11-29-2007

# Graceful exit via polymerization of pre-big-bang cosmology

Giuseppe De Risi
*University of Portsmouth*

Roy Maartens
*University of Portsmouth*

Parampreet Singh
*Pennsylvania State University*

# Graceful exit via polymerization of pre-big bang cosmology

Giuseppe De Risi[1,2], Roy Maartens[1], Parampreet Singh[3]
[1]*Institute of Cosmology & Gravitation, University of Portsmouth, Portsmouth PO1 2EG, UK*
[2]*Instituto Nazionale di Fisica Nucleare, 00186 Roma, Italy*
[3]*Institute for Gravitational Physics & Geometry, Pennsylvania State University, University Park PA 16802, USA*

We consider a phenomenological modification of the Pre-Big Bang scenario using ideas from the resolution of curvature singularities in Loop Quantum Cosmology. We show that non-perturbative Loop modifications to the dynamics, arising from the underlying polymer representation, can resolve the graceful exit problem. The curvature and the dilaton energy stay finite at all times, in both the string and Einstein frames. In the string frame, the dilaton tends to a constant value at late times after the bounce.



## I. INTRODUCTION

The problem of graceful exit from the pre- to the post-big bang branch, and the recovery of classical dynamics at late times, has remained a major issue for Pre-Big Bang (PBB) models [1] (for a review, see [2]). The equations derived from the low energy effective action of string theory, cannot provide a smooth transition between the pre-big bang phase and the standard post-big bang phase of decreasing curvature. There have been many attempts to solve this problem of overcoming the curvature singularity. If it is assumed that the curvature at the transition time is small enough to use the low energy equations, then a smooth transition can be achieved either by adding a nonlocal dilaton potential to the action [2, 3], or by considering an anisotropic universe dominated by some kind of matter with a suitable equation of state [4, 5, 6, 7]. If the curvature is very large, higher-order corrections to the low energy effective action have to be added, which can be derived from the loop expansion and from the $\alpha'$ expansion [8, 9, 10, 11, 12, 13]. Both of these approaches are based on ad hoc assumptions that have to be imposed by hand for a graceful exit solution. This is a consequence of our poor knowledge of the non-perturbative regime of string theory.

Curvature singularities have been recently studied in the framework of Loop Quantum Cosmology (LQC) [14], which is a canonical quantization of homogeneous cosmological spacetimes based on Loop Quantum Gravity (LQG). The gravitational phase variables are the matrix-valued Ashtekar connection $A_a^i$ and the conjugate triad $E_i^a$, and the elementary variables are the holonomies of the connection and the fluxes of the triad. In the quantum theory, holonomies represent fundamental excitations of quantum geometry which are polymer-like one-dimensional excitations. Holonomies also provide information about the connection which does not have a corresponding operator in LQG/C. For classical FRW cosmology, connection is proportional to $\dot{a}$ and thus holonomies encode information about the expansion (contraction) of the universe.

The quantum theory obtained from loop quantization turns out to be different from the Wheeler-de Witt quantization (the polymer representation is not equivalent to the usual Fock representation). Wheeler-de Witt quantization does not resolve the cosmological singularity, but in LQC a generic resolution of curvature singularities has been obtained. The resulting picture is of a universe which bounces when curvature reaches Planck scale and is classical at low curvatures. This picture is based on extensive analytical and numerical investigations for FRW flat [15, 16], closed [17] and open [18] models, Bianchi I models [19], de Sitter [20] and anti-de Sitter models [21]. Recent investigations for flat models have further revealed the genericity of the bounce for a dense subspace of the physical Hilbert space [22].

LQC therefore in principle allows us to incorporate non-perturbative effects in PBB models, at least at a phenomenological level. If string theory and LQG both encompass elements of an eventual Quantum Gravity theory, then it is interesting to explore the phenomenology that results when one applies ideas from one approach to models constructed in the framework of the other. We focus only on this phenomenology, and do not attempt to address the fundamental issue of the relation between string theory and LQG. Instead our approach is to extract the basic elements of LQC quantization that are relevant to understand singularity resolution, and apply them to PBB models.

We start with the massless dilaton $\phi$ in the Einstein frame and cast the problem as a Hamiltonian system in connection-triad phase space variables. Since there is no external time parameter in quantum gravity, subtleties arise in recovering the conventional notion of dynamics and identifying the post- and pre-big bang branches. These subtleties have been emphasized previously in the quantum cosmology of the PBB scenario [23]. To resolve them, we employ the ideas of relational dynamics used in LQC [15, 16, 17] and treat the dilaton, which is monotonic, as an internal clock. The change of scale factor (or other observables) with respect to the internal clock can then be found by solving the Hamiltonian constraint and computing for example, $da/d\phi$. Classically, as well in the PBB scenario, in the backward evolution of the post-big bang branch, the scale factor goes to zero as $\phi \to -\infty$, and it increases with an increase in $\phi$. Similarly, the forward evolution of the pre-big bang branch results in a decrease in the scale factor as $\phi$ increases, with the scale factor vanishing as $\phi \to \infty$.

The pre and post-big bang branches are distinguished by the behavior of the scale factor with respect to the dilaton. In classical general relativity and in PBB scenarios (without any tree-level corrections), the pre- and post-big bang branches are disjoint. A Wheeler-De Witt quantum cosmology analysis of the PBB scenario reveals that the pre- and post-big bang phases correspond to different branches of the wavefunction [23]. At an effective level, trajectories for the scale factor or the dilaton with respect to proper time can be obtained by recasting the equations via introduction of a parameter $t$, for example: $da/d\phi = (da/dt)/(d\phi/dt)$. The parameter $t$, which plays the role of classical external time, can be thought of as emerging by semi-classical approximations. We would employ this algorithm in our analysis, using the observation that the underlying loop quantum dynamics can be described by an effective Hamiltonian for states which are semi-classical at late times [16, 17]. As it will turn out, loop quantum geometric effects lead to a non-singular transition between the pre- and post-big bang branches, with respect to the internal clock $\phi$. Unlike the Wheeler-De Witt quantization, the pre- and post-big bang phases correspond to the same branch of the wavefunction in LQC [15, 16, 17]. Using the effective Hamiltonian, this dynamics translates to a non-singular evolution with respect to the time parameter $t$.

Since our approach captures the features of the polymer representation in LQG/C, we will refer to it as "polymerization", as in a recent work on the quantization of cosmological spacetimes [24]. Once the polymerization has been performed in the Einstein frame, we transform the dynamical equations to the string frame and study the solutions. We show that polymerization indeed cures the curvature singularity in both the Einstein and string frames. Furthermore, at late times there is a smooth transition to classical dynamics. Thus the graceful exit problem in the PBB scenario is overcome in this approach.

The paper is organized as follows. In Sec. II we start with the zeroth order effective action in the Einstein frame for the dilaton without potential, and use a Legendre transformation to cast the problem as a Hamiltonian system with connection-triad phase space variables. We then perform polymerization of the connection and show that generic solutions of the modified dynamical equations in the Einstein frame are non-singular. In Sec. III, we give the modified dynamical equations in the string frame. Interestingly, polymer modifications appear similar to quantum one loop corrections in the string frame. By numerically integrating the dynamical equations we show that singularity resolution is obtained also in the string frame. In Sec. IV we summarize our results and compare the polymer modifications leading to non-singular dynamics with previous attempts in PBB scenarios using one loop quantum corrections in the string frame action.

## II. POLYMERIZATION IN THE EINSTEIN FRAME

Our starting point is the low-energy effective action derived from string theory in the Einstein frame. We set the antisymmetric $B_{\mu\nu}$ field to zero, and assume stabilization of all moduli fields associated with compactification. In the absence of a dilaton potential, the action is (with units $c = \hbar = 1$ and $8\pi G = M_P^2$)

$$S = -\int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (1)$$

Variation with respect to the metric and the dilaton gives the equations of motion

$$M_P^2 G_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\,\partial_\sigma\phi\partial^\sigma\phi\,, \quad \nabla^\mu\nabla_\mu\phi = 0\,. \quad (2)$$

For the FRW metric these lead to

$$H^2 = \frac{1}{6M_P^2}\dot\phi^2\,,\ \dot H = -\frac{1}{2M_P^2}\dot\phi^2\,,\ \ddot\phi + 3H\dot\phi = 0\,, \quad (3)$$

of which only two equations are independent. In the PBB scenario, there are two disjoint solutions: an expanding post-big bang branch and the contracting pre-big bang branch, separated by a curvature singularity at $t = 0$, as shown in Fig. 1.

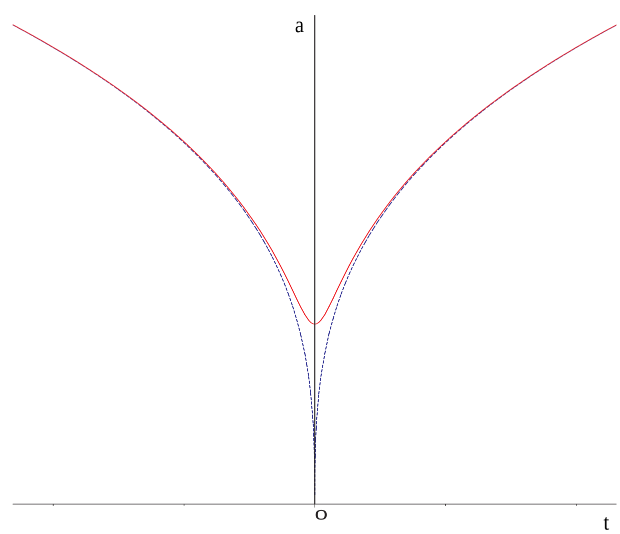


FIG. 1: The scale factor $a(t)$ in the Einstein frame: blue (dashed) curves are the standard singular PBB solutions; the red (solid) curve is the non-singular solution after polymer LQC modifications. (We set $M_P = 1 = \beta$ in all the plots.)

In order to implement effects of polymer quantization we cast the problem in the Hamiltonian framework with same phase space variables as in LQC. These are the gravitational degrees of freedom $(c, p)$, corresponding to the connection and the triad variables (the latter kinematically related to metric as $p = a^2$), and the matter degrees of freedom $(\phi, \pi_\phi)$. In the classical limit,

$$(c, p) = (\dot{a}, a^2)\,, \quad (\phi, \pi_\phi) = (\phi, a^3\dot{\phi})\,. \tag{4}$$

We obtain the Hamiltonian from the action (1) using a Legendre transformation:

$$\mathcal{H} = -3\, M_P^2\, c^2\, p^{1/2} + \frac{\pi_\phi^2}{2p^{3/2}}\,. \tag{5}$$

The dynamical equations (3) can be obtained via Hamilton's equations of motion: $\dot{x} = \{x, \mathcal{H}\}$.

In LQC, there is no analog of the connection operator. Information about the connection is instead obtained from the holonomies, whose elements are of the form $\exp(i\lambda c)$, where the underlying quantum geometry imposes [16]

$$\lambda = \beta\, \frac{L_P}{p^{1/2}}\,. \tag{6}$$

Here $\beta$ is a dimensionless constant, which in LQC is O(1). It determines the minimum eigenvalue of the area operator in LQG, $\beta^2 L_P^2$. The parameter $\lambda$ thus captures the discrete nature of quantum geometry in LQC. Dynamics in LQC is described by a quantum difference equation. However, using geometric methods of quantum mechanics one can find an effective Hamiltonian of the form (5) that provides an excellent approximation to the quantum evolution of the states which are semi-classical at late times [25]. One of the features of this Hamiltonian is the presence of higher order terms in $c$ which arise by expressing the Hamiltonian in the elements of holonomies. The effective Hamiltonian contains terms of the form $\sin(\lambda c)$, which can be thought of as capturing the non-local character of quantum curvature at leading order in an effective continuum spacetime [26].

To capture this key feature of LQC, we will adopt it at an effective level in the present scenario and perform a phenomenological polymerization,

$$c \;\rightarrow\; \frac{\sin(\lambda c)}{\lambda}\,, \tag{7}$$

which leads to

$$\mathcal{H}_{\rm eff} = -3M_P^2 \frac{\sin^2(\lambda c)}{\lambda^2}\, p^{1/2} + \frac{\pi_\phi^2}{2p^{3/2}}\,. \tag{8}$$

For small curvature, $\sin(\lambda c)/\lambda \approx c$, Eq. (8) reduces to Eq. (5). However significant departures from classical dynamics are expected when $\lambda c$ is large. As we show below the effective Hamiltonian (8) leads to classical dynamics where expected, and to a significant departure from it when the curvature scalar is large, $R = O(M_P^2)$, producing a non-singular evolution from the pre- to the post-big bang.

Let us first consider the Hamilton's equation for $p$,

$$\dot{p} = \{p, \mathcal{H}_{\rm eff}\} = \frac{2p^{1/2}}{\lambda}\, \sin(\lambda c)\cos(\lambda c)\,, \tag{9}$$

which leads to

$$H^2 = \frac{\dot{p}^2}{4p^2} = \frac{M_P^2}{\beta^2}\, \sin^2(\lambda c)\, \left[1 - \sin^2(\lambda c)\right]. \tag{10}$$

This can be cast in the form of a modified Friedman equation by using the Hamiltonian constraint $\mathcal{H}_{\rm eff} \approx 0$, and Eq. (4) to determine $\sin(\lambda c)$:

$$H^2 = \frac{1}{3M_P^2}\,\frac{\dot{\phi}^2}{2}\left(1 - \frac{\dot{\phi}^2/2}{M_*^4}\right), \quad M_* = \frac{3^{1/4}}{\sqrt{\beta}}\, M_P\,. \tag{11}$$

We obtain the Raychaudhuri equation from this by using the Klein-Gordon equation (or alternatively from Hamilton's equation for $\dot{c}$),

$$\dot{H} = -\frac{1}{2M_P^2}\dot{\phi}^2\left(1 - \frac{\dot{\phi}^2}{M_*^4}\right). \tag{12}$$

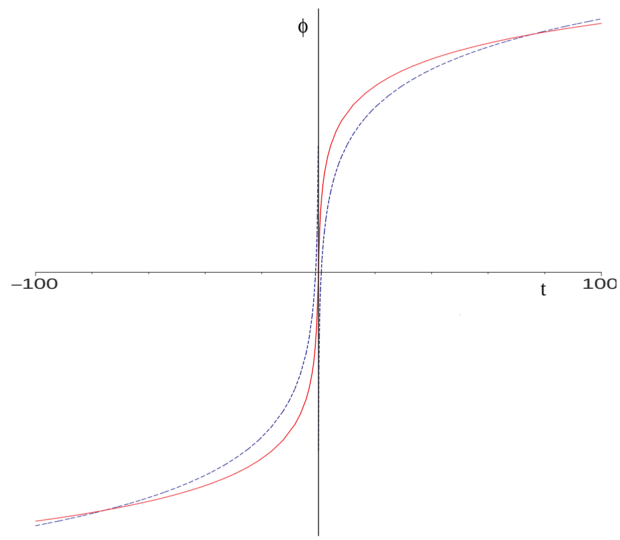


FIG. 2: The dilaton $\phi(t)$ for the standard PBB (blue, dashed) and the LQC polymerized (red, solid) models.

As is evident from the modified Friedman and Raychaudhuri equations, corrections originating from polymerization die away rapidly when $\dot{\phi}^2 \ll M_*^4$. However, when $\dot{\phi}^2$ is comparable to $M_*^4$, quantum gravity modifications become significant and lead to a bounce.

The Klein-Gordon equation implies $\dot{\phi} = \pi_\phi/a^3$, and substituting into Eq. (11) we find

$$\left(\frac{a}{a_0}\right)^6 = 1 + 3\frac{M_*^4}{M_P^2}\, t^2\,, \tag{13}$$

where $a_0 > 0$ is the minimum value of $a$, at $t = 0$. Then it follows that

$$\phi \;=\; \sqrt{\frac{2}{3}}\, M_P \sinh^{-1}\left(\frac{\sqrt{3}M_*^2}{M_P}\, t\right), \tag{14}$$

$$H \;=\; \frac{M_*^4 t}{M_P^2 + 3M_*^4 t^2}\,. \tag{15}$$

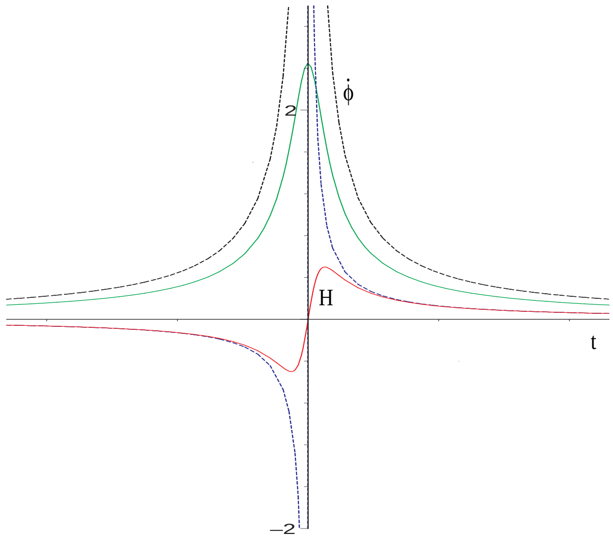


FIG. 3: The Hubble rate $H(t)$ for standard PBB (blue, dashed) and LQC polymerized (red, solid) models, and the dilaton velocity $\dot{\phi}(t)$ in standard (black, dashed) and polymerized (green, solid) models.

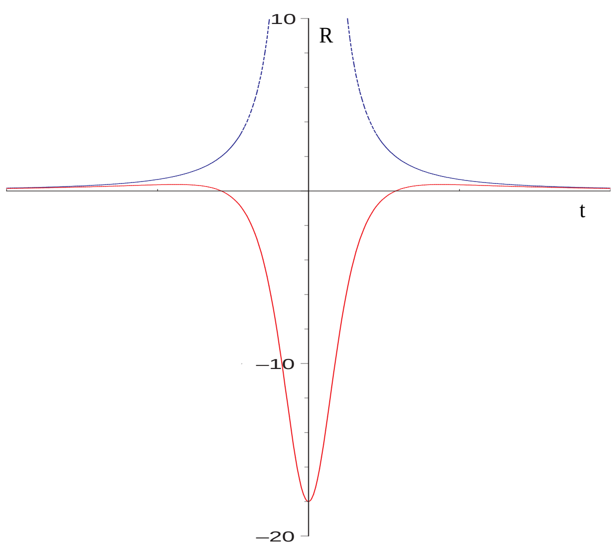


FIG. 4: The Ricci curvature $R(t)$ for LQC polymerized (red) and standard PBB (blue) models.

These solutions are regular for all time, as shown in Figs. 2 and 3 . The curvature stays finite through the bounce (Fig. 4). The scale factor undergoes a bounce from a phase of accelerated contraction as can be seen in Fig. 1. The kinetic energy of the dilaton is also regular and well behaved. The maximum dilaton kinetic energy and Ricci curvature are

$$\frac{\dot{\phi}_0^2}{2} = M_*^4\,, \quad |R_0| = \frac{6M_*^4}{M_P^2}\,. \tag{16}$$

It is important to note that the polymerization (7) is truly of non-perturbative character in the sense that $\sin(\lambda c)$ is an infinite series in powers of $\lambda$. However, by writing Eq. (10) in the form (11), this feature is concealed. Then the term in Eq. (11) that is quadratic in kinetic energy may appear to originate from a truncation of a series expansion in kinetic energy of the dilaton. No such truncation has been performed, and the appearance of the $\dot{\phi}^4$ term is an artifact of our expressing the non-perturbative Eq. (10) in the form of a modified Friedman equation (11).

## III. DYNAMICS IN THE STRING FRAME

We have seen that LQG-inspired polymerization of the connection variable leads to non-singular evolution between the pre-big bang and the post-big bang in the Einstein frame. A complete understanding requires analyzing the dynamics in the string frame, since a reliable solution of the graceful exit problem must work in both frames [2]. The string frame action is

$$\tilde{S} = -\int d^4x\sqrt{-\tilde{g}}\, e^{-\varphi/M_P}\left[\frac{M_P^2}{2}\tilde{R} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right]. \tag{17}$$

In an FRW universe, this action gives the equations of motion,

$$\tilde{H}^2 - \frac{\varphi'}{M_P}\tilde{H} + \frac{\varphi'^2}{6M_P^2} \;=\; 0\,, \tag{18}$$

$$\tilde{H}' + \frac{2\varphi'}{M_P}\tilde{H} - \frac{\varphi'^2}{2M_P^2} \;=\; 0\,, \tag{19}$$

$$\varphi'' + 3\varphi'\tilde{H} - \frac{\varphi'^2}{M_P} \;=\; 0\,. \tag{20}$$

Here a prime denotes a derivative with respect to proper time $\tilde{t}$ in the string frame, and $\tilde{H} = \tilde{a}'/\tilde{a}$. The transformation between the Einstein and string frames is given by

$$dt \;=\; e^{-\varphi(\tilde{t})/2M_P}\, d\tilde{t}\,, \tag{21}$$

$$a(t) \;=\; e^{-\varphi(\tilde{t})/2M_P}\,\tilde{a}(\tilde{t})\,,\;\; \phi(t) = \frac{1}{\sqrt{2}}\varphi(\tilde{t})\,. \tag{22}$$

Using these transformations, the LQC modified equations in the string frame are

$$\tilde{H}^2 - \frac{\varphi'}{M_P}\tilde{H} + \frac{\varphi'^2}{6M_P^2}\left[1 + e^{\varphi/M_P}\,\frac{\varphi'^2}{8M_*^4}\right] \;=\; 0\,, \tag{23}$$

$$\tilde{H}' + 2\frac{\varphi'}{M_P}\tilde{H} - \frac{\varphi'^2}{2M_P^2}\left[1 + e^{\varphi/M_P}\,\frac{\varphi'^2}{4M_*^4}\right] \;=\; 0\,, \tag{24}$$

$$\varphi'' + 3\varphi'\tilde{H} - \frac{\varphi'^2}{M_P} \;=\; 0\,. \tag{25}$$

Solutions of Eqs. (23)–(25) are illustrated in Figs. 5–8. The dynamical evolution in the string frame is such that

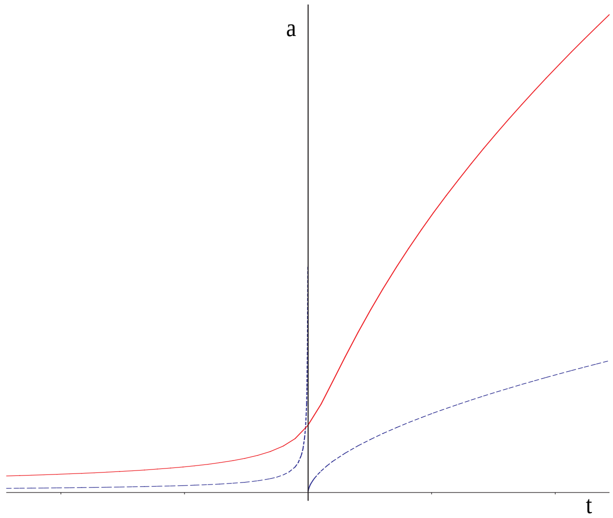


FIG. 5: As in Fig. 1, in the string frame.

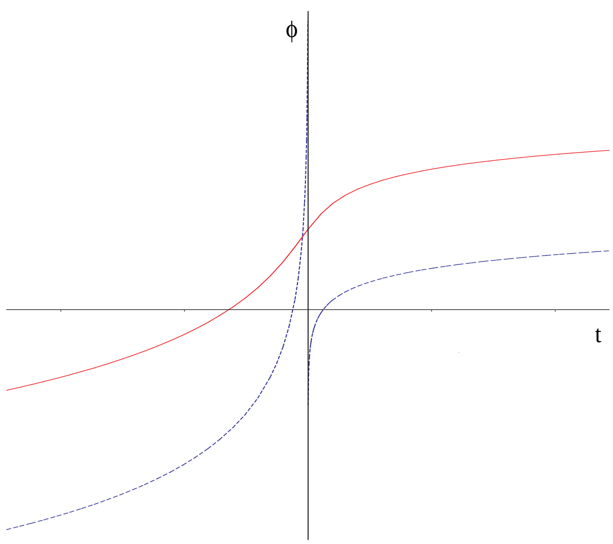


FIG. 6: As in Fig. 2, in the string frame.

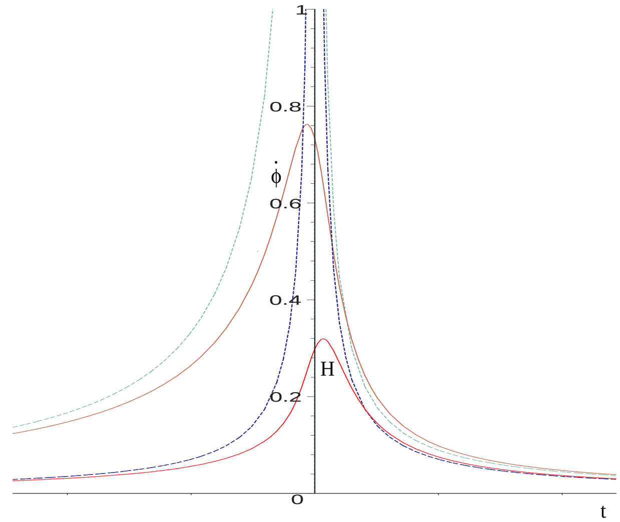


FIG. 7: As in Fig. 3, in the string frame.

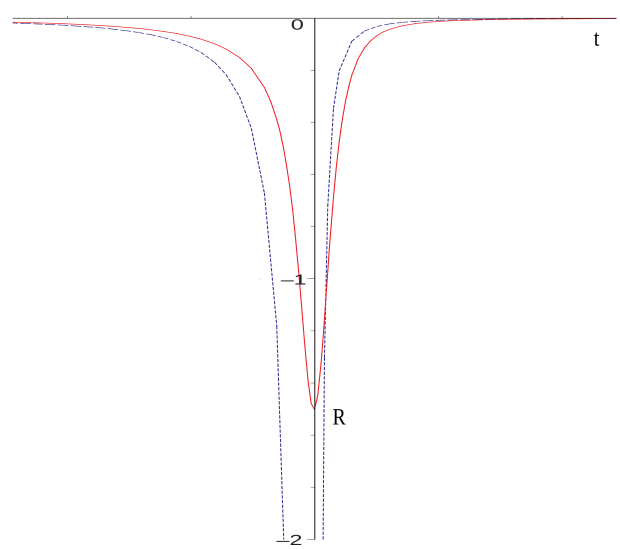


FIG. 8: As in Fig. 4, in the string frame.

the Hubble parameter (and hence the curvature) reaches a maximum and then starts to decrease, thus providing a graceful exit from the singularity problem. The dilaton monotonically grows with time as in the classical PBB scenario, but its slope decreases with time, and its derivative approaches zero. This opens up an interesting possibility, as it appears that polymer modifications can possibly alleviate the problem of late-time stabilization of the dilaton [2]. For a more complete investigation, one needs to study the dynamics with a non-perturbative potential, which is beyond the scope of the present work.

## IV. CONCLUSIONS

We have shown that LQC inspired polymerization applied to the PBB scenario leads to a solution of the graceful exit problem, i.e., a regularization of the singularity that divides the pre- and post-big bang branches in the PBB case. The polymerization was performed in the Einstein frame and the resulting modified dynamical equations were transformed to the string frame. The graceful exit is achieved both in Einstein and string frame without any fine tuning of parameters. In addition, the dilaton in the string frame approaches a constant value at late times. Thus a phenomenological approach using ideas from LQC within the string-theory based PBB leads to a resolution of a key problem in the standard PBB scenario.

We note that, unlike other attempts to construct graceful exit solutions, which introduce matter that violates the null energy condition to regularize the singularity [8, 29], our model regularizes the singularity via gravitational effects, without needing any null energy condition violating matter. Polymerization of the connection induces quantum gravitational effects which are only significant at large curvatures. As the curvature becomes small they die quickly and lead to classical dynamics at low curvature scales. Thus in comparison with previous attempts, our model does not suffer from the lack of classical solutions at late times and there is a generic graceful exit not only from the curvature singularity but also to the classical branch.

An important question which arises is whether polymerization has anything in common with quantum corrections considered so far in PBB. The appearance of the $\exp(\varphi/M_P)$ factor with the squared kinetic energy term is notable, since this is apparently similar to the term obtained by considering one-loop quantum corrections to the string frame action. With a correction

$$\mathcal{L}_q = 2\frac{(\nabla\varphi)^4}{M_P^4} \tag{26}$$

added to the string frame action [29], one obtains a first order correction to the energy-momentum tensor. This has the form of a perfect fluid with

$$\rho_q = 3\frac{\varphi'^4}{M_P^4}, \quad p_q = \frac{\varphi'^4}{M_P^4}\,, \tag{27}$$

so that the the equations of motion give

$$3\tilde{H}^2 - 3\frac{\varphi'}{M_P}\tilde{H} + \frac{\varphi'^2}{2M_P^2} = e^{\varphi/M_P}\rho_p, \tag{28}$$

$$2\tilde{H}' + 3\tilde{H}^2 - \frac{\varphi''}{M_P} + \frac{2\varphi'}{M_P}\tilde{H} - \frac{\varphi'^2}{2M_P^2} = e^{\varphi/M_P}p_q. \tag{29}$$

Notice that in this case, $w_q = p_q/\rho_q = 1/3$. By contrast, using Eqs. (23)–(25), we find that for the LQC polymerization,

$$\rho_{\rm poly} = -\frac{\varphi'^4}{16M_*^4}\,,\ p_{\rm poly} = -\frac{3\varphi'^4}{16M_*^4}\,,\ w_{\rm poly} = 3\,. \tag{30}$$

Thus it appears not to be possible to re-cast the LQC-inspired polymerization in terms of a one-loop quantum correction to the low energy string theory action.

In our phenomenological approach we have exported to PBB scenarios one of the key features of quantum geometry in non-perturbative LQC, as understood at the level of an effective continuum spacetime. In related work on string-based cyclic models [27], the results have been encouraging for the resolution of singularities, as in the present analysis. However, these investigations should be viewed as first steps in the direction of applying insights from non-perturbative quantum geometry approaches to string-based cosmological models. Various questions remain open, including an understanding of the relation of polymerization to an action framework. It is a priori not clear whether an action with a finite number of terms as understood in conventional treatments can be written to mimic polymerization at an effective Hamiltonian level. Another issue is to consider polymerization of the matter degrees of freedom along with the gravitational ones. Furthermore, the polymerization considered in our analysis does not capture effects arising from quantum properties of states, such as dispersions and skewness, which although expected to be small in magnitude may provide useful insights.

We conclude with a remark on the different possible ways to polymerize, alternative to Eqs. (6) and (7). Although it is tempting to consider various other choices, for example treating $\lambda$ as a constant or a function of $p$ different from Eq. (6), it turns out that such versions of polymerization are ruled out by physical constraints. Firstly, the alternatives do not lead to classical GR at low curvature scales, and secondly, they produce unphysical quantum gravity effects at gauge-dependent scales [16]. Polymerization based on the correct loop quantization of cosmological spacetimes [16] does not suffer from these problems. Apart from the functional dependence in $\lambda$, we could also consider polymerizations such as $c \to 2\sin(\lambda c/2)/\lambda$, which may arise from a different quantization scheme. Any such choice (or a similar trigonometric form) leads to similar qualitative results, only effecting the value of $\beta$ in Eq. (6).

**Acknowledgements:** We thank Alejandro Corichi, Aseem Paranjape, Luca Parisi and Kevin Vandersloot for helpful discussions, and Maurizio Gasperini for useful comments. GDR is supported by INFN. The work of RM is partly supported by STFC. PS is supported by NSF grant PHY-0456913 and the Eberly research funds of Penn State. He also thanks the Institute of Cosmology & Gravitation at Portsmouth for warm hospitality at the beginning of this work.

[1] M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993) [arXiv:hep-th/9211021].
[2] M. Gasperini and G. Veneziano, Phys. Rept. **373**, 1 (2003) [arXiv:hep-th/0207130].
[3] M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B **472**, 349 (1996) [arXiv:hep-th/9602087].
[4] M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B **272**, 277 (1991).
[5] M. Giovannini, Phys. Rev. D **57**, 7223 (1998) [arXiv:hep-th/9712122].
[6] M. Giovannini, Phys. Rev. D **59**, 083511 (1999) [arXiv:hep-th/9807049].
[7] G. De Risi and M. Gasperini, Phys. Lett. B **521**, 335 (2001) [arXiv:hep-th/0109137].
[8] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B **494**, 315 (1997) [arXiv:hep-th/9611039].
[9] R. H. Brandenberger, R. Easther and J. Maia, JHEP **9808**, 007 (1998) [arXiv:gr-qc/9806111].
[10] D. A. Easson and R. H. Brandenberger, JHEP **9909**, 003 (1999) [arXiv:hep-th/9905175].
[11] S. Foffa, M. Maggiore and R. Sturani, Nucl. Phys. B **552**, 395 (1999) [arXiv:hep-th/9903008].
[12] C. Cartier, E. J. Copeland and R. Madden, JHEP **0001**, 035 (2000) [arXiv:hep-th/9910169].
[13] C. Cartier, E. J. Copeland and M. Gasperini, Nucl. Phys. B **607**, 406 (2001) [arXiv:gr-qc/0101019].
[14] A. Ashtekar, arXiv:gr-qc/0702030.
For early works, see,
M. Bojowald, Living Rev. Rel. **8**, 11 (2005) [arXiv:gr-qc/0601085].
For a chronological review, see,
G. Date, arXiv:0704.0145v1 [gr-qc].
[15] A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. Lett. **96**, 141301 (2006) [arXiv:gr-qc/0602086].
[16] A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. D **74**, 084003 (2006) [arXiv:gr-qc/0607039].
[17] A. Ashtekar, T. Pawlowski, P. Singh and K. Vandersloot, Phys. Rev. D **75**, 024035 (2007) [arXiv:gr-qc/0612104].
[18] K. Vandersloot, Phys. Rev. D **75**, 023523 (2007) [arXiv:gr-qc/0612070].
[19] D.-W. Chiou, Phys. Rev. D **75**, 024029 (2007) [arXiv:arXiv:gr-qc/0609029].
[20] A. Ashtekar, E. Bentivegna and T. Pawlowski, "Dynamics of de Sitter universes in LQC", in preparation.
[21] E. Bentivegna and T. Pawlowski, "Anti de Sitter dynamics in LQC", in preparation.
[22] A. Ashtekar, A. Corichi and P. Singh, "Contrasting LQC and WDW theory using an exactly solvable model", in preparation.
[23] M. P. Dabrowski and C. Kiefer, Phys. Lett. B **397**, 185 (1997) [arxiv:hep-th/9701035].
[24] A. Corichi, T. Vukasinac and J. A. Zapata, arXiv:0704.0007v1 [gr-qc].
For a similar polymer quantization, see also,
V. Husain and O. Winkler, Phys. Rev. D **69**, 084016 (2004) [arXiv:gr-qc/0312094].
[25] J. Willis, "On the low energy ramifications and a mathematical extension of loop quantum gravity", Ph.D. thesis, Pennsylvania State University (2004).
[26] In general, there can be sub-leading corrections from terms of higher order arising due to quantum properties of the wavefunction. However, for states which correspond to large classical universes at late times these corrections turn out to be negligible [16, 17].
[27] M. Bojowald, R. Maartens and P. Singh, Phys. Rev. D **70**, 083517 (2004) [arXiv:hep-th/0407115];
P. Singh, K. Vandersloot and G. V. Vereshchagin, Phys. Rev. D **74**, 043510 (2006) [arXiv:gr-qc/0606032].
[28] J. Polchinski, "String Theory. Vol. 1: An Introduction to the Bosonic String" (Cambridge University Press, 1998).
[29] R. Brustein and R. Madden, Phys. Rev. D**57**, 712 (1998) [arXiv:hep-th/9708046].