A New Construction of Subgroups Inducing Isomorphic Representations.

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A new construction of subgroups inducing isomorphic representations

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A NEW CONSTRUCTION OF SUBGROUPS
INDUCING ISOMORPHIC REPRESENTATIONS

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ABSTRACT

This dissertation provides a method to construct infinite families of triples \((H_i, H_i', G_i)\) consisting of two non-conjugate subgroups \(H_i\) and \(H_i'\) of a finite group \(G_i\) whose trivial representations induce isomorphic representations of \(G_i\). Such triples are important in number theory, differential geometry, and group theory.
INTRODUCTION

This dissertation concerns the classical problem of constructing two non-conjugate subgroups $H$ and $H'$ of a finite group $G$ whose trivial representations induce isomorphic representations of $G$. We will refer to such a triple $(H, H', G)$ as a distinguished triple.

The first example of a distinguished triple was given by F. Gassmann in 1926. Even before then, it was known that distinguished triples — if they existed — were intimately connected to the problem of constructing two non-isomorphic algebraic number fields with the same laws of decomposition for rational primes. Gassmann’s triple involved two subgroups $H, H'$ of the symmetric group $G = S_6$. Realizing $S_6$ as the Galois group of some normal extension $N|\mathbb{Q}$ of the field of rational numbers and taking the fixed fields of Gassmann’s two subgroups $H, H' \subset S_6$ gives rise to two non-isomorphic algebraic number fields with the same laws of decomposition for rational primes.

More recently, in 1985, in a famous paper in the Annals of Mathematics, T. Sunada connected distinguished triples of finite groups to pairs of non-isometric isospectral Riemannian manifolds; that is, non-isometric Riemannian manifolds for which the eigenvalues of the Laplace operator coincide. These manifolds are higher-dimensional analogues of surfaces, immortalized in M. Kac’s famous query — Can you hear the shape of a drum?

In group theory, each distinguished triple $(H, H', G)$ gives rise to a non-trivial element in the kernel of the canonical map
\[ B(G) \rightarrow R(G) \]
from the Burnside ring $B(G)$ to the representation ring $R(G)$ of the finite group $G$. Namely, if $\pi^G_H$ denotes the permutation representation of $G$ acting by left translation on the left cosets of $H$ in $G$, then the formal difference $\pi^G_H - \pi^G_{H'}$ is a non-trivial element in $B(G)$ that maps to 0 in $R(G)$. All these examples motivate my study of distinguished triples.

There are many examples of distinguished triples spread throughout the literature. In Chapter 4 we present some examples. In Chapter 3 of this dissertation, we produce a very general yet simple method to canonically generate an infinite
family of triples \((H_i, H'_i, G_i)\) starting with a single distinguished triple \((H, H', G)\). This construction is an iterative procedure: from the given distinguished triple \((H, H', G)\) we canonically produce another triple \((H_1, H'_1, G_1)\) for which the trivial representations of \(H_1\) and \(H'_1\) induce isomorphic representations of \(G_1\); however this new triple need not be distinguished in that, sometimes, \(H_1\) and \(H'_1\) are conjugate in \(G_1\). When this degeneration happens, all further iterations produce triples with conjugate subgroups. Often, however, the first iteration produces a distinguished triple \((H_1, H'_1, G_1)\), and when this happens, all further iterations produce triples that are, first of all distinguished, and moreover, not copies of triples occurring in previous steps.

This dissertation is concerned with presenting this construction and giving examples of running it. It is the group theorists, number theorists, and differential geometers who may choose to exploit this new idea to construct new examples with exotic properties in their respective specialities.
CHAPTER 1

Preliminaries

The purpose of this chapter is to give some basic definitions and results that are referred to in the succeeding chapters. Our notation is also established here. All groups that appear in this dissertation are finite.

Section 1. A Review of Representation Theory

Let $G$ be a group and $V$ an $n$-dimensional vector space over the field $\mathbb{C}$ of complex numbers. Let $\text{Aut}(V)$ be the group of automorphisms of $V$. By fixing a basis of $V$, we can identify $\text{Aut}(V)$ with the matrix group $GL_n(\mathbb{C})$.

(1.1) Definition: A homomorphism $\rho : G \rightarrow \text{Aut}(V)$ is called a linear representation of $G$; the dimension $n$ of $V$ is called the degree of $\rho$.

(1.2) Basic Examples of Representations:

(i) Let $G$ be a group. The trivial representation of $G$ is the degree 1 representation $1_G : G \rightarrow \text{Aut}(\mathbb{C})$ defined by $1_G(g) = 1$ for all $g \in G$.

(ii) Let $G$ be a group of order $n$ and let $V = \mathbb{C}[G]$ be the group ring of $G$ over $\mathbb{C}$ with basis $\{e_g\}_{g \in G}$. Fix $s \in G$. Define $\rho : G \rightarrow \text{Aut}(V)$ by $\rho_s(e_g) = e_{sg}$. This representation is the regular representation of $G$.

(iii) Let $H \subseteq G$ and let $g_1H, \ldots, g_nH$ represent the cosets mod $H$. The action of $G$ on these cosets by left translation yields a group homomorphism $\pi : G \rightarrow S_n$ given by: for $g \in G$,

$$\pi_g(i) = j \text{ where } gg_iH = g_jH. \quad (1)$$

Embed $S_n$ in $GL_n(\mathbb{C})$ by the map $S_n \hookrightarrow GL_n(\mathbb{C})$ taking a permutation $\sigma$ to the $n \times n$ "permutation matrix" obtained by letting $\sigma$ permute the $n$ columns of the $n$-by-$n$ identity matrix. Hence there is the composition of maps $G \xrightarrow{\pi} S_n \xrightarrow{i} GL_n(\mathbb{C})$. This composition is called the representation of $G$ induced by the trivial representation of $H$, and is denoted by $1^G_H$. 

1
Let $\rho : G \to Aut(V)$ be a representation of degree $n$ of $G$.

(1.3) Definition: For $g \in G$, put $\chi_\rho(g) = \text{trace}(\rho_g)$. The function $\chi_\rho : G \to \mathbb{C}$ is called the character of the representation $\rho$.

Let $\rho$ and $\rho'$ be two representations of the same group $G$ in vector spaces $V$ and $V'$.

(1.4) Definition: $\rho$ and $\rho'$ are isomorphic if there is an $A \in GL_n(\mathbb{C})$ such that $A^{-1} \rho_g A = \rho'_g$ for all $g$ in $G$.

Of course $\rho \cong \rho'$ implies $\chi_\rho(g) = \chi_{\rho'}(g)$ for all $g$ in $G$. The converse is true and is a basic result, which we cite as the following proposition:

(1.5) Proposition: Two (complex) representations with the same character are isomorphic.

Proof: For a proof, see ([S], Theorem 4, Corollary 2, page 16).

Let $H \subseteq G$ and let $1_H$ be the trivial representation of $H$ and $1^G_H$ be the representation of $G$ induced by the trivial representation of $H$. The following proposition tells us how to compute $\chi_1^G_H$ from $\chi_1^H$. A proof is found in ([S], Theorem 12, page 30).

(1.6) Proposition: Let $|H|$ denote the order of $H$ and $R$ be a system of representations of $G/H$. For each $u \in G$, we have

$$\chi_1^G_H(u) = \sum_{r \in R \atop r^{-1}ur \in H} \chi_1^H(r^{-1}ur) = \frac{1}{|H|} \sum_{s \in G \atop s^{-1}us \in H} \chi_1^H(s^{-1}us).$$

In particular,

$$\chi_1^G_H(u) = \frac{|C_G(u)| \cdot |u^G \cap H|}{|H|}. \quad (2)$$

Here, $C_G(u)$ is the center of $u$ in $G$ and $u^G$ is the conjugacy class of $u$ in $G$.

It is both interesting and important that these character values have another interpretation.
(1.7) Lemma: $\chi_{1_H}(u) = \text{the number of fixed points of } u \text{ acting on } G/H$.

Proof: Let $g_1 H, \ldots, g_n H$ represent $G/H$. Now $u$ fixes $g_i H$ by left translation iff $ug_i H = g_i H$ iff $ug_i \in g_i H$ iff $u \in g_i H g_i^{-1}$. Hence the number of fixed points of $u$ is $\#\{i | u \in g_i H g_i^{-1}\}$. On the other hand, $\chi_{1_H}(u) = \text{trace } (1^G_H(u)) = \text{the sum of the diagonal elements of the permutation matrix } 1^G_H(u) = \text{the number of 1's on the diagonal of } 1^G_H(u)$. Now, the $i^{th}$ column of $1^G_H(u)$ has a 1 on the diagonal iff $ug_i H = g_i H$ (see (iii) of (1.2)) iff $u \in g_i H g_i^{-1}$ and thus

$$\chi_{1_H}(u) = \#\{i | u \in g_i H g_i^{-1}\}.$$  

(1.8) Lemma: Let $H, H' \subseteq G$. Then $1^G_H \cong 1^G_H'$, implies that $|H| = |H'|$.

Proof: Put $u =$ identity of $G$ in Lemma (1.7).

Notation: $a \cong b$ means that the elements $a$ and $b$ are conjugate in $G$.

The following lemma, found in [SH], gives several simple but very useful alternative ways to say that two subgroups induce isomorphic representations of $G$.

(1.9) Lemma: Let $H, H' \subseteq G$. Then the following statements are equivalent:

(a) $1^G_H \cong 1^G_{H'}$.

(b) $|u^G \cap H| = |u^G \cap H'|$ for all $u \in G$.

(c) There exists a bijection $\psi : H \rightarrow H'$ satisfying $\psi(h) \cong h$ for all $h \in H$.

Proof: c) $\Rightarrow$ b): Suppose such a bijection $\psi$ exists. Then $|H| = |H'|$. We will show $\chi_{1_H}(u) = \chi_{1_{H'}}(u)$. Using (2), we must show

$$|u^G \cap H| = |u^G \cap H'|.$$  

(3)

If both intersections are empty the equality holds, so we may as well assume $u \in u^G \cap H$. Then $\psi(u) \in \psi(H) = H'$ and $\psi(u) \in u^G$ by assumption. Hence $\psi(u) \in u^G \cap H'$. Since $\psi$ is injective, $|u^G \cap H| \leq |u^G \cap H'|$. Equality follows by symmetry. Hence b) holds. Now suppose b) holds. Since the $G$-conjugacy classes
$u^G \cap H$ in $H$ partition $H$, and similarly for $H'$, it follows that $|H| = |H'|$. Then a) follows from (2). If a) holds, then b) holds by (2) and Lemma (1.8). Finally, if b) holds, let $\psi : u^G \cap H \rightarrow u^G \cap H'$ be any bijection. Take $h \in H$. Then $h \in h^G \cap H$ and $\psi(h) \in h^G \cap H'$. Hence $\psi(h) \preceq h$, proving the lemma. 

Section 2. Triples of Groups $(H, H', G)$

When $H$ and $H'$ are subgroups of a group $G$, we say we have a triple of groups, denoted by $(H, H', G)$. The following definition distinguishes several important families of triples.

(1.10) Definition:

(i) $(H, H', G)$ is said to be a conjugate triple when $H$ is conjugate to $H'$ in $G$; otherwise the triple is non-conjugate.

(ii) $(H, H', G)$ is inductive if $1^G_H \cong 1^G_{H'}$.

(iii) $(H, H', G)$ is faithful if both group homomorphisms $\pi$ and $\pi'$ obtained from $G$ acting on $G/H$ and $G/H'$, respectively, are injective.

(iv) $(H, H', G)$ is distinguished if it is non-conjugate and inductive and $(G : H) = (G : H') = n \geq 7$.

Our goal, accomplished in Chapter 3, is to construct distinguished faithful triples. We take a first step by proving:

(1.11) Lemma: Let $(H, H', G)$ be an inductive triple. Then

(a) $(G : H) = (G : H')$. Call this common index $n$.

$G$ acts on $G/H$ and $G/H'$ yielding $\pi, \pi' : G \rightarrow S_n$, and

(b) $\ker(\pi) = \bigcap_{g \in G} H^g$.

(c) $\ker(\pi) = \ker(\pi')$.

Proof: (a) This follows immediately from Lemma (1.8).

(b) The kernel of $\pi$ is the intersection of the stabilizers of all the cosets $gH$,
and thus is the intersection of all the conjugates of $H$.

(c) Let $x \in \ker(\pi)$. Then $x \in gHg^{-1}$ for all $g \in G$ and this means that every conjugate of $x$ lies in $H$. Thus, if $x^G$ is the conjugacy class of $x$ in $G$, then $|x^G| = |x^G \cap H|$. Since $(H, H', G)$ is inductive, Lemma (1.9), (b) shows that $|x^G| = |x^G \cap H'|$, implying that $x^G \subseteq H'$. Thus $x$ lies in every conjugate of $H'$. So $\ker(\pi) \subseteq \ker(\pi')$. The opposite inclusion follows by symmetry. 

The next two lemmas will help us create faithful inductive triples from inductive triples.

(1.12) Lemma: Let $(H, H', G)$ be an inductive triple. Let $A$ denote the common kernel $\ker(\pi) = \ker(\pi')$. Then $H$ is conjugate to $H'$ in $G$ iff $H/A$ is conjugate to $H'/A$ in $G/A$.

Proof: $H/A$ is conjugate to $H'/A$ in $G/A$ iff there is a $gA$ in $G/A$ such that $(gA)HA(gA^{-1}) = H'A$. Now $A$ is normal in $G$ and is contained in both $H$ and in $H'$. Therefore $AHA$ simplifies to $H$ and the condition above becomes $gHg^{-1}A = H'$. But $g^{-1}A = g^{-1}Agg^{-1} = Ag^{-1}$ by normality of $A$ in $G$, so this further simplifies to $gHg^{-1} = H'$. This proves the lemma.

(1.13) Lemma: Let $H \subseteq G$ be a subgroup and let $A \subseteq H$ be normal in $G$. Then $\chi_{G/H}(g) = \chi_{H/A}(gA)$ for all $g \in G$.

Proof: We will use Lemma (1.7) and count fixed points. Let $g_1, g_2, ..., g_n \in G$ represent $G/H$. Then $g_1A, g_2A, ..., g_nA$ represent $(G/A)/(H/A)$. (If $g_iA \in g_jAHA$ for some $i \neq j$, then $g_j^{-1}g_i$ would lie in $AHA = H$, implying that $g_iH = g_jH$, a contradiction). A class $gA$ in $G/A$ fixes $(g_jA)(HA)$ by left translation iff $gA(g_jA)(HA) = (g_jA)HA$. But $Ag_j = g_jA$, using the normality of the kernel $A$, and $AAHA = H$. The condition becomes $gg_jH = g_jH$. This is equivalent to $gg_j \in g_jH$, and this occurs iff $g \in g_jHg_j^{-1}$.

Hence, with Lemma (1.7), $\chi_{G/H}(g) = \#\{j | g \in g_jHg_j^{-1}\}$ and similarly $\chi_{H/A}(g) = \#$ fixed points of $g$ acting on $g_1H, g_2H, ..., g_nH = \#\{j | g_jHg_j^{-1}\}$. 

Combining our earlier work, we now prove:

\textbf{(1.14) Lemma:} Let \((H, H', G)\) be a distinguished triple, that is, \(H\) and \(H'\) non-conjugate in \(G\) and \(1^G_H \cong 1^G_{H'}\). Let \(A := \ker(\pi) = \ker(\pi')\) be the common kernel. Then the new triple defined by \((H_0, H'_0, G_0) := (H/A, H'/A, G/A)\) is distinguished and faithful.

\textbf{Proof:} For all \(g \in G\) we have \(1^G_{H/A}(gA) = 1^G_{H'}(g) = 1^G_{H'}(g) = 1^G_{H'/A}(gA)\) by Lemma (1.13). Now, \(H/A\) and \(H'/A\) are non-conjugate in \(G/A\) by Lemma (1.12). And the permutation maps \(\pi_0\) and \(\pi'_0\) induced from \(\pi\) and \(\pi'\) are clearly faithful. Hence \((H_0, H'_0, G_0)\) is distinguished and faithful.

The next lemma is technical. Its purpose is to enable us to use Proposition (1.18) of the following section.

\textbf{(1.15) Lemma:} If \((H, H', G)\) is inductive, then \(\bigcup_{g \in G} H^g = \bigcup_{g \in G} H'^g\).

\textbf{Proof:} Choose \(x\) in \(\bigcup_{g \in G} H^g\). Then \(x\) lies in some conjugate \(H^g\) of \(H\). By hypothesis, there is a bijective conjugation \(\psi : H \rightarrow H'\) as in Lemma (1.9). Then \(\psi(x)\) lies in \(H'\) and is conjugate to \(x\). That is, some conjugate of \(x\) lies in \(H'\), so \(x\) lies in some conjugate of \(H'\). Hence \(x\) lies in \(\bigcup_{g \in G} H'^g\). The lemma follows by symmetry.

\textbf{Section 3. Subgroups of Permutation Groups}

This section collects together results concerning permutations and permutation groups that will be used in the sequel. First we present three well-known results by way of the following lemma:

\textbf{(1.16) Lemma:}

(i) For \(n \neq 4\), \(S_n\) has no subgroups of index \(k\) with \(2 < k < n\).

(ii) For \(n \neq 6\), subgroups of index \(n\) in \(S_n\) are conjugate.

(iii) \(A_n\) is the unique subgroup of index 2 in \(S_n\).

\textbf{Proof:} For a proof of (i), (ii) and (iii) see [H], (Satz 5.3, Satz 5.5 and Satz 5.1,
respectively).

Here is a consequence of Lemma (1.16):

(1.17) Corollary: Let \( n \geq 7 \) and let \( H \) and \( H' \) be two non-conjugate subgroups of index \( k \) in \( S_n \). Then \( k \nmid n \).

The next result is found in [K]. Its statement occurs as the reformulation of Satz 1 at the top of page 44 of [K].

(1.18) Proposition: Let \( G \) be either a symmetric group \( G \cong S_n \), for any \( n \), or an alternating group \( G \cong A_n \) with \( n \neq 5 \). Let \( H \) be the stabilizer of a point and \( H' \) be any subgroup of \( G \) (not necessarily of index \( n \)) with \( \bigcup_{g \in G} H^g = \bigcup_{g \in G} H'^g \). Then \( H \) and \( H' \) are conjugate in \( G \).

(1.19) Corollary: Let \( n \geq 7 \) and let \( G \cong S_n \) or \( G \cong A_n \). Let \( H, H' \) be subgroups of index \( n \) in \( G \) with \( 1^G_H \cong 1^G_{H'} \). Then \( H \) is conjugate to \( H' \) in \( G \).

Proof: If \( G \cong S_n \), the result follows from Lemma (1.16), (iii). So suppose \( G \cong A_n \). Then \( H, H' \) have index \( n \) in \( G \cong A_n \). We want to use Proposition (1.18). For this, we want \( H \) to be the stabilizer of a point, but this was not assumed in the statement of (1.19). However, \( G \) acts on the \( n \) cosets of \( H \), giving a new transitive permutation representation \( \pi : G \to S_n \) in which \( H \) is the stabilizer of a point. Now \( \pi \) is injective for \( \ker(\pi) \) is a normal subgroup of \( A_n \) with \( n \geq 7 \) so \( \ker(\pi) = \{1\} \). (If \( \ker(\pi) = G \) then \( \pi(G) \) would not be a transitive subgroup of \( S_n \)). So \( \pi \) is injective, so \( \pi(G) = H \) has index 2 in \( S_n \). So \( \pi(G) = A_n \), by Lemma (1.16), (iii).

Next, \( 1^G_H \cong 1^G_{H'} \), by assumption. So \( \bigcup_{g \in G} H^g = \bigcup_{g \in G} H'^g \) by Lemma (1.15). Now Proposition (1.18) applies and guarantees that \( H \) is conjugate to \( H' \) in \( G \).

We conclude Chapter 1 with the following result (see [H], Satz 5.5).

(1.20) Lemma: \( S_n \) has only inner automorphisms for \( n \neq 6 \).
Section 1. Permutation Equivalence

I now present three results which, although elementary, form the backbone of the construction of Chapter 3. The first is found in many books on permutation groups. The third lemma is probably known to specialists but I was not able to find a proof.

(2.1) Lemma: Let $H \subseteq G$ be a subgroup of index $n$. Let $r_1 H, r_2 H, \ldots, r_n H$ and $r'_1 H, r'_2 H, \ldots, r'_n H$ be two countings of the left cosets mod $H$. Let $\pi, \pi' : G \to S_n$ be the group homomorphisms defined by $G$ acting on $\{r_i H\}$ and $\{r'_i H\}$ respectively, where $i = 1, 2, \ldots, n$.

For each $i$ in $\{1, 2, \ldots, n\}$ write $r_i H = r'_j H$ for some $1 \leq j \leq n$. Sending $i$ to $j$ then defines a permutation $\sigma$ in $S_n$. Then $\sigma \pi(g) \sigma^{-1} = \pi'(g)$ for all $g \in G$. That is, $\pi$ and $\pi'$ are permutation equivalent, i.e., conjugate via some $\sigma \in S_n$.

Proof: Fix $g \in G$, and fix an index $j$. To show

$$\sigma \pi(g) \sigma^{-1}(j) = \pi'(j).$$

Choose $i$ with $\sigma(i) = j$. That is, $r_i H = r'_j H$. Then we want to show

$$\sigma \pi(g)(i) = \pi'(j).$$

Let $k = \pi(g)(i)$. So $gr_i H = r_k H$. Further, set $t = \sigma(k)$, so $r_t H = r'_t H$. So it is to be shown that $\pi'(j) = t$. That is, we want $gr_j H = r'_t H$, which means we want $gr_i H = r'_t H$. But $gr_i H = r_t H = r'_t H$, as desired. This proves the lemma.

(2.2) Lemma: Let $H_1, H_2 \subseteq G$ be of the same index $n$. Fix two countings $r_1 H_1, \ldots, r_n H_1$ and $s_1 H_2, \ldots, s_n H_2$ of $G/H_1$ and $G/H_2$, respectively, with $r_1 = 1$ and $s_1 = 1$. Let $\pi_1, \pi_2 : G \to S_n$ be the permutation representations with respect to the given countings. Then $H_1 \simeq H_2$ iff there exists a fixed $\sigma$ in $S_n$ with $\sigma \pi_1(g) \sigma^{-1} = \pi_2(g)$ for all $g$ in $G$. 

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Proof: Suppose \( H_1 \cong H_2 \). Then \( \gamma H_1 \gamma^{-1} = H_2 \) for some \( \gamma \) in \( G \). So \( \{ r_{i} \gamma^{-1} \}_{i=1}^{n} \) represents cosets mod \( H_2 \). (Put \( s_i' = r_{i} \gamma^{-1} \). If \( s_i' H_2 = s_j' H_2 \) then \( s_i' H_2 \gamma = s_j' H_2 \gamma \) so \( r_i \gamma^{-1} H_2 \gamma = r_j \gamma^{-1} H_2 \gamma \). Hence \( r_i H_1 = r_j H_1 \) so \( i = j \). I use this counting \( \{ s_i' = r_{i} \gamma^{-1} \}_{i=1}^{n} \) to define \( \pi_2 : G \to S_n \). Then \( \pi_2' \) and \( \pi_2 \) are permutation equivalent by Lemma (2.1). That is, there is a \( \sigma \in S_n \) with

\[
\sigma \pi_2 \sigma^{-1} = \pi_2(g) \text{ for all } g \in G.
\] (4)

Claim: \( \pi_1(g) = \pi_2'(g) \) for all \( g \in G \). To show \( \pi_1(g)(i) = \pi_2'(g)(i) \), \( i = 1, 2, \ldots, n \), for all \( g \) in \( G \).

Fix \( g \in G \) and fix \( i \) in \( \{1, 2, \ldots, n\} \). Then \( \pi_2'(g)(i) = k \) for some \( k \) with

\[
g(r_i \gamma^{-1}) H_2 = r_k \gamma^{-1} H_2,
\]
so \( gr_i \gamma H_2 \gamma = r_k \gamma^{-1} H_2 \gamma \). Hence \( gr_i H_1 = r_k H_1 \) so \( \pi_1(g)(i) = k \), as well. This proves the claim. The claim together with (4) show

\[
\sigma \pi_2 \sigma^{-1} = \pi_2(g) \text{ for all } g \in G.
\]

Conversely, suppose such a \( \sigma \) in \( S_n \) exists.

To show: \( H_1 \cong H_2 \).

By hypothesis, \( s_1 H_2, s_2 H_2, \ldots, s_n H_2 \) is a fixed counting of cosets mod \( H_2 \) with \( s_1 = 1 \). By hypothesis \( \pi_2(g) = \sigma \pi_1(g) \sigma^{-1} \). Therefore,

\[
H_2 = \{ g \in G | \pi_2(g)(1) = 1 \}
= \{ g \in G | \sigma \pi_1(g) \sigma^{-1}(1) = 1 \}.
\]

Put \( \sigma^{-1}(1) = i_0 \). Then

\[
H_2 = \{ g \in G | \sigma \pi_1(g)(i_0) = 1 \}
= \{ g \in G | \pi_1(g)(i_0) = \sigma^{-1}(1) = i_0 \}
= \{ g \in G | gr_{i_0} H_1 = r_{i_0} H_1 \}
= \{ g \in G | gr_{i_0} \in r_{i_0} H_1 \}
= \{ g \in G | g \in r_{i_0} H_1 r_{i_0}^{-1} \}
= r_{i_0} H_1 r_{i_0}^{-1} \]. Thus \( H_2 \) is conjugate to \( H_1 \).
In the next lemma we look at the images $\pi_1(G), \pi_2(G)$ in $S_n$. We no longer assume $\pi_1(g)$ is conjugate to $\pi_2(g)$ with the same element $g \in G$, but we do assume that the subgroups $\pi_1(G)$ and $\pi_2(G)$ are conjugate in $S_n$. There is an interesting conclusion:

**(2.3) Lemma:** Let $H_1, H_2 \subseteq G$ be of the same index $n$ and suppose that $\pi_1, \pi_2 : G \rightarrow S_n$ are injective. Then $\pi_1(G)$ is conjugate to $\pi_2(G)$ in $S_n$ iff there is an automorphism $\alpha : G \rightarrow G$ with $\alpha(H_1) = H_2$.

**Proof:** Suppose $\alpha : G \rightarrow G$ is an automorphism with $\alpha(H_1) = H_2$. If $\alpha$ is an inner automorphism, then $H_1$ and $H_2$ are conjugate in $G$. Hence $\pi_1(G)$ and $\pi_2(G)$ are conjugate in $S_n$, by Lemma (2.2). Now suppose $\alpha : G \rightarrow G$ is an outer automorphism. If $r_1, r_2, ..., r_n$ represent $G/H_1$, then $\alpha(r_1), \alpha(r_2), ..., \alpha(r_n)$ represent $G/H_2$ (if $\alpha(r_i)H_2 = \alpha(r_j)H_2$ then $\alpha(r_i)H_1 = \alpha(r_j)H_1$ so $r_iH_1 = r_jH_1$ so $i = j$). Put $r'_i = \alpha(r_i)$ $i = 1, 2, ..., n$. Use this new counting to define $\pi_2' : G \rightarrow S_n$.

Then $r'_1 = \alpha(r_1) = \alpha(1) = 1$. Now $\pi_2'$ is permutation equivalent to the given permutation representation $\pi_2$ by Lemma (2.1). So $\pi_2'(G)$ is conjugate to $\pi_2(G)$ in $S_n$; that is, there is $\sigma \in S_n$ with $\sigma^{-1}\pi_2'(g)\sigma = \pi_2'(g)$ for all $g \in G$. From now on we will deal with $\pi_1$ and $\pi_2'$.

**Claim:** $\pi_1(g) = \pi_2'(\alpha(g))$ for all $g$ in $G$.

Fix $1 \leq i \leq n$. Then $\pi_1(g)(i) = k$ for some $k = 1, 2, ..., n$ where $gr_iH_1 = r_kH_1$. So $\alpha(gr_iH_1) = \alpha(r_kH_1)$, and therefore $\alpha(g)\alpha(r_i)H_2 = \alpha(r_k)H_2$. So $\alpha(g)r'_iH_2 = r_k'H_2$ and this means $\pi_2'(\alpha(g))(i) = k$, proving the claim.

So $\pi_1(g) = \sigma^{-1}\pi_2(\alpha(g))\sigma$ for all $g$ in $G$ and it follows that $\pi_1(G)$ is conjugate to $\pi_2(G)$ in $S_n$.

Conversely, $\pi_2(G) = \sigma\pi_1(G)\sigma^{-1}$ for some $\sigma$ in $S_n$. Recall that $\pi_1$ and $\pi_2$ are assumed to be injective. Define $\alpha : G \rightarrow G$ by $\alpha(g) = \pi_2^{-1}(\sigma\pi_1(g)\sigma^{-1})$. That is, given $g \in G$, $\sigma\pi_1(g)\sigma^{-1} \in \pi_2(G)$ and $\alpha(g)$ is the (unique) element of $G$ with $\pi_2(\alpha(g)) = \sigma\pi_1(g)\sigma^{-1}$. Then $\alpha$ is an automorphism of $G$. This completes the proof of Lemma (2.3).
Section 2. Cycles and Fixed Points

This short section contains a lemma connecting cycles of a permutation to its fixed points.

(2.4) Lemma: Let \( h \in S_n \) and let \( c_i \) be the number of cycles of length \( i \) appearing when \( h \) is written as a product of disjoint cycles. Then

\[
\# \{ \text{fixed points of } h^i \} = \sum_{d \mid i} d \cdot c_d
\]

as \( d \) runs over all divisors of \( i \), including \( d = 1 \) and \( d = i \).

Proof: The fixed points of \( h^i \) are 1-cycles of \( h^i \). These arise from cycles of \( h \) whose \( i^{th} \) power is trivial, that is, from cycles of length \( d \mid i \). Each such \( d \)-cycle gives rise to \( d \) fixed points. \( \Box \)
CHAPTER 3
The Construction

In this chapter I develop an algorithm to generate an infinite family of triples
\((H_i, H'_i, G_i)\) starting with a distinguished triple \((H, H', G)\). Depending on the start-
ing triple this family occasionally degenerates into a single conjugate triple at the
first step. I will pinpoint when this degeneration occurs. Often, however, the family
is infinite, each triple in the family being a really new distinguished triple.

Start with \((H, H', G)\) a distinguished triple. By Lemma (1.14) we may assume
that \((H, H', G)\) is a faithful triple. Thus we have \(H\) and \(H'\) are not conjugate
in \(G\), their induced representations are equal, their common index is \(\geq 7\), and
\(\pi, \pi' : G \rightarrow S_n\) are injective. Define a new triple as follows:

\[ G_1 := S_n, \quad H_1 := \pi(G) \quad \text{and} \quad H' := \pi'(G). \]

(3.1) Theorem: \((H_1, H'_1, G_1)\) is an inductive faithful triple.

Proof: First I will show inductive. It is to be shown that \(1^{G_1}_{H_1} \cong 1^{G_1}_{H'_1}\).

We will find a bijective-locally conjugate map \(\varphi : H_1 \rightarrow H'_1\) and then invoke
Lemma (1.9). There is a natural bijection \(\varphi : H \rightarrow G \rightarrow H'_1\)
defined by \(\varphi(h) = \pi'(\pi^{-1}(h))\), for all \(h \in H_1\). This is a well-defined bijection. If we
write \(h = \pi(g)\) with \(g \in G\), then \(\varphi(h) = \pi'(g)\).

Claim: \(\varphi(h)\) is conjugate to \(h\) for all \(h \in H_1\).

Two elements of \(G_1 := S_n\) are conjugate if and only if they have the same cycle
structure. So we want to examine the cycle structure of the permutations \(\varphi(h)\) and
\(h\). As a permutation, \(h\) is just the element \(g \in G\) with \(\pi(g) = h\), \(g\) acting on cosets
mod \(H\). And \(\varphi(h)\) is the same element \(g \in G\), but now \(g\) acts on cosets mod \(H'\).

Let \(c_i = \text{the number of cycles of length } i\) in the canonical decomposition of
\(h\) \((i = 1, 2, ..., n)\), and similarly let \(c'_i\) be the number of cycles of length \(i\) in the
canonical decomposition of \(\varphi(h)\). Then \(h\) is conjugate to \(\varphi(h)\) in \(S_n\) iff \(c_i = c'_i\).
I proceed to show \( c_i = c'_i \) for \( i = 1, 2, \ldots, n \) with \( n \) fixed, by induction on \( i \). For \( i = 1 \), \( c_1 \) is the number of fixed points of \( h \). That is, \( c_1 \) is the number of fixed points of \( g \) acting on \( G/H \). But this is just the character value \( \chi_{_{G/H}}(g) \), by Lemma (1.7). Similarly, \( c'_1 = \chi_{_{G/H'}}(g) \). So \( c_1 = c'_1 \), since the characters are assumed to be equal.

Now suppose \( c_j = c'_j \) for all \( j \leq i \) for fixed \( i < n \). To show: \( c_{i+1} = c'_{i+1} \). Now by Lemma (2.4), we have \( \chi_{_{G/H}}(g^{i+1}) = \sum dc_d + (i + 1)c_{i+1} \) and \( \chi_{_{G/H'}}(g^{i+1}) = \sum dc'_d + (i + 1)c'_{i+1} \), in each case \( d \) running over the proper divisors of \( i + 1 \). With the induction hypothesis we see \( (i + 1)c_{i+1} = (i + 1)c'_{i+1} \) so \( c_{i+1} = c'_{i+1} \). Hence, by induction, \( c_i = c'_i \) for all \( i = 1, 2, \ldots, n \), proving the claim. It follows that the new triple \( (H_1, H'_1, G_1) \) is inductive.

It is now to be shown that \( \pi_1, \pi'_1 : G_1 \to S_n \) are faithful. Now \( ker(\pi_1) \) is a normal subgroup in \( G_1 \). But \( G_1 = S_n \) and \( n \geq 7 \) so the only possibilities are

\begin{enumerate}
  \item \( ker(\pi_1) \cong S_n \)
  \item \( ker(\pi_1) \cong A_n \), or
  \item \( ker(\pi_1) \cong \{1\} \).
\end{enumerate}

Case (i): \( ker(\pi_1) \cong S_n \). Since \( ker(\pi_1) \) is contained in both \( H_1 \) and \( H'_1 \), then \( H_1 = H'_1 = G_1 \). So \( \pi(G) = \pi'(G) = S_n \). But \( \pi \) is faithful so \( G \cong S_n \). Thus both \( H \) and \( H' \) have index \( n \) in \( G = S_n \), with \( n \geq 7 \). By Lemma (1.16), \( H, H' \) are conjugate in \( G \), a contradiction to "distinguished". Hence \( ker(\pi_1) \not\cong S_n \).

Case (ii): \( ker(\pi_1) \cong A_n \). Since \( ker(\pi_1) \) is contained in \( H_1 \) and in \( H'_1 \), then either \( H_1 = S_n \) or \( H_1 = A_n \). If \( H_1 = S_n \) then \( \pi(G) = S_n \) so \( G \cong S_n \). As argued in Case (i) above, this implies \( H \) and \( H' \) are conjugate in \( G \), a contradiction. If \( H_1 = A_n \) then \( G \cong A_n \), with \( n \geq 7 \) and \( H \) and \( H' \) are two subgroups of \( A_n \) of index \( n \), inducing isomorphic representations of \( A_n \). It follows from Corollary (1.19) that \( H \) and \( H' \) are conjugate in \( G \), a contradiction to "distinguished". Hence \( ker(\pi_1) \not\cong A_n \). It follows that \( ker(\pi_1) = \{1\} \). So \( \pi_1 \) is faithful. But \( ker(\pi_1) = ker(\pi'_1) \) by Lemma (1.11). Hence \( \pi_1 \) and \( \pi'_1 \) are faithful. Hence \( (H_1, H'_1, G_1) \) is an inductive faithful triple. \[\square\]
We now come to the main theorem.

(3.2) **Main Theorem:** Let \((H, H', G)\) be a distinguished faithful triple such that there is no automorphism \(\alpha\) of \(G\) with \(\alpha(H) = H'\). Then the same holds for \((H_1, H_1', G_1)\).

**Proof:** Theorem (3.1) shows that the triple \((H_1, H_1', G_1)\) obtained on the first iteration is inductive and faithful. Moreover, by Lemma (2.3), the new triple is non-conjugate. Now \(G_1 = S_n\) with \(n \geq 7\). Since \(S_n\) has only inner automorphisms for \(n \geq 7\), it follows that the new triple \((H_1, H_1', G_1)\) satisfies the hypotheses made on the original triple. This proves the main theorem. 

(3.3) **Corollary:** Let \((H, H', G)\) be a distinguished, faithful triple such that there is no automorphism \(\alpha\) of \(G\) with \(\alpha(H) = H'\). Then iterating the construction yields an infinite family of distinguished faithful triples, no two triples in this family being isomorphic.

**Proof:** Apply the main theorem iteratively. This produces an infinite family of distinguished faithful triples, and it remains to show that these triples are pairwise different. Let \(n_i\) be the index of \(H_i\) in \(G_i\). So \(n\) is the index of the starting subgroup \(H\) in \(G\), and \(n_1\) is the index of \(H_1 := \pi(G)\) in \(G_1 := S_n\), and \(n_2\) is the index of \(H_2 = \pi_1(S_n)\) in the symmetric group \(S_{n_1}\), etc. At each step, \(H_i\) and \(H_i'\) are non-conjugate subgroups of index \(n_i\) in the symmetric group \(S_{n_{i-1}}\). It follows from Corollary (1.17) that the index \(n_i\) exceeds the previous index \(n_{i-1}\). If we interpret \(n_0\) to be the starting index, \(n\), then this shows that the indices increase at each step, proving the Corollary.
This chapter contains examples of distinguished triples. One of these examples produces an infinite family of triples via my construction, and two examples degenerate under the construction. I begin with Gassmann's example.

**Example 1: Gassmann's Example**

Let $G = S_6$ and let $H$ and $H'$ be the subgroups

$$H = \{(1), (12)(34), (13)(24), (14)(23)\}$$

and

$$H' = \{(1), (12)(34), (12)(56), (34)(56)\}$$

Then $H$ and $H'$ have order 4 and therefore index 180 in $G$. Observe that $H$ is contained in the stabilizer of the "letter" (6) while the other subgroup $H'$ does not stabilize anything. Hence $H$ and $H'$ are not conjugate in $G = S_6$. But $H$ and $H'$ are bijectively-locally-conjugate in $G$. To see this, choose any bijection $\varphi : H \rightarrow H'$ such that $\varphi((1)) = (1)$. For $h \in H$ distinct from the identity element, $h$ and $\varphi(h)$ are both products of two disjoint transpositions, and therefore have the same cycle structure. This lets us conclude that $\varphi(h)$ is conjugate to $h$ in $G$. Hence $1^G_H \cong 1^G_{H'}$, by Lemma (1.9).

This triple is therefore distinguished. Moreover, it is faithful, for the kernel of $\pi$ is a subgroup of $H$ and is normal in $G = S_6$. That is, the kernel is too small to be anything other than 1.

Now, $S_6$ has one outer automorphism, up to inner automorphisms. As future work, it is my intention to describe this automorphism explicitly and decide whether this automorphism carries $H$ to $H'$. This will simultaneously decide whether running the construction degenerates.

**Example 2: Start with the cyclic group $(\mathbb{Z}/8\mathbb{Z}, +)$. The automorphism group of this group can be identified with the multiplicative group $(\mathbb{Z}/8\mathbb{Z})^*$. Thus $(\mathbb{Z}/8\mathbb{Z})^*$
acts on \((\mathbb{Z}/8\mathbb{Z},+)\) and we define \(G\) to be the semi-direct product:
\[
G = (\mathbb{Z}/8\mathbb{Z})^* \rtimes (\mathbb{Z}/8\mathbb{Z}).
\]
Thus \(G\) has order 32. Elements in \(G\) are pairs \((\sigma, g)\) which are multiplied explicitly by the formula
\[
(\sigma, g_1) \cdot (\tau, g_2) = (\sigma \tau, \tau(g_1) + g_2).
\]

Let \(H = \{(\sigma, 0) : \sigma \in (\mathbb{Z}/8\mathbb{Z})^*\}\) and \(H' = \{(1, 0), (3, 4), (5, 4), (7, 0)\}\). Then \(H\) is not conjugate to \(H'\) in \(G\) by easy direct calculation, not given here.

Claim: \(1^G_H \cong 1^G_{H'}\).

For this, define \(\varphi : H \to H'\) by
\[
\begin{align*}
(1, 0) & \to (1, 0) \\
(7, 0) & \to (7, 0) \\
(3, 0) & \to (3, 4) = (1, 2)(3, 0)(1, 2)^{-1} \\
(5, 0) & \to (5, 4) = (1, 7)(5, 0)(1, 7)^{-1}
\end{align*}
\]
Then \(\varphi\) is a bijective locally-conjugate map from \(H\) to \(H'\). Hence by Lemma (1.9), \(1^G_H \cong 1^G_{H'}\). Note that \((G : H) = 8\). Hence \((H, H', G)\) is distinguished. By computing the intersection of the conjugates of \(H\) directly, one sees that this triple is faithful.

This group has an outer automorphism taking \(H\) to \(H'\). Namely, \(G\) is generated by either of the following two sets
\[
< (1, 1), (3, 1), (5, 1) > \\
\downarrow ^\alpha \\
< (1, 3), (3, 7), (5, 7) >
\]
and the map \(\alpha\) between these generator sets defined by sending a generator to the generator directly below it can be checked to be an automorphism of \(G\) taking \(H\) to \(H'\). Hence, the construction becomes degenerate.

**Example 3:** This example is taken from [B]. Let \(n \geq 3\) and let
$G = SL_n(Z/pZ)$, where $p$ is a prime number. Set

$$H = \begin{pmatrix}
* & * & \ldots & * \\
0 & * & \ldots & * \\
0 & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \ldots & *
\end{pmatrix}$$

That is, $H = \{(a_{ij}) \in SL_n(Z/pZ) : a_{i1} = 0, i \neq 1\}$. And set

$$H' = \begin{pmatrix}
* & 0 & 0 & \ldots & 0 \\
* & * & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & * & \ldots & * \\
* & * & * & \ldots & *
\end{pmatrix}$$

That is, $H' = \{A^t : A \in H\}$.

Next, it is to be shown that $H$ and $H'$ induce isomorphic representations of $G$. I will check condition (b) of Lemma (1.9); namely, that $|g^G \cap H| = |g^G \cap H'|$. Now, a matrix $g$ in $G$ lies in $H$ iff $v = (1,0,\ldots,0)^t$ is an eigenvector of $g$. Moreover, a $G$-conjugate $\sigma g \sigma^{-1}$ lies in $H$ iff $\sigma^{-1}v$ is an eigenvector of $g$. This means that $|g^G \cap H|$ depends only on the number of eigenvectors of $g$ that are in the $G$-orbit of $v$. Now suppose that $n \geq 2$. Then the $G$-orbit of $v$ consists of all non-zero vectors in the $n$-dimensional space over $Z/pZ$, so the order of the intersection we are counting depends only on the number of eigenvectors of $g$. (Note that $g$ is an invertible transformation, so $0$ is not an eigenvector of $g$.)

Similarly, a conjugate of $g$ lies in $H'$ if and only if some conjugate of $g^t$ lies in $H$, so the order of the second intersection depends only on the number of eigenvectors of $g^t$. But $g$ and $g^t$ are conjugate in the general linear group $GL_n(Z/pZ)$ for they have the same rational canonical form. And therefore they have the same number of eigenvectors. So $|g^G \cap H| = |(g^t)^G \cap H| = |g^G \cap H'|$, showing that the trivial representations of $H$ and of $H'$ induce isomorphic representations of $G$.

Next I claim that $H$ is not conjugate to $H'$ in $G$. For this, observe that the column vector $(1,0,\ldots,0)^t$ is a common eigenvector for each element in $H$. If $H'$
were conjugate to $H$ then the elements in $H'$ would also have a common eigenvector, but when $n \geq 3$, they do not. However, the outer automorphism $\alpha : G \to G$ defined by $\alpha(g) = (g^t)^{-1}$ takes $H$ to $H'$. Hence, by Theorem (3.2), the first step of our construction on this example produces a degeneration. That is, $(H_1, H'_1, G_1)$ is a conjugate triple.

**Example 4**: Fix $p$, a prime number. Let $H = Z/pZ \oplus Z/pZ \oplus Z/pZ$ and let $H'$ be the “mod $p$ Heisenberg group”

$$H' = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b, c \in Z/pZ$.

Observe that $|H| = |H'| = p^3$. So $H$ and $H'$ are embedded in the symmetric group $S_{p^3}$ via the regular representation. Put $G := S_{p^3}$. Each non-zero element $\beta \in H \cup H'$ has order $p$.

Claim: $1^G_H \cong 1^G_{H'}$.

Take any bijection $\varphi : H \to H'$ with $\varphi(1_H) = 1_{H'}$. It now remains to show that $\varphi(h) \sim h$ for all $h \in H$. Take $h \neq 1_H$ in $H$. As a permutation, $h$ acts by left translation, so for $x \in H$ the $h$-orbit of $x$ is

$$x \to hx \to h^2x \to ... \to h^{p-1}x \to h^px = x.$$ 

This is a $p$-cycle. So $h$ cyclically permutes the $p^3$ elements of $H$ taken in blocks of $p$, so $h$ is a product of $p^2$ disjoint $p$-cycles. The same holds for $\varphi(h)$ ($\neq 1_{H'}$). Note both $1_H$ and $\varphi(1_H) = 1_{H'}$ are 1--cycles. Hence $h$ and $\varphi(h)$ have the same cycle structure for all $h \in H$ so are conjugate in $G = S_{p^3}$. Note that $H$ is abelian and $H'$ is non-abelian. So $H$ is not only not conjugate to $H'$, but as abstract groups, $H \not\cong H'$. Hence there is no automorphism $\alpha : G \to G$ with $\alpha(H) = H'$. By Theorem (3.2), this example will generate an infinite family of distinguished triples.
Bibliography


Vita

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