Continuous Functions in Pi-Primary Summands of $L(2)$ of Some Compact Solvmanifolds.

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Continuous functions in $\pi$-primary summands of $L^2$ of some compact solvmanifolds

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CONTINUOUS FUNCTIONS IN $\pi$-PRIMARY SUMMANDS
OF $L^2$ OF SOME COMPACT SOLVMANIFOLDS

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ABSTRACT

The compact quotients of three-dimensional solvable non-nilpotent Lie groups by discrete subgroups fall into two categories; those which are quotients of $S_R$ (semidirect product, $R$ acts on $R^2$ by rotation), and those which are quotients of $S_H = R \rtimes R^2$ (semidirect product, $R$ acts on $R^2$ via translation along a hyperbolic orbit).

Continuous functions in the primary summands of $L^2$ of each type of solvmanifold are examined. It is shown that there exists a Fejer theorem on solvmanifolds $M_R$ which are quotients of $S_R$; if $P_i : L^2(M_R) \rightarrow L^2(M_R)$ is orthogonal projection onto the $i$th primary summand of such a quotient, there exists a sequence of operators $S_n$ such that each $S_n$ is a finite linear combination of the $P_i$, and such that for each continuous function $f \in L^2(M_R)$, $S_n(f) \rightarrow f$ uniformly as $n \rightarrow \infty$. It is shown that no such theorem holds for quotients of $S_H$, for which it is known that orthogonal projections do not preserve continuity of functions. It is shown that if $f \in L^2(M_H)$ is continuous and $P_i(f)$ is essentially bounded, then $P_i(f)$ must be continuous. Together with a result of L. Richardson, this result implies that there are continuous $f \in L^2(M_H)$ for which $P_i(f)$ is essentially unbounded.

It is shown in Section 2 that continuous functions in primary summands of $L^2(M_H)$ must vanish on $M_H$, for all compact quotients of $M_H$ by discrete subgroups. There are five compact quotients of $S_R$; one quotient manifold is homeomorphic to the three-dimensional torus, and its continuous primary summand functions need not vanish. For three of the other quotients, it is shown that all continuous primary summand functions must vanish. For the remaining quotient it
is known that functions in certain subspaces of the primary summands must vanish, but it is not known whether all continuous primary summand functions must vanish in this case.
§0.1. Introduction

Let $G$ be a solvable, connected and simply connected Lie group, with Lie algebra $g$ and with cocompact discrete subgroup $\Gamma$. By a representation $\pi$ of $G$ we shall mean a strongly continuous, unitary representation of $G$ in some separable Hilbert space $H_\pi$; $\pi$ will be called irreducible if the space $H_\pi$ contains no proper closed nontrivial subspace invariant under $\pi$.

Let $M$ be the space of right cosets $\Gamma g$ of $\Gamma$ in $G$, endowed with the quotient topology. Then $G$ acts on $L^2(M)$ by right translation; i.e. $g \mapsto R(g)$, where $[R(g)f](\Gamma x) = f(\Gamma x g)$ for $f \in L^2(M)$ (here $M$ has the $G$-invariant probability measure inherited from Haar measure on $G$). $R$ is called the quasiregular representation of $G$ on $L^2(M)$.

It is well known that $L^2(M)$ decomposes into the direct sum $\bigoplus H_\pi$, where the spaces $H_\pi$ are mutually orthogonal $R(G)$-invariant subspaces, and $R$ on the space $H_\pi$ is a finite multiple of the irreducible representation $\pi$. ([GGP], section I.2). We let $(\Gamma \backslash G)^\wedge$ denote the set of irreducible representations appearing in the quasi-regular representation $R$ of $G$ on $L^2(M)$. Then the orthogonal projection $P_\pi$ of $L^2(M)$ onto $H_\pi$ is $L^2$-continuous and preserves $C^\infty(M)$ ([Aus-Bre 1], theorem 5), and is given by convolution with a bounded Borel measure $\sigma_\pi$.  

1
Now let $N$ be a nilpotent Lie group, connected and simply connected, with Lie algebra $\mathfrak{n}$ and cocompact discrete subgroup $\Gamma$.

If the coadjoint orbits of the action of $N$ on the dual $\mathfrak{n}^*$ are linear varieties, then $\Gamma \backslash N$ possesses the property that the orthogonal projections $P_{\pi}$ of $L^2(\Gamma \backslash N)$ onto $H_{\pi}$ preserve continuity ([Ri 1], [Bre 1]). These flat-orbit nilmanifolds share this property with compact quotients of the 3-dimensional solvable group $S_R$ by discrete subgroups. Here $S_R$ denotes the semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ via a one-parameter subgroup of rotations [Ri 1].

The work in chapter 2 was motivated by a theorem of L. Auslander and R. Tolimieri. Let $H_3$ be the 3-dimensional Heisenberg group, $\mathbb{R}^3$ endowed with the multiplication $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$, and let $\Gamma$ be the discrete group of integer points in $H_3$. Let $f$ be a continuous function in $H_{\pi} \subset L^2(\Gamma \backslash H_3)$, where $\pi$ is an irreducible, unitary, infinite dimensional representation in $(\Gamma \backslash H_3)^\wedge$. Then $f$ must have at least one zero on $\Gamma \backslash H_3$ ([Aus-Tol], Thm.II.2). This phenomenon arises from a rather surprising interaction between the representation theory of $H_3$ (which determines the primary summand $H_{\pi}$) and the topology of the manifold $\Gamma \backslash H_3$. In chapter 2 of this dissertation, we generalize this theorem to all compact nilmanifolds which are not $n$-tori. We then examine the question of whether a similar theorem holds for 3-dimensional compact solvmanifolds.

The central covariance of $\pi$-primary summand functions on $\Gamma \backslash H_3$ is a key element in Auslander and Tolimieri's proof that continuous $\pi$-primary summand functions have zeros. Since 3-dimensional non-nilpotent solvable Lie groups with cocompact discrete subgroups have trivial centers ([AGH], chapter 3), completely new techniques are needed to show that most 3-dimensional compact solvmanifolds do possess the property that their continuous $\pi$-primary functions (hereafter referred to as primary functions) must vanish, for infinite-dimensional $\pi$. A noteworthy exception is one compact quotient of $S_R$ which is actually homeomorphic.
to the 3-torus $T^3$; here one finds plenty of continuous primary functions which do not vanish, as one would expect. However, for three of four remaining compact quotients of $S_R$, it is shown that continuous primary functions must have zeros. For the fourth compact quotient of $S_R$, we have shown that continuous functions in certain subspaces of a primary summand $H_\pi$ must have zeros. As of this writing, however, it is conjectured but not known that all continuous primary summand functions on this manifold must have zeros.

Let $S_H$ be the semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$, where $\mathbb{R}$ acts upon $\mathbb{R}^2$ via the one-parameter subgroup $t \mapsto \begin{bmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{bmatrix}$ in $SL_2(\mathbb{R})$, where $\lambda + \lambda^{-1} = k + 1$ for any integer $k \geq 2$. It is shown in this dissertation that for all compact quotients of $S_H$, continuous primary functions must have zeros. This exhausts the compact solvmanifolds of dimension three.

Thus the interplay of topology and representation theory which produces zeros of continuous primary functions is seen to be more than a nilpotent phenomenon, but the extent of this interaction remains obscure. The possibility of a generalization of this theorem to a larger class of compact solvmanifolds deserves to be investigated. The study in section 1 of continuous functions in primary summands of 3-dimensional compact solvmanifolds was motivated by the fact that orthogonal projection onto $\pi$-primary summands preserves continuity of functions in $L^2$ of both compact quotients of flat-orbit nilmanifolds and compact quotients of the group $S_R$. In [Ri 2], L. Richardson proved a Fejer theorem for flat-orbit nilmanifolds. In chapter 1 of this dissertation, a Fejer theorem is proved for quotients of $S_R$ by compact discrete subgroups $\Gamma$ of $S_R$. Similar questions are examined for compact quotients of the 3-dimensional solvable group $S_H$. Let $\Gamma$ be a co-compact discrete subgroup of $S_H$. It is known that orthogonal projections $P_\pi$ of $L^2(\Gamma \backslash S_H)$ onto $H_\pi$ do not preserve continuity [Ri 1]. In chapter 1 of this essay, it is demonstrated that no standard Fejer theorem exists for solvmanifolds of this
type. However, the series determining the $L^2$ equivalence class of a projected function is seen to bear a resemblance to a standard lacunary Fourier series on $\mathbb{R}\setminus\mathbb{Z}$ (see [Zyg]). An adaptation of Sidon's theorem on convergence of lacunary Fourier series ([Zyg], thm. VI.6.1) is used to demonstrate that if the orthogonal projection $P_\pi f$ of a continuous function $f$ on $\Gamma\setminus S_H$ is an $L^\infty$ function, then $P_\pi f$ is actually continuous (Thm. 1.2.1). Together with Theorem 3.13 in [Ri 1], this implies that for each $H_\pi$ there is continuous $f \in L^2(\Gamma\setminus S_H)$ such that $P_\pi f$ is discontinuous and essentially unbounded.

This work suggests the possibility of further similarities in the harmonic analysis of flat-orbit nilmanifolds and compact quotients of $S_R$.

§0.2. Classification of 3-dimensional compact solvmanifolds

The following is a convenient exposition of results due to Auslander, Green and Hahn ([AGH], section 2.2).

**Lemma 0.1.** If $S$ is a connected, simply connected, 3-dimensional non-compact, non-nilpotent solvable Lie group with cocompact discrete subgroup $\Gamma \subset S$, then the dimension of the maximal nilpotent subgroup $N$ is 2.

The remainder of this section is devoted to proving the following.

**Theorem 0.2.** If $S$ satisfies the conditions of Lemma 1, then $S \cong S_H$ or $S_R$, where

$$S_H = \left\{ \begin{bmatrix} e^{kt} & x \\ e^{-kt} & y \\ 1 & t \end{bmatrix} : (t, x, y) \in \mathbb{R}^3 \right\}$$

for some $k$ such that $e^k + e^{-k} \in \mathbb{Z}$, $e^k + e^{-k} \neq 2$, and

$$S_R = \left\{ \begin{bmatrix} \cos \pi t & \sin \pi t & x \\ -\sin \pi t & \cos \pi t & y \\ 0 & 1 \end{bmatrix} : (t, x, y) \in \mathbb{R}^3 \right\}.$$
Proof of Theorem 0.2. We know that \( N \) is two-dimensional, from Lemma 1, and hence abelian. Consider the natural map \( \pi : S \rightarrow N \setminus S \). Since \( N \) is abelian, \( N \setminus S \) acts on \( N \) via \( (Ns)n = sns^{-1} \), for \( Ns \in N \setminus S \), \( n \in N \). It follows that \( S = \mathbb{R} \times \mathbb{R}^2 \), where \( \mathbb{R} \) is the additive group of real numbers acting upon \( \mathbb{R}^2 \). We give \( S \) the coordinates \( S = (t, u, v) : t \in \mathbb{R}, (u, v) \in \mathbb{R}^2 \) where \( (t, u, v)(t', u', v') = (t + t', (u, v) + \sigma(t)(u', v')) \) for \( \sigma \) some 1-parameter subgroup of \( GL_2(\mathbb{R}) \). \( \pi \) is an open map, and so \( \pi(\Gamma) \) is a discrete subgroup of \( N \setminus S \cong \mathbb{R} \). Thus \( \pi(\Gamma) \) is isomorphic to \( \mathbb{Z} \), and so is generated by a single element \( \theta \).

We then have that \( \Gamma \cap N \) is a lattice in \( N \) generated by two linearly independent basis vectors. If we coordinatize \( N \) so that \( \Gamma \cap N \) is generated by \((0,1)\) and \((1,0)\) in \( N \), then \( \theta \) acts on \( \mathbb{R}^2 \) as a linear transformation which preserves the lattice \( \mathbb{Z}^2 \). If \( \theta = N(t, u, v) \), then \( \theta \) acts by

\[
\theta \cdot (0, n, m) = (t, u, v)^{-1}(0, n, m)(t, u, v) \tag{0.1}
\]

and so the action of \( \pi(\Gamma) \) upon \( \Gamma \cap N \) is well defined.

\( \theta \) acts as a linear transformation \( T(\theta) \) which preserves \( \mathbb{Z}^2 \) and so has integral entries and a determinant of \( \pm 1 \).

It remains to be determined how, in light of these facts, \( \mathbb{R} \) acts upon \( \mathbb{R}^2 \). We must find a 1-parameter subgroup \( \phi \) such that \( \phi(1) = T(\theta) \).

Since \( T(\theta) \) must lie on a 1-parameter subgroup \( \phi \), \( \det T(\theta) = 1 \), so that \( T(\theta) \in SL_2(\mathbb{Z}) \). Since successive square roots of \( T(\theta) \) must also have determinant 1, \( \phi \) must be a 1-parameter subgroup of \( SL_2(\mathbb{R}) \).

There are three cases to consider.

Case 1. The eigenvalues of \( T(\theta) \) are real and positive.

Case 2. The eigenvalues of \( T(\theta) \) are real and negative.

Case 3. The eigenvalues of \( T(\theta) \) are complex conjugates in \( \mathbb{C} \sim \mathbb{R} \).
**Lemma 0.3.** Let $T$ be a diagonal matrix in $SL_2(\mathbb{R})$. If $T$ has a real square root which is not diagonal, then $T = \pm I$.

**Proof.** Suppose there is $T^{1/2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $b$ or $c$ nonzero, in $SL_2(\mathbb{R})$. Then we have $\begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & p^{-1} \end{bmatrix}$. Since not both $b$ and $c$ are zero, $a + d = 0$, and so $a^2 = d^2$. The diagonal terms $p, p^{-1}$ are thus equal, so that $T = \pm I$.

**Case 1.** The eigenvalues of $T(\theta)$ are real and positive.

**Case 1a.** Suppose the eigenvalues of $T(\theta)$ are positive, real and unequal. The previous lemma shows that if we write $AT(\theta)A^{-1} = \begin{bmatrix} p & 0 \\ 0 & 1/p \end{bmatrix}$, then $AT(\theta)A^{-1}$ has 2 square roots, $\pm \begin{bmatrix} p^{1/2} \\ (1/p)^{1/2} \end{bmatrix}$. The matrix $\begin{bmatrix} -p^{1/2} & -(1/p)^{1/2} \\ 0 & 0 \end{bmatrix}$ can have no further square root. Taking successive square roots we see that $A\phi(t)A^{-1} = \begin{bmatrix} p^t & 0 \\ 0 & p^{-t} \end{bmatrix}$, so that $\phi(t)$ is conjugate to this subgroup.

**Case 1b.** The eigenvalues of $T(\theta)$ are both 1.

(i) Suppose $T(\theta) = I$. Then clearly $T(\theta)$ lies on a compact, connected, 1-parameter subgroup. Since $S$ is assumed solvable, $\sigma$ cannot be trivial; so $\sigma(t)$ is a circle group, isomorphic to $\begin{bmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{bmatrix}$.

(ii) Suppose $T(\theta)$ is not diagonalizable. Then we have, for some $A \in GL_2(\mathbb{R})$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Suppose $T(\theta)^{1/2}$ is a real square root of $T(\theta)$. Then if $T(\theta)^{1/2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $\begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Since $(a + d) \cdot b = 1$, we have $c = 0$; thus $a^2 = d^2 = 1$ and so $a = d = 1$, since $a + d \neq 0$. Thus we have $2b = 1$, or $b = 1/2$. The only real square root of $T(\theta)$ is therefore $\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$. Arguing in this
way we see that $A\sigma(t)A^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, but this action of $\mathbb{R}$ upon $\mathbb{R}^2$ yields the 3-dimensional Heisenberg group $H_3$.

**Case 2.** The eigenvalues of $T(\theta)$ are real and negative.

**Case 2a.** Suppose the eigenvalues of $T(\theta)$ are unequal. If so, we may write $AT(\theta)A^{-1} = \begin{bmatrix} -p & -p^{-1} \end{bmatrix}$. If $T(\theta)^{1/2}$ were a real square root of $T(\theta)$, then $AT(\theta)^{1/2}A^{-1}$ would be a real square root of $\begin{bmatrix} -p & -p^{-1} \end{bmatrix}$. However, this matrix can have no real square root, since by Lemma 0.3 it has no nondiagonal square root, and it clearly can have no real diagonal square root. Therefore, such a matrix cannot lie on a 1-parameter subgroup of $SL_2(\mathbb{R})$.

**Case 2b.** Suppose the eigenvalues of $T(\theta)$ are $-1$. If $T(\theta) = -I$, then $\sigma(t)$ is a compact connected subgroup of $SL_2(\mathbb{R})$, and is therefore conjugate to the subgroup

$$Rot(\pi t) = \begin{bmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{bmatrix}.$$ 

If $T(\theta)$ is not diagonalizable, then if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a square root of $AT(\theta)A^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ (for some $A \in GL_2(\mathbb{R})$), we have

$$\begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$ 

Since $b(a + d) = 1$ and $c(a + d) = 0$ we have $c = 0$; so $a^2 = d^2 = 1$, a contradiction. Therefore $AT(\theta)A^{-1}$ and hence $T(\theta)$ cannot have a real square root.

**Case 3.** The eigenvalues of $T(\theta)$ are complex conjugates in $\mathbb{C}$.

Then $\omega + \bar{\omega} = 2\text{Re}\omega = \text{Tr}(T(\theta))$, the trace of $T(\theta)$.

If $\text{Re}(\omega) = \pm 1$ we are in cases 1b or 2b.

(i) If $\text{Tr}(T(\theta)) = 0$, the eigenvalues of $T(\theta)$ are $e^{i\pi/2}$ and $e^{-i\pi/2}$; thus $T(\theta)$ is conjugate to the map $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $\sigma(t)$ is conjugate to the 1-parameter subgroup $\begin{bmatrix} \cos \pi t/2 & \sin \pi t/2 \\ -\sin \pi t/2 & \cos \pi t/2 \end{bmatrix}$. 
(ii) If $\text{Tr}(T(\theta)) = 1$, the eigenvalues of $T(\theta)$ are $e^{i\pi/3}$ and $e^{-i\pi/3}$; thus $T(\theta)$ is conjugate to the map $T = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, and $\sigma(t)$ is conjugate to the 1-parameter subgroup $\begin{bmatrix} \cos \pi t/3 & \sin \pi t/3 \\ -\sin \pi t/3 & \cos \pi t/3 \end{bmatrix}$.

(iii) If $\text{Tr}(T(\theta)) = 1$, the eigenvalues of $T(\theta)$ are $e^{2i\pi/3}$ and $e^{-2i\pi/3}$, and $\sigma(t)$ is conjugate to the 1-parameter subgroup $\begin{bmatrix} \cos 2\pi t/3 & \sin 2\pi t/3 \\ -\sin 2\pi t/3 & \cos 2\pi t/3 \end{bmatrix}$.

Thus for each such $T(\theta)$, $T(\theta)^p = I$ for some $p \in \mathbb{Z}^+$, and so $\sigma(t)$ is a compact, connected subgroup of $SL_2(\mathbb{R})$; hence it is a circle group, isomorphic to the group $\begin{bmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{bmatrix}$.

In conclusion, the groups derived in case 1a are isomorphic to the group $S_H$. Pick $\lambda \in \mathbb{R}$ so that $\lambda + \lambda^{-1} \in \mathbb{Z}$, and $\lambda + \lambda^{-1} > 2$. Then if $\lambda'$ is any other positive number satisfying $\lambda' + \lambda'^{-1} > 2$, and $\lambda' + \lambda'^{-1} \in \mathbb{Z}$, then we see that the groups

$$S_\lambda = \left\{ \begin{bmatrix} \lambda^t \\ \lambda^{-t} \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ t \end{bmatrix} : (t, x, y) \in \mathbb{R} \right\}$$

and

$$S_{\lambda'} = \left\{ \begin{bmatrix} \lambda'^t \\ \lambda'^{-t} \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ t \end{bmatrix} : (t, x, y) \in \mathbb{R} \right\}$$

are isomorphic via the isomorphism

$$(t, x, y) \in S_\lambda \mapsto ((\log \lambda', \lambda) \cdot t, x, y) \in S_{\lambda'}.$$  

All the other groups derived are isomorphic to $S_R$, thus proving the main theorem of this section.

Therefore we have the following five compact quotients of $S_R$, with convenient coordinatization.

1. $\Gamma_{R,1}/S_{R,1} = M_{R,1}$; where $S_{R,1} = \mathbb{R} \ltimes \mathbb{R}^2$, $\mathbb{R}$ acts on $\mathbb{R}^2$ via the one-parameter subgroup $\sigma_1(t) = \begin{bmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{bmatrix}$, and $\Gamma_{R,1} = \{(p, m, n) \in S_{R,1} ; p, m, n \in \mathbb{Z} \}$.
\[ Z \]. Here \( \Gamma_{R,1} \) is isomorphic to the abelian group \( Z^3 \), and so \( M_{R,1} \cong T^3 \) ([Mos], Theorem A).

2. \( \Gamma_{R,2}\backslash S_{R,2} = M_{R,2} \); where \( S_{R,2} = R \ltimes R^2 \), \( R \) acts on \( R^2 \) via the one-parameter subgroup \( \sigma_2(t) = \begin{bmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{bmatrix} \), and \( \Gamma_{R,2} = \{(p,m,n) \in S_{R,2}; p,m,n \in Z\} \).

3. \( \Gamma_{R,3}\backslash S_{R,3} = M_{R,3} \); where \( S_{R,3} = R \ltimes R^2 \), \( R \) acts on \( R^2 \) via the one-parameter subgroup \( \sigma_3(t) \) with \( \sigma_3(1) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \). \( \sigma_3(t) \) is isomorphic to the subgroup \( \text{Rot}(2\pi t/3) = \begin{bmatrix} \cos 2\pi t/3 & \sin 2\pi t/3 \\ -\sin 2\pi t/3 & \cos 2\pi t/3 \end{bmatrix} \), and \( \Gamma_{R,3} = \{(p,m,n) \in S_{R,3}; p,m,n \in Z\} \).

4. \( \Gamma_{R,4}\backslash S_{R,4} = M_{R,4} \); where \( S_{R,4} = R \ltimes R^2 \), \( R \) acts on \( R^2 \) via the one-parameter subgroup \( \sigma_4(t) = \begin{bmatrix} \cos \pi t/2 & \sin \pi t/2 \\ -\sin \pi t/2 & \cos \pi t/2 \end{bmatrix} \), and \( \Gamma_{R,4} = \{(p,m,n) \in S_{R,4}; p,m,n \in Z\} \).

5. \( \Gamma_{R,6}\backslash S_{R,6} = M_{R,6} \); where \( S_{R,6} = R \ltimes R^2 \), \( R \) acts on \( R^2 \) via the one-parameter subgroup \( \sigma_6(t) \) in \( SL_2(R) \) with \( \sigma_6(1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \), and \( \Gamma_{R,6} = \{(p,m,n) \in S_{R,6}; p,m,n \in Z\} \).

We also have the following compact quotients of \( S_H \), with convenient coordinatizations.

Suppose \( k \in Z, k \geq 2 \). Define \( S_{H,k} = R \ltimes R^2 \), where \( R \) acts on \( R^2 \) via the one-parameter subgroup \( \sigma_k(t) \) in \( SL_2(R) \) with \( \sigma_k(1) = \begin{bmatrix} 1 & 1 \\ k-1 & k \end{bmatrix} \). Then \( S_{H,k} \cong S_H \) for each \( k \). Let \( \Gamma_{H,k} = \{(p,m,n) \in S_{H,k}; p,m,n \in Z\} \); then each \( \Gamma_{H,k}\backslash S_{H,k} = M_{H,k} \) is a distinct compact quotient of \( S_H \).

\[ \S 0.3. \text{Preliminaries} \]

Let \( G \) be a connected, simply connected Lie group with Lie algebra \( g \), and let \( g^* \) be the vector space of linear functionals on \( g \). We define a sequence of ideals
of the Lie algebra $\mathfrak{g}$ by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$; this is called the derived series of $\mathfrak{g}$, and $\mathfrak{g}$ is said to be solvable if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$. We define another sequence of ideals of the Lie algebra $\mathfrak{g}$ by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}]$; this is called the lower central series of $\mathfrak{g}$, and $\mathfrak{g}$ is said to be nilpotent if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$ (see [Hum], section 3). Throughout this dissertation, the term “nilmanifold” (“solvmanifold”) will refer to compact spaces $\Gamma \backslash G$, where $G$ is nilpotent (solvable) and $\Gamma$ is discrete and cocompact.

By a representation $\pi$ of $G$ we shall mean an equivalence class of strongly continuous homomorphisms of $G$ into the set of unitary linear transformations on some separable Hilbert space $H$. The representation $\pi$ will be called irreducible if $H$ contains no proper closed subspace invariant under $\pi$.

The adjoint representation of the group $G$ in the vector space $\mathfrak{g}$, written $\text{Ad}$, is defined as follows; for each element $x \in G$, $\text{Ad}(x) : \mathfrak{g} \rightarrow \mathfrak{g}$ is the differential at the identity of $G$ of the group automorphism $I(x)$, inner conjugation by $x \in G$. $\text{Ad}(x)$ satisfies

$$x(\exp X)x^{-1} = \exp[\text{Ad}(x)X] \quad (0.2)$$

for each $x \in G$, $X \in \mathfrak{g}$.

The coadjoint representation of $G$ is of central importance in the representation theory of nilpotent and solvable Lie groups. The set of equivalence classes of irreducible representations of a nilpotent Lie group $G$ is naturally parametrized by the orbit space $\mathfrak{g}^*/\text{Ad}^*G$; this is also true for the (completely) solvable Lie groups examined in this work. This parametrization, due to A. A. Kirillov, is freely drawn upon in this work; for details, see [CG], chapter II.

As described in section 0.2, there are two 3-dimensional, solvable, non-nilpotent Lie groups with cocompact discrete subgroups, the groups $S_H$ and $S_R$. Their Lie algebras are three-dimensional vector spaces spanned by the vectors $T$, $X$, and $Y$, where $\exp sT = (s,0,0)$, $\exp sX = (0,s,0)$, and $\exp sY = (0,0,s)$. 
There are 5 distinct, non-homeomorphic compact quotients of \( S_R \), and infinitely many distinct compact quotients of \( S_H \), as seen in section 0.2.

It will be convenient to use several different coordinatizations of \( S_R \) and \( S_H \).

The coordinatizations of \( S_R \) described at the end of section 0.2 will be called integral coordinatizations of \( S_r,p \). Let \( A \in GL_2(\mathbb{R}) \) be such that \( A\sigma_p(t)A^{-1} = R(2\pi t/p) = \begin{bmatrix} \cos 2\pi t/p & \sin 2\pi t/p \\ -\sin 2\pi t/p & \cos 2\pi t/p \end{bmatrix} \). If we recoordinatize \( N \) so that the action of \( \mathbb{R} \) on \( N \) is given by \( R(2\pi t/p) \), then \( \Gamma_{p,p} \cap N = A(\mathbb{Z}^2) \) (note that in the case of \( \Gamma_{1,1}, \Gamma_{1,2}, \) and \( \Gamma_{4,4}, A = I \)). In this coordinatization of \( S_r,p \), the nondegenerate coadjoint orbits of \( S_r,p \) are circular cylinders, \( x^2 + y^2 = \lambda^2 \), for some \( \lambda \in \mathbb{R} \). For the groups \( S_{r,3} \) and \( S_{r,6} \), the 2-torus \( N \cap \Gamma_{r,p} \setminus N \) will be a nonstandard torus in this coordinatization. We will call these coordinatizations the circular coordinatizations of \( S_r,p \).

The coordinatizations of the solvmanifolds \( S_{H,k} \) described at the end of section 0.2 will be referred to as the integral coordinatizations of \( S_{H,k} \). Let \( A \in GL_2(\mathbb{R}) \) be such that \( A\sigma_k(t)A^{-1} = \begin{bmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{bmatrix} \) where \( \lambda + \lambda^{-1} = k+1 \); if we recoordinatize \( N \) so that the action of \( \mathbb{R} \) on \( N \) is given by this one parameter subgroup, then \( \Gamma_{H,k} \cap N = A(\mathbb{Z}^2) \); the nondegenerate coadjoint orbits in this case are hyperbolic cylinders of the form \( xy = \lambda, \lambda \in \mathbb{R} \). The 2-torus \( N \cap \Gamma_{H,k} \setminus N \) in this coordinatization will be a nonstandard torus, for all \( k \geq 2 \). This coordinatization of \( S_{H,k} \) will be referred to as the hyperbolic coordinatization.

We will use the fact that, in the integral coordinatizations of \( S_{H,k} \), the nondegenerate coadjoint orbits satisfy the equation

\[
(k - 1)x^2 + (k - 1)xy - y^2 = \omega, \quad \omega \in \mathbb{R}.
\]

To see this we observe that the transformation \( A = \begin{bmatrix} \lambda - k & 1 \\ 1 - \lambda & 1 \end{bmatrix} \) in \( GL_2(\mathbb{R}) \) satisfies \( \phi_k(t)A = A\sigma_k(t) \), where \( \sigma_k \) is the one-parameter subgroup in \( SL_2(\mathbb{R}) \) satisfying

\[
\sigma_k(1) = \begin{bmatrix} 1 & 1 \\ k-1 & k \end{bmatrix}, \quad \text{and} \quad \phi_k(t) = \begin{bmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{bmatrix}, \quad \lambda + \lambda^{-1} = k+1.
\]
Thus if \([\sigma_k(t)](x,y) = (x(t),y(t))\), we have

\[
A(x(t),y(t)) = (x'(t),y'(t)) = ((\lambda - k)x(t) + y(t), (1 - \lambda)x(t) + y(t))
\]  

(0.6)

satisfying

\[
(x'(t))(y'(t)) = [(\lambda - k)(x(t)) + y(t)][(1 - \lambda)(x(t)) + y(t)]
\]

\[
= (\lambda - k)(1 - \lambda)x^2(t) + (1 - k)x(t)y(t) + y^2(t)
\]

(0.7)

\[
= C.
\]

Since this is

\[
(1 - k)x^2(t) + (1 - k)x(t)y(t) + y^2(t) = C
\]

(0.8)
equation (0.5) is established.

For each solvmanifold \(M_{H,k}(M_{R,i})\), the group \(S_{H,k}(S_{R,i})\) is a simply connected cover of \(M_{H,k}(M_{R,i})\) and \(\Gamma_{H,k}(\Gamma_{R,i})\) is the group of covering transformations of \(S_{H,k}(S_{R,i})\). Thus we have \(\Pi_1(M_{H,k}) = \Gamma_{H,k}, \Pi_1(M_{R,i}) = \Gamma_{R,i}\). The \(M_{H,k}\) and \(M_{R,i}\) are bundles over the circle with 2-torus fiber; the projection maps are

\[
\mu_H : M_{H,k} \longrightarrow \mathbb{Z}\mathbb{R}
\]

(0.9)

\[
\Gamma_{H,k}(t,u,v) \longmapsto \mathbb{Z} + t
\]

and

\[
\mu_R : M_{R,p} \longrightarrow \mathbb{Z}\mathbb{R}
\]

(0.10)

\[
\Gamma_{R,p}(t,u,v) \longmapsto \mathbb{Z} + t.
\]

A generalization of the following material may be found in [GGP], section 1.2.

Let \(M = \Gamma\backslash S\) be a solvmanifold, where \(S\) is a solvable Lie group, \(\Gamma\) a discrete subgroup of \(S\). We define a unitary representation \(R\) of \(S\) in the space \(L^2(M)\) as follows; for \(g \in S, f \in L^2(M), \) and \(\Gamma s \in \Gamma\backslash S\), we have

\[
[R(g)f](\Gamma s) = f(\Gamma sg).
\]

(0.11)

This is called the quasiregular representation of \(S\) on \(L^2(M)\). When \(M\) is a compact solvmanifold, \(L^2(M)\) decomposes canonically into the discrete direct sum
of subspaces $H_\pi$, which are invariant with respect to the action of $S$, and are such that when the action of $S$ is restricted to the subspace $H_\pi$, $S$ acts on $H_\pi$ as some finite multiple of an irreducible representation $\pi$ of $S$.

A convenient decomposition of $H_\pi$ into irreducible subspaces will be used throughout this dissertation; however, no canonical decomposition of $H_\pi$ into irreducible subspaces exists. The irreducible subspaces of the chosen decomposition of $H_\pi$ will be referred to as the constructible irreducible subspaces of $H_\pi$, and will be described in section 0.4.

$(\Gamma \backslash S)^\wedge$ will denote the set of unitary irreducible representations of $S$ appearing in the quasiregular representation $R$ of $S$ in $L^2(\Gamma \backslash S)^\wedge$.

$(\Gamma \backslash S)^\infty$ will denote the set of those representations $\pi \in (\Gamma \backslash S)^\wedge$ which are infinite dimensional.

§0.4. $\pi$-primary functions

In this section, we will describe those functions on $M_{H,k}(M_{R,p})$ which are primary functions. We will use integral coordinatizations of $S_{R,p}$ and $S_{H,k}$ (see section 0.3).

In the integral coordinatization of $S_{H,k}$, the coadjoint orbits satisfy

$$(k - 1)x^2 + (k - 1)xy - y^2 = \lambda$$

(0.12)

so that the orbits are saturated in the $T^*$-direction. We will call $\lambda$ an integral functional if $\lambda|_n = \alpha X^* + \beta Y^*$, $\alpha, \beta \in \mathbb{Z}$, and denote by $O_\pi$ the orbit of $\lambda$ in $s^*_{H,k}$.

Fix some nonzero integral functional $\lambda \in O_\pi$. We define the character $\chi_\lambda$ on the (abelian) nilradical $n$ as follows: if $\lambda|_n = \alpha X^* + \beta Y^*$, then

$$\chi_\lambda(0, r, s) = e^{2\pi i(\alpha r + \beta s)}$$

(0.13)

We seek a maximal subgroup $M$ of $S_{H,k}$ such that
(i) $M$ contains $N$:

(ii) $\chi_{\lambda}$ may be extended to a character of $M$, i.e. $\chi_{\lambda}[M,M] = 1$, where $[M,M]$ is the commutator subgroup of $M$. $M$ will be called a maximal subgroup subordinate to $\lambda$. We will call this extension $\tilde{\chi}$, the map extension of $\chi$ to $M$. To this end, it suffices to examine the values of $\chi_{\lambda}$ on terms in $[S_{H,k}, S_{H,k}]$ of the form

$$(t,0,0)(0,x,y)(-t,0,0)(0,-x,-y) = (0,(x,y) - \sigma_k(t)(x,y)).$$

(0.14)

Since $\sigma_k(t)$ has eigenvalues $\lambda^t$ and $\lambda^{-t}$, where $\lambda + \lambda^{-1} = k + 1$, and since $\lambda$ is nonzero,

$$\chi_{\lambda}(0,(x,y) - \sigma_k(t)(x,y)) = \exp 2\pi i \lambda((x,y) - \sigma_k(t)(x,y))$$

is 1 for all $(x,y)$ if and only if $t = 0$. Thus $N$ itself is maximal subordinate to $\lambda$ for all nonzero $\lambda$.

We define the Mackey space $M(\lambda)$ for $\lambda$ as follows:

$$M(\lambda) = \{f : S_{H,k} \rightarrow C \mid f \text{ is measurable, } |f| \in L^2(N \setminus S_{H,k}),$$

$$f(ng) = \chi_{\lambda}(n)f(g) \quad n \in N, \ g \in S_{H,k}\}.$$  

(0.15)

It is well known that the action of $S_{H,k}$ on $M(\lambda)$ by right translation is an irreducible representation $\pi$. ([Bre]).

Note that the functions $f$ in $M(\lambda)$ are left $\Gamma_{H,k} \cap N$ - invariant.

We define the homogenizing (lift) map $L : M(\lambda) \rightarrow L^2(\Gamma_{H,k} \setminus S_{H,k})$ as follows: for $f \in M(\lambda),$

$$Lf(\Gamma_{H,k}(t,x,y)) = \sum_{\gamma \in \Gamma_{H,k} \cap N \setminus \Gamma_{H,k}} (f \cdot \gamma)(t,x,y)$$

(0.16)

where $(f \cdot \gamma)(g) = f(\gamma g)$, for $\gamma, g \in S_{H,k}$.

Note that the sum in (0.16) is well defined with respect to equivalence classes of $\Gamma_{H,k} \cap N \setminus \Gamma_{H,k}$. For all $f \in M(\lambda)$, $Lf$ is a well-defined element of $L^2(\Gamma_{H,k} \setminus S_{H,k})$, 


and the map $L$ is $S_{H,k}$-equivariant, so that the image in $L^2(\Gamma_{H,k} \backslash S_{H,k})$ of $M(\lambda)$ is an irreducible $\pi$-subspace of $L^2(\Gamma_{H,k} \backslash S_{H,k})$, and therefore a subspace of the $\pi$-primary summand $H_\pi$.

Those subspaces of $H_\pi$ which are the images of homogenizing maps $L$ from Mackey spaces $M(\lambda)$ will be referred to as the constructible irreducible subspaces of $H_\pi$; there are finitely many such subspaces in each $H_\pi$. The maximum number of such mutually subspaces will be referred to as the multiplicity of $\pi$, $\text{mul}_\pi$. There is one constructible irreducible subspace in $H_\pi$ for each distinct $\Gamma_{H,k}$-orbit in the integral functionals of $O_\pi$, and so $\text{mul}_\pi$ is given by the number of such orbits.

The constructible irreducible subspaces are not canonically determined irreducible subspaces of $H_\pi$; nevertheless, the $\pi$-primary summand is the orthogonal direct sum of these spaces, and so we may represent a typical member of the $\pi$-primary summand $H_\pi$ as follows. Let $\{\lambda_i\}_{i=1}^{\text{mul}_\pi}$ be a set of representatives of distinct $\Gamma_{H,k}$-orbits in the integral functionals of $O_\pi$, $\lambda_i$ an integral functional for each $i$. Let $f_i \in M(\lambda_i)$; $L_i$, the lift map from $M(\lambda_i)$ to $L^2$. Then a typical member of $H_\pi$ has the form

$$F = \sum_{i=1}^{\text{mul}_\pi} L_i f_i = \sum_{i=1}^{\text{mul}_\pi} \left\{ \sum_{\gamma \in \Gamma_{H,k} \cap N \backslash \Gamma_{H,k}} (f_i \cdot \gamma) \right\} \quad (0.17)$$

If $f_i \in M(\lambda_i)$, then $f_i$ may be written

$$f_i(t,x,y) = \chi_{\lambda_i}(0,x,y)\tilde{f}_i(t) \quad (0.18)$$

where $\tilde{f}_i \in L^2(\mathbb{R})$. Thus, if we choose the elements $(n,0,0) \in \Gamma_{H,k}, n \in \mathbb{Z}$ to represent the equivalence classes of $\Gamma_{H,k} \cap N \backslash \Gamma_{H,k}$, we have
\[ Lf_i(\Gamma_{H,k}(t,x,y)) = \sum_{n \in \mathbb{Z}} [f_i(n,0,0)](t,x,y) \]
\[ = \sum_{n \in \mathbb{Z}} f_i((n,0,0) \cdot (t,x,y)) \]
\[ = \sum_{n \in \mathbb{Z}} f_i(n + t, \sigma_k(n)(x,y)) \]
\[ = \sum_{n \in \mathbb{Z}} \tilde{f}_i(n + t)\chi_{\lambda_i}(0,\sigma_k(n)(x,y)) \]
\[ = \sum_{n \in \mathbb{Z}} \tilde{f}_i(n + t)\chi_{\sigma_k^*(n)\lambda_i}(0,x,y). \]

Thus a typical member of \( H_\pi \) is of the form

\[ \sum_{i=1}^{\text{mul } \pi} \sum_{n \in \mathbb{Z}} \tilde{f}_i(n + t)\chi_{\sigma_k^*(n)\lambda_i}(0,x,y) \]

Recall from section 0.3 that all elements \( \sigma_k^*(n)\lambda_i \) satisfy equation (0.5).

Suppose we have \( \lambda \in \mathbb{R}^*, \lambda = \alpha X^* + \beta Y^* \) for some \( \alpha, \beta \in \mathbb{Z} \). Then

\[ \chi_{\lambda}(0,x,y) = \exp 2\pi i(\alpha x + \beta y). \]

If we set \( z_1 = e^{2\pi ix} \), \( z_2 = e^{2\pi iy} \), we may write \( \chi_{\lambda}(0,x,y) = e^{2\pi i(\alpha x + \beta y)} = z_1^\alpha z_2^\beta \), so that \( F \in H_\pi \) may be thought of as a function of \( t, z_1, z_2 \); i.e. as \( \tilde{F} \), where

\[ \tilde{F}(t,z_1,z_2) = \sum_{i=1}^{\text{mul } \pi} \sum_{n \in \mathbb{Z}} f_i(n + t)z_1^{\alpha n_i}z_2^{\beta n_i} \]

for \( \sigma_k^*(n)\lambda_i = \alpha_{n,i}X^* + \beta_{n,i}Y^* \).

Fixing \( t_0 \in \mathbb{R} \), we may define a cross section

\[ \tilde{F}_{t_0}(z_1,z_2) \equiv \tilde{F}(t_0,z_1,z_2) \]

so that \( \tilde{F}_t \) is a function from \( T^2 \) to \( \mathbb{C} \), for \( z_1, z_2 \) of modulus 1.

Let the integral functionals \( \{\lambda_i\}_{i=1}^{\text{mul } \pi} \) be a set of \( \Gamma_{H,k} \)-orbit representatives in \( \mathbb{R}^* \); we define \( H_{\pi_i} \) to be the image of the lift map

\[ L : M(\lambda_i) \longrightarrow L^2(\Gamma_{H,k} \backslash S_{H,k}) \]
(\(H_\pi\) is the \(i\)th constructible irreducible subspace of \(H_\pi\), the \(\pi\)-primary summand).

In the integral coordinatization of \(S_{R,p}\), the coadjoint orbits satisfy

(i) \(x^2 + y^2 = k^2\) for some \(k \in \mathbb{R}\) if \(p = 1, 2, 4\) \hspace{1cm} (0.24)

(ii) \(x^2 + xy + y^2 = k^2\) for some \(k \in \mathbb{R}\) if \(p = 3, 6\), \hspace{1cm} (0.25)

so that the orbits are saturated in the \(T^*\)-direction.

Fix some coadjoint orbit \(O_\pi \subset \mathfrak{g}_{R,p}^*\), and some nonzero integral functional \(\lambda \in O_\pi\).

We define the character \(\chi_\lambda\) on the nilradical \(N\) as follows:

\[
\chi_\lambda(0,r,s) = \exp 2\pi i \lambda(r,s) = \exp 2\pi i(\alpha r + \beta s) \tag{0.26}
\]

where \(\lambda = \alpha X^* + \beta Y^*\).

We seek a maximal extension of the character \(\chi_\lambda\) on \(M\); we examine the values of \(\chi_\lambda\) on terms of the form

\((t,0,0)(0,x,y)(-t,0,0)(0,-x,-y) = (0,(x,y) - \sigma_p(t)(x,y))\),

in the commutator \([S_{R,p}, S_{R,p}]\).

Since \(\sigma_p(t)\) has eigenvalues \(\exp \pm 2\pi it/p\), and since \(\sigma_p(p\mathbb{Z}) \equiv I\), we may extend the character \(\chi_\lambda\) to a character on the subgroup

\[M_p = \{(n,x,y); n = pk \text{ for some } k \in \mathbb{Z}, x,y \in \mathbb{R}\}\]. \hspace{1cm} (0.27)

Then \(M_p\) is called maximal subordinate to the functional \(\lambda\).

We define the Mackey space \(M(\lambda)\) for \(\lambda\) as follows.

\[M(\lambda) = \{f : S_{R,p} \rightarrow \mathbb{C} \mid f \text{ is measurable, } |f| \in L^2(\Gamma_{R,p}\backslash S_{Z,p}), \ f(mg) = \chi_\lambda(m)f(g) \mid m \in M_p, g \in S_{R,p}\}\] \hspace{1cm} (0.28)
The action of \( S_{R,p} \) on \( M(\lambda) \) by right translation is an irreducible representation \( \pi \), independent (up to equivalence) of the choice of \( \lambda \in \mathcal{O}_\pi \).

The functions \( f \in M(\lambda) \) are left \( \Gamma_{R,p} \cap M_p \)-invariant. We define the lift map \( L : M(\lambda) \rightarrow L^2(\Gamma_{R,p} \backslash S_{R,p}) \) as follows: for \( f \in M(\lambda) \),

\[
Lf(\Gamma_{R,p}(t,x,y)) = \sum_{\gamma \in \Gamma_{R,p} \cap M_p \backslash \Gamma_{R,p}} (f \cdot \gamma)(t,x,y)
\]

Note that the sum (0.29) is a sum of \( p \) terms, and is a left \( \Gamma_{R,p} \)-invariant function in \( L^2(\Gamma_{R,p} \backslash S_{R,p}) \). \( L \) is an \( S_{R,p} \)-equivariant map, so that the image in \( L^2(\Gamma_{R,p} \backslash S_{R,p}) \) of \( M(\lambda) \) is an irreducible \( \pi \)-subspace of \( L^2(\Gamma_{R,p} \backslash S_{R,p}) \), and therefore a subspace of the \( \pi \)-primary summand \( H_\pi \subset L^2(\Gamma_{R,p} \backslash S_{R,p}) \); this subspace will also be referred to as a constructible irreducible subspace of \( H_\pi \). The number of such mutually orthogonal subspaces of \( H_\pi \) is called \( \text{mul}_\pi \), and is equal to the number of disjoint \( \Gamma_{R,p} \)-orbits in the set of integral functions in \( \mathcal{O}_\pi \). \( H_\pi \) is the orthogonal direct sum of these subspaces.

We may represent a typical element of the \( \pi \)-primary summand \( H_\pi \) as follows. Let \( \{\lambda_i\}_{i=1}^{\text{mul}_\pi} \) be a set of representatives of distinct \( \Gamma_{R,p} \)-orbits of integral functionals in \( \mathfrak{n}^* \cap \mathcal{O}_\pi \). If \( f_i \in M(\lambda_i) \), a typical member of \( H_\pi \) has the form

\[
F = \sum_{i=1}^{\text{mul}_\pi} Lf_i = \sum_{i=1}^{\text{mul}_\pi} \left\{ \sum_{\gamma \in \Gamma_{R,p} \cap M_p \backslash \Gamma_{R,p}} f_i : \gamma \right\}
\]

(0.30)

If \( f_i \in M(\lambda_i) \), then \( f_i \) may be written

\[
f_i(t,x,y) = \chi_{\lambda_i}(0,x,y)f_i(t).
\]

(0.31)

where \( \tilde{f}_i \in L^2(p\mathbb{Z} \backslash \mathbb{R}) \) (\( \tilde{f}_i \) is to be thought of as a function on \( \mathbb{R} \) with period \( p \)).

Thus, if we choose the elements \( (n,0,0) \in \Gamma_{R,p} \), \( n = 0,1,2,\ldots,p-1 \) to represent
the equivalence classes of \( M_p \cap \Gamma_{R,p} \setminus \Gamma_{R,p} \), we have

\[
Lf_i(\Gamma_{R,p}(t,x,y)) = \sum_{n=0}^{p-1} (f_i \cdot (n,0,0))(t,x,y) \\
= \sum_{n=0}^{p-1} f_i(n+t, \sigma_k(n)(x,y)) \\
= \sum_{n=0}^{p-1} (\tilde{f_i}(n+t)\chi_{\lambda_i}(\sigma_k(n)(x,y))) \\
= \sum_{n=0}^{p-1} (\tilde{f_i}(n+t)\chi_{\sigma^*_k(n)\lambda_i}(x,y)).
\] (0.32)

Thus a typical member of \( H_{\pi} \) is of the form

\[
\sum_{i=1}^{\text{mul}(\pi)} \sum_{n \in \mathbb{Z}} \tilde{f_i}(n+t)\chi_{\sigma^*_k(n)\lambda_i}(0,x,y)
\] (0.33)

Recall that all elements \( \sigma^*_k(n)\lambda_i \) satisfy equation (0.24) or (0.25).

Suppose we have \( \lambda \in \mathfrak{n}^* \), \( \lambda = \alpha X^* + \beta Y^* \) for some \( \alpha, \beta \in \mathbb{Z} \). Then

\[
\chi_{\lambda}(0,x,y) = \exp 2\pi i (\alpha x + \beta y).
\] (0.34)

If we set \( z_1 = e^{2\pi i x} \), \( z_2 = e^{2\pi i y} \), we may define \( \tilde{F} \) and \( \tilde{F}_t \) as in equations (0.22) and (0.23) for \( F \in H_{\pi} \). If the integral functionals \( \{\lambda_i\}_{i=1}^{\text{mul} \pi} \) are a set of \( \Gamma_{R,p} \)-orbit representatives in \( \mathfrak{n}^* \), we define \( H_{\pi_i} \) to be the image of the lift map \( L(M(\lambda_i)) \rightarrow L^2(\Gamma_{R,p} \setminus S_{R,p}) \). \( H_{\pi_i} \) is then the \( i \)-th constructible irreducible subspace of \( H_{\pi} \).

We end this section with a fact, and a lemma.

1. \( S_{H,k}(S_{R,p}) \) in its integral coordinatization has the fundamental domain \([0,1]^3\); since \( T^3 \) has the same fundamental domain and since the invariant measure of the boundary is zero, the identification of fundamental domain produces a Borel isomorphism of the measure spaces and an isometry between \( L^2(T^3) \) and \( L^2(\Gamma_{H,k} \setminus S_{H,k}) \mid L^2(\Gamma_{R,p} \setminus S_{R,p}) \). Since each character \( f_{\alpha,\beta,\gamma}(t,x,y) = \exp 2\pi i (\alpha x + \beta y + \gamma t) \), \( \alpha, \beta, \gamma \in \mathbb{Z} \), appears in the summand \( H_{\pi} \) for which \( \alpha X^* + \beta Y^* \)
\( \beta Y^* \in \mathcal{O}_\pi \), we have that the \( \pi \)-primary summands \( H_\pi \), together with the constant functions, form a complete set of orthonormal subspaces in \( L^2(\Gamma_{H,k} \backslash S_{H,k}) \) \( [L^2(\Gamma_{R,p} \backslash S_{R,p})] \).

2. We define

\[ P_{\pi_i} : L^2(\Gamma_{R,p} \backslash S_{R,p}) \rightarrow H_{\pi_i} \]

to be the orthogonal projection of \( L^2(\Gamma_{R,p} \backslash S_{R,p}) \) onto \( H_{\pi_i} \). We have for \( f \in L^2(\Gamma_{R,p} \backslash S_{R,p}), \)

\[ P_{\pi_i}(f)(\Gamma_{R,p}(t,x,y)) = \sum_{N=0}^{p-1} f(t,\cdot,\cdot)^*(\sigma_p^*(N)\lambda_i)\chi_{\sigma_p^*(N)\lambda_i}(0,x,y) \quad (0.35) \]

where \( f(t,\cdot,\cdot)^* \) is the standard Fourier transform in the variables \( x \) and \( y \) for fixed \( t \) (note that for fixed \( t \), \( P_{\pi_i} f(t,x,y) \) is a function on \( (N \cap \Gamma_{R,p} \backslash N) \cong T^2 \)).

**Lemma 0.4.** Suppose \( f \) is continuous on \( M_{R,p} \). Then \( P_{\pi_i} f = L\tilde{f} \) for some continuous \( \tilde{f} \) in \( M(\lambda_i) \).

**Proof.** We need to demonstrate a continuous function \( \tilde{f} \in M(\lambda_i) \) such that \( L\tilde{f} = P_{\pi_i} f \).

Let

\[ \tilde{f}(t,x,y) = f(t,\cdot,\cdot)^*(\lambda_i)\chi_{\lambda_i}. \quad (0.36) \]

To see that \( L\tilde{f} = P_{\pi_i} f \), we must demonstrate

1. That \( \tilde{f} \in M(\lambda_i) \) and \( \tilde{f} \) is continuous in \((t,x,y)\);
2. That \( \tilde{f}((t+k),\sigma_p(k)(x,y)) = f(t,\cdot,\cdot)^*(\sigma_p^*(k)\lambda_i)\chi_{\sigma_p^*(k)\lambda_i}(x,y) \); ie. the \( k \)th terms in each sum are identical. By definition,

\[ \tilde{f}(t+k,\sigma_p(k)(x,y)) = f(t+k,\cdot,\cdot)^*(\lambda_i)\chi_{\lambda_i}(\sigma_p(k)(x,y)). \quad (0.37) \]

Since \( \chi_{\sigma_p^*(k)\lambda_i}(x,y) = \chi_{\lambda_i}(\sigma_p(k)(x,y)) \), to demonstrate part 2 we need only show that

\[ f((t+k),\cdot,\cdot)^*(\lambda_i) = f(t,\cdot,\cdot)^*(\sigma_p^*(k)\lambda_i). \quad (0.38) \]
By definition,
\[ \tilde{f}((t + k), x, y)^{\lambda_i} = f(t + k, x, y)^{\lambda_i} \]
\[ = \int_{N \cap \Gamma_{R,p} \setminus N} f(t + k, x, y) \chi_{\lambda_i}(x, y) \, dx \, dy. \] (0.39)

Since \( f \) is continuous on \( M_{R,p} \) and is therefore left \( \Gamma_{R,p} \)-invariant, we have
\[ \int_{N \cap \Gamma_{R,p} \setminus N} f(t + k, x, y) \chi_{\lambda_i} \, dx \, dy \]
\[ = \int_{N \cap \Gamma_{R,p} \setminus N} f(t, \sigma_p(-k)(x, y)) \chi_{\lambda_i}(x, y) \, dx \, dy \]
\[ = \int_{N \cap \Gamma_{R,p} \setminus N} f(t, x, y) \chi_{\lambda_i}(\sigma_p(k)(x, y)) \det \sigma_p(k) \, dx \, dy \]
\[ = \int_{N \cap \Gamma_{R,p} \setminus N} f(t, x, y) \chi_{\sigma_p^*(k) \lambda_i}(x, y) \, dx \, dy \]
\[ = f(t, x, y)^{\lambda_i} \tilde{\chi}_{\sigma_p^*(k) \lambda_i}. \]

\( \tilde{f} \) has the desired left-\( M_p \)-invariance, and is therefore in \( M(\lambda_i) \); since \( f \) itself is continuous in \( t \), \( f(t, x, y)^{\lambda_i} \) is continuous in \( t \) and therefore \( \tilde{f} \) is a continuous function in \( M(\lambda_i) \). This completes the proof of Lemma 0.4.
CHAPTER 1

CONVERGENCE THEOREMS ON SOLVMANIFOLDS

§1.1. Fejer theorems on compact solvmanifolds

Throughout this section, we will utilize circular coordinates for $M_{R,i}$, $i = 1, 2, 3, 4, 6$, as they are described in section 0.2.

First we prove the following:

Theorem 1.1. Let $P_{\lambda} : L^2(M_{R,i}) \rightarrow H_{\pi, \lambda}$ be orthogonal projection onto $H_{\pi, \lambda}$, given by the bounded Borel measure $\sigma_{\lambda}$. Then there exists a sequence of operators $S_n : L^2(M_{R,i}) \rightarrow L^2(M_{R,i})$, where

$$S_n = \sum \alpha_{\lambda,n} P_{\lambda} \quad \alpha_{\lambda,n} \in \mathbb{C} \quad (1.1)$$

such that

(i) the sum in (1.1) is finite for all $n$

(ii) for all $f \in C(M_{R,i})$, we have $S_n f \rightarrow f$ uniformly as $n \rightarrow \infty$.

Proof. Let $\{h_k\}_{k=1}^{\infty}$ be a compactly supported, $C^\infty$, rotation-invariant approximate identity on $N$, the nilradical of $S_{R,i}$. Let $D$ be a fundamental domain of $N \cap \Gamma_{R,i} \backslash N \cong T^2$ containing the identity of $N$ as an interior point; we choose the $h_k$ so that their supports are contained inside $D$, and so that for each $\varepsilon > 0$, there exists a $k \in \mathbb{N}$ such that if $N > k$, the support of $h_N$ is contained in an $\varepsilon$-ball around the origin in $N$.

Since each $h_k$ is $C^\infty$, we have

$$\phi_{R,k} = \sum_{|(N_1, N_2)| < R} \hat{h}_k(N_1, N_2) \exp 2\pi i(N_1 x_1 + N_2 y_2) \quad (1.2)$$
converging uniformly to \( h_k \) as \( R \to \infty \). Note also that since \( h_k \) is rotation-independent, \( \hat{h}_k(N_1, N_2) = \hat{h}_k(M_1, M_2) \) if \( N_1^2 + N_2^2 = M_1^2 + M_2^2 \), and that the sum is over a nonstandard lattice.

Define \( \phi_k = \phi_{R_k,k} \) for each \( k \) so that

\[
\|\phi_k - h_k\|_\infty < 1/k \quad \text{in} \quad N \cap \Gamma_{R,i} \setminus N, \quad \text{for each} \quad k \in \mathbb{N}.
\]

Now let \( f : M_{R,i} \to \mathbb{C} \) be a continuous function on \( M_{R,i} \), so that in particular \( f_t(x, y) \equiv f(t, x, y) \) is continuous on the 2-torus \( N \cap \Gamma_{R,i} \setminus N \). Recall that \( N \cap \Gamma_{R,i} \setminus N \) may be a nonstandard torus; \( N \) is coordinatized so that \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) as a 1-parameter group of rotations.

We define the function

\[
f_t \ast \phi_k(x, y) = \int_{N \cap \Gamma_{R,i} \setminus N} f_t(x - x', y - y') \phi_k(x', y') dx'dy'
\]

where the \((x', y')\) range over a fundamental domain for \( N \cap \Gamma_{R,i} \setminus N \).

We wish to show that \( f_t \ast \phi_k(x, y) \) converges uniformly to \( f_t \) for each \( t \) and, in fact, that \( f_t \ast \phi_k(x, y) \to f_t(x, y) = f(t, x, y) \) in the sup norm on \( M_{R,i} \).

First consider \( \sup_{\Gamma_{R,i}(t, x, y) \in M_{R,i}} |f_t - f_t \ast h_k| \). This is

\[
\sup |f_t(x, y) - \int_{N \cap \Gamma_{R,i} \setminus N} f_t(x' - x, y' - y) h_k(x', y') dx'dy'| = \sup |\int_{N \cap \Gamma_{R,i} \setminus N} f_t(x, y) - f_t(x' - x, y' - y) h_k(x', y') dx'(x', y')|.
\]

Since \( M_{R,i} \) is compact, \( f_t(x, y) \) is uniformly continuous in \((t, x, y)\). Let \( k \) be so large that \( |f_t(x, y) - f_t(x' - x, y' - y)| < \epsilon/2 \) on \( \text{supp} \{h_k\} \). Then clearly

\[
\sup_{\Gamma_{R,i}(t, x, y) \in M_{R,i}} \left| \int_{N \cap \Gamma_{R,i} \setminus N} [f_t(x, y) - f_t(x' - x, y' - y)] h_k(x', y') dx'(x', y') \right| < \frac{\epsilon}{2} \left| \int_{N \cap \Gamma_{R,i} \setminus N} h_k(x', y') dx'(x', y') \right| = \epsilon/2.
\]

On the other hand,

\[
\sup_{\Gamma_{R,i}(t, x, y) \in M_{R,i}} |f_t \ast h_k - f_t \ast \phi_k| \leq \sup_{t \in [0, 1]} \|f_t\|_\infty \|h_k - \phi_k\|_1 \text{ on } T^2
\]

\[
= \sup_{t \in [0, 1]} |f(t, x, y)| \|h_k - \phi_k\|_1
\]

\[
\leq \|f\|_\infty 1/k.
\]
Thus if $k$ is large enough, this term can also be made less than $\varepsilon/2$, so that $f_t * \phi_k \to f_t$ uniformly in the sup norm on $M_{R,i}$ as desired.

Define $\alpha_{\lambda,k} = \hat{h}_k(N_1, N_2)$ if $N_1^2 + N_2^2 = \lambda^2$. This is well defined since $h_k$ was initially rotation-invariant, and therefore $\hat{h}_k(N_1, N_2) = \hat{h}_k(M_1, M_2)$ if $N_1^2 + N_2^2 = m_1^2 + m_2^2$. Then we claim that, if we define $S_n$ as in (1.1), we have $S_n f(\Gamma_i(t, x, y)) = f_t * \phi_k(x, y)$, which as we have seen converges uniformly on $M_{R,i}$ to $f$. This is clear, since

$$f_t * \phi_k(x, y) = \sum_{|N_1, N_2| < R_k} \hat{h}_k(N_1, N_2) \hat{f}_t(N_1, N_2) \chi_{N_1, N_2}(x, y)$$

$$= \sum_{|N_1, N_2| < R_k} \alpha_{\lambda,k} f(t, \cdot, \cdot)^\wedge (N_1, N_2) \chi_{N_1, N_2}(x, y)$$

$$= \sum_{\lambda < R_k} \alpha_{\lambda,k} P_\lambda f(t, x, y).$$

This completes the proof of theorem 1.1.

The question of whether a similar Fejer theorem exists for $L^2(M_{H,k})$ arises. Theorem 1.4 demonstrates that no such theorem exists.

**Lemma 1.2.** ([Ru1], theorem 3.4.3). Suppose $\mu$ is a bounded Borel measure on $T^2$, and $\hat{\mu}$ (the Fourier-Stieltjes transform of $\mu$) has finite range $A = \{A_i \in \mathbb{C} : i \in 1, \ldots, n\}$. Then there are idempotent measures $\mu_i$, $i \in 1, \ldots, n$, such that $\mu = A_1 \mu_1 + \ldots + A_n \mu_n$.

Suppose that $S = \sum_{i=1}^N \alpha_i P_{\lambda_i}$, a bounded operator in $L^2$, maps $C(M_{H,k})$ to itself. Since each $P_{\lambda_i}$ is an orthogonal projection of $L^2(M_{H,k})$ onto a right $S_n$-invariant subspace of $L^2(M_{H,k})$, $g \cdot (P_{\lambda_i} f) = P_{\lambda_i} (g \cdot f)$, and so $g \cdot (S f) = S (g \cdot f)$ since $S$ is a finite sum of $P_{\lambda_i}$. If $S$ preserves continuity then by the closed graph theorem $S : C(M_{H,k}) \to C(M_{H,k})$ is bounded in the sup-norm ([Ri1]). If we define $S_0 : C(M_{H,k}) \to C$ by $S_0(f) = S f(\Gamma_k e)$, then $S_0$ is a continuous linear functional on $C(M_{H,k})$ with the sup-norm, and so by the Riesz-Markov-Kakutani
Theorem, there is a bounded $\Gamma$-biinvariant measure $\sigma_s$ on $M_{H,k}$ such that

$$S_0 f = \int_{M_{H,k}} f \, d\sigma_s.$$  \hfill (1.4)

Since $S$ is right $S_n$-invariant, we have

$$Sf(\Gamma g) = S_0(g \cdot f) = \int_{M_{H,k}} g \cdot f(\Gamma g_0) \, d\sigma_s(g_0) \quad \hfill (1.5)$$

and so the measure $\sigma_s$ determines the map $S$.

If we realize the $M_{H,k}$ so that the subgroup $\Gamma_k$ of $S_{H,k}$ has integral coordinates in $S_{H,k}$, then (from §5 in [Ri 1];

1. There exist one-parameter subgroups $d_1, d_2, d_3$ in $S_{H,k}$ such that $S_{H,k} = d_1(\mathbf{R})d_2(\mathbf{R})d_3(\mathbf{R})$, and $d_2(\mathbf{R})d_3(\mathbf{R}) = N$ is abelian and normal in $S_{H,k}$.

2. $\Gamma_k = d_1(\mathbf{Z})d_2(\mathbf{Z})d_3(\mathbf{Z})$, so that $F = d_1([0,1)d_2[0,1)d_3[0,1)$ is a fundamental domain for $M_{H,k}$.

We identity $F$ with $[0,1) \times [0,1) \times [0,1)$ via the map

$$(x_1, x_2, x_3) \mapsto d_1(x_1)d_2(x_2)d_3(x_3).$$

Note that $\Phi_F$ carries Borel sets to Borel sets, and so gives rise to an isometry

$$L^2(T^3) \longrightarrow L^2(M_{H,k})$$

and a map $\mu \longrightarrow \mu_F$ carrying Borel measures on $M_{H,k}$ to Borel measures on $T^3$.

**Lemma 1.3.** Suppose $\sigma_s$ corresponds to the map $S$. Then there is a bounded Borel measure $\sigma'$ on $N \cap \Gamma_k \setminus N$ such that $\sigma_s = \sigma' \times \delta_0$, where $\delta_0$ is the unit map at the identity of $d_1(\mathbf{R}) = \exp \mathbf{R}T$.

**Proof of Lemma 1.3.** Let $\phi(m,n,p) \in L^2(M_{H,k})$ be such that

$$\phi(m,n,p) \circ \Phi_F = \exp 2\pi i(mt + nx + py).$$
By Cor 3.5 in [Ri 1], \( \hat{\sigma}_{S\phi}(m,n,p) = S(\phi(m,n,p))(0) \). We see that
\[ S(\phi(m,n,p))(d_3(y)d_2(x)d_1(t)) = \exp(2\pi i m t)S(\phi(0,n,p))(d_3(y)d_2(x)d_1(t)) \]
almost everywhere, since \( S \) is a finite sum of \( P_{\lambda} \), which behave like \( \delta_0 \) in the \( t \)-variable.

Theorem 3.6 in [Ri1] then implies that if
\[ S\phi(m,n,p)(d_3(0)d_2(0)d_1(0)) \neq S\phi(0,n,p)(d_3(0)d_2(0)d_1(0)) \]
then \( S(\phi(m,n,p)) \) and \( e^{2\pi i m t}S(\phi(0,n,p)) \) must differ on a set of positive measure, so that we have \( \hat{\sigma}_{S\phi}(m,n,p) = \hat{\sigma}_{S\phi}(0,n,p) \). Thus \( \hat{\sigma}_{S\phi} \) is independent of \( m \), and so \( \sigma_{S\phi} \) (and hence \( \sigma_F \)) may be decomposed into a cartesian product measure \( \sigma_\tau = \sigma' \times \sigma_0 \) as required. This completes the proof of Lemma D.

If \((0,m,p)\) is in the \( \text{Ad}^\ast \)-orbit of \( \pi_n, O_{\pi_n} \), then \( S\phi(0,m,p)(d_3(0)d_2(0)d_1(0)) = \sum_{i=1}^{K} \alpha_i P_{\lambda_i} \phi(0,m,p)(d_3(0)d_2(0)d_1(0)) = \alpha_N P_{\lambda_1} \phi(0,m,p)(d_3(0)d_2(0)d_1(0)) = \alpha_N \phi(0,m,p)(d_3(0)d_2(0)d_1(0)) = 1 \cdot \alpha_n \), since \( \phi(0,m,p) \) is in the \( \pi_n \)-primary summand. Therefore for all \((0,m,p) \in O_{\pi_n} \),
\[ \hat{\sigma}'_F(0,m,p) = \alpha_n, \text{ so } \hat{\sigma}'_F \text{ has finite range.} \]

By Lemma C, then, we have
\[ \sigma' = \sum_{i=1}^{K} \alpha_i \mu_i, \]
where the \( \mu_i \) are idempotent measures. But the supports of each \( \hat{\mu}_i \) do not lie in the coset ring of \((N \cap \Gamma_k \setminus N)^\wedge\), a contradiction ([Ru1]). This proves the following:

**Theorem 1.4.** Let \( S : L^2(M_{H,k}) \rightarrow L^2(M_{H,k}) \) be of the form \( S = \sum_{i=1}^{k} \alpha_i P_{\lambda_i} \), for some set \( \{P_{\lambda_i}\}_{i=1}^{k} \) on orthogonal projections onto primary summands.

Then there exists \( f \in C(M_{H,k}) \) such that \( Sf \) is not continuous; therefore \( S \) does not preserve continuity.

Thus there exists no Fejer theorem on \( M_{H,k} \); i.e., there is no sequence of operators
\[ S_n = \sum_{k=1}^{M} \alpha_{n,k} P_{\pi_n,k}, \quad n \in \mathbb{N}. \]
such that $S_n$ preserves continuity of functions in $L^2(M_{H,k})$, and such that for each continuous $f$, $S_n f$ converges uniformly to $f$.

§1.2. A Sidon theorem for primary summand functions in $H_\pi \subset L^2(M_{H,k})$.

Theorem 1.3 is an adaptation of Sidon's theorem on absolute convergence of lacunary Fourier series (see [Zyg], Thm. 6.6.1).

Suppose $S_{H,k} = \mathbb{R} \ltimes \mathbb{R}^2$ is coordinatized so that $\mathbb{R}$ acts on $\mathbb{R}^2$ via the 1-parameter subgroup $\sigma_H(t) = \begin{bmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{bmatrix}$. Then the coadjoint orbits will be "hyperbolic cylinders", saturated in the $t$-direction, given by the equations

$$xy = k, \quad k \in \mathbb{R}. \quad (2.1)$$

Let $\pi \in (\Gamma_{H,k} \setminus S_{H,k})^\wedge$ be an infinite-dimensional representation. Let

$P_\pi : L^2(\Gamma_{H,k} \setminus S_{H,k}) \to L^2(\Gamma_{H,k} \setminus S_{H,k})$ be the orthogonal projection onto the $\pi$-primary summand of $L^2$; $P_\pi$ does not preserve continuity of functions [Ril1]. Let $(\alpha, \beta)$ be a fixed lattice point in the coadjoint orbit $O_\pi$, lying in the plane $RX^* + RY^*$, noting that with the chosen coordinatization of $S_{H,\lambda}$, the torus $N \cap \Gamma_{H,k} \setminus N$ will be a nonstandard torus, and so $(\alpha, \beta) \in O_\pi$ satisfying $\chi_{(\alpha, \beta)}(N \cap \Gamma_{H,k}) = 1$ will not have integer coordinates. We will call the set of $(\alpha', \beta')$ satisfying $\chi_{(\alpha', \beta')} (N \cap \Gamma_{H,k}) = 1$ the lattice $L^*$. Let

$\text{mul}(\pi)$ be the multiplicity of $\pi$ in the $\pi$-primary summand. Let $T = \{t_i\}_{i=1}^{\text{mul}(\pi)}$ be a set in $[0,1)$ such that $(\alpha_i, \beta_i) = \sigma(t_i)(\alpha, \beta)$ are representatives of the complete set of disjoint $\Gamma_{H,k}$-orbits in $L^* \cap O_\pi$.

Let $\{p_i\}_{i \in \mathbb{Z}^+}$ be the sequence of positive numbers of the form $n + t_i$, $(t_i \in T, \ n \in \mathbb{Z})$, arranged in increasing order; $p_i < p_{i+1}$. Let $\{q_i\}_{i \in \mathbb{Z}^+}$ be the sequence of negative numbers of that form, with $q_{i+1} < q_i$. Note that $\{p_i\}_{i \in \mathbb{Z}^+} \cup \{q_i\}_{i \in \mathbb{Z}^+}$ is a discrete set in $\mathbb{R}$, and that $q = \min_{i \in \mathbb{Z}} |p_{i+1} - p_i|, |q_{i+1} - q_i| > 0$. Note also that
since $\lambda > 1$, $\lambda^2 > 1$. Let $P_\pi$ be orthogonal projection onto a $\pi$-primary summand $H_\pi$.

**Theorem 1.3.** Suppose $f \in P_\pi(L^2(\Gamma_{H,k} \setminus S_{H,k}))$, $\pi \in (\Gamma_{H,k} \setminus S_{H,k})^\wedge$. If $f$ is in $L^\infty(\Gamma_{H,k} \setminus S_{H,k})$, then for almost all fixed $t = t_0$, we have

$$f^+_t(x,y) = f^+(t_0,x,y) = \sum_{i \in \mathbb{Z}^+} f(t_0, \cdot, \cdot)^\wedge(\sigma(p_i)(\alpha, \beta))\chi_{\sigma(p_i)(\alpha, \beta)}(x,y) \quad (2.1)$$

and

$$f^-_t(x,y) = f^-(t_0,x,y) = \sum_{i \in \mathbb{Z}^-} f(t_0, \cdot, \cdot)^\wedge(\sigma(q_i)(\alpha, \beta))\chi_{\sigma(p_i)(\alpha, \beta)}(x,y) \quad (2.2)$$

absolutely and uniformly convergent to $f^+(t_0,x,y)$ and $f^-(t_0,x,y)$, where $f^+ + f^- = f$.

**Proof.** First we make a few simplifying assumptions. We suppose without loss of generality that $\pi^* \cap \mathcal{O}_\pi$ is a first quadrant curve; and we may discard finitely many terms in (2.1) and (2.2) without affecting the result.

A sequence $\{\lambda_i\}_{i \in \mathbb{N}}$ of positive real numbers is said to be lacunary if $\lambda_{i+1}/\lambda_i > \mu$ for some real $\mu > 1$, for all $i$.

**Lemma 1.4.** The sequence $\{||\sigma(p_i)(\alpha, \beta)||\}_{i > N}$ of norms is lacunary for $N$ large enough.

**Proof of Lemma 1.4.** we have

$$\frac{||\sigma(P_{i+1})(\alpha, \beta)||^2}{||\sigma(P_{i})(\alpha, \beta)||^2} = \frac{\lambda^{2P_{i+1}} \alpha^2 + \lambda^{-2P_{i+1}} \beta^2}{\lambda^{2P_{i}} \alpha^2 + \lambda^{-2P_{i}} \beta^2} \geq \frac{\lambda^{2P_{i+1}} \alpha^2}{\lambda^{2P_{i}} \alpha^2 + 1} \quad \text{if } i \text{ is large enough.} \quad (2.3)$$

This last term is $\lambda^{2(P_{i+1}-P_i)}_{1+1/\lambda^{2P_i} \alpha^2}$. Since $1/\lambda^{2P_i} \alpha^2 \to 0$ as $i \to \infty$, we have, for some $\mu > 1$, $\lambda^{2(P_{i+1}-P_i)}_{1+1/\lambda^{2P_i} \alpha^2} > \mu^2 > 1$ (since $|P_{i+1} - P_i|$ is bounded away from zero).
Thus for large enough \( i_0 \) we have \( \frac{\|\sigma(P_{i+1})(\alpha, \beta)\|}{\|\sigma(P_i)(\alpha, \beta)\|} > \mu > 1 \) for some constant \( \mu, \ i > i_0 \). This finishes the proof of Lemma 2.2. Note that the same argument shows that \( \|\sigma(q_j)(\alpha, \beta)\| \) is lacunary, for large enough \( j > j_0 \). We begin by discarding those terms \( f(t_0, \cdot, \cdot) (\sigma(p_i)(\alpha, \beta)) \) and \( f(t_0, \cdot, \cdot) (\sigma(q_j)(\alpha, \beta)) \) for which \( i < i_0, \ j < j_0 \).

Note also that the line \( x = y \) bisects the first quadrant in \( n^* \). Since \( \mathcal{O}_\pi \cap n^* \) is a 1st quadrant hyperbola, there is an \( i' \) such that the sets \( \{\sigma(q_i)(\alpha, \beta)\} \) and \( \{\sigma(p_i)(\alpha, \beta)\} \) are on opposite sides of this line if \( i > i' \). We discard also those terms for which \( i < i' \).

Let \( r \in \mathbb{Z}^+ \), and let \( s \in \{0, 1, \ldots, r-1\} \) be fixed. We define

\[
P^{(s)}_{\ell}(x, y) = \prod_{k=1}^{\ell} \left( 1 + \frac{\varepsilon_k}{2} \chi_{\sigma(P_{kr+s})(\alpha, \beta)}(x, y) + \frac{\bar{\varepsilon}_k}{2} \chi_{-\sigma(P_{kr+s})(\alpha, \beta)}(x, y) \right) \tag{2.4}
\]

where \( \{\varepsilon_i\}_{i=1}^{\ell} \) are complex numbers of modulus 1, and \( \chi_{\sigma(P_{kr+s})(\alpha, \beta)} = \exp 2\pi i (\sigma(P_{kr+s})(\alpha, \beta))(x, y) \). If we define \( \theta(\varepsilon_i) \in \mathbb{R} \) so that \( e^{2\pi i \theta(\varepsilon_i)} = \varepsilon_i \), then

\[
\varepsilon_k \chi_{\sigma(P_{kr+s})(\alpha, \beta)} = \exp 2\pi i (\sigma(P_{kr+s})(\alpha, \beta) + \theta(\varepsilon_i)). \tag{2.5}
\]

Thus we have

\[
1 + \frac{\varepsilon_k}{2} \chi_{\sigma(P_{kr+s})(\alpha, \beta)} + \frac{\bar{\varepsilon}_k}{2} \chi_{-\sigma(P_{kr+s})(\alpha, \beta)} = 1 + \cos[2\pi \sigma(P_{kr+s})(\alpha, \beta)(x, y) + \theta(\varepsilon_i)] > 0. \tag{2.6}
\]

Therefore, \( |P^{(s)}_{\ell}(x, y)| = P^{(s)}_{\ell}(x, y) \) for each \( i, s \). Assume for now that \( \varepsilon_i = 1 \) for all \( i \). Multiplying \( P^{(s)}_{\ell}(x, y) \) out gives the expression

\[
1 + \frac{1}{2} \sum_{k=1}^{\ell} \left[ \chi_{\sigma(P_{kr+s})(\alpha, \beta)} + \chi_{-\sigma(P_{kr+s})(\alpha, \beta)} \right] + \left( \frac{1}{2} \right)^2 \sum_{i,j=1}^{\ell} \left[ \chi_{\sigma(P_{ir+s})(\alpha, \beta)} + \chi_{-\sigma(P_{ir+s})(\alpha, \beta)} \right] \left[ \chi_{\sigma(P_{jr+s})(\alpha, \beta)} + \chi_{-\sigma(P_{jr+s})(\alpha, \beta)} \right] \]

\[
+ \ldots + \left( \frac{1}{2} \right)^{\ell} \sum_{i=1}^{\ell} \left[ \chi_{\sigma(P_{ir+s})(\alpha, \beta)} + \chi_{-\sigma(P_{ir+s})(\alpha, \beta)} \right]. \tag{2.7}
\]
When multiplied out these terms yield expressions of two types.

For each subset \( \{n_k > n_{k-1} > \ldots > n_1\} \) of the set \( \{1, \ldots, \ell\} \), we have a term of the type

\[
\left( \frac{1}{2} \right)^k \chi[\sigma(P_{n_k r+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n_1 r+s})(\alpha, \beta)]
\]

(2.8)

and one of the type

\[
\left( \frac{1}{2} \right)^k \chi[-\sigma(P_{n_k r+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n_1 r+s})(\alpha, \beta)]
\]

(2.9)

Note that the signs of the leading \((n_k)\)-terms differ. Note also that each term

\( \pm \sigma(P_{n_k r+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n_1 r+s})(\alpha, \beta) \)

is in \( \mathcal{L}^* \), the lattice subgroup of \( \mathfrak{n}^* \) dual to \( \mathcal{L} = \Gamma_{H,k} \cap N \setminus N \), and that \( \chi(\alpha, \beta) \perp \chi(\alpha_2, \beta_2) \) if \( (\alpha_1, \beta_1) \neq (\alpha_2, \beta_2) \).

**Lemma 1.5.** Let \( \{n_k > \ldots > n_1\} \) be a subset of \( \{1, \ldots, i\} \). For \( r \) large enough (and depending only upon \( \mu \)) we have

\[
\sigma(P_{n_k r+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n_1 r+s})(\alpha, \beta) = \sigma(P_t)(\alpha, \beta), \quad t \in \mathbb{Z},
\]

if and only if \( k = 1 \) and \( P_{n_1 r+s} = P_t \).

**Proof.** First we prove 2 facts.

1. There exists \( \varepsilon > 0 \) such that the intervals \( (\|\sigma(P_i)(\alpha, \beta)\| \cdot (1 - \varepsilon), \|\sigma(P_i)(\alpha, \beta)\|(1 + \varepsilon)) \) are disjoint in \( \mathbb{R} \), for all \( i \).

   If we choose \( \varepsilon < \frac{\mu - 1}{\mu + 1} \), then

   \[
   \frac{(1 + \varepsilon)}{(1 - \varepsilon)} < \mu < \frac{\|\sigma(P_{i+1})(\alpha, \beta)\|}{\|\sigma(P_i)(\alpha, \beta)\|}.
   \]

   (2.10)

   Thus we have

   \[
   \|\sigma(P_i)(\alpha, \beta)\|(1 + \varepsilon) < \|\sigma(P_{i+1})(\alpha, \beta)\||(1 - \varepsilon), \quad \text{proving 1.}
   \]

2. \( R \) may be chosen large enough so that

\[
\|\sigma(P_{n_k r+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n_1 r+s})(\alpha, \beta)\| \in (\|\sigma(P_{n_k r+s})(\alpha, \beta)\|(1 - \varepsilon), \|\sigma(P_{n_k r+s})(\alpha, \beta)\||(1 + \varepsilon)).
\]
We pick \( r \) large enough that
\[
\left( \frac{\mu^r - 2}{\mu^r - 1}, \frac{\mu^r}{\mu^r - 1} \right) \subseteq (1 - \varepsilon, 1 + \varepsilon).
\] (2.11)
then we have
\[
\|\sigma(P_{nkr+s})(\alpha, \beta)\| - \ldots - \|\sigma(P_{n1r+s})(\alpha, \beta)\|
\]
\[
< \|\sigma(P_{nkr+s})(\alpha, \beta)\| \pm \ldots \pm \|\sigma(P_{n1r+s})(\alpha, \beta)\| \quad (2.12)
\]
< \|\sigma(P_{nkr+s})(\alpha, \beta)\| + \ldots + \|\sigma(P_{n1r+s})(\alpha, \beta)\|

Since by lacunarity we have
\[
\|\sigma(P_{nkr+s})(\alpha, \beta)\| > \mu^{(nk-n_i)r}\|\sigma(P_{n1r+s})(\alpha, \beta)\|, \quad (2.13)
\]
and since \( n_k - n_i > k - i \), we can state that
\[
\|\sigma(P_{nkr+s})(\alpha, \beta)\|(1 - \mu^{-r} - \mu^{-2r} - \ldots - \mu^{-kr})
\]
\[
< \|\sigma(P_{nkr+s})(\alpha, \beta)\| \pm \ldots \pm \|\sigma(P_{nkr+s})(\alpha, \beta)\| \quad (2.14)
\]
< \|\sigma(P_{nkr+s})(\alpha, \beta)\|(1 + \mu^{-r} + \ldots + \mu^{-kr})

Since \( 1 - \mu^{-r} - \ldots - \mu^{-kr} > \frac{\mu^r - 2}{\mu^r - 1} \), and \( 1 + \mu^{-r} + \ldots + \mu^{kr} < \frac{\mu^r}{\mu^r - 1} \), we have the desired result for Claim 2.

We proceed to finish the proof of Lemma 1.5. By Claim 1, we have that
\( \sigma(P_{nkr+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n1r+s})(\alpha, \beta) \neq \sigma(P_t)(\alpha, \beta) \) if \( t \neq n_kr + s \). If \( t = n_kr + s \), this implies that \( \pm \sigma(P_{nkr+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n1r+s})(\alpha, \beta) = 0 \), an impossibility, by Claim 2, unless \( k = 1 \) and \( P_t = P_{n1r+s} \).

This completes the proof of Lemma 1.5.

Lemma 1.6.
\[
\sigma(P_{nkr+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n1r+s})(\alpha, \beta) \neq \sigma(q_j)(\alpha, \beta)
\]
for any \( q_j < 0 \).

Proof. The main idea is as follows; if
\( \sigma(P_{nkr+s})(\alpha, \beta) \pm \ldots \pm \sigma(P_{n1r+s})(\alpha, \beta) = \sigma(q_j)(\alpha, \beta) \) for some \( j \), then
\[ \| \sigma(P_{nkr+s})(\alpha, \beta) - \sigma(q_j)(\alpha, \beta) \| = \| \sigma(P_{nkr+s})(\alpha, \beta) - \sigma(P_{nkr+s})(\alpha, \beta) \| + \cdots \]

\[ \pm \sigma(P_{nkr+s})(\alpha, \beta) \| \]  

must be greater than the minimal distance from \( \sigma(P_{nkr+s})(\alpha, \beta) \) to the line \( y = x \), since \( \sigma(P_{nkr+s})(\alpha, \beta) \) and \( \sigma(q_j)(\alpha, \beta) \) lie on opposite sides of \( y = x \).

We show that for \( R \) large enough this condition cannot hold.

The minimal distance from \( \sigma(P_{nkr+s})(\alpha, \beta) \) to the line \( y = x \) is

\[ \frac{1}{\sqrt{2}} | \lambda^{2(P_{nkr+s})} \alpha - \lambda^{-2(P_{nkr+s})} \beta | , \]

as is easily verified.

Let \( M = \inf_{(a, b) \in \mathcal{L}^* \cap \mathcal{O}_*} \| (a, b) \| . \)

This is positive since \( \mathcal{L}^* \) is discrete.

Let \( R \) be large enough so that \( \mu^2 R < \left( \frac{2 \alpha \beta}{M^2} \right)^2 \); clearly \( R \) may be chosen so. Since \( \mu^2 R < \left( \frac{\| \sigma(P_{nkr+s})(\alpha, \beta) \|}{\| \sigma(P_{nkr+s})(\alpha, \beta) \|} \right)^2 \), and \( \frac{2 \alpha \beta}{M^2} > \left( \frac{2 \alpha \beta}{\| \sigma(P_{nkr+s})(\alpha, \beta) \|} \right)^2 \), we have

\[ \frac{\| \sigma(P_{nkr+s})(\alpha, \beta) \|^2 - 2 \alpha \beta}{\| \sigma(P_{nkr+s})(\alpha, \beta) \|^2} > \mu^2 R - \frac{2 \alpha \beta}{M^2} > 2 \left( \frac{\mu^2 R - 1}{M^2} \right) . \]

(2.15)

Since \( \| \sigma(P_{nkr+s})(\alpha, \beta) \|^2 - 2 \alpha \beta = | \lambda^{P_{nkr+s}} \alpha - \lambda^{-P_{nkr+s}} \beta |^2 \) we have (from 2.15) that

\[ \frac{1}{\sqrt{2}} | \lambda^{P_{nkr+s}} \alpha - \lambda^{-P_{nkr+s}} \beta | > \left( \frac{\mu^2 R - 1}{M^2} \right) \]

\[ \| \sigma(P_{nkr+s})(\alpha, \beta) \| \pm \cdots \pm \| \sigma(P_{nkr+s})(\alpha, \beta) \| \]

(2.16)

a contradiction. This concludes Lemma 1.6.

We have demonstrated thus far that

\[ \chi \sigma(P_{nkr+s})(\alpha, \beta) \pm \cdots \pm \chi \sigma(P_{nkr+s})(\alpha, \beta) \]

(2.17)

unless \( k = 1 \) and \( P_t = P_{nkr+s} \), and also that

\[ \chi \sigma(P_{nkr+s})(\alpha, \beta) \pm \cdots \pm \chi \sigma(q_i)(\alpha, \beta) \]

(2.18)

for all \( q_i \) under consideration.
We next examine terms appearing in $P^{(s)}_i$ of the form (2.9), i.e.

$$X - \sigma(P_{n_k r+s})(\alpha, \beta) \pm \cdots \pm \sigma(P_{n_1 r+s})(\alpha, \beta)$$

since $(\alpha, \beta)$ and therefore $\sigma(P_{n_k r+s})(\alpha, \beta)$ are first quadrant points, the leading term $-\sigma(P_{n_k r+s})(\alpha, \beta)$ is a third quadrant point.

For such points $(a, b)$, the minimum distance to the first quadrant is $\|(a, b)\|$. 

**Lemma 1.7.** For $R$ large enough, and dependent only upon

$$\mu, -\sigma(P_{n_k r+s})(\alpha, \beta) \pm \cdots \pm \sigma(P_{n_1 r+s})(\alpha, \beta)$$

cannot be a first quadrant point and so cannot be in $O \cap L^*$. 

**Proof.** Suppose $(\alpha', \beta') = -\sigma(P_{n_k r+s})(\alpha, \beta) \pm \cdots \pm \sigma(P_{n_1 r+s})(\alpha, \beta)$ is in the first quadrant. Then the distance from $(\alpha', \beta')$ to the point

$$-\sigma(P_{n_k r+s})(\alpha, \beta)$$

must exceed $\|\sigma(P_{n_k r+s})(\alpha, \beta)\|$, the minimum distance from $-\sigma(P_{n_k r+s})(\alpha, \beta)$ to the first quadrant. Note that

$$\|(\alpha', \beta') - (-\sigma(P_{n_k r+s})(\alpha, \beta))\|$$

$$= \|\pm \sigma(P_{n_k r-s})(\alpha, \beta) \pm \cdots \pm \sigma(P_{n_1 r+s})(\alpha, \beta)\|$$

$$\leq \|\sigma(P_{n_k r-s})(\alpha, \beta)\| \left(\frac{\mu R}{\mu R - 1}\right)$$

$$< \|\sigma(P_{n_k r+s})(\alpha, \beta)\| \mu^{-(n_k - n_{k-1}) R} \cdot \frac{\mu^R}{\mu R - 1}.$$ 

(2.19)

Since $n_k - n_{k-1} \geq 1$, we have

$$\|\sigma(P_{n_k r+s})(\alpha, \beta)\| \mu^{-(n_k - n_{k-1}) R} \cdot \frac{\mu^R}{\mu R - 1} < \|\sigma(P_{n_k r+s})(\alpha, \beta)\| \cdot \frac{1}{\mu R - 1}. $$

(2.20)

If $\frac{1}{\mu R - 1} < \delta < 1$, then we see that $(\alpha', \beta')$ cannot possibly be a first quadrant point, proving Lemma 1.7.

Referring to Lemmas 1.5-1.7 and to expressions 2.7-2.9, we see that the only terms in $P^{(s)}_i$ that are not orthogonal to the function $f_{t_0}(x, y)$ are the terms

$$\sum_{i=1}^{\ell} \frac{\varepsilon_i}{2} \chi_{\sigma(P_{n_k r+s})(\alpha, \beta)}.$$ 

(2.21)
if $R$ is chosen large enough (here we drop the assumption that $\epsilon_i = 1$ for all $i$.)

Let $A_{jr+s} = f(t_0, ' , ' , ^\wedge (\sigma(P_{jr+s})(\alpha, \beta)) \subset nC$. Let $\epsilon_j$ be such that $\epsilon_j \cdot A_{jr+s} = |A_{jr+s}|$. Recall from (2.4) that $|P_\ell(s)(x, y)| = P_\ell(s)(x, y)$. We have on the one hand that for almost all $t_0$,

$$|\int_{T^2} f(t_0, x, y)P_\ell(s)(x, y)dx \, dy| \leq \|f\|_\infty \int_{T^2} P_\ell(s)(x, y)dx \, dy$$

$$\leq \|f\|_\infty$$

(2.22)

since $\int_{T^2} P_\ell(s)(x, y)dx \, dy = \int_{T^2} dx \, dy = 1$.

On the other hand,

$$\int_{T^2} f(t_0, x, y)P_\ell(s)(x, y)dx \, dy = \int_{T^2} f(t_0, x, y) \sum_{i=1}^{\ell} \frac{\epsilon_i}{2} \chi_{\sigma(P_{jr+s})(\alpha, \beta)}$$

$$= \frac{1}{2} \sum_{j=1}^{\ell} |A_{jr+s}|.$$  

(2.23)

Since for all $i$ we have

$$\sum_{j=1}^{i} |A_{jr+s}| \leq 2\|f\|_\infty,$$

(2.24)

by summing (2.24) over all values of $s$ from 0 through $r - 1$, we have that

$$\sum_{j \in \mathbb{Z}^+} |A_j| = \sum_{j \in \mathbb{Z}^+} |f(t_0, ' , ' , ^\wedge (\sigma(P_j)(\alpha, \beta))|$$

$$< 2r\|f\|_\infty,$$  

for almost all $t_0$.

This shows that over one leg of the hyperbola $O_\pi$, the function $f_{t_0}$ converges absolutely and uniformly; clearly the same proof may be used to show that for almost all $t_0$,

$$\sum_{j \in \mathbb{Z}^+} |f(t_0, ' , ' , ^\wedge (\sigma(q_j)(\alpha, \beta))| < 2r\|f\|_\infty.$$  

(2.26)

This completes the proof of Theorem 1.3.

**Corollary 1.8.** Suppose $\pi \in (\Gamma_{H,k} \setminus S_{H,k})^\wedge$, and that $f \in L^2(\Gamma_{H,k} \setminus S_{H,k})$ is continuous on $\Gamma_{H,k} \setminus S_{H,k}$. If $P_\pi f$ is $L^\infty$, then $P_\pi f$ is continuous.

**Proof.** Let $R = \{(t, 0, 0) \in S_{H,k} : t \in \mathbb{R}\}$. Then $R$ is a subgroup of $S_{H,k}$. 


Suppose \( f \in L^2(\Gamma_{H,k}\backslash S_{H,k}) \) is continuous. Then for \( (\alpha, \beta) \in \mathcal{L}^\ast \), we have

\[
f(t, \cdot, \cdot)^\wedge(\alpha, \beta) = \int_{T^2} f(t, x, y) \chi_{-(\alpha, \beta)}(x, y) \, dx \, dy
\]

continuous on \( R \).

If \( P_{\pi} f \) is in \( L^\infty \), then by Theorem 1.3,

\[
\sum_{(\alpha, \beta) \in \mathcal{L}_{\pi} \cap \mathcal{L}^*} |f(t, \cdot, \cdot)^\wedge(\alpha, \beta)| < K \|f\|_\infty
\]

where \( K \) is a constant depending on the lacunary constant \( \mu \).

Since the inequality is independent of \( t \), we have that

\[
P_{\pi} f = \sum_{(\alpha, \beta) \in \mathcal{L}_{\pi} \cap \mathcal{L}^*} f(t, \cdot, \cdot)^\wedge(\alpha, \beta) \chi_{-(\alpha, \beta)}(x, y)
\]

(2.28)

is the uniformly convergent sum of functions continuous on \( S_{H,k} \). Since \( P_{\pi} f \) also possesses left \( \Gamma_{H,k} \)-invariance, \( P_{\pi} f \in C(\Gamma_{H,k}\backslash S_{H,k}) \).

**Corollary 1.9.** There exists \( f \), continuous on \( M_{H,k} \), such that \( P_{\pi} f \) is essentially unbounded.

**Proof.** Example 5.3 in [Ri1] shows that for each orthogonal projection \( P_{\pi} \), there must exist an \( f \in C(M_{H,k}) \) such that \( P_{\pi} f \) is not continuous. By Theorem 1.3, however, \( P_{\pi} f \) must then be essentially unbounded.

**Corollary 1.10.** If \( f \in P_{\pi}(L^2(\Gamma_{H,k}\backslash S_{H,k})) \) is \( L^\infty \), then for a.e. fixed \( t_0 \) we have that

\[
f(t_0, \cdot, \cdot): N \cap \Gamma_{H,k} \backslash N \rightarrow \mathbb{C}
\]

is a continuous function on \( N \cap \Gamma_{H,k} \backslash N \cong T^2 \).

**Proof.** By Theorem 1.3, we have that for \( f \in P_{\pi}(L^2(\Gamma_{H,k}\backslash S_{H,k})) \) essentially bounded, the inequality in (2.27) holds, for a.a. \( t_0 \).

Thus for a.a. fixed \( t_0 \),

\[
f(\Gamma_{H,k}(t_0, x, y)) = \sum f(t_0, \cdot, \cdot)^\wedge(\alpha, \beta) \chi_{(\alpha, \beta)}(x, y)
\]
is a uniformly convergent sum of functions that are continuous on $N \cap \Gamma_{H,k} \setminus N$, and so is itself continuous on $N \cap \Gamma_{H,k} \setminus N$. 
SECTION 2

ZEROS OF FUNCTIONS ON SOLVMANIFOLDS

§2.1. Zeros of continuous functions in $H_\pi$ of a compact nilmanifold

Theorem 2.0. Let $N$ be a nilpotent Lie group with cocompact discrete subgroup $\Gamma$, $\Gamma \backslash N$ not isomorphic to $T^n$ for any $n \in \mathbb{N}$. If $f$ is a continuous function in $H_\pi \subseteq L^2(\Gamma \backslash N)$ for $\pi \in (\Gamma \backslash N)^\wedge_\infty$, then $f$ has at least one zero on $\Gamma \backslash N$.

Proof. We proceed by induction on $\dim N$.

We begin with $\dim N = 3$, where the 3-dimensional Heisenberg group $H_3$ is the only example of a nilpotent group with quotient manifolds that are not isomorphic to $T^3$. Theorem 2.1 for this case was proved by L. Auslander and R. Tolimieri in ([A-T], Thm. II.2.).

Lemma 2.1. Let $\Gamma'$ be a uniform subgroup of $H_3$. Then if $\Gamma = \{(p,m,n) \in H_3 : p,m,n \in \mathbb{Z}\}$, $\Gamma'$ contains a subgroup isomorphic to $\Gamma$, and its index in $\Gamma'$ is finite.

This lemma follows immediately from the results of A. I. Malcev in [Mal].

We are given $\Gamma \backslash N$ compact, and the map

$$\Phi : L^2(\Gamma' \backslash N) \rightarrow L^2(\Gamma \backslash N)$$

defined by $\Phi f(\Gamma' x) = f(\Gamma' x)$ is a well-defined, $N$-equivariant isometry of $L^2(\Gamma' \backslash N)$ with its image in $L^2(\Gamma \backslash N)$.

Suppose $\pi \in (\Gamma' \backslash N)^\wedge_\infty$ and that $f$ is a continuous function in $H_\pi \subseteq L^2(\Gamma' \backslash N)$.
Then $\Phi f$ is continuous in $L^2(\Gamma \backslash N)$. Since $\Phi$ is an $N$-equivariant isometry, $\Phi(H_\pi)$ is contained in the $\pi$-primary summand of $L^2(\Gamma \backslash N)$. By Theorem II.2 in [A-T], then, $\Phi f$ must have a zero. This completes the first step of the induction.

Suppose that the Lie algebra center $z(n)$ has a nontrivial subspace on which $\lambda_\pi$ is zero, where the character $\chi_{\lambda_\pi}$ induces to $\pi$; then $k = z(n) \cap \ker \lambda_\pi$ is a nonzero, rational subspace of $n$, and if $K = \exp k$, then functions in $H_\pi$ are $K$-invariant. Therefore $\pi$ is actually a representation of a lower dimensional group $\overline{N} = N/K$, $H_\pi$ may be imbedded in $L^2(\overline{\Gamma} \backslash \overline{N})$ where $\overline{\Gamma}$ is the image in $\overline{N}$ of $\Gamma$, and thus continuous functions in $H_\pi$ must have zeros by the induction hypothesis.

Therefore we suppose that $z(n)$ is 1-dimensional, and that $\chi_\pi$ inducing $\pi$ is nontrivial on $z(n)$.

Suppose $\{X_1, \ldots, X_n\}$ is a strong Malcev basis through $z(n)$, such that $z(n) = \mathbb{R}X_n$, and such that

$$
\Gamma = \exp Z \cdot X_n \cdot \exp ZX_{n-1} \cdots \exp ZX_1.
$$

(see [C-G], Thm. V.1.6.).

Suppose $F$ is a continuous, nonvanishing function in $H_\pi$. Then $F(x_1, \ldots, x_n) = \exp 2\pi ip x_n F(x_1, \ldots, x_{n-1}, 0)$, since $\pi \in (\Gamma \backslash N)^\wedge$, $p \in \mathbb{Z}$. Consider the function

$$
G(x_1, \ldots, x_n) = \frac{F(x_1, \ldots, x_n)}{|F(x_1, \ldots, x_n)|}.
$$

This function is continuous and nonvanishing on $\Gamma \backslash N$, and possesses the same $Z(N)$-covariance as $F$. Let $\Gamma_p$ be defined as follows;

$$
\Gamma_p = \exp \frac{Z}{p}X_n \cdot \exp ZX_{n-1} \cdot \exp ZX_{n-2} \cdots \exp ZX_1.
$$

(2.2)

Since $F$ is left $\Gamma_p$-invariant, so is $G$, and both are defined on $\Gamma_p \backslash N$; note $\Gamma_p$ is uniform in $N$, since $\Gamma \subseteq \Gamma_p$. Let

$$
\mu : N \longrightarrow Z(N) \backslash N
$$
be the natural map, and let $\tilde{N}$, $\tilde{\Gamma}_p$ be the images of $N$ and $\Gamma_p$ under $\mu$.

Define
\[
\Omega : \Gamma_p\backslash N \rightarrow \tilde{\Gamma}_p\backslash \tilde{N} \times T 
\]
by
\[
\Gamma_p(x_1, \ldots, x_n) \mapsto (\tilde{\Gamma}_p(x_1, \ldots, x_{n-1}), G(x_1, \ldots, x_n)). \tag{2.3}
\]
Then $\Omega$ is continuous on $\Gamma_p\backslash N$ since $G$ is; it is $1-1$ since $G$ takes on the value 1 exactly once on every fiber over $\tilde{\Gamma}_p\backslash \tilde{N}$. $\Omega$ is clearly onto, and since $\Gamma_p\backslash N$ is compact, $\Omega$ is a homeomorphism of $\Gamma_p\backslash N$ and $\tilde{\Gamma}_p\backslash \tilde{N} \times T$ (note: $\tilde{\Gamma}_p\backslash \tilde{N}$ is compact since $Z(N)$ is a rational subgroup).

However, if $\Gamma_p$ is a $k$-step nilpotent group, then $\tilde{\Gamma}_p \times Z$ is a $k-1$-step nilpotent group (recall that $\Gamma_p$ is not actually abelian). Therefore, since these groups are respectively the fundamental groups of $\Gamma_p\backslash N$ and $\tilde{\Gamma}_p\backslash \tilde{N} \times T$, we have a contradiction.

§2.2. Homotopy classes of functions on solvmanifolds

In section 2.3 we use homotopy classes of functions from solvmanifolds to the circle to show that $\pi$-primary summand functions which are continuous must have zeros.

In this section, we demonstrate that functions nonvanishing on the solvmanifolds $M_{R,p}$, $p = 2, 3$, and $M_{H,k}$ for all $k \geq 2$, must be nullhomotopic on 2-torus fibers of the bundles $M_{R,p} \rightarrow T$ and $M_{H,k} \rightarrow T$, where $T$ is the circle group. (Note: this is also true of the bundles $M_{R,p}$, $p = 4$ and 6, but this fact is not used in section 2.3).

We consider first the solvmanifolds $M_{R,p}$, $p = 2, 3$.

**Theorem 2.2.** For the manifolds $M_{R,p}$, $p = 2, 3$, the functions $f_p : M_{R,p} \rightarrow T$ defined by $f_p(\Gamma_p(t,x,y)) = e^{2\pi i t}$ are continuous and generate the groups of homotopy classes of functions from $M_{R,p}$ to $T$. 
Proof. We first state a few relevant facts. ([G-H]).

Denote by $[M,T]$ the set of homotopy equivalence classes of continuous functions from $M$ to $T$.

1. For all solvmanifolds under consideration, we have that $H^1(M) = [M,T]$ via the map

$$* : [M,T] 	o H^1(M);$$

$$f \mapsto f_*(\omega)$$

where $\omega$ is a generating cocycle in $H^1(T)$, and $f_*(\omega)$ is the class in $H^1(M)$ of the cocycle $\sigma$ which satisfies

$$\sigma(\gamma) = \omega(f \circ \gamma)$$

for all 1-simplices $\gamma$.

2. For all solvmanifolds under consideration, we have $H^1(M) \cong \text{Hom}(H_1(M),\mathbb{Z})$ via the isomorphism

$$\alpha : H^1(M) \to \text{Hom}(H_1(M),\mathbb{Z}); \quad \alpha(\sigma) = \bar{\sigma}$$

where for a cycle $\gamma \in H_1(M)$, $\bar{\sigma}(\gamma) = [\sigma, \gamma]$.

This follows from the existence of the exact sequence

$$0 \to \text{Ext}_\mathbb{Z}(H_{n-1}(M),\mathbb{Z}) \to H^n(M) \xrightarrow{\alpha_n} \text{Hom}(H_n(M),\mathbb{Z}) \to 0$$

for all $n \in \mathbb{Z}^+$ (Universal Coefficient Theorem). $H_0(M)$ is always a projective $\mathbb{Z}$-module, and so $\text{Ext}_\mathbb{Z}(H_0(M),\mathbb{Z})$ is zero. Therefore $\alpha$ is an isomorphism.

We begin by demonstrating that for $M_{R,p}, p = 2, 3$, we have $H^1(M_{R,p}) \cong \mathbb{Z}$, generated by the cocycle $\lambda_1$ for which $\tilde{\lambda}_1(\gamma_1) = 1$ (here $\gamma_1$ is the 1-simplex $t \in [0,1] \to \Gamma_p(t,0,0)$). We also define the simplices

$$\gamma_2 : t \in [0,1] \to \Gamma_p(0,t,0)$$

$$\gamma_3 : t \in [0,1] \to \Gamma_p(0,0,t)$$
and note that $\gamma_1, \gamma_2$ and $\gamma_3$ generate the group $\pi_1(M_{R,p})$. Furthermore, $\pi_1(M_{R,p})$ is isomorphic to $\Gamma_p$.

**Case 1:** $H_1(M_{R,2}) = \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z}_2 \cdot \gamma_2 \oplus \mathbb{Z}_2 \cdot \gamma_3$.

This follows from the fact that $[\pi_1(M_{R,2}), \pi_1(M_{R,2})]$ is generated by the elements $\gamma_2^2$ and $\gamma_3^2$ in $\pi_1(M_{R,2})$.

**Case 2:** $H_1(M_{R,3}) = \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z}_2 \cdot \gamma_2$.

Here we use the fact that $[\pi_1(M_{R,3}), \pi_1(M_{R,3})]$ is generated in $\pi_1(M_{R,3})$ by the elements $\gamma_2\gamma_3$ and $\gamma_3^3$.

We now compute $H^1(M_{R,p})$, $p = 2, 3$, using the fact that $H^1(M_{R,p}) \cong \text{Hom}(H_1(M_{R,p}), \mathbb{Z})$.

**Case 1.** $H^1(M_{R,2}) \cong \text{Hom}(\mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z}_2 \cdot \gamma_2 \oplus \mathbb{Z} \cdot \gamma_3, \mathbb{Z}) = \text{Hom}(\mathbb{Z} \cdot \gamma_1, \mathbb{Z}) \cong \mathbb{Z} \cdot \lambda_1$, where $\lambda_1$ is the cocycle in $H^1(M_{R,2})$ satisfying $\lambda_1(\gamma_1) = 1$, $\lambda_1(\gamma_2) = \lambda_1(\gamma_3) = 0$.

**Case 2.** $H^1(M_{R,3}) \cong \text{Hom}(\mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z}_2 \cdot \gamma_2, \mathbb{Z}) = \text{Hom}(\mathbb{Z} \cdot \gamma_1, \mathbb{Z}) \cong \mathbb{Z} \cdot \lambda_1$, where $\lambda_1$ is the cocycle in $M_{R,3}$ satisfying $\lambda_1(\gamma_1) = 1$, $\lambda_1(\gamma_2) = \lambda_1(\gamma_3) = 0$.

Finally, if we suppose that $\omega$ is any cocycle generating $H^1(T)$, then $[M_{R,p}, T]$ is generated by any continuous function $f$ on $M_{R,p}$ satisfying $f_*(\omega) = \lambda_1$. Therefore we must have $f_*(\omega)(\gamma_1) = 1$, $f_*(\omega)(\gamma_2) = f_*(\omega)(\gamma_3) = 0$.

Since $f_p$ in the statement of Theorem 2.2 satisfies these conditions and is continuous on $M_{R,p}$, $f_p$ generates $[M_{R,p}, T]$ for $p = 2, 3$. This completes the proof of Theorem 2.2.

**Theorem 2.3.** For the manifolds $M_{H,k}$, $k = 3, 4, 5, \ldots$ the functions

$$f_k : M_{H,k} \longrightarrow T$$

defined by

$$f_k(\Gamma_k(t,x,y)) = e^{2\pi it}$$

are continuous on $M_{H,k}$ and generate the groups of homotopy classes of functions from $M_{H,k}$ to $T$. 
Proof. Facts 1 and 2 following the statement of Theorem 2.2 also apply here. We begin by demonstrating that for $M_{H,k}$, $k \geq 3$, we have $H^1(M_{H,k}) \cong \mathbb{Z}$, generated by the cocycle $\lambda_1$ for which $\tilde{\lambda}_1(\gamma_1) = 1$ (here $\gamma_1$ is the 1-simplex $t \in [0,1] \mapsto \Gamma_k(t,0,0)$).

Again we define the simplices

$$\gamma_2 : t \in [0,1] \mapsto \Gamma_k(0,t,0)$$

$$\gamma_3 : t \in [0,1] \mapsto \Gamma_k(0,0,t)$$

and note that $\gamma_1$, $\gamma_2$ and $\gamma_3$ generate the group $\pi_1(M_{H,k})$. We also have $\pi_1(M_{H,k}) \cong \Gamma_k$.

Case 1. $H_1(M_{H,2}) = \mathbb{Z} \cdot \gamma_1$.

This follows from the fact that $[\pi_1(M_{H,2}), \pi_1(M_{H,2})]$ is generated in $\pi_1(M_{H,2})$ by the elements $\gamma_2$ and $\gamma_2 \gamma_3$, which together generate all terms of the form $\gamma_2^M \gamma_3^N$.

Case 2. $H_1(M_{H,k}) = \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot \gamma_2$ for $k \geq 3$.

In the case $k \geq 3$, we have $[\pi_1(M_{H,k}), \pi_1(M_{H,k})]$ generated by $\gamma_2^{k-1}$ and $\gamma_3$ in $\pi_1(M_{H,k})$.

We now complete $H^1(M_{H,k})$ for $k \geq 2$.

Case 1. $H^1(M_{H,2}) \cong \text{Hom}(H_1(M_{H,2}), \mathbb{Z}) = \text{Hom}(\mathbb{Z} \cdot \gamma_1, \mathbb{Z}) \cong \mathbb{Z} \cdot \lambda_1$, where $\lambda_1$ is the cocycle satisfying $\tilde{\lambda}_1(\gamma_1) = 1$, $\tilde{\lambda}_1(\gamma_2) = \tilde{\lambda}_1(\gamma_3) = 0$.

Case 2. $H^1(M_{H,k}) \cong \text{Hom}(H_1(M_{H,k}), \mathbb{Z}) = \text{Hom}(\mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot \gamma_2, \mathbb{Z})$ for $k \geq 3$. However, this is $\text{Hom}(\mathbb{Z} \cdot \gamma_1, \mathbb{Z}) \cong \mathbb{Z} \cdot \lambda_1$, where $\lambda_1$ is the cocycle satisfying $\tilde{\lambda}_1(\gamma_1) = 1$, $\tilde{\lambda}_1(\gamma_2) = \tilde{\lambda}_1(\gamma_3) = 0$.

If we suppose again that $\omega$ is a cocycle generating $H^1(T)$, then $[M_{H,k}, T]$ is generated by any continuous $f$ on $M_{H,k}$, $k \geq 2$, satisfying $f_*(\omega) = \lambda_1$. The rest of the argument proceeds as for Theorem 2.2.
§2.3. Zeros of continuous functions in $H_\pi \subseteq L^2(M_{R, p})$

We begin this section by demonstrating that $M_{R,1} \cong T^3$ possesses $\pi$-primary functions which are nonvanishing, as one would expect.

Suppose $\lambda_\pi \in \mathcal{L}^* \cap \mathcal{O}_\pi$; then the character $\chi_{\lambda_\pi}$ defined on a maximal subgroup $M$ of $S_{R,1}$ gives rise to the Mackey space $M(\lambda_\pi)$, as demonstrated in the introduction (0.3).

Functions in the image of the lift map $L : M(\lambda_\pi) \rightarrow L^2(M_{R,1})$ are of the form $f : M_{R,1} \rightarrow \mathbb{C}$, $f(\Gamma_1(t,x,y)) = \tilde{f}(t)\chi_{\lambda_\pi}(0,x,y)$ for some $L^2$ function $\tilde{f}$ with period 1. Clearly $f$ is continuous if $\tilde{f}$ is.

Thus we see that the $\pi$-primary summand contains the character

$$\psi(\Gamma_1(t,x,y)) = e^{2\pi i \lambda_\pi(x,y)} \quad (2.14)$$

which is continuous and nonvanishing on $M_{R,1}$.

We wish to emphasize here that the manifold $M_{R,1} \cong T^3$ is the only compact 3-dimensional non-abelian solvmanifold known to possess continuous, nonvanishing $\pi$-primary summand functions. We demonstrate in what follows that for $M_{R,p}$, $p = 2, 4, 6$, continuous functions in the infinite-dimensional $\pi$-primary summands must have zeros. In the case of $M_{R,3}$, continuous functions in certain constructible subspaces which span the $\pi$-primary summands are known to have zeros, but the complete answer for $M_{R,3}$ is not known.

We begin by examining the situation for $M_{R,2}$.

**Theorem 2.4.** Suppose $f$ is a continuous function in $H_\pi \subseteq L^2(M_{R,2})$, for $\pi \in (\Gamma_{R,2}\setminus S_{R,2})/_\infty$. Then $f$ has at least one zero on $M_{R,2}$.

**Proof.** Recall that $S_{R,2} = \mathbb{R} \ltimes \mathbb{R}^2$, with $\mathbb{R}$ acting on $\mathbb{R}^2$ via the 1-parameter subgroup $\sigma(t) = \text{Rot}(\pi t)$, and that $\Gamma_2$ is the subgroup of integer points in $S_{R,2}$. 
The coadjoint orbits are therefore circular cylinders in $\mathfrak{g}_{R,2}^{*}$, saturated in the $T^*$-direction. The coadjoint orbit associated with $\pi \in (M_{R,2})_{\infty}$ is that containing an integral functional $\lambda_\pi = \alpha X^* + \beta Y^*$ for which $\chi_{\lambda_\pi}$ induces $\pi$.

Let $P_{\pi_i}$ be orthogonal projection of $L^2(M_{R,2})$ onto the irreducible subspace $H_{\pi,i}$ which is the image of the lift map $L_i$ from $M(\lambda_\pi)$ to $L^2(M_{R,2})$. We note that if $f$ is continuous on $M_{R,2}$, then $P_{\pi_i}f$ is continuous on $M_{R,2}$ [Ri1], and

$$P_{\pi_i}f = \sum_{j=0}^{1} f(t, \cdot)^\vee (\sigma(j)(\alpha, \beta)) \chi_{\sigma(j)(\alpha, \beta)}$$

(2.15)

is equal to $L_i f'$ for some continuous $f'$ in $M(\lambda_\pi)$. (Lemma 0.4)

Thus, in order to prove Theorem 2.4, it suffices to look at sums of functions of the form $\sum_{i=1}^{\text{mul}(\pi)} L_i f_i$, for $f_i$ continuous in $M(\lambda_\pi)$, where $M(\lambda_\pi)$ lifts to $H_{\pi,i}$.

Let $S = \{\lambda : \alpha^2 + \beta^2 = \lambda^2, (\alpha, \beta) \in \mathbb{Z}^2, (\alpha, \beta) \neq (0,0)\}$. Order the elements of $S$ so that $\lambda_k > \lambda_{k-1}$. The proof of Theorem 2.4 is by induction on the elements of $S$.

Case 1. $\lambda_1 = 1, \text{mul}(\pi_{\lambda_1}) = 2$.

This case falls into the category of odd $\lambda_N$, which is treated in the induction step.

For the induction step, we suppose that if $f$ is continuous on $M_{R,2}$, $f \in H_{\pi_k}$ for $\lambda_k < \lambda_n$, then $f$ must have a zero.

Case 1: $\lambda_n^2$ is odd.

Let $\{(\alpha_i, \beta_i)\}_{i=1}^{\text{mul}(\pi_{\lambda_n})}$ be a complete set of $\Gamma_2$-orbit representatives in $O_{\lambda_N} \cap \mathcal{L}^*$; for convenience we may take them to be in the set $\{(\alpha, \beta) : \alpha, \beta \in \mathbb{Z}, \alpha + \beta > 0\}$. Note that $\alpha + \beta$ is odd whenever $\alpha^2 + \beta^2 = \lambda_N^2$ is odd.

Let $P = \{p = \alpha_i + \beta_i\}$. Order $P$ so that $p_k > p_{k-1}$. Note that the $p$ are positive.

Let $Q_k = \{i : \alpha_i + \beta_i = p_k\}, k = 1, \ldots, m$. Note that each $Q_k$ has cardinality 2; this follows from the fact that if $\alpha_i + \beta_i = \alpha_j + \beta_j$ and $\alpha_i^2 + \beta_i^2 = \alpha_j^2 + \beta_j^2$ (orbit
condition), then if \( \alpha_i \neq \alpha_j \), we must have \( \alpha_i = \beta_j \). Thus \( Q_k \) has cardinality 2, one for each of \((\alpha, \beta)\) and \((\beta, \alpha)\). Let \( \{f_i : \mathbb{R} \to \mathbb{C}\}_{i=1}^{\text{mul}(\pi_\alpha)} \) be a set of continuous functions with period 2.

Then as demonstrated in section 0.4 a typical continuous \( \phi \) in \( H_{\pi_\alpha} \) has the form (setting \( z_1 = e^{2\pi i x}, z_2 = e^{2\pi i y} \))

\[
\phi(t, z_1, z_2) = \left\{ \sum_{Q_k} \left[ \sum_{i \in Q_k} f_i(t)z_1^{\alpha_i}z_2^{\beta_i} + f_i(t + 1)z_1^{-\alpha_i}z_2^{-\beta_i} \right] \right\}. \tag{2.16}
\]

Fix \( t \) and define \( \phi_t(z_1, z_2) \equiv \phi(t, z_1, z_2) \), so that \( \phi_t : N \cap \Gamma_2 \setminus N \cong T^2 \to \mathbb{C} \).

Suppose \( \phi \) is nonvanishing on \( M_{R,2} \). Then \( \phi_t : T^2 \to \mathbb{C} \) is nullhomotopic, by the results of section 2. Consider \( \tilde{\phi}_t \), the restriction of \( \phi_t \) to the closed curve \( z_1 = z_2 \) in \( T^2 \). Then \( \tilde{\phi}_t \), defined by

\[
\tilde{\phi}_t(z) = \phi_t(z, z) = \sum_{\{Q_k\}} \left[ \sum_{i \in Q_k} f_i(t)z^{p_k} + \sum_{i \in Q_k} f_i(t + 1)z^{-p_k} \right] \tag{2.17}
\]

is a curve in \( \mathbb{C} \setminus \{0\} \) which has winding number zero.

Clearly \( \tilde{\phi}_t \) may be viewed as the restriction to \( T \) of a meromorphic function \( \Phi_t : \mathbb{C} \to \mathbb{C} \),

\[
\Phi_t(\omega) = \sum_{\{Q_k\}} \left[ \sum_{i \in Q_k} f_i(t)\omega^{p_k} + \sum_{i \in Q_k} f_i(t + 1)\omega^{-p_k} \right] \tag{2.18}
\]

We claim that \( \Phi_t \) has a pole of odd order at \( \omega \equiv 0 \). If not, then the coefficients in \( \Phi_t \) of negative exponents are zero; but since \( \Phi_t \) contains only terms with odd exponents, \( \Phi_t \) has no constant term, and thus we would have a polynomial \( \Phi_t \) with \( \Phi_t(0) = 0 \). Thus \( \Phi_t \) would wind at least once on the circle \( T \), so \( \tilde{\phi}_t \) would wind, a contradiction. Let \( \gamma \geq 1 \) be the order of the pole at zero. Then we may write

\[
\Phi_t(z) = z^{-\gamma} \left\{ \sum_{\{Q_k\}} \sum_{i \in Q_k} f_i(t)z^{p_k+\gamma} + \sum_{i \in Q_k} f_i(t + 1)z^{-p_k+\gamma} \right\}
\]

\[
= z^{-\gamma} p(z) \tag{2.19}
\]

where \( p \) is a polynomial in \( z \) with even exponents.
But then the zeros of \( p(z) \) (and so the zeros of \( \Phi_t \)) occur in pairs of equal modulus, so that the number of zeros inside the unit circle is even. Define \( \Gamma : [0,1] \rightarrow T \) by \( \Gamma(t) = e^{2\pi it} \). Then we have

\[
\text{Ind}_\Gamma \Phi_t = N_{\Phi_t} - P_{\Phi_t} \neq 0 \tag{2.20}
\]

where \( N_{\Phi_t} \) and \( P_{\Phi_t} \) are, respectively, the number of zeros and poles of \( \Phi_t \) inside \( \Gamma \) ([Ru2], chapter 10). Thus the winding number of \( \tilde{\phi}_t \) cannot be zero, and we arrive at a contradiction.

Case 2: \( \lambda^2_n \equiv 0 \) mod 4.

Note that if \( \alpha^2 + \beta^2 = \lambda^2_N \equiv 0 \) mod 4, then both \( \alpha \) and \( \beta \) must be even.

Let \( \{ (\alpha_i, \beta_i) \}_{i=1}^{\text{mul}(\pi_n)} \) be a set of \( \Gamma_{R,2} \)-orbit representatives, and let \( \{ f_i : \mathbb{R} \rightarrow \mathbb{C} \}_{i=1}^{\text{mul}(\pi_n)} \) be continuous functions with period 2. Then a typical \( \phi \in H_{\pi \lambda_n} \) may be written

\[
\phi(t, z_1, z_2) = \sum_{i=1}^{\text{mul}(\pi_n)} f_i(t) z_1^{\alpha_i} z_2^{\beta_i} + f_i(t + 1) z_1^{-\alpha_i} z_2^{-\beta_i}, \tag{2.21}
\]

for \( z_1 = e^{2\pi i x} \), \( z_2 = e^{2\pi i y} \), as demonstrated in section 0.4.

We define

\[
\psi(t, z_1, z_2) = \sum_{i=1}^{\text{mul}(\pi_n)} f_i(t) z_1^{\alpha_i/2} z_2^{\beta_i/2} + f_i(t + 1) z_1^{-\alpha_i/2} z_2^{-\beta_i/2}. \tag{2.22}
\]

Note that since the \( \alpha_i \) and \( \beta_i \) are all divisible by 2, \( \psi \) has integer exponents.

Therefore, \( \psi \) is continuous, \( \Gamma_{R,2} \)-invariant, and lives in \( H_{\pi \lambda_k} \) where \( \lambda^2_k = \lambda^2_n/4 \). By the induction hypothesis, \( \psi \) has a zero. Since \( \phi(t, z_1, z_2) = \psi(t, z_1^2, z_2^2) \), \( \phi \) must also have a zero.

Case 3: \( \lambda^2_n \equiv 2 \) mod 4.

Let \( \{ (\alpha_i, \beta_i) \}_{i=1}^{\text{mul}(\pi_n)} \) be a complete set of \( \Gamma_{R,2} \)-orbit representatives from \( O_{\pi \lambda_n} \cap \mathcal{L}^* \), satisfying \( \alpha_i > 0 \) for each \( i \). Note that \( \alpha_i^2 + \beta_i^2 = \lambda^2_n = 2 \) mod 4 implies that \( \alpha_i \) and \( \beta_i \) are both odd for each \( i \).
Let $P = \{p = \alpha_i \text{ for some } i\}$. Order $P$ so that $p_k > p_{k-1}$. Let $Q_k = \{i : \alpha_i = p_k\}$, and note that each $Q_k$ has cardinality 2. Let $\{f_i : \mathbb{R} \rightarrow \mathbb{C} \}_{i=1}^{\text{mul}(\pi_n)}$ be a set of continuous functions of period 2. Then a typical continuous $\phi$ in $H_{\pi_n}$ has the form of (2.16).

Fix $t$ and define $\phi_t(z_1, z_2) = \phi(t, z_1, z_2)$, $\phi_t : N \cap \Gamma_2 \setminus N \cong \mathbb{T}^2 \rightarrow \mathbb{C}$. Suppose that $\phi$ is nonvanishing, so that $\phi_t$ must be nullhomotopic on $N \cap \Gamma_2 \setminus N$.

Define $\bar{\phi}_t(z) = \phi_t(z, 1)$, the restriction of $\phi_t$ to the curve $z_2 = 1$ in $N \cap \Gamma_2 \setminus N$. Then we have

$$\bar{\phi}_t(z) = \sum_{Q_k} \left\{ \left[ \sum_{i \in Q_k} f_i(t) \right] z^{p_k} + \left[ \sum_{i \in Q_k} f_i(t + 1) \right] z^{-p_k} \right\}.$$

Since each $\alpha_i$ is odd, each $p_k$ is odd; by choice of $\alpha_i$, we have $p_k > 0$ for all $k$.

From here we proceed, as in case 1, to demonstrate that $\bar{\phi}_t$ must wind on the circle $\mathbb{T}$, a contradiction. This completes case 3 and finishes the proof of Theorem 2.4.

In Theorem 2.5, we show that continuous functions in the $\pi$-primary summands of $L^2(M_{R,4})$ and $L^2(M_{R,6})$ must have zeros.

**Theorem 2.5.** Let $f$ be a continuous function in $H_{\pi} \subseteq L^2(M_{R,i})$, for $i = 4$ or 6, $\pi \in (\Gamma_{R,i} \setminus S_{R,i})_{\infty}$. Then $f$ has at least one zero on $M_{R,i}$.

**Proof.** Define the groups

$$\Gamma'_4 = \{(m, n, p) \in \Gamma_{R,4} \subseteq S_{R,4} : M = 2k, \text{ for some } k \in \mathbb{Z}\}$$

$$\Gamma'_6 = \{(m, n, p) \in \Gamma_{R,6} \subseteq S_{R,6} : M = 3k, \text{ for some } k \in \mathbb{Z}\}$$

Then $\Gamma'_4$ and $\Gamma'_6$ are subgroups of $\Gamma_{R,4}$ and $\Gamma_{R,6}$ respectively, of finite index; thus $\Gamma'_i$ is cocompact in $S_{R,i}$ for each $i$, and it is straightforward to verify that $\Gamma'_i \cong \Gamma_{R,2}$, $i = 4, 6$.

We prove Theorem 2.5 for $M_{R,4}$; the proof for $M_{R,6}$ is analogous in every respect.
Since $\Gamma'_4 \cong \Gamma_{R,2}$ and is cocompact in $S_{R,4}$, we have $\Pi_1(\Gamma'_4 \setminus S_{R,4}) \cong \Gamma_{R,2}$. Therefore we have $\Gamma'_4 \setminus S_{R,4} \cong M_{R,2}$ ([Mos], Theorem A).

Functions which are $\Gamma_{R,4}$-periodic are $\Gamma'_4$-periodic, so $L^2(M_{R,4})$ embeds isometrically in $L^2(M_{R,2})$. Furthermore this embedding is $S_{R,4}$-equivariant with respect to the quasi-regular representation, and so takes $\pi$-spaces to $\pi$-spaces.

Let $\Phi$ be the isometric embedding of $L^2(M_{R,4})$ in $L^2(M_{R,2})$. Then if $f$ is a continuous function in $H_\pi \subseteq L^2(M_{R,4})$, $\Phi f$ is continuous in $H_\pi \subseteq L^2(M_{R,2})$ and so must have a zero. However, if $\Phi f(\Gamma_{R,2}(t,x,y)) = 0$, then since $f$ is $\Gamma_4$-invariant, $f(\Gamma_{R,4}(t,x,y)) = 0$; thus $f$ must have a zero. This completes the proof of Theorem 2.5.

We finish this section with a theorem summarizing what is known for $M_{R,3}$.

**Theorem 2.6.** Let $f$ be a continuous element of a constructible, irreducible subspace of a $\pi$-primary summand $H_\pi \subseteq L^2(M_{R,3})$ (see section 0.3). Then $f$ has at least one zero on $M_{R,3}$.

**Proof.** Suppose $\lambda_\pi \in L^* \cap O_\pi$, an integral functional in $\mathcal{S}_{R,3}^*$; then the character $\chi_{\lambda_\pi}$ defined on a maximal subgroup $M$ of $S_{R,3}$ gives rise to the Mackey space, $M(\lambda_\pi)$. The constructible irreducible subspace corresponding to $\lambda_\pi = (\alpha, \beta)$ is the image of the lift map $L : M(\lambda_\pi) \rightarrow L^2(M_{R,3})$, an $S_{R,3}$-invariant isometry.

A typical continuous element of this constructible irreducible subspace of $H_\pi$ has the form

$$\tilde{f}(\Gamma_3(t,x,y)) = \tilde{f}(t,z_1,z_2)$$

$$= f(t)z_1^\alpha z_2^\beta + f(t+1)z_1^\beta z_2^{-(\alpha+\beta)} + f(t+2)z_1^{-(\alpha+\beta)}z_2^\alpha$$

for $z_1 = e^{2\pi i x}$, $z_2 = e^{2\pi i y}$, and $f : \mathbb{R} \rightarrow \mathbb{C}$ a continuous function of period 3, as demonstrated in section 0.4.

Suppose $\tilde{f}$ is nonvanishing on $M_{R,3}$. Then $\tilde{f}$ must be nullhomotopic when restricted to $T^2$-fibers of the bundle $M_{R,3} \rightarrow T$. 


We examine the functions \( \overline{f} \) on a case-by-case basis.

**Case 1.** \( \alpha \equiv \beta \equiv 1 \mod 3 \), or \( \alpha \equiv \beta \equiv 2 \mod 3 \).

We define \( \phi_t(z) = \overline{f}(t,z,1) \) for fixed \( t \in \mathbb{R} \); \( \phi_t \) must have winding number zero on \( T \), since \( \overline{f}(t,z_1,z_2) \) is null-homotopic on \( T^2 \) for fixed \( t \). We have

\[
\phi_t(z) = f(t)z^\alpha + f(t + 1)z^\beta + f(t + 2)z^{-(\alpha + \beta)}. \tag{2.24}
\]

Clearly one of \( \alpha, \beta \) and \( -(\alpha + \beta) \) must be negative. If we view \( \phi_t \) as the restriction to the set \( T = \{|z| = 1, \ z \in \mathbb{C} \} \) of the meromorphic function

\[
\Phi_t(\omega) = f(t)\omega^\alpha + f(t + 1)\omega^\beta + f(t + 2)\omega^{-(\alpha + \beta)} \tag{2.25}
\]

we see that \( \Phi_t \) has a pole at \( \omega = 0 \).

Let \( \gamma \) be the order of the pole at zero. Then we may write

\[
\Phi_t(\omega) = \omega^{-\gamma}\{f(t)\omega^{\alpha+\gamma} \cdot f(t + 1)^{\beta+\gamma} + f(t + 2)^{-(\alpha + \beta)+\gamma}\}
\]

\[
= \omega^{-\gamma}p(\omega) \tag{2.26}
\]

where \( p(\omega) \) is a polynomial. Note that the exponents of \( p(\omega) \) must all be divisible by 3. Since \( \alpha + \beta \equiv 1 \) or 2 mod 3, we must have \( \gamma \equiv 1 \) or 2 mod 3, so that \( \alpha + \gamma \equiv \beta + \gamma \equiv -(\alpha + \beta) + \gamma \equiv 0 \mod 3 \). Thus the zeros of \( p(\omega) \) are grouped as triples of equal modulus; in particular, the number of zeros of \( p(\omega) \) (and hence of \( \Phi_t(\omega) \)) inside \( T \) is a multiple of 3. However, the pole of \( \Phi_t \) at \( \omega = 0 \) is not a multiple of 3, and therefore, referring to (2.20), we see that \( \Phi_t \) must wind on the curve \( T \), and therefore that \( \phi_t \) cannot be nullhomotopic, a contradiction.

**Case 2.** \( \alpha \equiv 1 \mod 3 \), \( \beta \equiv 2 \mod 3 \).

We define \( \phi_t(z) = \overline{f}(t,z,z^{-1}) = f(t)z^{\alpha-\beta} + f(t + 1)z^{2\beta+\alpha} + f(t + 2)z^{-(2\alpha+\beta)} \) as in case 1 and note that \( \phi_t \) is \( \overline{f} \) restricted to the curve \( z_2 = z_1^{-1} \) in the \( T^2 \)-fiber over \( \Gamma_3(t,0,0) \).
Clearly one of $\alpha - \beta$, $2\beta + \alpha$, and $-(2\alpha + \beta)$ must be negative, since $(\alpha - \beta) + (2\beta + \alpha) = 2\alpha + \beta$. All are congruent to $2 \mod 3$, so none can be zero. If we view $\phi_t$ as the restriction to $T$ of the meromorphic function $\Phi_t(\omega) = f(t)\omega^{\alpha-\beta} + f(t + 1)\omega^{2\beta + \alpha} + f(t + 2)\omega^{-(2\alpha + \beta)}$, we see that $\Phi_t$ has a pole at zero. Let $\gamma$ be the order of the pole at zero; then $-\gamma \equiv 2 \mod 3$, and we may write
\[
\Phi_t(\omega) = \omega^{-\gamma} \left\{ f(t)\omega^{\alpha-\beta+\gamma} + f(t + 1)\omega^{2\beta+\alpha+\gamma} + f(t + 2)\omega^{-(2\alpha+\beta)+\gamma} \right\}
\] (2.27)
where $p(\omega)$ is a polynomial. Note that since $-\gamma \equiv 2 \mod 3$, and since all of $\alpha - \beta$, $2\beta + \alpha$, and $-(2\alpha + \beta)$ are congruent to $2 \mod 3$, the function $p(\omega)$ contains only terms with exponents divisible by $3$, and so the number of zeros inside $T$ is a multiple of $3$. Since $\gamma$ is not a multiple of $3$, we have $\text{Ind}_T \Phi_t \neq 0$, and again we arrive at a contradiction.

**Case 3:** $\alpha \equiv 2 \mod 3$, $\beta \equiv 1 \mod 3$.

Proceeding as before, we define $\phi_t(z) = f(t, z, z^{-1}) = f(t)z^{\alpha-\beta} + f(t + 1)z^{2\beta+\alpha} + f(t + 2)z^{-(2\alpha+\beta)}$, and note that $\phi_t$ is $\bar{f}$ restricted to the curve $z_2 = z_1^{-1}$ in the $T^2$-fiber over $\Gamma_3(t, 0, 0)$.

Again, one of $\alpha - \beta$, $2\beta + \alpha$, and $-(2\alpha + \beta)$ must be negative, and all are congruent to $1 \mod 3$, so that none is zero. If we view $\phi_t$ as the restriction to $T$ of the meromorphic function
\[
\Phi_t(\omega) = f(t)\omega^{\alpha-\beta} + f(t + 1)\omega^{2\beta+\alpha} + f(t + 2)\omega^{-(2\alpha+\beta)}
\]
we see that $\Phi_t$ has a pole at zero. Let $\gamma$ be the order of the pole at zero; then $-\gamma \equiv 1 \mod 3$, and we may write
\[
\Phi_t(\omega) = \omega^{-\gamma} \left\{ f(t)\omega^{\alpha-\beta+\gamma} + f(t + 1)\omega^{2\beta+\alpha+\gamma} + f(t + 2)\omega^{-(2\alpha+\beta)+\gamma} \right\}
\]
where $p(\omega)$ is a polynomial. Note that since $-\gamma \equiv 1 \mod 3$, and since all of $\alpha - \beta$, $2\beta + \alpha$, and $-(2\alpha + \beta)$ are as well, the function $p(\omega)$ contains only terms
with exponents divisible by 3. The number of zeros inside $T$ is therefore a multiple of 3. Since $\gamma$ is not a multiple of 3, we have $\text{Ind}_T \Phi_t \neq 0$, and we arrive at a contradiction.

**Case 4**: $\alpha \equiv 0 \mod 3$, $\beta \equiv 1 \mod 3$ or $\alpha \equiv 2 \mod 3$, $\beta \equiv 0 \mod 3$.

If $(\alpha, \beta)$ is such that $\alpha \equiv 0 \mod 3$ and $\beta \equiv 1 \mod 3$, then the $\Gamma_3$-orbit containing $(\alpha, \beta)$ also contains the point $(\beta, -(\alpha + \beta))$, which is of the type dealt with in case 2. Since the function $\tilde{f}$ is independent of the base point chosen from the $\Gamma_3$-orbit, $\tilde{f}$ must have a zero. Similarly, if $(\alpha, \beta)$ is of the type $\alpha \equiv 2 \mod 3$, $\beta \equiv 0 \mod 3$, then the point $(-\alpha + \beta, \alpha)$ is of the type dealt with in case 2.

**Case 5**: $\alpha \equiv 0 \mod 3$, $\beta \equiv 2 \mod 3$; or $\alpha \equiv 1 \mod 3$, $\beta \equiv 0 \mod 3$.

If $(\alpha, \beta)$ is such that $\alpha \equiv 0 \mod 3$, $\beta \equiv 2 \mod 3$, then $(\beta, -(\alpha, \beta))$ is in the same $\Gamma_3$-orbit as $(\alpha, \beta)$ and is of the type dealt with in case 3. If $(\alpha, \beta)$ is of the type $\alpha \equiv 1 \mod 3$, $\beta \equiv 0 \mod 3$, then the point $(-\alpha, \beta, \alpha)$ is of the type dealt with in case 3.

**Case 6**: $\alpha \equiv \beta \equiv 0 \mod 3$.

We have $\alpha = 3^k \alpha'$ and $\beta = 3^k \beta'$ for some $k \neq 0$ and some pair $(\alpha', \beta')$, not both congruent to 0 mod 3. We may therefore define

$$\bar{g}(t, z_1, z_2) = f(t)z_1^{\alpha'}z_2^{\beta'} + f(t + 1)z_1^{\beta'}z_2^{-(\alpha' + \beta')} + f(t + z)z_1^{-(\alpha' + \beta')}z_2^{\alpha'},$$

which is clearly the lift of a function in the Mackey space $M(\lambda_\pi')$ for $\lambda_\pi' = (\alpha', \beta')$; since not both $\alpha'$ and $\beta'$ are congruent to 0 mod 3, $\bar{g}$ must have a zero, since one of cases 1-5 applies. Since $\bar{g}$ has a zero, and since we have $\bar{g}(t, z_1^{2k}, z_2^{3k}) = \tilde{f}(t, z_1, z_2)$, $\tilde{f}$ must have a zero on $M_{R,3}$.

This completes the proof of Theorem 2.6.
§2.4. Zeros of continuous functions in $H^0 = C(M_{H,k}) \cap H^\infty$

In this section, we demonstrate that functions in a uniformly dense subspace $K_\pi$ of continuous functions in $H^0 = C(M_{H,k}) \cap H^\infty$ must have zeros on $M_{H,k}$, for all hyperbolic solvmanifolds $M_{H,k}$. It then follows easily that all continuous functions on $H^\infty$ must have zeros (Lemma 2.9).

If $\{\lambda_{\pi_i}\}_{i=1}^{\text{mul}(\pi)}$ is a complete set of $\Gamma_{H,k}$-orbit representatives from $\mathcal{O}_\pi \cap L^*$, then the set $\{L_i : M(\lambda_{\pi_i}) \to L^2(M_{H,k})\}_{i=1}^{\text{mul}(\pi)}$ is a complete set of lift maps into the constructible irreducible subspaces of $H^\infty$. Let $T_i : L^2(\mathbb{R}) \to M(\lambda_{\pi_i})$ be the isometry intertwining the Schrödinger model of $\pi$ on $L^2(\mathbb{R})$ and the induced model on $M(\lambda_{\pi_i})$; i.e. $T_i f = \tilde{f}$, where $\tilde{f}(t, x, y) = \chi_{\lambda_{\pi_i}}(x, y)f(t)$.

Then $L'_i = L_i \circ T_i$ lifts $L^2(\mathbb{R})$ into the $i$th constructible irreducible subspace of $H^\infty$. We define

$$K_\pi = L'_1(C^\infty_0(\mathbb{R})) \oplus \cdots \oplus L'_i(C^\infty_0(\mathbb{R})) \subseteq H^\infty.$$

**Lemma 2.7.** $K_\pi$ is uniformly dense in $H^0$.

**Proof.** We first demonstrate that $K_\pi$ is uniformly dense in $H^\infty = C^\infty(M_{H,k}) \cap H^\infty$.

If $S_\pi(\mathbb{R})$ are the smooth vectors for the Schrödinger model of $\pi$ on $L^2(\mathbb{R})$, then we have

$$\overline{L'_i(C^\infty_0(\mathbb{R}))} \supseteq L'_i(S_\pi(\mathbb{R}))$$

in the sup-norm on $M_{H,k}$ (Lemma 5, [Bre2]).

Since $L'_i$ intertwines $\pi$ on $L^2(\mathbb{R})$ and $H^\infty_{\pi,i}$, $L'_i(S_\pi(\mathbb{R})) = H^\infty_{\pi,i}$ by preservation of smooth vectors under intertwining maps.

We have also that $\bigoplus_{i} H^\infty_{\pi,i} = H^\infty$, since orthogonal projection onto $S_{H,k}$-invariant subspaces preserves infinite differentiability ([Aus-Bre], section 2). Therefore if $\phi \in H^\infty_\pi$, and $\phi = \sum_{i=1}^{\text{mul}(\pi)} \phi_i$, where $\phi_i = P_{\pi,i}(\phi)$, then for each $i \in$
1, \ldots, \text{mul}(\pi) \) there is an \( f_i \in L_i(C_0^\infty(\mathbb{R})) \) such that \( \|\phi_i - f_i\|_\infty < \epsilon/(\text{mul } \pi) \). Therefore \( \|\phi_i - \sum f_i\|_\infty < \epsilon \), and so \( K_\pi \) is uniformly dense in \( H_\pi^\infty \).

We finish the proof of Lemma 4.1 by demonstrating that \( H_\pi^\infty \) is uniformly dense in \( H_\pi^0 \).

Let \( F \) be a fundamental domain for \( \Gamma_k \setminus S_{H,k} \) containing the identity; define a \( C^\infty \) approximate identity \( \{\varepsilon_n\}_{n=1}^{\infty} \) so that

1. \( 0 \leq \varepsilon_n < \infty \) for each \( n \in \mathbb{Z}^+ \).
2. Support \( \varepsilon_n \) is contained in the interior of \( F \) for each \( n \in \mathbb{Z}^+ \).

We define, for \( \phi \in H_\pi^0 \),

\[
\phi \ast \varepsilon_n(t,x,y) = \int_{F \subseteq S_{H,k}} \phi((t,x,y)(t',x',y')^{-1})\varepsilon_n(t',x',y')dt'dx'dy'. \tag{2.29}
\]

Then \( \phi \ast \varepsilon_n \) is \( C^\infty \) for each \( n \in \mathbb{N} \), since \( \varepsilon_n \) is \( C^\infty \) and \( S_{H,k} \) is unimodular, and in fact \( \phi \ast \varepsilon_n \in H_\pi^\infty \) since \( \phi \ast \varepsilon_n \) is the uniform limit of linear combinations of right translates of \( \phi \).

We now claim that \( \phi \ast \varepsilon_n \) converges uniformly to \( \phi \) on \( M_{H,k} \). We have

\[
\|\phi \ast \varepsilon_n - \phi\|_\infty \leq \sup_{(t,x,y)} \int_F |\phi((t,x,y)(t',x',y')^{-1}) - \phi(t,x,y)|\varepsilon_n(t',x',y')dt'dx'dy'. \tag{2.30}
\]

Choose \( N \) so that for \( n \in \mathbb{N} \), we have \( |\phi((t,x,y)(t',x',y')^{-1}) - \phi(t,x,y)| < \epsilon \) for all \( (t,x,y) \in F \) and all \( (t',x',y') \in \text{support}(\varepsilon_n) \). Then \( \|\phi \ast \varepsilon_n - \phi\|_\infty \leq \int \varepsilon \cdot \varepsilon_n(t',x',y')dt'dx'dy' = \epsilon \), which completes the proof of uniform convergence.

Thus we have \( H_\pi^\infty \subseteq H_\pi^0 \) uniformly dense in \( H_\pi^0 \), completing the proof of Lemma 2.7.

Recall from section 0.2 that \( S_{H,k} = \mathbb{R} \ltimes \mathbb{R}^2 \), with \( \mathbb{R} \) acting on \( \mathbb{R}^2 \) via the 1-parameter subgroup \( \sigma_k : \mathbb{R} \rightarrow SL_2(\mathbb{R}) \) satisfying \( \sigma_k(1) = \begin{bmatrix} 1 & 1 \\ k-1 & k \end{bmatrix} \). \( \sigma_k \) is conjugate to the 1-parameter subgroup \( \sigma(t) = \begin{bmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{bmatrix} \), where \( \lambda + \lambda^{-1} = k+1 \), \( \lambda \in \mathbb{R} \). The coadjoint orbits in \( \mathfrak{sl}_{H,k} \) are therefore hyperbolic cylinders,
saturated in the $T^*$-direction, and satisfying the equation

$$(k - 1)x^2 + (k - 1)xy - y^2 = \omega$$

(2.31)

for some $\omega \in \mathbb{R}$. Note that two coadjoint orbits satisfy (2.31) for each value of $\omega$, each being a connected component of the set in $S_{H,}^*$ satisfying (2.31). The coadjoint orbit associated with $\pi \in (\Gamma_{H,k} \backslash S_{H,k})^\infty$ is that containing an integral functional $\lambda_\pi = \alpha X^* + \beta Y^*$ for which $\chi_{\lambda_\pi}$ induces $\pi$.

**Theorem 2.8.** Suppose $f$ is a continuous function in $H_\pi \subseteq L^2(M_{H,k})$, for $\pi \in (\Gamma_{H,k} \backslash S_{H,k})^\infty$. Then $f$ has at least one zero on $M_{H,k}$.

**Proof.** We begin by proving theorem 4.2 for functions $f \in K_\pi \subseteq H_\pi$. We have $K_\pi = L'_1(C_{0}^{\infty}(\mathbb{R})) \oplus \cdots \oplus L'_1(\text{null}(\pi))$, so that a typical element of $K_\pi$ has the form

$$\phi(\Gamma_k(t,x,y)) = \sum_{i=1}^{\text{mul}(\pi)} \sum_{n \in \mathbb{Z}} f_i(t + n) \exp 2\pi i (\alpha_{n,i} x + \beta_{n,i} y)$$

(2.32)

where $\{(\alpha_{0,i}, \beta_{0,i})\}_{i=1}^{\text{mul}(\pi)}$ is a set of distinct $\Gamma_k$-orbit representatives in $O_\pi \cap L^*$, $(\alpha_{n,i}, \beta_{n,i}) = \sigma_k(n)(\alpha_{0,i}, \beta_{0,i})$, and where for each $i = 1, \ldots, \text{mul}(\pi)$, and for fixed $t$, the sum over $n$ in (2.32) is finite. Suppose $\phi$ is nonvanishing on $M_{H,k}$. By setting $z_1 = e^{2\pi i x}, z_2 = e^{2\pi i y}$, we may define

$$\tilde{\phi}(t,z_1,z_2) = \phi(\Gamma_k(t,x,y)) = \sum_{i=1}^{\text{mul}(\pi)} \sum_{n \in \mathbb{Z}} f_i(t + n) z_1^{\alpha_{n,i}} z_2^{\beta_{n,i}}$$

(2.33)

For any fixed $t$, we must have $\tilde{\phi}$ nullhomotopic on $N \cap \Gamma_k \cong T^2 \setminus N$ by Theorem 2.3.

We note at this point that if $O_\pi$ satisfies $(k - 1)\alpha^2 + (k - 1)\alpha\beta - \beta^2 = \omega$ for $\omega > 0$, then either all points in $O_{(\alpha,\beta)}$ satisfy $\alpha > 0$, or they all satisfy $\alpha < 0$. If not, then since $O_{(\alpha,\beta)}$ is connected, $O_{(\alpha,\beta)}$ must intersect the $y$-axis, so that $\alpha = 0$ and $-\beta^2 = \omega$, a contradiction. Similarly, if $O_{(\alpha,\beta)}$ satisfies $(k - 1)\alpha^2 + (k - 1)\alpha\beta - \beta^2 = \omega$ for $\omega < 0$, then either all points in $O_{(\alpha,\beta)}$ satisfy $\beta > 0$, or they all satisfy $\beta < 0$. 
Suppose \( \mathcal{O}(\alpha, \beta) \) satisfies \((k-1)\alpha^2 + (k-1)\alpha\beta - \beta^2 = \omega \) for \( \omega > 0 \).

**Case 1.** Suppose all \((\alpha, \beta) \in \mathcal{O}(\alpha, \beta)\) satisfy \( \alpha > 0 \). Then we have

\[
\phi(t, z_1, z_2) = \sum_{i=1}^{\text{mul}_\pi} \sum_{n \in \mathbb{Z}} f_i(t + n) z_1^{\alpha_{n,i}} z_2^{\beta_{n,i}}
\]

where \(\alpha_{n,i} > 0\) for all \(n, i\). Fixing \(t\), we have

\[
\phi_t(z) = \phi(t, z, 1) = \sum_{i=1}^{\text{mul}_\pi} \sum_{n \in \mathbb{Z}} f_i(t + n) z^{\alpha_{n,i}}
\]

a curve of winding number zero on the circle \(T\). The sum in (2.34) is always finite.

We may consider \(\phi_t\) to be the restriction of a polynomial \(\Phi_t\) on \(\mathbb{C}\) to \(T\), i.e. \(\Phi_t(\omega) = \sum_{i=1}^{\text{mul}_\pi} \sum_{n \in \mathbb{Z}} f_i(t + n) \omega^{\alpha_{n,i}}\). Since \(\alpha_{n,i}\) is never zero, \(\Phi_t\) has no constant term and so has a zero at \(\omega = 0\). Since \(\Phi_t\) has no pole inside \(T\), we see, referring to (2.20) that \(\Phi_t\) and hence \(\phi_t\) cannot have winding number zero on \(T\). Therefore \(\phi\) cannot be null-homotopic on \(N \cap \Gamma_k \setminus N\) for fixed \(t\), and so \(\phi\) cannot be nonvanishing.

**Case 2.** Suppose all \((\alpha, \beta) \in \mathcal{O}(\alpha, \beta)\) satisfy \(\alpha < 0\). Then set \(\phi_t(z) = \phi(t, z^{-1}, 1)\) for fixed \(t\) and proceed as in Case 1.

Clearly, if \(\mathcal{O}(\alpha, \beta)\) satisfies \((k-1)\alpha^2 + (k-1)\alpha\beta - \beta^2 = \omega \) for \(\omega < 0\), we examine

**Case 3.** Suppose all \((\alpha, \beta) \in \mathcal{O}(\alpha, \beta)\) satisfy \(\beta > 0\). Then set \(\phi_t(z) = \phi(t, 1, z)\) for fixed \(t\) and proceed as in Case 1.

**Case 4.** Suppose all \((\alpha, \beta) \in \mathcal{O}(\alpha, \beta)\) satisfy \(\beta > 0\). Then set \(\phi_t(z) = \phi(t, 1, z^{-1})\) and proceed as in Case 1.

Thus we have proof that all \(f \in K_\pi\) have zeros, and so we finish with the following corollary.

**Corollary 2.9.** If all functions \(f \in K_\pi\) satisfy \(f(\Gamma_k(t, x, y)) = 0\) at some point \(z_n = \Gamma_k(t_n, x_n, y_n)\), then every function \(\phi \in H^0_\pi\) has at least one zero.
Proof. Suppose $\phi \in H^0_\pi$. Let $\{\phi_n\}_{n=1}^\infty \subseteq K_{\pi}$, $\phi_n \longrightarrow \phi$ uniformly on $M_{H,k}$.

Let the sequence $\{z_n\}_{n=1}^\infty$ of points on $M_{H,k}$ satisfy $\phi_n(z_n) = 0$ for each $n$, and suppose $z$ is the limit in $M_{H,k}$ of some subsequence $\{z_{\sigma(n)}\}$ of $\{z_n\}$.

If we choose $n$ large enough that for $k > n$, $\|\phi_k - \phi\|_\infty < \varepsilon/2$ and $|\phi(z_n) - \phi(z)| < \varepsilon/2$ (by uniform continuity), then

$$|\phi_k(z_k) - \phi(z)| < |\phi_k(z_k) - \phi(z_k)| + |\phi_k(z_k) - \phi(z)| < \varepsilon.$$ 

However, since $\phi_k(z_k) = 0$, we have $|\phi(z)| < \varepsilon$ for all $\varepsilon > 0$, so that $\phi(z) = 0$. 
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Major Field: Mathematics (Analysis)

Title of Dissertation: Continuous functions in $\pi$-primary summands of $L^2$ of some compact solvmanifolds.

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