

3-26-2020

On Selected Subclasses of Matroids

Tara Elizabeth Fife

Louisiana State University at Baton Rouge

Follow this and additional works at: https://digitalcommons.lsu.edu/gradschool_dissertations



Part of the [Discrete Mathematics and Combinatorics Commons](#)

Recommended Citation

Fife, Tara Elizabeth, "On Selected Subclasses of Matroids" (2020). *LSU Doctoral Dissertations*. 5192.
https://digitalcommons.lsu.edu/gradschool_dissertations/5192

This Dissertation is brought to you for free and open access by the Graduate School at LSU Digital Commons. It has been accepted for inclusion in LSU Doctoral Dissertations by an authorized graduate school editor of LSU Digital Commons. For more information, please contact gradetd@lsu.edu.

ON SELECTED SUBCLASSES OF MATROIDS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Tara Elizabeth Fife

B.S. in Math., Brigham Young University-Idaho, 2014

M.S., Louisiana State University, 2015

May 2020

This work is dedicated to all those who strive to help those who are in need.

All of us, at some time or other, need help. Whether we're giving or receiving help, each one of us has something valuable to bring to this world. That's one of the things that connects us as neighbors—in our own way, each one of us is a giver and a receiver.

— Fred Rogers

The World According to Mister Rogers: Important Things to Remember

Acknowledgments

It is not possible to thank everyone who has made a significant difference in my life or even all those who have significantly helped me in my education. However, I would like to mention a few individuals who have provided significant help in regards to my education.

Jerry Heath, my middle school geography teacher, taught his students to see the best in themselves. My high school science teacher, Elaine Garbarino Asmus, was excellent in inspiring me through her example to pursue higher education. Ben Woodruff, my undergraduate advisor, helped convince me to study mathematics because he saw how much I liked it. My undergraduate experience would have been far less enriching if it had not been for my friends who I made there. In particular, Timothy O’Leary and Caley Johns were always there for me, and Lindsey Beecher and Kelsey Wells helped me by providing advice and perspective.

I am very grateful to my LSU family. I enjoyed working with and experiencing graduate school with my cohort. Of particular note, Ryan Leigon and Erika Gerhold worked with me on countless homework assignments during my first year, Joshua Finkelstein has made an effort to stay in contact after he left, and Irfan Alam was always there to help with anything that I could need. Many of the more senior graduate students and postdocs, including Eric Bucher, Kristen Wetzler, Simon Pfeil, Kimberly Sevin D’souza, Joshua Fallon, Trey Trampel, Riley Casper, Mike Wong, and Kevin Grace became my friends and provided a preview of what I would soon experience in the academic world. My professors here were excellent, both in teaching material and in providing perspective.

In particular, I would like to thank Bogdan Oporowski, Shea Vela–Vick, Milen Yakimov, Anas Mahmoud, Dan Cohen, Stefan van Zwam, and Stephen Shipman for their assistance.

I am pleased to have become a part of the matroid community. I appreciate all those who have collaborated with me. I am obliged to Deb Chun and Tyler Moss as well as Carolyn Chun and Joe Bonin for hosting me on research projects. I am glad for my academic siblings, especially Dennis Hall, Jesse Taylor, Jagdeep Singh, Zach Gershkoff, and Cameron Crenshaw, who I have had the chance to know well. In addition, Charles Semple, Geoff Whittle, Nick Brettell, Rutger Campbell, Dillon Mayhew, and Jorn van der Pol have been particularly welcoming.

Of course, I'm most grateful to my advisor. When I decided that I wanted to work with James, I expected that he would be an excellent adviser. Yet, he has far surpassed my expectations, both in his kindness and generosity, and in his tenacity in helping his students to develop into mathematicians. He and his wife Judith have helped me to feel at home in Baton Rouge.

Thanks James.

Contents

Acknowledgments	iv
Abstract	vii
Chapter 1. Introduction	1
1.1. Graph theory and connections to matroid theory	1
1.2. Nested matroids	7
1.3. Matroid operations and constructions	9
Chapter 2. Laminar Matroids	15
2.1. Introduction to laminar matroids	15
2.2. Canonical presentation	18
2.3. Excluded minors	25
2.4. Constructing laminar matroids	30
2.5. Which laminar matroids are binary or ternary?	35
Chapter 3. Generalized Laminar Matroids	39
3.1. Introduction	39
3.2. Preliminaries	41
3.3. Excluded minors	46
3.4. Intersections with other classes of matroids	58
Chapter 4. Unbreakable Frame Matroids	63
4.1. Introduction to unbreakable matroids	63
4.2. Preliminaries	65
4.3. Beginning the proof of the main theorem	68
4.4. Finishing the proof of the main theorem	75
4.5. A Partial Converse to the Main Theorem	87
Appendix. Copyright Information	92
Bibliography	102
Vita	105

Abstract

Matroids were introduced by Whitney to provide an abstract notion of independence. In this work, after giving a brief survey of matroid theory, we describe structural results for various classes of matroids.

A connected matroid M is unbreakable if, for each of its flats F , the matroid M/F is connected. Pfeil showed that a simple graphic matroid $M(G)$ is unbreakable exactly when G is either a cycle or a complete graph. We extend this result to describe which graphs are the underlying graphs of unbreakable frame matroids.

A laminar family is a collection \mathcal{A} of subsets of a set E such that, for any two intersecting sets, one is contained in the other. For a capacity function c on \mathcal{A} , let \mathcal{I} be $\{I : |I \cap A| \leq c(A) \text{ for all } A \in \mathcal{A}\}$. Then \mathcal{I} is the collection of independent sets of a (laminar) matroid on E . We characterize the class of laminar matroids by their excluded minors and present a way to construct all laminar matroids using basic operations.

A flat of a matroid M is Hamiltonian if it has a spanning circuit. A matroid M is nested if its Hamiltonian flats form a chain under inclusion; M is laminar if, for every 1-element independent set X , the Hamiltonian flats of M containing X form a chain under inclusion. We generalize these notions to define the classes of k -closure-laminar and k -laminar matroids. The second class is always minor-closed, and the first is if and only if $k \leq 3$. We give excluded-minor characterizations of the classes of 2-laminar and 2-closure-laminar matroids.

Chapter 1. Introduction

In this dissertation, it will be assumed that the reader is familiar with the basic theory of matroid theory as detailed in Oxley [25]. That book will be followed for terminology and notation for matroids and graphs. However, this introductory chapter reviews several basic definitions and includes some material that will play a key role throughout the rest of the dissertation.

1.1. Graph theory and connections to matroid theory

In this section, we introduce several fundamentals of graph theory. We also define a matroid and some classes of matroids derived from graphs. A more thorough introduction to graph theory may be found in [11]. A *graph*, $G = (V, E)$, consists of a set $V(G) = V$ of *vertices* together with a multiset $E(G) = E$ of *edges*, each consisting of a pair $\{u, v\}$ of vertices. We say that u and v are *incident* with e , and if $u \neq v$, then u and v are *adjacent*. If $u = v$, then we say that e is a *loop*. If two non-loop edges are incident to the same vertices, then they are *parallel*. A graph is *simple* if it has no loops or parallel edges. A *complete graph* is a simple graph where every two vertices are adjacent. We will consider only *finite graphs*, that is, graphs where both V and E are finite. Graphs G_1 and G_2 are *isomorphic*, written $G_1 \cong G_2$, if there are bijections $\psi : V(G_1) \rightarrow V(G_2)$ and $\theta : E(G_1) \rightarrow E(G_2)$ such that a vertex $v \in V(G_1)$ is incident to an edge $e \in E(G_1)$ if and only if $\psi(v)$ is incident to $\theta(e)$ in G_2 .

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $E' \subseteq E(G)$, let V' be the set of vertices incident to some $e \in E'$. Then the graph (V', E') is

the *induced graph* $G[E']$. Now, for $V' \subseteq V(G)$, let E' be the set of edges incident to some $v \in V'$. Then the graph (V', E') is the *induced graph* $G[V']$.

A *walk* is a sequence $W = \{v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n\}$, where each $e_i \in E$, each $v_i \in V$, and where each e_i is incident to both v_{i-1} and v_i . If the v_i 's are distinct, then so are the e_i 's, and W is a *path*. Such a path is a (v_0, v_n) -*path* and is said to *join* v_0 and v_n . The vertices v_0 and v_n are *endpoints* of the path and the vertices $\{v_1, v_2, \dots, v_{n-1}\}$ are called *interior vertices*. Two paths P_1 and P_2 are *internally disjoint* if any vertex $v \in V(P_1) \cap V(P_2)$ is an endpoint in both paths. A graph is *connected* if each pair of vertices can be joined by a path; otherwise it is *disconnected*. In a graph G , the maximal connected subgraphs of G are the *connected components* of G . If P is a (u, v) -path and e is an edge of G that joins u and v , then the subgraph whose vertex set is $V(P)$ and whose edge set is $E(P) \cup e$ is called a *cycle*. If $e \in E(G)$ is not in any cycle of G , then it is a *cut edge* of G . We will abuse notation by referring to a path or a cycle and its edge set interchangeably. A *forest* is a graph that contains no cycles. A graph is a *tree* if it is a connected forest. A subgraph T of G is a *spanning tree* of G if it is a tree and each vertex of G is incident to an edge in T .

Let \mathcal{I} be the collection of subsets I of E such that G has no cycles using only edges from I . Then \mathcal{I} is the set of *independent sets* of the *cycle matroid* of G . Whitney [37] generalized this definition of independence in graphs, along with the usual definition of linear independence to introduce an abstract notion of independence on a finite set. Specifically, a *matroid* M is a pair (E, \mathcal{I}) , where E is a finite set, and \mathcal{I} is a collection of subsets of E having the following three properties.

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_2| > |I_1|$, then there is some element $e \in I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

For a matroid M , the set E is called the *ground set* and the members of \mathcal{I} are called *independent sets*. The ground set of a matroid is denoted by $E(M)$, and the collection of independent sets of M is denoted by $\mathcal{I}(M)$. If $e \in E$, then we say that e is an *element* of M and we write $e \in M$. Two matroids M_1 and M_2 are *isomorphic* if there is a bijection $\psi : E(M_1) \rightarrow E(M_2)$ such that for all $X \in \mathcal{I}(M_1)$, the set $\psi(X)$ is independent in M_2 if and only if X is independent in M_1 . In this case, we write $M_1 \cong M_2$. A *basis* of M is a maximal independent set of M . The collection of bases of M is denoted by $\mathcal{B}(M)$. By (I3), we know that all bases of M have the same size. The *rank*, $r(M)$, of M is the size of a basis of M . A set is called *dependent* if it is not independent, and a *circuit* of M is a minimal dependent set. The collection of circuits of M is denoted by $\mathcal{C}(M)$. If $\{e\}$ is a circuit, then e is called a *loop*. If e is not contained in any circuits, then e is a *coloop*. If $\{e, f\}$ is a circuit, then e and f are *parallel*. If every circuit containing e also contains f , then e and f are in *series*.

Let A be a matrix over a field \mathbb{F} . Suppose that the columns of A are labeled e_1, e_2, \dots, e_n . Let $E = \{e_1, e_2, \dots, e_n\}$. Let \mathcal{I} be the collection of subsets I of E such that I labels a linearly independent set of columns of A . Then (E, \mathcal{I}) is a matroid. Such a matroid is said to be *representable* over \mathbb{F} . In particular, M is *binary* if it is representable over $GF(2)$, and M is *ternary* if it is representable over $GF(3)$. If M is representable over every field, then M is *regular*.

It is well known, and easily checked, that cycle matroids satisfy (I1)-(I3). A circuit of $M(G)$ corresponds to a cycle of G . To get a basis of $M(G)$, choose a spanning tree for each connected component of G , and take the union of their edge sets. A matroid is *graphic* if it is isomorphic to the cycle matroid of some graph. We now describe a few other classes of matroids that are derived from graphs.

A Θ -*graph* is a graph consisting of two vertices that are joined by three internally disjoint paths. A *handcuff* is a graph that consists either of two cycles that share a single vertex, or two vertex-disjoint cycles together with a minimal path that meets each of the cycles in a single vertex. Let \mathcal{C} be the collection of Θ -graphs and handcuffs of G . Then \mathcal{C} is the collection of circuits of a matroid $B(G)$, which we call the *bicircular* matroid of G .

A *biased graph* (G, Ψ) consists of a graph G and a set Ψ of cycles of G such that if C_1 and C_2 are in Ψ and the induced graph $G[C_1 \cup C_2]$ is a Θ -graph, then the third cycle in $G[C_1 \cup C_2]$ is also in Ψ . The cycles in Ψ are called *balanced*; all other cycles are *unbalanced*. Such a collection Ψ is said to *satisfy the Θ property*. We say that G is *balanced* if it has no unbalanced cycles; otherwise G is *unbalanced*.

From a biased graph (G, Ψ) , we obtain a matroid $M(G, \Psi)$ whose ground set is $E(G)$ and whose set of circuits consists of the members of Ψ together with those Θ -graphs and handcuffs in which all cycles are unbalanced. A matroid M is a *frame matroid* if $M \cong M(G, \Psi)$ for some biased graph (G, Ψ) . Note that $M(G, \Psi)$ is the cycle matroid of G when Ψ consists of all of the cycles of G , while $M(G, \Psi)$ is the bicircular matroid of G when Ψ is empty.

Recall that H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We define two particular subgraphs. For $v \in V(G)$, the *deletion* $G - v$ of v from G is the graph

$(V-v, E')$, where E' is the set of edges of G not incident to v . For $e \in E(G)$, the subgraph $(V, E - e)$ is the *deletion* of e from G and is denoted by $G - e$ or $G \setminus e$. The *contraction* of e , denoted by G/e is the graph obtained by identifying the vertices incident to e and removing e from the edge set of G . In particular, if e is a loop, then G/e is equal to $G \setminus e$. It is well known that, for any $e, f \in E$, we have $G/e \setminus f = G \setminus f/e$ and $G \setminus e \setminus f = G \setminus f \setminus e$, as well as $G/e/f = G/f/e$. So, for disjoint sets $X \subseteq E$ and $Y \subseteq E$, we can denote by $G/X \setminus Y$ the graph that is derived from G by, in any order, contracting all the edges of X and deleting all the edges of Y . A vertex is isolated if it is not incident to any edges. If H is obtained from $G/X \setminus Y$, for some X and Y , by deleting any number of isolated vertices, then H is a *minor* of G . A minor H of G is a *proper minor* if $H \neq G$.

The definition of minors for matroids is similar to that for graphs, although matroids do not have vertices. For the matroid $M = (E, \mathcal{I})$, the *deletion* $M \setminus e$ of e from M is the matroid $(E - e, \mathcal{I} - e)$, where $\mathcal{I} - e$ is the set of independent sets of M avoiding e . Suppose e is not a loop. Then the *contraction* M/e of e from M is the matroid $(E - e, \mathcal{I}')$, where \mathcal{I}' is the set $\{I \subseteq E - e : I \cup e \in \mathcal{I}\}$. If e is a loop, we define $M/e = M \setminus e$. As for graphs, for disjoint sets $X \subseteq E$ and $Y \subseteq E$, the matroid $M/X \setminus Y$ is well-defined and is obtained by contracting the elements of X and deleting the elements of Y from M , in any order. The matroid $M/X \setminus Y$ is a *minor* of M , and a minor N of M is a *proper minor* if $N \neq M$. For $X \subseteq E$, the *restriction* of M to X is $M|X = M \setminus (E - X)$, and the *rank* $r_M(X)$ of X is the rank of $M|X$. If there is no ambiguity, we will often write $r(X)$ instead of $r_M(X)$. Deletion and contraction in graphs corresponds to deletion and contraction in matroids, in that, $M(G) \setminus e = M(G \setminus e)$ and $M(G)/e = M(G/e)$.

Let \mathcal{M} be a collection of graphs or a collection of matroids. Then \mathcal{M} is *minor-closed*

if, for every $M \in \mathcal{M}$ and every minor M' of M , we have $M' \in \mathcal{M}$. If \mathcal{G} is a minor-closed class of graphs, then a graph G is an *excluded minor* of \mathcal{G} if $G \notin \mathcal{G}$, but all of its proper minors are in \mathcal{G} . Similarly, if \mathcal{M} is a minor-closed class of matroids, then a matroid M is an *excluded minor* of \mathcal{M} if $M \notin \mathcal{M}$, but all of its proper minors are in \mathcal{M} . A common question in both structural graph theory and structural matroid theory involves determining if a class is minor-closed and, if so, finding its collection of excluded minors. The Robertson–Seymour theorem [33] showed that, for any infinite collection \mathcal{G} of graphs, there are graphs H and G in \mathcal{G} with H isomorphic to a minor of G . In particular, this means that every minor-closed class of graphs, other than the class of all graphs, has a finite number of excluded minors, up to isomorphism. However, many classes of matroids have an infinite number of excluded minors.

Recall that $G = (V, E)$ is *connected* if, for any two vertices $u, v \in V$, there is a path from u to v . For $V' \subseteq V$, we denote by $G - V'$, the graph $G(V - V', E')$, where $E' = \{e \in E : e \text{ is not incident with a vertex in } V'\}$. If, for all $V' \subseteq V(G)$ with $|V'| < n$, the graph $G - V'$ is connected, then G is *n -connected*, or more specifically, *n -vertex-connected*. If G is n -connected, and s and t are distinct vertices of G , then, by Menger’s Theorem (see, for example, [25, Theorem 8.5.1]), there are n internally disjoint paths from s to t .

The matroid definition of connectivity is more difficult to state. We call

$$\lambda_M(X) = r(X) + r(E - X) - r(M)$$

the *connectivity function* of M . If there is no ambiguity, then we will often write $\lambda(X)$ instead of $\lambda_M(X)$. For a positive integer k , if $\lambda(X) < k$, then both X and $E - X$ are called *k -separating*. If X is k -separating, and $\min\{|X|, |E - X|\} \geq k$, then $(X, E - X)$ is called

a *k-separation* of M . If, for all $k \in \{1, 2, \dots, n-1\}$, there are no k -separations of M , then M is *n-connected* [36]. If M is 2-connected, then we call M *connected*. Let G be a graph with $|V(G)| \geq 3$ and no isolated vertices. Then $M(G)$ is 2-connected exactly when G is 2-connected and loopless (see, for example, [25, Corollary 8.1.6]). Tutte showed that a simple graph G without isolated vertices and with $|V(G)| \geq 4$ is 3-connected exactly when $M(G)$ is 3-connected. If M_1 and M_2 are matroids on disjoint ground sets, E_1 and E_2 , then the *direct sum*, $M_1 \oplus M_2$, of M_1 and M_2 is the matroid on $E_1 \cup E_2$ whose collection of independent sets is $\{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1), I_2 \in \mathcal{I}(M_2)\}$. A *connected component* of M is a maximal set $N \subseteq E(M)$ such that $M|N$ is connected. We can write all matroids as direct sums of their connected components. In Chapter 2, we shall describe a decomposition of 2-connected matroids into 3-connected matroids, circuits, and cocircuits.

There are several other types of connectivity; we discuss one type which is relevant to this work. For $X \subseteq E(M)$, the *closure* of X is $cl_M(X) = \{e \in E(M) : r(X \cup e) = r(X)\}$. If $cl_M(F) = F$, then F is a *flat* of M . A connected matroid M is *unbreakable* if M/F is connected for all flats F of M . Pfeil [28] characterized the graphs G for which $M(G)$ is unbreakable. In Chapter 3, we consider for which graphs G there is a set of balanced cycles Ψ for which $M(G, \Psi)$ is unbreakable.

1.2. Nested matroids

In this section, we introduce a class of matroids that is of particular importance in motivating the material done in both Chapters 2 and 3. We also introduce some related classes of matroids. For a positive integer n , let $[n] = \{1, 2, 3, \dots, n\}$. For any integer r

with $0 \leq r \leq n$, the *uniform matroid*, $U_{r,n}$, of rank r on $[n]$ is the matroid where $I \subseteq [n]$ is independent when $|I| \leq r$. Uniform matroids are not graphic unless $r \in \{0, 1, n-1, n\}$.

Let $\mathcal{J} = (J_1, J_2, \dots, J_n)$ be a collection of, not necessarily distinct, subsets of E . A *transversal* of \mathcal{J} is a set $\{e_1, e_2, \dots, e_n\}$, where $e_i \in J_i$ for each $i \in [n]$, and the e_i 's are distinct. (See also [25, Section 1.6].) If X is a transversal of \mathcal{J} , then we also say X is a transversal of $(J_i : i \in [n])$. A subset X of E is a *partial transversal* of $(J_i : i \in [n])$ if it is a transversal of $(J_i : i \in K)$ for some $K \subseteq [n]$. Let \mathcal{I} be the collection of partial transversals of \mathcal{J} . Then \mathcal{I} is the collection of independent sets of a matroid, $M = M[\mathcal{J}]$. Such a matroid is called a *transversal matroid*, and \mathcal{J} is a *presentation* of M . Uniform matroids are transversal. Graphic matroids need not be transversal. The class of transversal matroids is not minor-closed. The smallest minor-closed class of matroids that contains the class of transversals is the class of *gammoids* [24]. We omit a more direct definition of gammoids. All gammoids, and hence all transversal and uniform matroids, are representable over all sufficiently large fields.

A transversal matroid is *nested* if it has a nested presentation, that is, a transversal presentation (B_1, B_2, \dots, B_n) such that $B_1 \subseteq B_2 \subseteq \dots \subseteq B_n$. These matroids were introduced by Crapo [8] and have appeared under a variety of names including freedom matroids [9], generalized Catalan matroids [4], shifted matroids [1], and Schubert matroids [34] (see [3]).

Oxley, Prendergast, and Row [26] showed that the class of nested matroids is minor-closed and they determined the excluded minors for this class. Let M be a matroid of rank $r \geq 1$. The *truncation* $T(M)$ of M is the matroid whose collection of independent sets is $\{I \in \mathcal{I}(M) : |I| \leq r-1\}$. The rank of $T(M)$ is one less than $r(M)$. For

$0 \leq k \leq r - 1$, we define $T_k(M)$, the *truncation to rank k* of M , as the matroid obtained by repeatedly truncating M until the resulting matroid has rank k .

Theorem 1.2.1. *A matroid is nested if and only if, for all $r \geq 2$, it has no minor isomorphic to the matroid that is obtained by truncating, to rank r , the direct sum of two r -element circuits.*

Let M be a matroid with $e \notin E(M)$. An *extension* of M is any matroid M' on $E(M) \cup e$ such that $M' \setminus e = M$. The *free extension* of M is $T(M \oplus U_{1,1})$. Crapo [8] showed that nested matroids coincide with the class of matroids that can be obtained from the empty matroid by applying the operations of adding a coloop and taking a free extension (see also [4, Theorem 3.14]). A straightforward modification of this result yields the following characterization of nested matroids.

Theorem 1.2.2. *The class of nested matroids coincides with the class of matroids that can be obtained from the empty matroid by adding coloops and truncating.*

In Chapter 2, we give structural results on a class of matroids which is strikingly similar to nested matroids, both in terms of its excluded minors and in terms of the construction given above. In Chapter 3, we discuss further generalizations of these classes.

1.3. Matroid operations and constructions

In this section, we describe some additional operations and constructions on matroids that are of importance to this work.

Duality is of particular significance to matroid theory and graph theory (see [25, Chapter 2]). For a matroid M , we define the *dual* M^* of M as the matroid whose collection of bases is $\{E(M) - B : B \in \mathcal{B}(M)\}$. A *planar embedding* of a graph G is a drawing

of G in the plane such that vertices correspond to distinct points in the plane, edges correspond to simple curves that connect their endpoints but meet no other vertices, and each point of intersection of two such curves is an end of both edges. If G has a planar embedding, then it is *planar*. A *plane graph* is a planar graph with a planar embedding. For the definition of the dual of a plane graph, see [Section 5.2][25]. We note that if G is a plane graph, then $(M(G))^* = M(G^*)$, where G^* is the dual of G . For a connected plane graph, $G = (G^*)^*$, and, for all matroids, $M = (M^*)^*$. For any graph G , the matroid $(M(G))^*$ is denoted $M^*(G)$. It is called the *bond matroid* of G . A matroid is *cographic* if it is isomorphic to the bond matroid of some graph.

The regular matroid R_{10} , and the matroid R_{12} are the vector matroids of the following matrices, A_{10} and A_{12} , respectively, over $GF(2)$.

$$A_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Seymour's Decomposition Theorem (see, for example, [Section 13.1][25]) implies that every 3-connected regular matroid is either graphic or cographic or has an R_{10} or R_{12} -minor.

Let r be the rank function of M . Then the *corank* r_M^* of M is the rank function of M^* . We get

$$r^*(X) = r(E - X) + |X| - r(M).$$

This tells us that $\lambda_M(X) = r(X) + r^*(X) - |X|$ and that $(X, E(M) - X)$ is k -separating in M^* exactly when it is k -separating in M . Thus, M^* is n -connected exactly when M is n -connected.

Now let M_1 and M_2 be arbitrary matroids with $p_1 \in E(M_1)$ and $p_2 \in E(M_2)$ where p_i is not a loop or a coloop of M_i , for $i \in \{1, 2\}$. Assume that $E(M_1)$, $E(M_2)$, and $\{p\}$ are disjoint sets. Let \mathcal{C}_P be defined as follows.

1.3.1.

$$\begin{aligned} \mathcal{C}_P &= \mathcal{C}(M_1 \setminus p_1) \cup \{(C_1 - p_1) \cup p : p_1 \in C_1 \in \mathcal{C}(M_1)\} \\ &\cup \mathcal{C}(M_2 \setminus p_2) \cup \{(C_2 - p_2) \cup p : p_2 \in C_2 \in \mathcal{C}(M_2)\} \\ &\cup \{(C_1 - p_1) \cup (C_2 - p_2) : p_i \in C_i \in \mathcal{C}(M_i) \text{ for each } i\}. \end{aligned}$$

Then \mathcal{C}_P is the collection of circuits of a matroid, namely, the *parallel connection*

$P(M_1, M_2) = P((M_1, p_1), (M_2, p_2))$ of M_1 and M_2 with respect to the *basepoints* p_1

and p_2 . Often we view $P(M_1, M_2)$ as being formed from two matroids M_1 and M_2 with

$\{p\} = E(M_1) \cap E(M_2)$. In this context, p is the *basepoint* of the connection, and when p is

not a loop or a coloop in either M_1 or M_2 , we have $\mathcal{C}(P(M_1, M_2))$ is defined as in 1.3.1. If

p is a loop in M_1 , then

$$P(M_1, M_2) = P(M_2, M_1) = M_1 \oplus (M_2/p).$$

If P is a coloop in M_1 , then

$$P(M_1, M_2) = P(M_2, M_1) = (M_1 \setminus p) \oplus M_2.$$

Suppose that M and N are matroids, each with at least two elements, and that $\{p\} = E(M) \cap E(N)$, where p is not a loop or a coloop in either matroid. Then the *2-sum* $M \oplus_2 N$ is $P(M, N) \setminus p$. The element p is the *basepoint* of the 2-sum, and M and N are the *parts* of the 2-sum. Equivalently, $M \oplus_2 N$ is the matroid on $(E(M) \cup E(N)) - p$ with circuits

$$\mathcal{C}(M \setminus p) \cup \mathcal{C}(N \setminus p) \cup \{(C \cup D) - p : p \in C \in \mathcal{C}(M) \text{ and } p \in D \in \mathcal{C}(N)\}.$$

Recall that a matroid is disconnected if it can be written as a direct sum of smaller matroids. The following analog was proved independently by Bixby, Cunningham, and Seymour, see [25, Section 8.3].

Theorem 1.3.2. *A 2-connected matroid M is not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2 , each of which has at least three elements and is isomorphic to a proper minor of M .*

This tells us that every matroid that is not 3-connected can be constructed from 3-connected proper minors of itself by a sequence of the operations of direct sum and 2-sum. A *matroid-labeled tree* is a tree T with vertex set $\{M_1, M_2, \dots, M_k\}$, for some positive integer k , such that

- (i) each M_i is a matroid;
- (ii) if M_{j_1} and M_{j_2} are joined by an edge e_i of T , then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$, and e_i is not a loop or a coloop of either M_{j_1} or M_{j_2} ; and
- (iii) if M_{j_1} and M_{j_2} are non-adjacent, then $E(M_{j_1}) \cap E(M_{j_2}) = \emptyset$.

Let e be an edge of a matroid-labeled tree T , joining N_1 and N_2 . We denote by T/e the matroid-labeled tree formed by contracting the edge e and relabeling the resulting vertex $N_1 \oplus_2 N_2$. Since the 2-sum is associative, for any $S \subseteq E(T)$, the matroid-labeled tree T/S is well-defined. A *tree decomposition* of a 2-connected matroid M is a matroid-labeled tree T such that if $V(T) = \{M_1, M_2, \dots, M_k\}$ and $E(T) = \{e_1, e_2, \dots, e_{k-1}\}$, then

$$(i) \quad E(M) = (E(M_1) \cup E(M_2) \cup \dots \cup E(M_k)) \cup \{e_1, e_2, \dots, e_{k-1}\};$$

$$(ii) \quad |E(M_i)| \geq 3 \text{ for all } i \text{ unless } |E(M)| < 3, \text{ in which case } k = 1 \text{ and } M_1 = M;$$

and

$$(iii) \quad M \text{ is the matroid that labels the single vertex } T/\{e_1, e_2, \dots, e_k\}.$$

We will use tree decompositions in Chapter 2 to characterize a class of matroids.

Let M be a matroid. A *hyperplane* of M is a flat H of M with $r(H) = r(M) - 1$.

1. If H is both a circuit and a hyperplane of M , then H is a *circuit-hyperplane*. If H is a circuit-hyperplane of M , then

$$(\mathcal{C}(M) - \{H\}) \cup \{H \cup e : e \in E(M) - H\}$$

is the collection of circuits of a matroid. We say that the latter matroid is obtained from M by *relaxing* the circuit-hyperplane H . Equivalently, for a circuit-hyperplane H of M , the relaxation of H is the matroid whose set of bases is $\mathcal{B}(M) \cup H$. We will consider one example here. Let $r \geq 2$ be an integer. Let W_r be the simple graph consisting of a cycle C of size r , together with a vertex v , and edges from v to each vertex in C . Then W_r is called a *wheel graph*. The matroid $\mathcal{W}_r = M(W_r)$ is a *wheel*. The cycle C is both a circuit and a hyperplane of \mathcal{W}_r . The matroid obtained from \mathcal{W}_r by relaxing C is denoted \mathcal{W}^r and

is called a *whirl*. In particular, \mathcal{W}^2 is the matroid $U_{2,4}$. Tutte's Wheels-and-Whirls Theorem (see, for instance, [25, Theorem 8.8.4]) states the following.

Theorem 1.3.3. *Let M be a 3-connected matroid M having at least one element. Then M has no element e for which $M \setminus e$ or M/e is 3-connected if and only if M has rank at least three and is isomorphic to a wheel or a whirl*

Chapter 2. Laminar Matroids

2.1. Introduction to laminar matroids

The results in this in this chapter are based on joint work with James Oxley [14].

Given a set E , a family \mathcal{A} of subsets of E is *laminar* if, for every two sets A and B in \mathcal{A} with $A \cap B \neq \emptyset$, either $A \subseteq B$ or $B \subseteq A$. Let \mathcal{A} be a laminar family of subsets of a finite set E . Let c be a function from \mathcal{A} into the set of real numbers. Define \mathcal{I} to be the set of subsets I of E such that $|I \cap A| \leq c(A)$ for all A in \mathcal{A} . It is well known (see, for example, [16, 17, 19, 22]) and easily checked that \mathcal{I} is the set of independent sets of a matroid on E . However, we include the proof for completeness, as one does not seem to appear in print.

Theorem 2.1.1. *Let \mathcal{A} be a laminar family of subsets of a finite set E , and let c be a function from \mathcal{A} into the set of real numbers. Then $\{I \subseteq E : |I \cap A| \leq c(A) \text{ for all } A \in \mathcal{A}\}$ is the collection of independent sets of a matroid.*

Proof. We show that $\mathcal{I} = \{I \subseteq E : |I \cap A| \leq c(A) \text{ for all } A \in \mathcal{A}\}$ satisfies (I1) – (I3). It is clear that $\emptyset \in \mathcal{I}$, and that if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$. We show (I3) using induction on $|\mathcal{A}|$. If $|\mathcal{A}| = 0$, then \mathcal{I} is the collection of independent sets of $U_{n,n}$, where $n = |E|$, so (I3) holds. Now, suppose that $k = |\mathcal{A}| \geq 1$ and that for all laminar families of size less than k , (I3) holds. Choose I_1 and I_2 in \mathcal{I} with $|I_2| > |I_1|$. Let \mathcal{P} be the maximal elements of \mathcal{A} . Since \mathcal{A} is laminar, $\mathcal{P} \cup \{f \in E : f \notin A \text{ for any } A \in \mathcal{P}\}$ partitions E . Now, since $|I_2| \geq |I_1|$, either there is an $e \in I_2 - I_1$ with $e \in \{f \in E : f \notin A \text{ for any } A \in \mathcal{P}\}$, or there is a $P \in \mathcal{P}$ with $|I_2 \cap P| > |I_1 \cap P|$. In the former case, $(I_1 \cup e) \in \mathcal{I}$, as $(I_1 \cup e) \cap A = (I_1 \cap A)$ for all $A \in \mathcal{A}$. In the latter, let $\mathcal{A}' = \mathcal{A} - P$, and c' be the restriction of c to the domain

This chapter is adapted from: Tara Fife, and James Oxley, Laminar matroids, *European J. Combin.* **62** (2017), 206–216. It is reprinted with permission from The European Journal of Combinatorics

\mathcal{A}' . Then, for $i \in \{1, 2\}$ and for all $A \in \mathcal{A}'$, we have $|I_i \cap A| \leq c(A) = c'(A)$. So by our induction step, there is an $e \in I_2 - I_1$ with $|(I_1 \cup e) \cap A| \leq c'(A) = c(A)$ for all $A \in \mathcal{A}'$. Since $|I_1 \cap P| < |I_2 \cap P| \leq c(P)$, we have $|(I_1 \cup e) \cap P| \leq c(P)$. Thus (I3) holds for \mathcal{A} , and (E, \mathcal{I}) is indeed a matroid. \square

We call c a *capacity function* for the matroid (E, \mathcal{I}) and write this matroid as $M(E, \mathcal{A}, c)$. A matroid M is *laminar* if it is isomorphic to $M(E, \mathcal{A}, c)$ for some set E , laminar family \mathcal{A} , and capacity function c . We call (E, \mathcal{A}, c) a *presentation* for M .

Laminar matroids have appeared quite frequently in the literature during the last fifteen years. Interest in them has focused on how certain optimization problems, particularly the matroid secretary problem, behave for such matroids [2, 6, 12, 19, 21, 35]. Huynh [18] gave an overview of this work. With the exception of the thesis of Finkelstein [16], where it is shown, for example, that every laminar matroid is a gammoid, there appears to have been little work done on exploring the matroid properties of the class of laminar matroids. Here we do just that. In particular, we give three characterizations of this class of matroids beginning with the following.

Theorem 2.1.2. *A matroid is laminar if and only if, for all circuits C_1 and C_2 with $C_1 \cap C_2 \neq \emptyset$, either $cl(C_1) \subseteq cl(C_2)$, or $cl(C_2) \subseteq cl(C_1)$.*

As we shall see, it is not difficult to show that the class of laminar matroids is minor-closed. For each $r \geq 3$, let Y_r be the matroid that is obtained by truncating, to rank r , the parallel connection of two r -element circuits. Observe that the deletion from Y_r of the basepoint of the parallel connection is isomorphic to $U_{r, 2r-2}$. From the last result, Y_r is

not laminar. Indeed, the collection of such matroids is the set of excluded minors for the class of laminar matroids.

Theorem 2.1.3. *A matroid is laminar if and only if it has no minor isomorphic to any member of $\{Y_r : r \geq 3\}$.*

As the reader will observe, these excluded minors are strikingly similar to the excluded minors for the class of nested matroids. In particular, each excluded minor for nested matroids can be obtained from an excluded minor for laminar matroids by the contraction of a single element. This shows us that all nested matroids are laminar, a fact that we shall show directly.

Our third characterization of the class of laminar matroids is a constructive one that reveals how nested matroids and laminar matroids differ.

Theorem 2.1.4. *The class of laminar matroids coincides with the class of matroids that can be constructed by beginning with the empty matroid and using the following operations.*

- (i) *Adding a coloop to a previously constructed matroid.*
- (ii) *Truncating a previously constructed matroid.*
- (iii) *Taking the direct sum of two previously constructed matroids.*

In the next section, we prove Theorem 2.1.2 and show that every laminar matroid has a unique presentation with no superfluous information. In Section 2.3, we prove Theorem 2.1.3, while, in Section 2.4, we prove Theorem 2.1.4 and determine all of the laminar matroids whose duals are also laminar. Finkelstein [16] showed that all laminar matroids are gammoids. Hence, by a result of [29], all laminar matroids are representable over all sufficiently large fields. In Section 2.5, we characterize binary laminar matroids and ternary laminar matroids.

2.2. Canonical presentation

In this section, we obtain a presentation for a laminar matroid that has no redundant information. It is clear that, for a capacity function c of a laminar matroid, we lose no generality in assuming that the range of c is the set of non-negative integers. The following lemma is an immediate consequence of the definition of laminar matroids.

Lemma 2.2.1. *If I is independent in $M(E, \mathcal{A}, c)$ and $A \in \mathcal{A}$, then I is independent in $M(E, \mathcal{A} - \{A\}, c|_{\mathcal{A} - \{A\}})$.*

Throughout this section, we shall assume that M is the laminar matroid $M(E, \mathcal{A}, c)$. Here and throughout this chapter, whenever we write $c(A)$, it will be implicit that $A \in \mathcal{A}$. We say that a set $A \in \mathcal{A}$ is *essential* if $M(E, \mathcal{A}, c) \neq M(E, \mathcal{A} - \{A\}, c|_{\mathcal{A} - \{A\}})$. When M has no loops, we say that (E, \mathcal{A}, c) is a *canonical presentation* for M if every $A \in \mathcal{A}$ is essential. When M has a loop, we say that a presentation of M is *canonical* if it can be written as $(E, \mathcal{A} \cup \{A_0\}, c)$, where $A_0 = cl(\emptyset)$ and $(E - A_0, \mathcal{A}, c|_{\mathcal{A}})$ is a canonical presentation of $M \setminus A_0$.

We omit the proof of the following well-known observation (see, for example, [22, Section 2.4]).

Lemma 2.2.2. *Suppose A and B are members of \mathcal{A} such that $B \subsetneq A$ and $c(B) \geq c(A)$. Then B is not essential.*

Let A and H be members of a laminar family \mathcal{A} and suppose that $A \subsetneq H$. If there is no $G \in \mathcal{A}$ such that $A \subsetneq G \subsetneq H$, then we say that A is a *child* of H . For A in \mathcal{A} , denote by $\chi(A)$ the set of children of A , and let $S(A) = \{e : e \in A - \cup_{F \in \chi(A)} F\}$. Observe that, in $M|A$, either all of the elements of $S(A)$ are coloops or all such elements are free.

We define $b(A) = |S(A)| + \sum_{F \in \chi(A)} c(F)$. When A is essential, we now bound the capacity of A in terms of $b(A)$.

Lemma 2.2.3. *If $c(A) \geq b(A)$, then A is not essential.*

Proof. Let I be independent in $M(E, \mathcal{A} - \{A\}, c|_{\mathcal{A} - \{A\}})$. Then $|I \cap F| \leq c(F)$ for all $F \in \chi(A)$, and $|I \cap S(A)| \leq |S(A)|$. Since the set $S(A)$ together with children of A partitions A , we see that

$$|I \cap A| = |I \cap S(A)| + \sum_{F \in \chi(A)} |I \cap F| \leq |S(A)| + \sum_{F \in \chi(A)} c(F) = b(A).$$

Hence A is not essential. □

The last lemma generalizes the following elementary fact about canonical presentations.

Corollary 2.2.4. *If (E, \mathcal{A}, c) is a canonical presentation for M , then $|A| > c(A)$ for all A in \mathcal{A} .*

Proof. For $A \in \mathcal{A}$, let $d(A) = |A' \in \mathcal{A} : A' \subseteq A|$. If $d(A) = 1$, then A has no children, and, by 2.2.3, we have $|A| > c(A)$. Now suppose that, for all $A' \in \mathcal{A}$ with $d(A') \leq k - 1$, we have $|A'| > c(A')$, and that $d(A) = k$. Then, by 2.2.3, we have $|A| > c(A)$, so the statement holds by induction. □

With the goal of showing the uniqueness of canonical presentations, next we exhibit some relationships between circuits and canonical presentations. In particular, the next lemma will show that if M has no loops, then $cl(C) \in \mathcal{A}$ for each circuit C and $c(cl(C)) = |C| - 1$.

Lemma 2.2.5. *Let C be a circuit of M . Assume that (E, \mathcal{A}, c) is canonical. Then*

(i) \mathcal{A} contains a member A_C of capacity $|C| - 1$ such that $C \subseteq A_C$; and

(ii) if $|C| \geq 2$, then $A_C = cl(C) - cl(\emptyset)$.

Proof. Part (i) holds if $|C| = 1$. Assume that $|C| \geq 2$, and that $e \in C$. Then, since C is dependent, but $C - e$ is independent, we must have $e \in A$ for some $A \in \mathcal{A}$ where $|(C - e) \cap A| \leq c(A)$, but $|C \cap A| > c(A)$. Then $c(A) = |(C - e) \cap A|$. Now, $C \cap A$ is dependent, since $|C \cap A| > c(A)$. Thus $C \cap A = C$, so $C \subseteq A$ and $c(A) = |C| - 1$. Hence (i) holds.

To prove (ii), assume that $|C| \geq 2$. Let f be an element of $cl(C) - cl(\emptyset)$. By (i), $C \subseteq A_C$. Suppose $f \in cl(C) - C$. Then there is some circuit D with $f \in D \subseteq C \cup f$. Then, by (i), $D \subseteq A_D \in \mathcal{A}$ and $|D| - 1 = c(A_D)$. Since f is not a loop, $C \cap D$, and hence $A_C \cap A_D$, is non-empty. As \mathcal{A} is a laminar family, this implies that $A_C \subseteq A_D$, or $A_D \subseteq A_C$. But, since $c(A_D) = |D| - 1 \leq |C| - 1 = c(A_C)$, we deduce, from Lemma 2.2.2, that $A_D \subseteq A_C$. Thus $f \in A_C$ as desired. Hence $cl(C) - cl(\emptyset) \subseteq A_C$.

Now, suppose that $f \in A_C - C$. Since $f \in A_C$, by the definition of a canonical presentation, $f \notin cl(\emptyset)$. Arbitrarily choose an element e of C . Then, since $|(C - e) \cup f| = |C| > c(A_C)$ and $(C - e) \cup f \subseteq A_C$, we have that $(C - e) \cup f$ is dependent, so $f \in cl(C)$. \square

This lemma has the following consequence.

Corollary 2.2.6. *If C and D are intersecting circuits of an arbitrary matroid N such that $cl(C) \not\subseteq cl(D)$ and $cl(D) \not\subseteq cl(C)$, then N is not laminar.*

Proof. Neither C nor D is a loop because loops are in the closure of all sets. Assume that N is laminar and let (E, \mathcal{A}, c) be a canonical presentation of N . Since C meets D , we deduce that A_C meets A_D . But neither is a subset of the other. \square

Theorem 2.2.7. *A laminar matroid M has a unique canonical presentation. Indeed, when M is loopless, $\mathcal{A} = \{cl(C) : C \text{ is a circuit of } M\}$ and $c(cl(C)) = r(C) = |C| - 1$.*

The core of the proof of this theorem is contained in the next result.

Lemma 2.2.8. *Let (E, \mathcal{A}, c) be a canonical presentation for a loopless laminar matroid M . If $A \in \mathcal{A}$, then A is dependent. Moreover, if C is a maximum-sized circuit contained in A , then $A_C = A$ so $c(A) = |C| - 1$.*

Proof. By Corollary 2.2.4, $c(A) < |A|$. Thus A is dependent. Now choose A to be a minimal counterexample to Lemma 2.2.8. As $C \subseteq A \cap A_C$, either $A \subsetneq A_C$ or $A_C \subsetneq A$. In the first case, by Lemma 2.2.2, $c(A) \leq c(A_C) - 1 = |C| - 2$. Hence A cannot contain an independent set of size $|C| - 1$. This is a contradiction since $C \subseteq A$. Thus $A_C \subsetneq A$. Now A has a child A' such that $A_C \subseteq A' \subsetneq A$. The choice of A implies that $A' = A_C$.

Let A_1, A_2, \dots, A_n be the children of A other than A_C and write A_0 for A_C . Then, for each i , our choice of A means that $c(A_i) = |C_i| - 1$, where C_i is a maximum-sized circuit contained in A_i . Arbitrarily choose e_i in C_i . Then $C_i - e_i$ is a basis for A_i . Clearly $S(A) = A - (A_0 \cup A_1 \cup \dots \cup A_n)$.

By Lemma 2.2.3, $|S(A)| + \sum_{i=0}^n c(A_i) = b(A) \geq c(A) + 1$. Now $|\bigcup_{i=0}^n (C_i - e_i) \cup S(A)| = b(A)$ so $\bigcup_{i=0}^n (C_i - e_i) \cup S(A)$ contains a subset X such that $|X| = c(A) + 1$. As $|X \cap A| = |X| > c(A)$, we see that X is dependent. Thus X contains a circuit Z and $cl(Z) = A_Z$. Then $c(A_Z) = |Z| - 1 \leq |X| - 1 = c(A)$. As A_Z and A meet, it follows

by Lemma 2.2.2 that $A_Z \subseteq A$. Now $Z \not\subseteq S(A)$, otherwise A_Z is a proper subset of A that is in \mathcal{A} but is not contained in a child of A . Thus either Z meets C_i and C_j for some distinct i and j , or Z meets C_i and $S(A)$. In each case, by Corollary 2.2.6, $cl(C_i) \subseteq cl(Z)$, or $cl(Z) \subseteq cl(C_i)$. If Z meets $S(A)$, then $Z \not\subseteq cl(C_i)$ so $cl(C_i) \subsetneq cl(Z)$. The last inclusion also holds if Z meets C_j since $cl(C_i)$ and $cl(C_j)$ are disjoint. As $cl(C_i)$ is a child of A , and $cl(C_i) \subseteq cl(Z) \subsetneq A$, it follows that $cl(Z) = cl(C_i)$, a contradiction. \square

Proof of Theorem 2.2.7. It suffices to prove the result for loopless matroids. Suppose that (E, \mathcal{A}, c) is a canonical presentation for M , and let M be loopless. Then $\{A_C : C \text{ is a circuit of } M\} \subseteq \mathcal{A}$. Now take A in \mathcal{A} . Then, by Lemma 2.2.8, $A = A_C$ where C is a maximum-sized circuit contained in A . Thus the theorem holds. \square

We omit the proof of the following elementary result (see, for example, [25, Exercise 1.1.5]).

Lemma 2.2.9. *Let N be a matroid, C be a circuit of N , and e be a non-loop element of $cl(C) - C$. Then N has circuits D and D' such that $e \in D \cap D'$ and $(D \cup D') - e = C$.*

Next we prove our first main result.

Proof of Theorem 2.1.2. By Corollary 2.2.6, if C and D are intersecting circuits in a laminar matroid, then $cl(D) \subseteq cl(C)$ or $cl(C) \subseteq cl(D)$. To prove the converse, let N be a matroid in which, for every two intersecting circuits, the closure of one is contained in the closure of the other. We may assume that N is loopless. Let $\mathcal{A}' = \{cl(C) : C \in \mathcal{C}(N)\}$.

Suppose $A_1, A_2 \in \mathcal{A}'$ and $A_1 \cap A_2 \neq \emptyset$. Let C_1 and C_2 be circuits so that $cl(C_i) = A_i$ for each i in $\{1, 2\}$. If $C_1 \cap C_2 \neq \emptyset$, then, by the given condition, $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.

Now suppose that $C_1 \cap C_2 = \emptyset$ and $e \in cl(C_1) \cap cl(C_2)$. Since e is not a loop, Lemma 2.2.9

implies that, for each i in $\{1, 2\}$, there are circuits D_i and D'_i of M such that $e \in D_i \cap D'_i$ and $D_i \cup D'_i = C_i \cup e$. Now $e \in D_1 \cap D_2$ so our hypothesis implies, without loss of generality, that $cl(D_1) \subseteq cl(D_2)$.

If $cl(D'_1)$ is contained in either $cl(D_2)$ or $cl(D'_2)$, then C_1 and hence $cl(C_1)$ is contained in $cl(C_2)$. But otherwise, both $cl(D_2)$ and $cl(D'_2)$ are subsets of $cl(D'_1)$, so $cl(C_2) \subseteq cl(C_1)$. We conclude that \mathcal{A}' is a laminar family.

For each A in \mathcal{A}' , let $c'(A) = r_N(A)$, and let $N' = M(E, \mathcal{A}', c')$. We shall show that every circuit of N is dependent in N' , and every circuit of N' is dependent in N . From this, it will follow immediately that $N = N'$ (see, for example, [25, Lemma 2.1.22]). Suppose C is a circuit of N . Then $|C \cap cl(C)| = |C| > r_N(C) = c'(cl(C))$, so C is dependent in N' . Now let D be a circuit of N' . Then \mathcal{A}' contains a set A' such that $c'(A') < |D \cap A'|$. But, for all d in D , as $D - d$ is independent in N' , it follows that $|(D - d) \cap A'| \leq c'(A')$. Hence $D \subseteq A'$ and $c'(A') < |D|$. But $c'(A') = r_N(cl(C'))$ for some circuit C' of N . Thus D is dependent in N otherwise $c'(A') \geq |D|$, a contradiction.

We conclude that $N = N'$, so N is laminar and the theorem holds. \square

The next two results are immediate consequences of Theorem 2.1.2.

Corollary 2.2.10. *A matroid is laminar if and only if, for every pair C_1, C_2 of non-spanning circuits with $C_1 \cap C_2 \neq \emptyset$, either $cl(C_1) \subseteq cl(C_2)$ or $cl(C_2) \subseteq cl(C_1)$.*

Corollary 2.2.11. *Every matroid with at most one non-spanning circuit is laminar.*

Our third corollary of Theorem 2.1.2 requires some more proof.

Corollary 2.2.12. *Let M be a loopless laminar matroid and (E, \mathcal{A}, c) be its canonical presentation. Suppose $|E| \geq 2$. Then the following are equivalent.*

- (i) M is connected;
- (ii) $E \in \mathcal{A}$;
- (iii) M has a spanning circuit; and
- (iv) M has a spanning circuit using e for each $e \in M$.

Proof. Since M is loopless, clearly (iv) implies (iii), which implies (i). Suppose $E \in \mathcal{A}$. Then, by Lemmas 2.2.5 and 2.2.8, as M is loopless, $E = A_C = cl(C)$ where C is a maximum-sized circuit of M . Hence (ii) implies (iii).

Now, suppose M is connected but $E \notin \mathcal{A}$. If M has an element e that is in no member of \mathcal{A} , then e is a coloop of M , a contradiction. Thus, if F_1, F_2, \dots, F_k are the maximal members of \mathcal{A} , then $F_1 \cup F_2 \cup \dots \cup F_k = E$. We shall show that $k = 1$. Assume $k > 1$. For each i in $\{1, 2, \dots, k\}$, let C_i be a maximum-sized circuit contained in F_i . Then, by Lemmas 2.2.5 and 2.2.8, $F_i = cl(C_i)$. As M is connected, it has a circuit D meeting C_1 and C_2 . By Corollary 2.2.6, either $cl(D) \subseteq cl(C_i)$ for some i in $\{1, 2\}$, or $cl(D)$ contains both F_1 and F_2 . Since F_1 and F_2 are disjoint, the latter holds. By Theorem 2.2.7, $A_D \in \mathcal{A}$, and A_D contains F_i and F_2 , a contradiction. Hence $E \in \mathcal{A}$. We conclude that (i) implies (ii).

Finally, suppose that C is a spanning circuit of M , and that $e \notin C$. Then $E \in \mathcal{A}$. If $e \notin A$ for all $A \in (\mathcal{A} - \{E\})$, then $C \triangle \{e, f\}$ is clearly a circuit of M for all $f \in C$. Assume that $e \in A'$, for some child A' of A . As $r(A') < r(C)$, there is some $f \in C - A'$. Then $C \triangle \{e, f\}$ is a circuit. We conclude that (iii) implies (iv), so the corollary holds. \square

The next result follows immediately from the last result and Theorem 2.2.7.

Corollary 2.2.13. *Let M be a loopless laminar matroid and (E, \mathcal{A}, c) be its canonical presentation. Then the members of \mathcal{A} are connected flats of M .*

Corollary 2.2.14. *In a laminar matroid with (E, \mathcal{A}, c) as its canonical presentation, if F is a connected flat of M with $|F| \geq 2$, then $F \in \mathcal{A}$.*

Proof. Since F is a connected flat of M , by Corollary 2.2.12 $M|F$ has a spanning circuit C in $M|F$. But C is also a circuit of M , so $F = cl(C) \in \mathcal{A}$. \square

2.3. Excluded minors

In this section, we show that the class of laminar matroids is a minor-closed, and we prove our excluded-minor characterization.

Lemma 2.3.1. *Every minor of a laminar matroid is laminar.*

Proof. Let M be a laminar matroid and (E, \mathcal{A}, c) be its canonical presentation. Suppose $e \in E$. Clearly $\{A - e : A \in \mathcal{A}\}$ is a laminar family; we denote it by $\mathcal{A} - e$. Observe that, if A and A' are members of \mathcal{A} with $A \subsetneq A'$, then $|A' - A| \geq 2$. To see this, note that, by Lemmas 2.2.2 and 2.2.3 and Corollary 2.2.4, $c(A) + 2 \leq c(A') + 1 \leq b(A') \leq c(A) + |A' - A|$.

For each $A' \in \mathcal{A} - e$, choose the unique $A \in \mathcal{A}$ with $A - e = A'$, and let $c'(A') = c(A)$.

We shall show that

2.3.1.1. $M \setminus e = M(E - e, \mathcal{A} - e, c')$.

Suppose that I is independent in $M \setminus e$. Then $|I \cap A| \leq c(A)$ for all A in \mathcal{A} . As $e \notin I$, it follows that $|I \cap (A - e)| \leq c'(A - e)$ for all $A - e$ in $\mathcal{A} - e$. Thus I is independent in $M(E - e, \mathcal{A} - e, c')$.

Now, suppose that J is independent in $M(E - e, \mathcal{A} - e, c')$. Then $|J \cap (A - e)| \leq c'(A - e)$ for all $(A - e) \in (\mathcal{A} - e)$. Now, for each $A \in \mathcal{A}$, we have $|J \cap A| = |J \cap (A - e)| \leq c'(A - e) = c(A)$, so J is independent in M and, hence, is independent in $M \setminus e$. We conclude that 2.3.1.1 holds.

To show that M/e is laminar, we may assume that e is not a loop as otherwise the result holds by 2.3.1.1. Now, define c'' on $\mathcal{A} - e$ by

$$c''(A - e) = \begin{cases} c(A) - 1 & \text{if } e \in A; \\ c(A) & \text{if } e \notin A. \end{cases}$$

We will show that

2.3.1.2. $M/e = M(E - e, \mathcal{A} - e, c'')$.

Suppose that I is independent in M/e . Then $I \cup e$ is independent in M . Thus, $|(I \cup e) \cap A| \leq c(A)$ for all $A \in \mathcal{A}$. Now

$$\begin{aligned} |I \cap (A - e)| &= |(I \cup e) \cap (A - e)| = \begin{cases} |(I \cup e) \cap A| - 1 & \text{if } e \in A; \\ |(I \cup e) \cap A| & \text{if } e \notin A; \end{cases} \\ &\leq \begin{cases} c(A) - 1 & \text{if } e \in A; \\ c(A) & \text{if } e \notin A; \end{cases} \\ &= c''(A - e). \end{aligned}$$

Thus I is independent in $M(E - e, \mathcal{A}', c'')$.

Now suppose that J is independent in $M(E - e, \mathcal{A} - e, c'')$. Then $|J \cap A'| \leq c''(A')$

for all $A' \in \mathcal{A}'$. Let $A \in \mathcal{A}$ be such that $A' = A - e$. Then

$$\begin{aligned} |(J \cup e) \cap A| &= \begin{cases} |J \cap (A - e)| + 1 & \text{if } e \in A; \\ |J \cap (A - e)| & \text{if } e \notin A; \end{cases} \\ &\leq \begin{cases} c''(A - e) + 1 & \text{if } e \in A; \\ c''(A - e) & \text{if } e \notin A; \end{cases} \\ &= c(A). \end{aligned}$$

We conclude that $J \cup e$ is independent in M , so J is independent in M/e . Thus 2.3.1.2

holds and, hence, so does the theorem. \square

We note that, from the well-known matroid P_6 , one can see that the presentations in 2.3.1.1 and 2.3.1.2 need not be canonical. In particular, the canonical presentation of P_6 , with the labeling in Figure 2.1, is $\mathcal{A} = \{\{a, b, c\}, \{a, b, c, d, e, f\}\}$, with $c(\{a, b, c\}) = 2$ and $c(\{a, b, c, d, e, f\}) = 3$. From 2.3.1.1, we get the presentation $\{\{b, c\}, \{b, c, d, e, f\}\}$, with $c(\{b, c\}) = 2$ and $c(\{b, c, d, e, f\}) = 3$ for $P_6 \setminus a$. We, get from 2.3.1.2, the presentation $\{\{a, b, c\}, \{a, b, c, e, f\}\}$, with $c(\{a, b, c\}) = 2 = c(\{a, b, c, e, f\})$ for P_6/d . Neither of these presentations is canonical.

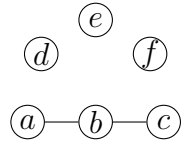


Figure 2.1. A geometric presentation for P_6

We now prove our second main result.

Proof of Theorem 2.1.3. Recall that, for each $r \geq 3$, the matroid Y_r is obtained from the parallel connection of two r -element circuits C_1 and C_2 across the basepoint p by truncating this parallel connection to rank r . Since the only new circuits created by truncation are spanning, it follows that C_1 and C_2 are the only non-spanning circuits of Y_r .

2.3.2.1. Y_r is an excluded minor for the class of laminar matroids for all $r \geq 3$.

To see this, first observe that, by Corollary 2.2.10, Y_r is not laminar. Let $e \in E$. Without loss of generality, we may assume that $e \in C_1$. The only potential non-spanning circuit of $M \setminus e$ is C_2 . Thus $M \setminus e$ is laminar by Corollary 2.2.11.

The contraction M/p has $C_1 - p$ and $C_2 - p$ as its only non-spanning circuits. Since these circuits are disjoint, it follows by Corollary 2.2.10 that M/p is laminar. Now assume that $e \neq p$ and $e \in C_1$. Then M/e has $C_1 - e$ as its only non-spanning circuit. Thus M/e is laminar by Corollary 2.2.11. We conclude that 2.3.2.1 holds.

Now let N be an excluded minor for the class of laminar matroids. Since N is not laminar, by Theorem 2.1.2, N contains two intersecting circuits C_1 and C_2 such that $C_1 \cap C_2 \neq \emptyset$ and neither $cl(C_1) \subseteq cl(C_2)$ nor $cl(C_2) \subseteq cl(C_1)$. Choose such a pair of circuits $\{C_1, C_2\}$ such that $|C_1 \cup C_2|$ is minimal.

Since N is an excluded minor, $|C_1 \cap C_2| = 1$ otherwise, for e in $C_1 \cap C_2$, the matroid N/e has $C_1 - e$ and $C_2 - e$ as intersecting circuits with the closure of neither containing the other, so N/e is not laminar, a contradiction. Similarly, $E(N) = C_1 \cup C_2$ otherwise deleting an element of $E(N) - (C_1 \cup C_2)$ would yield a non-laminar matroid.

2.3.2.2. Let C be a circuit of N such that C meets $C_1 - cl(C_2)$ and $C_2 - cl(C_1)$. Then C is spanning.

To see this, note that, as $C \not\subseteq cl(C_1)$ and $C \not\subseteq cl(C_2)$, Theorem 2.1.2 implies that $C_1 \subseteq cl(C)$ and $C_2 \subseteq cl(C)$. Hence C is spanning.

2.3.2.3. $cl(C_i) = C_i$ for each i in $\{1, 2\}$.

It suffices to prove this assertion for $i = 1$. Suppose $e \in cl(C_1) - C_1$. By Lemma 2.2.9, N has circuits D and D' with $e \in D \cap D'$ and $C_1 \cup e = D \cup D'$. Both $C_1 - D$ and $C_1 - D'$ are non-empty so $|C_1 \cup C_2|$ exceeds both $|C_2 \cup D|$ and $|C_2 \cup D'|$. Hence, by the minimality assumption, either $cl(C_2)$ is contained in one of $cl(D)$ or $cl(D')$; or $cl(C_2)$ contains both $cl(D)$ and $cl(D')$. This gives a contradiction since, in the first case, $cl(C_2) \subseteq cl(C_1)$ while, in the second, $cl(C_1) \subseteq cl(C_2)$. Thus $cl(C_1) = C_1$.

2.3.2.4. $|C_1| = r(N) = |C_2|$.

Take e in $C_1 - C_2$. As $cl(C_2) = C_2$, it follows that $C_1 - e$ and C_2 are intersecting circuits of N/e . Hence, by Theorem 2.1.2, $cl_{N/e}(C_2) \subseteq cl_{N/e}(C_1 - e)$, or $cl_{N/e}(C_1 - e) \subseteq cl_{N/e}(C_2)$. The first possibility gives the contradiction that $cl_N(C_2) \subseteq cl_N(C_2 \cup e) \subseteq cl_N(C_1)$. Hence $cl(C_1) \subseteq cl(C_2 \cup e)$, so $cl(C_2 \cup e) = E(N)$. Thus $C_2 \cup e$ spans N while the circuit C_2 does not, so $r(N) = |C_2|$. By symmetry, 2.3.2.4 holds.

2.3.2.5. *The only non-spanning circuits of N are C_1 and C_2 .*

Let D be a non-spanning circuit of N that differs from C_1 and C_2 . Then D meets each of $C_1 - C_2$ and $C_2 - C_1$. As $cl(C_2) = C_2$ and $cl(C_1) = C_1$, we deduce by 2.3.2.2 that D is spanning. Thus 2.3.2.5 holds.

Since $cl(C_1) = C_1$, we deduce that $r(N) \geq 3$. Recalling that a matroid of given rank is uniquely determined by a list of its non-spanning circuits (see, for example, [25, Proposition 1.4.14]), we deduce that $N \cong Y_r$ for some $r \geq 3$. \square

The following is an immediate consequence of Theorem 2.1.3.

Corollary 2.3.3. *Every matroid of rank at most two is laminar.*

2.4. Constructing laminar matroids

In this section, we begin by proving our third characterization of laminar matroids, Theorem 2.1.4. We then show that all nested matroids are laminar and we determine precisely which laminar matroids have duals that are also laminar.

Proof of Theorem 2.1.4. We first show that the class of laminar matroids is closed under adding coloops, truncating, and taking direct sums. If $M(E, \mathcal{A}, c)$ is a laminar matroid, then we see that $M(E, \mathcal{A}, c) \oplus U_{1,1} = M(E \cup e, \mathcal{A}, c)$. Further, when $r(M(E, \mathcal{A}, c)) > 0$, one easily checks that $T(M(E, \mathcal{A}, c)) = M(E, \mathcal{A}', c')$ where $\mathcal{A}' = \mathcal{A} \cup \{E\}$, while $c'(E) = r_M(E) - 1$, and $c'(A) = c(A)$, for all A in $\mathcal{A}' - \{E\}$. Finally, let $(E_1, \mathcal{A}_1, c_1)$ and $(E_2, \mathcal{A}_2, c_2)$ be canonical presentations for laminar matroids on disjoint sets E_1 and E_2 . Then $M(E_1, \mathcal{A}_1, c_1) \oplus M(E_2, \mathcal{A}_2, c_2) = M(E_1 \cup E_2, \mathcal{A}_1 \cup \mathcal{A}_2, c)$, where c coincides with c_1 when restricted to \mathcal{A}_1 and with c_2 when restricted to \mathcal{A}_2 .

Let M be a laminar matroid having (E, \mathcal{A}, c) as its canonical presentation. To prove that every laminar matroid can be constructed from the empty matroid in the manner described, we proceed by induction on $|E(M)|$. The result is immediate if $|E(M)| \leq 1$. Assume it holds if $|E(M)| < k$ and let $|E(M)| = k \geq 2$. If $M(E, \mathcal{A}, c)$ is disconnected, then $M(E, \mathcal{A}, c)$ is the direct sum of its components, each of which can be constructed.

Hence we may assume that M is connected. Thus M is loopless. Moreover, by Corollary 2.2.12, $E \in \mathcal{A}$. Let A_1, A_2, \dots, A_n be the children of E in \mathcal{A} . Then, by Theorem 2.2.7, each A_i is a flat of M and $c(A_i) = r(A_i)$.

Suppose first that $E - \cup_{i=1}^n A_i$ is non-empty and let e be in this set. Then e is free in M . Now $M \setminus e$ can be constructed in the manner described. Since M can be obtained from $M \setminus e$ by adjoining e as a coloop and then truncating the resulting matroid, we deduce that M can be constructed in the desired manner. We may now assume that $E = \cup_{i=1}^n A_i$.

For each i in $\{1, 2, \dots, n\}$, let M_i be $M(A_i, \mathcal{A}_i, c_i)$, where $\mathcal{A}_i = \mathcal{A} \cap 2^{A_i}$, and c_i is the restriction of c to \mathcal{A}_i . Evidently

$$M_i = M|_{A_i}. \quad (2.4.1)$$

Since $E = \cup_{i=1}^n A_i$, it follows that $n \geq 2$.

We show next that, for all i in $\{1, 2, \dots, n\}$,

$$r(M_i) < r(M) < r(M_1) + r(M_2) + \dots + r(M_n). \quad (2.4.2)$$

The first inequality follows by Theorem 2.2.7, Lemma 2.2.2, and Corollary 2.2.12 since $r(A_i) = c(A_i) < c(E) = r(M)$. The second inequality is an immediate consequence of the fact that M is connected. We conclude that (2.4.2) holds.

Now let $r = r(M)$ and let M' be the truncation of $M_1 \oplus M_2 \oplus \dots \oplus M_n$ to rank r . Then, by (2.4.1) and (2.4.2), for all i in $\{1, 2, \dots, n\}$,

$$M|_{A_i} = M_i = M'|_{A_i}. \quad (2.4.3)$$

To complete the proof of the theorem, observe that the following are equivalent for a subset X of E .

- (i) X is a non-spanning circuit of M' ;
- (ii) X is a non-spanning circuit of $M_1 \oplus M_2 \oplus \cdots \oplus M_n$;
- (iii) X is a circuit of M_i for some i in $\{1, 2, \dots, n\}$;
- (iv) X is a non-spanning circuit of M .

Since $r(M') = r(M)$, we deduce that $M' = M$. □

Now let M be a nested matroid having (B_1, B_2, \dots, B_n) as a presentation with $\emptyset \neq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_n$. Let $B_0 = E(M) - B_n$. Then one easily checks that $M = M(E, \mathcal{B}, c)$ where $\mathcal{B} = \{B_0, B_1, \dots, B_n\}$ and $c(B_i) = i$ for all i . In fact, all nested matroids are laminar.

Lemma 2.4.1. *Let N be a transversal matroid having a nested presentation (B_1, B_2, \dots, B_n) .*

Then N is laminar with the set \mathcal{A} in its canonical presentation consisting of $cl(\emptyset)$ together with the unique maximal subset of $\{B_1, B_2, \dots, B_n\}$ no two members of which are equal.

Proof. Clearly we may assume that each B_i is non-empty. As above, let $B_0 = E(N) - B_n$ and take $c(B_0) = 0$. The members of B_1, B_2, \dots, B_n need not be distinct. Pass through this list of sets deleting each B_i for which $B_i = B_{i+1}$. Let \mathcal{B} be the resulting collection of distinct sets $B_{i_1}, B_{i_2}, \dots, B_{i_k}$. Each of these sets is properly contained in its successor. Define $c(B_{i_t}) = r_N(B_{i_t})$. Then it is straightforward to check that $N = M(E(N), \mathcal{B} \cup \{B_0\}, c)$. □

Corollary 2.4.2. *A loopless laminar matroid M with canonical presentation (E, \mathcal{A}, c) is nested if and only if \mathcal{A} is totally ordered under set inclusion.*

Proof. If M is nested, then it is an immediate consequence of the last lemma that \mathcal{A} is indeed totally ordered. Conversely, let $\mathcal{A} = \{A_0, A_1, \dots, A_n\}$ where $A_i \subsetneq A_{i+1}$ for all $i < n$.

Let \mathcal{B} be the family of sets consisting of $c(A_1)$ copies of A_1 along with $c(A_i) - c(A_{i-1})$ copies of A_i for all i in $S\{2, 3, \dots, n\}$. The corollary now follows without difficulty. \square

The last result is reminiscent of the following result of [26], which was elegantly restated by Bonin and de Mier [3]. Recall that a *cyclic flat* of a matroid is a flat that is a union of circuits.

Lemma 2.4.3. *A matroid is nested if and only if its collection of cyclic flats is totally ordered under set inclusion.*

The next result determines precisely which matroids have the property that both M and M^* are laminar, noting that nested matroids are a fundamental class of such matroids.

Proposition 2.4.4. *The following are equivalent for a matroid M .*

- (i) *Both M and M^* are laminar.*
- (ii) *Each component of M is either a nested matroid or is a truncation to some non-zero rank of the direct sum of two uniform matroids of positive rank.*

The proof of this proposition will use the following.

Lemma 2.4.5. *Let (E, \mathcal{A}, c) be the canonical presentation of a connected laminar matroid M . If $A \in \mathcal{A}$, then $E - A$ is a connected flat of M^* .*

Proof. Suppose $e \in A$ and $e \in cl^*(E - A)$. Then $e \notin cl(A - e)$. This gives a contradiction to Theorem 2.2.7 as A is the closure of a circuit of M . We deduce that $E - A$ is flat of M^* .

Now assume $M^*|(E - A)$ is disconnected. Then so is M/A . Let X and Y be distinct components of M/A . As M has no coloops, each of X and Y has at least two elements. If both $X \cup A$ and $Y \cup A$ are connected flats of M , then, by Corollary 2.2.13, both are in \mathcal{A} and we contradict the fact that \mathcal{A} is laminar. Thus we may assume that $M|(X \cup$

A) is disconnected. Then the last matroid has X and A as its components. Now M has a circuit C that meets both A and X . This circuit must also meet $E - (X \cup A)$ otherwise $M|(X \cup A)$ is connected. Now $C - A$ is a union of circuits of M/A . Since no such circuit meets both X and $E - (X \cup A)$, there is a circuit D of M/A contained in $C \cap X$. As $r(A) + r(X) = r(A \cup X)$, it follows that D is a circuit of M that is properly contained in C , a contradiction. We conclude that $E - A$ is a connected flat in M^* . \square

Proof of Proposition 2.4.4. Since, by Theorem 2.1.4, the class of laminar matroids is closed under direct sums, it suffices to prove the proposition in the case that M is connected. If M is nested, then, by [26], so is M^* . Now suppose that M is the truncation to some positive rank r of the direct sum of uniform matroids M_1 and M_2 where $r(M_1) \geq r(M_2) > 0$. Since each uniform matroid is laminar and the class of laminar matroids is closed under direct sums and truncation, M is laminar. To see that M^* is also laminar, suppose first that $r = r(M_1)$. Then M has $E(M_2)$ as its unique proper non-empty cyclic flat. Hence M^* has $E(M_1)$ as its unique proper non-empty cyclic flat so, by Lemma 2.4.3, M^* is nested and hence is laminar. We may now assume that $r > r(M_1)$. For each i in $\{1, 2\}$, let $E_i = E(M_i)$. Then, by determining the hyperplanes of M , it follows that the only non-spanning circuits of M^* consist, for each permutation (i, j) of $\{1, 2\}$ of those subsets of E_i with exactly $|E_i| - r + r_j + 1$ elements. Thus $M^*|E_i$ is uniform for each i , and M^* is obtained by truncating the direct sum of $M^*|E_1$ and $M^*|E_2$ to rank $|E(M)| - r$. We deduce, from above, that M^* is laminar. We conclude that if (ii) holds, then both M and M^* are laminar.

To prove the converse, let M be a connected laminar matroid such that M^* is laminar but M is not nested. Let (E, \mathcal{A}, c) be the canonical presentation of M . Since M is not nested, \mathcal{A} contains two disjoint sets, A_1 and A_2 . Without loss of generality, we may assume that A_1 and A_2 are minimal members of \mathcal{A} . Let (E, \mathcal{A}^*, c^*) be the canonical presentation of M^* . By Lemmas 2.4.5 and 2.2.14, $E - A_1$ and $E - A_2$ are in \mathcal{A}^* . Thus $E - A_1$ and $E - A_2$ are disjoint. Hence $E = A_1 \cup A_2$. Since the only non-spanning circuits of M are contained in A_1 or A_2 , and $M|_{A_i}$ is uniform matroid for each i , the result follows. \square

The last proof was inspired by the work of Bonin and de Mier [3] on lattice path matroids.

2.5. Which laminar matroids are binary or ternary?

In this section, we determine precisely which laminar matroids are binary or ternary. We begin by showing that the class of laminar matroids is closed under the operation of parallel extension.

Lemma 2.5.1. *Every parallel extension of a laminar matroid is laminar.*

Proof. Let M be a laminar matroid having (E, \mathcal{A}, c) as its canonical presentation. By Theorem 2.1.4, it suffices to prove the result when M is loopless. Let e be an element of M and let f be an element not in E . Suppose first that e is in a 2-circuit C of M . Then $A_C \in \mathcal{A}$. Now add f to every member of \mathcal{A} that contains e leaving the capacity of each such set unchanged. It is straightforward to check that this process yields a laminar matroid that is an extension of M and has $\{e, f\}$ as a circuit. If e is not in a 2-circuit of M , then add $\{e, f\}$ to \mathcal{A} as a set of capacity 1, and add f to every member of \mathcal{A} that contains

e leaving the capacity of each such set unchanged. Again, it is straightforward to check that this gives a laminar matroid that is a parallel extension of M . \square

The characterizations of the laminar matroids that are binary or ternary involve the matroid Y_r . We recall that, for $r \geq 3$, this matroid is the truncation to rank r of the parallel connection, across the basepoint p , of two r -element circuits. In particular, Y_3 is isomorphic to a single-element deletion of $M(K_4)$. Moreover, as noted in the introduction, for all r ,

$$Y_r \setminus p \cong U_{r, 2r-2}. \quad (2.5.1)$$

Theorem 2.5.2. *The following are equivalent for a matroid N .*

- (i) *Each component of N has rank at most one or can be obtained from a circuit by a sequence of parallel extensions.*
- (ii) *N is graphic and laminar.*
- (iii) *N is regular and laminar.*
- (iv) *N is binary and laminar.*
- (v) *N has no minor in $\{Y_3, U_{2,4}\}$*

Proof. It is clear that each of (i)–(iv) implies its successor. We complete this proof by showing that (v) implies (ii), and that (ii) implies (i). To show that (v) implies (ii), suppose that N has no Y_3 - or $U_{2,4}$ -minor. Then N has no minor in $\{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})\}$. Thus N is graphic. Moreover, by (2.5.1), Y_r has a $U_{2,4}$ -minor for all $r \geq 4$. It follows, by Theorem 2.1.3, that N is laminar. Thus (v) implies (ii).

To show that (ii) implies (i), suppose that N is a graphic laminar matroid. Then, by Theorem 2.1.4 and Lemma 2.5.1, we may assume that $N = M(G)$ for some simple 2-connected graph G having at least three vertices. Now G does not have $K_4 \setminus e$ as a minor. Take a maximal-length cycle C in G . Then either $E(G) = E(C)$, or $G \setminus E(C)$ has a path joining distinct non-consecutive vertices of C . In the latter case, G has $K_4 \setminus e$ as a minor. This contradiction completes the proof. \square

To prove the characterization of ternary laminar matroids, we shall use a result of Cunningham and Edmonds (in [10]) that decomposes a 2-connected matroid into circuits, cocircuits, and 3-connected matroids.

Theorem 2.5.3. *Let M be a 2-connected matroid. Then M has a tree decomposition T in which every vertex label is 3-connected, a circuit, or a cocircuit, and there are no two adjacent vertices that are both labeled by circuits or are both labeled by cocircuits. Moreover, T is unique to within relabeling of its edges.*

The tree decomposition of M whose uniqueness is guaranteed by the last theorem is called the *canonical tree decomposition* of M .

Theorem 2.5.4. *The following are equivalent for a matroid N .*

- (i) N is ternary and laminar.
- (ii) N has no minor in $\{U_{2,5}, U_{3,5}, Y_3\}$.
- (iii) Each component of $\text{si}(N)$ has rank at most one, is $U_{2,4}$ or $U_{2,4} \oplus_2 U_{2,4}$, or can be obtained from an n -circuit for some $n \geq 3$ by 2-summing on copies of $U_{2,4}$ across k distinct elements of the circuit for some k in $\{0, 1, \dots, n\}$.

Proof. It is clear that (i) implies (ii). Moreover, it follows from (2.5.1) that (ii) implies (i).

To show that (ii) implies (iii), suppose that N has no minor in $\{U_{2,5}, U_{3,5}, Y_3\}$.

Clearly, we may also assume that N is simple and 2-connected and $r(N) \geq 2$. By Tutte's Wheels-and-Whirls Theorem (see, for example, [25, Theorem 8.8.4]), every 3-connected matroid with at least four elements has a $U_{2,4}$ - or $M(K_4)$ -minor. As N has no minor in $\{U_{2,5}, U_{3,5}, Y_3\}$, the only possible 3-connected minor of N with at least four elements is $U_{2,4}$. Consider the canonical tree decomposition T for N . By Theorem 2.5.2, we may assume that N is non-binary. Then T has a vertex labeled by a copy of $U_{2,4}$ and every other vertex of T is labeled by a circuit, a cocircuit, or a copy of $U_{2,4}$. Moreover, as N is simple, no leaf of T is labeled by a cocircuit. If T has an interior vertex labeled by a cocircuit, then this vertex has neighbors x and y each of which is labeled by a circuit or a copy of $U_{2,4}$. Thus N has as a minor the parallel connection of two copies of $U_{2,3}$, so N has a Y_3 -minor, a contradiction. Now every $U_{2,4}$ -labeled vertex of T is a leaf since if a $U_{2,4}$ -labeled vertex has two neighbors, then each is labeled by a circuit or a copy of $U_{2,4}$, so N has Y_3 as a minor. It follows that N is $U_{2,4}$ or $U_{2,4} \oplus_2 U_{2,4}$, or N can be obtained from an n -circuit C by 2-summing on copies of $U_{2,4}$ across k distinct elements of C for some k with $0 \leq k \leq n$. We deduce that (ii) implies (iii).

Finally, suppose (iii) holds. Then one easily checks that N has no minor in $\{U_{2,5}, U_{3,5}, Y_3\}$, so (iii) implies (ii), and the theorem holds. \square

Chapter 3. Generalized Laminar Matroids

3.1. Introduction

The results of this chapter are based on joint work with James Oxley [15]. In Chapter 2, we characterized laminar matroids both constructively and via excluded minors. Here we exploit some of the similarities between nested and laminar matroids to define two natural infinite families of classes of matroids, each having the classes of nested and laminar matroids as their smallest members. Every matroid belongs to a member of each of these families.

We say that a flat in a matroid is *Hamiltonian* if it has a spanning circuit. A restatement of Lemma 2.4.3 is that a matroid is nested if and only if its Hamiltonian flats form a chain under inclusion. This immediately yields the following result.

Proposition 3.1.1. *A matroid is nested if and only if, for all circuits C_1 and C_2 , either $C_1 \subseteq cl(C_2)$, or $C_2 \subseteq cl(C_1)$.*

This parallels the following characterization of laminar matroids from Chapter 2.

Theorem 3.1.2. *A matroid is laminar if and only if, for all circuits C_1 and C_2 with $|C_1 \cap C_2| \geq 1$, either $C_1 \subseteq cl(C_2)$, or $C_2 \subseteq cl(C_1)$.*

Using circuit elimination, it can quickly be shown that we get a similar description in terms of Hamiltonian flats.

Corollary 3.1.3. *A matroid is laminar if and only if, for every 1-element independent set X , the Hamiltonian flats containing X form a chain under inclusion.*

In light of these results, for any non-negative integer k , we define a matroid M to

This chapter is adapted from: Tara Fife, and James Oxley, Generalized laminar matroids, *Europ. J. Combin.* **79** (2019), 111–122. It is reprinted with permission from The European Journal of Combinatorics

be *k-closure-laminar* if, for any k -element independent subset X of $E(M)$, the Hamiltonian flats of M containing X form a chain under inclusion. We say that M is *k-laminar* if, for any two circuits C_1 and C_2 of M with $|C_1 \cap C_2| \geq k$, either $C_1 \subseteq cl(C_2)$ or $C_2 \subseteq cl(C_1)$. The following observation is straightforward.

Lemma 3.1.4. *A matroid M is k -closure-laminar if and only if, whenever C_1 and C_2 are circuits of M with $r(cl(C_1) \cap cl(C_2)) \geq k$, either $C_1 \subseteq cl(C_2)$, or $C_2 \subseteq cl(C_1)$.*

Observe that the class of nested matroids coincides with the classes of 0-laminar matroids and 0-closure-laminar matroids, while the class of laminar matroids coincides with the classes of 1-laminar matroids and 1-closure-laminar matroids. For all distinct circuits C_1 and C_2 , we have

$$r(cl(C_1) \cap cl(C_2)) \geq r(C_1 \cap C_2) = |C_1 \cap C_2|.$$

So, it is easy to see that k -closure-laminar matroids are also k -laminar. For $k \geq 2$, consider the matroid that is obtained from a $(k+1)$ -element circuit C by attaching, via parallel connection, a single triangle at each of two different elements of C . This matroid is k -laminar, but not k -closure-laminar. Thus, for all $k \geq 2$, the class of k -laminar matroids strictly contains the class of k -closure-laminar matroids. Our hope is that, for small values of k , the classes of k -laminar and k -closure-laminar matroids will enjoy some of the computational advantages of laminar matroids.

In Lemma 3.2.2, we show that the class of k -laminar matroids is minor-closed. This implies the previously known fact that the class of k -closure-laminar matroids is minor-closed for $k \in \{0, 1\}$. We show that the latter class is also minor-closed for $k \in \{2, 3\}$. Somewhat surprisingly, for all $k \geq 4$, the class of k -closure-laminar matroids is not minor-

closed. This is shown in Section 3.2. In Section 3.3, we prove the main results of the chapter, namely the excluded-minor characterizations of the classes of 2-laminar matroids and 2-closure-laminar matroids. In Section 3.4, we consider the intersection of the classes of k -laminar and k -closure-laminar matroids with other well-known classes of matroids. In particular, we show that these intersections with the class of paving matroids coincide. Moreover, although all nested and laminar matroids are representable, we note that, for all $k \geq 2$, the classes of k -laminar and k -closure-laminar matroids both contain members that are not representable.

3.2. Preliminaries

In this section, we establish some basic properties of k -laminar and k -closure-laminar matroids. The first result summarizes some of these properties. Its straightforward proof is omitted.

Proposition 3.2.1. *Let M be a matroid and k be a non-negative integer.*

- (i) *If M is k -closure-laminar, then M is k -laminar.*
- (ii) *If M is k -closure-laminar, then M is $(k + 1)$ -closure-laminar.*
- (iii) *If M is k -laminar, then M is $(k + 1)$ -laminar.*
- (iv) *M is k -closure-laminar if and only if, whenever C_1 and C_2 are non-spanning circuits of M with $r(cl(C_1) \cap cl(C_2)) \geq k$, either $C_1 \subseteq cl(C_2)$, or $C_2 \subseteq cl(C_1)$.*
- (v) *M is k -laminar if and only if, whenever C_1 and C_2 are non-spanning circuits of M with $|C_1 \cap C_2| \geq k$, either $C_1 \subseteq cl(C_2)$ or $C_2 \subseteq cl(C_1)$.*
- (vi) *If M has at most one non-spanning circuit, then M is k -laminar and k -closure-laminar.*

Clearly, for all k , the classes of k -laminar and k -closure-laminar matroids are closed under deletion. Next, we investigate contractions of members of these classes.

Lemma 3.2.2. *The class of k -laminar matroids is minor-closed.*

Proof. Let M be a k -laminar matroid and $e \in E(M)$. If C_1 and C_2 are circuits of M/e , then there are circuits C'_1 and C'_2 of M with $C'_i \in \{C_i, C_i \cup e\}$ for each $i \in \{1, 2\}$. Hence if $|C_1 \cap C_2| \geq k$, then $|C'_1 \cap C'_2| \geq k$. Consequently, we may assume that $C_1 \subseteq C'_1 \subseteq cl_M(C'_2)$. Since $cl_{M/e}(C_2) = cl_M(C_2 \cup e) - e$, and $e \notin C_1$, we get $C_1 \subseteq cl_{M/e}(C_2)$. Thus the class of laminar matroids is closed under contraction. As this class is certainly closed under deletion, the lemma holds. \square

As we will see, the class of k -closure-laminar matroids is not closed under contraction when $k \geq 4$. The next lemma will be useful in proving that the classes of 2-closure-laminar and 3-closure-laminar matroids are minor-closed.

Lemma 3.2.3. *Let C be a circuit of a k -laminar matroid M such that $|C| \geq 2k - 1$. If $e \in E(M) - cl(C)$ and $r(cl(C \cup e) - cl(C)) \geq 2$, then $cl(C \cup e)$ is a Hamiltonian flat of M .*

Proof. Take an element f of $cl(C \cup e) - (cl(C) \cup cl(\{e\}))$. Then M has a circuit D such that $\{e, f\} \subseteq D \subseteq C \cup \{e, f\}$. As $f \notin cl(\{e\})$, we may choose an element d in $D - \{e, f\}$. By circuit elimination, M has a circuit D' such that $f \in D' \subseteq (C \cup D) - d$. Then $e \in D'$ as $f \notin cl(C)$. Applying circuit elimination again gives a circuit C' contained in $(D \cup D') - e$. As $f \notin cl(C)$, it follows that $C' = C$. Hence $C - D \subseteq D'$. As $|C| \geq 2k - 1$, either $|D \cap C|$ or $|D' \cap C|$ is at least k . Since neither D nor D' is contained in $cl(C)$, it follows that C is contained in $cl(D)$ or $cl(D')$. Thus D or D' is a spanning circuit of $cl(C \cup e)$, so this flat is Hamiltonian. \square

Theorem 3.2.4. *The classes of 2-closure-laminar and 3-closure-laminar matroids are minor-closed.*

Proof. For some k in $\{2, 3\}$, let e be an element of a k -closure-laminar matroid M , and let C_1 and C_2 be distinct circuits in M/e with $r_{M/e}(cl_{M/e}(C_1) \cap cl_{M/e}(C_2)) \geq k$. We aim to show that $cl_{M/e}(C_1) \subseteq cl_{M/e}(C_2)$ or $cl_{M/e}(C_2) \subseteq cl_{M/e}(C_1)$. This is certainly true if $r_M(C_1) = k$ or $r_M(C_2) = k$, so assume each of $|C_1|$ and $|C_2|$ is at least $k + 2$. As $k \in \{2, 3\}$, it follows that $|C_i| \geq 2k - 1$ for each i .

3.2.4.1. *For each i in $\{1, 2\}$, there is a circuit D_i of M such that $cl_M(C_i \cup e) - cl(\{e\}) = cl_M(D_i) - cl(\{e\})$.*

To see this, first note that C_i or $C_i \cup e$ is a circuit of M . In the latter case, we take $D_i = C_i \cup e$. In the former case, by Lemma 3.2.3, the result is immediate unless $cl(C_i \cup e) = cl(C_i) \cup cl(\{e\})$, in which case we can take $D_i = C_i$. Thus 3.2.4.1 holds.

Now $r(cl_M(C_1 \cup e) \cap cl_M(C_2 \cup e)) \geq k + 1$ as $r(cl_{M/e}(C_1) \cap cl_{M/e}(C_2)) \geq k$. Hence, by 3.2.4.1, $r(cl_M(D_1) \cap cl_M(D_2)) \geq k$. Thus $cl_M(D_i) \subseteq cl_M(D_j)$ for some $\{i, j\} = \{1, 2\}$. Hence $cl_M(C_i \cup e) - cl_M(\{e\}) \subseteq cl_M(C_j \cup e) - cl_M(\{e\})$, so $cl_{M/e}(C_i) - cl_M(\{e\}) \subseteq cl_{M/e}(C_j) - cl_M(\{e\})$. As each element of $cl_M(\{e\}) - e$ is a loop in M/e , we deduce that $cl_{M/e}(C_i) \subseteq cl_{M/e}(C_j)$. Thus the theorem holds. \square

Theorem 3.2.5. *For all $k \geq 4$, the class of k -closure-laminar matroids is not minor-closed.*

The proof of this theorem will use Bonin and De Mier's characterization of matroids in terms of their collections of cyclic flats [5, Theorem 3.2]. A *partially ordered set* is a set P together with *partial order*, a binary relation, \leq that is reflexive, transitive, and antisymmetric on P , (see also [25, Section 1.7]). The partially ordered set is a *lattice* if for each $x, y \in P$ there are elements $x \vee y$ and $x \wedge y$ of P such that $x \vee y \geq x$ and $x \vee y \geq y$;

and if $z \geq y$ and $z \geq y$, then $z \geq x \vee y$; and $x \wedge y \leq x$ and $x \wedge y \leq y$; and if $z \leq y$ and $z \leq y$, then $z \leq x \wedge y$.

Theorem 3.2.6. *Let \mathcal{Z} be a collection of subsets of a set E and let r be an integer-valued function on \mathcal{Z} . There is a matroid for which \mathcal{Z} is the collection of cyclic flats and r is the rank function restricted to the sets in \mathcal{Z} if and only if*

(Z0) \mathcal{Z} is a lattice under inclusion;

(Z1) $r(0_{\mathcal{Z}}) = 0$;

(Z2) $0 < r(Y) - r(X) < |Y - X|$ for all sets X, Y in \mathcal{Z} with $X \subsetneq Y$; and

(Z3) for all sets X, Y in \mathcal{Z} ,

$$r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|.$$

Proof of Theorem 3.2.5. Let A, B , and C be disjoint sets with $A = \{a_1, a_2, \dots, a_{k-1}\}$, $B = \{b_1, b_2, \dots, b_{k-1}\}$, and $C = \{c_1, c_2, \dots, c_{k-1}\}$. Let $D = \{e, a_1, b_1, c_1\}$ where $e \notin A \cup B \cup C$. Let $E = A \cup B \cup C \cup D$ and let \mathcal{Z} be the following collection of subsets of E having the specified ranks and cardinalities.

Table 3.1. The lattice \mathcal{Z}

Rank t	Cardinality	Members of \mathcal{Z} of rank t
0	0	\emptyset
k	$k + 1$	$A \triangle D, B \triangle D, C \triangle D$
$2k - 3$	$2k - 2$	$A \cup B, A \cup C, B \cup C$
$2k - 2$	$2k$	$A \cup B \cup D, A \cup C \cup D, B \cup C \cup D$
$2k - 1$	$3k + 1$	E

We will show that \mathcal{Z} is the collection of cyclic flats of a matroid M on E . We then show that M is k -closure-laminar but that M/e is not. Observe that $A \triangle D, B \triangle D, C \triangle D, A \cup B, A \cup C$, and $B \cup C$ form an antichain, and that, for example, $A \cup B \cup D$ contains

exactly three members of this antichain, $A \cup B$, $A \triangle D$, and $B \triangle D$. It is straightforward to see that \mathcal{Z} is a lattice obeying (Z1) (see Figure 3.1). We can quickly check that \mathcal{Z} obeys (Z2). To check that \mathcal{Z} obeys (Z3), we see by symmetry that we need only check (Z3) when (X, Y) is one of $(A \cup C, A \triangle D)$, $(A \cup C, B \triangle D)$, $(A \cup C, B \cup C \cup D)$, $(A \cup C, B \cup C)$, $(A \triangle D, B \triangle D)$, $(A \triangle D, B \cup C \cup D)$, and $(A \cup B \cup D, A \cup C \cup D)$. Calculating $r(X) + r(Y) - r(X \vee Y) - r(X \wedge Y) - |(X \cap Y) - (X \wedge Y)|$ for each of these pairs, we find that the first and fifth give 0, and the other five give $k - 4$. Hence \mathcal{Z} obeys (Z3) for all $k \geq 4$, so M is a matroid. As noted in [5], its circuits are the minimal subsets S of E such that \mathcal{Z} contains an element Z containing S with $|S| = r(Z) + 1$.

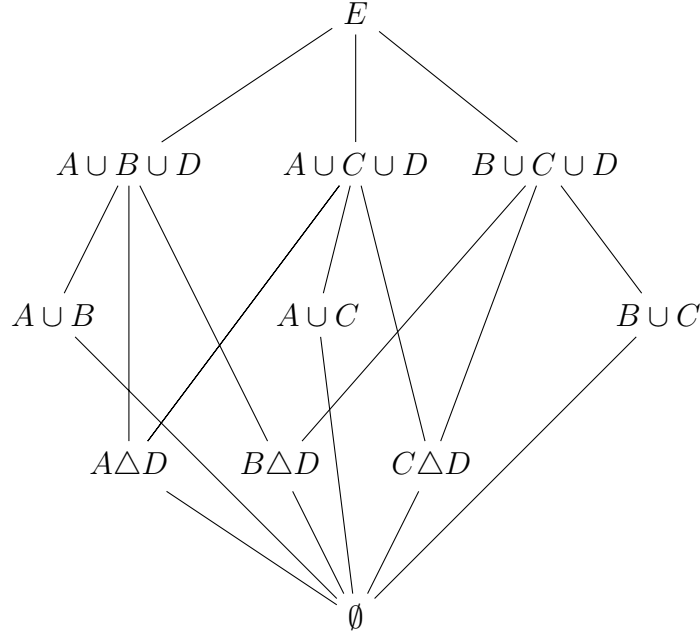


Figure 3.1. The lattice \mathcal{Z} of cyclic flats of the matroid in the proof of Theorem 3.2.5.

To show that M is k -closure-laminar, we first note that $A \cup C \cup D$ is non-Hamiltonian for $2 = |A \cup C \cup D| - r(A \cup C \cup D)$, yet there is no element of $A \cup C \cup D$ that is in all three non-spanning circuits of $M|(A \cup C \cup D)$. By symmetry, $A \cup B \cup D$ and $B \cup C \cup D$

are non-Hamiltonian. All of the other cyclic flats of M are Hamiltonian. By symmetry, if (X, Y) is a pair of incomparable Hamiltonian flats of M , then we may assume that (X, Y) is $(A \cup C, A \triangle D)$, $(A \cup C, B \triangle D)$, $(A \cup C, B \cup C)$, or $(A \triangle D, B \triangle D)$. For each such pair, we need to check that $r(X \cap Y) \leq k - 1$. For the first and third pairs, $|X \cap Y| = k - 1$; for the second and fourth pairs, $|X \cap Y| = 2$. Hence M is indeed k -closure-laminar. To see that M/e is not k -closure-laminar, note that $A \cup C$ and $B \cup C$ are circuits of this matroid. Moreover, $cl_{M/e}(A \cup C) = A \cup C \cup b_1$ and $cl_{M/e}(B \cup C) = B \cup C \cup a_1$. Then $cl_{M/e}(A \cup C) \cap cl_{M/e}(B \cup C) = C \cup \{a_1, b_1\}$. The last set has rank k in M/e as $(C \triangle D) - e$ is the only circuit of M/e contained in it. Thus M/e is not k -closure-laminar as neither $cl_{M/e}(A \cup C)$ nor $cl_{M/e}(B \cup C)$ is contained in the other. \square

3.3. Excluded minors

We now note some excluded minors for the classes of k -laminar and k -closure-laminar matroids. For $n \geq k + 2$, let $M_n(k)$ be the truncation to rank n of the cycle matroid of the graph consisting of two vertices that are joined by three internally disjoint paths P , X_1 , and X_2 of lengths k , $n - k$, and $n - k$, respectively. In particular, $M_4(2) \cong M(K_{2,3})$. Observe that, when $k = 0$, the path P has length 0 so its endpoints are equal. Thus $M_n(0)$ is the truncation to rank n of the direct sum of two n -circuits. Let $M^-(K_{2,3})$ be the unique matroid that is obtained by relaxing a circuit-hyperplane of $M(K_{2,3})$. For $n \geq k + 3 \geq 5$, let $N_n(k)$ be the truncation to rank n of the graphic matroid that is obtained by attaching two $(n - k)$ -circuits to distinct elements of a $(k + 2)$ -circuit via parallel connection. For $n \geq k + 2 \geq 4$, let $P_n(k)$ be the truncation to rank n of the graphic matroid that is obtained by attaching two $(n - k + 1)$ -circuits to distinct elements

of a $(k + 1)$ -circuit via parallel connection. Thus $P_n(k)$ is a single-element contraction of $N_{n+1}(k)$. Moreover, $P_4(2)$ is isomorphic to the matroid that is obtained by deleting a rim element from a rank-4 wheel.

Lemma 3.3.1. *For all $n \geq k + 2$, the matroid $M_n(k)$ is an excluded minor for the classes of k -laminar matroids and k -closure-laminar matroids.*

Proof. We may assume that $k \geq 2$, as the lemma holds for $k = 0$ and for $k = 1$ by results in [26] and Chapter 2. Clearly $M_n(k)$ is not k -laminar so is not k -closure-laminar. If we delete an element of $M_n(k)$, then we get a matroid with at most one non-spanning circuit. By Proposition 3.2.1(vi), such a matroid is k -closure-laminar and hence is k -laminar. If we contract an element of P from $M_n(k)$, we get a matroid that is k -closure-laminar since in it the closures of the only two non-spanning circuits meet in $k - 1$ elements. Instead, if we contract an element of X_1 or X_2 , we again get a matroid with exactly one non-spanning circuit. Thus the lemma holds. \square

Similar arguments give the following result.

Lemma 3.3.2.

- (i) *The matroid $M^-(K_{2,3})$ is an excluded minor for the classes of 2-laminar and 2-closure-laminar matroids.*
- (ii) *For all $n \geq k + 3 \geq 5$, the matroid $N_n(k)$ is an excluded minor for the class of k -laminar matroids.*
- (iii) *For all $n \geq k + 2 \geq 4$, the matroid $P_n(k)$ is an excluded minor for the class of k -closure-laminar matroids.*

The main results of this chapter show that we have now identified all of the excluded minors for the classes of 2-laminar and 2-closure-laminar matroids. We will use the following basic results. We omit the elementary proof of the second one.

Lemma 3.3.3. *Let C be a circuit of a matroid M . If there is a partition $\{A, B\}$ of C and distinct elements x and y for which $A \cup x$, $B \cup x$, $A \cup y$, and $B \cup y$ are all circuits, then x and y are parallel.*

Proof. By submodularity and the fact that C is a circuit,

$$r(\text{cl}(A) \cap \text{cl}(B)) \leq |A| + |B| - (|C| - 1) = 1.$$

From this, the lemma is immediate. □

Lemma 3.3.4. *Let C and D be distinct circuits of a matroid M .*

- (i) *If $D \not\subseteq \text{cl}(C)$, then $|D - \text{cl}(C)| \geq 2$.*
- (ii) *If $|D - C| = 1$ and D' is a circuit contained in $C \cup D$ other than C or D , then $C - D \subseteq D'$.*

Lemma 3.3.5. *Let M be an excluded minor for \mathcal{M} , where \mathcal{M} is either the class of 2-laminar or the class of 2-closure-laminar matroids. Let C_1 and C_2 be circuits of M neither of which is contained in the closure of the other such that $|C_1 \cap C_2| \geq 2$ when \mathcal{M} is the class of 2-laminar matroids while $r(\text{cl}(C_1) \cap \text{cl}(C_2)) \geq 2$ otherwise. Then*

- (i) *$E(M) = C_1 \cup C_2$ if \mathcal{M} is the class of 2-laminar matroids;*
- (ii) *$E(M) = C_1 \cup C_2 \cup (\text{cl}(C_1) \cap \text{cl}(C_2))$ if \mathcal{M} is the class of 2-closure-laminar matroids;*
- (iii) *M has $\text{cl}(C_1)$ and $\text{cl}(C_2)$ as hyperplanes, so $|C_1| = |C_2|$; and*
- (iv) *if C is a circuit of M that meets both $C_1 - \text{cl}(C_2)$ and $C_2 - \text{cl}(C_1)$, then either C is spanning, or C contains $C_1 \triangle C_2$.*

Proof. For (ii), if $f \in E(M) - (C_1 \cup C_2 \cup (\text{cl}(C_1) \cap \text{cl}(C_2)))$, then $C_i \subseteq \text{cl}_{M \setminus f}(C_j)$ for some $\{i, j\} = \{1, 2\}$. Thus $C_i \subseteq \text{cl}_M(C_j)$, a contradiction. Hence (ii) holds. Part (i) follows similarly. Certainly $C_2 - \text{cl}(C_1)$ contains an element e . As $e \notin \text{cl}(C_1)$, if $\{x, y\}$ is an

independent subset of $cl(C_1) \cap cl(C_2)$, then $\{x, y\}$ is independent in M/e . It follows, since $M/e \in \mathcal{M}$, that either $C_1 \subseteq cl_{M/e}(C_2 - e)$ or $(C_2 - e) \subseteq cl_{M/e}(C_1)$. The former yields a contradiction. Hence $C_2 \subseteq cl_M(C_1 \cup e)$, so $cl_M(C_1 \cup e) = E(M)$. Thus $cl(C_1)$ is a hyperplane of M . By symmetry, so is $cl(C_2)$. Hence $|C_1| = |C_2|$, so (iii) holds.

Now let C be a circuit of M that meets both $C_1 - cl(C_2)$ and $C_2 - cl(C_1)$. As $C - cl(C_2)$ is non-empty, $|C - cl(C_2)| \geq 2$, so $|C \cap C_1| \geq 2$. Suppose C is non-spanning. As $cl(C_1)$ is a hyperplane and C meets $C_2 - cl(C_1)$, it follows that $cl(C) \not\subseteq cl(C_1)$ and $cl(C_1) \not\subseteq cl(C)$. Since $|C \cap C_1| \geq 2$, if $E(M) - (C \cup C_1)$ contains an element e , then, as $M \setminus e \in \mathcal{M}$, we get a contradiction. Therefore $E(M) = C \cup C_1$. By symmetry, $E(M) = C \cup C_2$. Thus C contains $C_1 \triangle C_2$, so (iv) holds. \square

Theorem 3.3.6. *The excluded minors for the class of 2-laminar matroids are*

$M^-(K_{2,3})$, $M_n(2)$ for all $n \geq 4$, and $N_n(2)$ for all $n \geq 5$.

Proof. Suppose that M is an excluded minor for the class of 2-laminar matroids. Then M has circuits C_1 and C_2 with $|C_1 \cap C_2| \geq 2$ such that neither C_1 nor C_2 is contained in the closure of the other. Thus each of $C_1 - cl(C_2)$ and $C_2 - cl(C_1)$ contains at least two elements. By Lemma 3.3.5(i), $E(M) = C_1 \cup C_2$. Moreover, $|C_1 \cap C_2| = 2$, otherwise we could contract an element of $C_1 \cap C_2$ and still get a matroid that is not 2-laminar. Let $\{a, b\} = C_1 \cap C_2$.

Suppose $g \in cl(C_1) - C_1$. Leading up to 3.3.6.5, we shall prove four preliminary results.

3.3.6.1. *Suppose D is a circuit contained in $C_1 \cup g$ and containing $\{g, a, b\}$. Then $D \subseteq cl(C_2)$.*

To see this, note that, as $\{a, b, g\} \subseteq D \cap C_2$, we deduce, since M/g is 2-laminar having $D - g$ and $C_2 - g$ as circuits, that $D - g \subseteq cl_{M/g}(C_2 - g)$ or $C_2 - g \subseteq cl_{M/g}(D - g)$. The latter implies that $C_2 \subseteq cl(D)$. But $cl(D) \subseteq cl(C_1)$, so this yields a contradiction. Hence 3.3.6.1 holds.

3.3.6.2. *If D_1 and D_2 are circuits contained in $C_1 \cup g$ and containing $\{g, a, b\}$, then $D_1 = D_2$.*

Suppose $D_1 \neq D_2$. By Lemma 3.3.4(ii), $D_1 \cup D_2 = C_1 \cup g$. By 3.3.6.1, $D_i \subseteq cl(C_2)$ for each i in $\{1, 2\}$. Thus $C_1 \subseteq cl(C_2)$. This contradiction implies that 3.3.6.2 holds.

3.3.6.3. *If D_1 and D_2 are distinct circuits contained in $C_1 \cup g$ and each contains g , then D_1 or D_2 contains $\{a, b\}$.*

Assume that this fails. By Lemma 3.3.4(ii), $D_1 \cup D_2 = C_1 \cup g$. We may suppose that $D_1 \cap \{a, b\} = \{a\}$ and $D_2 \cap \{a, b\} = \{b\}$. For each i in $\{1, 2\}$, assume that D_i avoids some element d_i of $C_1 - C_2$. Now $cl(D_i) \subseteq cl(C_1)$, so $C_2 \not\subseteq cl(D_i)$. As $M \setminus d_i$ is 2-laminar, it follows that $D_i \subseteq cl(C_2)$ for each i . Thus $C_1 \subseteq cl(C_2)$, a contradiction. It follows that we may assume that $C_1 - C_2 \subseteq D_1$, so $D_1 = (C_1 - b) \cup g$.

Now $\{b, g\} \subseteq D_2$. As $\{a, b, g\} \subsetneq C_2$, we see that D_2 contains an element w of $C_1 - \{a, b\}$. Then $w \in D_1 \cap D_2$ and $b \in D_2 - D_1$. Hence there is a circuit D_3 contained in $(D_1 \cup D_2) - w$ and containing b . As g must be in D_3 , it follows by Lemma 3.3.4(ii) that $D_3 \cup D_2 = C_1 \cup g$. Thus $a \in D_3$, so $\{a, g\} \subseteq D_3$. Hence, by 3.3.6.1, $D_3 \subseteq cl(C_2)$. As $\{g, b\} \subseteq D_2 \cap D_3$ and $M \setminus e$ is 2-laminar for e in $C_2 - cl(C_1)$, we deduce that $D_3 \subseteq cl(D_2)$ otherwise $D_2 \subseteq cl(D_3) \subseteq cl(C_2)$, so $C_1 \subseteq cl(C_2)$, a contradiction. We conclude that D_2 spans C_1 . As $a \notin D_2$, we see that $D_2 = (C_1 - a) \cup g$.

Next take an element v of $D_3 - \{g, a, b\}$. Then $v \in D_1$ and $b \in D_3 - D_1$. Thus M has a circuit D_4 that contains b and is contained in $(D_1 \cup D_3) - v$. Clearly $g \in D_4$ and $D_4 \neq D_3$. Thus, by 3.3.6.2, $\{a, b\} \not\subseteq D_4$. Hence $D_4 \cap \{a, b\} = \{b\}$. But D_2 and D_4 are distinct circuits contained in $C_1 \cup g$ and each contains g . Yet $D_2 \cup D_4$ avoids a , which contradicts Lemma 3.3.4(ii). Thus 3.3.6.3 holds.

3.3.6.4. *Suppose D_1 and D_2 are distinct circuits contained in $C_1 \cup g$ and each contains g . If $\{a, b\} \subseteq D_1$, then $D_1 \cap D_2 = \{g\}$, so $D_2 = C_1 \triangle D_1$.*

By 3.3.6.1, $D_1 \subseteq cl(C_2)$. Now suppose that $\{a, b\} \cap D_2 \neq \emptyset$. Then, by 3.3.6.2, we may assume that $\{a, b\} \cap D_2 = \{b\}$. As $D_1 \cup D_2 = C_1 \cup g$, we see that $D_2 \not\subseteq cl(C_2)$, otherwise $C_1 \subseteq cl(C_2)$. If D_2 does not contain $C_1 - C_2$, then, for f in $(C_1 - C_2) - D_2$, the matroid $M \setminus f$ is 2-laminar and so $C_2 \subseteq cl(D_2) \subseteq cl(C_1)$, a contradiction. Thus $C_1 - C_2 \subseteq D_2$, so $D_2 = (C_1 - a) \cup g$. Take x in $(D_1 \cap D_2) - \{g, b\}$. Then M has a circuit D_3 contained in $(D_1 \cup D_2) - x$ and containing a . Clearly $g \in D_3$, so D_3 and D_2 are distinct circuits contained in $C_1 \cup g$ and each contains g . It follows by 3.3.6.3 that D_3 or D_2 contains $\{a, b\}$. Thus, by 3.3.6.2, D_3 or D_2 is D_1 . But $D_3 \neq D_1$ as $x \in D_1 - D_3$; and $D_2 \neq D_1$ by assumption. We conclude that $\{a, b\} \cap D_2 = \emptyset$.

Finally, suppose that $h \in (D_1 \cap D_2) - g$. Then, as $M \setminus e$ is 2-laminar for e in $C_2 - cl(C_1)$, we deduce that $D_1 \subseteq cl(D_2)$ or $D_2 \subseteq cl(D_1)$. As $D_1 \subseteq cl(C_2)$ and $C_1 \not\subseteq cl(C_2)$, it follows that $D_1 \subseteq cl(D_2)$, so D_2 spans C_1 . But D_2 avoids $\{a, b\}$, so $r(D_2) \leq |C_1| - 2$, a contradiction. Thus 3.3.6.4 holds.

On combining 3.3.6.1–3.3.6.4, we obtain the following.

3.3.6.5. *If $g \in cl(C_1) - C_1$, then there are circuits G and G' that meet in $\{g\}$ such that*

$G \cup G' = C_1 \cup g$ and $\{a, b\} \subseteq G \subseteq cl(C_2)$. Furthermore, G , G' , and C_1 are the only circuits contained in $C_1 \cup g$, so $cl(G) - C_2 = G - C_2$.

As there are at least two elements in each of $C_2 - cl(C_1)$, $C_1 - cl(C_2)$, and $C_1 \cap C_2$, it follows that $r(M) \geq 4$. Next we show

3.3.6.6. $|cl(C_1) - C_1| = |cl(C_2) - C_2| \leq 1$.

Suppose that $cl(C_1) - C_1 = \{g_1, g_2, \dots, g_t\}$ where $t \geq 2$. For each i in $\{1, 2, \dots, t\}$, let G_i and G'_i be the associated circuits given by 3.3.6.5 whose union is $C_1 \cup g_i$, where $\{a, b, g_i\} \subseteq G_i$ and $G'_i = C_1 \triangle G_i$. By 3.3.6.1, $G_i \subseteq cl(C_2)$. For distinct i and j in $\{1, 2, \dots, t\}$, as G_i and G_j are distinct circuits contained in the 2-laminar matroid $M|cl(C_1)$, and $|G_i \cap G_j| \geq 2$, the closures of G_1, G_2, \dots, G_t form a chain under inclusion. Say $cl(G_1) \supseteq cl(G_2) \supseteq \dots \supseteq cl(G_t)$. Since $cl(G_i) - C_2 = G_i - C_2$, it follows that $G_1 - C_2 \supseteq G_2 - C_2 \supseteq \dots \supseteq G_t - C_2$. Now let $\{f_1, f_2, \dots, f_s\} = cl(C_2) - C_2$. For each f_i , there are circuits F_i and F'_i whose union is $C_2 \cup f_i$ such that $\{a, b\} \subseteq F_i$ and $F'_i = C_2 \triangle F_i$. Moreover, we may assume that $cl(F_1) \supseteq cl(F_i)$ for all i .

By 3.3.6.5, for all i ,

$$F_i - \{a, b, f_i\} \subseteq cl(C_1) - C_1 = \{g_1, g_2, \dots, g_t\} \subseteq cl(G_1).$$

Thus $F_i - f_i \subseteq cl(G_1)$ so $f_i \in cl(G_1)$. Hence, by 3.3.6.5, $f_i \in G_1$. Moreover, as $F_i \subseteq \{f_i, a, b, g_1, g_2, \dots, g_t\}$ and $\{a, b, g_1, g_2, \dots, g_t\} \subseteq cl(G_1)$, we see that $cl(F_i) \subseteq cl(G_1)$. Since $\{f_1, f_2, \dots, f_s\} \subseteq G_1$, we deduce, since $G_1 \subseteq cl(C_2)$, that $G_1 = \{g_1, a, b, f_1, f_2, \dots, f_s\}$. As $cl(F_1) \subseteq cl(G_1)$, it follows by symmetry that $cl(F_1) = cl(G_1)$. Moreover, symmetry also gives that $F_1 = \{f_1, a, b, g_1, g_2, \dots, g_t\}$. Since G_1 and F_1 are both circuits spanning the

same set, they have the same cardinality, so $t = s$; that is,

$$|cl(C_1) - C_1| = |cl(C_2) - C_2|.$$

By Lemma 3.3.3, since $\{g_1, g_2, \dots, g_t\}$ is independent, we get that $G_i - g_i \neq G_j - g_j$ for distinct i and j . Thus

$$t + 3 = |G_1| > |G_2| > \dots > |G_t| \geq 4$$

where the last inequality follows because G_t is not a proper subset of C_2 .

Now suppose that $|G_2| = |G_1| - 1$ where $(G_1 - g_1) - (G_2 - g_2) = \{f_i\}$. Choose $e \in C_1 - cl(C_2)$. As $f_i \in G'_2 - G'_1$, strong circuit elimination on G'_1 and G'_2 , both of which contain e , yields a circuit D containing f_i and avoiding e . Since D avoids $\{a, b\}$, it follows that $\{g_1, g_2\} \subseteq D$. As $e \notin C_2 \cup D$, we deduce that $D \subseteq cl(C_2)$, otherwise we obtain the contradiction that $C_2 \subseteq cl(D) \subseteq cl(C_1)$. But $(G'_2 - g_2) - (G'_1 - g_1) = \{f_i\}$, so $D \subseteq G'_2 \cup g_1$, and $(G'_2 \cup g_1) \cap cl(C_2) = \{f_i, g_1, g_2\}$. As $D \subseteq cl(C_2)$, it follows that $D \subseteq \{f_i, g_1, g_2\}$. This is a contradiction to 3.3.6.5 because $D \notin \{F_i, F'_i\}$. We deduce that $|G_2| \leq |G_1| - 2$. Thus $|G_2| \leq t + 3 - 2 = t + 1$. Hence $|G_t| \leq 3$, a contradiction. We conclude that 3.3.6.6 holds.

By Lemma 3.3.5(iii), $cl(C_1)$ and $cl(C_2)$ are hyperplanes of M , and $|C_1| = |C_2|$.

Suppose that $cl(C_1) = C_1$. Then, by 3.3.6.6, $cl(C_2) = C_2$. As $E(M) = C_1 \cup C_2$, every circuit of M other than C_1 or C_2 must meet both $C_1 - C_2$ and $C_2 - C_1$. Assume M has such a circuit C that is non-spanning. Then, by Lemma 3.3.5(iii), $C_1 \triangle C_2 \subseteq C$. As $|C_1 \cap C_2| = 2$ but C is non-spanning, it follows that $C = C_1 \triangle C_2$. Thus $r(C) = r(M) - 1$, so $r(M) - 1 = |C_1| + |C_2| - 5$. But $r(M) - 1 = r(C_1) = |C_1| - 1$. Hence $|C_2| = 4$, so $|C_1| = 4$. It follows easily that $M \cong M(K_{2,3}) \cong M_4(2)$. Now suppose that every circuit other than C_1

or C_2 is spanning. Then, letting $|C_1| = n$, we see that $|C_2| = n$ and $r(M) = r(C_1) + 1 = n$.

It follows that $M \cong M(K_{2,3}^-)$ when $n = 4$, while $M \cong M_n(2)$ when $n \geq 5$.

By 3.3.6.6, we may now suppose that $cl(C_1) - C_1 = \{g\}$. Then $cl(C_2) - C_2 = \{f\}$, say. By 3.3.6.5, $\{a, b, g, f\}$ is a circuit of M as are both $G' = (C_1 - \{a, b, f\}) \cup \{g\}$ and $F' = (C_2 - \{a, b, g\}) \cup \{f\}$. All circuits of M other than $C_1, C_2, \{a, b, g, f\}, G'$, and F' must meet both $C_1 - cl(C_2)$ and $C_2 - cl(C_1)$. Hence, by Lemma 3.3.5(iv), every such circuit is spanning as $C_1 \triangle C_2$ properly contains G' . Again letting $|C_1| = n$, we see that $|C_2| = n$ and $r(M) = n$. Thus $M \cong N_n(2)$ for some $n \geq 5$. \square

Theorem 3.3.7. *The excluded minors for the class of 2-closure-laminar matroids are $M^-(K_{2,3})$, $M_n(2)$ for all $n \geq 4$, and $P_n(2)$ for all $n \geq 4$.*

Proof. Let M be an excluded minor for the class of 2-closure-laminar matroids. Clearly M is simple. Now M has two circuits C_1 and C_2 with $r(cl(C_1) \cap cl(C_2)) \geq 2$ such that neither is a subset of the closure of the other. By Lemma 3.3.5(ii),

3.3.7.1. $E(M) = C_1 \cup C_2 \cup (cl(C_1) \cap cl(C_2))$.

Clearly $|C_1 \cap C_2| \leq 2$, otherwise we could contract an element of $C_1 \cap C_2$ and still have a matroid that is not 2-closure-laminar. We break the rest of the proof into three cases based on the size of $C_1 \cap C_2$.

3.3.7.2. $C_1 \cap C_2 \neq \emptyset$.

Assume the contrary. Let $\{x, y\}$ be a subset of $cl(C_1) \cap cl(C_2)$. To show 3.3.7.2, we first establish that

3.3.7.3. $\{x, y\} \not\subseteq C_2$.

Suppose $\{x, y\} \subseteq C_2$. As $M|(C_1 \cup \{x, y\})$ is connected, there is a circuit D_1 with $\{x, y\} \subseteq D_1 \subseteq C_1 \cup \{x, y\}$. Then, for c in $C_1 - D_1$, the matroid $M \setminus c$ is 2-closure-laminar. Now $C_2 \not\subseteq cl(D_1)$ since $cl(D_1) \subseteq cl(C_1)$. Thus $D_1 \subseteq cl(C_2)$, so $C_1 \cap cl(C_2)$ is non-empty. Choose an element z in $C_1 \cap cl(C_2)$. Now M has circuits C_x and C'_x , with $x \in C_x \cap C'_x$, and $C_x \cup C'_x = C_1 \cup x$. It also has circuits C_y , and C'_y with $y \in C_y \cap C'_y$ and $C_y \cup C'_y = C_1 \cup y$. We may assume that $z \in C_x \cap C_y$. Then $\{x, z\} \subseteq cl(C_x) \cap cl(C_2)$. As $C_1 - (C_x \cup C_2)$ is non-empty, this implies that $C_x \subseteq cl(C_2)$ since $C_2 \not\subseteq cl(C_x)$ because $cl(C_x) \subseteq cl(C_1)$. Similarly, $C_y \subseteq cl(C_2)$.

Suppose $(C_x - x) \cap (C'_x - x)$ is non-empty and choose e in this set. Then, as $\{e, x\} \subseteq C_x \cap C'_x$ and $y \notin C_x \cup C'_x$, either $C_x \subseteq cl(C'_x)$ or $C'_x \subseteq cl(C_x)$. In the latter case, $C'_x \subseteq cl(C_x) \subseteq cl(C_2)$, so $C_1 \subseteq cl(C_2)$, a contradiction. Thus $C_x \subseteq cl(C'_x)$. But then C_1 and C'_x have the same rank, and hence the same size. Then $C'_x = C_1 \triangle \{x, c\}$ for some $c \in C_1$. Now consider the 2-closure-laminar matroid $M \setminus c$. In it, C'_x and C_2 are circuits as $c \notin C_2$. Then $r_{M \setminus c}(cl_{M \setminus c}(C'_x) \cap cl_{M \setminus c}(C_2)) \geq 2$ so $C'_x \subseteq cl_{M \setminus c}(C_2)$ or $C_2 \subseteq cl_{M \setminus c}(C'_x)$. As $C_x \subseteq cl_M(C_2)$ and $cl_{M \setminus c}(C'_x) \subseteq cl_M(C_1)$, we obtain the contradiction that $C_1 \subseteq cl(C_2)$ or $C_2 \subseteq cl(C_1)$. We conclude that $C_x \cap C'_x = \{x\}$. Likewise $C_y \cap C'_y = \{y\}$.

If there is some element f in $C_x \cap C'_y$, then, as $f \in C_x$, we have $f \in cl(C_2)$. But then $\{f, y\} \subseteq cl(C'_y) \cap cl(C_2)$. Thus either $C'_y \subseteq cl(C_2)$ or $C_2 \subseteq cl(C'_y)$. The former cannot occur as $C_y \subseteq cl(C_2)$; nor can the latter as $cl(C'_y) \subseteq cl(C_1)$. Hence $C_x \cap C'_y = \emptyset$. Likewise, $C_y \cap C'_x = \emptyset$. But then $C_x \triangle \{x, y\} = C_y$ and $C'_x \triangle \{x, y\} = C'_y$. Hence, by Lemma 3.3.3, x and y are parallel, a contradiction. Thus 3.3.7.3 holds.

Next we suppose that $x \in C_2$ and $y \notin C_2$. Choose a circuit D with $\{x, y\} \subseteq D \subseteq C_2 \cup y$. Then $D \subseteq cl(C_1)$, since $C_2 - (D \cup C_1) \neq \emptyset$ and $cl(D) \cap cl(C_1)$ has rank at least

two, while $cl(D) \subseteq cl(C_2)$. Now $(D \cap C_2) - x$ certainly contains some element d . Then $\{x, d\} \subseteq cl(C_1) \cap cl(C_2)$. Applying 3.3.7.3 gives a contradiction.

We may now assume that $\{x, y\} \cap C_2 = \emptyset$. Let D be a circuit with $\{x, y\} \subseteq D \subseteq C_2 \cup \{x, y\}$. Then $D \subseteq cl(C_1)$ as $C_2 - (D \cup C_1) \neq \emptyset$. By replacing y by an element of $D \cap C_2$, we revert to the case eliminated in the last paragraph. Hence 3.3.7.2 holds.

Now, we consider the case when $|C_1 \cap C_2| = 1$. Let $C_1 \cap C_2 = \{x\}$ and choose y in $(cl(C_1) \cap cl(C_2)) - (C_1 \cap C_2)$. Suppose $y \notin C_1 \cup C_2$. Then M has circuits D_1 and D_2 containing $\{x, y\}$ and contained in $C_1 \cup y$ and $C_2 \cup y$, respectively. Without loss of generality, as $E(M) - (D_1 \cup D_2)$ is non-empty, we may assume that $D_1 \subseteq cl(D_2)$. Since $|D_1| \geq 3$, there is an element z of $D_1 - \{x, y\}$. Then z is in $cl(D_2)$ and so is in $cl(C_2)$. Thus $\{x, z\} \subseteq C_1 \cap cl_{M \setminus y}(C_2)$ and we obtain a contradiction. It follows, by 3.3.7.1, that $C_1 \cup C_2 = E(M)$.

We may now assume that $y \in C_1 \cap cl(C_2)$. Then M has a circuit D such that $\{x, y\} \subseteq D \subseteq C_2 \cup y$. Clearly $D \subseteq cl(C_2)$. To see that $D \subseteq cl(C_1)$, we note that $C_2 - (D \cup C_1) \neq \emptyset$, and $C_1 \not\subseteq cl(D)$ as $D \subseteq cl(C_2)$. We now have $D - x \subseteq cl_{M/x}(C_1 - x) \cap cl_{M/x}(C_2 - x)$. Thus $r_{M/x}(D - x) \leq 1$, otherwise, for some $\{i, j\} = \{1, 2\}$, we have $cl_{M/x}(C_i - x) \subseteq cl_{M/x}(C_j - x)$, a contradiction. As $y \in D - x$, we see that $r_M(D) = 2$, so $D = \{x, y, y'\}$ for some y' . We deduce that

3.3.7.4. $\{x, y, y'\}$ is the only circuit of $M|(C_2 \cup y)$ containing $\{x, y\}$.

We show next that

3.3.7.5. $(C_2 - \{x, y'\}) \cup y$ is the only circuit of $M|(C_2 \cup y)$ containing y but not x .

By Lemma 3.3.4(ii), every circuit D' of M that contains y , avoids x , and is contained in $C_2 \cup y$ must contain $(C_2 - \{x, y'\}) \cup y$. If $y' \in D'$, then $D' = (C_2 \cup y) - x$. Using D' and D , we find a circuit D'' containing x and contained in $(C_2 \cup y) - y'$. As D'' must also contain y , we see that $\{x, y\} \subseteq D''$ and we showed in 3.3.7.4 that M has no such circuit. We conclude that 3.3.7.5 holds.

By 3.3.7.5 and symmetry, M has $(C_1 - \{x, y\}) \cup y'$ as a circuit, say C'_1 . Let C'_2 be the circuit $(C_2 - \{x, y'\}) \cup y$. Next we note that

3.3.7.6. $cl(C_2) - C_2 = \{y\}$ and $cl(C_1) - C_1 = \{y'\}$.

Assume there is an element y_1 in $(cl(C_2) - C_2) - y$. Then $\{y, y_1\}$ is a subset of $cl_{M/x}(C_1 - x) \cap cl_{M/x}(C_2 - x)$ that is independent in M/x . Thus $C_i - x \subseteq cl_{M/x}(C_j - x)$ for some $\{i, j\} = \{1, 2\}$, so $C_i \subseteq cl_M(C_j)$, a contradiction. It follows that $cl(C_2) - C_2 = \{y\}$. By symmetry, $cl(C_1) - C_1 = \{y'\}$.

By Lemma 3.3.5(iii), M has $cl(C_1)$ and $cl(C_2)$ as hyperplanes, and $|C_1| = |C_2|$. Let C be a circuit of M that is not C_1, C_2, C'_1, C'_2 , or D . If $y \in C \subseteq C_2 \cup y$, then, by 3.3.7.4 and 3.3.7.5, C is D or C'_2 . We deduce that C meets both $C_1 - cl(C_2)$ and $C_2 - cl(C_1)$. Then, by Lemma 3.3.5(iii), either C is spanning, or C contains $C_1 \triangle C_2$. But $|C_1 \cap C_2| = 1$ so C is spanning. We conclude that C_1, C_2, C'_1, C'_2 , and D are the only non-spanning circuits of M . Hence $M \cong P_n(2)$ for some $n \geq 4$.

Finally, suppose $|C_1 \cap C_2| = 2$. Then M is not 2-laminar so it has as a minor one of the matroids identified in Theorem 3.3.6. But M cannot have a $N_n(2)$ -minor for any $n \geq 5$ as this matroid has $P_{n-1}(2)$, an excluded minor for the class of 2-closure-laminar matroids,

as a proper minor. Thus M has as a minor $M^-(K_{2,3})$ or $M_n(2)$ for some $n \geq 4$. The result follows by Lemmas 3.3.1 and 3.3.2. \square

Our methods for finding the excluded minors for the classes of k -laminar and k -closure-laminar matroids for $k = 2$ do not seem to extend to larger values of k .

3.4. Intersections with other classes of matroids

We now discuss how the classes of k -closure-laminar and k -laminar matroids relate to some other well-known classes of matroids. Finkelstein [16] showed that all laminar matroids are gammoids, so they are representable over all sufficiently large fields [29, 20]. An immediate consequence of the following easy observation is that, for all $k \geq 2$, if M is a k -closure-laminar matroid or a k -laminar matroid, then M need not be representable and hence M need not be a gammoid. To see this, note that the non-Pappus matroid, see Figure 3.2, has rank three, but it is not representable.

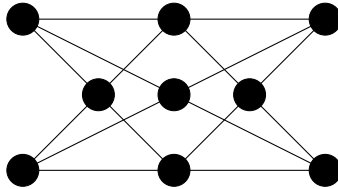


Figure 3.2. A geometric presentation for the non-Pappus matroid.

Proposition 3.4.1. *If $r(M) \leq k + 1$, then M is k -laminar and k -closure-laminar.*

We use the next lemma to describe the intersection of the classes of 2-laminar and 2-closure-laminar matroids with the classes of binary and ternary matroids. Recall that $M_4(2) \cong M(K_{2,3})$ and that the definitions of $P_n(2)$ and $N_n(2)$ require that $n \geq 4$ and $n \geq 5$, respectively.

Lemma 3.4.2. *The matroid $M^-(K_{2,3})$ is ternary and non-binary; $P_n(2)$ has a $U_{n,2n-3}$ -minor; $N_n(2)$ has a $U_{n,2n-4}$ -minor; and $M_n(2)$ has a $U_{n,2n-3}$ -minor when $n \geq 5$.*

Proof. The first part follows because $M^-(K_{2,3})$ can be obtained from $U_{2,4}$ by adding elements in series to two elements of the latter. Next we note that we get $U_{n,2n-3}$ from $P_n(2)$ by deleting the basepoints of the parallel connections involved in its construction. Deleting the basepoints of the parallel connections involved in producing $N_n(2)$ gives $U_{n,2n-4}$. Finally, when $n \geq 5$, we get $U_{n,2n-3}$ from $M_n(2)$ by deleting an element of the path P . \square

The next two results follow without difficulty by combining the last lemma with Theorems 3.3.6 and 3.3.7 as the set of excluded minors for $\mathcal{M} \cap \mathcal{N}$ where \mathcal{M} and \mathcal{N} are minor-closed classes of matroids consists of the minor-minimal matroids that are excluded minors for \mathcal{M} or \mathcal{N} (see, for example, [25, Lemma 14.5.1]). Recall that $N_5(2)$ and $P_4(2)$ are the matroids obtained by adjoining, via parallel connection, two triangles across distinct elements of a 4-circuit and a triangle, respectively.

Corollary 3.4.3. *A matroid M is binary and 2-laminar if and only if it has no minor isomorphic to $U_{2,4}$, $M(K_{2,3})$, or $N_5(2)$.*

Corollary 3.4.4. *A matroid M is binary and 2-closure-laminar if and only if it has no minor isomorphic to $U_{2,4}$, $M(K_{2,3})$, or $P_4(2)$.*

Similarly, we find the excluded minors for the classes of ternary 2-laminar matroids and ternary 2-closure-laminar matroids by noting that deleting an element from F_7^* produces $M(K_{2,3})$, so F_7^* is not 2-laminar.

Corollary 3.4.5. *A matroid M is ternary and 2-laminar if and only if it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, F_7 , $M^-(K_{2,3})$, $M(K_{2,3})$, or $N_5(2)$.*

Corollary 3.4.6. *A matroid M is ternary and 2-closure-laminar if and only if it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, F_7 , $M^-(K_{2,3})$, $M(K_{2,3})$, or $P_4(2)$.*

Next we describe the intersection of the class of graphic matroids with the classes of 2-laminar and 2-closure-laminar matroids both constructively and via excluded minors.

Corollary 3.4.7. *A matroid M is graphic and 2-laminar if and only if it has no minor isomorphic to $U_{2,4}$, $M(K_{2,3})$, F_7 , $M^*(K_{3,3})$, or $N_5(2)$.*

Corollary 3.4.8. *A matroid M is graphic and 2-closure-laminar if and only if it has no minor isomorphic to $U_{2,4}$, $M(K_{2,3})$, F_7 , or $P_4(2)$.*

Lemma 3.4.9. *Let M be a simple, connected, graphic matroid. Then M is 2-laminar if and only if M is a coloop, M is isomorphic to $M(K_4)$, or M is the cycle matroid of a graph consisting of a cycle with at most two chords such that, when there are two chords, they are of the form (u, v_1) and (u, v_2) where v_1 is adjacent to v_2 .*

Proof. Clearly each of the specified matroids is 2-laminar. Now let G be a simple, 2-connected graph. Suppose first that G is not outerplanar. By a theorem of Chartrand and Harary [7], either G is K_4 or G has $K_{2,3}$ as a minor. In the latter case, $M(G)$ is not 2-laminar. Now suppose that G is outerplanar. If G has two chords that are not of the form (u, v_1) and (u, v_2) where v_1 is adjacent to v_2 , then $M(G)$ has $N_4(2)$ as a minor, and so is not 2-laminar. □

Proposition 3.4.10. *Let M be a simple, connected, graphic matroid. Then M is 2-closure-laminar if and only if M is a coloop, M is isomorphic to $M(K_4)$, or M is the cycle matroid of a cycle with at most one chord.*

Proof. This follows from Lemma 3.4.9 by noting that the cycle matroid of a cycle with two chords of the form (u, v_1) and (u, v_2) where v_1 is adjacent to v_2 has $P_4(2)$ as a minor. □

Recall that a connected matroid is unbreakable if M/F is connected for each rank one flat F of M .

Lemma 3.4.11. *A connected laminar matroid is unbreakable.*

Proof. Let (E, \mathcal{A}, c) be a canonical presentation for M . If F is a rank one flat of M containing e , then $M/F = M/e - (F - e)$, so it suffices to show that for all $e \in M$, the matroid M/e has a spanning circuit. By 2.3.1.2, we have that $M/e = M(E - e, \mathcal{A} - e, c'')$, where $\mathcal{A} - e = \{A - e : A \in \mathcal{A}\}$ and

$$c''(A - e) = \begin{cases} c(A) - 1 & \text{if } e \in A; \\ c(A) & \text{if } e \notin A. \end{cases}$$

However, this presentation need not be canonical. By Lemma 2.2.12, M has a spanning circuit C with $e \in C$. To see that $(C - e)$ is a circuit of M/e , note that $c(A) - |C \cap A| = c(A - e) - |(C - e) \cap (A - e)|$, for all $A \in \mathcal{A}$. So, by Lemma 2.2.12 M/e is indeed connected. □

The parallel connection of two triangles is not unbreakable, so for all $k \geq 2$, a connected k -laminar matroid and, hence, a connected k -closure-laminar matroid need not be unbreakable.

We now show that the intersections of the classes of k -laminar and k -closure-laminar matroids with the class of paving matroids coincide.

Theorem 3.4.12. *Let M be a paving matroid, and k be a non-negative integer. Then M is k -laminar if and only if M is k -closure-laminar.*

Proof. By Proposition 3.2.1(i), it suffices to prove that if M is not k -closure-laminar, then M is not k -laminar. We use the elementary observation that, since M is paving, for every flat F , either $F = E(M)$, or $M|F$ is uniform. Suppose that C_1 and C_2 are circuits of M for which $r(cl(C_1) \cap cl(C_2)) \geq k$ but neither $cl(C_1)$ nor $cl(C_2)$ is contained in the other. Then neither $cl(C_1)$ nor $cl(C_2)$ is spanning. Hence both $M|cl(C_1)$ and $M|cl(C_2)$ are uniform. Let X be a basis of $cl(C_1) \cap cl(C_2)$. Then M has circuits C'_1 and C'_2 containing X such that $cl(C'_i) = cl(C_i)$ for each i . Thus M is not k -laminar. \square

It is well known that the unique excluded minor for the class of paving matroids is $U_{0,1} \oplus U_{2,2}$. Using this, in conjunction with Theorems 3.3.6 and 3.4.12, it is not difficult to obtain the following.

Corollary 3.4.13. *The following are equivalent for a matroid M .*

- (i) M is 2-laminar and paving;
- (ii) M is 2-closure-laminar and paving;
- (iii) M has no minor in $\{U_{0,1} \oplus U_{2,2}, M^-(K_{2,3})\} \cup \{M_n(2) : n \geq 4\}$.

Chapter 4. Unbreakable Frame Matroids

4.1. Introduction to unbreakable matroids

The results in this chapter are based on joint work with Dillon Mayhew, James Oxley, and Charles Semple which has not yet been published[13]. Sets X and Y in a matroid are *skew* if $r(X) + r(Y) = r(X \cup Y)$. In particular, two cycles C and D of G , are skew in $M(G)$ if $|V(C) \cap V(D)| \leq 1$. A matroid M is *unbreakable* if M is connected and, for every flat F of M , the contraction M/F is connected. Pfeil [28] showed that a matroid is unbreakable if and only if its dual has no two skew circuits. Indeed, unbreakable matroids grew out of an attempt to find a matroid analogue of graphs with no two vertex-disjoint circuits.

Frame matroids, which were introduced by Zaslavsky [38, 39] as bias matroids, are a fundamental class of matroids. Geometrically, such matroids coincide with the restrictions of those matroids in which each non-loop element lies on a line joining two elements of a fixed basis. Frame matroids include graphic, bicircular, and signed-graphic matroids. Irene Pivotto gave a good introduction to frame matroids and related classes of matroids in a three-part blog post [30, 31, 32]. Frame matroids are also discussed in [25, Section 6.10].

Pfeil [28] determined all unbreakable regular matroids. Both of the one-element matroids, $U_{0,1}$ and $U_{1,1}$, are unbreakable.

Theorem 4.1.1. *A regular matroid M with at least two elements is unbreakable if and only if M is loopless and $\text{si}(M)$ is isomorphic to $M^*(K_{3,3})$, R_{10} , or the cycle matroid of a complete graph or cycle with at least three vertices.*

In particular, this theorem shows that a loopless graphic matroid M is unbreakable if and only if $\text{si}(M)$ is isomorphic to the cycle matroid of a complete graph or a cycle. The purpose of this chapter is to prove the following generalization of Theorem 4.1.1.

Theorem 4.1.2. *Let $M(G, \Psi)$ be a 3-connected unbreakable frame matroid and assume that G has no isolated vertices. Then either $|V(G)| \leq 6$, or the simple graph associated with G is obtained from a complete graph by deleting the edges of a path of length at most two.*

For 3-connected unbreakable bicircular matroids, we can be even more explicit.

Theorem 4.1.3. *Let M be the bicircular matroid of a graph G having no isolated vertices. If M is 3-connected and unbreakable, then either $|V(G)| \leq 6$, or the simple graph associated with G is complete.*

This theorem is a consequence of the following more general result, which is itself a corollary of Theorem 4.1.2.

Theorem 4.1.4. *Let $M(G, \Psi)$ be a 3-connected unbreakable frame matroid and assume that G has no isolated vertices. If Ψ contains no 3-cycles, then either $|V(G)| \leq 6$, or the simple graph associated with G is complete.*

To see that we cannot sharpen the bound $|V(G)| \leq 6$ in the last three theorems, we consider the bicircular matroid M of the 9-edge graph that is obtained from a 6-cycle by adding an edge in parallel to every second edge. Then M^* is the rank-3 matroid that is obtained from a 3-element basis by freely adding two points on each line that is spanned by two of the basis elements. This matroid is clearly 3-connected having no two skew circuits. Hence M is 3-connected and unbreakable.

In Section 4.3, we prove a more specific version of Theorem 4.1.2 when the underlying graph has a 2-vertex cut. We conclude the proof of Theorem 4.1.2 and prove Theorem 4.1.4 in Section 4.4. In the next section, we note some preliminaries that will be used in these proofs. In Section 4.5, we prove the following result, which can be viewed as a partial converse to Theorem 4.1.2.

Theorem 4.1.5. *Let H be a simple graph with at least seven vertices that is complete or can be obtained from a complete graph by deleting one edge or two adjacent edges. Then there is a 3-connected unbreakable matroid $M(G, \Psi)$ such that H is the simple graph associated with G .*

4.2. Preliminaries

This section contains a number of lemmas that we will use in the proofs of the main results. The first was proved by Pfeil [28].

Lemma 4.2.1. *If M is an unbreakable matroid and F is a flat of M , then M/F is also unbreakable.*

Zaslavsky [40] proved that the class of frame matroids is closed under taking minors. It will be useful to recall how one shows this. Let M be a frame matroid, $M(G, \Psi)$, and let e be an edge of G . Then $M \setminus e = M(G \setminus e, \Psi \setminus e)$ where $\Psi \setminus e$ is the collection of cycles in Ψ that do not contain e . Contraction is not as easy to describe. Suppose first that $\{e\}$ is a balanced loop. Then e is a loop in the matroid M , so $M(G, \Psi)/e = M(G, \Psi) \setminus e$. Next, let $\{e\}$ be an unbalanced loop at the vertex v . Then $M(G, \Psi)/e = M(G', \Psi')$, where G' and Ψ' are constructed as follows. First delete e from G and declare all remaining loops at v to be in Ψ' . Then, for every edge joining v to some other vertex u , replace that edge

by a loop at u and declare that this loop is not in Ψ' . Finally, take each cycle in Ψ that avoids v and add it to Ψ' . The last possibility for the edge e is that it joins distinct vertices u and v of G . In that case, $M(G, \Psi)/e = M(G/e, \Psi'')$ where Ψ'' consists of the minimal sets of the form $C - e$ where $C \in \Psi$.

In a biased graph, (G, Ψ) , a subgraph H of G is *balanced* if every cycle in H is balanced. Zaslavsky [40] proved the following result.

Proposition 4.2.2. *Let (G, Ψ) be a biased graph and X be a subset of $E(G)$. Then*

- (i) *X is independent in $M(G, \Psi)$ if and only if $G[X]$ has no balanced cycles and no component with more than one cycle; and*
- (ii) *the rank of X in $M(E, \mathcal{A}, c)$ is given by $r(X) = |V(G[X])| - k'(G[X])$, where $k'(G[X])$ is the number of balanced components of $G[X]$.*

The next result is a straightforward consequence of the previous result.

Lemma 4.2.3. *Let (G, Ψ) be a biased graph and L be its set of balanced loops. If $U \subseteq V(G)$, then $E(G[U]) \cup L$ is a flat of $M(G, \Psi)$.*

We conclude this section with four lemmas that will be useful in the proof of Theorem 4.1.2.

Lemma 4.2.4. *Let $M(G, \Psi)$ be a simple frame matroid. Let H be a vertex-induced subgraph of G and let C be a shortest unbalanced cycle in H . If $|C| \geq 3$, then C is induced.*

Proof. Suppose $G[C] \setminus E(C)$ has an edge e . As $|C| \geq 3$ and $M(G, \Psi)$ is simple, e is not in a 2-cycle with an edge of C . Thus $|C| \geq 4$ and e is a chord of C . Then $G[C \cup e]$ is a Θ -graph. By the choice of C , the two shorter cycles in this Θ -graph are balanced, so C must also be balanced, a contradiction. □

For a vertex z in a graph G , we denote by E_z the set of edges meeting z .

Lemma 4.2.5. *Let (G, Ψ) be a biased graph and C be an unbalanced cycle of G . If w is in $V(G) - V(C)$ and w is adjacent to at least two vertices of C , then there is an unbalanced cycle C_w with $w \in V(C_w)$ and $E(C_w) \subseteq E(C) \cup E_w$.*

Proof. Let H be the subgraph of G induced by $C \cup \{f, g\}$ where f and g join w to distinct vertices of C . Then H is a Θ -graph containing the unbalanced cycle C . Thus, at least one of the cycles using w is unbalanced, so the lemma holds. \square

Lemma 4.2.6. *Let (G, Ψ) be a biased graph and C be an unbalanced cycle of G with $|C| \geq 3$. If G has a vertex w that is adjacent to each vertex of $V(C) - w$, then there is an unbalanced 3-cycle C_w with $w \in V(C_w)$ and $E(C_w) \subseteq E(C) \cup E_w$.*

Proof. If $w \notin V(C)$, then, by Lemma 4.2.5, we have an unbalanced cycle C' with $w \in V(C')$ and $E(C') \subseteq E(C) \cup E_w$. If $w \in V(C)$, let $C' = C$. Let u and v be the neighbors of w in the subgraph C' . Let H be a subgraph of G induced by a set of edges consisting of C' along with exactly one edge between w and each vertex in $V(C') - \{u, v, w\}$. Let C_w be a shortest unbalanced cycle that uses w and is contained in H . Clearly $|C_w| \geq 3$. As H is simple, $M(G, \Psi)|_{E(H)}$ is simple, so, by Lemma 4.2.4, C_w is an induced cycle of H . As w is adjacent to every vertex of $V(C') - w$, it follows that C_w is a 3-cycle. \square

Lemma 4.2.7. *Let M be a frame matroid $M(G, \Psi)$ where, G is a connected graph. Let C be an unbalanced cycle and let T be a tree in G such that $V(C) \subseteq V(T)$ and $G - V(T)$ is disconnected. Then M is not unbreakable.*

Proof. If we contract the edges of T from G , then the composite vertex that results by identifying all of the vertices of T is a cut vertex in the resulting graph, and this vertex

meets at least one unbalanced loop. Contracting such a loop yields a biased graph with more than one component having an edge. We deduce that $M/E(G[V(T)])$ is disconnected. This implies that M is not unbreakable for this is certainly true if M has any balanced loops and otherwise holds by Lemma 4.2.3 since $E(G[V(T)])$ is a flat of $M(G, \Psi)$.

□

4.3. Beginning the proof of the main theorem

The purpose of this section is to prove the first of two theorems the combination of which yields the main result of the chapter. It is commonplace in matroid theory to use $\text{si}(M)$ to denote the simple matroid associated with a matroid M . It will be convenient here to use the same notation for graphs. Thus, for a graph G , we denote by $\text{si}(G)$ the graph that is obtained from G by deleting all the loops of G , deleting any isolated vertices of G , and deleting all but one edge from each parallel class of G . As with matroids, we will not be concerned with the edges labels on $\text{si}(G)$ but only with the isomorphism type of this graph. We will not need or define a simplification on a biased graph. Paths and cycles will occur frequently in the proof. If D is a path or cycle, we will frequently use D to denote its edge set $E(D)$. Its vertex set will be denoted by $V(D)$.

Throughout this and the next section, we shall assume that M is a 3-connected unbreakable frame matroid $M(E, \mathcal{A}, c)$ and that G has no isolated vertices. We shall also assume that M is not graphic since the case where M is graphic is dealt with by Theorem 4.1.1. Then G has at least one unbalanced cycle. Moreover, $|E(G)| \geq 4$ so M has no 1- or 2-circuits. Thus we have the following result.

Lemma 4.3.1. *All 1- and 2-cycles in G are unbalanced.*

Because M is connected, G is certainly connected. Thus, by Proposition 4.2.2(ii),
 $r(M) = |V(G)|$.

Lemma 4.3.2. *G has no vertex that meets fewer than three edges.*

Proof. Suppose G has a vertex u for which the set E_u of edges meeting u has size at most two. Then $r_M(E(G) - E_u) \leq |V(G - u)| = |V(G)| - 1$, so E_u contains a cocircuit of M .

This contradicts the fact that M is 3-connected having at least four elements. \square

Next we show the following.

Lemma 4.3.3. *G is 2-connected.*

Proof. Suppose that G has a cut vertex v . Let A_1 be a component of $G - v$. Let A be the graph induced by the vertex set $V(A_1) \cup v$, and let B be the graph induced by the edge set $E(G) - E(A)$. By Proposition 4.2.2, $r(M) = |V(G)| = |V(A)| + |V(B)| - 1$.

As $r(E(A)) \leq |V(A)|$ and $r(E(B)) \leq |V(B)|$, we see that $r(E(A)) + r(E(B)) - r(M) \leq |V(A)| + |V(B)| - (|V(A)| + |V(B)| - 1) = 1$. By Lemma 4.3.2, we deduce that $(E(A), E(B))$ is a 2-separation of M , a contradiction. \square

Lemma 4.3.4. *If $\text{si}(G)$ is a cycle, then $|V(G)| \leq 6$.*

Proof. By Lemmas 4.3.1 and 4.3.2, every vertex x of G must meet an unbalanced cycle C_x of size at most two. Fix a vertex v of G and such an unbalanced cycle C_v . If $|V(G)| \geq 7$, then there is a vertex u that has distance at least three from each of the vertices in C_v .

Then, for each choice of C_u , the matroid $M/cl(C_v \cup C_u)$ is disconnected, a contradiction. \square

The proof of Theorem 4.1.2 will distinguish the cases when G is 3-connected and when it is not, beginning with the latter.

Theorem 4.3.5. *Let $M(G, \Psi)$ be a 3-connected unbreakable frame matroid, M , and assume that G is 2-connected, but not 3-connected. Then G has at most six vertices.*

Proof. Assume that $|V(G)| \geq 7$. Let $\{u, v\}$ be a vertex cut in G . Let A_1 and B_1 be disjoint non-empty graphs each a disjoint union of components of $G - \{u, v\}$ such that $A_1 \cup B_1 = G - \{u, v\}$. Let (A, B) be a partition of $E(G)$ with $A \subseteq G[V(A_1) \cup \{u, v\}]$ and $B \subseteq G[V(B_1) \cup \{u, v\}]$. Hence, each edge joining u and v , and each unbalanced loop incident to u or v , can lie in A or B . Assume initially that each such edge lies in B . Because M is 3-connected, Proposition 4.2.2(ii) implies that $G[A]$ is unbalanced. By symmetry, we deduce that each of A and B contains an unbalanced cycle that contains no edge joining u and v and is not an unbalanced loop incident to u or v .

Next we show the following.

4.3.5.1. *Suppose B has a path P_B joining u and v that does not use all of the vertices of B . Let C_A be an unbalanced cycle in A , and let P^u and P^v be internally disjoint paths from u and v to C_A with each such path using a single vertex of C_A . Then*

- (i) P^u and P^v each have at most one edge;
- (ii) $V(C_A) \cup \{u, v\} = V(A)$;
- (iii) $|V(A) - \{u, v\}| \leq 2$; and
- (iv) if $|V(A) - \{u, v\}| = 2$, then no edge in P^u or P^v is in a 2-cycle.

Parts (i) and (ii) will follow from Lemma 4.2.7. When C_A uses both u and v , we let T consist of all but one edge of C_A . By Lemma 4.2.7, $G - V(T)$ must be connected, so

$V(C_A) = V(A)$. When C_A contains v but not u , let T be a tree whose edges consist of P_B and all but one edge of C_A . Then $V(C_A) \cup \{u\} = V(A)$, so P^u has just one edge. Finally, suppose C_A contains neither u nor v . Let T be a tree whose edge set consists of P_B , P^u , and all but one edge of C_A . Then $V(A) = V(C_A) \cup V(P^u) \cup \{v\}$. Thus P^v consists of a single edge. By symmetry, P^u also consists of a single edge. Hence $V(A) - \{u, v\} = V(C_A)$. Thus (i) and (ii) hold.

To prove (iii), we add the assumption that C_A is a shortest unbalanced cycle in A and assume that $|V(A) - \{u, v\}| \geq 3$. Then, by (ii), $|C_A| \geq 3$. Let x and y be the vertices in $V(C_A) \cap V(P^u)$ and $V(C_A) \cap V(P^v)$, respectively, and choose w in $V(C_A) - \{x, y\}$. Then, by Lemma 4.3.2, w is incident with an edge f not in C_A . By Lemma 4.2.4, the other endpoint of f is not in $V(C_A)$. Hence, by (ii), it is in $\{u, v\}$, and, without loss of generality, we may assume that it is u . Thus $u \neq x$, so P_u has a single edge.

Consider the Θ -graph H with edge set $C_A \cup P^u \cup f$ and let D be the cycle in this Θ -graph avoiding y . As the next step towards proving 4.3.5.1(iii), we show that

4.3.5.2. D is balanced.

Suppose first that $y = v$. Then $|D| \leq |C_A|$, so, by minimality, D is balanced unless equality holds here. In the exceptional case, the neighbors of v on C_A are w and x . Since $|V(C_A) - \{u, v\}| = |V(A) - \{u, v\}| \geq 3$, there is an internal vertex t of the (x, w) -path in C_A avoiding v . As t does not have degree two and C_A has no chords, A has an edge joining t and u . Thus u is adjacent to every vertex of $V(D) - u$. It follows that D is balanced, otherwise, by Lemma 4.2.6, G has an unbalanced 3-cycle containing u and avoiding v . This contradicts the choice of C_A .

We may now assume that $y \neq v$. If D is unbalanced, then, using it as the unbalanced cycle in (ii), we obtain a contradiction since D avoids y . We conclude that 4.3.5.2 holds.

Because D is balanced but C_A is not, the third cycle, J , in the Θ -graph H must be unbalanced. Taking the subgraph of G whose edge set is $J \cup P_B \cup P^v$ gives us a Θ -graph containing cycles J' and J'' that avoid x and w , respectively. Consider a tree that is obtained from J' by deleting an edge. By Lemma 4.2.7, J' is balanced. Similarly, J'' is balanced, so J is balanced, a contradiction. Thus 4.3.5.1(iii) holds.

To prove 4.3.5.1(iv), assume that $V(A) - \{u, v\} = \{s, t\}$. Suppose that there are at least two edges joining x and u . Let T be a tree consisting of one of these edges together with the path P_B . Then, by Lemma 4.2.7, M is not unbreakable, a contradiction. We deduce that there is at most one edge between x and u . By symmetry, there is at most one edge between y and v . Hence 4.3.5.1(iv) holds. This completes the proof of 4.3.5.1.

Previously, for a 2-vertex cut $\{u, v\}$ in G , we defined subgraphs A and B whose union is G . If both A and B contain (u, v) -paths that do not use all of their vertices, then, by 4.3.5.1(iii), $|V(G)| \leq 6$. If each of $\text{si}(A)$ and $\text{si}(B)$ is a path, then $\text{si}(G)$ is a cycle, so, by Lemma 4.3.4, $|V(G)| \leq 6$. Thus we may assume that exactly one of $\text{si}(A)$ and $\text{si}(B)$ is a path. It follows that we may also assume that G has no edge joining u and v .

Now, we choose the vertex cut $\{u, v\}$ and the subgraphs A and B such that $\text{si}(A)$ is not a path and $|V(A)|$ is a minimum subject to this requirement. Then $\text{si}(B)$ is a path. Let C_A and C_B be shortest unbalanced cycles in A and B , respectively, such that neither cycle uses an edge joining u to v and neither cycle is an unbalanced loop incident to either u or v . Subject to this, choose $|V(C_A) \cap \{u, v\}|$ to be a maximum. By 4.3.5.1(iii),

$|V(B)| \leq 4$. Let P_A^u and P_A^v be disjoint paths from C_A to u and v , respectively, chosen so that $|P_A^u| + |P_A^v|$ is a minimum. Let x and y be the vertices of C_A that are also in P_A^u and P_A^v , respectively. Next we note that

4.3.5.3. $V(A) = V(C_A) \cup V(P_A^u) \cup V(P_A^v)$.

To see this, note that if $V(A) \neq V(C_A) \cup V(P_A^u) \cup V(P_A^v)$ and T is a tree whose edge set is $P_A^u \cup P_A^v$ together with all but one edge of C_A , then $G - V(T)$ is disconnected, a contradiction to Lemma 4.2.7. Thus 4.3.5.3 holds.

Next we show that

4.3.5.4. C_A does not use both u and v .

Suppose otherwise. Then, since G has no edge joining u and v , the cycle C_A has a vertex not in $\{u, v\}$. By Lemma 4.3.2, this vertex has degree at least three, so the cycle C_A is not an induced cycle of $G[A]$, a contradiction to Lemma 4.2.4. Hence 4.3.5.4 holds.

4.3.5.5. If $u \notin V(C_A)$, then P_A^u has a single edge.

Suppose not, letting u' be the neighbour of u on the path P_A^u . Observe that $\{u', v\}$ cannot be a vertex cut of G otherwise $\text{si}(A - u)$ is a path and so $\text{si}(A)$ is a path, a contradiction. Thus u is adjacent to some vertex w of $V(A) - \{u', v\}$. By the choice of P_A^u , we see that $w \in V(P_A^v)$. The union of an edge joining u and w with the edge set of P_A^v and all but one edge of C_A is a tree T such that $G - V(T)$ is disconnected, a contradiction to Lemma 4.2.7. We conclude that 4.3.5.5 holds.

4.3.5.6. $|C_A| \geq 3$.

This follows by 4.3.5.3, 4.3.5.5, and symmetry, otherwise $|V(G)| \leq 6$.

The choice of C_A implies that A has no unbalanced 2-cycles, nor unbalanced loops.

Hence, by Lemma 4.3.2, we have the following.

4.3.5.7. *Every vertex of C_A must be adjacent to a vertex outside of $V(C_A)$.*

By 4.3.5.5, we may now assume that $u \notin V(C_A)$. As the next step towards proving Theorem 4.3.5, we now show the following.

4.3.5.8. *For $|C_A| \geq 4$, suppose s and t are distinct vertices of C_A that are neighbours of u in G . Then C_A has an edge joining s and t .*

Let f and g be edges joining u to s and t , respectively. Then, in the Θ -graph H with edge set $C_A \cup \{f, g\}$, at least one cycle meeting u is unbalanced. Let C'_A be such an unbalanced cycle. As $|C'_A| \leq |C_A|$, equality must hold, so there is an (s, t) -path P^{st} in C_A of length two such that the edge set of C'_A is $\{f, g\} \cup (C_A - P^{st})$. Note that $v \in V(C_A)$, otherwise, as C'_A is an unbalanced cycle in A of length $|C_A|$ that uses u , we have a contradiction to the choice of C_A . By replacing C_A by C'_A in 4.3.5.4, we deduce that $v \notin V(C'_A)$. Thus v is the internal vertex of P^{st} . As the 4-cycle C''_A with vertex set $\{f, g\} \cup P^{st}$ uses u and v , it must be balanced. By Lemma 4.3.2 and 4.3.5.3, every vertex in $V(C_A) - \{v, s, t\}$ is adjacent to u . Then, as u is also adjacent to s and t , Lemma 4.2.6 gives us an unbalanced 3-cycle in A , a contradiction. We conclude that 4.3.5.8 holds.

Suppose $v \in V(C_A)$. Then, by 4.3.5.7, every vertex of $V(C_A) - v$ is adjacent to u . Thus, by 4.3.5.8, $|V(C_A)| \leq 3$, so $|V(G)| \leq 6$. We may now assume that $v \notin V(C_A)$. By 4.3.5.3 and 4.3.5.5, $V(C_A) = V(A) - \{u, v\}$. We show next that

4.3.5.9. $|V(B) - \{u, v\}| = 1$.

Recall that $|V(B)| \leq 4$. Suppose that $|V(B) - \{u, v\}| = 2$. By 4.3.5.1(iv), with the roles of A and B reversed, the 2-cycle C_B is vertex-disjoint from $\{u, v\}$. Then $cl(C_B)$ consists of all of the edges in the parallel class containing C_B , and $cl(C_A \cup C_B) = C_A \cup cl(C_B)$. Now, contracting the edges of $C_A \cup cl(C_B)$ from G produces a 2-vertex disconnected graph in which each of u and v meets an unbalanced loop. We deduce that $M/cl(C_A \cup C_B)$ is disconnected, a contradiction. Thus 4.3.5.9 holds.

As $|V(G)| \geq 7$, we deduce that $|C_A| \geq 4$. By 4.3.5.7, every vertex of C_A must be adjacent to u or v . Moreover, by symmetry, 4.3.5.8 holds when u is replaced by v . Using 4.3.5.8 for both u and v , we deduce that $|C_A| = 4$, and two consecutive vertices of C_A are adjacent to u , but not v , while the other two are adjacent to v , but not u . Also, by 4.3.5.8, no vertex of C_A is adjacent to both u and v .

We may assume that either C_B meets u or that C_B is an unbalanced loop incident to neither u nor v . In the former case, G has a tree T that uses one edge of C_B and otherwise consists of a path, in A , of length three that uses u , exactly one of the neighbours of u in C_A , and both of the neighbours of v on C_A . Deleting the vertices of T from G disconnects the graph, a contradiction to Lemma 4.2.7. In the latter case, contracting the edges in C_A and C_B yields a graph that has three unbalanced loops at each of u and v . Since $E(C_A) \cup E(C_B)$ is a flat, this completes the proof of Theorem 4.3.5. \square

4.4. Finishing the proof of the main theorem

In this section, we shall complete the proof of Theorem 4.1.2 by dealing with the case when G has no 2-vertex cut. In particular, we prove the following.

Theorem 4.4.1. *Let $M(G, \Psi)$ be a 3-connected unbreakable frame matroid, where G is 3-connected and $|V(G)| \geq 7$. Then $\text{si}(G)$ can be obtained from a complete graph by deleting the edges of a path of length at most two.*

We begin with some preparatory results.

Lemma 4.4.2. *Let M be a 3-connected unbreakable frame matroid $M(G, \Psi)$, where G is 3-connected and unbalanced. Then the following hold for any pair $\{x, y\}$ of nonadjacent vertices of G .*

- (i) $G - \{x, y\}$ is balanced.
- (ii) Every unbalanced cycle in G uses at least one of x and y .
- (iii) There is at least one unbalanced cycle in G that avoids x and at least one unbalanced cycle that avoids y .
- (iv) If C_y is a shortest unbalanced cycle in G containing y and avoiding x , and $|C_y| \geq 3$, then C_y is an induced subgraph of G .

Hence

- (v) for every unbalanced cycle C in G , the graph $\text{si}(G - V(C))$ is complete.

Proof. To show (i), suppose $G - \{x, y\}$ has an unbalanced cycle C . Let T be a spanning tree of $G - \{x, y\}$ using all but one edge of C . Then, as x and y are nonadjacent, $G - V(T)$ is disconnected, contradicting Lemma 4.2.7. Thus (i) holds. Part (ii) is a restatement of (i), and (v) is an immediate consequence of (ii).

To prove part (iii), assume that $G - x$ is balanced. Then the edge set W of $G - \{x, y\}$ is a flat of M . The graph G/W has three vertices including a cut vertex that results from identifying all the vertices in W . In G/W , all the cycles incident with y are bal-

anced, so this cut vertex actually induces a separation in M/W , a contradiction. Thus (iii) holds.

Part (iv) follows from Lemma 4.2.4 applied to $H = G - x$. To see this, note that C_y is a shortest unbalanced cycle of H since, by (i), $G - \{x, y\}$ is balanced. \square

For the rest of the section, u and v will denote a fixed pair of non-adjacent vertices of G , and W will denote the edge set $E(G - \{u, v\})$. By Lemma 4.4.2(iii), we can choose shortest unbalanced cycles C_u and C_v avoiding v and u , respectively. By Lemma 4.4.2(ii), C_u and C_v contain u and v , respectively. Our strategy will be to show that such cycles are small, and exploit the fact that, by Lemma 4.4.2(v) both $\text{si}(G - V(C_u))$ and $\text{si}(G - V(C_v))$ are complete graphs to show that $\text{si}(G)$ is almost a complete graph.

Lemma 4.4.3. *Let M be a 3-connected unbreakable frame matroid $M(G, \Psi)$ where, G is 3-connected and has at least one unbalanced cycle. Suppose that $|C_u| \geq |C_v|$.*

- (i) *Suppose that $|C_u| \geq 4$ and C is an unbalanced cycle of G that avoids u . Then C uses all but at most one vertex of $V(C_u) - u$. Moreover, if there is a vertex in $(V(C_u) - u) - V(C)$, then it must be adjacent to u .*
- (ii) $|C_u| \leq 4$.
- (iii) *Either $|C_v| \in \{1, 2\}$, or the subgraph of G induced by $C_u \cup C_v$ is one of the graphs shown in Figure 4.1.*
- (iv) *If $w \in V(G)$ is in an unbalanced cycle of size at most three, then w is nonadjacent to at most two other vertices.*
- (v) *Every vertex w of G is nonadjacent to at most three other vertices.*
- (vi) *If $|C_u| = 4$, then $|V(G)| \leq 6$ and $\text{si}(G)$ has at most seven edges fewer than the complete graph on $|V(G)|$ vertices.*

Proof. As noted above, C_u and C_v use u and v , respectively. To see (i), suppose that $|C_u| \geq 4$. First observe that, by Lemma 4.4.2(v), the subgraph of $\text{si}(G)$ induced by

$V(G) - V(C)$ must be complete. Thus all vertices in $C_u - u$ are either adjacent to u or in C . By Lemma 4.2.4, $G[V(C_u)]$ must be a cycle. Let u' and u'' be the neighbors of u in C_u . As u' and u'' are nonadjacent, by Lemma 4.4.2(ii), $\{u', u''\} \cap V(C) \neq \emptyset$ and since all the vertices in $V(C_u) - \{u, u', u''\}$ are in $V(C)$, statement (i) holds.

To show part (ii), suppose that $|C_u| \geq 5$ and that u_1, u_2, u_3 , and u_4 are distinct vertices in $C_u - u$, with u_1 adjacent to u and u_2 , and with u_4 adjacent to u and u_3 . We note that u_2 is not necessarily adjacent to u_3 . By (i), we may suppose that $V(C_u) - \{u, u_1\} \subseteq V(C_v)$. Now, as G is 3-connected, u_3 must have a neighbor w with $w \notin V(C_u)$. As u_3 and both of its neighbors in C_u are also in C_v , Lemma 4.2.4 implies that $w \notin V(C_v)$. Now, as $\{w, u\}$ avoids C_v , by Lemma 4.4.2(i), w is adjacent to u . Consider the Θ -graph H formed from C_u along with edges joining w to u_3 and to u . As C_u is a shortest unbalanced cycle using u and avoiding v , the cycle in H with vertex set $\{u, u_4, u_3, w\}$ is balanced. Let C'_u be the cycle in H that avoids u_4 . It must be unbalanced. Clearly, $|C_u| = |C'_u|$. Thus C'_u is also a shortest unbalanced cycle using u and avoiding v . Hence, by Lemma 4.2.4, C'_u has no chords.

We now note that $u_1 \in V(C_v)$; otherwise, by Lemma 4.4.2(ii), u_1 is adjacent to w , so C'_u had a chord, a contradiction. Now, let w' be a vertex adjacent to u_2 with $w' \notin V(C_u)$. By symmetry with w , we see that $w' \notin V(C_v)$, and w' is adjacent to u . As C'_u has no chords, $w \neq w'$. Since C_v avoids w and w' , we deduce by Lemma 4.4.2(v) that these vertices must be adjacent. Now, C'_u together with edges joining w' to u_2 and to w forms a Θ -graph H' . The cycle in H' using u_3 and w' avoids both u and v so must be balanced. Thus the third cycle C''_u in H' , whose vertex set is $\{u, u_1, u_2, w', w\}$, is unbalanced. But

this cycle has an edge joining u and w' as a chord. As $|C_u''| = 5 \leq |C_u|$, we deduce by Lemma 4.2.4 that C_u'' is balanced, a contradiction. We conclude that (ii) holds.

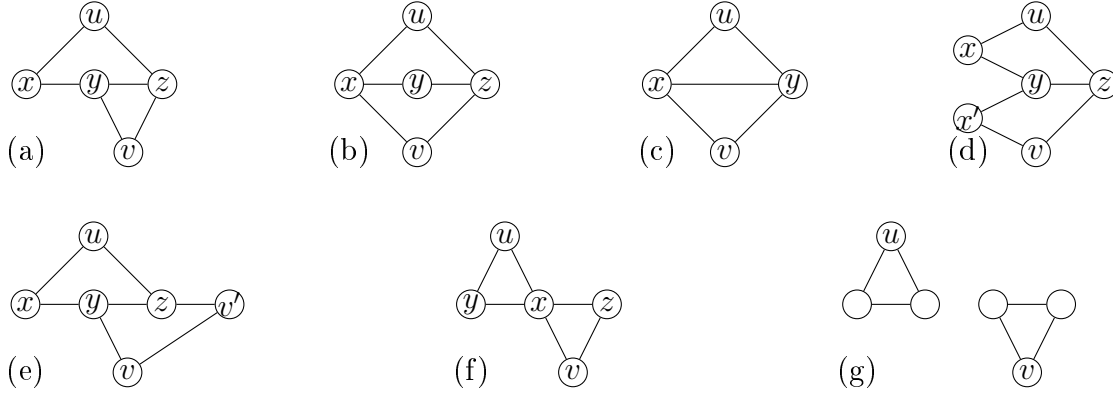


Figure 4.1. The possibilities for $G[C_u \cup C_v]$ in Lemma 4.4.3(iii).

Part (iii) follows from parts (i) and (ii) by straightforward case checking. Next we show (iv). Let S be the set of vertices that are not adjacent to w . Assume that $|S| \geq 3$. By Lemma 4.4.2(iii), there is a shortest unbalanced cycle C_s avoiding w . As $w \in V(G) - V(C_s)$, by Lemma 4.4.2(v), $S \subseteq V(C_s)$, so $|V(C_s)| \geq 3$. Choose a vertex s in $V(S)$. Let C_w be a shortest unbalanced cycle avoiding s . Any cycle containing $\{w, s\}$ has size at least four, and, by hypothesis, w is in some cycle of size at most three. Hence C_w is a shortest cycle containing w , so $|C_w| \leq 3 \leq |C_s|$. Thus, taking $(s, w) = (u, v)$, we deduce from (ii) that $|C_s| \leq 4$.

Suppose that $|C_s| = 4$. Then, by (i), $|C_w| \geq 3$, so $|C_w| = 3$. Then, by (iii), $G[C_s \cup C_w]$ is isomorphic to the graph in Figure 4.1(a). Since all of the vertices nonadjacent to w are in C_s but w is adjacent to two of the vertices of C_s , we obtain the contradiction that $|S| \leq 2$.

We may now suppose that $|C_s| < 4$, so $S = V(C_s)$ and $|C_s| = 3$. As $|C_w| \leq 3$, we see that $V(C_s) \cap V(C_w) = \emptyset$. Suppose $t \in V(G) - (V(C_s) \cup V(C_w))$. By Lemma 4.4.2(v),

$\text{si}(G - V(C_w))$ is complete, so t is adjacent to every vertex of C_s . By Lemma 4.2.6, there is an unbalanced 3-cycle C'_s with $t \in V(C'_s) \subseteq V(C_s) \cup \{t\}$. By Lemma 4.4.2(v) again, $\text{si}(G - V(C'_s))$ is complete. Thus the vertex of $V(C_s) - V(C'_s)$ is adjacent to w , a contradiction. We deduce that $V(G) = V(C_s) \cup V(C_w)$. By definition, no vertex of C_s is adjacent to w . By Lemma 4.3.2, w has degree at least three, so $G[V(C_w)]$ has at least three edges incident with w . By Lemma 4.4.2, if $|C_w| \geq 3$, then C_w is induced, a contradiction. Thus $|C_w| \leq 2$, so w is adjacent to at most one vertex. Hence G is not 3-connected, a contradiction. We conclude that (iv) holds.

Now, we show part (vi). Since $|V(C_u)| = 4$, by (i), there can be no unbalanced cycles of G of size less than three because any such unbalanced cycle must avoid u and so must use at least two vertices of C_u as well as v . Thus G is simple.

4.4.3.1. *No vertex w in $V(G) - V(C_u)$ is adjacent to u and each of its neighbors in C_u .*

By Lemma 4.4.2(iv), C_u has no chords, so u is adjacent to exactly two vertices, u' and u'' , of C_u . Suppose w is adjacent to each of the vertices in $\{u, u', u''\}$. By definition of C_u , every 3-cycle with vertex set in $V(C_u) \cup w$ is balanced. In particular, the cycles with vertex sets $\{w, u, u'\}$ and $\{w, u, u''\}$ are balanced. Hence so is the cycle with vertex set $\{w, u', u, u''\}$. As C_u is unbalanced, the cycle with vertex set $w \cup (V(C_u) - u)$ is unbalanced. But this cycle avoids $\{u, v\}$, a contradiction to Lemma 4.4.2(ii). Thus 4.4.3.1 holds.

By symmetry, if $|C_v| = 4$, then no vertex of $V(G) - V(C_v)$ is adjacent to v and both of its neighbors in C_v .

We complete the proof of (vi) by considering the cases in (iii) where $|C_u| = 4$. First suppose that $G[C_u \cup C_v]$ is as shown in Figure 4.1(d). As x and z are nonadjacent, a

shortest unbalanced cycle C_x avoiding z must contain x . By Lemma 4.4.2(v), $G - V(C_x)$ is complete, so $x' \in V(C_x)$ and two members of $\{u, v, y\}$ are in $V(C_x)$, as the subgraph induced by $\{u, v, y\}$ has no edges. Thus, by symmetry, we may assume that $V(C_x)$ is $\{x, x', u, v\}$ or $\{x, x', u, y\}$. Suppose G has a vertex w that is not in $V(C_u) \cup V(C_v)$. By Lemma 4.4.2(v), each of $G - V(C_u)$, $G - V(C_v)$, and $G - V(C_x)$ is complete. Thus w is adjacent to u , x , and z , a contradiction to 4.4.3.1. We conclude that $V(G) = V(C_u) \cup V(C_v)$, so $|V(G)| = 6$. Hence G is obtained from the complete graph on $\{u, x, y, z, x', v\}$ by deleting the edges (u, v) , (u, y) , (v, y) , (x, x') , (z, x) , and (z, x') as well as possibly (x, v) . We note that each of the choices of $V(C_x)$ has (u, x') as an edge. Hence, when $G[C_u \cup C_v]$ is as shown in Figure 4.1(d), G is simple, $|V(G)| = 6$, and G has at most seven fewer edges than K_6 .

Now suppose that $G[C_u \cup C_v]$ is as shown in Figure 4.1(a). By Lemma 4.4.2(v), $G - V(C_v)$ is complete. Thus every vertex of G not in $V(C_u) \cup V(C_v)$ must be adjacent to u . As u has degree at least three, there is such a vertex w . Moreover, w is adjacent to x . Suppose w is adjacent to z . The choice of C_u means that the cycles with vertex sets $\{u, w, z\}$ and $\{u, w, x\}$ are balanced. Hence, so is the cycle with vertex set $\{w, x, u, z\}$. As C_u is unbalanced, so is the cycle with vertex set $\{w, x, y, z\}$. This is a contradiction since this cycle avoids $\{u, v\}$. We deduce that w is not adjacent to z .

Next we show that w is not adjacent to y . Assume the contrary. Let C_w be a shortest unbalanced cycle avoiding z . Then $V(C_w)$ contains w . Also, by Lemma 4.4.2(v), $V(C_w)$ must contain x together with either u or $\{v, y\}$. The cycle with vertex set $\{u, w, x\}$ avoids v and so is balanced. By Lemma 4.2.4, C_w has no chords, so $V(C_w)$ cannot contain

$\{w, x, u\}$. Thus it contains $\{w, x, y, v\}$. But the cycle with vertex set $\{w, x, y\}$ implies that C_w has a chord, a contradiction. Hence w is not adjacent to y .

Now let C_w be a shortest unbalanced cycle avoiding y . Then $w \in V(C_w)$. Also, by Lemma 4.4.2(v), $u \in V(C_w)$ and either x or z is in $V(C_w)$. As C_w is chordless and $|V(C_u)| = 4$, we deduce that $\{w, u, x\} \not\subseteq V(C_w)$. By (ii), as $|C_w| \leq 4$ and w and z are nonadjacent, $V(C_w) = \{u, z, w, w'\}$ for some w' . Since (w, z) is not an edge of G , the edges of C_w are $(u, z), (z, w'), (w', w)$, and (w, u) . As C_w has no chords, u and w' are not adjacent. If $w' \neq v$, then, as $w' \notin V(C_v)$, we get w' is adjacent to u , a contradiction. Thus $V(C_w) = \{u, z, v, w\}$. Then, by (iv), y must be adjacent to every vertex in $V(G) - \{u, w\}$. But w was an arbitrarily chosen vertex in $V(G) - ((V(C_u) \cup V(C_v)))$ and we showed that y is not adjacent to w . We deduce that $V(G) = V(C_u) \cup V(C_v) \cup w$, so $|V(G)| = 6$. Moreover, since G has (w, v) as an edge, G has at most six edges fewer than the complete graph on $\{u, x, y, z, v, w\}$.

Next assume that $G[C_u \cup C_v]$ is as shown in Figure 4.1(b). Then y is adjacent to a vertex w not in $V(C_u) \cup V(C_v)$. By Lemma 4.4.2(v), we may assume that w is adjacent to u and then, by symmetry, that the cycle with vertex set $\{u, w, y, z\}$ is unbalanced. Replacing C_u by this cycle reduces us to the case when $G[C_u \cup C_v]$ is as shown in Figure 4.1(d), which was dealt with above.

Next suppose that $G[C_u \cup C_v]$ is as shown in Figure 4.1(e). Assume first that u and v' are not adjacent. Observe that, because C_u is a shortest unbalanced cycle avoiding v , it is also a shortest unbalanced cycle avoiding v' ; otherwise, there is an unbalanced 3-cycle using v and not v' that, because it cannot use u , violates the choice of C_v . As C_v is a shortest unbalanced cycle avoiding u , by interchanging the labels on v and v' , we reduce

to the previously considered case in Figure 4.1(d). We may now assume that u and v' are adjacent. By symmetry, v and x are also adjacent. Now let D be a shortest unbalanced cycle avoiding z . Then, by Lemma 4.4.2(v), $v \in V(D)$ and $x \in V(D)$. Moreover, y or u is in $V(D)$. The former does not occur as $V(D) \neq \{x, y, v\}$ and D has no chords. Thus $y \notin V(D)$, so $u \in V(D)$. Also, by Lemma 4.4.2(v), $v' \in V(D)$. Since $|D| \leq 4$, we deduce that $V(D) = \{u, x, v, v'\}$. Suppose G has a vertex w that is not in $V(C_u) \cup V(C_v)$. Then, as each of $G - V(C_u)$, $G - V(C_v)$, and $G - V(D)$ is complete, w is adjacent to u, x, y , and z . By Lemma 4.2.6, G has an unbalanced 3-cycle C_w using w and two vertices in $\{u, x, y, z\}$. Then C_w violates the choice of C_u . We deduce that $V(G) = V(C_u) \cup V(C_v)$. Moreover, $|E(G)| \geq 9$. We conclude that (vi) holds.

To prove (v), again we let S be the set of vertices that are not adjacent to w . Suppose that $|S| \geq 4$. Take s in S , and let C and D be shortest unbalanced cycles avoiding s and w , respectively. Then $S \subseteq V(D)$, so $|D| \geq 4$. Moreover, $w \in C$. By (iv), we may assume that $|C| \geq 4$. By (ii), $|D| = 4 = |C|$. Then, by (iii) and (vi), $|V(G)| \geq 6$. But w is adjacent to some vertex in D , so $|S| \leq 3$, a contradiction. Thus (v) holds and the proof of the lemma is complete. \square

We can now complete the proof of the main theorem of this section.

Proof of Theorem 4.4.1. We begin by proving the following.

4.4.4.1. *Let H be a simple 3-connected graph on at least seven vertices. Let u, v_1, v_2 , and v_3 be distinct vertices of H such that H has none of the edges $(u, v_1), (u, v_2)$, and (u, v_3) . Then $H \neq \text{si}(G)$.*

Assume the contrary. Let C_v be a shortest unbalanced cycle avoiding u . Then, by Lemma 4.4.2(v), $\{v_1, v_2, v_3\} \subseteq V(C_v)$. Let C_u be a shortest unbalanced cycle avoiding v_1 . Then $u \in V(C_u)$. By Lemma 4.4.3(vi), as $|V(G)| \geq 7$, neither $|C_v|$ nor $|C_u|$ is 4. Thus $|C_v| = 3$ and $|C_u| \leq 3$. Then $V(C_u) \cap V(C_v) = \emptyset$ as u is not adjacent to some vertex in $\{v_1, v_2, v_3\}$. Moreover, there is a vertex y that is not in $V(C_u) \cup V(C_v)$. By Lemma 4.4.2(v), y is adjacent to each vertex of $G - V(C_u)$. In particular, y is adjacent to each vertex of C_v . by Lemma 4.4.2(v), $\text{si}(G - V(C_y))$ is complete. But this is a contradiction as C_y avoids u and at least one of v_1, v_2 , and v_3 . We conclude that 4.4.4.1 holds.

Next we show the following.

4.4.4.2. *Suppose that H is a simple 3-connected graph with at least seven vertices. If, for distinct vertices u, v, s , and t , neither (u, v) nor (s, t) is an edge of H , then $\text{si}(G) \neq H$.*

Suppose that $\text{si}(G) = H$. Let C_u, C_v, C_s , and C_t be shortest unbalanced cycles avoiding, respectively, v, u, t , and s . By Lemma 4.4.2(v), these cycles use, respectively, u, v, s , and t . Moreover, by Lemma 4.4.3(ii) and (vi), all of these cycles have at most three edges. Hence each We would like circuits C_1 and C_2 in $\{C_u, C_v, C_s, C_t\}$ with $\{u, v, s, t\} \subseteq C_1 \cup C_2$. Now C_u uses u and avoids v , while C_v uses v and avoids u . Both C_u and C_v use exactly one vertex of $\{s, t\}$. We can take $C_1 = C_u$ and $C_2 = C_v$ unless, by symmetry, C_u and C_v both contain s . Now C_t uses u or v but not both. Taking $C_2 = C_t$, we let C_1 be C_v or C_u , respectively. By potentially relabeling s and t , we may assume that $V(C_1)$ and $V(C_2)$ meet $\{u, v, s, t\}$ in $\{u, s\}$ and $\{v, t\}$, respectively.

Continuing with the proof of 4.4.4.2, we now show the following.

4.4.4.3. *There is a vertex $y \in V(G) - (V(C_1) \cup V(C_2))$ that is adjacent to each vertex of $G - y$.*

Clearly $|V(C_1) \cap V(C_2)| \leq 1$. We may assume that $V(C_1) \cap V(C_2) = \{w\}$, say. Then $|C_1| = 3 = |C_2|$. As $|V(G)| \geq 7$, there are distinct vertices $x_1, x_2 \in V(G) - (V(C_1) \cup V(C_2))$. Assume that w is adjacent to neither x_1 nor x_2 . Let C_x be a shortest unbalanced cycle avoiding w . Then, by Lemma 4.4.2(v), $\{x_1, x_2\} \subseteq V(C_x)$ and $V(C_x)$ meets each of $\{u, v\}$ and $\{s, t\}$. Hence $|C_x| \geq 4$. Thus, by Lemma 4.4.3(ii) and (vi), we get the contradiction that $|V(G)| \leq 6$. Hence w is adjacent to x_1 , say. By Lemma 4.4.2(v), x_1 is also adjacent to each vertex in $G - V(C_1)$ and to each vertex in $G - V(C_2)$. Thus x_1 is adjacent to each vertex of $G - x_1$, so 4.4.4.3 holds with $y = x_1$.

For a vertex z , recall that E_z is the set of edges meeting z . Let X be the set of edges that only meet vertices in $\{u, v, s, t\}$. We show next that

4.4.4.4. *$G \setminus X$ has all of its cycles balanced.*

Suppose that $G \setminus X$ has an unbalanced cycle C . Since y is adjacent to each vertex of $C - y$, by Lemma 4.2.6, G has an unbalanced 3-cycle C_y with $C_y \subseteq C \cup E_y$. Let f be the edge of C_y that is not incident with y . Then $f \in C$ and, by assumption, f does not join two vertices of $\{u, v, s, t\}$. Thus C_y avoids at least three vertices in $\{u, v, s, t\}$. But, by Lemma 4.4.2(ii),

Since y is adjacent to each vertex of the unbalanced cycle C_1 , by Lemma 4.2.6, there is an unbalanced 3-cycle C' using y and exactly two vertices of C_1 . Because neither (u, v) nor (s, t) is an edge of G , Lemma 4.4.2(ii) implies that $V(C')$ contains u and s . Hence $V(C') = \{y, u, s\}$. By symmetry, there is an unbalanced cycle C'' with vertex

set $\{y, v, t\}$. Now let F be the flat of M that is spanned by the edges meeting y and one of u, v, s , and t . The biased graph G' corresponding to M/F has unbalanced loops at y corresponding to the edges (u, s) and (v, t) of G . Any other edge of X either corresponds to an unbalanced loop at y in G' , or is in F . As $G \setminus X$ has every cycle balanced, letting $X' = X - F$, we deduce that $G' \setminus X'$ has only balanced cycles. Thus M/F has no circuit that meets both $X - F$ and $E(G' \setminus X')$. As the last two sets are non-empty, this contradicts the fact that M is unbreakable. We conclude that 4.4.4.2 holds.

Consider the complement of $\text{si}(G)$ in K_n . By 4.4.4.1, this complement has no vertex of degree three or more and, by 4.4.4.2, has no two-edge matching. Thus the complement is a path of length at most two. Thus Theorem 4.4.1 holds. \square

Proof of Theorem 4.1.2. This follows by combining Theorems 4.3.5 and 4.4.1. \square

Proof of Theorem 4.1.4. Assume that $|V(G)| \geq 7$. By Theorem 4.1.2, $\text{si}(G)$ is the complement in K_n of a path of length at most two. But, as every 3-cycle of G is unbalanced, by Lemma 4.4.2(v), G has no pair of nonadjacent vertices. Hence $\text{si}(G)$ is complete. \square

In Theorems 4.1.2, 4.1.3, and 4.1.4, we impose the condition that M is 3-connected. To extend these results to the case when M is not 3-connected will require considerably more work. The following two results of Pfeil [28] will certainly help in this analysis.

Lemma 4.4.5. *If a matroid M has a free element, then M is unbreakable.*

Lemma 4.4.6. *For matroids M_1 and M_2 , the 2-sum, $M_1 \oplus_2 M_2$ is unbreakable if and only if the basepoint p of the 2-sum is a free element in both M_1 and M_2 .*

4.5. A Partial Converse to the Main Theorem

In a private communication, Peter Nelson asked how many arbitrary edges could be removed from the complete graph and still have the simplification of the underlying graph of some unbreakable 3-connected frame matroid. To answer Peter's question, we use Theorems 4.1.2 and 4.1.5. The latter is proved in this section. This proof will use the following result.

Lemma 4.5.1. *Let $M = M(G, \Psi)$. Suppose that M is connected having at least two elements and that, for each unbalanced cycle C of G , the graph $\text{si}(G - V(C))$ is complete and C has a vertex that is adjacent to each vertex of $V(G) - V(C)$. If F is a flat of M containing an unbalanced cycle of G , then M/F is connected.*

Proof. We start by showing the following.

4.5.1.1. *Let H be a graph with no balanced loops such that each vertex meets an unbalanced loop. If $\text{si}(H)$ is complete, then $M(H, \Psi)$ is unbreakable.*

First, we note that if e is an edge of H , and H' is the graph corresponding to $M(H, \Psi)/cl(\{e\})$, then H' has no balanced loops, $\text{si}(H')$ is complete, and each vertex of H' is incident to an unbalanced loop. Because H' satisfies the same hypotheses as H , it suffices to show that $M(H, \Psi)$ is connected. If H has only one vertex, then $r(M(H, \Psi)) = 1$, and the statement clearly holds. Thus assume H has at least two vertices. If e and f are unbalanced loops at different vertices of H , then there is a circuit consisting of e , f , and a path connecting the vertices incident to e and f . Thus all of the unbalanced loops of H are in the same connected component of $M(H, \Psi)$. If f is an edge incident to the vertices x and y , then as there are unbalanced loops e_x and e_y incident to

x and y , we have that $\{e_x, e_y, f\}$ is a circuit of $M(H, \Psi)$. We conclude that $M(H, \Psi)$ is connected, so 4.5.1.1 holds.

Now, as M is connected, G has no balanced loops. Let C be an unbalanced cycle of G , and let v be a vertex of C that is adjacent to every vertex of $G - V(C)$. If G' is the graph corresponding to $M/cl(C)$, then G' has no balanced loops, $si(G')$ is complete, and each vertex of G' is incident to at least one unbalanced loop derived from an edge incident to v . Thus, by 4.5.1.1, $M/cl(C)$ is unbreakable. Now, let F be a flat containing C . Then $M/F = (M/cl(C))/(F - cl(C))$, where $F - cl(C)$ is a flat of $M/cl(C)$. Hence, M/F is connected. \square

Lemma 4.5.2. *Let $M = M(G, \Psi)$ and F be a flat of M that does not contain any unbalanced cycles. Let the biased graph (G', Ψ') correspond to M/F . Suppose that every cycle using the vertex u of G is unbalanced and that C is a 3-cycle of G using u . Then $C - F$ is a union of disjoint unbalanced cycles of G' at least one of which is incident to u . Furthermore, if C' is a 3-cycle incident to u and edge-disjoint from C , then $C' - F$ is in the same connected component of M/F as $C - F$.*

Proof. Since F contains no unbalanced cycles, it follows that $G' \cong G/F$. Thus, in G' , the set $C - F$ is a disjoint union of cycles. We want each of these cycles to be unbalanced. Because F is a flat, every loop in $C - F$ must be unbalanced. Thus the desired result holds unless $C - F$ contains an balanced 2-cycle, say $\{a, b\}$. Consider the exceptional case. Let c be the third edge of C . Then $c \notin F$, otherwise $\{a, b\}$ is unbalanced. It follows that F contains an (s, t) -path P , where s and t are the endvertices of c , and P does not meet the third vertex of C . Then $G[C \cup P]$ is a Θ -graph. As the cycle $P \cup \{a, b\}$ meets u , it is unbal-

anced. Thus $\{a, b\}$ is an unbalanced cycle of G/P and hence of G/F , a contradiction. We conclude that the first part of the lemma holds. For the second part, because $G[C' - F]$ is connected and each cycle of G' in $C - F$ is unbalanced, the result is immediate. \square

Lemma 4.5.3. *Let J be a complete graph with m vertices where $m \geq 5$. Let v be a vertex of J and Ψ be the set of cycles that avoid v . Then $M(J, \Psi)$ is a 3-connected frame matroid.*

Proof. Clearly Ψ satisfies the Θ -property, so $M(J, \Psi)$ is a frame matroid M . Moreover, M is simple and connected. Let (X, Y) be a 2-separation of M . As $M \setminus E_v$ is the cycle matroid of K_{m-1} , it is 3-connected, so we may assume that $|Y \cap (E(M \setminus E_v))| \leq 1$. Thus $r(X) \geq m - 2$. If $|X \cap E_v| \geq 2$, then $r(X) = r(M)$, a contradiction. Thus $|X \cap E_v| = t$ for some t in $\{0, 1\}$, so $|Y \cap E_v| = m - 1 - t$. Hence $r(X) = m - 2 + t$ and $r(Y) \geq m - 1 - t$, so

$$r(X) + r(Y) - r(M) \geq m - 2 + t + m - 1 - t - m = m - 3 \geq 2,$$

a contradiction. Hence $M(J, \Psi)$ is 3-connected. \square

The last lemma fails for $m = 4$ since, in that case, $M(J, \Phi)$ is a 6-element rank-4 matroid with a triangle so it is not 3-connected. Our proof of Theorem 4.1.5 will also use the following result of Oxley and Wu [27].

Lemma 4.5.4. *For $n \geq 2$, let X and Y be subsets of the ground set of a matroid M that has no circuits with fewer than n elements. Suppose that $M|X$ and $M|Y$ are both vertically n -connected and that $r(X) + r(Y) - r(X \cup Y) \geq n - 1$. Then $M|(X \cup Y)$ is vertically n -connected.*

We are now ready to prove the main result of this section.

Proof of Theorem 4.1.5. For some $n \geq 7$, we have that $H \in \{K_n, K_n \setminus e, K_n \setminus \{f, g\}\}$, where f and g are adjacent edges. The result holds when $H = K_n$, as $M(K_n)$ is 3-connected and, by Theorem 4.1.1, $M(K_n)$ is unbreakable.

Now let $H = K_n \setminus e$, where e joins u and v . Let Ψ be the set of all cycles of H that avoid $\{u, v\}$. It is easily checked that Ψ satisfies the Θ -property. We show first that $M(H, \Psi)$ is an unbreakable matroid M . Let F be a flat of M . If F contains an unbalanced cycle, then, by Lemma 4.5.1, M/F is connected. Now assume that F contains no unbalanced cycle. Let $W = E(H - \{u, v\})$. Then M/W is a rank-3 matroid consisting of two disjoint $(n-2)$ -point lines, and one easily checks that M/W is unbreakable. Thus, if F contains W , then $M/F = (M/W)/(F - W)$, and, since $F - W$ is a flat of M/W , it follows that M/F is connected. We may now assume that F does not contain W . By Lemma 4.5.2, for each w in $\{u, v\}$, the elements of $(E_w \cup W) - F$ are in the same connected component of M/F . Since $W - F \neq \emptyset$, the components containing $(E_u \cup W) - F$ and $(E_v \cup W) - F$ are the same, so M/F is connected. We conclude that M is unbreakable.

To see that M is 3-connected, first note that, as G is simple, so is M . By Lemma 4.5.3, each of $M \setminus E_u$ and $M \setminus E_v$ is 3-connected. As $r(E - E_u) + r(E - E_v) - r(E) = n - 1 + n - 1 - n \geq n - 2 \geq 2$, we deduce by Lemma 4.5.4 that M is indeed 3-connected.

Finally, we assume that $H = K_n \setminus \{f, g\}$ where $f = (u, v_1)$ and $g = (u, v_2)$. Let Ψ consist of all cycles that avoid both u and the edge h that joins v_1 and v_2 . It is straightforward to check that Ψ satisfies the Θ -property. Let $M = M(H, \Psi)$ and write E_1 and E_2 for E_{v_1} and E_{v_2} , respectively. Let $W = E(H) - (E_u \cup E_1 \cup E_2)$.

We show first that M is unbreakable. Consider a flat F of M . By Lemma 4.5.1, M/F is certainly connected if F contains an unbalanced cycle. Now $M(H, \Psi)/W$ consists

of the following four matroids freely placed in rank four: an $(n-3)$ -point line, a point, and two $(n-3)$ -element parallel classes. It is easily checked that $M(H, \Psi)/W$ is unbreakable. Hence if $W \subseteq F$, then M/F is connected.

Now, suppose that F does not contain an unbalanced cycle and that $W \not\subseteq F$. Let (H', Ψ') be the biased graph corresponding to $M(H, \Psi)/F$. As F does not contain an unbalanced cycle, $H' = H/F$. Then, by Lemma 4.5.2, $(E_u \cup W) - F$ is contained in a connected component of M/F . If $h \notin F$, then h is in an unbalanced cycle of H' with at most three elements. Since $(E_u \cup W) - F$ contains an unbalanced cycle, h is in the same component of M/F as $(E_u \cup W) - F$.

Let j be an element of M/F that is not in the same component as $(E_u \cup W) - F$. Then, without loss of generality, j meets v_1 . As $W \not\subseteq F$, there is a 3-cycle D in H' that contains j and an edge w of $W - F$ and is edge-disjoint from E_u . As j and w are in different components of M/F , we deduce that D is unbalanced. There is an unbalanced cycle D' of H' that uses u and is edge-disjoint from D . As H' is connected, it follows that M/F has a circuit containing $D \cup D'$, a contradiction. We conclude that M is unbreakable.

Lastly, we show that M is 3-connected. Certainly M is connected and simple. Both $M \setminus (E_u \cup E_1)$ and $M \setminus (E_u \cup E_2)$ are the cycle matroids of complete graphs so they are 3-connected. Since the ground sets of these matroids meet in W , and $(E_u \cup E_1) \cap (E_u \cup E_2) = E_u \cup h$, it follows by Lemma 4.5.4 that $M \setminus (E_u \cup h)$ is 3-connected. Moreover, by Lemma 4.5.3, $M \setminus (E_1 \cup E_2)$ is 3-connected. As $(E_u \cup h) \cap (E_1 \cup E_2) = \{h\}$, it follows by Lemma 4.5.4 that $M \setminus h$ is 3-connected. Since $r(M) = r(M \setminus h)$ and M is simple, it follows that M is 3-connected. □

Appendix. Copyright Information

Publishing Agreement for the article used in Chapter 2

Article: Laminar Matroids

Corresponding author: Ms. Tara Fife

E-mail address: tfife2@lsu.edu, fi.tara@gmail.com

Journal: European Journal of Combinatorics

Our reference YEUJC2650

PII: S0195-6698(17)30002-1

DOI: 10.1016/j.ejc.2017.01.002

Your Status

I am one author signing on behalf of all co-authors of the manuscript

Data Protection & Privacy

I do wish to receive news, promotions and special offers about products and services from Elsevier Ltd and its affiliates worldwide.

Assignment of Copyright

I hereby assign to Elsevier Ltd the copyright in the manuscript identified above (where Crown Copyright is asserted, authors agree to grant an exclusive publishing and distribution license) and any tables, illustrations or other material submitted for publication as part of the manuscript (the "Article"). This assignment of rights means that I have granted to Elsevier Ltd, the exclusive right to publish and reproduce the Article, or any part of the Article, in print, electronic and all other media (whether now known or later developed), in any form, in all languages, throughout the world, for the full term

of copyright, and the right to license others to do the same, effective when the Article is accepted for publication. This includes the right to enforce the rights granted hereunder against third parties. Supplemental Materials

With respect to Supplemental Materials that I wish to make accessible through a link in the Article or on a site or through a service of Elsevier Ltd, Elsevier Ltd shall be entitled to publish, post, reformat, index, archive, make available and link to such Supplemental Materials on a non-exclusive basis in all forms and media (whether now known or later developed) and to permit others to do so. "Supplemental Materials" shall mean additional materials that are not an intrinsic part of the Article, including but not limited to experimental data, e-components, encodings and software, and enhanced graphical, illustrative, video and audio material.

Reversion of rights

Articles may sometimes be accepted for publication but later rejected in the publication process, even in some cases after public posting in "Articles in Press" form, in which case all rights will revert to the author (see <http://www.elsevier.com/locate/withdrawalpolicy>).

Revisions and Addenda

I understand that no revisions, additional terms or addenda to this Journal Publishing Agreement can be accepted without Elsevier Ltd's express written consent. I understand that this Journal Publishing Agreement supersedes any previous agreements I have entered into with Elsevier Ltd in relation to the Article from the date hereof.

Author Rights for Scholarly Purposes

I understand that I retain or am hereby granted (without the need to obtain further permission) the Author Rights (see description below), and that no rights in patents, trademarks or other intellectual property rights are transferred to Elsevier Ltd.

The Author Rights include the right to use the Preprint, Accepted Manuscript and the Published Journal Article for Personal Use, Internal Institutional Use and for Scholarly Sharing.

In the case of the Accepted Manuscript and the Published Journal Article the Author Rights exclude Commercial Use (unless expressly agreed in writing by Elsevier Ltd), other than use by the author in a subsequent compilation of the author's works or to extend the Article to book length form or re-use by the author of portions or excerpts in other works (with full acknowledgment of the original publication of the Article).

Author Representations / Ethics and Disclosure / Sanctions

I affirm the Author Representations noted below, and confirm that I have reviewed and complied with the relevant Instructions to Authors, Ethics in Publishing policy, Declarations of Interest disclosure and information for authors from countries affected by sanctions (Iran, Cuba, Sudan, Burma, Syria, or Crimea). Please note that some journals may require that all co-authors sign and submit Declarations of Interest disclosure forms. I am also aware of the publisher's policies with respect to retractions and withdrawal (<http://www.elsevier.com/locate/withdrawalpolicy>).

For further information see the publishing ethics page at <http://www.elsevier.com/publishingethics> and the journal home page. For further information on sanctions, see <https://www.elsevier.com/trade-sanctions>

Author representations

- The Article I have submitted to the journal for review is original, has been written by the stated authors and has not been previously published.
- The Article was not submitted for review to another journal while under review by this journal and will not be submitted to any other journal.
- The Article and the Supplemental Materials do not infringe any copyright, violate any other intellectual property, privacy or other rights of any person or entity, or contain any libellous or other unlawful matter.
- I have obtained written permission from copyright owners for any excerpts from copyrighted works that are included and have credited the sources in the Article or the Supplemental Materials.
- Except as expressly set out in this Journal Publishing Agreement, the Article is not subject to any prior rights or licenses and, if my or any of my co-authors' institution has a policy that might restrict my ability to grant the rights required by this Journal Publishing Agreement (taking into account the Author Rights permitted hereunder, including Internal Institutional Use), a written waiver of that policy has been obtained.
- If I and/or any of my co-authors reside in Iran, Cuba, Sudan, Burma, Syria, or Crimea, the Article has been prepared in a personal, academic or research capacity and not as an official representative or otherwise on behalf of the relevant government.
- If I am using any personal details or images of patients, research subjects or other individuals, I have obtained all consents required by applicable law and complied with the publisher's policies relating to the use of such images or personal information. See <http://www.elsevier.com/patientphotographs> for further information.
- Any software contained in the Supplemental Materials is free from viruses, contaminants or worms.
- If the Article or any of the Supplemental Materials were prepared jointly with other authors, I have informed the co-author(s) of the terms of this Journal Publishing Agreement and that I am signing on their behalf as their agent, and I am authorized to do so.

For information on the publisher's copyright and access policies, please see

<http://www.elsevier.com/copyright>.

[For more information about the definitions relating to this agreement click here.](#)

I have read and agree to the terms of the Journal Publishing Agreement.

18th January 2017

T-copyright-v21/2016

Publishing Agreement for the article used in Chapter 3

Article: Generalized Laminar Matroids

Corresponding author: Ms. Tara Fife

E-mail address: tfife2@lsu.edu

Journal: European Journal of Combinatorics

Our reference YEUJC2934

PII: S0195-6698(18)30205-1

DOI: 10.1016/j.ejc.2018.12.005

Your Status

I am one author signing on behalf of all co-authors of the manuscript

Data Protection & Privacy

I do not wish to receive news, promotions and special offers about products and services from Elsevier Ltd and its affiliates worldwide.

Assignment of Copyright

I hereby assign to Elsevier Ltd the copyright in the manuscript identified above (where Crown Copyright is asserted, authors agree to grant an exclusive publishing and distribution license) and any tables, illustrations or other material submitted for publication as part of the manuscript (the "Article"). This assignment of rights means that I have granted to Elsevier Ltd, the exclusive right to publish and reproduce the Article,

or any part of the Article, in print, electronic and all other media (whether now known or later developed), in any form, in all languages, throughout the world, for the full term of copyright, and the right to license others to do the same, effective when the Article is accepted for publication. This includes the right to enforce the rights granted hereunder against third parties. Supplemental Materials

“Supplemental Materials” shall mean materials published as a supplemental part of the Article, including but not limited to graphical, illustrative, video and audio material.

With respect to any Supplemental Materials that I submit, Elsevier Ltd shall have a perpetual worldwide, non-exclusive right and license to publish, extract, reformat, adapt, build upon, index, redistribute, link to and otherwise use all or any part of the Supplemental Materials in all forms and media (whether now known or later developed), and to permit others to do so.

Research Data

"Research Data" shall mean the result of observations or experimentation that validate research findings and that are published separate to the Article, which can include but are not limited to raw data, processed data, software, algorithms, protocols, and methods.

With respect to any Research Data that I wish to make accessible on a site or through a service of Elsevier Ltd, Elsevier Ltd shall have a perpetual worldwide, non-exclusive right and license to publish, extract, reformat, adapt, build upon, index, redistribute, link to and otherwise use all or any part of the Research Data in all forms and media (whether now known or later developed) and to permit others to do so. Where I have selected a specific end user license under which the Research Data is to be made

available on a site or through a service, the publisher shall apply that end user license to the Research Data on that site or service.

Reversion of rights

Articles may sometimes be accepted for publication but later rejected in the publication process, even in some cases after public posting in "Articles in Press" form, in which case all rights will revert to the author (see <https://www.elsevier.com/about/our-business/policies/article-withdrawal>).

Revisions and Addenda

I understand that no revisions, additional terms or addenda to this Journal Publishing Agreement can be accepted without Elsevier Ltd's express written consent. I understand that this Journal Publishing Agreement supersedes any previous agreements I have entered into with Elsevier Ltd in relation to the Article from the date hereof.

Author Rights for Scholarly Purposes

I understand that I retain or am hereby granted (without the need to obtain further permission) the Author Rights (see description below), and that no rights in patents, trademarks or other intellectual property rights are transferred to Elsevier Ltd.

The Author Rights include the right to use the Preprint, Accepted Manuscript and the Published Journal Article for Personal Use and Internal Institutional Use. They also include the right to use these different versions of the Article for Scholarly Sharing purposes, which include sharing:

- the Preprint on any website or repository at any time;
- the Accepted Manuscript on certain websites and usually after an embargo period;

- the Published Journal Article only privately on certain websites, unless otherwise agreed by Elsevier Ltd.

In the case of the Accepted Manuscript and the Published Journal Article the Author Rights exclude Commercial Use (unless expressly agreed in writing by Elsevier Ltd), other than use by the author in a subsequent compilation of the author's works or to extend the Article to book length form or re-use by the author of portions or excerpts in other works (with full acknowledgment of the original publication of the Article).

Author Representations / Ethics and Disclosure / Sanctions

I affirm the Author Representations noted below, and confirm that I have reviewed and complied with the relevant Instructions to Authors, Ethics in Publishing policy, Declarations of Interest disclosure and information for authors from countries affected by sanctions (Iran, Cuba, Sudan, Burma, Syria, or Crimea). Please note that some journals may require that all co-authors sign and submit Declarations of Interest disclosure forms. I am also aware of the publisher's policies with respect to retractions and withdrawal (<https://www.elsevier.com/about/our-business/policies/article-withdrawal>).

For further information see the publishing ethics page at <https://www.elsevier.com/about/our-business/policies/publishing-ethics> and the journal home page. For further information on sanctions, see <https://www.elsevier.com/about/our-business/policies/trade-sanctions>

Author representations

- The Article I have submitted to the journal for review is original, has been written by the stated authors and has not been previously published.
- The Article was not submitted for review to another journal while under review by this journal and will not be submitted to any other journal.

- The Article and the Supplemental Materials do not infringe any copyright, violate any other intellectual property, privacy or other rights of any person or entity, or contain any libellous or other unlawful matter.
- I have obtained written permission from copyright owners for any excerpts from copyrighted works that are included and have credited the sources in the Article or the Supplemental Materials.
- Except as expressly set out in this Journal Publishing Agreement, the Article is not subject to any prior rights or licenses and, if my or any of my co-authors' institution has a policy that might restrict my ability to grant the rights required by this Journal Publishing Agreement (taking into account the Author Rights permitted hereunder, including Internal Institutional Use), a written waiver of that policy has been obtained.
- If I and/or any of my co-authors reside in Iran, Cuba, Sudan, Burma, Syria, or Crimea, the Article has been prepared in a personal, academic or research capacity and not as an official representative or otherwise on behalf of the relevant government or institution.
- If I am using any personal details or images of patients, research subjects or other individuals, I have obtained all consents required by applicable law and complied with the publisher's policies relating to the use of such images or personal information. See <https://www.elsevier.com/about/our-business/policies/patient-consent> for further information.
- Any software contained in the Supplemental Materials is free from viruses, contaminants or worms.
- If the Article or any of the Supplemental Materials were prepared jointly with other authors, I have informed the co-author(s) of the terms of this Journal Publishing Agreement and that I am signing on their behalf as their agent, and I am authorized to do so.

Governing Law and Jurisdiction

This Agreement will be governed by and construed in accordance with the laws of the country or state of Elsevier Ltd ("the Governing State"), without regard to conflict of law principles, and the parties irrevocably consent to the exclusive jurisdiction of the courts of the Governing State.

For information on the publisher's copyright and access policies, please see
<http://www.elsevier.com/copyright>.

[For more information about the definitions relating to this agreement click here.](#)

I have read and agree to the terms of the Journal Publishing Agreement.

27th December 2018

T-copyright-v22/2017

Definition of Personal Use by Elsevier

“Personal use” is defined by Elsevier as follows (see
<https://authors.elsevier.com/authorform/staticpage/definitions.do?lang=English>).

Use by an author in the author's classroom teaching (including distribution of copies, paper or electronic) or presentation by an author at a meeting or conference (including distribution of copies to the delegates attending such meeting), distribution of copies (including through e-mail) to known research colleagues for their personal use, use in a subsequent compilation of the author's works, inclusion in a thesis or dissertation, preparation of other derivative works such as extending the Article to book-length form, or otherwise using or re-using portions or excerpts in other works (with full acknowledgment of the original publication of the Article).

Bibliography

- [1] Ardila, F., The Catalan matroid, *J. Combin. Theory Ser. A* **104** (2003), 49–62.
- [2] Babaioff, M., Immorlica, N., and Kleinberg, R., Matroids, secretary problems, and online mechanisms, Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 434–443, ACM, New York, 2007.
- [3] Bonin, J. and de Mier, A., Lattice path matroids: Structural properties. *European J. Combin.* **27** (2006), 701–738.
- [4] Bonin, J., de Mier, A., and Noy, M., Lattice path matroids: enumerative aspects and Tutte polynomials, *J. Combin. Theory Ser. A* **104** (2003) 63–94.
- [5] Bonin, J. and De Mier, A., The lattice of cyclic flats of a matroid, *Ann. Combin.* **12** (2008), 155–170.
- [6] Chakraborty, S. and Lachish, O., Improved competitive ratio for the matroid secretary problem, *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, 1702–1712, ACM, New York, 2012.
- [7] Chartrand, G. and Harary, F., Planar permutation graphs, *Ann. Inst. Henri Poincaré* **3** (1967), 433–438.
- [8] Crapo, H., Single-element extensions of matroids. *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 55–65.
- [9] Crapo, H. and Schmitt, W., A free subalgebra of the algebra on matroids, *European J. Combin.* **26** (2005), 1066–1085.
- [10] Cunningham, W., *A combinatorial decomposition theory*, Ph.D. thesis, University of Waterloo, 1973.
- [11] Diestel, R., *Graph Theory*, Fifth edition, Graduate Texts in Mathematics, 173. Springer, Berlin, 2018.
- [12] Feldman, M., Svensson, O., and Zenklusen, R., A simple $O(\log\log(\text{rank}))$ -competitive algorithm for the matroid secretary problem, *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, 1189–1201, SIAM, Philadelphia, PA, 2015.
- [13] Fife, T., Mayhew, D., Oxley, J., and Semple, C., The unbreakable frame matroids, submitted.
- [14] Fife, T. and Oxley, J., Laminar matroids, *European J. Combin.* **62** (2017), 206–216.

- [15] Fife, T. and Oxley, J., Generalized laminar matroids, *Europ. J. Combin.* **79** (2019), 111–122.
- [16] Finkelstein, L., Two algorithms for the matroid secretary problem, Master’s thesis, Technion, Israel Institute of Technology, Haifa, 2011.
- [17] Gabow, H. and Kohno, T., A network-flow-based scheduler: design, performance history and experimental analysis, Second Workshop on Algorithmic Engineering and Experiments (ALENEX 00) (San Francisco, CA), *ACM J. Exp. Algorithmics* **6** (2001), 1–14.
- [18] Huynh, T., The matroid secretary problem, The Matroid Union, Posted on September 7, 2015, <http://matroidunion.org/?p=1487>
- [19] Im, S. and Wang, Y., Secretary problems: laminar matroid and interval scheduling, *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*, 1265–1274, SIAM, Philadelphia, PA, 2011.
- [20] Ingleton A. W. and Piff, M. J., Gammoids and transversal matroids, *J. Combin. Theory Ser. B* **15** (1973), 51–68.
- [21] Jaillet, P., Soto, J., and Zenklusen, R., Advances on matroid secretary problems: free order model and laminar case, *Integer programming and combinatorial optimization*, 254–265, Lecture Notes in Comput. Sci., 7801, Springer, Heidelberg, 2013.
- [22] Kaparis, K. and Letchford, A., On laminar matroids and b -matchings, submitted.
- [23] Lachish, O., $O(\log \log \text{rank})$ competitive ratio for the matroid secretary problem, *55th Annual IEEE Symposium on Foundations of Computer Science–FOCS 2014*, 326–335, IEEE Computer Soc., Los Alamitos, CA, 2014.
- [24] Mason, J. H., On a class of matroids arising from paths in graphs, *Proc. Lond Math. Soc. (3)* **25** (1972), 55–74.
- [25] Oxley, J., *Matroid Theory*, Second edition, Oxford University Press, New York, 2011.
- [26] Oxley, J., Prendergast, K., and Row, D., Matroids whose ground sets are domains of functions, *J. Austral. Math. Soc. Ser. A* **32** (1982), 380–387.
- [27] J. Oxley, J. and Wu, H., On matroid connectivity, *Discrete Math.* **146** (1995), 321–324.
- [28] Pfeil, S., On Properties of Matroid Connectivity, Ph.D. Thesis, Louisiana State University, 2016.

- [29] Piff, M. J. and Welsh, D. J. A., On the vector representation of matroids, *J. Comb. Theory Ser. B* **15** (1973), 51–68.
- [30] Pivotto, I., Biased graphs and their matroids I, The Matroid Union, Posted on August 26, 2013, <http://matroidunion.org/?p=161>
- [31] Pivotto, I., Biased graphs and their matroids II, The Matroid Union, Posted on October 7, 2013, <http://matroidunion.org/?p=279>
- [32] Pivotto, I., Biased graphs and their matroids III, The Matroid Union, Posted on October 28, 2013, <http://matroidunion.org/?p=357>
- [33] Robertson, N. and Seymour, P. D., Graph minors. XX. Wagner’s conjecture. *J. Combin. Theory Ser. B* **92**, (2004) 325–357.
- [34] Sohoni, M., Rapid mixing of some linear matroids and other combinatorial objects, *Graphs Combin.* **15** (1999), 93–107.
- [35] Soto, J., Matroid secretary problem in the random assignment model, *SIAM J. Comput.* **42** (2013), 178–211.
- [36] Tutte, W. T., Connectivity in matroids, *Canad. J. Math.* **18**, 1301–1324.
- [37] Whitney, H., On the abstract properties of linear dependence, *Amer. J. Math.* **57** (1935), 509–533
- [38] Zaslavsky, T., Signed graphs, *Discrete App. Math.* **4** (1982), 47–74.
- [39] Zaslavsky, T., Biased graphs. I. Bias, balance, and gains, *J. Combin. Theory Ser. B* **47** (1989), 32–52.
- [40] Zaslavsky T., Biased graphs. II. The three matroids, *J. Combin. Theory Ser. B* **51** (1991), 46–72

Vita

Tara Fife was born in 1992 in Idaho. She finished her undergraduate studies at Brigham Young University-Idaho in 2014 and earned a master of science degree in mathematics from Louisiana State University in December 2015. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May, 2020.