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Cosmological dynamics in spin-foam loop quantum cosmology: challenges and prospects

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We explore the structure of the spin foam-like vertex expansion in loop quantum cosmology and discuss properties of the corresponding amplitudes, with the aim of elucidating some of the expansion's useful properties and features. We find that the expansion is best suited for consideration of conceptual questions and for investigating short-time, highly quantum behavior. In order to study dynamics at cosmological scales, the expansion must be carried to very high order, limiting its direct utility as a calculational tool for such questions. Conversely, it is unclear that the expansion can be truncated at finite order in a controlled manner.

I. INTRODUCTION

Efforts to quantize Einstein's general relativity have fallen into two broad classes, canonical approaches rooted in Dirac's quantization of constrained systems, and covariant approaches based on the sum-over-histories formulation of quantum theory. Each of these approaches has respective strengths and challenges. Understanding the link between these distinct quantizations of gravity and mapping the physical predictions between the canonical and the covariant approaches is an important open problem. Within the framework of the loop quantization of gravity, both paths to quantization have been pursued. In the canonical loop quantum gravity approach, one aims to obtain a physical Hilbert space with an inner product on physical states by implementing spatial diffeomorphism and Hamiltonian constraints at the quantum level. On the other hand, in the covariant "spin-foam" formulation, the physical inner product and the resulting physics are tied to summing over spin foam amplitudes associated with a suitable discretization of the spacetime manifold. Both these approaches capture elements of a discrete quantum geometry, yielding rich physical predictions.¹ However, the precise relation between the canonical and covariant quantization approaches in loop quantum gravity (LQG) is not yet clearly established. The primary reason lies in the mathematical complexities inherent in the quantization of gravitational spacetimes, an as-yet incomplete task in both approaches. This naturally leads to the following questions: Can this relationship be understood for simpler, yet still non-trivial spacetimes, which *can* be successfully quantized? If so, how do we understand the qualitative aspects of physics established in one approach in the framework of the other approach?

To gain insight on these questions, cosmological spacetimes provide a useful setting. In the last decade, techniques of loop quantum gravity have been applied to the successful quantization of various homogeneous cosmological spacetimes, and the physical Hilbert space is known rigorously in loop quantum cosmology (LQC) [4]. A key prediction of LQC is the existence of a "bounce" when spacetime curvature becomes Planckian [5]. The existence of a bounce away from the curvature singularity has been established in numerical simulations in a variety of models, even for highly quantum states (for reviews see Refs. [6] and [7].) Non-perturbative quantum gravitational effects studied in a range of isotropic and anisotropic spacetimes point towards resolution of all strong curvature singularities as a generic feature of LQC [8]. Further, for a particular choice of lapse in the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime sourced with a massless scalar field, the model can be solved exactly, establishing a robust picture of the bounce for all the states in the physical Hilbert space [9]. Using this solvable model of LQC (sLQC), self-consistent quantum probabilities for the bounce can be calculated and the resolution of the singularity demonstrated rigorously within the consistent histories formulation of quantum theory [10].

Interestingly, the solvable model can be used to give loop quantum cosmology a spin-foam-like sum-over-histories formulation [11]. The theory may even be written as a standard path integral [12], starting from the Hamiltonian in sLQC and obtaining a Lagrangian in the phase space variables.² This work provides a concrete platform from which to explore the physics of canonical loop quantum cosmology in the covariant spin-foam language. We will refer to this formulation as spin-foam loop quantum cosmology, not to be confused with spin-foam cosmology [14–16], which aims

¹ For recent textbook-level introductions to these approaches, with many references to earlier literature, see [1–3].

² Note that the starting point of this spin-foam-like formulation of loop quantum cosmology is not a covariant action in four dimensions, but rather turns out to be an infinite series in curvature invariants [13].

to investigate cosmological issues in the fully covariant spin-foam formulation, albeit from a very different starting point and including certain types of inhomogeneities.

Solvable LQC can be deparameterized, with the scalar field serving as a physical matter “clock”. The quantum theory of sLQC can therefore be analyzed, equivalently, using either a “relativistic” (Klein-Gordon) or “non-relativistic” (Schrödinger) representation. The latter allows a direct connection with conventional Schrödinger quantum mechanics and its sum-over-histories formulation. Nonetheless, a key difference from the conventional path integral approach is that in sLQC one is dealing with polymer quantization rather than standard Fock quantization. The path integral for LQC resembles the vertex expansion of spin-foam models obtained by summing over suitably chosen dual triangulations which capture the discretization of spacetime. In work so far on the path integral formulation of LQC [11, 17–19] the emphasis has been on establishing the formal structure linking the two approaches. This has helped clarify certain technical issues in the spin-foam paradigm using results from the canonical picture [11]. On the other hand, using the covariant picture, insights into the choice of regulator and a local vertex expansion have been obtained [19]. Though these developments have been useful, it has not been clear how effectively the spin-foam like vertex expansion in LQC can be employed to shed light in a practical way on cosmological dynamics in the spin-foam motivated language. If it can be used, then one would like to understand features of the physical evolution in the covariant picture.

In this manuscript, within the framework of spin-foam LQC we explore some features of the vertex expansion of the theory’s “transition amplitudes” (equivalently, inner product). The vertex expansion is composed of a sum over amplitudes for spacetime histories which undergo M discrete volume transitions. The M th term in the vertex expansion in the spin-foam context refers to a dual triangulation of the spacetime manifold with M vertices. We study the way amplitudes for different terms in the vertex expansion scale with volume, and the way this scaling is affected by the degeneracies in the volume transitions. The behavior for the small M cases is found by explicit computation. We argue that in order to capture cosmologically relevant physical evolution, large orders M in the expansion are required. We show in particular that low orders in the vertex expansion can at best capture the dynamics of only short intervals $\Delta\phi$ in the matter field ϕ and small changes in cosmological volume. In that regard the vertex expansion in LQC is much like an expansion of a standard quantum propagator in powers of Et/\hbar . Truncating the expansion at low order yields an approximation that is useful to study physical evolution only for short time intervals. A different expansion in LQC will therefore be necessary in order to study cosmological dynamics.

Moreover, we also find that the while the quantum amplitudes do satisfy the quantum constraint, the vertex expansion of those amplitudes does not satisfy the constraint in a controlled way if it is truncated at any finite order, once again calling into question the utility of the vertex expansion in studying the theory’s cosmological dynamics.

The manuscript is organized as follows. In the next section, we summarize the main results from solvable LQC and discuss the quantum Hamiltonian constraint and its different representations. In this manuscript we focus on the “relativistic” representation, which results in a representation for the quantum transition amplitudes that is directly analogous to the Hadamard propagator of standard quantum field theory, as discussed in Sec. III. Some useful properties and a reformulation of the vertex expansion are discussed in Sec. IV. In Sec. V we discuss various results. We conclude with a summary in Sec. VI.

II. SOLVABLE LOOP QUANTUM COSMOLOGY

The quantization described in our analysis is based on the spacetime metric for the spatially flat, homogeneous isotropic spacetime, given by

$$ds^2 = -N^2 dt^2 + q_{ab} dx^a dx^b = -N^2 dt^2 + a^2(t) d\mathbf{x}^2, \quad (2.1)$$

where $N(t)$ is the lapse function. The spatial topology is taken to be \mathbb{R}^3 and a fiducial spatial cell \mathcal{V} is fixed in order to define a symplectic structure after integration over the spatial volume of \mathcal{V} . We take the matter to be a massless, minimally coupled scalar field ϕ . The classical cosmological dynamics for this spacetime and matter content yields expanding and contracting solutions which are disjoint and singular for any given value of the scalar field momentum p_ϕ . The loop quantization for this metric for lapse $N = 1$ was first rigorously performed in Refs. [5]. The quantization results in resolution of the cosmological singularity at the level of the physical Hilbert space. The big bang and big crunch singularities in the expanding and contracting branches are avoided, and replaced by a non-singular “bounce” of the universe at the Planck scale and effective unitary evolution if the scalar field ϕ is taken as a physical clock. Significant control over the physical Hilbert space can be achieved by choosing the lapse $N = a^3$, in which case the model becomes exactly solvable (“sLQC”) [9]. In the following we will concern ourselves with this particular choice, and summarize some of the basic features of the construction.

At the classical level, after imposition of the symmetries of this spacetime, the symmetry reduced gravitational phase space variables are the connection c and the triad p , and the classical Hamiltonian constraint can be written as

$$C = p_\phi^2 - 3\pi G v^2 b^2 \approx 0. \quad (2.2)$$

Here v is related to the physical volume $V = |p|^{3/2}$ of the fiducial cell \mathcal{V} as $v = V/2\pi G$, and $b = c/|p|^{1/2}$. The modulus sign arises due to the two possible orientations of the triad. The phase space variables are (v, b) and (ϕ, p_ϕ) , satisfying

$$\{v, b\} = 2\gamma, \quad \text{and} \quad \{\phi, p_\phi\} = 1, \quad (2.3)$$

with $\gamma \approx 0.2375$ as the Barbero-Immirzi parameter. Upon quantization, the action of the volume operator is multiplicative on states $\Psi(\nu)$,

$$\hat{V} \Psi(\nu) = 2\pi\gamma l_p^2 |\nu| \Psi(\nu), \quad (2.4)$$

where $\nu = v/\gamma\hbar$. Unlike V , there is no corresponding operator \hat{b} in the loop quantization. Rather, its action is captured via holonomies of the connection, through the translation operator acting on the volume eigenkets $|\nu\rangle$:

$$\widehat{\exp(i\lambda b)}|\nu\rangle = |\nu - 2\lambda\rangle, \quad (2.5)$$

where λ captures the minimum non-zero area in quantum geometry given by $\lambda^2 = 4\pi\sqrt{3}\gamma l_p^2$. This action of the holonomy operators is responsible for the discrete quantum evolution discussed below.

Using the action of $\hat{p}_\phi = -i\hbar\partial_\phi$ on states $\Psi(\nu, \phi)$, the quantum Hamiltonian constraint can be written in a Klein-Gordon form,

$$\hat{C}\Psi(\nu, \phi) = -(\partial_\phi^2 + \Theta(\nu))\Psi(\nu, \phi). \quad (2.6)$$

Here Θ is a positive definite and essentially self-adjoint operator, with continuous eigenvalues $\omega_k = \kappa|k|$, where $\kappa = \sqrt{12\pi G}$:

$$\Theta|k\rangle = \omega_k^2|k\rangle. \quad (2.7)$$

The symmetric eigenfunctions of Θ are $e_k^{(s)}(\nu) \equiv \langle\nu|k\rangle$. They are orthogonal and satisfy the completeness relations

$$\sum_{\nu=4\lambda n} e_k^{(s)}(\nu)^* e_{k'}^{(s)}(\nu) = \delta^{(s)}(k, k'), \quad (2.8)$$

$$\int_{-\infty}^{+\infty} dk e_k^{(s)}(\nu) e_k^{(s)}(\nu')^* = \delta_{\nu, \nu'}^{(s)}. \quad (2.9)$$

(See [20] for further details.)

On physical states $\Psi(\nu, \phi)$ the action of Θ is

$$\Theta \Psi(\nu, \phi) = -\frac{3\pi G}{4\lambda^2} \left\{ \sqrt{|\nu(\nu+4\lambda)|} |\nu+2\lambda| \Psi(\nu+4\lambda, \phi) - 2\nu^2 \Psi(\nu, \phi) + \sqrt{|\nu(\nu-4\lambda)|} |\nu-2\lambda| \Psi(\nu-4\lambda, \phi) \right\}. \quad (2.10)$$

The corresponding matrix elements may be written

$$\Theta_{\nu\nu'} = \left(\frac{\kappa}{4\lambda}\right)^2 \sqrt{|\nu \cdot \nu'|} |\nu + \nu'| \left\{ \delta_{\nu, \nu'} - \frac{1}{2} [\delta_{\nu+4\lambda, \nu'} + \delta_{\nu-4\lambda, \nu'}] \right\}. \quad (2.11)$$

In this simple model, Θ is a ϕ -independent spatial Laplacian operator whose action links the physical wavefunction with uniform discreteness in volume. The physical Hilbert space can be decomposed into disjoint sectors of positive and negative frequency symmetric solutions to the quantum constraint which satisfy

$$\mp i\partial_\phi \Psi(\nu, \phi) = \sqrt{\Theta} \Psi(\nu, \phi). \quad (2.12)$$

A physical state has support only on a lattice $\epsilon \in [0, 4)$ related to ν via $\nu = 4n\lambda + \epsilon$ where $n \in \mathbb{Z}$. Each lattice is left invariant by the unitary dynamical evolution in ϕ . This results in a super-selection of the physical Hilbert space: $\mathcal{H}_{\text{phys}} = \oplus \mathcal{H}_\epsilon$. In our analysis we will work with the $\epsilon = 0$ lattice which includes the case of zero volume – the classical big bang singularity. The classical singularity is resolved in the quantum theory leading to a “bounce” at small volume [5, 9], where quantum geometry leads to an effective repulsive force. The bounce can also be understood via properties of the eigenfunctions of Θ numerically [5] as well as analytically [20]. The eigenfunctions are found to decay exponentially near the classical singularity unless $k \lesssim |\nu|/2\lambda$. The exponential fall-off is determined by the value of the scalar field momentum which is a direct measure of the bounce volume.

Eq.(2.12) is analogous to the Schrödinger equation of ordinary non-relativistic quantum mechanics. In this representation, positive and negative frequency states can be expanded in terms of the symmetric eigenfunctions of the Θ operator as

$$\Psi^\pm(\nu, \phi) = \int_{-\infty}^{\infty} dk \tilde{\Psi}(k) e_k^{(s)}(\nu) e^{\pm i\omega_k \phi}, \quad (2.13)$$

where $\Psi(k)$ is the wave profile. The physical states have finite norm with respect to the Schrödinger inner product computed at a fiducial (but immaterial) $\phi = \phi_o$,

$$\langle \Psi_1 | \Psi_2 \rangle = \sum_{\nu=4n} \Psi_1^*(\nu, \phi_o) \Psi_2(\nu, \phi_o). \quad (2.14)$$

A unitarily equivalent representation is the “relativistic” one where the physical states can be written as

$$\underline{\Psi}^\pm(\nu, \phi) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\omega_k}} \tilde{\Psi}(k) e_k^{(s)}(\nu) e^{\pm i\omega_k \phi}. \quad (2.15)$$

In this representation the inner product is the Klein-Gordon product

$$\langle \underline{\Psi}^\pm | \underline{\Phi}^\pm \rangle = \mp i \sum_{\nu=4n\lambda} \underline{\Psi}^\pm(\nu, \phi)^* \overleftrightarrow{\partial}_\phi \underline{\Phi}^\pm(\nu, \phi), \quad (2.16)$$

which for the current simple model (zero potential) turns out to be precisely equal to the Schrödinger inner product. Here $\overleftrightarrow{\partial}_\phi = \overrightarrow{\partial}_\phi - \overleftarrow{\partial}_\phi$. Note that the $\sqrt{2\omega_k}$ in the measure could alternatively be absorbed into a renormalization of the eigenfunctions and corresponding completeness relations. To make the connection with the covariant description and full quantum gravity, this “relativistic” representation is more natural. Further, it is a useful representation to work with for more general models which do not deparameterize. In the following analysis, we will primarily employ the relativistic representation.

III. PROPAGATORS

An object of primary interest in our discussion is the “extraction amplitude” – essentially, the inner product – which in the relativistic representation turns out to be in essence the Hadamard propagator of ordinary quantum field theory, and which can be interpreted as a propagator in LQC as well. On the other hand, in the non-relativistic representation the extraction amplitude is the Newton-Wigner function. In the following, we summarize the construction behind these amplitudes and obtain the composition laws which enable them to be viewed as propagators. Our discussion will be based on the earlier analyses of Refs. [11, 17].

Let us start with the Hadamard function, a two point function in the relativistic representation, given by the physical inner product between the eigenstates $|\nu, \phi\rangle$, which can be obtained using group averaging:

$$\begin{aligned} G_H(\nu_f, \phi_f; \nu_i, \phi_i) &\equiv \langle \nu_f, \phi_f | \nu_i, \phi_i \rangle \\ &= \int_{-\infty}^{\infty} d\alpha \text{kin} \langle \nu_f, \phi_f | e^{i\alpha \hat{C}} | \nu_i, \phi_i \rangle_{\text{kin}}, \end{aligned} \quad (3.1)$$

where \hat{C} is given by (2.6), α is the group averaging parameter, and $|\nu, \phi\rangle_{\text{kin}}$ denote the states in the kinematical Hilbert space. Since \hat{p}_ϕ commutes with the ϕ -independent Θ , the Hadamard function can be written as a product of two amplitudes,

$$A_H(\Delta\phi; \alpha) = \langle \phi_f | e^{i\alpha p_\phi^2} | \phi_i \rangle, \quad \text{and} \quad A_\Theta(\nu_f, \nu_i; \alpha) = \langle \nu_f | e^{-i\alpha \Theta} | \nu_i \rangle, \quad (3.2)$$

such that

$$G_H(\nu_f, \phi_f; \nu_i, \phi_i) = \int_{-\infty}^{\infty} d\alpha A_H(\Delta\phi; \alpha) A_\Theta(\nu_f, \nu_i; \alpha). \quad (3.3)$$

Using $\langle \phi | p_\phi \rangle = \exp(ip_\phi \phi / \hbar) / \sqrt{2\pi}$ and the resolution of the identity we get

$$A_H(\Delta\phi; \alpha) = \int_{-\infty}^{\infty} dp_\phi \langle \phi_f | e^{i\alpha p_\phi^2} | p_\phi \rangle \langle p_\phi | \phi_i \rangle = \int_{-\infty}^{\infty} \frac{dp_\phi}{2\pi} e^{i\alpha p_\phi^2} e^{ip_\phi \Delta\phi}. \quad (3.4)$$

Similarly, the gravitational part of the amplitude can be written as

$$A_{\Theta}(\nu_f, \nu_i; \alpha) = \int_{-\infty}^{\infty} dk \langle \nu_f | e^{-i\alpha\Theta} | k \rangle \langle k | \nu_i \rangle = \int_{-\infty}^{\infty} dk e^{-i\alpha\omega_k^2} e_k^{(s)}(\nu_f) e_k^{(s)*}(\nu_i), \quad (3.5)$$

where in the last step we have used the eigenvalue equation for Θ .

Using the above expressions for A_{H} and A_{Θ} , we can separate the Hadamard function into positive and negative frequency parts. To prove this, let us rearrange the integrals in G_{H} as

$$G_{\text{H}}(\nu_f, \phi_f; \nu_i, \phi_i) = \int_{-\infty}^{\infty} dk e_k^{(s)}(\nu_f) e_k^{(s)*}(\nu_i) \int_{-\infty}^{\infty} \frac{dp_{\phi}}{2\pi} e^{ip_{\phi}\Delta\phi} \int_{-\infty}^{\infty} d\alpha e^{-i\alpha(p_{\phi}^2 - \omega_k^2)}. \quad (3.6)$$

Performing the integration over the group averaging parameter yields a sum of Dirac delta functions which separate the terms with positive and negative frequencies, resulting in

$$\begin{aligned} G_{\text{H}}(\nu_f, \phi_f; \nu_i, \phi_i) &= \int_{-\infty}^{\infty} \frac{dk}{2\omega_k} [e^{+i\omega_k\Delta\phi} + e^{-i\omega_k\Delta\phi}] e_k^{(s)}(\nu_f) e_k^{(s)*}(\nu_i) \\ &= +\langle \nu_f, \phi_f | \nu_i, \phi_i \rangle_+ + -\langle \nu_f, \phi_f | \nu_i, \phi_i \rangle_- \\ &= G_{\text{H}}^+(\nu_f, \phi_f; \nu_i, \phi_i) + G_{\text{H}}^-(\nu_f, \phi_f; \nu_i, \phi_i). \end{aligned} \quad (3.7)$$

Here the ‘‘Wightman functions’’ G_{H}^{\pm} give the physical inner products between the positive/negative frequency eigenstates. Using the Klein-Gordon inner product, it is straightforward to show that they satisfy the following composition law:

$$\begin{aligned} G_{\text{H}}^{\pm}(\nu_f, \phi_f; \nu_i, \phi_i) &= G_{\text{H}}^{\pm}(\nu_f, \phi_f; \nu, \phi) \circ G_{\text{H}}^{\pm}(\nu, \phi; \nu_i, \phi_i) \\ &= \mp i \sum_{\nu} G_{\text{H}}^{\pm}(\nu_f, \phi_f; \nu, \phi) \overleftrightarrow{\partial}_{\phi} G_{\text{H}}^{\pm}(\nu, \phi; \nu_i, \phi_i), \end{aligned} \quad (3.8)$$

where we have used the completeness relation for the eigenfunctions $e_k^{(s)}(\nu)$. (The first line defines the relativistic composition operator \circ .) This composition law allows us to view the Hadamard two point function as a transition amplitude or propagator of the dynamics from ϕ_i to ϕ_f . The propagation action from the state $\Psi^{\pm}(\nu', \phi')$ to $\Psi^{\pm}(\nu, \phi)$ is

$$\Psi^{\pm}(\nu, \phi) = G^{\pm}(\nu, \phi; \nu', \phi') \circ \Psi^{\pm}(\nu', \phi') = \mp i \sum_{\nu'} G_{\text{H}}^{\pm}(\nu, \phi; \nu', \phi') \overleftrightarrow{\partial}_{\phi'} \Psi^{\pm}(\nu', \phi'). \quad (3.9)$$

One can similarly define a Newton-Wigner two-point function [11, 17],

$$G_{\text{NW}}(\nu_f, \phi_f; \nu_i, \phi_i) \equiv \int_{-\infty}^{\infty} d\alpha \text{kin} \langle \nu_f, \phi_f | e^{i\alpha\hat{C}} 2|p_{\phi}| | \nu_i, \phi_i \rangle_{\text{kin}}. \quad (3.10)$$

Using the observation that the gravitational part of the amplitude A_{Θ} is identical to the one in the Hadamard case, it is easily shown that the positive frequency/negative frequency pieces of the Newton-Wigner function are related to those of the Hadamard propagator by

$$G_{\text{NW}}^{\pm}(\nu_f, \phi_f; \nu_i, \phi_i) = \mp 2i \partial_{\phi_f} G_{\text{H}}^{\pm}(\nu_f, \phi_f; \nu_i, \phi_i). \quad (3.11)$$

Note that G_{NW} is given by essentially the same expression as for the Hadamard propagator, but without the $1/2\omega_k$ in the measure.

It is straightforward to check that it satisfies the following ‘‘non-relativistic’’ composition law:

$$\begin{aligned} G_{\text{NW}}^{\pm}(\nu_f, \phi_f; \nu_i, \phi_i) &= G_{\text{NW}}^{\pm}(\nu_f, \phi_f; \nu, \phi) \circ_{\text{NW}} G_{\text{NW}}^{\pm}(\nu, \phi; \nu_i, \phi_i) \\ &= \sum_{\nu} G_{\text{NW}}^{\pm}(\nu_f, \phi_f; \nu, \phi) G_{\text{NW}}^{\pm}(\nu, \phi; \nu_i, \phi_i). \end{aligned} \quad (3.12)$$

The Newton-Wigner propagator naturally propagates states in the Schrödinger inner product,

$$\Psi^{\pm}(\nu, \phi) = \sum_{\nu'} G_{\text{NW}}^{\pm}(\nu, \phi; \nu', \phi') \Psi^{\pm}(\nu', \phi'). \quad (3.13)$$

The propagation relations for the Hadamard and Newton-Wigner functions show that the propagation action is unchanged in both the ‘‘non-relativistic’’ and ‘‘relativistic’’ representations.

With these properties established, we are now equipped to employ the Hadamard propagator to compute the vertex amplitude in spin-foam loop quantum cosmology in the next section.

IV. THE VERTEX EXPANSION IN SPIN FOAM LOOP QUANTUM COSMOLOGY

In order to compute the transition amplitude from a kinematical state $|\nu_i, \phi_i\rangle$ to $|\nu_f, \phi_f\rangle$, we need to compute the amplitude corresponding to the gravitational part given by Eq. (3.2). To evaluate it, following [11] we use the ideas of the sum-over-histories formulation of quantum theory with Θ playing the role of the Hamiltonian. The main idea behind this construction is as follows. The “time” interval α is divided into N equal parts of length ε . Each interval of time is labelled by a volume element ν via insertions of resolutions of the identity in volume. This provides a time discrete history, $\Delta\phi_N$, identified with $N - 1$ volumes between ν_i and ν_f . The gravitational part of the amplitude can then be written as

$$A_\Theta(\nu_f, \nu_i; \alpha) = \sum_{\nu_{N-1}, \dots, \nu_1} \langle \nu_N | e^{-i\varepsilon\Theta} | \nu_{N-1} \rangle \dots \langle \nu_1 | e^{-i\varepsilon\Theta} | \nu_0 \rangle = \sum_{\Delta\phi_N} U_{\nu_N \nu_{N-1}} \dots U_{\nu_1 \nu_0}. \quad (4.1)$$

where $\nu_0 = \nu_i$ and $\nu_N = \nu_f$. In ordinary quantum mechanics, the transition amplitude is then computed by taking $N \rightarrow \infty$ which removes any dependence on $\varepsilon = \alpha/N$. Here this limit is tricky, since each term in the above product yields an ε term in the first order. The total product is thus proportional to ε^N , which vanishes in the naive limit $N \rightarrow \infty$. To take this limit, the above sum is instead reorganized according to the number M of discrete volume *transitions* to a distinct volume, regardless of “when” (i.e. at what value of ϕ) they occur. The gravitational amplitude can then be written as a sum over amplitudes for individual paths $(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0)$ with M transitions [11],

$$A_\Theta(\nu_f, \nu_i; \alpha) = \sum_{M=0}^N \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0; \alpha). \quad (4.2)$$

This reorganization of the sum allows taking the limit $N \rightarrow \infty$, which then results in the transition amplitude between $|\nu_i, \phi_i\rangle$ and $|\nu_f, \phi_f\rangle$ after integration over α and p_ϕ . Note that in this reorganization in terms of volume transitions, while by construction no two *consecutive* volumes will be the same, for any given history $(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0)$ individual volumes may be repeated. Take p to be the number of *unique* volumes $(w_{p-1}, w_{p-2}, \dots, w_1, w_0)$ appearing in the path (so $p \leq M + 1$), where $w_0 = \nu_0$.³ The degeneracy of each volume w_k in the given path will be denoted d_k , so $\sum_{k=0}^{p-1} d_k = M + 1$.

The Hadamard propagator can then be written as

$$G_H^\pm(\nu_f, \phi_f; \nu_i, \phi_i) = \sum_{M=0}^{\infty} \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_M^\pm(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0; \Delta\phi), \quad (4.3)$$

where the “path amplitude” A_M^\pm associated with the path $(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0)$ is given in terms of the matrix elements $\Theta_{\nu\nu'} = \langle \nu | \Theta | \nu' \rangle$ by [11]⁴

$$A_M^\pm(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0; \Delta\phi) = \Theta_{\nu_M \nu_{M-1}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \times \prod_{k=0}^{p-1} \frac{1}{(d_k - 1)!} \left(\frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{d_k - 1} \sum_{l=0}^{p-1} \frac{e^{\pm i \sqrt{\Theta_{w_l w_l}} \Delta\phi}}{2 \sqrt{\Theta_{w_l w_l}} \prod_{\substack{j=0 \\ j \neq l}}^{p-1} (\Theta_{w_l w_l} - \Theta_{w_j w_j})}. \quad (4.4)$$

Eq. (4.3) is the “vertex expansion” defining the spin-foam formulation of loop quantum cosmology in analogy with the sum over amplitudes for transitions between fixed “initial” and “final” boundary surfaces in the covariant spin-foam formulation of loop quantum gravity. In covariant LQG, the interpolating manifold is given a triangulation with edges colored by spins, and an amplitude assigned to each such colored triangulation. The full transition amplitude is then given in a sum-over-histories prescription by summing over all possible colorings and (dual) triangulations. In spin-foam LQC, the sum over internal volumes is analogous to the sum over colorings, and the sum over the number of volume transitions analogous to the sum over dual triangulations. (See Fig. 1.)

³ Note this is a slight change from the notation of [11, 17] because we choose to number the unique volumes w_k such that $w_0 = \nu_0$. The labeling of the w_k will therefore agree with that of the complete path in the case that all volumes in the path are distinct. However, in some situations there may be some virtue to ditching this correspondence and instead choosing to order the w_i from smallest to largest, or something of that nature, in which case the w_i will be a list of consecutive volumes. One must be cautious with this notation, recognizing that the set $\{w_i\}$ is specific to each individual path. *Given* the path $(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0)$, one must then *calculate* p and the corresponding $\{w_i\}$ and their degeneracies $\{d_i\}$, and then proceed to evaluate the corresponding path amplitude (which depends on *all three* sets of numbers.)

⁴ There is a similar alternative expansion for G_{NW}^\pm in terms of the matrix elements of $H = \sqrt{\Theta}$.

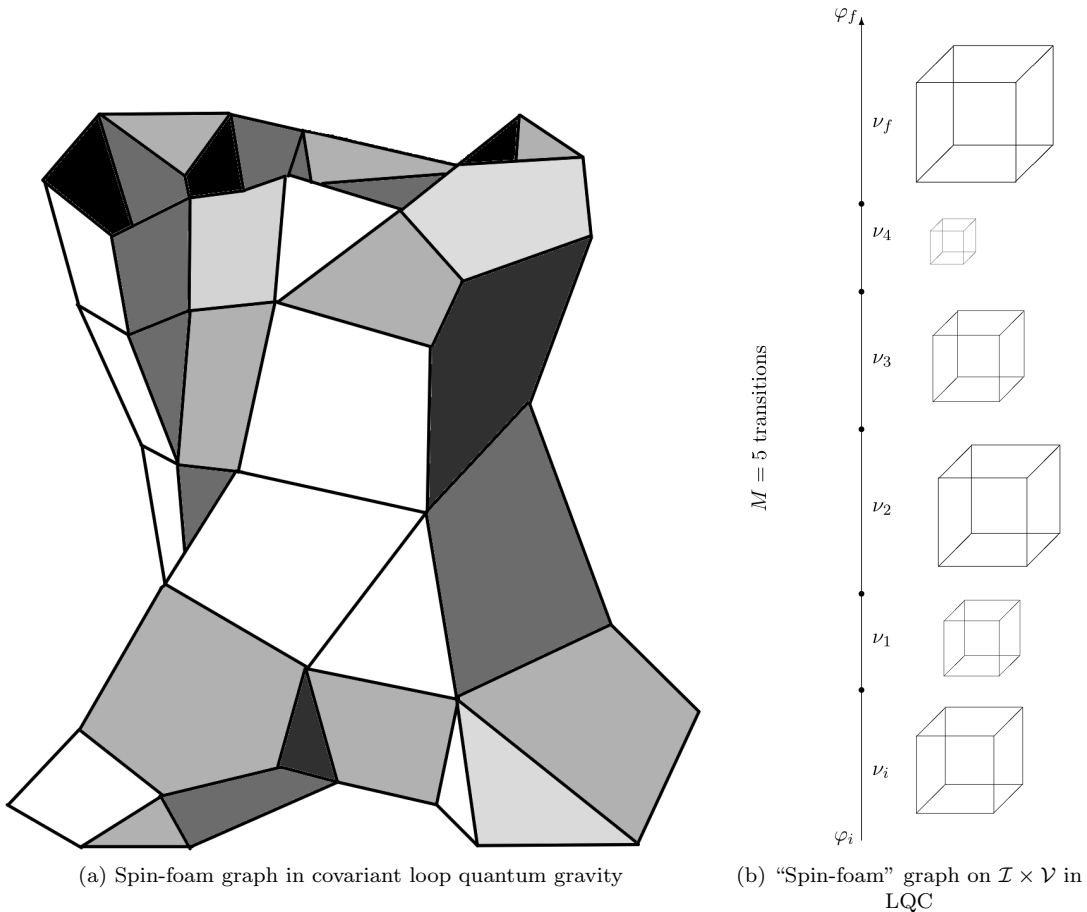


FIG. 1. In covariant loop quantum gravity, amplitudes for transitions between fixed “initial” and “final” boundary surfaces are defined by a sum-over-histories prescription. The spacetime manifold is given a (dual) triangulation with edges colored by spins and an amplitude is assigned to each such colored dual triangulation. The full transition amplitude is then calculated a la Feynman by summing over all possible colorings and (dual) triangulations – the “vertex expansion” of the amplitudes of spin-foam loop quantum gravity [2]. An analogous graph is shown for spin-foam loop quantum cosmology. Shown is an example of a particular cosmological history with $M = 5$ volume transitions. Here $\nu_i = \nu_0$ and $\nu_f = \nu_5 = \nu_M$. The dots (“vertices”) denote the transitions; the value of M is analogous to the choice of dual triangulation. The “edges” of the graph are labeled by the volumes, analogous to the spin colorings. This is analogous to an individual graph (triangulated spacetime manifold) connecting fixed boundary surfaces (here fixed ν_i and ν_f) in spin-foam loop quantum gravity. The vertex expansion assigns an amplitude to each such history. The complete transition amplitude (equivalently, inner product) is the sum of all such amplitudes.

It is sometimes convenient to regard the path amplitude for any path not satisfying the condition that each transition must be to a different volume as simply zero, rather than restricting the sum in Eq. (4.2) or (4.3). So instead we could write

$$G_{\text{H}}^{\pm}(\nu_f, \phi_f; \nu_i, \phi_i) = \sum_{M=0}^{\infty} \sum_{\nu_{M-1}, \dots, \nu_1} P_M^{\pm}(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0; \Delta\phi) \quad (4.5)$$

as a sum over *all* paths connecting $\nu_i = \nu_0$ to $\nu_f = \nu_M$, where now

$$P_M^{\pm}(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0; \Delta\phi) = \Omega_{\nu_M \nu_{M-1}} \dots \Omega_{\nu_2 \nu_1} \Omega_{\nu_1 \nu_0} \times \prod_{k=0}^{p-1} \frac{1}{(d_k - 1)!} \left(\frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{d_k - 1} \sum_{l=0}^{p-1} \frac{e^{\pm i \sqrt{\Theta_{w_l w_l}} \Delta\phi}}{2\sqrt{\Theta_{w_l w_l}} \prod_{\substack{j=0 \\ j \neq l}}^{p-1} (\Theta_{w_l w_l} - \Theta_{w_j w_j})}. \quad (4.6)$$

Here we have defined the off-diagonal part of Θ as $\Omega = \Theta - \Theta^D$, encompassing the off-diagonal “transition” matrix

elements in the vertex expansion, which as we will see gives a non-zero contribution to Eq. (4.6) only for paths for which all transitions are to distinct, neighboring volumes. The diagonal part is denoted by Θ^D .

The case $M = 0$ (no transitions) is slightly special. In this case $A_0^\pm = P_0^\pm$, and can be calculated directly to be

$$P_0^\pm(\nu_f, \nu_0; \Delta\phi) = \frac{e^{\pm i\sqrt{\Theta_{\nu_0\nu_0}}\Delta\phi}}{2\sqrt{\Theta_{\nu_0\nu_0}}}\delta_{\nu_f, \nu_0}. \quad (4.7)$$

(Recall that in our notation $\nu_0 = \nu_i$ and $\nu_M = \nu_f$.)

V. FEATURES OF VERTEX AMPLITUDES IN SPIN FOAM LOOP QUANTUM COSMOLOGY

A. Vertex amplitudes and their calculation

In order to simplify investigation of various properties of the propagator, recalling $\nu = 4\lambda n$ it is useful to express the matrix elements $\langle \nu | \Theta | \nu' \rangle = \Theta_{\nu, \nu'}$ of Eq. (2.11) in terms of n instead of ν . That is,

$$\begin{aligned} \Theta_{n, n'} &= \kappa^2 \sqrt{|n \cdot n'|} |n + n'| \cdot \left\{ \delta_{n, n'} - \frac{1}{2} [\delta_{n, n'+1} + \delta_{n, n'-1}] \right\} \\ &= 2\kappa^2 n^2 \delta_{n, n'} - \kappa^2 n^2 \beta(n, n'), \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} \beta(n, n') &= \frac{1}{2} \sqrt{\left| \frac{n'}{n} \right|} \left| 1 + \frac{n'}{n} \right| [\delta_{n, n'+1} + \delta_{n, n'-1}] \\ &= f_+(n) \delta_{n, n'-1} + f_-(n) \delta_{n, n'+1}, \end{aligned} \quad (5.2)$$

with

$$f_\pm(n) = \sqrt{1 \pm \frac{1}{n}} \left(1 \pm \frac{1}{2n} \right). \quad (5.3)$$

It is to be noted that the only non vanishing matrix elements are $\Theta_{n, n} = 2\kappa^2 n^2$, and $\Theta_{n, n\pm 1} = -\kappa^2 n^2 f_\pm(n)$. In the vertex expansion, the off-diagonal matrix elements corresponding to transitions to distinct volumes can then be written as

$$\begin{aligned} \langle n | \Omega | n' \rangle &= -\kappa^2 \frac{1}{2} \sqrt{|n \cdot n'|} |n + n'| \cdot [\delta_{n, n'+1} + \delta_{n, n'-1}] \\ &= -\kappa^2 n^2 \beta(n, n') \\ &= -\kappa^2 n^2 [f_+(n) \delta_{n, n'-1} + f_-(n) \delta_{n, n'+1}]. \end{aligned} \quad (5.4)$$

All matrix elements $\Omega_{n, n'}$ are zero except for $\Omega_{n, n\pm 1} = -\kappa^2 n^2 f_\pm(n)$. Crucially, this means that the only paths which have non-zero amplitude are ones for which each transition is to a neighboring volume precisely one unit greater or smaller than the one before. Thus, the product of off-diagonal terms can be written as

$$\begin{aligned} \Omega_{\nu_M \nu_{M-1}} \dots \Omega_{\nu_2 \nu_1} \Omega_{\nu_1 \nu_0} &= (-1)^M \kappa^{2M} \prod_{i=1}^M n_i^2 \beta(n_i, n_{i-1}) \\ &= (-1)^M \kappa^{2M} \prod_{i=1}^M n_i^2 [f_+(n_i) \delta_{n_i, n_{i-1}-1} + f_-(n_i) \delta_{n_i, n_{i-1}+1}]. \end{aligned} \quad (5.5)$$

Substituting into Eq. (4.6) and expressing everything in terms of n instead of $\nu = 4\lambda n$, the path amplitude P_M^\pm becomes

$$P_M^\pm(n_M, n_{M-1}, \dots, n_1, n_0) = \frac{(-1)^M}{2^{2M-p+2} \sqrt{2\kappa}} \prod_{i=1}^M n_i^2 \beta(n_i, n_{i-1}) \prod_{k=0}^{p-1} \frac{1}{(d_k - 1)!} \left(\frac{1}{m_k} \frac{\partial}{\partial m_k} \right)^{d_k - 1} \sum_{l=0}^{p-1} \frac{e^{\pm i\sqrt{2\kappa} m_l \Delta\phi}}{\prod_{\substack{j=0 \\ j \neq l}}^{p-1} (m_l^2 - m_j^2)}, \quad (5.6)$$

where we have used $\frac{\partial}{\partial \Theta_{nn}} = (4\kappa^2 n)^{-1} \frac{\partial}{\partial n}$. Here we have expressed the full path of $M + 1$ volumes as $(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0) = 4\lambda \times (n_M, n_{M-1}, \dots, n_1, n_0)$, and the set of p unique volumes appearing in that path as $(w_{p-1}, w_{p-2}, \dots, w_1, w_0) = 4\lambda \times (m_{p-1}, m_{p-2}, \dots, m_1, m_0)$. For paths with no degeneracies (all $d_k = 1$, so $p = M + 1$) – the so-called “Wheeler-DeWitt” paths – this simplifies to

$$P_M^\pm(n_M, n_{M-1}, \dots, n_1, n_0) = \frac{(-1)^M}{2^{M+1} \sqrt{2\kappa}} \prod_{i=1}^M n_i^2 \beta(n_i, n_{i-1}) \sum_{l=0}^M \frac{e^{\pm i \sqrt{2\kappa} m_l \Delta \phi}}{m_l \prod_{\substack{j=0 \\ j \neq l}}^M (m_l^2 - m_j^2)}, \quad (5.7)$$

It is perhaps worth observing that for even modest values of n , $f_\pm(n) \approx 1$. Moreover, since at most one of the two delta functions in $\beta(n_i, n_{i-1})$ can be non-zero on any section of any path, the factors of β are appreciably different from 1 on supported paths only for small n (cf. Eq. (5.5)), and therefore except at Planck-scale volumes essentially serve only to enforce the neighbor-neighbor volume transitions.

B. Structure of volume histories in the vertex expansion

The restriction on the paths arising from the quantum constraint that each transition is to a neighboring volume only implies that the supported paths are the 2^M “tree” graphs starting from $\nu_i = \nu_0$, as illustrated in Fig. 2. Therefore the list of p unique volumes $(w_{p-1}, w_{p-2}, \dots, w_1, w_0)$ appearing in a given path will always be a listing of a consecutive/contiguous range of volumes. Moreover, this implies that at order M in the vertex expansion, the absolute *maximum* change in volume that can be captured at that order is bounded above, $\Delta\nu \leq 4\lambda M$ ($\Delta n \leq M$). Among other things, this immediately implies that very high orders in the vertex expansion will be necessary to accurately capture cosmological dynamics. Low orders can only hope to capture highly quantum behaviors. In conjunction with qualitative considerations to be discussed later, this will imply a constraint on just how large M must be in order to faithfully describe cosmological evolution.

Since all paths in the classical theory either expand or contract, paths in the vertex expansion which steadily either increase or decrease in volume – the paths on the boundaries of the tree of possible paths emanating from ν_i – may be referred to as “Wheeler-DeWitt” or “classical” paths.

As has been emphasized, because of the nearly diagonal structure of the matrix elements of Θ (or Ω), only transitions to *neighboring* volumes have non-zero path amplitudes, enforced by the factors of β in Eq. (5.6). This is a direct consequence of the structure of the quantum evolution operator Θ which only links neighboring volumes in a uniform discrete grid. It is important to recall that this structure followed from restricting to $j = 1/2$ in the trace over $SU(2)$ connections in the field strength tensor in constructing the quantum the Hamiltonian constraint [5]. If higher j 's are included, we expect the quantum constraint to be increasingly non-local in volume.⁵ (The expansion of the Newton-Wigner propagator, which involves $\sqrt{\Theta}$, will also be more non-local, but not sufficiently to fundamentally alter the general conclusions arrived at here [20].)

C. Volume scaling of vertex amplitudes

For later use, we would like to understand how the path amplitudes at each order in the vertex expansion scale with volume (n). To that end, let us begin by writing them down explicitly for the first few orders in the vertex expansion (small M).

For $M = 0$, no degeneracy is possible, $d_0 = 1$. From Eq. (4.7),

$$P_0^\pm(n_0) = \frac{1}{2\sqrt{2\kappa}} \frac{e^{\pm i \sqrt{2\kappa} n_0 \Delta \phi}}{n_0} \delta_{n_f, n_0}. \quad (5.8)$$

(Recall that $n_0 = n_i$ and $n_M = n_f$.)

For the case $M = 1$, again $d_0 = d_1 = 1$, and

$$P_1^\pm(n_1, n_0) = -\frac{1}{2^2 \sqrt{2\kappa}} n_1^2 \beta(n_1, n_0) \left[\frac{e^{\pm i \sqrt{2\kappa} n_0 \Delta \phi}}{n_0(n_0^2 - n_1^2)} + \frac{e^{\pm i \sqrt{2\kappa} n_1 \Delta \phi}}{n_1(n_1^2 - n_0^2)} \right]. \quad (5.9)$$

⁵ Evidence of this structure exists in the case of $j = 1$ [21] and higher spins [22].

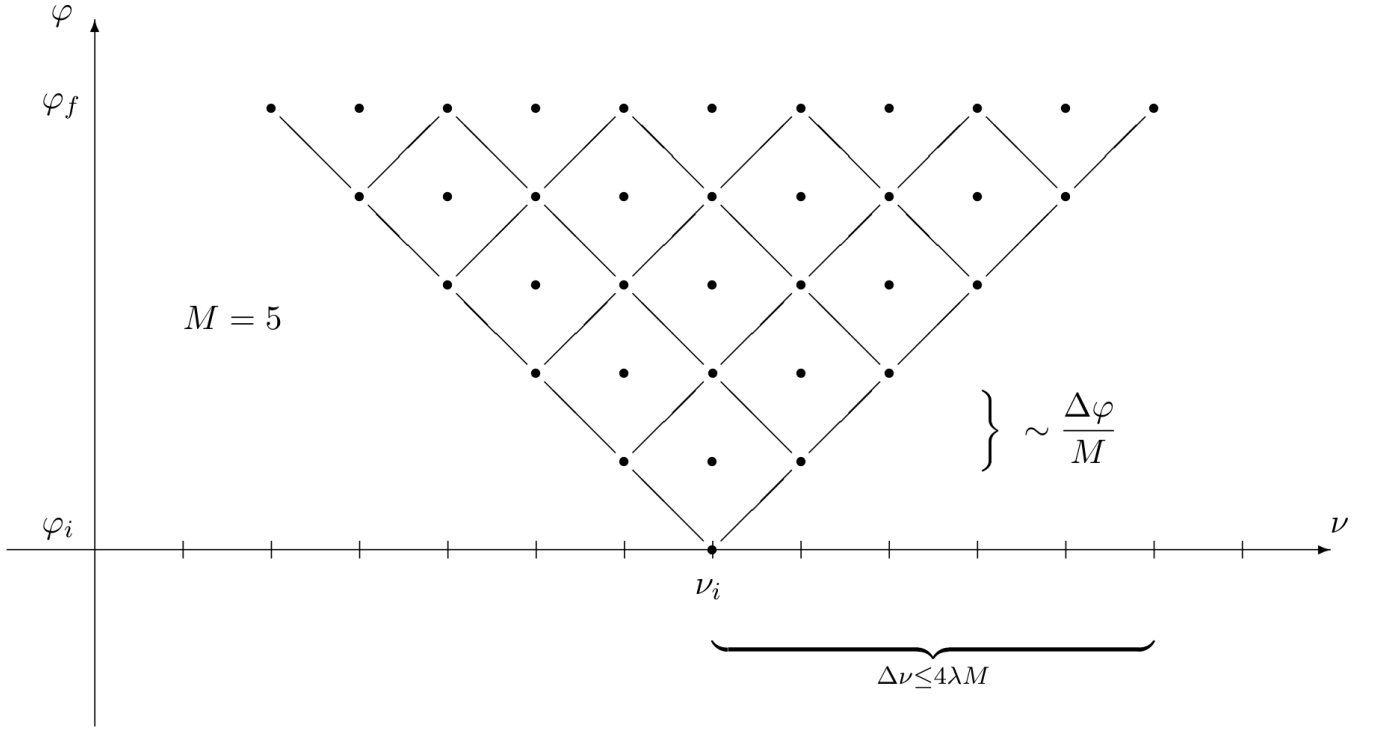


FIG. 2. The set of all possible cosmological histories that have non-zero amplitudes starting from volume $\nu_i = 4\lambda n_i$ with $M = 5$ volume transitions. At order M in the vertex expansion, there are 2^M possible distinct supported paths emanating from a given ν_i . Note that the path amplitudes assign non-zero values only to paths which connect adjacent volumes. Accordingly, the maximum change in cosmological volume at order M in the vertex expansion is bounded above, $\Delta\nu \leq 4\lambda M$ ($\Delta n \leq M$). In spite of appearances in the figure, the “times” (φ) at which the transitions occur are not determined, though the “typical” interval will be of order $\Delta\varphi/M = (\varphi_f - \varphi_i)/M$. As we will show in Sec. V E, this implies additionally that the vertex expansion must be carried to orders $M \gtrsim n\kappa\Delta\varphi$ in order for the expansion to include trajectories sufficiently fine-grained to accurately capture the relevant dynamics. An additional feature of the set of supported paths is that at each “time” step (that is, after a given number of transitions), not all possible volumes are accessible. The fact that the volume changes by $\pm 4\lambda$ at every transition means that after an odd/even number of transitions, only volumes that are an odd/even multiple of 4λ different from ν_i are encountered on a supported path.

Since $f_{\pm}(n) \approx 1 \pm \frac{1}{n}$ for large n ,

$$n_1^2 \beta(n_1, n_0) \approx n_0^2 \left[\left(1 + \frac{1}{2n_0}\right) \delta_{n_1, n_0-1} + \left(1 - \frac{1}{2n_0}\right) \delta_{n_1, n_0+1} \right]. \quad (5.10)$$

Therefore, on supported paths (so $n_1 = n_0 \pm 1$), P_1^{\pm} scales at large n_0 as

$$P_1^{\pm}(n_1, n_0) \approx -\frac{\pm}{2^3 \sqrt{2\kappa}} \left(1 \mp \frac{1}{2n_0}\right) e^{\pm \sqrt{2} i \kappa n_0 \Delta\phi} \left[1 - \frac{e^{\mp \sqrt{2} i \kappa \Delta\phi}}{1 \mp \frac{1}{n_0}}\right]. \quad (5.11)$$

(The signs depend on the branch of the supported paths you’re on.) For $M = 2$, all $d_k = 1$ (the “Wheeler-DeWitt” paths), one finds

$$P_2^{\pm}(n_2, n_1, n_0) = \frac{1}{2^3 \sqrt{2\kappa}} n_2^2 \beta(n_2, n_1) n_1^2 \beta(n_1, n_0) \times \left[\frac{e^{\pm i \sqrt{2} \kappa n_0 \Delta\phi}}{n_0(n_0^2 - n_1^2)(n_0^2 - n_2^2)} + \frac{e^{\pm i \sqrt{2} \kappa n_1 \Delta\phi}}{n_1(n_1^2 - n_0^2)(n_1^2 - n_2^2)} + \frac{e^{\pm i \sqrt{2} \kappa n_2 \Delta\phi}}{n_2(n_2^2 - n_0^2)(n_2^2 - n_1^2)} \right]. \quad (5.12)$$

When n_0 is large, on supported Wheeler-DeWitt paths ($n_1 = n_0 \pm 1$, $n_2 = n_0 \pm 2$), we obtain similarly

$$P_2^\pm(n_2, n_1, n_0) \approx \frac{1}{2^3 \sqrt{2\kappa}} n_0 \left(1 \pm \frac{1}{1 \pm n_0}\right) \left(1 \pm \frac{1}{n_0}\right)^3 e^{\pm i \sqrt{2\kappa} n_0 \Delta\phi} \times \left[\frac{1}{\left(1 \pm \frac{2}{n_0}\right) \left(1 \pm \frac{4}{n_0}\right)} - \frac{e^{\pm i \sqrt{2\kappa} \Delta\phi}}{\left(1 \pm \frac{1}{n_0}\right) \left(\pm 1 + \frac{1}{n_0}\right) \left(\pm 2 + \frac{1}{n_0}\right)} + \frac{e^{\pm i 2 \sqrt{2\kappa} \Delta\phi}}{2 \left(1 \pm \frac{2}{n_0}\right) \left(\pm 1 + \frac{1}{n_0}\right) \left(\pm 4 + \frac{1}{n_0}\right)} \right]. \quad (5.13)$$

(The various \pm signs in this relation are mostly uncorrelated with one another.)

We see that P_M^\pm for the Wheeler-DeWitt paths with $M = 0, 1, 2$ scale with volume like n_0^{M-1} in the large- n_0 limit. This scaling with volume may at first seem surprising, because one might conclude by naive counting of powers of n and m in Eq. (5.6) that the P_M^\pm scale roughly as $1/n$ with volume. However, the situation is a bit more subtle than that. Without becoming bogged down in the details, the subtleties arise because of the difference of squares in the denominator of Eq. (5.6) and the fact that the unique volumes appearing in the list for each supported path are consecutive. Therefore that difference always contains terms which are close neighbors, which therefore scale as n rather than n^2 , as taken into account explicitly in the calculations above. This alters the conclusions drawn from power-counting.

Indeed, the pattern of volume scaling n_0^{M-1} of the P_M^\pm for the Wheeler-DeWitt paths in the large- n_0 limit continues for all M . To see this, we need to assess the volume scaling of both volume-dependent factors in Eq. (5.7). Let us begin with the simpler of the two, the product of transition matrix elements that leads to $\prod n^2 \beta$. The expanding/contracting Wheeler-DeWitt paths can be indexed as $n_i = n_0 \pm i$, $i = 0 \dots M$. Then

$$\begin{aligned} \prod_{i=1}^M n_i^2 \beta(n_i, n_{i-1}) &= \prod_{i=1}^M (n_0 + i)^2 \prod_{j=1}^M f_{\mp}(n_j) \\ &= n_0^{2M} \prod_{i=1}^M \left(1 \pm \frac{i}{n_0}\right)^2 \prod_{j=1}^M f_{\mp}(n_0(1 \pm j/n_0)). \end{aligned} \quad (5.14)$$

Now,

$$\prod_{i=1}^M \left(1 \pm \frac{i}{n_0}\right)^2 = (\pm)^M \frac{1}{n_0^M} \frac{\Gamma(M+1 \pm n_0)}{\Gamma(1 \pm n_0)}. \quad (5.15)$$

Using Stirling's formula⁶ it is straightforward to show that this approaches unity in the limit $n_0 \gg M$. In a similar manner, the product of f_{\mp} (recalling $\prod_i \sqrt{x_i} = \sqrt{\prod_i x_i}$) also approaches unity in the same limit, and as expected Eq. (5.14) scales as n_0^{2M} in the large- n limit.

The volume scaling of the sum appearing in Eq. (5.7) is determined by the volume products in the denominators. For simplicity we will imagine an expanding Wheeler-DeWitt path, but the choice doesn't actually make any difference to the analysis. Let us concentrate on a single such factor for an arbitrary choice of l , where of course $0 \leq l \leq M$. We parameterize the volumes appearing in the product relative to $n_l = n_0 + l$ by a set of integers k_j^l , so that $n_j = n_l + k_j^l$, where $k_j^l \in \{-l, -l+1, \dots, -1, 1, \dots, M-l\}$. In that case the volume products in the denominators are

$$\begin{aligned} n_l \prod_{\substack{j=0 \\ j \neq l}}^M (n_l^2 - n_j^2) &= n_l \prod_{j=0}^{l-1} (n_l^2 - n_j^2) \prod_{j=l+1}^M (n_l^2 - n_j^2) \\ &= n_l^{M+1} (-2)^M \prod_{j=0}^{l-1} k_j^l \prod_{j=l+1}^M k_j^l \prod_{j=0}^{l-1} \left(1 + \frac{k_j^l}{2n_l}\right) \prod_{j=l+1}^M \left(1 + \frac{k_j^l}{2n_l}\right) \\ &= n_l^{M+1} (-2)^M (-)^l \prod_{j=1}^l j \prod_{j=1}^{M-l} j \prod_{j=1}^l \left(1 - \frac{j}{2n_l}\right) \prod_{j=1}^{M-l} \left(1 + \frac{j}{2n_l}\right) \\ &= n_l^{M+1} (-2)^M (-)^l \Gamma(l+1) \Gamma(M-l+1) \frac{1}{(2n_l)^l} \frac{\Gamma(2n_l)}{\Gamma(2n_l-l)} \frac{1}{(2n_l)^{M-l}} \frac{\Gamma(M-l+1+2n_l)}{\Gamma(1+2n_l)} \\ &= (-)^{M+l} \Gamma(l+1) \Gamma(M-l+1) n_l \frac{\Gamma(2n_l)}{\Gamma(2n_l-l)} \frac{\Gamma(M-l+1+2n_l)}{\Gamma(1+2n_l)}. \end{aligned} \quad (5.16)$$

⁶ As well as $\lim_{n \rightarrow \infty} (1+x/n)^n = e^x$, and the reflection identity for the case of contracting paths.

The crucial point is now that application once again of Stirling's formula to all of the factors involving n_l in this expression reveals that the volume denominators Eq. (5.16) scale as $n_l^{M+1} \sim n_0^{M+1}$ in the limit $n_l \gg M$. Thus, as was to be shown, the *ratios* of the leading transition prefactor Eq. (5.14) to each of the volume denominators Eq. (5.16) scale as $n^{2M}/n^{M+1} = n^{M-1}$ in the limit $n \gg M$, rather than the $1/n$ that would be expected from naive power-counting.⁷

These considerations can become quite involved, and we do not attempt to repeat this analysis for the (much more numerous) paths with degeneracies here. The arguments we make for the Wheeler-DeWitt paths are sufficient for our purposes below.

D. Satisfaction of the constraint in the vertex expansion

An important property of the Hadamard propagator is that it satisfies the constraint, or in other words, the action of \hat{C} on the propagator vanishes:

$$\hat{C}G_H^\pm(\nu_f, \phi_f; \nu_i, \phi_i) = 0. \quad (5.17)$$

This is easily seen from, for example, Eq. (3.7). Can it also be shown from the vertex expansion? The answer is that it can. However, we shall argue that satisfaction of the constraint strictly holds *only* if one includes *all* the terms in the vertex expansion from $M = 0 \dots \infty$. A pertinent question is to ask in what sense a *truncated* series in M is a solution of the quantum Hamiltonian constraint. In Ref. [11], this issue was addressed using a bookkeeping perturbation parameter (λ) introduced via $\Theta = \Theta^D + \lambda\Omega$. (Their bookkeeping λ is distinct from the λ related to the area gap we use in this paper.) It was then shown that (5.17) is satisfied in the vertex expansion [11] in the sense that

$$(\partial_\phi^2 + \Theta^D) P_M(n_f, \phi_f; n_i, \phi_i) + \Omega P_{M-1}(n_f, \phi_f; n_i, \phi_i) = 0, \quad (5.18)$$

so that the diagonal piece of the constraint at any order M in the vertex expansion is cancelled by the off-diagonal piece from one order down. Here the P_M appearing in these expressions and following are the *sums* over all paths at order M of the individual path amplitudes appearing in Eq. (4.5). That is,

$$P_M(\nu_f, \phi_f; \nu_i, \phi_i) = \sum_{\nu_{M-1}, \dots, \nu_1} P_M^\pm(\nu_M = \nu_f, \nu_{M-1}, \dots, \nu_1, \nu_0 = \nu_i; \Delta\phi). \quad (5.19)$$

If the maximum number of volume transitions is M^* , then in [11] it is argued that

$$(\partial_\phi^2 + \Theta^D + \Omega) \sum_{M=0}^{M^*} \lambda^M P_M(n_f, \phi_f; n_i, \phi_i) + \Omega P_{M-1}(n_f, \phi_f; n_i, \phi_i) = \mathcal{O}(\alpha^{M^*+1}). \quad (5.20)$$

Here the term which is approximated away is the off-diagonal piece originating from Ω acting on the path amplitude for the M^* th volume transition. Though this relation appears to suggest that the constraint is satisfied to order M^* , it is important to note that λ is not small. In fact, as a bookkeeping parameter it is strictly equal to unity [11]. It is therefore unclear what is the error in truncating the vertex expansion at finite M^* .

To understand the truncated vertex expansion and whether it satisfies the constraint within a controlled approximation, let us begin with the $M = 0$ term. In this case, the off-diagonal term Ω is trivially zero and Eq. (5.18) yields

$$(\partial_\phi^2 + \Theta^D) P_0^+(n_1, n_0) = 0. \quad (5.21)$$

Using the explicit expression of P_0^+ , it is straightforward to verify that the L.H.S of (5.21) indeed vanishes:

$$\partial_\phi^2 P_0^+(n_1, n_0) = -\frac{1}{\sqrt{2}} \kappa n_0 e^{i\sqrt{2}\kappa n_0 \Delta\phi} \delta_{n_1, n_0} = -\sum_{n'_1} \Theta_{n_1, n'_1}^D P_0^+(n'_1, n_0) = -\Theta^D P_0^+(n_1, n_0). \quad (5.22)$$

Hence, the L.H.S of Eq. (5.21) is identically zero. The constraint in this particular case is satisfied exactly since $\Omega = 0$.

⁷ The underlying reason for the difference is now actually easy to see. In each of the M difference-of-squares factors in the product, $n_l^2 - n_j^2 = (n_l + n_j)(n_l - n_j)$. In the limit $n \gg M$, all of the M difference terms scale like n rather than n^2 , and so the power counting should properly give $1/n \times n^M = n^{M-1}$ in that limit. The detailed analysis bears out this expectation.

Now let us consider Eq. (5.18) for $M = 1$. Then,

$$(\partial_\phi^2 + \Theta^D) P_1^+(n, n_0) = -\Omega P_0^+(n, n_0) \quad (5.23)$$

After a straightforward computation, the L.H.S yields

$$\begin{aligned} (\partial_\phi^2 + \Theta^D) P_1^+(n_1, n_0) &= (\partial_\phi^2 + 2\kappa^2 n_1^2) P_1^+(n_1, n_0) \\ &= \frac{\kappa}{2\sqrt{2}} \left(\frac{(n_0 - 1)^2}{n_0} f_+(n_0 - 1) \delta_{n_1, n_0 - 1} + \frac{(n_0 + 1)^2}{n_0} f_-(n_0 + 1) \delta_{n_1, n_0 + 1} \right) e^{i\sqrt{2}\kappa n_0 \Delta\phi}. \end{aligned} \quad (5.24)$$

Similarly, we can compute the off-diagonal part for $M = 0$ which contributes in Eq. (5.18) for $M^* = 1$. It turns out to be

$$\begin{aligned} \Omega P_0^+(n_1, n_0) &= \sum_{n'_1} \Omega_{n_1, n'_1} P_0^+(n'_1, n_0) \\ &= -\frac{\kappa}{2\sqrt{2}} \frac{n_1^2}{n_0} \beta(n_1, n_0) e^{i\sqrt{2}\kappa n_0 \Delta\phi} \\ &= -\frac{\kappa}{2\sqrt{2}} \left(\frac{(n_0 - 1)^2}{n_0} f_+(n_0 - 1) \delta_{n_1, n_0 - 1} + \frac{(n_0 + 1)^2}{n_0} f_-(n_0 + 1) \delta_{n_1, n_0 + 1} \right) e^{i\sqrt{2}\kappa n_0 \Delta\phi}. \end{aligned} \quad (5.25)$$

Hence Eq. (5.18) is again satisfied, up to the error term which is the off-diagonal piece for $M = 1$. To estimate the error involved, let us compute this term:

$$\begin{aligned} \Omega P_1^+(n_1, n_0) &= -\kappa^2 n_1^2 \sum_{n'_1} \beta(n_1, n'_1) P_1^+(n'_1, n_0) \\ &= \frac{1}{4\sqrt{2}} \kappa n_1^2 \left[\frac{(n_0 - 1)^2}{2n_0 - 1} f_+(n_0 - 1) \left(\frac{1}{n_0} - \frac{e^{-i\sqrt{2}\kappa n_0 \Delta\phi}}{n_0 - 1} \right) \beta(n_1, n_0 - 1) \right. \\ &\quad \left. - \frac{(n_0 + 1)^2}{2n_0 + 1} f_-(n_0 + 1) \left(\frac{1}{n_0} - \frac{e^{i\sqrt{2}\kappa n_0 \Delta\phi}}{n_0 + 1} \right) \beta(n_1, n_0 + 1) \right]. \end{aligned} \quad (5.26)$$

For large n (volume), this scales as n^2 . On the other hand, the rest of the terms (5.24) and (5.25) scale as n . In particular, the ratio of the remainder term and any of the terms in Eq. (5.17) for $M^* = 1$ scales as follows:

$$\frac{\Omega P_1^+(n_1, n_0)}{(\partial_\phi^2 + \Theta^D) P_1^+(n_1, n_0)} \sim n. \quad (5.27)$$

If truncation at $M^* = 1$ was to be viable in general this ratio would need to be much smaller than unity. However, the relative error in the truncation seems to grow with volume n .

This is not a phenomenon restricted to low- M . We again restrict attention to the Wheeler-DeWitt paths to make our point. Let us examine the scaling with n for a general truncation order M . We have seen explicitly that for small M , $(\partial_\phi^2 + \Theta^D) P_M$ scales as n^M . However, ΩP_M scales as n^{M+1} . Hence their ratio scales as n , as was found for $M = 1$ in the above computation. This scaling continues at all orders M in the limit $n \gg M$. This is, in fact, required by Eq. (5.18). We already know that when $n \gg M$, the P_M scale as n^{M-1} . Then ΩP_{M-1} will scale as n^{M+1} , and Eq. (5.18) therefore requires that $(\partial_\phi^2 + \Theta^D) P_M$ does as well. (In other words, Eq. (5.18) demands that $(\partial_\phi^2 + \Theta^D)$ changes the volume scaling of P_M by one order lower than does Ω .) Thus, for an arbitrary truncation order M^* , the relative truncation error is

$$\frac{\Omega P_{M^*}^+(n_M, n_0)}{(\partial_\phi^2 + \Theta^D) P_{M^*}^+(n_M, n_0)} \sim n, \quad (5.28)$$

at least in the limit $n \gg M$ for Wheeler-DeWitt paths.

This relation implies that there is a regime ($n \gg M$) in which the term which is ignored can be at least of the same order as the terms in Eq. (5.18), at least for some paths in the expansion. On inclusion of this term, the constraint does not annihilate the truncated Hadamard propagator at any given order M in the vertex expansion.⁸ Hence the

⁸ Repeating this exercise in the non-relativistic representation, the same conclusion is expected to hold for the Newton-Wigner propagator.

truncated vertex expansion does not faithfully captures the vanishing of the quantum constraint: a naively truncated expansion does *not* appear to be a solution to the constraint at any finite order within a controlled approximation. A consequence of this is that the quantum bounce will not necessarily manifest at any truncated order in the expansion. We have already seen that large orders in the vertex expansion are in any event required to capture large changes in volume. Hence, for small M^* the quantum bounce may not be evident in the vertex expansion.

It remains to consider whether it is possible to reorganize the vertex expansion in spin foam LQC in such a manner as to arrive at an expansion that may be truncated at finite order and remain a solution to the quantum constraint in a controlled way. We do not take up this question here.

E. Qualitative considerations

It is important to understand in this “covariant” description of LQC to what extent the vertex expansion discussed in the previous sections can be used as a tool to probe quantum cosmological dynamics. For example, can we employ the vertex expansion to probe the quantum bounce of a loop quantized universe starting from a macroscopic volume? Unfortunately, what we find is that in order to probe long evolutions it is necessary to carry the vertex expansion to very (very) high orders.

We have already seen that large orders in the vertex expansion are required in order to capture large changes in volume. Just how large must the order of the expansion be?

Our starting point is the effective Hamiltonian in LQC [5], which for the case of a spatially flat homogeneous and isotropic spacetime with massless scalar matter is given by

$$C_{\text{eff}} = p_\phi^2 - \frac{\sqrt{3}}{4} \gamma \hbar \nu^2 \sin^2(\lambda b) . \quad (5.29)$$

Solving Hamilton’s equations for ν and ϕ , we obtain

$$\nu = \nu_0 \cosh(\kappa(\phi - \phi_0)) , \quad (5.30)$$

where ν_0 and ϕ_0 are constants of integration. Consequently,

$$\kappa d\phi = \frac{d\nu}{\sqrt{\nu^2 - \nu_0^2}} . \quad (5.31)$$

With $\nu = 4n\lambda$ and in the limit of large volume $n \gg n_0$, we find

$$\kappa d\phi \approx n^{-1} dn . \quad (5.32)$$

This tells us that semiclassical cosmological trajectories require a “time” interval $d\phi$ of order $1/n\kappa$ in order for the volume to change by one Planck unit, $dn = 1$.

The vertex expansion expresses amplitudes for transitions between (ν_i, ϕ_i) and (ν_f, ϕ_f) as a sum of amplitudes for paths with M Planck-scale volume transitions. Since these amplitudes are zero unless the volumes are adjacent, we have already seen that at a minimum, it must be that $M \geq |\nu_f - \nu_i|/4\lambda = |n_f - n_i|$.⁹ Additionally, with $\phi_f - \phi_i = \Delta\phi$, for paths with M transitions the “typical” time ϕ available for each transition is of order $\Delta\phi/M$. In order for the vertex expansion at order M to effectively capture the dynamics, it must be that the paths included are sufficiently fine-grained, so that the time required for each transition is less than $1/n\kappa$. In other words, it must be that $\Delta\phi/M \lesssim 1/n\kappa$, so that

$$M \gtrsim n\kappa\Delta\phi . \quad (5.33)$$

This inequality implies that in order to probe long intervals of cosmological “time” ϕ using the vertex expansion, we must continue the expansion to sufficiently large values of M . As a corollary, the vertex expansion at low orders in M is at best a short-time expansion useful for investigating highly quantum phenomena. Therefore, the vertex expansion at low M cannot be used to accurately probe quantum cosmological dynamics unless *both* $\Delta\phi$ *and* the change in volume $|\nu_f - \nu_i|$ are small.

⁹ In point of fact, it must be that M is greater than the largest difference in volumes exhibited by the trajectory of interest. For example, to capture a quantum bounce, M must be greater than $(1/4\lambda)$ times the difference between the initial volume and the bounce volume. If the initial volume is macroscopic, this will be a large M indeed.

VI. DISCUSSION

We have explored the spin-foam-like “vertex expansion” of the transition amplitudes of solvable loop quantum cosmology in some detail, an expansion that is in many ways analogous to the vertex expansion of covariant spin-foam loop quantum gravity. The hope expressed in Refs. [11] is that insights from an exactly solvable model in the canonical picture might help shed some light on some of the difficult conceptual issues arising in the full covariant theory. Here, we have instead focused on the features and utility of the vertex expansion as a tool for investigating LQC itself.

We have found that the vertex expansion has many of the features of an expansion in Et/\hbar of a standard quantum mechanical propagator. That is, that low orders in the expansion can accurately capture the quantum dynamics only for short “times” $\Delta\phi$. Very large orders in the expansion are required in order to probe dynamics on cosmologically relevant scales. Moreover, truncating the expansion at any finite order does not appear to be a well-controlled approximation, and is likely to obscure evidence of the marquee feature of LQC, the quantum bounce at small volume, even though the bounce may be seen clearly in the propagator itself, for example, from Eq. (3.7) [10, 23]. Ironically, therefore, while the vertex expansion for LQC may be illuminating for certain conceptual issues, in its current form it is unlikely to prove a useful tool for direct calculations of cosmologically relevant questions.¹⁰ The vertex expansion may, nevertheless, prove useful for investigating highly quantum, short-time questions of Planck-scale dynamics.

This feature of the vertex expansion has its origins in the essentially local dynamics, Eq. (2.10), of solvable loop quantum cosmology (sLQC). In turn, as noted above this highly local dynamics is connected with the restriction to $j = 1/2$ in the quantization of sLQC. The inclusion of higher spins should yield a dynamics that links more distant volumes. An example is the case of the quantum Hamiltonian constraint obtained using the $j = 1$ representation in LQC [21]. In this particular case, the quantum evolution operator Θ links wavefunctions at nine steps in volume with a maximum non-local volume difference equaling 8λ instead of 4λ . It is expected that for higher j , a more non-local quantum difference equation will be obtained [22]. In such models, lower orders M in the corresponding vertex expansion may accurately capture the dynamics of longer “time” intervals.

Alternately, it may turn out to be possible to re-order the vertex expansion in such a way that low orders in the expansion accurately capture semi-classical dynamics and key global quantum features such as the bounce – one that can exhibit large volume changes at low orders in the expansion, and can be truncated consistently at finite order in a controlled way. This would be more akin to the WKB (\hbar) expansion in ordinary quantum theory. Indeed, in keeping with this line of thought, in Ref. [12] the transition amplitudes are expressed as an ordinary path integral over phase space variables, and the semi-classical dynamics and quantum bounce become evident at lowest order in the standard way (stationary action). Nonetheless, a controlled order-by-order expansion could be of some use. Indeed, it is likely there exist physically distinct expansions depending on the choice of order parameter, for example, l_p vs. λ , with different physical meanings and uses, but we do not take up that possibility here.¹¹

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¹⁰ It is worth reiterating that the spin-foam-like vertex expansion we have been discussing is conceptually distinct from the approach to spin-foam cosmology from the full covariant theory [14–16].

¹¹ The expansion in terms of the matrix elements of $\sqrt{\Theta}$ discussed in e.g. Ref. [11], on the other hand, will have very similar properties to those discussed in this paper, but will be more non-local because of the square-root.

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