some Consequences of Theorems 1.3, 1.4 and 1.7

The most natural consequences of our limit shapes concern the structure of unimodal sequences of size n at the scale of $\sqrt{n}$. For example, Theorem 1.3 implies the following corollary concerning the number of parts in strongly unimodal sequences.

Corollary 2.7. *Let $\epsilon > 0$ be arbitrary. The number of parts of 100% of strongly unimodal sequences of size n as $n \to \infty$ lies in the interval*

$$\sqrt{n}\left(2\frac{\sqrt{6}}{\pi}\log 2-\epsilon, 2\frac{\sqrt{6}}{\pi}\log 2+\epsilon\right).$$

We leave the statement of similar corollaries to the reader.

Recall that the *rank* of a semi-strict unimodal sequence is the number of parts to the right of the peak minus the number of parts to the left of the peak. Bringmann–Jennings-Shaffer–Mahlburg proved that the limiting distribution of this statistic is a point mass with mean $\frac{\sqrt{n}\log n}{\pi}$ ([14], Prop. 1.2 part (3)). Theorem 1.7 anticipates this result, as follows.

After Theorem 1.7, we see that a typical semi-strict unimodal sequence of size n is made up of a distinct parts partition of size roughly $\frac{n}{3}$ and an unrestricted partition of size roughly $\frac{2n}{3}$. It follows from Theorem 1.7 that this distinct parts partition has roughly $\frac{\sqrt{6}}{\pi}\log(2)\sqrt{n}$ parts. Recalling Theorem 1.1, a typical partition of size m has roughly $\frac{\sqrt{3}}{\pi\sqrt{2}}\sqrt{m}\log m$ parts. Hence, we should expect the limiting

rank of 100% semi-strict unimodal sequences to be

$$\frac{\sqrt{3}}{\pi\sqrt{2}}\sqrt{\frac{2n}{3}}\log\left(\frac{2n}{3}\right)-\frac{\sqrt{6}}{\pi}\log(2)\sqrt{n}\sim\frac{\sqrt{n}\log n}{\pi},$$

as proved in [14] using the Method of Moments.

Theorem 1.7 also leads to a limit shape for overpartitions, combinatorial objects having many similarities to classical partitions. As defined in [19], an overpartition is a partition in which the last occurrence of a part may (or may not be) marked. For example, the overpartitions of size 3 are $(\overline{3})$, (3), $(\overline{2},\overline{1})$, $(\overline{2},1)$, $(2,\overline{1})$, $(2,1)$, $(1,1,\overline{1})$, $(1,1,1)$. We denote the set of overpartitions of size n by $\overline{\mathscr{P}}(n)$ with cardinality $\overline{p}(n)$.

From [3] equation 1.7, we have the generating function identity

$$\frac{1+q}{q}\sum_{n\geq 1}dm(n)q^n=\prod_{j\geq 1}\frac{1+q^j}{1-q^j}=\sum_{n\geq 0}\overline{p}(n)q^n,$$

thus $dm(n+1)+dm(n)=\overline{p}(n)$. We now give a short bijective proof of this equality and use it to derive a limit shape for overpartitions. If $\lambda\in\mathscr{D}_m(n)$, let the peak and parts to its left be marked. If $\lambda\in\mathscr{D}_m(n+1)$, let only the parts to its left be marked and subtract 1 from the peak.

If we plot diagrams for overpartitions as Vershik does for partitions in [41]—in the first quadrant as weakly decreasing columns of squares and without distinguishing marked parts—then our bijection leads to a map between diagrams of semi-strict unimodal sequences and a transfer of limit shapes. Since $dm(n+1)\sim dm(n)$, it is easy to see that a limit shape for overpartitions is obtained immediately by adding horizontal components of the limit shape for semi-strict unimodal sequences.

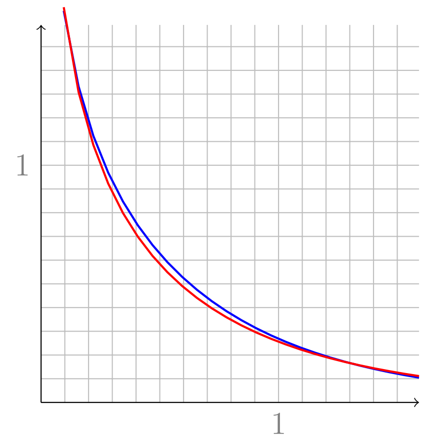

Figure 2.2. $f_{\overline{p}}(x)$, the limit shapes for overpartitions (in blue) and $f_p(x)$, the limit shape for unrestricted partitions (in red)

Corollary 2.8 (Overpartitions)**.** *Let $\epsilon > 0$ be arbitrary and let $(g)^{-1}$ denote the inverse function of g, so*

$$\begin{aligned} f_{\overline{p}}(x) :&= \left(\left(-\frac{2}{\pi} \log\left(1 - e^{-\frac{\pi}{2}x}\right) \right)^{-1} - \left(-\frac{2}{\pi} \log\left(e^{-\frac{\pi}{2}x} - 1\right) \right)^{-1} \right)^{-1} \\ &= \frac{2}{\pi} \log\left(\frac{1 + e^{-\frac{\pi}{2}x}}{1 - e^{-\frac{\pi}{2}x}} \right). \end{aligned}$$

Then

$$\lim_{n\to\infty} \frac{1}{\overline{p}(n)} \left\{ \lambda \in \overline{\mathscr{P}}(n) : \widetilde{\varphi}(\lambda) \subset N_\epsilon(f_{\overline{p}}) \right\} = 1.$$

If we represent marked parts in a diagram by shading the top square, we see that conjugation is also an involution on overpartitions. Thus, $y = f_{\overline{p}}(x)$ is symmetric in x and y, as expected.

Remark 2.9. *In [18], Corteel-Hitczenko proved that the expected weight of overlined parts is asymptotic to $\frac{n}{3}$. In view of Theorem 1.7 and the map between semi-strict unimodal sequences and overpartitions, we obtain the following refinement: For any $\epsilon > 0$ and "for 100% of overpartitions as $n \to \infty$", the total weight of marked parts lies between $\frac{n}{3} - \epsilon\sqrt{n}$ and $\frac{n}{3} + \epsilon\sqrt{n}$.*

Chapter 3: Distinct Parts Partitions with Bounded Largest Part

3.1 Proof Outline and Probabilistic Model

Throughout this chapter x will be a positive real number. We prove Theorem 1.9 through three propositions. Proposition 3.1 anticipates the asymptotic behavior of $\log q_t(n)$ through classical saddle-point bounds, while Propositions 3.2 and 3.3 complete the proof using Fristedt's probabilistic machinery.

3.1.1 Saddle-point Bounds

This section elaborates further on the ideas behind Lemmas 2.1 and 2.2 that will be used here also. Again, we begin with the trivial inequality,

$$q_t(n) \leq x^{-n} \mathcal{Q}_{t,n}(x). \tag{3.1}$$

As explained in the book of Flajolet and Sedgewick ([26], p. 550), the right-hand side of (3.1), as a function of $x \in (0, \infty)$ has positive second derivative with respect to x and tends to $+\infty$ when $x \to 0$ and $x \to \infty$. Thus, there is a unique saddle-point $x = x_n$ on the positive real axis for the function $|z^{-n} \mathcal{Q}_{t,n}(z)|$ of a complex variable z. In fact, x will approach 1 as $n \to \infty$, from below when $t > 2$ and from above when $t < 2$. As in Lemmas 2.1 and 2.2, we expect that this x actually satisfies

$$\log q_t(n) \sim \log \left(x^{-n} \mathcal{Q}_{t,n}(x) \right),$$

and we will ultimately prove that this is the case. But for now, we ascertain an upper bound for $q_t(n)$ by finding the asymptotic behavior of the right-hand side of the above.

More explicitly, we will set $x = e^{-\frac{y}{\sqrt{n}}}$ for $y \in \mathbb{R}$ and write

$$\log\left(x^{-n}\mathcal{Q}_t(x)\right) = \sqrt{n} f_n(y), \qquad \text{where} \qquad f_n(y) := y + \frac{1}{\sqrt{n}} \log \mathcal{Q}_t(x).$$

One computes

$$f_n'(y) = 1 - \frac{1}{n} \sum_{k \le t\sqrt{n}} \frac{kx^k}{1+x^k}, \tag{3.2}$$

so that the saddle-point occurs at (or very near) x when $f_n'(y) \sim 0$. We will show in Proposition 3.2 that this is accomplished by choosing $y = \beta$; indeed, the sum in (3.2) is just a Riemann sum for the integral (1.5) defining β. With β in hand, an application of Euler-MacLaurin summation leads to the following.

Proposition 3.1. *With $x = e^{-\frac{\beta}{\sqrt{n}}}$, we have*

$$\log\left(x_n^{-n}\mathcal{Q}_{t,n}(x_n)\right) = B(t)\sqrt{n} - \log\left(1+e^{-\beta t}\right)\{t\sqrt{n}\} + \log\left(\sqrt{\frac{1+e^{-\beta t}}{2}}\right) + o(1), \tag{3.3}$$

where $B(t)$ is as defined in Theorem 1.9.

As observed above, Proposition 3.1 implies $\log q_t(n) \ll B(t)\sqrt{n}$, but we will see later that the two are actually asymptotic.

3.1.2 Probabilistic Model

From probability theory we will require the elementary notions of expectation, variance and distribution of discrete random variables, as well Fourier inversion of characteristic functions (which in this context is equivalent to an application of Cauchy's Theorem from complex analysis). We will also mention central and local limit theorems. All of these topics are covered in most standard probability texts; for instance see [9].

We now repair the inequality (3.1) by introducing a certain probability measure depending on x as

$$P_x(N = k) = \frac{q_t(k)x^k}{\mathcal{Q}_{t,n}(x)}, \qquad \text{so that} \qquad q_t(n) = x^{-n}\mathcal{Q}_t(x)P_x(N = n). \tag{3.4}$$

We define P_x and the random variable N below. As in section 3.1.1, we will eventually choose $x = e^{-\frac{\beta}{\sqrt{n}}}$. At any rate, $P_x(N = n) \leq 1$ and we will see that it does not affect the exponential part of the asymptotic for $q_t(n)$.

Our probability measure P_x is similar to the ones introduced by Fristedt [25], who invented an early variant of a *Boltzmann model* for partitions and used it to prove many far-reaching results on the structure of partitions. When applied to partitions, Boltzmann sampling algorithms select partitions of size roughly n, roughly uniformly and in nearly linear time, assuming n is large. (See [23] for more on Boltzmann sampling for combinatorial structures.)

Following Fristedt, we define a probability measure P_x on the set of partitions λ generated by $\mathcal{Q}_{t,n}$ by setting

$$P_x(\lambda) := \frac{x^{|\lambda|}}{\mathcal{Q}_{t,n}(x)},$$

where $|\lambda|$ is the size of the partition λ, i.e. the sum of its parts. Here, P_x depends on n, but we will refrain from notating this because x will depend on n as in Section 3.1.1.

Let $\{X_k\}_{k=1}^{t\sqrt{n}}$ be random variables giving the multiplicity of k in a partition λ. Since our partitions have distinct parts, X_k is Bernoulli and one computes

$$P_x(X_k = 0) = \frac{1}{1+x^k} \qquad \text{and} \qquad P_x(X_k = 1) = \frac{x^k}{1+x^k}.$$

It is also straightforward to show that the X_k's are independent under P_x. Now set $N := \sum_{k \leq t\sqrt{n}} kX_k$, a random variable representing the size of a partition. Using independence, its expectation and variance under P_x are

$$\mathrm{E}_x(N) = \sum_{k \leq t\sqrt{n}} \frac{kx^k}{1+x^k}, \qquad \sigma_n^2 := \mathrm{Var}_x(N) = \sum_{k \leq t\sqrt{n}} \frac{k^2x^k}{(1+x^k)^2}. \tag{3.5}$$

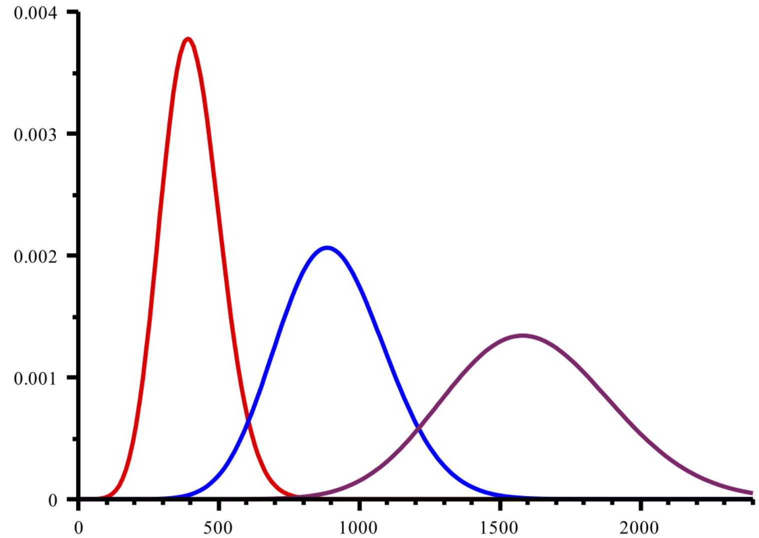

Figure 3.1. Plots of $P_x(N = k)$ when $t = 3$ and $n = 400$ (red), 900 (blue) and 1600 (purple), generated using Maple.

Returning to (3.2), we see that $f_n'(y) \sim 0$ if and only if $\mathrm{E}_x(N) \sim n$, so the choice $y = \beta$ ensures that the expectation of N is asymptotically n under P_x with $x = \exp\left(-\frac{\beta}{\sqrt{n}}\right)$. Thus, we prove the following.

Proposition 3.2. *With* $x = \exp\left(-\frac{\beta}{\sqrt{n}}\right)$, *we have*

$$\mathrm{E}_x(N) = n + O(\sqrt{n}), \tag{3.6}$$

and

$$\sigma_n^2 = \mathrm{Var}_x(N) = \frac{t}{(1 + e^{\beta t})\beta'(t)} n^{\frac{3}{2}} + O(n). \tag{3.7}$$

In fact, we will show that $\frac{N-n}{\sigma_n}$ is asymptotically normally distributed under P_x (see Figure 1), and so a sort of central limit theorem holds for the X_k. Heuristically, this suggests that $P_x(N = n) \sim \frac{1}{\sqrt{2\pi}\sigma_n}$, as follows: N takes only integer values, so we expect

$$P_x(N = n) = P_x\left(-\frac{1}{2} \leq N - n \leq \frac{1}{2}\right)$$

$$
\begin{aligned}
&= P_x\left(-\frac{1}{2\sigma_n} \le \frac{N-n}{\sigma_n} \le \frac{1}{2\sigma_n}\right) \\
&\approx \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{2\sigma_n}}^{\frac{1}{2\sigma_n}} e^{-\frac{u^2}{2}} du \\
&\sim \frac{1}{\sqrt{2\pi}\sigma_n}.
\end{aligned}
$$

Since $q_t(n) = x^{-n}\mathcal{Q}_{t,n}(x)P_x(N=n)$, the above local limit theorem, together with Proposition 3.1, implies Theorem 1. Our final proposition is a formal statement of the asymptotic normality of $\frac{N-n}{\sigma_n}$ together with the above heuristic.

Proposition 3.3. *With* $x = \exp\left(-\frac{\beta}{\sqrt{n}}\right)$, *we have*

$$
\lim_{n\to\infty} P_x\left(\frac{N-n}{\sigma_n} \le v\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} e^{-\frac{u^2}{2}} du, \qquad \textit{for } v \in \mathbb{R}. \tag{3.8}
$$

Moreover,

$$
P_x(N=n) \sim \frac{1}{\sqrt{2\pi}\sigma_n}. \tag{3.9}
$$

The proof of Proposition 3.3 proceeds via Fourier inversion of the characteristic function for N; it is here that the circle method is hidden and here that we need our most technical estimates.

3.2 The Functions $\beta(t), B(t)$ and $A(t)$.

In this section, we prove the claimed limits in Remark 1.10 and record some additional properties of the functions $\beta(t)$, $B(t)$ and $A(t) := A_n(t)\left(1+e^{-\beta t}\right)^{\{t\sqrt{n}\}}$. Here and in later sections, we require properties of the dilogarithm function, $\mathrm{Li}_2(z)$, which we defined in Section 1.1. (See also [1], §27.7, where $f(x) = \mathrm{Li}_2(1-x)$.)

Proposition 3.4. *The function* $\beta = \beta(t)$ *satisfies the following properties.*

(a) *We have*

$$
\begin{cases}
\beta(t) < 0 & \textit{if } \sqrt{2} < t < 2, \\
\beta(t) = 0 & \textit{if } t = 2 \\
\beta(t) > 0 & \textit{if } t > 2.
\end{cases}
$$

(b) β *is well-defined by* (1.5)*; in particular,* β *is strictly increasing with*

$$\beta'(t) = \begin{cases} \frac{\beta t}{2(1+e^{\beta t})-t^2} & \text{for } t \neq 2, \\ \\ \frac{3}{2} & \text{for } t = 2. \end{cases} \tag{3.10}$$

(c) *The following limits hold:*

$$\lim_{t\to\sqrt{2}^+} \beta(t) = -\infty \qquad \lim_{t\to\infty} \beta(t) = \frac{\pi}{2\sqrt{3}}. \tag{3.11}$$

Proof. If $t > 2$, then we must have $\beta(t) > 0$, for if not,

$$1 = \int_0^t \frac{u}{1+e^{\beta u}} du > \frac{1}{2} \int_0^t u du = \frac{t^2}{4},$$

which leads to the contradiction $2 > t$. A similar argument proves the remaining statements in part (a).

For $t \neq 2$, we rewrite (1.5) as

$$\beta^2(t) = \int_0^{\beta(t)t} \frac{u}{1+e^u} du, \tag{3.12}$$

and take the derivative of both sides to get

$$\beta'(t) = \frac{\beta t}{2(1+e^{\beta t}) - t^2}, \qquad \text{for } t \neq 2.$$

To find $\beta'(2)$, we use the first two terms of the Taylor series for the integrand in (3.12) to write

$$\beta^2 = \frac{\beta^2 t^2}{4} - \frac{\beta^3 t^3}{12} + O\left(\beta^5 t^5\right),$$

for t near 2 (so β near 0). This implies

$$\beta = \frac{3}{t} - \frac{12}{t^3} + O(\beta^3 t^2),$$

and thus by L'Hospital's Rule,

$$\beta'(2) = \lim_{t\to 2} \frac{\beta(t)}{t-2} = \lim_{t\to 2} \frac{\frac{3}{t} - \frac{12}{t^3} + O(\beta^3 t^2)}{t-2} = \lim_{t\to 2} \frac{-3}{t^2} + \frac{36}{t^4} = \frac{3}{2}.$$

We see that $\beta'(t) > 0$ for $t > 2$ by observing

$$1 = \int_0^t \frac{u}{1+e^{\beta u}} du > \frac{1}{1+e^{\beta t}} \int_0^t u du = \frac{1}{1+e^{\beta t}} \cdot \frac{t^2}{2}. \tag{3.13}$$

A similar argument shows that $\beta'(t) > 0$ for $\sqrt{2} < t < 2$ also. Thus, part (b) is proved.

The first limit in (3.11) is easy to see, for

$$1 = \int_0^t \frac{u}{1+e^{\beta u}} du \le \int_0^t u du = \frac{t^2}{2},$$

and thus as $t \to \sqrt{2}^+$, we must have $\beta(t) \to -\infty$. We evaluate the second limit in (3.11) by expressing the integral in (3.12) in terms of the dilogarithm. Thus, (3.12) implies that for $t > 2$, we have

$$\beta(t)^2 = \int_0^{t\beta(t)} \frac{u}{1+e^u} du = \mathrm{Li}_2(1 - e^{-\beta(t)t}) - \frac{1}{2}\mathrm{Li}_2(1 - e^{-2\beta(t)t}). \tag{3.14}$$

Hence, $\lim_{t\to\infty} \beta(t)^2 = \frac{\pi^2}{6} - \frac{\pi^2}{12}$, so $\lim_{t\to\infty} \beta(t) = \frac{\pi}{2\sqrt{3}}$, and part (c) is proved. □

Proposition 3.5. *The function $B(t)$ in (1.6) is strictly increasing, and we have the following limits for $B(t)$ and $A(t) := A_n(t)\left(1 + e^{-\beta t}\right)^{\{t\sqrt{n}\}}$:*

$$\lim_{t\to\infty} B(t) = \frac{\pi}{\sqrt{3}} \qquad \textit{and} \qquad \lim_{t\to\infty} A(t) = \frac{1}{4\sqrt[4]{3}}.$$

Proof. We compute

$$\begin{aligned} B'(t) &= 2\beta'(t) - \frac{te^{-\beta(t)t}}{1+e^{-\beta(t)t}}\left(\beta'(t)t + \beta(t)\right) + \log\left(1 + e^{-\beta(t)t}\right) \\ &= \beta'(t)\left(2 - \frac{t^2 e^{-\beta(t)t}}{1+e^{-\beta(t)t}}\right) - \frac{\beta(t)te^{-\beta(t)t}}{1+e^{-\beta(t)t}} + \log\left(1 + e^{-\beta(t)t}\right) \\ &= \log\left(1 + e^{-\beta(t)t}\right). \end{aligned}$$

Thus, $B(t)$ is a strictly increasing function, and one easily sees that $\lim_{t\to\infty} B(t) = \frac{\pi}{\sqrt{3}}$.

Finally, we can rewrite $A(t)$ using $\beta'(t)$ found in (3.10), and get

$$A(t) = \frac{1}{2}\sqrt{\frac{\beta(t)\left(1+e^{-\beta(t)t}\right)}{\pi\left(2-\frac{t^2}{1+e^{\beta(t)t}}\right)}},$$

from which it is clear that $\lim_{t\to\infty} A(t) = \frac{1}{4\sqrt[4]{3}}$. □

3.3 Proofs of Propositions 3.1, 3.2 and 3.3

Recall that x depends on n and β as $x = e^{-\frac{\beta}{\sqrt{n}}}$. In the proofs of Propositions 3.1 and 3.3, we will need to separate the cases $x > 1$, $x < 1$ and $x = 1$, which after Proposition 3.4 correspond to $\sqrt{2} < t < 2$, $t > 2$ and $t = 2$, respectively. With this in mind, we define

$$\gamma(t) := -\beta(t), \qquad \text{for } \sqrt{2} < t < 2, \tag{3.15}$$

so that $\gamma(t) > 0$ and $x^{-1} = e^{-\frac{\gamma}{\sqrt{n}}} < 1$.

It is also necessary to account for the fact that $t\sqrt{n}$ is not always an integer. Thus, we define

$$t_n := \frac{\lfloor t\sqrt{n}\rfloor}{\sqrt{n}} = t - \frac{\{t\sqrt{n}\}}{\sqrt{n}}, \tag{3.16}$$

so that $t_n\sqrt{n} \in \mathbb{N}$, and a sum over $k \le t\sqrt{n}$ is really a sum from $k = 1 \ldots t_n\sqrt{n}$. Also, we may replace any differentiable function $f(t_n)$ with $f(t) + o(1)$. We will often do this below when $f(t_n)$ is part of the constant term.

Proof of Proposition 3.1. Case 1: $t > 2$. The first iteration of Euler-MacLaurin summation ([34], Appendix B) picks off the claimed main term and constant term:

$$\begin{aligned}
&\log \mathcal{Q}_t(x) \\
&= \sum_{k=1}^{t_n\sqrt{n}} \log\left(1+e^{\frac{-\beta k}{\sqrt{n}}}\right) \\
&= \int_1^{t\sqrt{n}} \log\left(1+e^{\frac{-\beta u}{\sqrt{n}}}\right) du - \int_{t_n\sqrt{n}}^{t\sqrt{n}} \log\left(1+e^{\frac{-\beta u}{\sqrt{n}}}\right) du \\
&\quad + \frac{1}{2}\left(\log\left(1+e^{\frac{-\beta}{\sqrt{n}}}\right) + \log\left(1+e^{-\beta t_n}\right)\right) - \int_1^{t_n\sqrt{n}} \frac{\frac{\beta}{\sqrt{n}}e^{-\frac{\beta}{\sqrt{n}}u}}{1+e^{-\frac{\beta}{\sqrt{n}}u}}\left(\{u\}-\frac{1}{2}\right) du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{n}}{\beta}\int_{\frac{\beta}{\sqrt{n}}}^{\beta t}\log\left(1+e^{-v}\right)dv - \frac{\sqrt{n}}{\beta}\int_{\beta t_n}^{\beta t}\log\left(1+e^{-v}\right)dv \\
&\quad + \frac{1}{2}\log\left(1+e^{\frac{-\beta}{\sqrt{n}}}\right) + \frac{1}{2}\log\left(1+e^{-\beta t}\right) + o(1) - \int_{\frac{\beta}{\sqrt{n}}}^{\beta t}\frac{e^{-v}}{1+e^{-v}}\left(\left\{\frac{\sqrt{n}v}{\beta}\right\}-\frac{1}{2}\right)du \\
&= \frac{\sqrt{n}}{\beta}\left(\mathrm{Li}_2\left(-e^{-\beta t}\right) - \mathrm{Li}_2\left(-e^{-\frac{\beta}{\sqrt{n}}}\right)\right) - \{t\sqrt{n}\}\log\left(1+e^{-\beta t}\right) \\
&\quad + \frac{1}{2}\log\left(1+e^{\frac{-\beta}{\sqrt{n}}}\right) + \frac{1}{2}\log\left(1+e^{-\beta t}\right) + o(1) - \int_{\frac{\beta}{\sqrt{n}}}^{\beta t}\frac{e^{-v}}{1+e^{-v}}\left(\left\{\frac{\sqrt{n}v}{\beta}\right\}-\frac{1}{2}\right)dv.
\end{aligned} \tag{3.17}
$$

The latter integral is $o(1)$ because it is the product of an L^1 function and a bounded oscillating function—as in the proof of the Riemann-Lebesgue Lemma, we prove this first when $\frac{e^{-v}}{1+e^{-v}}$ is replaced by a step function, then we approximate $\frac{e^{-v}}{1+e^{-v}}$ in L^1 by step functions. For the rest of the expression, we apply the following identity for the dilogarithm ([1], 27.7.6):

$$
\mathrm{Li}_2(-x) = -\frac{\pi^2}{12} + \mathrm{Li}_2(1-x) - \frac{1}{2}\mathrm{Li}_2(1-x^2) - \log x \cdot \log(1+x). \tag{3.18}
$$

Thus, recalling (3.14), we obtain the following from (3.17), applying the relevant Taylor series for the logarithm and dilogarithm:

$$
\begin{aligned}
&\frac{\sqrt{n}}{\beta}\left(\beta^2 + t\beta\log\left(1+e^{-\beta t}\right) - \mathrm{Li}_2\left(1-e^{-\frac{\beta}{\sqrt{n}}}\right) + \frac{1}{2}\mathrm{Li}_2\left(1-e^{-\frac{2\beta}{\sqrt{n}}}\right)\right. \\
&\quad \left. -\frac{\beta}{\sqrt{n}}\log\left(1+e^{-\frac{\beta}{\sqrt{n}}}\right)\right) + \frac{1}{2}\log\left(1+e^{\frac{-\beta}{\sqrt{n}}}\right) + \frac{1}{2}\log\left(1+e^{-\beta t}\right) + o(1) \\
&= \sqrt{n}\left(\beta + t\log\left(1+e^{-\beta t}\right)\right) - 1 + 1 - \log(2) + \frac{1}{2}\log(2) + \frac{1}{2}\log\left(1+e^{-\beta t}\right) + o(1) \\
&= \sqrt{n}\left(\beta + t\log\left(1+e^{-\beta t}\right)\right) + \log\left(\sqrt{\frac{1+e^{-\beta t}}{2}}\right) + o(1).
\end{aligned}
$$

Combining this with $\log\left(x^{-n}\right) = \beta\sqrt{n}$ proves Proposition 3.1 when $t > 2$.

Case 2: $\sqrt{2} < t < 2$. We first write

$$
\log\left(x^{-n}\mathscr{D}_{t,n}(x)\right)
$$

$$= -\gamma\sqrt{n} + \sum_{k=1}^{t_n\sqrt{n}} \log\left(1 + e^{\frac{\gamma k}{\sqrt{n}}}\right)$$

$$= -\gamma\sqrt{n} + \sum_{k=1}^{t_n\sqrt{n}} \left(\frac{\gamma k}{\sqrt{n}} + \log\left(1 + e^{-\frac{\gamma k}{\sqrt{n}}}\right)\right)$$

$$= -\gamma\sqrt{n} + \frac{\gamma t_n(t_n\sqrt{n}+1)}{2} + \sum_{k=1}^{t_n\sqrt{n}} \log\left(1 + e^{-\frac{\gamma k}{\sqrt{n}}}\right)$$

$$= \gamma\sqrt{n}\left(\frac{t_n^2}{2} - 1\right) + \frac{\gamma t_n}{2} + \sum_{k=1}^{t_n\sqrt{n}} \log\left(1 + e^{-\frac{\gamma k}{\sqrt{n}}}\right)$$

$$= \gamma\sqrt{n}\left(\frac{t^2}{2} - 1\right) - \gamma t\{t\sqrt{n}\} + \frac{\gamma t}{2} + \sum_{k=1}^{t_n\sqrt{n}} \log\left(1 + e^{-\frac{\gamma k}{\sqrt{n}}}\right) + o(1). \tag{3.19}$$

We then analyze the sum with Euler-MacLaurin summation as before:

$$\sum_{k=1}^{t_n\sqrt{n}} \log\left(1 + e^{-\frac{\gamma}{\sqrt{n}}}\right)$$
$$= \frac{\sqrt{n}}{\gamma}\left(\mathrm{Li}_2(-e^{-\gamma t}) - \mathrm{Li}_2(-e^{-\frac{\gamma}{\sqrt{n}}})\right) + \frac{1}{2}\log\left(1 + e^{-\frac{\gamma}{\sqrt{n}}}\right) + \frac{1}{2}\log(1 + e^{-\gamma t})$$
$$- \{t\sqrt{n}\}\log(1 + e^{-\gamma t}) + o(1). \tag{3.20}$$

This time we rewrite (1.5), the integral definition for $\beta = -\gamma$, to get

$$\gamma^2 = \int_0^{t\gamma} \frac{u}{1 + e^{-u}} du = -\frac{\pi^2}{12} + \frac{\gamma^2 t^2}{2} + \gamma t \log(1 + e^{-\gamma t}) - \mathrm{Li}_2(-e^{-\gamma t}). \tag{3.21}$$

We then apply this and the dilogarithm identity (3.18) to get the following from (3.20), in a manner similar to Case 1:

$$\frac{\sqrt{n}}{\gamma}\left(\gamma^2\left(\frac{t^2}{2} - 1\right) + \gamma t\log(1 + e^{-\gamma t}) - \mathrm{Li}_2(1 - e^{-\frac{\gamma}{\sqrt{n}}}) + \frac{1}{2}\mathrm{Li}_2(1 - e^{-2\frac{\gamma}{\sqrt{n}}})\right.$$
$$\left.-\frac{\gamma}{\sqrt{n}}\log(1 + e^{-\frac{\gamma}{\sqrt{n}}})\right) + \frac{1}{2}\log(1 + e^{-\gamma t}) - \{t\sqrt{n}\}\log(1 + e^{-\gamma t}) + o(1)$$
$$= \sqrt{n}\left(\gamma\left(\frac{t^2}{2} - 1\right) + t\log(1 + e^{-\gamma t})\right) + \log\sqrt{\frac{1 + e^{-\gamma t}}{2}}$$
$$- \{t\sqrt{n}\}\log(1 + e^{-\gamma t}) + o(1).$$

Combining with (3.19), we have the following expression for $\log\left(x^{-n}\mathscr{D}_{t,n}(x)\right)$:

$$\sqrt{n}\left(\gamma\left(t^2 - 2\right) + t\log(1 + e^{-\gamma t})\right) + \log\sqrt{\frac{1 + e^{-\gamma t}}{2}} + \frac{\gamma t}{2}$$

$$- \{t\sqrt{n}\}\left(\gamma t + \log(1+e^{-\gamma t})\right) + o(1)$$

$$= \sqrt{n}\left(-2\gamma + t\log(1+e^{\gamma t})\right) + \log\sqrt{\frac{1+e^{\gamma t}}{2}} - \{t\sqrt{n}\}\log(1+e^{\gamma t}) + o(1).$$

Replacing $\gamma(t)$ with $-\beta(t)$ completes the proof when $\sqrt{2} < t < 2$.

Case 3: $t = 2$. Here, we have $\beta = 0$ and $x = 1$, and so

$$x^{-n}\mathscr{D}_{2,n}(x) = 2^{t_n\sqrt{n}} = 2^{2\sqrt{n}-\{2\sqrt{n}\}} = e^{2\log 2\sqrt{n} - \log 2\{2\sqrt{n}\}},$$

as required. $\square$

Proof of Proposition 3.2. The proof when $t = 2$ (and so $x = 1$) is straightforward. Now let $t \neq 2$. We need only recognize the Riemann-Sums:

$$\begin{aligned}
\mathrm{E}_x(N) &= \sum_{k \le t\sqrt{n}} k \frac{e^{-\frac{\beta k}{\sqrt{n}}}}{1+e^{-\frac{\beta k}{\sqrt{n}}}} \\
&= n \sum_{k \le t\sqrt{n}} \frac{k}{\sqrt{n}} \frac{e^{-\frac{\beta k}{\sqrt{n}}}}{1+e^{-\frac{\beta k}{\sqrt{n}}}} \cdot \frac{1}{\sqrt{n}} \\
&= n\left(\int_0^t \frac{ue^{-\beta u}}{1+e^{-\beta u}} du + O\left(\frac{1}{\sqrt{n}}\right)\right) \\
&= n + O(\sqrt{n}),
\end{aligned}$$

by (1.5). We calculate the variance similarly, using integration by parts to evaluate the integral. We also use the fact that $\frac{e^{-u}}{(1+e^{-u})^2} = \frac{e^u}{(1+e^u)^2}$. Thus,

$$\begin{aligned}
\mathrm{Var}_x(N) &= \sum_{k \le t\sqrt{n}} k^2 \frac{e^{\frac{-\beta k}{\sqrt{n}}}}{\left(1+e^{-\frac{\beta k}{\sqrt{n}}}\right)^2} \\
&= n^{\frac{3}{2}} \sum_{k \le t\sqrt{n}} \left(\frac{k}{\sqrt{n}}\right)^2 \frac{e^{\frac{-\beta k}{\sqrt{n}}}}{\left(1+e^{-\frac{\beta k}{\sqrt{n}}}\right)^2} \cdot \frac{1}{\sqrt{n}} \\
&= n^{\frac{3}{2}}\left(\int_0^t \frac{u^2 e^{-\beta u}}{(1+e^{-\beta u})^2} du + O\left(\frac{1}{\sqrt{n}}\right)\right) \\
&= \frac{n^{\frac{3}{2}}}{\beta^3}\int_0^{\beta t} \frac{u^2 e^u}{(1+e^u)^2} du + O(n)
\end{aligned}$$

$$= \frac{n^{\frac{3}{2}}}{\beta^3}\left(-\frac{u^2}{1+e^u}\bigg|_0^{\beta t} + 2\int_0^{\beta t}\frac{u}{1+e^u}du\right) + O(n)$$
$$= \frac{n^{\frac{3}{2}}}{\beta^3}\left(-\frac{\beta^2 t^2}{1+e^{\beta t}} + 2\beta^2\right) + O(n).$$

by (3.12). Combining and recalling (3.10) finishes the proof. □

The proof of Proposition 3.3 is the most technical and will require the following two lemmas.

Lemma 3.6. *Let*

$$f_x(s) := \log\left(\frac{1+e^{is}x}{1+x}\right) - is\frac{x}{1+x} + \frac{s^2}{2}\frac{x}{(1+x)^2}. \tag{3.22}$$

There exists a constant $c > 0$ such that for any $x \in (0,1)$ and any $s \in \mathbb{R}$, we have

$$|f_x(s)| \leq c\frac{x|s|^3}{(1-x)^3}.$$

Lemma 3.7. *Let $\epsilon \in \left(0, \frac{1}{2}\right]$ and let $\|\alpha\|$ denote the distance between α and the nearest integer. Then*

$$\inf_{\frac{\epsilon}{n}\leq\alpha\leq\frac{1}{2}}\sum_{k\leq n}\|k\alpha\|^2 \gg n.$$

We append the proofs of these lemmas to Section 3.4. The proof of Lemma 3.6 is similar to the proof of Lemma 1 in [36]. Roth and Szekeres [37] proved Lemma 3.7 for $\frac{1}{2n} \leq x \leq \frac{1}{2}$ when $\{k\}$ is replaced by a much more general sequence, but with a weaker lower bound.

Proof of Proposition 3.3. To determine the asymptotic behavior of $P_x(N = n)$, we will apply Fourier inversion to the characteristic function for N:

$$\phi_x(s) := E_x(e^{isN}) = \sum_{k\geq 0} P_x(N=k)e^{isk} = \frac{1}{\mathscr{D}_{t,n}(x)}\sum_{k\geq 0}\left(\text{Coeff}[x^k]\mathscr{D}_{t,n}(x)\right)x^k e^{isk}$$
$$= \frac{\mathscr{D}_{t.n}(xe^{is})}{\mathscr{D}_{t,n}(x)}.$$

Note that ϕ_x depends on n, although we refrain from notating this. We have

$$P_x(N=n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_x(s)e^{-ins}ds = \frac{1}{2\pi\sigma_n}\int_{-\pi\sigma_n}^{\pi\sigma_n}\phi_x\left(\frac{u}{\sigma_n}\right)e^{-i\frac{nu}{\sigma_n}}du. \tag{3.23}$$

We break up this integral as

$$\left(\int_{|u|\le\frac{\sigma_n}{\sqrt{n}}v_0} + \int_{\frac{\sigma_n}{\sqrt{n}}v_0\le|u|\le\pi\sigma_n}\right)\phi_x\left(\frac{u}{\sigma_n}\right)e^{-i\frac{nu}{\sigma_n}}du, \tag{3.24}$$

where v_0 is a sufficiently small constant, depending on t and chosen below. Note that $\sigma_n \asymp n^{\frac{3}{4}}$, so $\frac{\sigma_n}{\sqrt{n}} \to \infty$. We show that the integral on the right in (3.24) tends to 0, while for the left integral we show that, pointwise in u,

$$\lim_{n\to\infty}\phi_x\left(\frac{u}{\sigma_n}\right)e^{-i\frac{nu}{\sigma_n}} = e^{-\frac{u^2}{2}}. \tag{3.25}$$

We then show that for some $A' > 0$, the integrand $\phi_x\left(\frac{u}{\sigma_n}\right)$ is dominated by $e^{-A'u^2} \in L^1(\mathbb{R})$. Thus, applying the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_{|u|\le\frac{\sigma_n}{\sqrt{n}}v_0}\phi_x\left(\frac{u}{\sigma_n}\right)e^{-i\frac{nu}{\sigma_n}}du = \int_{\mathbb{R}}e^{-\frac{u^2}{2}} = \sqrt{2\pi}, \tag{3.26}$$

which when combined with (3.23) proves that $P_x(N=n) \sim \frac{1}{\sqrt{2\pi}\sigma_n}$. A similar application of the Dominated Convergence Theorem also implies (3.8), since the characteristic function of $\frac{N-n}{\sigma_n}$ is

$$\mathrm{E}_x\left(e^{iu\frac{N-n}{\sigma_n}}\right) = \mathrm{E}_x\left(e^{i\frac{u}{\sigma_n}N}\right)e^{-iu\frac{n}{\sigma_n}} = \phi_x\left(\frac{u}{\sigma_n}\right)e^{-iu\frac{n}{\sigma_n}}.$$

To carry out this plan, we separate the cases $t > 2$, $\sqrt{2} < t < 2$ and $t = 2$.

Case 1: $t > 2$. Recalling the expectation and variance in (3.5), Proposition 3.2 implies

$$\begin{aligned}\log\left(\phi_x\left(\frac{u}{\sigma_n}\right)e^{-i\frac{nu}{\sigma_n}}\right) &= \log\left(\mathcal{Q}_t(xe^{i\frac{u}{\sigma_n}})\right) - \log\left(\mathcal{Q}_t(x)\right) - i\frac{nu}{\sigma_n}\\ &= \sum_{k\le t\sqrt{n}}\log\left(\frac{1+x^ke^{i\frac{ku}{\sigma_n}}}{1+x^k}\right) - i\frac{nu}{\sigma_n}\end{aligned}$$

$$
\begin{aligned}
&= i\frac{u}{\sigma_n}\left(\sum_{k\le t\sqrt{n}}\frac{kx^k}{1+x^k}-n\right)-\frac{u^2}{2\sigma_n^2}\left(\sum_{k\le t\sqrt{n}}\frac{k^2x^k}{(1+x^k)^2}\right)\\
&\quad+\sum_{k\le t\sqrt{n}} f_{x^k}\left(\frac{ku}{\sigma_n}\right)\\
&= i\frac{u}{\sigma_n}(\mathrm{E}_x(N)-n)-\frac{u^2}{2}+\sum_{k\le t\sqrt{n}} f_{x^k}\left(\frac{ku}{\sigma_n}\right)\\
&= uO\left(n^{-\frac{1}{4}}\right)-\frac{u^2}{2}+\sum_{k\le t\sqrt{n}} f_{x^k}\left(\frac{ku}{\sigma_n}\right), \qquad (3.27)
\end{aligned}
$$

where f_{x^k} is as in Lemma 3.6. Using Proposition 3.2 and Lemma 3.6, we have

$$
\begin{aligned}
\left|\sum_{k\le t\sqrt{n}} f_{x^k}\left(\frac{ku}{\sigma_n}\right)\right| &\le \frac{cu^3}{\sigma_n^3}\sum_{k\le t\sqrt{n}} k^3\frac{x^k}{(1-x^k)^3}\\
&= \frac{cu^3n^2}{\sigma_n^3}\sum_{k\le t\sqrt{n}}\left(\frac{k}{\sqrt{n}}\right)^3\frac{e^{-\frac{\beta k}{\sqrt{n}}}}{\left(1-e^{-\frac{\beta k}{\sqrt{n}}}\right)^3}\cdot\frac{1}{\sqrt{n}}\\
&= \frac{cu^3n^2}{\sigma_n^3}\left(\int_0^t\frac{v^3e^{-\beta v}}{(1-e^{-\beta v})^3}dv+O\left(\frac{1}{\sqrt{n}}\right)\right)\\
&= u^3O\left(n^{-\frac{1}{4}}\right),
\end{aligned}
$$

since the integral converges. This proves (3.25).

Next, we find a dominating function in the range $|u|\le\frac{\sigma_n}{\sqrt{n}}v_0$. Here, we will set $v:=\frac{\sqrt{n}}{\sigma_n}u$, so $|v|\le v_0$. Recognizing Riemann sums, the following holds for such v uniformly.

$$
\begin{aligned}
\log\phi_x\left(\frac{v}{\sqrt{n}}\right) &= \sum_{k\le t\sqrt{n}}\left(\log\left(1+e^{-\frac{\beta}{\sqrt{n}}k+i\frac{vk}{\sqrt{n}}}\right)-\log\left(1+e^{-\frac{\beta}{\sqrt{n}}k}\right)\right)\\
&= \sqrt{n}\int_0^t\left(\log\left(1+e^{-\beta w+ivw}\right)-\log\left(1+e^{-\beta w}\right)\right)dw+o(\sqrt{n})\\
&= \frac{\sqrt{n}}{\beta-iv}\left(\frac{\pi^2}{12}+\mathrm{Li}_2\left(-e^{-\beta t+ivt}\right)\right)-\frac{\sqrt{n}}{\beta}\left(\frac{\pi^2}{12}+\mathrm{Li}_2\left(-e^{-\beta t}\right)\right)\\
&\quad+o(\sqrt{n}). \qquad (3.28)
\end{aligned}
$$

The Taylor series for $\mathrm{Li}_2(z)$ about $z = -e^{-\beta(t)t}$ is

$$\mathrm{Li}_2\left(-e^{-\beta t}\right) + \log\left(1+e^{-\beta t}\right)\left(ze^{\beta t}+1\right) - \frac{1}{2}\left(\frac{e^{-\beta t}}{1+e^{-\beta t}} - \log\left(1+e^{-\beta t}\right)\right)\left(ze^{\beta t}+1\right)^2 + O\left((ze^{\beta t}+1)^3\right).$$

Substituting $z = -e^{-\beta t + ivt}$, we obtain the following:

$$\begin{aligned}
&\mathrm{Li}_2\left(-e^{-\beta t+ivt}\right) \\
&= \mathrm{Li}_2\left(-e^{-\beta t}\right) + \log\left(1+e^{-\beta t}\right)\left(1-e^{ivt}\right) \\
&\qquad - \frac{1}{2}\left(\frac{e^{-\beta t}}{1+e^{-\beta t}} - \log\left(1+e^{-\beta t}\right)\right)\left(1-e^{ivt}\right)^2 + O(v^3) \\
&= \mathrm{Li}_2\left(-e^{-\beta t}\right) - it\log\left(1+e^{-\beta t}\right)v \\
&\qquad + \left(\frac{t^2\log\left(1+e^{-\beta t}\right)}{2} + \frac{t^2}{2}\cdot\frac{e^{-\beta t}}{1+e^{-\beta t}} - \frac{t^2\log\left(1+e^{-\beta t}\right)}{2}\right)v^2 + O(v^3) \\
&= \mathrm{Li}_2\left(-e^{-\beta t}\right) - it\log\left(1+e^{-\beta t}\right)v + \frac{1}{2}\cdot\frac{t^2}{1+e^{\beta t}}v^2 + O(v^3). \qquad (3.29)
\end{aligned}$$

Also, note that

$$\frac{1}{\beta - iv} = \frac{1}{\beta} + \frac{i}{\beta^2}v - \frac{1}{\beta^3}v^2 + O(v^3). \qquad (3.30)$$

Thus, from (3.29) and (3.30), we choose v_0 small enough so that the dominating term for the real part of (3.28) is

$$\begin{aligned}
&\sqrt{n}v^2\left(\frac{1}{\beta}\cdot\frac{1}{2}\cdot\frac{t^2}{1+e^{\beta t}} + \frac{t\log\left(1+e^{-\beta t}\right)}{\beta^2} - \frac{1}{\beta^3}\mathrm{Li}_2\left(-e^{-\beta t}\right) - \frac{1}{\beta^3}\cdot\frac{\pi^2}{12}\right) \\
&= \sqrt{n}\frac{v^2}{\beta}\left(\frac{1}{2}\cdot\frac{t^2}{1+e^{\beta t}} - 1\right),
\end{aligned}$$

where we used the dilogarithm identity (3.18) with the alternate definition of β given in (3.14). By (3.13), this is $-A\sqrt{n}v^2$ for some $A > 0$. Hence, for some $A > 0$,

$$\left|\phi_x\left(\frac{v}{\sqrt{n}}\right)\right| \ll e^{-A\sqrt{n}v^2} \qquad \text{for } |v| \le v_0.$$

This implies $\left|\phi_x\left(\frac{u}{\sigma_n}\right)\right| \ll e^{-A'u^2}$ for some A' in the required range. Thus, (3.26) is proved.

For the remaining range, $\frac{\sigma_n}{\sqrt{n}} v_0 \le |u| \le \pi \sigma_n$, we will use the substitution $w := \frac{u}{\sigma_n}$ and bound $\phi_x(w)$ for $\frac{v_0}{\sqrt{n}} \le |w| \le \pi$. Following the analysis of Roth and Szekeres ([37], p. 253), we write

$$\left| \frac{1 + x^k e^{iwk}}{1 + x^k} \right|^2 = \frac{1}{(1 + x^k)^2} \left(1 + 2x^k \cos(wk) + x^{2k}\right) = 1 - \frac{2x^k(1 - \cos(wk))}{(1 + x^k)^2}.$$

Note that the expression on the far left is positive almost everywhere; therefore, $0 < \frac{2x^k(1-\cos(wk))}{(1+x^k)^2} < 1$ almost everywhere. Thus, it is safe to expand the logarithm as follows.

$$\begin{aligned}
\log |\phi_x(w)| &= \frac{1}{2} \sum_{k \le t\sqrt{n}} \log \left(1 - \frac{2x^k(1 - \cos(wk))}{(1 + x^k)^2}\right) \\
&\le -\frac{1}{2} \sum_{k \le t\sqrt{n}} (1 - \cos(wk)) \frac{2x^k}{(1 + x^k)^2} \\
&\ll - \sum_{k \le t\sqrt{n}} (1 - \cos(wk)) \\
&\le - \sum_{k \le t\sqrt{n}} \left\| \frac{wk}{2\pi} \right\|^2 .
\end{aligned}$$

Since $\frac{v_0}{\sqrt{n}} \le |w| \le \pi$, the latter is $\ll -\sqrt{n}$ by Lemma 3.7, taking $\epsilon \le t v_0$, so that $\frac{\epsilon}{t\sqrt{n}} \le \frac{v_0}{\sqrt{n}}$. This implies that the right integral in (3.24) tends to 0, so Proposition 3.3 is proved for $t > 2$.

Case 2: $\sqrt{2} < t < 2$. Below, we use the fact that $\frac{x^k}{(1+x^k)^2} = \frac{x^{-k}}{(1+x^{-k})^2}$. Thus,

$$\begin{aligned}
&\log \left(\phi_x\left(\frac{u}{\sigma_n}\right) e^{-i\frac{un}{\sigma_n}}\right) \\
&= i\frac{u}{\sigma_n} \left(\sum_{k \le t\sqrt{n}} \frac{kx^k}{1 + x^k} - n\right) - \frac{u^2}{2\sigma_n^2} \sum_{k \le t\sqrt{n}} \frac{k^2 x^k}{(1 + x^k)^2} + i\frac{u}{\sigma_n} \frac{t_n\sqrt{n}(t_n\sqrt{n} + 1)}{2} + \\
&\quad + \sum_{k \le t\sqrt{n}} \left(\log\left(\frac{1 + x^{-k} e^{-ik\frac{u}{\sigma_n}}}{1 + x^{-k}}\right) - i\frac{u}{\sigma_n} \frac{kx^k}{1 + x^k} + \frac{u^2}{2\sigma_n^2} \frac{k^2 x^k}{(1 + x^k)^2}\right) \\
&= o(1) - \frac{u^2}{2} + i\frac{u}{\sigma_n} \frac{t_n\sqrt{n}(t_n\sqrt{n} + 1)}{2} + \sum_{k \le t\sqrt{n}} f_{x^{-k}}\left(-k\frac{u}{\sigma_n}\right)
\end{aligned}$$

$$
\begin{aligned}
&- i\frac{u}{\sigma_n}\sum_{k\le t\sqrt{n}} k\underbrace{\left(\frac{x^k}{1+x^k}+\frac{x^{-k}}{1+x^{-k}}\right)}_{=1}\\
&= o(1) - \frac{u^2}{2} + \sum_{k\le t\sqrt{n}} f_{x^{-k}}\left(-k\frac{u}{\sigma_n}\right)\\
&= -\frac{u^2}{2} + o(1),
\end{aligned}
$$

where Lemma 3.6 was used as before to show that the sum is $o(1)$. This proves (3.25).

To find a dominating function in the range $|u| \le \frac{\sigma_n}{\sqrt{n}} v_0$, we once again set $v := \frac{\sqrt{n}}{\sigma_n} u$, and write

$$
\left|\phi_x\left(\frac{v}{\sqrt{n}}\right)\right| = \left|e^{i\frac{t_n(t_n\sqrt{n}+1)}{2}v}\prod_{k\le t\sqrt{n}}\frac{1+x^{-k}e^{-ik\frac{v}{\sqrt{n}}}}{1+x^{-k}}\right| = \left|\phi_{x^{-1}}\left(\frac{-v}{\sqrt{n}}\right)\right|.
$$

Thus, we may perform an analysis similar to Case 1 with $\beta \to \gamma$ and conclude that the dominating part of $\mathrm{Re}\left(\log\phi_x\left(\frac{v}{\sqrt{n}}\right)\right)$ is

$$
\sqrt{n}(-v)^2\left(\frac{1}{\gamma}\cdot\frac{1}{2}\cdot\frac{t^2}{1+e^{\gamma t}} + \frac{t\log\left(1+e^{-\gamma t}\right)}{\gamma^2} - \frac{1}{\gamma^3}\mathrm{Li}_2\left(-e^{-\gamma t}\right) - \frac{1}{\gamma^3}\cdot\frac{\pi^2}{12}\right).
$$

We now apply the identity (3.18) for the dilogarithm with (3.21) to get

$$
\begin{aligned}
\sqrt{n}\frac{v^2}{\gamma}\left(\frac{t^2}{2\left(1+e^{\gamma t}\right)} + 1 - \frac{t^2}{2}\right) &= \sqrt{n}\frac{v^2}{\gamma}\left(\frac{2\left(1+e^{-\gamma t}\right)-t^2}{2\left(1+e^{-\gamma t}\right)}\right)\\
&= \sqrt{n}v^2\left(\frac{-t^2}{\beta'(t)\cdot 2\left(1+e^{\beta t}\right)}\right),
\end{aligned}
$$

which is negative by Proposition 3.4. Thus, as in Case 1, $\left|\phi_x\left(\frac{u}{\sigma_n}\right)\right| \ll e^{-A'u^2}$ for some A' in the required range, so (3.26) is proved.

As in Case 1, a similar application of Lemma 3.7 to $\phi_{x^{-1}}(-w)$ shows that the right integral in (3.24) tends to 0, so Proposition 3.3 is proved for $\sqrt{2} < t < 2$.

Case 3: $t = 2$. For fixed u in the range $|u| \le \frac{\sigma_n}{\sqrt{n}} v_0$, where v_0 will be specified below, we write

$$
\phi_1\left(\frac{u}{\sigma_n}\right)e^{-i\frac{un}{\sigma_n}} = \prod_{k\le 2\sqrt{n}}\frac{1+e^{ik\frac{u}{\sigma_n}}}{2}\cdot e^{-i\frac{un}{\sigma_n}} = \prod_{k\le 2\sqrt{n}}\cos\left(k\frac{u}{2\sigma_n}\right)\cdot e^{i\frac{u}{\sigma_n}\left(n-\frac{t_n\sqrt{n}(t_n\sqrt{n}+1)}{2}\right)}
$$

$$= \prod_{k \leq 2\sqrt{n}} \cos\left(k \frac{u}{2\sigma_n}\right) + o(1),$$

since

$$\frac{t_n\sqrt{n}(t_n\sqrt{n}+1)}{2} = \frac{t_n^2 n}{4} + O(\sqrt{n}) = n + O(\sqrt{n}).$$

Note that, over the summation range, $k = O(\sqrt{n})$ uniformly, so $\frac{k}{\sigma_n} = O\left(\frac{1}{\sqrt[4]{n}}\right)$ uniformly. Thus, the following holds for fixed u, where v_0 is chosen so that the logarithms below are defined:

$$\begin{aligned}
\log\left(\phi_1\left(\frac{u}{\sigma_n}\right)e^{-i\frac{un}{\sigma_n}}\right) &= \sum_{k \leq 2\sqrt{n}} \log\left(\cos\left(k\frac{u}{2\sigma_n}\right)\right) + o(1) \\
&= \sum_{k \leq 2\sqrt{n}} \log\left(1 - k^2\frac{u^2}{4\sigma_n^2} + O\left(\frac{1}{n}\right)\right) + o(1) \\
&= \sum_{k \leq 2\sqrt{n}} \left(-k^2\frac{u^2}{4\sigma_n^2} + O\left(\frac{1}{n}\right)\right) + o(1) \\
&= -\frac{u^2 \cdot t_n^3 n^{\frac{3}{2}}}{4\sigma_n^2 \cdot 3} + o(1) \\
&= -\frac{u^2}{2} + o(1),
\end{aligned}$$

since $\sigma_n^2 \sim \frac{2}{3}n^{\frac{3}{2}}$ by Propositions 3.2 and 3.4. This proves (3.25).

To find a dominating function in the range $|u| \leq \frac{\sigma_n}{\sqrt{n}}v_0$, we write $\frac{v}{\sqrt{n}} := \frac{u}{\sigma_n}$ once again, and we choose v_0 small so that the logarithms below are defined. Thus,

$$\begin{aligned}
\mathrm{Re}\left(\log\phi_1\left(\frac{v}{\sqrt{n}}\right)\right) &= \sum_{k \leq 2\sqrt{n}} \log\cos\left(v\frac{k}{2\sqrt{n}}\right) \\
&= 2\sqrt{n}\int_0^1 \log\cos\left(vw\right)dw + O(\{2\sqrt{n}\}) + O\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{2\sqrt{n}}{v}\int_0^v \log\cos(w)dw + O(1).
\end{aligned}$$

It is not difficult to calculate the following Taylor series about $v = 0$ (the knowledge that the function is even is helpful):

$$\frac{1}{v}\int_0^v \log\cos(w)dw = -\frac{1}{6}v^2 + O(v^4).$$

Thus, choosing v_0 small enough, we have $\operatorname{Re}\left(\log \phi_1\left(\frac{v}{\sqrt{n}}\right)\right) \ll e^{-A\sqrt{n}v^2}$ for some A, which implies $\left|\phi_1\left(\frac{u}{\sigma_n}\right)\right| \ll e^{-A'u^2}$ for some A' in the required range for u, so (3.26) is proved.

Applying Lemma 2 as in Case 1, one can bound $\phi_1(w)$ for $w = \frac{u}{\sigma_n}$ in the required range, and show that the right integral in (3.24) tends to 0. This proves Proposition 3.3 for $t = 2$. □

3.4 Bounding Logarithmic Series: Proofs of Lemmas 3.6 and 3.7

This section completes the proof of Theorem 1.9 (and hence the proof of Theorem 1.7) by proving Lemmas 3.6 and 3.7.

Proof of Lemma 3.6. The proof is very similar to Lemma 1 in [36]. For $|s| \le \frac{1-x}{2}$, we have

$$\begin{aligned}\log\left(\frac{1+xe^{is}}{1+x}\right) &= \sum_{j\ge 1}\frac{1}{j}\left((-x)^j - (-x)^j e^{isj}\right)\\ &= -\sum_{j\ge 1}\frac{(-x)^j}{j}\sum_{k\ge 1}\frac{(is)^k j^k}{k!}\\ &= -\sum_{k\ge 1}\frac{(is)^k}{k!}\sum_{j\ge 1}(-x)^j j^{k-1}, \qquad (3.31)\end{aligned}$$

where swapping the order of summation in (3.31) is valid due to absolute convergence for

$|s| \le \frac{1-x}{2}$. Indeed,

$$\left|\sum_{k\ge 1}\frac{(is)^k}{k!}\sum_{j\ge 1}(-x)^j j^{k-1}\right| \le \sum_{k\ge 1}\frac{s^k}{k}\sum_{j\ge 1}x^j\frac{n(n+1)\cdots(j+k-2)}{(k-1)!} \le \sum_{k\ge 1}\frac{x}{k}\left(\frac{s}{1-x}\right)^k,$$

which converges.

Note that the $k = 1$ and $k = 2$ terms in (3.31) are, respectively,

$$-is\sum_{j\ge 1}(-x)^j = is\frac{x}{1+x} \qquad \text{and} \qquad \frac{s^2}{2}\sum_{j\ge 1}(-x)^j j = -\frac{s^2}{2}\frac{x}{(1+x)^2}.$$

Thus, by (3.31), we obtain

$$
\begin{aligned}
|f_x(s)| &\le \sum_{k\ge 3} \frac{|s|^k}{k!} \sum_{j\ge 1} j^{k-1} x^j \\
&\le \sum_{k\ge 3} \frac{x}{k} \left(\frac{|s|}{1-x}\right)^k \\
&\le \frac{x|s|^3}{3(1-x)^3} \cdot \frac{1}{1-\frac{|s|}{1-x}} \\
&\le \frac{2x|s|^3}{3(1-x)^3}.
\end{aligned}
$$

For $|s| \ge \frac{1-x}{2}$, we have

$$
\left| -i\frac{x}{1+x}s + \frac{1}{2}\frac{x}{(1+x)^2}s^2 \right| \le \frac{x|s|^3}{(1-x)|s|^2} + \frac{x|s|^3}{(1-x)^2|s|} \le (4+2)\frac{x|s|^3}{(1-x)^3},
$$

so it remains to prove that for $|s| \ge \frac{1-x}{2}$,

$$
\left|\log\left(\frac{1+xe^{is}}{1+x}\right)\right| \le c' \frac{x|s|^3}{(1-x)^3},
$$

for some $c' > 0$. For $|s| \ge \frac{1}{4}$, we have

$$
\begin{aligned}
\left|\log\left(\frac{1+xe^{is}}{1+x}\right)\right| &\le \sum_{m\ge 1} \frac{x^m}{m} \left|1-e^{ism}\right| \\
&\le -2\log(1-x) \\
&\le 2\frac{x}{1-x} \\
&\le 2\cdot 4^3 \frac{x|s|^3}{(1-x)^3}.
\end{aligned}
$$

Finally, for $\frac{1-x}{2} \le |s| \le \frac{1}{4}$ (which implies $x \ge \frac{1}{2}$ and $\frac{|s|}{1-x} \ge \frac{1}{2}$), we have

$$
\begin{aligned}
\left|\log\left(\frac{1+xe^{is}}{1+x}\right)\right| &\le \left|\log\left(1+\frac{x}{1+x}(e^{is}-1)\right)\right| \\
&\le \left|\log\left(1+ie^{\frac{s}{2}}S\right)\right|,
\end{aligned}
$$

where $S = \frac{x}{1+x} \cdot 2\sin\left(\frac{s}{2}\right)$ satisfies

$$
|S| \le \frac{x|s|}{1+x} \le \frac{x|s|}{1-x} \le 4\frac{x|s|^3}{(1-x)^3}.
$$

The left-most inequality above implies $|S| \leq \frac{1}{4}$. Thus,

$$\left|\log\left(1 + ie^{\frac{s}{2}}S\right)\right| \leq \frac{|S|}{1-|S|} \leq \frac{4}{3}|S| \leq \frac{4}{3} \cdot 4\frac{x|s|^3}{(1-x)^3},$$

and we are done.

$\square$

Proof of Lemma 3.7. Let $f_n(\alpha) := \sum_{k \leq n} ||k\alpha||^2$. We prove first that

$$\inf_{\frac{1}{2n} \leq \alpha \leq \frac{1}{2}} f_n(\alpha) \gg n. \tag{3.32}$$

The extension of (3.32) to the range $\left[\frac{\epsilon}{n}, \frac{1}{2}\right]$ for $\epsilon \in \left(0, \frac{1}{2}\right]$ follows from

$$\inf_{\frac{\epsilon}{n} \leq \alpha \leq \frac{1}{2n}} f_n(\alpha) = \sum_{k \leq n} k^2 \frac{\epsilon^2}{n^2} \gg n.$$

Now, note that $f_n(\alpha)$ is piecewise a parabola of the form

$$\sum_{k \leq n} k^2 \left(\alpha - \frac{\ell_k}{k}\right)^2, \qquad \gcd(\ell_k, k) = 1.$$

Thus, we see by taking the derivative of f that its minimum in $\left[\frac{1}{2n}, \frac{1}{2}\right]$ occurs at a rational number (or possibly more than one). Therefore, it suffices to show that there is a constant c, independent of n, such that $f_n(\alpha) \geq cn$ for all rational $\alpha \in \left[\frac{1}{2n}, \frac{1}{2}\right]$. In what follows, we will be rather wasteful with our estimates, but for clarity we will produce explicit constants at each step.

Naturally,

$$f_n\left(\frac{1}{2}\right) \geq \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{4} \geq \frac{n}{16}.$$

Now let $\alpha = \frac{a}{b}$ with $\gcd(a,b) = 1$ and $3 \leq b \leq n$. For each $j \in [1, b-1]$, we have

$$\#\{k \leq n : \ ka \equiv j \pmod{b}\} \geq \left\lfloor \frac{n}{b} \right\rfloor.$$

Thus,

$$f_n(\alpha) = \sum_{k \leq n} \left\|k\frac{a}{b}\right\|^2 \geq 2 \cdot \sum_{j < \frac{b}{2}} \left\lfloor \frac{n}{b} \right\rfloor \frac{j^2}{b^2} \geq 2 \cdot \left\lfloor \frac{n}{b} \right\rfloor \frac{1}{b^2} \frac{b^3}{2 \cdot 6 \cdot 2^3} \geq \frac{1}{96} n.$$

Now assume $b > n$, $\gcd(a, b) = 1$, and $\frac{1}{2n} \le \frac{a}{b} \le \frac{1}{2}$. If $\frac{b}{2} \le na < b$, then clearly

$$f_n(\alpha) = \sum_{k \le n} \left\| k\frac{a}{b} \right\|^2 \ge \frac{a^2}{b^2} \sum_{k \le \frac{n}{2}} k^2 \ge \frac{1}{2^2 n^2} \frac{n^3}{2 \cdot 6 \cdot 2^3} = \frac{n}{384}.$$

Now assume $\frac{n}{\frac{b}{a}} \ge 1$ and note that a generates the additive group $\pmod{b}$. Partition the set $\{ka\}_{k=1}^{n}$ into subsets between multiples of b as

$$\left\{a, 2a, \dots, \left\lfloor \frac{b}{a} \right\rfloor a\right\} \cup \left\{(\left\lfloor \frac{b}{a} \right\rfloor + 1)a, \dots, (2\left\lfloor \frac{b}{a} \right\rfloor + \eta_2)a\right\} \cup \dots,$$

where the $\eta_j \in \{0, 1\}$. There are at least $\left\lfloor \frac{n}{\lfloor \frac{b}{a} \rfloor} \right\rfloor \ge 1$ such sets, and each contains a sequence of $\frac{\left\lfloor \frac{b}{a} \right\rfloor}{2} \ge 1$ elements that are at least $a, 2a, \dots, \frac{\left\lfloor \frac{b}{a} \right\rfloor}{2} a$, respectively. Hence, we have

$$f_n(\alpha) \ge \left\lfloor \frac{n}{\lfloor \frac{b}{a} \rfloor} \right\rfloor \sum_{j \le \frac{\left\lfloor \frac{b}{a} \right\rfloor}{2}} \frac{j^2 a^2}{b^2} \ge \frac{n}{2\left\lfloor \frac{b}{a} \right\rfloor} \frac{a^2}{b^2} \frac{\left\lfloor \frac{b}{a} \right\rfloor^3}{2 \cdot 6 \cdot 2^3} \ge \frac{n}{768},$$

since $\frac{b}{a} \ge 2$ implies $\frac{a}{b} \left\lfloor \frac{b}{a} \right\rfloor \ge \frac{1}{2}$. Thus, (3.32) is proved and with it, Lemma 3.7. $\square$

Chapter 4: Partition Inequalities

4.1 Notation

In this Chapter, we will use *frequency notation* to refer to partitions, where $\lambda = \left(\ldots, 2^{f_2}, 1^{f_1}\right)$ means that λ contains exactly f_i parts equal to i. For example, $(5^2, 2, 1^3)$ is the partition $5+5+2+1+1+1$.

Recall that the *M-modular diagram* of a partition $\lambda = \{\lambda_1, \ldots, \lambda_\ell\}$ is a modification of the Ferrer's diagram, wherein each λ_j is first written as $Mq + r$ for $0 \le r < M$, and then is represented as a row of q M's and a single r at the end of the row. (See [2], p. 13.) These r's we will refer to as *ends* or *r-ends*. For example, the 10-modular diagram of $\lambda = (53^2, 46, 36, 16, 11, 1)$ has three 6-ends, two 3-ends and two 1-ends:

$$\begin{array}{cccccc}
10 & 10 & 10 & 10 & 10 & 3 \\
10 & 10 & 10 & 10 & 10 & 3 \\
10 & 10 & 10 & 10 & 6 & \\
10 & 10 & 10 & 6 & & \\
10 & 6 & & & & \\
10 & 1 & & & & \\
1 & & & & &
\end{array}$$

We will also speak of *attaching* and *removing* a column from an M-modular diagram. These operations are best defined with an example:

$$
\begin{array}{lllll}
10 & 10 & 10 & 10 & 10 \\
10 & 10 & 10 & 10 & 10 \\
10 & 10 & 10 & 10 & \\
10 & 10 & 10 & & \\
10 & 10 & 10 & & \\
10 & 10 & 10 & & \\
10 & & & &
\end{array}
\quad \underbrace{\cup}_{\substack{\textbf{attach}\\ \textbf{the column}}} \quad
\begin{array}{l}
10 \\ 10 \\ 10
\end{array}
\longrightarrow
\begin{array}{llllll}
10 & 10 & 10 & 10 & 10 & 10 \\
10 & 10 & 10 & 10 & 10 & 10 \\
10 & 10 & 10 & 10 & 10 & \\
10 & 10 & 10 & & \uparrow & \\
10 & 10 & 10 & & & \\
10 & 10 & 10 & & & \\
10 & & & & &
\end{array}
$$

$$
\begin{array}{lllll}
10 & 10 & 10 & 10 & 10 \\
10 & 10 & 10 & 10 & 10 \\
10 & 10 & 10 & 10 & \\
10 & 10 & 10 & & \\
10 & 10 & 10 & & \\
10 & 10 & 10 & & \\
10 & & \uparrow & &
\end{array}
\quad \underbrace{\setminus}_{\substack{\textbf{remove}\\ \textbf{the column}}} \quad
\begin{array}{l}
10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10
\end{array}
\longrightarrow
\begin{array}{llll}
10 & 10 & 10 & 10 \\
10 & 10 & 10 & 10 \\
10 & 10 & 10 & \\
10 & 10 & & \\
10 & 10 & & \\
10 & 10 & & \\
10 & & &
\end{array}
$$

We shall only attach or remove columns consisting entirely of M's, and it is easy to see that these operations preserve M-modular diagrams.

4.2 Proof of Theorem 1.12

We provide a combinatorial proof via injection that is nearly identical to that of Theorem 5.1 in [7], but we highlight a technical difference that arises in the general version. In keeping with [7], we let $\nu_j = \nu_j(\lambda)$ denote the number of parts of λ congruent to $j \pmod{M}$. (The modulus never varies and will be clear from context.)

Proof. First let $L = 1$. We will prove the general case as a consequence of this one. For each n, we seek an injection

$$\varphi_1 : \left\{(a^k, b^\ell) \vdash n : k, \ell \geq 0\right\} \hookrightarrow \left\{(1^k, c^\ell) \vdash n : k, \ell \geq 0\right\}.$$

Let $d :=\gcd(a, b)$. Explicitly, φ_1 is as follows:

$$\varphi_1\left(a^k, b^\ell\right) = \begin{cases} \left(1^{\ell+a(k-\ell)}, c^\ell\right) & \text{if } k \geq \ell, \qquad\qquad \text{(Case 1)} \\ \left(1^{k+b(\ell-k)}, c^k\right) & \text{if } \ell > k \text{ and } \frac{a}{d} \nmid (\ell-k), \qquad \text{(Case 2)} \\ \left(1^{k+1+b(\ell-k-1)-a}, c^{k+1}\right) & \text{if } \ell > k \text{ and } \frac{a}{d} \mid (\ell-k). \qquad \text{(Case 3)} \end{cases}$$

This definition can be motivated by noting that each pre-image consists either of k pairs (a, b) and $k - \ell$ excess a's, or of ℓ pairs (a, b) and $\ell - k$ excess b's. (There can also be no excess.) The pairs are mapped as $(a, b) \mapsto (1, c)$. The excess a's or b's are treated by the following cases.

Case 1. For the $k - \ell$ excess a's, $(a) \mapsto (1^a)$.

Case 2. For the $\ell - k$ excess b's, $(b) \mapsto (1^b)$.

Case 3. For all but the last two excess b's, $(b) \mapsto (1^b)$. For the last two b's,

$(b^2) \mapsto (1^{b-a+1}, c)$.

Note that in Case 3 there are at least two excess b's, for if not, $\frac{a}{d} = 1$ and then

$a \mid b$,

a contradiction. Also, by hypothesis, $b > \frac{c}{2}$, so that $2b > c$.

Let $(1^{\nu_1}, c^{\nu_c})$ be a partition in the image of φ_1. The cases are separated as follows:

Case 1. $a \mid (\nu_1 - \nu_c)$,

Case 2. $a \nmid (\nu_1 - \nu_c)$ and $b \mid (\nu_1 - \nu_c)$,

Case 3. $\nu_1 - \nu_c \equiv -b \pmod a$ and $\nu_1 - \nu_c \equiv -a \pmod b$.

This concludes the proof for $L = 1$.

Now let $L \geq 2$. Again we define an injection

$$\varphi_L : \{\lambda \vdash n : \lambda_j \in \{a, b, \ldots, LM + a, LM + b\}\}$$

$$\hookrightarrow \{\lambda \vdash n : \lambda_j \in \{1, c, \ldots, LM+1, LM+c\}\}.$$

Let λ be a partition in the left set. Then λ consists of the triple

$$\left(\lambda_{(a)}, \lambda_{(b)}, (a^k, b^\ell)\right),$$

where $\lambda_{(a)}$ is the M-modular diagram obtained by subtracting a from every part of the form $Mj + a$; $\lambda_{(b)}$ is defined similarly. We apply φ_1 to (a^k, b^ℓ) and reattach the 1-ends and c-ends based on the case into which (a^k, b^ℓ) falls.

Case 1: $k \geq \ell$. Attach the 1-ends to $\lambda_{(a)}$ and the c-ends to $\lambda_{(b)}$. The map φ_1 guarantees exactly $\#\lambda_{(b)}$ c-ends. Likewise, there are at least as many 1-ends as there are parts of $\lambda_{(a)}$; any excess 1's are attached as parts to $\lambda_{(a)}$. The required image of λ is then the union of these two partitions.

Cases 2 and 3: $\ell > k$. Attach the 1-ends to $\lambda_{(b)}$ and the c-ends to $\lambda_{(a)}$ as before. φ_1 guarantees at least $\#\lambda_{(a)}$ c-ends. In Case 2 we are guaranteed at least $\#\lambda_{(a)}$ 1-ends because $b > 1$ implies

$$k + b(\ell - k) > \ell.$$

In Case 3, $\frac{a}{d} > 1$ implies $\ell - k > 1$, so

$$k + 1 + b(\ell - k - 1) - a = \ell + (b-1)(\ell - k - 1) - a \geq \ell,$$

and we are guaranteed at least $\#\lambda_{(a)}$ 1-ends.

Given the image of λ, we may clearly recover $\lambda_{(a)}$ and $\lambda_{(b)}$ based on its 1-ends and c-ends and the fact that φ_1 is an injection. Thus, φ_L is an injection. □

Remark 4.1. The condition $a \nmid b$ in Theorem 1.12 is necessary to avoid cases like

$$\frac{1}{(q, q^5; q^6)_L} - \frac{1}{(q^2, q^4; q^6)_L},$$

in which the coefficient of q^4 is -1.

Remark 4.2. If we had copied the proof of Theorem 5.1 in [7] exactly, then the conditions “$\frac{a}{d}\mid$” and “$\frac{a}{d}\nmid$” would be replaced by “$a\mid$” and “$a\nmid$”. But this is not an injection because Case 2 is only correctly separated from the other two when $\gcd(a,b)=1$. For example, this direct version of Berkovich-Garvan's map gives:

$$\begin{cases}4^7,6^4\\ \\ 4^4,6^6\end{cases} \longrightarrow (1^{16},9^4), \qquad \text{instead of our} \qquad \begin{cases}4^7,6^4\\ \\ 4^4,6^6\end{cases} \longrightarrow \begin{cases}1^{16},9^4\\ \\ 1^7,9^5\end{cases}.$$

In the first example, the partitions fall into cases 1 and 2. The second example corrects the overlap and places the partitions into cases 1 and 3.

We demonstrate the injection of Theorem 1.12 with an example.

Example 4.3. Here, $(n,M,L,a,b,c)=(52,10,2,4,6,9)$. Numbers above arrows indicate the case into which a pre-image falls.

$$\begin{array}{ccc|ccc}
16^3,4 & \xrightarrow{3} & 11^3,9^2,1 & 14^2,6^4 & \xrightarrow{3} & 19^2,9,1^5\\
16^2,14,6 & \xrightarrow{3} & 19,11^2,9,1^2 & 14^2,6^2,4^3 & \xrightarrow{1} & 11^2,9^2,1^{12}\\
16^2,6^2,4^2 & \xrightarrow{3} & 11^2,9^3,1^3 & 14^2,4^6 & \xrightarrow{1} & 11^2,1^{30}\\
16^2,4^5 & \xrightarrow{1} & 19^2,1^{14} & 14,6^5,4^2 & \xrightarrow{3} & 19,9^3,1^6\\
16,14^2,4^2 & \xrightarrow{1} & 19,11^2,1^{11} & 14,6^3,4^5 & \xrightarrow{1} & 11,9^3,1^{14}\\
16,14,6^3,4 & \xrightarrow{3} & 19,11,9^2,1^4 & 14,6,4^8 & \xrightarrow{1} & 11,9,1^{32}\\
16,14,6,4^4 & \xrightarrow{1} & 19,11,9,1^{13} & 6^8,4 & \xrightarrow{2} & 9,1^{43}\\
16,6^6 & \xrightarrow{2} & 11,1^{41} & 6^6,4^4 & \xrightarrow{3} & 9^5,1^7\\
16,6^4,4^3 & \xrightarrow{3} & 11,9^4,1^5 & 6^4,4^7 & \xrightarrow{1} & 9^4,1^{16}\\
16,6^2,4^6 & \xrightarrow{1} & 19,9^2,1^{15} & 6^2,4^{10} & \xrightarrow{1} & 9^2,1^{34}\\
16,4^9 & \xrightarrow{1} & 19,1^{33} & 4^{13} & \xrightarrow{1} & 1^{52}\\
14^3,6,4 & \xrightarrow{1} & 11^3,9,1^{10} & & &
\end{array}$$

4.3 Proofs of Theorems 1.13 and 1.14

We begin by recalling the main steps in McLaughlin's proof of Theorem 7 from [33]; our proof is based on a combinatorial reading. First, Cauchy's Theorem ([2],

Th. 2.1) is used with some algebraic manipulation to write, for fixed m,

$$\sum_{n\geq 0} c(m,n)q^n = \sum_{\substack{0\leq k<\frac{m}{2}\\ M|m-2k}} \frac{q^{kM}}{(q^M;q^M)_{m-k}(q^M;q^M)_k} \times \left(q^{a(m-2k)} + q^{(M-a)(m-2k)} - q^{b(m-2k)} - q^{(M-b)(m-2k)}\right).$$

It then happens that the factor in parentheses is equal to

$$q^{a(m-2k)}\left(1-q^{(b-a)(m-2k)}\right)\left(1-q^{(M-b-a)(m-2k)}\right).$$

But the conditions on a, b and M that lead to the condition $M \mid (m-2k)$ in the sum imply that both factors above are canceled in $\frac{1}{(q^M;q^M)_{m-k}}$. This gives nonnegativity.

The key steps in the proof are the decomposition of the sum over k and the nonnegativity of

$$\frac{(1-q^r)(1-q^s)}{(q;q)_n} \qquad \text{for } 1\leq r<s\leq n.$$

Both of these have simple combinatorial explanations, which we employ with M-modular diagrams to piece together a proof of Theorem 1.13. Our proof naturally leads to the finite versions with any $L \geq 1$ instead of ∞. The proof of Theorem 1.14 is then a slight modification.

Proof of Theorem 1.13. Let $\mathcal{P}(n,m,j,A)$ denote the set of partitions of n into m parts congruent to $\pm j$ modulo M such that the largest part is at most A. (We have suppressed the modulus M from the notation.) Let $\mathcal{P}_k(n,m,j,A)$ be the subset of partitions $\lambda \in \mathcal{P}(n,m,j,A)$ with either $\nu_j(\lambda)=k$ or $\nu_{M-j}(\lambda)=k$.

Clearly, we have the disjoint union $\mathcal{P}(n,m,j,A) = \bigsqcup_{0\leq k\leq \frac{m}{2}} \mathcal{P}_k(n,m,j,A)$. Thus, to show

$$\mathcal{P}(nM,m,b,LM-b) \hookrightarrow \mathcal{P}(nM,m,a,LM+a),$$

we may provide injections

$$\varphi_k : \mathcal{P}_k(nM,m,b,LM-b) \hookrightarrow \mathcal{P}_k(nM,m,a,LM+a)$$

for each $k \in [0, \frac{m}{2}]$.

Each $\lambda \in \mathcal{P}(nM, m, b, LM-b)$ consists of a triple

$$\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^{\nu_b}, (M-b)^{\nu_{M-b}})\right),$$

where $\lambda_{(b)}$ is the M-modular diagram with ν_b nonnegative parts created by removing the b-ends. The M-modular diagram $\lambda_{(M-b)}$ is defined analogously by removing the $(M-b)$-ends.

When $k = \frac{m}{2}$, we simply map

$$\varphi_{\frac{m}{2}}\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^{\frac{m}{2}}, (M-b)^{\frac{m}{2}})\right) := \left(\lambda_{(b)}, \lambda_{(M-b)}, (a^{\frac{m}{2}}, (M-a)^{\frac{m}{2}})\right).$$

The required partition is then obtained by reattaching the a-ends to $\lambda_{(b)}$ and reattaching the $(M-a)$-ends to $\lambda_{(M-b)}$.

Now assume $k < \frac{m}{2}$. Note that

$$0 \equiv nM \equiv b\nu_b(\lambda) - b\nu_{M-b}(\lambda) \pmod{M}, \tag{4.1}$$

which implies $\nu_b(\lambda) - \nu_{M-b}(\lambda) \equiv 0 \pmod{M}$ because $\gcd(b, M) = 1$. Thus, we assume without loss of generality that $M \mid (m-2k)$.

Let $y := \frac{(b-a)(m-2k)}{M}$ and $z := \frac{(M-b-a)(m-2k)}{M}$. These are positive integers.

Case 1: $\nu_{M-b}(\lambda) = k$. There are k pairs of $(b, M-b)$ and $m-2k$ excess b's. We map

$$\varphi_k\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^{m-k}, (M-b)^k)\right) := \left(\lambda_{(b)} \cup \underbrace{\begin{bmatrix} M \\ \vdots \\ M \end{bmatrix}}_{y \text{ rows}}, \lambda_{(M-b)}, (a^{m-k}, (M-a)^k)\right)$$

$$=: \left(\lambda'_{(b)}, \lambda_{(M-b)}, (a^{m-k}, (M-a)^k\right).$$

Here $\lambda'_{(b)}$ is the M-modular diagram formed by attaching the above column to $\lambda_{(b)}$. Note that $a < b < M$ implies $0 < y < m - k$, so that $\lambda'_{(b)}$ is still an M-modular diagram with $m - k$ nonnegative parts.

To obtain the required partition, attach the a-ends to $\lambda'_{(b)}$ and the $(M - a)$-ends to $\lambda_{(M-b)}$. It is evident that there are m parts. Size is preserved, as

$$\begin{aligned}
&|\lambda'_{(b)}| + |\lambda_{(M-b)}| + (m-k)a + k(M-a) \\
&= |\lambda_{(b)}| + My + |\lambda_{(M-b)}| + a(m-2k) + kM \\
&= |\lambda_{(b)}| + (b-a)(m-2k) + |\lambda_{(M-b)}| + a(m-2k) + kM \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| + b(m-2k) + kM \\
&= |\lambda|.
\end{aligned}$$

Moreover, it is clear that the operations are reversible, so that, within Case 1, φ_k is an injection.

Case 2a: $\nu_b(\lambda) = k$ and $\lambda_{(M-b)}$ does not contain a column of height y.[4] There are k pairs of $(b, M-b)$ and $m - 2k$ excess $(M-b)$'s. We map

$$\varphi_k\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^k, (M-b)^{m-k})\right) := \left(\lambda_{(b)}, \lambda_{(M-b)} \cup \underbrace{\begin{bmatrix} M \\ \vdots \\ M \end{bmatrix}}_{z \text{ rows}}, (a^{m-k}, (M-a)^k)\right)$$

$$=: \left(\lambda_{(b)}, \lambda'_{(M-b)}, (a^{m-k}, (M-a)^k\right),$$

where $\lambda'_{(M-b)}$ is defined by attaching the above column. Note again that $b, a < \frac{M}{2}$ implies $0 < z < m - k$, so that $\lambda'_{(M-b)}$ is still an M-modular diagram with $m - k$ nonnegative parts. Furthermore, $b - a \neq M - b - a$, so $\lambda'_{(M-b)}$ still does not contain a column of height y.

[4] Or equivalently, the y-th part of $\lambda_{(M-b)}$ equals the $(y+1)$-st part.

To obtain the required partition, attach the a-ends to $\lambda'_{(M-b)}$ and the $(M-a)$-ends to $\lambda_{(b)}$. It is evident that there are m parts. Size is preserved, as

$$\begin{aligned}
&|\lambda_{(b)}| + |\lambda'_{(M-b)}| + (m-k)a + k(M-a) \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| + Mz + a(m-2k) + kM \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| + (M-b-a)(m-2k) + a(m-2k) + kM \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| + (M-b)(m-2k) + kM \\
&= |\lambda|.
\end{aligned}$$

Moreover, it is clear that the operations are reversible, so that, within Case 2a, φ_k is an injection.

Case 2b: $\nu_b(\lambda) = k$ and $\lambda_{(M-b)}$ contains a column of height y.[5] In this case we send

$$\left(\lambda_{(b)}, \lambda_{(M-b)}, (b^k, (M-b)^{m-k})\right) \mapsto \left(\lambda_{(b)}, \lambda_{(M-b)} \setminus \underbrace{\begin{bmatrix} M \\ \vdots \\ M \end{bmatrix}}_{y \text{ rows}}, (a^k, (M-a)^{m-k}) \right)$$

$$=: \left(\lambda_{(b)}, \lambda'_{(M-b)}, (a^k, (M-a)^{m-k}\right),$$

where $\lambda'_{(M-b)}$ is defined by removing the above column. As before, we still may consider $\lambda'_{(M-b)}$ an M-modular diagram with $m-k$ nonnegative parts.

To obtain the required partition, attach the a-ends to $\lambda_{(b)}$ and the $(M-a)$-ends to $\lambda'_{(M-b)}$. It is evident that there are m parts. Size is preserved, as

$$\begin{aligned}
&|\lambda_{(b)}| + |\lambda'_{(M-b)}| + ka + (m-k)(M-a) \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| - My + kM + (M-a)(m-2k) \\
&= |\lambda_{(b)}| + |\lambda_{(M-b)}| - (b-a)(m-2k) + kM + (M-a)(m-2k)
\end{aligned}$$

[5]Or equivalently, the y-th part of $\lambda_{(M-b)}$ is strictly greater than the $(y+1)$-st part.

$$= |\lambda_{(b)}| + |\lambda_{(M-b)}| + (M-b)(m-2k) + kM$$
$$= |\lambda|.$$

Moreover, it is clear that the operations are reversible, so that, within Case 2b, φ_k is an injection.

Let $\left(\lambda_{(a)}, \lambda_{(M-a)}, a^{\nu_a}, (M-a)^{\nu_{M-a}}\right)$ lie in the image of φ_k. Then cases are separated as follows.

Case 1: $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ contains a column of height y.

Case 2a: $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ does not contain a column of height y.

Case 2b: $\nu_a < \nu_{M-a}$.

Finally, note that in each case φ_k adds at most M to the largest part of what becomes $\lambda_{(a)}$, so indeed φ_k maps $\mathcal{P}_k(nM, m, b, LM-b)$ into $\mathcal{P}_k(nM, m, a, LM+a)$ as required. This completes the proof of the first statement.

When M is even and a is odd, we can use exactly the same injections, assuming because of (4.1) that $m - 2k \equiv \frac{M}{2} \pmod{M}$. We note that $\gcd(b, M) = 1$ implies that b is also odd, so y and z are still integers. $\square$

Remark 4.4. We note that the extra factor $\frac{1}{(1-q^{LM+a})}$ in the left term of Theorem 1.13 is necessary. For example, in

$$\frac{1}{(zq^2, zq^5; q^7)_2} - \frac{1}{(zq^3, zq^4; q^7)_2},$$

the coefficients of $z^7q^{70}, z^{13}q^{70}, z^{16}q^{70}$, and $z^{18}q^{70}$ are all negative.

The proof of Theorem 1.14 is similar, but now cases are determined by columns that occur twice.

Proof of Theorem 1.14. We define injections φ'_k to be the same as φ_k, except that in Cases 2a and 2b we condition on whether or not a partition contains *two* columns of height y. This ensures that φ'_k preserves distinct parts partitions:

Case 1. Note that $\lambda_{(b)}$ is a distinct parts partition into $m-k$ nonnegative parts (so 0 occurs at most once). As such, $\lambda_{(b)}$ must contain a column of height y. (Recall that $y < m-k$.) Attaching another such column means that $\lambda'_{(b)}$ still has distinct nonnegative parts. Attaching the ends as above also preserves distinct parts.

Case 2a. Again attaching the column to $\lambda_{(M-b)}$ preserves distinct parts because $z < m-k$. The fact that $M-b-a \neq b-a$ implies that $\lambda'_{(M-b)}$ still does not contain two columns of height y.

Case 2b. Since $\lambda_{(M-b)}$ contains two columns of height y, removing one such column preserves distinct parts.

Cases are separated as follows.

Case 1: $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ contains two columns of height y.

Case 2a: $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ does not contain two columns of height y.

Case 2b: $\nu_a < \nu_{M-a}$.

This concludes the proof. $\square$

Remark 4.5. Unlike in Theorem 1.13, it appears that the extra factor $\left(1+q^{LM+a}\right)$ in the left term of Theorem 1.14 is often not needed for nonnegativity. A computational search up to $M \leq 12$, $L \leq 20$ and $nM \leq 250$ reveals that for

$$\sum_{m,n\geq 0} d'(m,n)z^m q^n := (-zq^a, -zq^{M-a}; q^M)_L - (-zq^b, -zq^{M-b}; q^M)_L,$$

we have some $d'(m,nM) < 0$ only when $(a,b,M) = (1,2,5)$.

In fact, we can condition on more than just 2 columns to prove the following new result.

Proposition 4.6. *Let $d \geq 0$, $1 \leq a < b < \frac{M}{2}$ and $\gcd(b, M) = 1$. Let $p^{(d)}(n, m, j, A)$ denote the number of partitions of n into m parts congruent to $\pm j \pmod{M}$, whose parts are at most A such that the gap between successive parts is greater than dM. Then for all $n, m \geq 0$,*

$$p^{(d)}(nM, m, a, LM + a) \geq p^{(d)}(nM, m, b, LM - b).$$

If in addition a is odd, then we also have

$$p^{(d)}\left(nM + \frac{M}{2}, m, a, LM + a\right) \geq p^{(d)}\left(nM + \frac{M}{2}, m, b, LM - b\right).$$

Substituting $d = 0$ and $d = 1$ gives Theorems 1.13 and 1.14 respectively.

Proof. Let $\lambda = \left(\lambda_{(b)}, \lambda_{(M-b)}, (b^{\nu_b}, (M-b)^{\nu_{M-b}})\right)$ be a partition counted by $p^{(d)}(nM, m, b, LM - b)$. Then the M modular diagrams $\lambda_{(b)}$ and $\lambda_{(M-b)}$ are partitions into nonnegative multiples on M such that the difference in successive parts is at least $(d+1)M$. Our injections $\varphi_k^{(d)}$ are the same as before, except that we condition in cases 2 or 3 on whether or not $\lambda_{(M-b)}$ contains $d+2$ columns of height y. $\square$

4.4 Applications to Kanade-Russell's Conjectures

In [31], Kanade and Russell conjectured several new Rogers-Ramanujan-type product-sum identities—three arising from the theory of affine Lie algebras, and several companions. Bringmann, Jennings-Shaffer and Mahlburg were able to prove many of these [13], and they reduced the sum-sides of the four conjectures below from triple series to a single series. Here, $\mathcal{KR}_j$ is Identity j in [31], and $H_j(x)$ is the sum side as denoted in [13].

$$\mathcal{KR}_4: \qquad H_4(1) = \frac{1}{(q,q^4,q^5,q^9,q^{11};q^{12})_\infty},$$

$$\mathcal{KR}_{4a}: \qquad H_5(1) = \frac{1}{(q,q^5,q^7,q^8,q^9;q^{12})_\infty},$$

$$\mathcal{KR}_6: \qquad H_8(1) = \frac{1}{(q,q^3,q^7,q^8,q^{11};q^{12})_\infty},$$

$$\mathcal{KR}_{6a}: \qquad H_9(1) = \frac{1}{(q^3,q^4,q^5,q^7,q^{11};q^{12})_\infty}.$$

The pairs of sum-sides, $(H_4(1), H_5(1))$ and $(H_8(1), H_9(1))$, are composed of two generating functions for partitions that satisfy the same set of gap conditions, but H_5 and H_9 have an additional condition on the smallest part (see [31]). Hence, as with the Rogers-Ramanujan sum-sides, we have the inequalities

$$H_4(1) - H_5(1) \succeq 0 \qquad \text{and} \qquad H_8(1) - H_9(1) \succeq 0,$$

which, *if the conjectures are true*, imply the following result.

Proposition 4.7. *The following inequalities hold.*

$$\frac{1}{(q,q^4,q^5,q^9,q^{11};q^{12})_\infty} - \frac{1}{(q,q^5,q^7,q^8,q^9;q^{12})_\infty} \succeq 0, \tag{4.2}$$

$$\frac{1}{(q,q^3,q^7,q^8,q^{11};q^{12})_\infty} - \frac{1}{(q^3,q^4,q^5,q^7,q^{11};q^{12})_\infty} \succeq 0. \tag{4.3}$$

Proof. (4.3) is an immediate consequence of Theorem 1.12, since for every $L \geq 0$,

$$\frac{1}{(q,q^8;q^{12})_L} - \frac{1}{(q^4,q^5;q^{12})_L} \succeq 0.$$

Multiplying both sides by the positive series $\frac{1}{(q^3,q^7,q^{11};q^{12})_\infty}$ and taking the limit as $L \to \infty$ finishes the proof of (4.3).

Andrews' Anti-telescoping Method [4] works seamlessly to show (4.2). Define

$$P(j) := (q,q^4,q^{11};q^{12})_j \qquad \text{and} \qquad Q(j) := (q,q^7,q^8;q^{12})_j,$$

and note that the following implies (4.2):

$$\frac{1}{P(L)} - \frac{1}{Q(L)} \succeq 0 \quad \text{for all } L \geq 0. \tag{4.4}$$

Now we write

$$\begin{aligned}
\frac{1}{P(L)} - \frac{1}{Q(L)} &= \frac{1}{Q(L)} \left(\frac{Q(L)}{P(L)} - 1 \right) \\
&= \frac{1}{Q(L)} \sum_{j=1}^{L} \left(\frac{Q(j)}{P(j)} - \frac{Q(j-1)}{P(j-1)} \right) \\
&= \sum_{j=1}^{L} \frac{1}{\frac{Q(L)}{Q(j-1)} P(j)} \left(\frac{Q(j)}{Q(j-1)} - \frac{P(j)}{P(j-1)} \right),
\end{aligned}$$

whose j-th term is

$$\begin{aligned}
&\frac{(1-q^{12j-11})q^{12(j-1)}}{(q^{12j-11}, q^{12j-5}, q^{12j-4}; q^{12})_{L-j+1} (q, q^4, q^{11}; q^{12})_j} \times \left(-q^7 - q^8 + q^4 + q^{11} \right) \\
= &\frac{(1-q^{12j-11})q^{12(j-1)}}{(q^{12j-11}, q^{12j-5}, q^{12j-4}; q^{12})_{L-j+1} (q, q^4, q^{11}; q^{12})_j} \times q^4 (1-q^3)(1-q^4). \qquad (4.5)
\end{aligned}$$

The terms $(1-q^4)$ and $(1-q^{12j-11})$ are cancelled in the denominator, and we can write $\frac{1-q^3}{1-q} = 1 + q + q^2$. Hence, (4.5) is nonnegative for every j, proving (4.4) and then (4.2). □

Another pair of identities in [31] with an Ehrenpreis Problem set-up is the following.

$$\begin{aligned}
\mathcal{KR}_5 : \qquad & H_6(1) = \frac{1}{(q^2;q^4)_\infty} \prod_{n \geq 0} \left(1 + q^{4n+1} + q^{2(4n+1)} \right), \\
\mathcal{KR}_{5a} : \qquad & H_7(1) = \frac{1}{(q^2;q^4)_\infty} \prod_{n \geq 0} \left(1 + q^{4n+3} + q^{2(4n+3)} \right)
\end{aligned}$$

Both identities were proved in [13], and there is an obvious injection proving

$$\frac{1}{(q^2;q^4)_\infty} \prod_{n \geq 0} \left(1 + q^{4n+1} + q^{2(4n+1)} \right) - \frac{1}{(q^2;q^4)_\infty} \prod_{n \geq 0} \left(1 + q^{4n+3} + q^{2(4n+3)} \right) \succeq 0,$$

namely, sending each $(4n+3)$ to the pair $(4n+1, 2)$.

Finally, we discuss the Ehrenpreis problems among $\mathcal{KR}_1$, $\mathcal{KR}_2$ and $\mathcal{KR}_3$. These were proved in [13], and their respective sum-sides were denoted $H_1(x)$, $H_2(x)$ and $H_3(x)$. Using the methods of [31], we have found slightly different conditions for the partitions enumerated on the sum-side:

1. No consecutive parts are allowed.
2. Odd parts do not repeat.
3. Even parts appear at most twice.
4. We have $(\lambda_j, \lambda_{j+1}, \lambda_{j+2}) \notin \{(2k, 2k, 2k+2), (2k, 2k, 2k+3), (2k+1, 2k+3, 2k+5), (2k-2, 2k, 2k)\}$ for any j and k.[6]

Note that our fourth condition is changed slightly from Kanade and Russell's in [31], page 5. The sum-side of $\mathcal{KR}_2$ has the further restriction that the part 1 may not appear, and in the sum-side of $\mathcal{KR}_3$, the parts 1, 2 and 3 may not appear. Hence, $H_1(1) \succeq H_2(1) \succeq H_3(1)$ and it follows from Theorem 1.1 of [13] that

$$\frac{1}{(q, q^4, q^6, q^8, q^{11}; q^{12})_\infty} \succeq \frac{(-q^3, -q^9; q^{12})_\infty}{(q^2, q^4, q^8, q^{10}; q^{12})_\infty} \succeq \frac{1}{(q^4, q^5, q^6, q^7, q^8; q^{12})_\infty}.$$

The inequality between the far left and right products is a consequence of Theorem 5.3 of [7], but a direct proof of the other two inequalities remains open.

[6] As in [31], we have written parts in increasing order.

Chapter 5: Conclusion

5.1 Asymptotic Statistics for Unimodal Sequences

The methods of Chapter 2 for deriving limit shapes of unimodal sequences seem to be widely applicable, and one could readily apply them to derive limit shapes for the other types of unimodal sequences described by Bringmann and Mahlburg in [16]. It may also be profitable to further explore DeSalvo-Pak-type transfer of limit shapes related to other unimodal sequences.

While limit shapes give very strong results "at the level of $\sqrt{n}$", we still understand very little of the structure of unimodal sequences in comparison to partitions. The probabilistic methods of Fristedt [25] and Chapter 3 seem to not easily extend to unimodal sequences because often the generating functions involved cannot be written as products, which is what leads to independence in the X_k's. It would be interesting to develop methods that circumvent these apparent difficulties.

5.2 Partition Inequalities

As we pointed out in the introduction, the inequality

$$\frac{1}{(q, q^4; q^5)_\infty} - \frac{1}{(q^2, q^3; q^5)_\infty} \succeq 0$$

was the start of Andrews-Baxter's "motivated proof" of the Rogers-Ramanujan identities [5]. They defined $G_1 := (q, q^4; q^5)_\infty^{-1}$ and $G_2 := (q^2, q^3; q^5)_\infty^{-1}$, and then recursively

$$G_i := \frac{G_{i-2} - G_{i-1}}{q^{i-2}}, \qquad \text{for } i \geq 3. \tag{5.1}$$

They then observed computationally that $G_i = 1 + \sum_{n\geq i} g_{i,n}q^n \succeq 0$. Thus, as $i \to \infty$ the coefficient of q^n in G_i is eventually 0. This was their "Empirical Hypothesis," and proving it leads easily to a new proof of the Rogers-Ramanujan identities.

Note that, starting from the sum-sides of G_1 and G_2, the recursive definition (5.1) and the Empirical Hypothesis are completely natural. For example, if $\mathcal{RR}$ denotes the set of gap-2 partitions, then by the Rogers-Ramanujan Identities,

$$G_1 - G_2 = \sum_{\substack{\lambda\in\mathcal{RR} \\ \lambda\ni 1}} q^{|\lambda|} = q\left(1 + \sum_{\substack{\lambda\in\mathcal{RR} \\ \lambda_j\geq 3}} q^{|\lambda|}\right),$$

and so

$$G_2 - G_3 = \sum_{\substack{\lambda\in\mathcal{RR} \\ \lambda_j\geq 2 \\ \lambda_j\ni 2}} q^{|\lambda|} = q^2\left(1 + \sum_{\substack{\lambda\in\mathcal{RR} \\ \lambda_j\geq 4}} q^{|\lambda|}\right),$$

and so on.

For $\mathcal{KR}_4$, $\mathcal{KR}_{4a}$, $\mathcal{KR}_6$ and $\mathcal{KR}_{6a}$, we can expect the more complicated conditions on the sum-sides to lead to more complicated recurrences. For example, the recurrence below was shown for $\mathcal{KR}_4$ ([13], equation 4.2).

$$H_4(x) = (1 + xq)H_4(xq^2) + xq^2(1 + xq^3)H_4(xq^4) + x^2q^6(1 - xq^4)H_4(xq^6).$$

Combinatorial proofs of the above and the similar recurrences in [13] may give insight into an "Empirical Hypothesis" for $\mathcal{KR}_4$, $\mathcal{KR}_{4a}$, $\mathcal{KR}_6$ and $\mathcal{KR}_{6a}$. Indeed, the techniques for "motivated proofs" have been expanded to accommodate identities with gap-conditions more complicated than those of $\mathcal{RR}$, notably in [20], [29] and [32]. At the very least, it would be profitable to expand the techniques behind "motivated proofs," especially if they could be applied to asymmetric products like those of Kanade-Russell.

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, New York: Dover, 1972.

[2] G.E. Andrews, *The Theory of Partitions,* Cambridge University Press, 1984.

[3] G.E. Andrews, "Concave and convex compositions", *Ramanujan J.*, **31** (2013), 67-82.

[4] G.E. Andrews, "Differences of partition functions: the anti-telescoping method." In H.M. Farkas, R.C. Gunning, M.I. Knopp, and B.A. Taylor, editors, *From Fourier Analysis and Number Theory to Radon Transforms and Geometry,* 1-20, *Dev. Math.*, **28**, Springer, New York, 2013.

[5] G.E. Andrews and R. Baxter, "A motivated proof of the Rogers-Ramanujan identities," *Amer. Math. Monthly*, **96** (1989), 401-409.

[6] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Second Edition, Springer-Verlag, New York-Heidelberg, 1976.

[7] A. Berkovich and F.G. Garvan, "Dissecting the Stanley partition function," *J. Combin. Number Theory Ser. A*, **112** (2005), no. 2, 277-291.

[8] A. Berkovich and K. Grizzell, "On the class of dominant and subordinate products," *Mathematics*, **1** (2013), no. 2, 65-75.

[9] P. Billingsley, *Probability and Measure,* John Wilet & Sons Inc., 1995.

[10] W. Bridges, "Distinct parts partitions with bounded largest part", preprint (2020); available at https://arxiv.org/abs/2004.12036

[11] W. Bridges, "Limit shapes for unimodal sequences," preprint (2020); available at https://arxiv.org/abs/2001.06878

[12] W. Bridges, "Partition inequalities and applications to sum-product conjectures of Kanade-Russell," accepted for publication in *Integers*; available at https://arxiv.org/abs/1903.07587

[13] K. Bringmann, C. Jennings-Shaffer, and K. Mahlburg, "Proofs and reductions of various conjectured partition identities of Kanade and Russell", accepted for publication in *J. Reine Angew. Math.*; available at https://arxiv.org/abs/1809.06089

[14] K. Bringmann, C. Jennings-Shaffer, and K. Mahlburg, "The asymptotic distribution of the rank for unimodal sequences", preprint (2019); available at https://arxiv.org/abs/1910.10790

[15] K. Bringmann, C. Jennings-Shaffer, K. Mahlburg, and R. Rhoades, R., "Peak positions of strongly unimodal sequences", *Trans. Amer. Math. Soc.*, **373** (2019), 7087-7109.

[16] K. Bringmann and K. Mahlburg, "Asymptotic formulas for stacks and unimodal sequences", *J. Combin. Th. Ser. A*, **126** (2014), 194-215.

[17] E. R. Canfield, "From recursions to asymptotics: on Szekeres' formula for the number of partitions", *Electron. J. Combin.*, **4** (1997), Research Paper 6, approx. 16 pp.

[18] S. Corteel and P. Hitczenko, "Multiplicity and number of parts of overpartitions", *Ann. Comb.*, **8** (2004), 287-301.

[19] S. Corteel and J. Lovejoy, "Overpartitions", *Trans. Amer. Math. Soc.*, **356** (2004), 1623-1635.

[20] B. Coulson, S. Kanade, J. Lepowsky, R. McRae, F. Qi, M. Russell, and C. Sadowski, "A motivated proof of the Göllnitz-Gordon-Andrews identities," *Ramanujan J.*, **47** (2017), no. 1, 97-129.

[21] S. DeSalvo and I. Pak, "Limit shapes via bijections", *Combin. Probab. Comput.*, **28** (2019), 187-240.

[22] A. Dembo, A. Vershik, and O. Zeitouni, "Large deviations for integer partitions", *Markov Process. Related Fields*, **6** (2000), 147-179.

[23] P. Duchon, P. Flajolet, G. Louchard and G. Schaeffer, "Boltmann samplers for the random generation of combinatorial structures, *Combin. Probab. Comput.*, **13** (2004), 577-625.

[24] P. Erdős and J. Lehner, "The distribution of the number of summands in the partitions of a positive integer", *Duke Math. J.*, **8** (1941), 335-345.

[25] B. Fristedt, "The structure of random partitions of large integers", *Trans. Amer. Math. Soc.*, **337** (1993), 703-735.

[26] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[27] G.H. Hardy and S. Ramanujan, "Asymptotic formulae in combinatory analysis," *Proc. London Math. Soc.* (2), **17** (1918), 75-115.

[28] K. Kadell, "An Injection for the Ehrenpreis Rogers-Ramanujan problem," *J. Combin. Theory Ser. A*, **86** (1999), no. 2, 390-394.

[29] S. Kanade, J. Lepowsky, M. Russell and A. Sills, “Ghost series and a motivated proof of the Andrews-Bressoud identities,” *J. Combin. Theory Ser. A*, **146** (2017), 33-62.

[30] S. Kanade and M. Russell, “IdentityFinder and some new identities of Rogers-Ramanujan type,” *Exp. Math.*, **24** (2015), no. 4, 419-423.

[31] S. Kanade and M. Russell, “Staircases to analytic sum-sides for many new integer partition identities of Rogers-Ramanujan type”, *Electron. J. Combin.* **26** (2019), no. 1, Paper 1.6, 33 pp.

[32] J. Lepowsky and M. Zhu, “A motivated proof of Gordon’s identities”, *Ramanujan J.*, **29** (2012), no. 1-3, 199-211.

[33] J. McLaughlin, “Refinements of some partition inequalities,” *Integers* **16** (2016), Paper No. A66, 11 pp.

[34] H.L. Montgomery and R.C. Vaughn, *Multiplicative Number Theory: I. Classical Theory*, Cambridge University Press, 2006.

[35] F. V. Petrov, “Two elementary approaches to the limit shapes of young diagrams”, trranslation from Russian in *J. Math. Sci. (N.Y.)*, **166** (2010), 63-74.

[36] D. Romik, “Partitions of n into $t\sqrt{n}$ parts”, *European J. Combin.*, **26** (2005), 1-17.

[37] K. Roth and G. Szekeres, “Some asymptotic formulae in the theory of partitions, ” *Quart. J. Math. Oxford Ser.* (2), **5** (1954), 241-259.

[38] R. Stanley, “Log-concave and unimodal sequences in algebra, combinatorics and geometry.” In *Graph theory and its applications: East and West (Jinan, 1986),* 500-535, *Ann New York Acad. Sci.*, **576**, New York, 1989.

[39] G. Szekeres, “An asymptotic formula in the theory of partitions”, *Quart. J. Math., Oxford Ser.* (2), **2** (1951) 85-108.

[40] G. Szekeres, “Some asymptotic formulae in the theory of partitions (II)”, *Quart. J. Math., Oxford Ser.* (2), **4** (1953), 96-111.

[41] A. M. Vershik, “Statistical mechanics of combinatorial partitions, and their limit configurations”, translation from Russian in *Funct. Anal. Appl.* **30** (1996), 90-105.

[42] E.M. Wright, “Stacks”, *Quart. J. Math. Oxford Ser.* (2), **19** (1968), 313-320.

[43] E.M. Wright, “Stacks (II)”, *Quart. J. Math. Oxford Ser.* (2), **22** (1971), 107-116.

[44] E.M. Wright, Stacks (III), *Quart. J. Math. Oxford Ser.* (2), **23** (1972), 153-158.

Vita

Walter McFarland Bridges was born on October 28, 1992 in Gadsden, Alabama. He and his family moved to Hendersonville, North Carolina in 2004, where he was a student teacher at Hendersonville High School while completing his degree in Mathematics Secondary Education from Appalachian State University in 2014. He came to Louisiana State University in 2016 to pursue graduate studies in Mathematics and is currently a candidate for a Doctor of Philosophy in Mathematics.