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Phase Estimation in Linear and Nonlinear Interferometers

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PHASE ESTIMATION IN LINEAR AND NONLINEAR INTERFEROMETERS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

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by
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In the loving memory of my late grandmother, Meera Devi Upadhyay. Without her effort and support, I would not have been where I am today.
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# Table of Contents

ACKNOWLEDGMENTS ........................................................................ iii

LIST OF FIGURES ........................................................................ vi

ABSTRACT ...................................................................................... viii

CHAPTER
1 INTRODUCTION TO QUANTUM OPTICS AND QUANTUM METROLOGY ........................................ 1
1.1 Brief history of light ................................................................. 1
1.2 Theoretical foundations of quantum optics ................................. 4
1.3 Brief review of quantum metrology ......................................... 12
1.4 Discussion .............................................................................. 17

2 PHASE ESTIMATION WITH A MULTIMODE INTERFEROMETER .............. 18
2.1 Background ........................................................................... 18
2.2 Multimode interferometer ....................................................... 19
2.3 Scattershot metrology with probabilistic photon sources ............... 30
2.4 Discussion .............................................................................. 31

3 PHASE ESTIMATION WITH A SU(1,1) INTERFEROMETER ............ 33
3.1 Introduction to SU(1,1) interferometer .................................... 33
3.2 Phase estimation with SU(1,1) interferometer ......................... 36
3.3 Discussion .............................................................................. 50

4 QUANTUM FISHER INFORMATION IN A SU(1,1) INTERFEROMETER 52
4.1 Overview of QFI analysis in a Mach-Zehnder interferometer .......... 52
4.2 SU(1,1) and phase configurations for single phase estimation .......... 55
4.3 Single phase estimation with an arbitrary and a vacuum state .......... 58
4.4 Two-parameter phase estimation ............................................ 62
4.5 Discussion .............................................................................. 65

5 SUMMARY .................................................................................... 67

REFERENCES .................................................................................. 69

APPENDIX
A QFI FOR MULTIPARAMETER PHASE ESTIMATION ....................... 77

B PHASE ESTIMATION WITH SU(1,1) INTERFEROMETER ................ 79
B.1 Coherent State and Displaced-Squeezed-Vacuum State ............... 79
B.2 Thermal State and Squeezed-Vacuum State ............................ 82

C QFI FOR SU(1,1) INTERFEROMETER .......................................... 84
List of Figures

1.1 Various states of light shown in term of its quadratures $\hat{X}_1$ and $\hat{X}_2$ in phase space. ................................................................. 11

1.2 A schematic of a Mach-zehnder interferometer which can be used to measure phase difference $\varphi$. ......................................................... 12

2.1 Architecture of the proposed parallel QuFTI optical interferometer, which simultaneously measures $d$ independent unknown phases $\{\varphi_j\}_{j=1}^d$. ....................................................... 20

2.2 Reck decomposition of a three-mode unitary $\hat{V}$. Any unitary can be decomposed using $\frac{n(n-1)}{2}$ beam splitters and phase shifters. ............ 21

2.3 A cascade of sequential QUMI. Each of the interferometer measures a single phase. ................................................................. 27

2.4 Total variance $(\Delta \vec{\varphi}^2)$ with different metrological strategies to estimate $(d = m - 1)$ phases. ......................................................... 29

2.5 The total variance $(\Delta \vec{\varphi}_{\text{avg}})$ for scattershot four-mode, three-phase parallel QuFTI. ................................................................. 31

3.1 A schematic of a SU(1,1) interferometer. ........................................... 34

3.2 Phase sensitivity $\Delta \phi$ with a thermal state and a squeezed vacuum state (black) and HL (blue). ........................................................ 43

3.3 Phase sensitivity $\Delta \phi$ (black) as a function of $r$, along with HL (blue) and SNL (red). ................................................................. 44

3.4 The effect on the phase sensitivity with the increase in the squeezing parameter $r$. ................................................................. 46

3.5 Phase sensitivity $\Delta \phi$ as a function of the gain parameter $g$ of the OPA. ......................................................................................... 47

3.6 Phase sensitivity $\Delta \phi$ with a coherent and a DSV state with on-off detector. ..................................................................................... 48

3.7 The phase sensitivity $\Delta \phi$ in the presence of photon loss. ................. 50

4.1 A Mach-Zehnder interferometer with different phase configurations. ................................................................. 53

4.2 Schematic of a SU(1,1) interferometer with different phase shifts. ....... 56
4.3 Comparison of the QFI of the phase-averaged state. &nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nbsp;&nb
Abstract

Phase estimation has a wide range of applications. Over the years, several strategies have been studied to improve precision in phase estimation. These strategies include using exotic quantum states to quantum detection schemes. This dissertation summarizes my effort in improving the precision of phase estimation with a linear and nonlinear interferometer.

Chapter 1 introduces quantum optics and quantum metrology. I introduce all relevant quantum states of light used. We also look into tools and terminologies of quantum metrology such as Fisher information, shot-noise limit, Heisenberg limit, etc., along with examples of phase estimation with a Mach-Zehnder interferometer.

In Chapter 2, I discuss multiple phase estimation using a multimode interferometer. Building upon previous work, our scheme consists of a multimode interferometer with single-photon inputs. By using a quantum Fisher information analysis, we show that our scheme gives a constant improvement over other schemes. We also show that our scheme with photon-number-resolving detection approaches the quantum Cramér-Rao bound. Moreover, we also consider the probabilistic nature of photon emission at the input, and we study its effect on phase sensitivity.

I discuss phase estimation with SU(1,1) interferometer in Chapter 3. We look at phase sensitivity in this interferometer with different input states. Namely, we consider two different phase estimation scheme, one using thermal and squeezed states, and others using coherent and displaced squeezed states with parity and on-off as a detection scheme. We also look into the effect of photon loss inside the interferometer.

In Chapter 4, we revisit phase estimation in SU(1,1) interferometer from the perspective of quantum Fisher information. I discuss in detail a longstanding confusion regarding the use of quantum Fisher information in SU(1,1) interferometer. We show that phase averaging or quantum Fisher information matrix method is needed in general for calculating the phase sensitivity which resolves inconsistencies reported in previously published articles.
Chapter 1
Introduction to Quantum Optics and Quantum Metrology

In this chapter, I briefly review concepts in quantum optics and quantum metrology that are essential for our work in the following chapters. First, we look into human’s journey of more than 2000 years to understand the nature of light. Then, we delve into quantum optics. For completeness, I start with the quantization of electromagnetic fields and then we look into different types of light along with their phase space representations. After that, I formally introduce quantum metrology and the task of phase estimation. We look in detail several concepts such as Fisher information both classical and quantum. I also define other terminologies such as shot-noise limit, Heisenberg limit and the quantum and classical Cramér-Rao bound.

1.1 Brief history of light

Since the dissertation is concerned with the applications of “Quantum light”, let us start with a brief history of Mankind’s quest to understand light. Now it is very easy to answer if somebody asks, What is Light? We can instantly say it is electromagnetic radiation. But it took thousands of years for people to come up with this definition and understand its true nature. In the ancient world, many philosophers related light to fire which was easy to relate as can it be seen with eyes. Philosophers of ancient India and Greece thought of light to consist of a small ball of fire. In 300 BC, Euclid wrote Optica in which he studied the properties of light [1]. He correctly postulated that light travels in a straight line and described the laws of reflection and also studied them mathematically. In 55 BC, Lucretius proposed a particle theory of light, which was not generally accepted [2]. Ptolemy in the 2nd century wrote about the laws of refraction [3].

However most of the knowledge from ancient Greece was lost and several centuries later, Ibn al-Haytham, a middle eastern scientist made significant contributions for which he is
considered as one of the founders of modern optics. Ptolemy and Aristotle had suggested that light shone from the eye to illuminate objects whereas Ibn al-Haytham postulated that light travels to the eye in rays from different points on an object [4]. Rene Descartes suggested objects has to be illuminated to be seen and rejected the ideas of Ibn al-Haytham [5]. In 1637 he published a theory of refraction of light that assumed, incorrectly, that light traveled faster in a denser medium than in a less dense medium. Although Descartes was incorrect about the relative speed, he was correct in assuming that light behaved like a wave and refraction can explain the different speed of light in different media. Pierre Gassendi proposed a particle theory of light. Isaac Newton studied Gassendi’s work at an early age and adapted his view to Descartes’ theory. He stated in “Hypothesis of Light” that light was composed of particle of matter. One of Newton’s argument against the wave-nature of light was that waves bend around obstacles while light only travels in straight line. However, he did describe diffraction by suggesting that light particle does create a localized wave in aether. Newton published his final theory in “Opticks” in 1704 [6].

In parallel, Christiaan Huygens worked out a mathematical wave theory of light in 1678 [7]. He proposed that light is emitted in all directions as waves in a medium called luminiferous aether. The wave theory predicted that light waves could interfere with each other like sound waves. Thomas Young demonstrated diffraction in an experiment, further confirming the wave nature of light. He also proposed that different colors were caused by different wavelengths [8]. In 1816, Francois Arago and Augustin-Jean Fresnel showed that polarization of light can be explained by wave theory if light were a transverse wave. There was only one weakness on the wave theory of light [9]. It required a medium or hypothetical substance luminiferous aether for light to propagate. However, the famous experiment by Michelson and Morley in the 1880s conclusively disproved the existence of such a substance [10].

In 1845, Michael Faraday observed that the plane of linearly polarised light is rotated when the light waves travel along the magnetic field direction in the presence of a trans-
parent dielectric (Faraday rotation). This was the first evidence that light was related to electromagnetism. He also proposed that light is a high-frequency electromagnetic vibration, which could propagate even in the absence of a medium [11]. James Clerk Maxwell was influenced by his work and he studied electromagnetic radiation and light. Maxwell discovered that self-propagating electromagnetic waves can travel through space at a constant speed, which happened to be equal to the previously measured speed of light. Maxwell thus concluded that light is a form of electromagnetic radiation on series of paper in the early 1860s papers “On Physical Lines of Force” [12]. In 1873, he published “A Treatise on Electricity and Magnetism” with a full mathematical description of the behavior of electric and magnetic fields, now known as Maxwell’s equations [13]. Soon after that Heinrich Hertz confirmed Maxwell’s theory experimentally [14].

In 1901, Max Planck, in an attempt to describe blackbody radiation, suggested that even though light is a wave, these waves can only gain or lose energy in a finite amount. Planck called this a “quantum” of light energy [15]. In 1905, Albert Einstein used the idea of light quanta to explain the photoelectric effect and suggested that these light quanta exist in nature [16]. In 1923, Arthur Holly Compton showed that the wavelength shift seen when low intensity X-rays scattered from electrons can only be explained by particle theory [17]. In 1926, Gilbert N. Lewis named these “light quanta” as photons and Dirac published his seminal paper on the quantum theory of radiation one year later [18]. Modern quantum optics was essentially born in 1956 with the work of Hanbury Brown and Twiss [19]. The invention of the laser in 1960 led to new interest. In 1960s, Glauber and others described new states of light which have different statistical properties to those of classical light along with quantum description of coherence. Several experiments were conducted in the 1970s and 1980s confirming the prediction of the theorists, and with rapid development in technology, the field of Quantum Optics was firmly established with many applications [20, 21].
1.2 Theoretical foundations of quantum optics

To describe quantum states of light, we need to adopt a quantum description. A state of light is defined by a state vector $|\psi\rangle$ for a pure state and by a density matrix $\rho$ for a mixed state. These quantities contain all the information about the underlying physical system (light). In our study, relevant quantum optics states are coherent state, squeezed state, vacuum state, thermal state, Fock state and displaced-squeezed state (also called coherent squeezed state). To retrieve information from these systems, we need to make a measurement on the system. If $A$ denotes a Hermitian observable of the system that we want to measure, then the expectation value of the measurement is given by $\langle A \rangle = \langle |\psi\rangle A |\psi\rangle$ for pure states and by $\langle A \rangle = \text{tr}(\rho A)$ in general. In our study, relevant measurement correspond to photon number resolving detection, parity detection and on-off detection, which I discuss in the following sections.

1.2.1 Quantization of electromagnetic field

The following sections follow from “Introductory Quantum Optics” by Gerry and Knight [22]. For completeness, let us start with the quantization of classical field equation of electromagnetism in free space without any radiation source, namely Maxwell’s equations, given by:

\begin{align*}
\nabla \cdot \vec{E} &= 0, \\
\nabla \cdot \vec{B} &= 0, \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\
\nabla \times \vec{B} &= \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}.
\end{align*}

Since the electromagnetic field does not have a source, we can work in Coulomb gauge, i.e.,

\begin{equation}
\nabla \cdot \vec{A} = 0,
\end{equation}
Using this Gauge condition, we can get the vector wave equation for the vector potential \( \vec{A}(\vec{r},t) \):

\[
\nabla^2 \vec{A}(\vec{r},t) = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2}, \tag{1.6}
\]

We can separate out the vector potential \( \vec{A}(\vec{r},t) \) into two terms and restrict the field to a finite volume \( V = L^3 \) considering a cubical cavity of side \( L \) such that \( \vec{A}(\vec{r},t) \) can be written in terms of a discrete set of orthogonal mode functions \( \vec{u}_k(\vec{r}) \) corresponding to frequency \( \omega_k \) as:

\[
\vec{A}(\vec{r},t) = \sum_k c_k \vec{u}_k(\vec{r}) e^{-i\omega_k t} + \sum_k c_k^\dagger \vec{u}_k^*(\vec{r}) e^{i\omega_k t}, \tag{1.7}
\]

Substituting the above equation in Eq. 1.6, we get,

\[
\left( \nabla^2 + \frac{\omega_k^2}{c^2} \right) \vec{u}_k(\vec{r}) = 0, \tag{1.8}
\]

The solution of the above equation takes the form:

\[
\vec{u}_k(\vec{r}) = \frac{1}{L^{3/2}} \hat{\epsilon}_\lambda e^{i\vec{k} \cdot \vec{r}}, \tag{1.9}
\]

where \( \hat{\epsilon}_\lambda \) is the unit polarization vector perpendicular to the wave vector \( \vec{k} \) and the components of the \( \vec{k} \) is given by:

\[
k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L}, \tag{1.10}
\]

Thus, the vector potential \( \vec{A}(\vec{r},t) \) can be written as:

\[
\vec{A}(\vec{r},t) = \sum_k \left( \frac{\hbar}{2\omega_k \epsilon_0} \right)^{1/2} \left[ a_k \vec{u}_k(\vec{r}) e^{-i\omega_k t} + a_k^\dagger \vec{u}_k^*(\vec{r}) e^{i\omega_k t} \right]. \tag{1.11}
\]
The associated electric field can then be written as:

$$\vec{E}(\vec{r}, t) = \sum_k \left( \frac{\hbar}{2\omega_k \epsilon_0} \right)^{1/2} [a_k \vec{u}_k(\vec{r}) e^{-i\omega_k t} + a_k^\dagger \vec{u}_k^*(\vec{r}) e^{i\omega_k t}] \right].$$  \hspace{1cm} (1.12)

We can now quantize the electric field, which we can do by extending the amplitudes $a_k$ and $a_k^\dagger$ to be mutually adjoint operators obeying the commutation relations:

$$[\hat{a}_k, \hat{a}_{kl}] = [\hat{a}_k^\dagger, \hat{a}_{kl}^\dagger] = 0 \text{ and } [\hat{a}_k, \hat{a}_{kl}^\dagger] = \delta_{kk,l}. \hspace{1cm} (1.13)$$

The operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ are called annihilation and creation operators for a quantum mechanical harmonic oscillator respectively. An ensemble of independent harmonic oscillators typically describe the modes of the electric field. The Hamiltonian of the field is then given by:

$$H = \frac{1}{2} \int_v \left[ \epsilon_0 \vec{E}^2 + \frac{1}{u_0} \vec{B}^2 \right] d^3 r = \sum_k \hbar \omega_k \left( \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right). \hspace{1cm} (1.14)$$

which in the second part represents the sum of two terms. The first is the number of photons in each mode of the radiation field multiplied by the energy of a photon $\hbar \omega_k$ and the second term is the energy of the vacuum fluctuations in each mode of the field.

1.2.2 Field quadratures

Other types of operators can be constructed by pairing annihilation and creation operators that is widely used in quantum optics to describe the noise of the electromagnetic radiation. These are called “Quadrature Operators”. These operators are basically dimensionless quantities corresponding to the position and momentum operators. The two Hermitian quadrature operators $\hat{X}_1$ and $\hat{X}_2$ are defined as:

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2}, \hspace{1cm} (1.15)$$

$$\hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}. \hspace{1cm} (1.16)$$
They satisfy the following commutation and Heisenberg uncertainty relation given by:

\[
[\hat{X}_1, \hat{X}_2] = \frac{i}{2}, \quad \langle (\Delta \hat{X}_1)^2 \rangle \langle (\Delta \hat{X}_2)^2 \rangle \geq \frac{1}{16}.
\] (1.17)

### 1.2.3 Fock states

One of the exotic quantum states is a Fock state, which has a well-defined number of particles. These states, denoted by \(|n\rangle\), are the eigenstates of the Hamiltonian \(H\) with eigenvalue \(n\), where \(n\) is the number of photons in the field. The energy of a single photon in a mode of frequency \(\omega\) is \(E = h\omega\). Here, for simplicity, we assume that all these states are single mode. The operator \(\hat{n} = \hat{a}^\dagger \hat{a}\) is known as the photon number operator and acts on the state \(|n\rangle\) as:

\[
\hat{n}|n\rangle = n|n\rangle \quad (n = 0, 1, 2, 3...).
\] (1.18)

giving the total number of photons in state \(|n\rangle\). These states are orthonormal, i.e. \(\langle m|n \rangle = \delta_{nm}\) and form a complete set \(\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}\). Hence, any arbitrary state can be expanded as a combination of these states. The action of the creation and annihilation operators on the number state \(|n\rangle\) is given by:

\[
\hat{a}|n\rangle = \sqrt{n}|n - 1\rangle,
\] (1.19)

\[
\hat{a}^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle.
\] (1.20)

The number state \(|n\rangle\) can be obtained from the ground state \(|0\rangle\) by successive operation of the creation operator as follows:

\[
|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle.
\] (1.21)

For Fock states, the mean value of the quadrature operators vanish, i.e.,

\[
\langle n|\hat{X}_1|n\rangle = \langle n|\hat{X}_2|n\rangle = 0.
\] (1.22)
These states have equal uncertainties in both quadratures given by:

\[
\langle (\Delta \hat{X}_1)^2 \rangle = \langle (\Delta \hat{X}_2)^2 \rangle.
\] (1.23)

The vacuum state \( (n = 0) \) minimizes the uncertainty product. Fock states are highly nonclassical, and they have a well-defined photon number but a completely random phase distribution. An intuitive way to picture these quantum states is using a phase space diagrams which is basically the pictorial view of the quadrature operators introduced in the last section. A phase space diagram graphically shows the uncertainty a given state has in the two quadratures. In Figure 1.1, the blue ring represents the Fock state. This radius depends on the photon number chosen.

### 1.2.4 Coherent states

Coherent states \( |\alpha\rangle \) are a theoretical model of the output of a laser. They are defined as the eigenstates of the annihilation operator given by:

\[
\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.
\] (1.24)

and \( |\alpha|^2 \) is the amplitude of the coherent state. From the previous section, we know that Fock states (number states) form a complete basis, and hence we can write a coherent state \( |\alpha\rangle \) as a superposition of Fock states as:

\[
|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.
\] (1.25)

These states can be generated by displacing the vacuum state \( |0\rangle \) with a displacement operator \( \hat{D}(\alpha) \) as:

\[
|\alpha\rangle = \hat{D}(\alpha)|0\rangle.
\] (1.26)
And the displacement operator \( \hat{D}(\alpha) \) is defined as: \( \hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \). The probability of finding \( n \) photons in a coherent state \( |\alpha\rangle \) is given by a Poisson distribution:

\[
P_n = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}.
\] (1.27)

A coherent state, like the vacuum state, is also a minimum uncertainty state, and they have equal uncertainties in both \( \hat{X}_1 \) and \( \hat{X}_2 \) quadratures, i.e. \( \langle (\Delta \hat{X}_1)^2 \rangle = \langle (\Delta \hat{X}_2)^2 \rangle = \frac{1}{4} \).

The coherent state, unlike Fock states, are non-orthogonal, i.e.,

\[
\langle \beta | \alpha \rangle = \exp \left[ \frac{1}{2} (\beta^* \alpha - \beta \alpha^*) \right] \exp \left[ -\frac{1}{2} |\beta - \alpha|^2 \right].
\] (1.28)

Although they are not orthogonal, they can be used as a basis set as they span the Hilbert space and there are always enough states to express any state in terms of coherent states. This property is termed as “over completeness”. The yellow circle shows the coherent state in Figure 1.1.

**1.2.5 Thermal states**

Another type of state of light we consider is a thermal state of light. Thermal states are mixed states and are given by a density operator as:

\[
\rho = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{1 + \bar{n}} \right)^n |n\rangle \langle n|.
\] (1.29)

The probability of finding \( n \) photons in the field is given by:

\[
P_n = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}}.
\] (1.30)

The thermal state is shown in red in Figure 1.1. The thermal state also shares the origin with the vacuum state but it is larger giving a hint about the noise in its quadratures.
1.2.6 Squeezed states

We have seen in the previous sections and from the quadrature diagrams that there is some uncertainty for quantum-mechanical light fields. This minimum area is upper bounded by the Heisenberg uncertainty relations. But the only restriction that the uncertainty principle puts is on the total area and not on the shape of the uncertainty area. That is, we can decrease the uncertainty in one quadrature at the expense of increasing it in the other quadrature. States with these properties are called squeezed states $|\xi\rangle$, and can be generated by applying the single-mode unitary squeezing operator $\hat{S}$ on the vacuum state $|0\rangle$, i.e. $\hat{S}(\xi)|0\rangle = |\xi\rangle$ and the operator $\hat{S}$ is given by:

$$\hat{S}(\xi) = \exp\left\{\left(\frac{1}{2}\xi \hat{a}^{\dagger}^2 - \frac{1}{2}\xi^* \hat{a}^2\right)\right\}, \xi = re^{i\phi}. \quad (1.31)$$

Here $\xi$ is called the squeezing parameter. The mean number of photons in the squeezed state is $\langle \hat{a}^\dagger \hat{a} \rangle = \sinh^2 r$. We can also write the squeezed state in terms of Fock state as:

$$|\xi\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} e^{in\phi} (\tanh r)^n \frac{\sqrt{(2n)!}}{(n!)^2 2^n} |2n\rangle. \quad (1.32)$$

It is clear that $|\xi\rangle$ only contains even number of photons in the superposition. It can be shown easily that for these states the quadrature uncertainty is given by:

$$\langle (\Delta \hat{X}_1)^2 \rangle = \frac{1}{4} e^{2r}, \quad (1.33)$$

$$\langle (\Delta \hat{X}_2)^2 \rangle = \frac{1}{4} e^{-2r}. \quad (1.34)$$

The squeezed coherent state is shown in green in Figure 1.1 in the lower quadrant. It has unequal quadrature noise. It is squeezed to reduce the noise in the $\hat{X}_2$ quadrature at the expense of increasing it in the conjugate $\hat{X}_1$ quadrature. Finally, we consider a two-mode squeezed vacuum state $|\xi\rangle_{\text{TMSV}}$ that can be generated by applying the two-mode unitary
squeezing operator:
\[ \hat{S}(\xi) = \exp\left(\xi \hat{a}^{\dagger} \hat{b}^{\dagger} - \xi^{*} \hat{a} \hat{b}\right). \]  

(1.35)

on the two-mode vacuum state, i.e. \( \hat{S}(\xi)|0, 0\rangle = |\xi\rangle_{TMSV} \), where \( \hat{a} \) and \( \hat{b} \) denote the annihilation operators for the two modes. The two mode squeezed vacuum state \( |\xi\rangle_{TMSV} \) can be written in terms of the Fock states \( |n\rangle \) as:

\[ |\xi\rangle_{TMSV} = \frac{1}{\cosh r} \sum_{n=0}^{\infty} e^{in\phi} (\text{tanh } r)^n |n_a, n_b\rangle. \]  

(1.36)

We won’t specifically use this state in our work but these states are generated with nonlinear processes such as in four-wave mixing (FWM) and optical parametric oscillator (OPA). We will use OPA in Chapter 3 in SU(1,1) interferometer.

Figure 1.1: Various states of light shown in term of its quadratures \( \hat{X}_1 \) and \( \hat{X}_2 \) in phase space. The vacuum state (black) is at the origin of phase space along with the thermal state. The coherent (yellow) and squeezed coherent state (green) is displaced from the origin. The Fock state (blue) appears as a ring around the origin.
1.3 Brief review of quantum metrology

In the previous section, I discussed different states of light. Now, we are ready to look into the field of quantum metrology. First, I define the important task of phase estimation and introduce Mach-Zehnder interferometer for achieving this task. We also look in depth the concept of Fisher information, both classical and quantum. Using this, we define both classical and quantum Cramér-Rao bound which lower bounds the precision of estimation. Finally, to get a feel of quantum metrology, we look at examples of phase estimation in Mach-Zehnder interferometer using different states of light and also introduce metrics such as shot noise limit and Heisenberg limit.

![Figure 1.2: A schematic of a Mach-zehnder interferometer which can be used to measure phase difference $\varphi$.](image)

1.3.1 Introduction to quantum metrology

One of the fundamental tasks in quantum metrology is phase estimation. The task is as follows: we want to estimate a physical parameter (phase) using light and we want to achieve maximum precision possible so as to minimize the uncertainty about the value
of the phase. To achieve this task, we use an optical interferometer, which makes use of interference of light. Figure 1.2 shows a Mach-Zehnder interferometer. It consists of two input ports in which different states of light can be fed in. The input light is mixed in a 50:50 beam splitter. The light coming out of the beam splitter interacts with the phase $\varphi$ and the information about this phase is encoded in the state of light. Finally, the light interacts with the second beam splitter and are detected. Although our goal is to measure the accuracy in measuring $\varphi$, we do not measure this directly. What we measure is the statistics of the light at the output and using these statistics, we can infer about the phase $\varphi$. Over the past several years, several strategies choosing different input states and detection schemes have been studied with the goal of minimizing uncertainty in estimating $\varphi$ [23]. The precision of estimating $\Delta \varphi$ can be obtained using simple error-propagation formula. However, for a more firm theoretical footing, we use Fisher information, which I discuss in the following sections.

### 1.3.2 Classical Fisher information and classical Cramér-Rao bound

The central task in estimation theory is to determine quantities which may or may not be directly observed. In other words, we want to determine a parameter $\varphi$ based on the $k$ measurements of $x_i$, $i = 1, 2, ..., k$ which might vary from shot to shot [24, 25].

Let $p(x|\varphi)$ denote conditional probability of obtaining measurement result $x$ given that our unknown parameter has value $\varphi$. Our goal is to construct an estimator $\hat{\varphi}(x)$ which is a function that outputs the most accurate estimate of the parameter $\varphi$ based on the given data set. Since the given data set itself is probabilistic, $\hat{\varphi}(x)$ is also a probabilistic quantity. To quantify this discrepancy, quadratic cost $(\hat{\varphi}(x) - \varphi)$ can be considered. Since the value of the cost changes on every trial of the experiment, we are interested in the mean squared error (MSE) given by [24]:

$$
(\Delta \hat{\varphi})^2 = \int p(x|\varphi)(\hat{\varphi}(x) - \varphi)^2 dx. 
$$

(1.37)
One of the main tasks in estimation theory is to find estimators, which minimize the MSE, i.e. they give the best possible precision. The estimator that minimizes Eq. (1.37) is called efficient estimator and there might exist many efficient estimators for the same estimation problem [24]. There are certain features to look for when deciding on an estimator called consistency. An estimator $\hat{\varphi}$ is consistent if in the limit of a large number of repetitions for every $\varphi$, it returns the true value of the parameter:

$$\lim_{k \to \infty} \hat{\varphi}(x) = \varphi.$$  

(1.38)

Another desired feature for estimator is unbiasedness. An estimator is called unbiased if on average it returns the correct value of the parameter for every $\varphi$,

$$\int p(x|\varphi)\hat{\varphi} = \varphi.$$  

(1.39)

An optimal unbiased estimator is the one that minimizes $(\Delta \hat{\varphi})^2$ for all $\varphi$. Finding an optimal estimator is very difficult, however one can always construct the Cramér-Rao Bound (CRB) that lower-bounds the MSE of any unbiased estimator $\hat{\varphi}$ as[23, 24]:

$$(\Delta \hat{\varphi})^2|_{\varphi} \geq \frac{1}{F_{\varphi}}.$$  

(1.40)

where $F_{\varphi}$ is the classical Fisher information (FI) given by [24]:

$$F_{\varphi} = \int \frac{1}{p(x|\varphi)}\left(\frac{dp(x|\varphi)}{d\varphi}\right)^2 dx = \int \left(\frac{d}{d\varphi}\ln p(x|\varphi)\right)^2 dx.$$  

(1.41)

Intuitively, the bigger the FI, the higher is the precision of estimation. In experiments, many repetitions are made resulting in a set of outcomes $\vec{x} = (x_1, x_2, \ldots, x_k)$ which are independent and identically distributed (i.i.d) random variables with probability distribution $p(x|\varphi)$. Therefore, a joint density is given by $p(\vec{x}|\varphi) = \prod_{i=1}^{k} p(x_i|\varphi)$, which factorizes as a product of individual probability and because of this property, FI becomes additive on
i.i.d variables,

\[ F^{(k)}_\phi = k F_\phi, \quad (\Delta \tilde{\phi})^2 |_\phi \geq \frac{1}{k F_\phi}. \]  

(1.42)

where \( F^{(k)}_\theta \) and \( F_\theta \) denotes FI for \( k \) repetitions and single experiment respectively.

The CRB is always saturable asymptotically in the limit of an infinite number of repetitions. Since in actual experiments, measurements are repeated many times to get statistically meaningful results, the CRB does provide a good approximate bound even though, in principle, it is not saturable for a finite number of trials.

1.3.3 Quantum Fisher information and quantum Cramér-Rao bound

In a quantum setting, the parameter \( \phi \) is encoded in a quantum state \( \rho_\phi \). Measurement \( M_x \) is performed on the state with result \( x \) and probability \( p_\phi(x) = \text{Tr}(\rho_\phi M_x) \). Designing optimal estimation strategy corresponds to accurately inferring the parameter \( \phi \) from the data but it also encompasses maximizing over all possible measurement to maximize the precision. This task is very difficult to accomplish in general. However, as in classical estimation, it is relatively easy to obtain useful lower bounds on the minimum MSE. The quantum Cramér-Rao bound (QCRB) is a generalization of the classical CRB, which lower bounds the variance of estimation for all possible unbiased estimators and most general measurement \([26, 27, 28]\). The QCRB is given by:

\[ (\Delta \tilde{\phi})^2 \geq \frac{1}{F_Q}. \]  

(1.43)

where \( F_Q \) is called the quantum Fisher information (QFI). Similar to CFI, QFI is also additive and is given by:

\[ F_Q = \text{Tr}(\rho_\phi L^2_\phi). \]  

(1.44)

where \( L_\phi \) is a Hermitian operator called symmetric logarithmic derivative (SLD). The calculation of QFI, in contrast to classical Fisher information is difficult since it requires
finding SLD which is given by an indirect operator equation:

\[
\frac{dp_\varphi}{d\varphi} = \frac{1}{2}(\rho_\varphi L_\varphi + L_\varphi \rho_\varphi).
\]  \hspace{1cm} (1.45)

This suggest that QFI is solely determined by the dependence of \(\rho_\varphi\) on the estimated parameter and hence allows us to analyze the sensitivity of a scheme without considering any particular measurements. For pure states, \(\rho_\varphi = |\psi_\varphi\rangle\langle\psi_\varphi|\), the QFI simplifies to [23]:

\[
F_Q(\psi_\varphi) = 4(\langle\dot{\psi}_\varphi^*|\dot{\psi}_\varphi\rangle - |\langle\dot{\psi}_\varphi^*|\psi_\varphi\rangle|^2), \quad |\dot{\psi}_\varphi\rangle = \frac{d|\psi_\varphi\rangle}{d\varphi}.
\]  \hspace{1cm} (1.46)

For a pure state and unitary evolution when the parameter is encoded on the state by generating Hamiltonian \(\hat{H}\), the QFI is proportional to the variance of \(\hat{H}\):

\[
F_Q(|\psi_\varphi\rangle) = 4(\Delta H)^2 = 4(\langle\psi|\hat{H}^2|\psi\rangle - \langle\psi|\hat{H}|\psi\rangle).
\]  \hspace{1cm} (1.47)

### 1.3.4 Quantum metrology with various states of light

I want to end this chapter with a few examples of doing phase estimation using various states of light. This also ultimately bring up the shot noise and the Heisenberg limit that we have not yet formally defined. Let us start with input states \(|\psi_{in}\rangle = |\alpha\rangle|0\rangle\), that is a coherent state in one arm and a vacuum state in the other. The precision of estimation with this input light is given by:

\[
\Delta \varphi = \frac{1}{\sqrt{\bar{n}}},
\]  \hspace{1cm} (1.48)

where \(\bar{n} = |\alpha|^2\) is the average photon number in the coherent state. This is called the shot noise limit and is the characteristic of the classical light field. This limit can be beaten if we use quantum states of light. For example, let’s say our input states are \(|\psi_{in}\rangle = |\alpha\rangle|\xi\rangle\), a coherent state in one arm and a squeezed vacuum state in the other. In the limit of large
photon number, the precision of estimation of this scheme is given by:

\[ \Delta \varphi = \frac{1}{\bar{n}^{3/4}} \quad (1.49) \]

As we can see, this strategy has better precision of estimation than just the coherent state. This precision can be further improved by using squeezed light in both ports of the interferometer, i.e. \( |\psi_{\text{in}}\rangle = |\xi\rangle |\xi\rangle \). This beats the Heisenberg limit given by:

\[ \Delta \varphi = \frac{1}{\bar{n}} \quad (1.50) \]

where \( \bar{n} \) is the average total photon number. This is called the Heisenberg limit. Not only squeezed states, \( N00N \) states can also achieve this precision of estimation which suggests entanglement is necessary for sub-shot-noise scaling. In fact, the main motivation of the quantum metrology strategies studied over the last several decades has been to demonstrate and achieve Heisenberg scaling in estimating phases.

1.4 Discussion

In this chapter, we looked at various states of light and its use in Mach-Zehnder interferometer. We also looked into estimation theory and discussed both classical and quantum Fisher information along with their corresponding Cramér-Rao bounds. In the following chapters, we use these basic concepts for phase estimation with both linear and nonlinear interferometers.
Chapter 2
Phase Estimation with a Multimode Interferometer

In Chapter 1, I introduced the basic concepts of quantum optics along with different states of light. I also discussed the task of phase estimation in quantum metrology. We looked in detail the concepts of both classical and quantum Fisher information. Now, we are ready to further delve deeper in the field of metrology using a multimode interferometer.\textsuperscript{1} First, I discuss motivations and previous works on the use of multimode interferometers. Then, I introduce and discuss in details our multimode interferometer scheme proposed in Ref. \cite{29} for multiple phase estimation and present results for doing multiple phase estimation simultaneously.

2.1 Background

We have already seen a few examples of phase estimation in Chapter 1 using different states of light. The main motivation of studying quantum metrology is to use quantum resources to achieve precision in phase estimation, which is otherwise impossible classically \cite{23, 30, 31, 32, 33, 34}. Although optical interferometers have been used as early as in the work of Mach and Zehnder, and Michelson and Morley to measure the relative phase shifts, many recent developments have been made in both experiment and in theory \cite{31, 35, 36, 37}. Technological advancements in generating and detecting quantum resources with high efficiency have been the primary factor in keeping researchers interested in this domain \cite{38}. There has been substantial progress in developing on-demand single photon sources and high-efficiency detectors \cite{39}. Researchers have also studied waveguides, which can be integrated into an all-optical chip and allows for an impressive level of fidelity in comparison to networks utilizing nonlinear optical elements and photon-number-resolving

detectors [40].

Although multimode interferometers, which are natural extension of two-mode interferometers, have been studied as early as in 1997 Ref. [41], they have been gaining much attention over the last few years [42, 43, 44, 45, 46]. One of the main reason is the possibility of generating number-path entanglement as various studies have shown entanglement to be a necessity for sub-shot-noise sensitivity. Motes et al., motivated by the possibility of generating number-path entanglement by the use of a multimode interferometer, introduced a scheme with single-photon inputs and bucket detectors, and showed sub-shot-noise sensitivity for a single phase estimation [47, 48].

With its multimode structure, it is intuitive that these interferometers can be used for multiple phase estimation simultaneously, which has implications for the wider research community, such as imaging. A quantum advantage in imaging would be of significant value in biology, especially for samples that are sensitive to light [38, 49, 50, 51, 52]. The earliest known theoretical work on multiparameter phase estimation comes from Ref. [53]. They studied the simultaneous estimation of multiple phases and showed that their scheme beats classical strategies. However, they used a very complicated state of light as an input. Motivated by a $N00N$ state, which attains Heisenberg limit for a two-mode interferometer, they generalized their input state to a multimode “$N00N$” state, making their scheme notoriously difficult to implement practically. In Ref [54], the authors studied multiple phase estimation (specifically, two and three phases) using three-and four-mode interferometer with single-photon inputs and they showed sub-shot-noise sensitivity.

### 2.2 Multimode interferometer

We now introduce our multimode interferometer for doing multiple phase estimation simultaneously. Our multimode interferometer resembles the architecture in Ref. [47]. Figure 2.1 shows a schematic of our multimode interferometer. It consists of a unitary $\hat{V}$, called a quantum Fourier transform matrix, and its conjugate $\hat{V}^{\dagger}$. A single photon is input in each port of the interferometer. These photons are first evolved passively using
Figure 2.1: Architecture of the proposed parallel QuFTI optical interferometer, which simultaneously measures $d$ independent unknown phases $\{\phi_j\}_{j=1}^d$. The interferometer consists of $m$ modes with an input of $m$ single photons, $|1\rangle^\otimes m$. The unitary $\hat{V}$ (and its conjugate) is a quantum Fourier transform implemented with a network of beam splitters and phase shifters.

A unitary $\hat{V}$. Then, the photons encounter multiple phases placed in the arms of the interferometer and interact with the phases. The phase information gets encoded in the photon statistics. After the photons pass through $\hat{V}^\dagger$, measurement is done at the output. The entire evolution unitary $\hat{U}$ is given by $\hat{U} = \hat{V}\hat{\Phi}\hat{V}^\dagger$. The elements of $\hat{V} = \{V_{ij}\}$ is given by:

$$V_{ij} = \frac{1}{\sqrt{m}} e^{2\pi(i-1)(j-1)/m}. \quad (2.1)$$

Mathematically, the phases are represented by $\hat{\Phi}$, and $\hat{\Phi} = \{\Phi_{kt}\}$ is a $m \times m$ diagonal matrix of $d$ independent phases $\vec{\phi} = \{\phi_j\}_{j=1}^d$, which we would like to measure. $\hat{\Phi}$ has the
Figure 2.2: Reck decomposition of a three-mode unitary $\hat{V}$. Any unitary can be decomposed using $\frac{n(n-1)}{2}$ beam splitters and phase shifters.

Form:

$$\Phi_{k\ell} = \begin{cases} 
\delta_{k\ell} \cdot e^{i\varphi_k} & k \leq d \\
\delta_{k\ell} & k > d 
\end{cases}$$

(2.2)

The above architecture is identical to that of Ref. [47], except for the form of $\Phi$. We will refer to this device as “parallel QuFTI”, where “QuFTI” stands for Quantum Fourier Transform Interferometer.

The unitary $\hat{V}$ can be implemented experimentally in a laboratory using $\frac{n(n-1)}{2}$ beam splitters and phase shifters [55]. As an example, the Reck decomposition of a three-mode $\hat{V}$ is shown in Figure 2.2.
We need to know the output state to get the information about the phases. The output state $|\psi_{\text{out}}\rangle$ of the interferometer is given by:

$$|\psi_{\text{out}}\rangle = \hat{U} |\psi_{\text{in}}\rangle = \sum_i \gamma^{(i)} |n_1^{(i)}, \ldots, n_m^{(i)}\rangle = \sum_i \gamma^{(i)} |n^{(i)}\rangle. \quad (2.3)$$

Here, $|n^{(i)}\rangle = |n_1^{(i)}, \ldots, n_m^{(i)}\rangle$ denotes the possible output photon configurations with $m$ total photons, i.e. $\sum_j n_j^{(i)} = m$. In general, for a total of $N$ photon inputted into the interferometer with $m$ input/output modes, the total number of possible configurations ($s$) is given by:

$$s = \binom{m + N - 1}{m}. \quad (2.4)$$

which is the number of ways to configure $N$ indistinguishable photons into $m$ distinguishable bins. For example, consider a two-mode interferometer, if the input is $|1, 1\rangle$, then the possible output configurations are $|2, 0\rangle$, $|0, 2\rangle$, and $|1, 1\rangle$. The sum in Eq. (2.3) is necessary because all possible configurations with a non-zero probability of occurring have to be included. The coefficients $\gamma^{(i)}$ of every output configuration are related to the matrix permanent of matrices closed related to $\hat{U}$ [56]. For a $n \times n$ matrix $M$ with complex entries $m_{i,j} \in \mathbb{C}$. The permanent of $M$ is defined as:

$$\text{perm}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^n m_{i,\sigma(i)}. \quad (2.5)$$

where $S_n$ is the symmetric group on $n$ elements.

To calculate $\gamma^{(i)}$, for the photon configuration $i$ and the associated matrix permanent $\text{perm}(W^{(i)})$, if we denote the $j$th row vector of $\hat{U}$ as $u_j$, then $W^{(i)}$ consists of $n_j^{(i)}$ rows of $u_j$. The matrix permanents are invariant under row interchange so the ordering of the rows is not important.
The amplitude coefficient $\gamma^{(i)}$ is given by:

$$
\gamma^{(i)} = \frac{\text{perm}(W^{(i)})}{\sqrt{n_1^{(i)}! \ldots n_m^{(i)}!}}.
$$

(2.6)

Although determinant of a matrix is easy to calculate, many might find it surprising that the calculation of a permanent of a matrix is very difficult. The computational complexity of the matrix permanents lies in the class $\#P - hard$, pronounced “sharp-P hard”. This already difficult task is further complicated in our scenario as our matrix can be complex.

Our goal is to use the multimode interferometer described above to estimate $d$ unknown phases. A straightforward generalization of the Cramér-Rao bound (CRB) (discussed in Chapter 1) would be to calculate the error of each parameter separately. However, this does not give a proper bound on the precision of estimation as it ignores the fact that various components of $\vec{\phi}$ may be correlated. Although each of the $\varphi_i$ are independent, one arm is used as a reference to measure the remaining phases establishing correlations. To quantify the error of this estimation procedure, a covariance matrix Cov($\vec{\phi}$) is used. CRB can now be generalized to include multiparameter cases [25, 26] by making a lower bound on the covariance matrix. The precision of estimation for our scheme is given by the inequality:

$$
|\Delta \vec{\phi}|^2 \equiv \sum_{j=1}^{d} \Delta \varphi_j^2 \equiv \text{Tr}[\text{Cov}(\vec{\phi})] \geq \frac{1}{\nu} \text{Tr}[\mathcal{F}_{\vec{\phi}}^{-1}],
$$

(2.7)

where $\nu$ is the number of independent trials and $\mathcal{F}_{\vec{\phi}} = \{\mathcal{F}_{i,j}^{\text{cl}}\}$ is the classical Fisher information (CFI) matrix given by:

$$
\mathcal{F}_{i,j}^{\text{cl}} = \sum_x \frac{1}{p(x|\vec{\phi})} \frac{\partial p(x|\vec{\phi})}{\partial \varphi_i} \frac{\partial p(x|\vec{\phi})}{\partial \varphi_j},
$$

(2.8)

and $p(x|\vec{\phi}) = |\langle x | \psi_{\text{out}} \rangle|^2$ is the probability of observing outcome $x$ conditioned on $\vec{\phi}$. Because of the dependence of the Fisher information on $\vec{\phi}$, it may be the case that the
measurement precision is best near certain values of \( \vec{\phi} \) as found in Refs. [48, 57].

We want to find the best precision we can get with our input states without considering any particular measurement strategy. This information is given by the multivariable generalization of the quantum Fisher information (QFI) discussed in Chapter 1. More precisely, the QFI depends only on the input states and the evolution unitary \( \hat{V} \). With the QFI, we can construct the quantum Cramér-Rao bound (QCRB) [26, 28] that lower-bounds the uncertainty in estimating \( \vec{\phi} \), which is independent of any measurement scheme and depends only on the probe state. The QCRB is identical to Eq. (2.7), except the classical Fisher information matrix \( \mathcal{F}_{i,j}^{\text{class}} \) is replaced by the quantum Fisher information (QFI) matrix [26] given by:

\[
\mathcal{F}_{i,j}^{\text{qfim}} = \frac{1}{2} \langle \psi_{\text{out}} | (L_i L_j + L_j L_i) | \psi_{\text{out}} \rangle,
\]

where \( L_i = 2(\partial_{\phi_i} \psi_{\text{out}}) \langle \psi_{\text{out}} | + | \psi_{\text{out}} \rangle (\partial_{\phi_i} \psi_{\text{out}}) \). Subsequently, we will refer to \( \mathcal{F}_{\vec{\phi}} = \{ \mathcal{F}_{i,j}^{\text{qfim}} \} \) to mean the QFI matrix. It is worth noting that the dimension of both the CFI matrix and the QFI matrix is equal to the number of phases we are estimating (which is \( d \) in our case).

It was shown by Humphreys et al. [53] that for an arbitrary pure input states of a multi-mode Fock states, the QFI matrix of the estimated phases is given as:

\[
\mathcal{F}_{\vec{\phi}} = 4 \sum_i |\gamma^{(i)}|^2 |n^{(i)}\rangle \langle n^{(i)}| - 4 \sum_{i,j} |\gamma^{(i)}|^2 |\gamma^{(j)}|^2 |n^{(i)}\rangle \langle n^{(j)}|,
\]

where \( \gamma^{(i)} \) are defined in Eq. (2.6). The QFI matrix is calculated as:

\[
[F_{\vec{\phi}}]_{l,n} = 4 \langle \hat{b}^\dagger_l \hat{b}_l \hat{b}^\dagger_n \hat{b}_n \rangle - 4 \langle \hat{b}^\dagger_l \hat{b}_l \rangle \langle \hat{b}^\dagger_n \hat{b}_n \rangle,
\]

where \( \hat{b}_l^\dagger = \sum_j V_{ij} \hat{a}^\dagger_j \) [58]. For a \( k \)-photon Fock state in every mode, the QFI for our setup
is given by:

\[
\mathcal{F}_\phi = 4k(k+1) \begin{pmatrix}
\frac{m-1}{m} & -\frac{1}{m} & \ldots & -\frac{1}{m} \\
-\frac{1}{m} & \frac{m-1}{m} & \ldots & -\frac{1}{m} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{m} & \frac{1}{m} & \ldots & \frac{m-1}{m}
\end{pmatrix}.
\] (2.12)

To construct the quantum Cramér-Rao bound, we need the inverse of the quantum Fisher information matrix which is given by [59]:

\[
\mathcal{F}_\phi^{-1} = \frac{1}{4k(k+1)} \begin{pmatrix}
\frac{m-d+1}{m-d} & \frac{1}{m-d} & \ldots & \frac{1}{m-d} \\
\frac{1}{m-d} & \frac{m-d+1}{m-d} & \ldots & \frac{1}{m-d} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{m-d} & \frac{1}{m-d} & \ldots & \frac{m-d+1}{m-d}
\end{pmatrix}.
\] (2.13)

Substituting the trace of \( \mathcal{F}_\phi^{-1} \) into Eq. (2.7) and recalling that the matrix is \( d \times d \), we arrive at the bound:

\[
|\Delta \phi|^2 \geq \frac{1}{\nu} \frac{1}{4k(k+1)} \frac{d(m-d+1)}{(m-d)}.
\] (2.14)

This is the quantum Cramér-Rao bound for our scheme, which we refer to as “parallel QuFTI”. This bound gives information about the ultimate precision that we can achieve from our choice of the input states and the evolving unitary \( \hat{V} \). We can see that larger \( k \) (higher Fock state) gives better sensitivity. More precisely, since \( |\Delta \phi|^2 \) scales inversely with \( k^2 \), this suggests an asymptotic improvement approaching the Heisenberg limit. However, higher Fock states are notoriously difficult to create [60]; therefore, for our analysis in the rest of the paper, we stick to \( k = 1 \) as on-demand single-photon sources are becoming experimentally viable quickly.
2.2.1 Resource counting

Our goal is to demonstrate that our proposed multimode interferometer performs better than other previously reported strategies. For this, we need to compare the precision of estimation of our proposed scheme with other existing schemes. We use two existing schemes for this purpose. The first scheme, which we refer to as “sequential QUMI”, was proposed by Olson et al. [48]. The second benchmark that we use is the classical strategy, where the inputs are uncorrelated coherent states $\otimes_{i=1}^{m} |\alpha_i\rangle$. For a fair comparison, we will use the same number of photons as a resource in each of these schemes. From Eq. (2.14), for $k = 1$, we have:

$$|\Delta \vec{\phi}_1|^2 = \frac{1}{\nu_1} \frac{(m - d + 1)d}{8(m - d)}.$$  \hspace{1cm} (2.15)

where $\nu_1$ is the number of measurements for the parallel QuFTI. The precision of estimation of sequential QUMI is given by:

$$|\Delta \vec{\phi}_2|^2 = \frac{1}{\nu_2} \left( \frac{1}{\sqrt{8 \left( 1 - \frac{1}{m} \right)}} \right)^2.$$  \hspace{1cm} (2.16)

Here, $\nu_2$ is the number of repetitions of the experiment. This scheme measures a single phase at a time. Hence it requires $d$ interferometers to measure $d$ phases. For each phase estimation, it uses $m$ number of photons. Thus, this cascade of interferometers shown in Figure 2.3, uses a total of $md$ number of photons for the measurement of $d$ phases, with an assumption that the repetition $\nu_2$ is 1.

For our parallel QuFTI, a single measurement requires $m$ photons. Thus, for a fair comparison with the sequential QUMI, we require $\nu_1 = d \nu_2$. Hence, Eq. (2.15) becomes

$$|\Delta \vec{\phi}_1|^2 = \frac{1}{\nu_1} \frac{(m - d + 1)d}{8(m - d)} = \frac{1}{\nu_2} \frac{(m - d + 1)}{8(m - d)}.$$  \hspace{1cm} (2.17)

Finally, for the classical strategy, we let the average photon number of the input $\bar{n} = \sum_{i=1}^{m} |\alpha_i|^2 = md$, so that a fair comparison requires $\nu_3 = \nu_2$. Hence, the precision of
estimation is given by [53]:

\[
|\Delta \bar{\phi}_3|^2 = \frac{1}{\nu_3} \frac{d^2}{md} = \frac{1}{\nu_2} \frac{d}{m}.
\] (2.18)

Expressing the phase sensitivity of the various schemes using the same number of photons \((md\nu_2)\) allows for easy and fair comparison. For example, in the case of \(d = 1\), the sequential QUMI and parallel QuFTI are identical strategies. In this case, comparison against the classical strategy reproduces the previous result from Ref. [48], which showed an improvement over the classical strategy for \(m < 7\). However, as we increase \(d\) along with \(m\), we see that the parallel QuFTI continues to improve relative to the classical strategy. Setting \(d = m - 1\) yields the maximum improvement over the classical case and our parallel QuFTI achieves an asymptotic improvement of a factor of four in the total variance (defined as the square of phase sensitivity) as can be seen in Figure 2.4.
2.2.2 Measurement strategies

We have discussed the advantage of our scheme in the context of quantum Fisher information. However, this may be too good to be true in an experimental implementation as QFI may contain hidden resources which may hinder any advantage. Thus, one should always provide an actual detection scheme that can be implemented in an experiment for useful comparison [61]. There is also a consensus that the theoretical bound derived from QFI cannot be achieved experimentally in multiparameter estimation scenario; however, it can be reached arbitrarily close [38].

For the sequential QUMI and the classical strategies, single-photon detectors (SPDs) and homodyne detection are the QCRB-saturating measurement schemes, respectively [23, 48]. Unfortunately, SPDs perform quite poorly for estimating multiple phases simultaneously. In fact, it do not even beat the classical strategy. Hence, for the parallel QuFTI, we consider detection schemes consisting of photon-number-resolving-detectors (PNRDs), and a combination of SPDs and PNRDs. Due to the rapid technological advances in single-photon detectors, we believe our scheme can be implemented experimentally with high-efficiency PNRDs, which can be done either in tungsten transition edge sensor [62] or titanium-based transition edge sensor [63].

Figure 2.4 shows the sensitivity of our multimode interferometer. We can see that the QFI is constant over all modes. We also see that a detection scheme corresponding to an array of $m$ photon-number-resolving detectors (PNRDs) nearly achieves the QCRB of our parallel QuFTI. To compute the precision of estimation of these specific detection schemes, we used a numerical method to find the minimum of the classical Fisher information ($F^{\text{clas}}$). However, numerically computing these values for a large number of modes was problematic due to the complex landscape optimization of the Fisher information. In addition to the overhead of calculating the matrix permanents, the optimization showed a sensitive dependence on the phases, making it a numerically intensive task and limited our computation.
Figure 2.4: Total variance ($\Delta\varphi^2$) with different metrological strategies to estimate $(d = m-1)$ phases. The QCRB for the parallel QuFTI strategy (pink, Eq. (2.17)) gives the lower bound on the variance for any measurement scheme. The one-PNRD (purple) and PNRD (orange) are obtained from numerically optimizing $\varphi$ from the classical CRB (Eq. (2.7)). For comparison, the classical (coherent state) strategy (blue, Eq. (2.18)) and sequential QUMI (red, Eq. (2.16)) are shown.

One might argue that PNRDs are far more costly and challenging to implement experimentally than SPDs, along with other issues such as low counting rate [64]. To make the multiple phase estimation with our multimode interferometer more experiment-friendly, we now introduce a much less demanding experimental setup. This proposed scheme provides much higher sensitivity than the single-photon-detection scheme and on par with the PNRD for a lower number of modes. We propose to use a PNRD in only one of the output port and SPDs in the others. A single PNRD can be approximated by mixing the target mode with a series of vacuum modes using beam splitters and placing SPDs at the output of each of these modes [65]. Our calculation assumes a number resolving detector on the first mode, but it can be placed on any arm as desired. As can be seen in Figure 2.4, interestingly, we can achieve the same sensitivity for a three-mode interferometer as using the photon number resolving detector in all the detection arms. This can be explained by the
symmetry of the QFT. For a small number of modes, regardless of the phases, any cyclic permutation of event outcomes are equally likely (for instance, if $m = 3$, the outcomes, (1,2,0), (0,1,2), and (2,0,1) occur with the same frequency). Of course, with the increase in the number of modes, the number of distinguishable events reduces, and we expect the sensitivity to worsen if we do not include more PNRDs.

2.3 Scattershot metrology with probabilistic photon sources

Our parallel QuFTI scheme is readily implementable in a laboratory with available technology for a small number of modes with few PNRDs. One of the main requirements needed for our scheme is the generation of indistinguishable photons. There have been many proposals and development of single-photon sources using atoms [66], molecules [67], color centers in diamond [68], quantum dots [39, 69], and spontaneous parametric down conversion (SPDC) [57]. Because many of these techniques produce single photons probabilistically, an input state consisting of $m$ photons is not always guaranteed. With the rapid development on single photon sources, it can be expected that truly on-demand sources will be available in the near future. We nonetheless consider a “scattershot” input state to take into account the probabilistic nature of photon generation. A similar approach was recently proposed and demonstrated to improve the sampling efficiency for BOSON SAMPLING [45, 57]. Our calculation shows that our scheme can still provide a sub-shot-noise sensitivity even when the photon sources are not necessarily reliable on-demand sources.

In a scattershot scenario, photon pairs are emitted from a source (for instance, a SPDC) with some non-unit probability. The detection event of a photon heralds the injection of the twin photon into a specific port of the interferometer. In this way, at a given time, one can keep track of the modes which received an input photon and the total number of photons present inside the interferometer. Using this strategy, with knowledge of the input, we can still measure the phase, although with a lower sensitivity than with an input with a full array of $m$ deterministic photon sources.

Let us consider $m$ SPDC sources with the probability $p_i$ to generate a particular input
Figure 2.5: The total variance ($\Delta \vec{\varphi}_{\text{avg}}$) for scattershot four-mode, three-phase parallel QuFTI using all NRD detection scheme (blue) and one-NRD detection scheme (green) with photon source efficiency $p$ compared to the minimum variance for a loseless classical (coherent) source (black) with average photon number $\bar{n} = 4$.

configuration. For each input configuration, we can compute the associated variance $\Delta \vec{\varphi}_i^2$ from the classical Fisher information, so that the average variance $\Delta \vec{\varphi}_{\text{avg}}^2$ is given by:

$$\langle \Delta \vec{\varphi}_{\text{avg}}^2 \rangle = \sum_{i=1}^{\text{total number of input configurations}} p_i (\Delta \vec{\varphi}_i^2)^{-2},$$

(2.19)

where the summation is over the total number of input configurations.

We consider a four-mode, three-phase parallel QuFTI with probabilistic photon sources. For simplicity, we assumed that all sources have an equal probability $p$ of emitting a heralded photon. We can see in Figure 2.5 that even a source with an efficiency of around 50% beats the lossless coherent source, assuming that a full PNRD measurement is performed. Moreover, a source of 65% efficiency with a single PNRD also achieves supersensitivity.

2.4 Discussion

In this chapter, we looked at our proposed multimode scheme for measuring multiple phases simultaneously. Our scheme offers better sensitivity compared to sequential QUMI
and classical strategies. Sequential QUMI only offers an advantage for modes $m < 7$. This is because, with increasing modes, the probability that the photons interact with the single-phase decreases rapidly. Another main difference between these two schemes is the use of PNRDs in our case. We found that SPDs do not work in the case of multiparameter phase estimation. We regard this to be a modest experimental overhead, considering the important task and applications of estimating multiple phases simultaneously.

In terms of the phase sensitivity, we learned that the quantum Cramér-Rao bound for our scheme shows an asymptotic constant factor improvement. This can be approximately obtained with PNRDs, and this is also possible for a small number of modes with an array of single photon detectors and one number-resolving detector. We also showed that super sensitivity is possible in our scattershot metrology scheme even with inefficient but heralded single-photon sources.
Chapter 3
Phase Estimation with a SU(1,1) Interferometer

In the last chapter, I discussed multimode interferometer for doing multiple phase estimation simultaneously. This interferometer falls under the linear interferometer category. In linear interferometers, the input states evolve passively without any energy contribution from the interferometer. There is another type of interferometer called SU(1,1) interferometer, which consists of an active element that contributes to the input states, increasing the mean photon number beyond what is injected. In this chapter,\textsuperscript{1} we look into this interferometer for doing a single phase estimation. First, I discuss in detail the historical development and background on SU(1,1) interferometer. Then, I discuss results from two of our paper (Refs. [70, 71]) on phase estimation in SU(1,1) interferometer; using a thermal and a squeezed vacuum state, and a coherent and a displaced-squeezed-vacuum state, as inputs.

3.1 Introduction to SU(1,1) interferometer

A SU(1,1) interferometer was originally proposed by Yurke et al. in 1986, building upon the foundational work of Wódkiewicz and Eberly [72, 73]. These interferometers are so named because the group SU(1,1) characterizes these interferometers. A SU(1,1) interferometer is similar to a Mach-Zehnder interferometer in architecture. While Mach-Zehnder consists of two beam splitters, the SU(1,1) consists of an active element, such as a four-wave mixer (FWM) or an optical parametric amplifier, as can be seen in Figure 3.1. A strong laser (pump) is used to activate the nonlinear response of the active element. This pump is blocked after the second OPA, and it is not counted as a resource in doing phase estimation. Analogous to Mach-Zehnder interferometer, after the first optical parametric

\textsuperscript{1}This Chapter is based on the contents of: S Adhikari, N Bhusal, C You, H Lee and J P Dowling. Phase estimation in an SU (1, 1) interferometer with displaced squeezed states. OSA Continuum, 1, 438-450, 2018, and X Ma, C You, S Adhikari, E S Matekole, R T Glasser, H Lee and J P Dowling. Sub-shot-noise-limited phase estimation via SU (1, 1) interferometer with thermal states. Optics Express, 26, 18492, 2018. Reprinted by permission of the Optical Society of America.
Figure 3.1: A schematic of a SU(1,1) interferometer. Two OPAs with the same squeezing parameter $g$ is used. The pump field between the two OPAs has a $\pi$ phase difference. Parity measurement is performed in mode $b$, and the on-off detection is done in both modes, $a$ and $b$.

amplifier (OPA), one of the arms undergoes a $\phi$ phase shift and the other arm is used as a reference. The modes are recombined in the second OPA, and the output depends on the phase shift $\phi$. From the statistics of the light collected at the detector, inference about the phase can be made. In the Yurke et al. scheme, vacuum states were injected in both input arms, and an intensity measurement was performed [72]. They showed that sub-shot-noise sensitivity can be achieved using the nonlinearity of the OPA or the FWM. With technological developments, these interferometers have gained considerable attention recently, and variations of these interferometers have been proposed. Plick et al. studied coherent states in both input arms of the interferometer with simple intensity measurement. The bright coherent beams boost the mean photon number of the squeezed light, creating a very
sensitive device [74]. They showed that the sensitivity is far below the shot noise limit, as was experimentally demonstrated by Ou in 2012 [75]. Experimental implementation with very bright coherent beams is difficult. To lower the intensity of the beams, Li et al. introduced a squeezed vacuum state, replacing one of the two coherent input states [76]. They implemented homodyne as a detection scheme, which is a convenient measurement for experimental detection of squeezing. In another work, Li et al. introduced parity detection with the same inputs as before, and showed that the sensitivity approaches the Heisenberg limit under optimal conditions [77]. Hu et al. studied phase estimation with a coherent state and a displaced-squeezed-vacuum state (DSV), with homodyne detection, and showed Heisenberg-like scaling in the optimal case [78]. It was shown that the DSV states perform better than Li et al.’s scheme, which used a coherent and squeezed vacuum with homodyne detection. In recent years, an analysis of the effect of loss on these interferometers has also been performed [79, 80, 81]. More recently, Szigeti et al. introduced a modification of the SU(1,1) interferometer, where all the input particles participate in the phase measurement and showed how this can be implemented in spinor Bose-Einstein condensates and hybrid atom-light systems [82]. Over the last years, besides the theoretical progress, several experimental realizations have also been performed. A supersensitive phase measurement with a truncated SU(1,1) interferometer has been demonstrated recently [83]. That scheme consists of only one amplifier, and they used a seeded four-wave mixing in $^{85}$Rb vapor as the nonlinear interaction, along with a balanced homodyne as measurement. Similarly, an unseeded SU(1,1) interferometer, composed of two cascaded degenerate parametric amplifiers with direct detection at the output, was investigated in Ref. [84]. It achieves phase super sensitivity, beating shot noise limit by 2.3 dB. Du et al. studied a direct phase estimation of FWM-based SU(1,1) interferometer, which showed a 3 dB improvement in sensitivity over MZI [85].
3.2 Phase estimation with SU(1,1) interferometer

We have a pretty good idea of the history of the SU(1,1) interferometer from the last section. Now, let us formally describe the operation of this interferometer in detail. We start by defining the elements of the SU(1,1) interferometer. The first OPA is denoted by operator $\hat{T}_{OPA1}$ and is given by:

$$\hat{T}_{OPA1} = \begin{pmatrix} \mu_1 & \nu_1 \\ \nu_1^* & \mu_1 \end{pmatrix}, \quad (3.1)$$

Similarly, the second OPA is given by:

$$\hat{T}_{OPA2} = \begin{pmatrix} \mu_2 & \nu_2 \\ \nu_2^* & \mu_2 \end{pmatrix}, \quad (3.2)$$

where $\mu_j = \cosh (g_j)$, $\nu = e^{i\theta_j} \sinh (g_j)$, and $\theta_j$ and $g_j$ are the phase and parametrical strength of the OPAs $(j = 1, 2)$. Similarly, the phase shift is given by:

$$\hat{T}_\phi = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}. \quad (3.3)$$

Hence, the entire transformation of the SU(1,1) interferometer is $\hat{T} = \hat{T}_{OPA2} \hat{T}_\phi \hat{T}_{OPA1}$.

Now, let us look at how the operators evolve through the SU(1,1) interferometer. Let $\hat{a}_0(\hat{a}_0^\dagger)$, $\hat{b}_0(\hat{b}_0^\dagger)$ and $\hat{a}_1(\hat{a}_1^\dagger)$, $\hat{b}_1(\hat{b}_1^\dagger)$ be the annihilation(creation) operators for the two input modes and after the first OPA respectively. The input mode operators are transformed by the first OPA as:

$$\begin{pmatrix} \hat{a}_1 \\ \hat{b}_1^\dagger \end{pmatrix} = \hat{T}_{OPA1} \begin{pmatrix} \hat{a}_0 \\ \hat{b}_0^\dagger \end{pmatrix}. \quad (3.4)$$

Similarly, the relation between the output annihilation(creation) operators ($\hat{a}_2(\hat{a}_2^\dagger), \hat{b}_2(\hat{b}_2^\dagger)$)
and the input operators is given by:

\[
\begin{pmatrix}
\hat{a}_2 \\
\hat{b}_2^\dagger
\end{pmatrix}
= \hat{T}
\begin{pmatrix}
\hat{a}_0 \\
\hat{b}_0^\dagger
\end{pmatrix}.
\] (3.5)

In our work, it is assumed that the first and second OPA has a $\pi$ phase difference ($\theta_1 = 0$ and $\theta_2 = \pi$), and same parametrical strength ($g_1 = g_2 = g$). In this scenario, the second OPA will undo what the first one did (namely $\hat{a}_2 = \hat{a}_0$ and $\hat{b}_2 = \hat{b}_0$) when the phase shift $\phi = 0$, which is referred to as a balanced situation.

### 3.2.1 SU(1,1) interferometer in symplectic formalism

In the last section, we discussed the working of the SU(1,1) interferometer. We discussed the evolution of mode operators through various components of the SU(1,1) interferometer. These mode operators are primarily linked with the state vector and the density matrix formalism. There is another equivalent formalism developed by Eugene Wigner in 1932 [86]. It is called a Wigner function and gives a quasi-probability distribution for a given state of light in phase space. Any Wigner function that is Gaussian in form has many nice properties that make calculations much more straightforward, as these states can be fully characterized by their mean and covariance matrices [87]. In our work, all the states that we use are Gaussian states, and we use this formalism to our advantage.

In this formalism (referred to as symplectic or characteristic), we evolve the mean and covariance matrices of our states through various components of the SU(1,1) interferometer and construct the Wigner functions at the output, from which we calculate the phase sensitivity.

Let $\hat{a}$ ($\hat{a}^\dagger$), $\hat{b}$ ($\hat{b}^\dagger$) be the annihilation (creation) operators of the upper and lower modes respectively. We define the quadrature operators of the modes as [88]:

\[
\hat{x}_{a_k} = \frac{1}{\sqrt{2}}(\hat{a}_k + \hat{a}_k^\dagger), \quad \hat{p}_{a_k} = -\frac{i}{\sqrt{2}}(\hat{a}_k - \hat{a}_k^\dagger),
\] (3.6)

Let $\hat{a}$ ($\hat{a}^\dagger$), $\hat{b}$ ($\hat{b}^\dagger$) be the annihilation (creation) operators of the upper and lower modes respectively. We define the quadrature operators of the modes as [88]:

\[
\hat{x}_{a_k} = \frac{1}{\sqrt{2}}(\hat{a}_k + \hat{a}_k^\dagger), \quad \hat{p}_{a_k} = -\frac{i}{\sqrt{2}}(\hat{a}_k - \hat{a}_k^\dagger),
\] (3.6)
\[ \hat{x}_{bk} = \frac{1}{\sqrt{2}} (\hat{b}_k + \hat{b}_k^\dagger), \quad \hat{p}_{ak} = -\frac{i}{\sqrt{2}} (\hat{b}_k - \hat{b}_k^\dagger). \] (3.7)

A column vector of quadrature operators can be written as:

\[ X_k = (\hat{X}_{k,1}, \hat{X}_{k,2}, \hat{X}_{k,3}, \hat{X}_{k,4})^T = (\hat{x}_{ak}, \hat{p}_{ak}, \hat{x}_{bk}, \hat{p}_{bk})^T. \] (3.8)

The mean and the covariance of quadrature operators are given by:

\[ \bar{X}_k = (\langle \hat{X}_{k,1} \rangle, \langle \hat{X}_{k,2} \rangle, \langle \hat{X}_{k,3} \rangle, \langle \hat{X}_{k,4} \rangle)^T, \] (3.9)

\[ \Gamma_{mn}^k = \text{Tr}[ (\Delta \hat{X}_{k,m} \Delta \hat{X}_{k,n} + \Delta \hat{X}_{k,n} \Delta \hat{X}_{k,m}) \rho ], \] (3.10)

where \( \Delta \hat{X}_{k,m} = \hat{X}_{k,m} - \langle \hat{X}_{k,m} \rangle \), \( \Delta \hat{X}_{k,n} = \hat{X}_{k,n} - \langle \hat{X}_{k,n} \rangle \), and \( \rho \) is a density matrix of the input state. Using the mean \( \bar{X}_0 \) and the covariance matrix \( \Gamma_0 \), the Wigner function of the input states can be written as:

\[ W(X_0) = \exp \left[ - (X_0 - \bar{X}_0)^T \cdot (\Gamma_0)^{-1} \cdot (X_0 - \bar{X}_0) \right] \frac{1}{\sqrt{|\Gamma_0|}}. \] (3.11)

The symplectic representation of the components of the SU(1,1) interferometer, namely, the first OPA, the phase shifter, and the second OPA, are described in phase space by:

\[ S_{\text{OPA1}} = \begin{pmatrix} \cosh(g) & 0 & \sinh(g) & 0 \\ 0 & \cosh(g) & 0 & -\sinh(g) \\ \sinh(g) & 0 & \cosh(g) & 0 \\ 0 & -\sinh(g) & 0 & \cosh(g) \end{pmatrix}, \] (3.12)
\[
S_\phi = \begin{pmatrix}
\cos(\phi) & -\sin(\phi) & 0 & 0 \\
\sin(\phi) & \cos(\phi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
(3.13)

\[
S_{\text{OPA}2} = \begin{pmatrix}
\cosh(g) & 0 & -\sinh(g) & 0 \\
0 & \cosh(g) & 0 & \sinh(g) \\
-\sinh(g) & 0 & \cosh(g) & 0 \\
0 & \sinh(g) & 0 & \cosh(g)
\end{pmatrix},
\]
(3.14)

As before, we have assumed same parametrical strength \((g_1 = g_2 = g)\) and a \(\pi\) phase difference between the first and the second OPA. The transformation of \(\bar{X}_0\) and \(\Gamma_0\) through the SU(1,1) interferometer is given by \(S = S_{\text{OPA}2}S_\phi S_{\text{OPA}1}\). Hence, the output mean \(\bar{X}_2\) and covariance \(\Gamma_2\) of the quadrature operators in the SU(1,1) interferometer is obtained by:

\[
\bar{X}_2 = S\bar{X}_0,
\]
(3.15)

\[
\Gamma_2 = S\Gamma_0 S^T.
\]
(3.16)

Finally, the Wigner function of the output state is:

\[
W(X_2) = \frac{\exp[-(X_2 - \bar{X}_2)^T \cdot (\Gamma_2)^{-1} \cdot (X_2 - \bar{X}_2)]}{\sqrt{\det(\Gamma_2)}}.
\]
(3.17)

Technically, we do not even need to calculate the input Wigner function. We only need to calculate the evolution of the input means and covariance matrices through the interferometer.

### 3.2.2 Measurement Strategies

In the last section, we learned to evolve the states through the interferometer in characteristic formalism. Now, we have the output Wigner function, and we need to employ
measurement scheme to get information about the phase $\phi$. We use two measurement schemes to obtain the uncertainty in estimating $\phi$.

Our first measurement scheme is parity detection. It was first proposed by Bollinger et al. to study spectroscopy and was later adopted by Gerry in quantum optics for phase estimation [89, 90]. Parity detection is a single-mode measurement, and the parity operator on output mode $b$ is given by:

$$\hat{\Pi}_b = (-1)^{\hat{b}_1^\dagger \hat{b}_2}.$$

(3.18)

The parity measurement satisfies $\langle \hat{\Pi}_b \rangle = \pi W(0)$ [91]. That is, the expectation of the parity is given by the value of the Wigner function at the origin of the phase space. This property makes it simpler to calculate the parity signal. Since parity is a single mode measurement, we only need the Wigner function in the mode where the measurement is done.

Using parity measurement, the phase sensitivity $\Delta \phi$ can be characterized using error propagation formula:

$$\Delta \phi = \frac{\langle \Delta \hat{\Pi}_b \rangle}{\left| \frac{\partial \langle \hat{\Pi}_b \rangle}{\partial \phi} \right|},$$

(3.19)

where $\langle \Delta \hat{\Pi}_b \rangle = \sqrt{\langle \hat{\Pi}_b^2 \rangle - \langle \hat{\Pi}_b \rangle^2}$ with $\langle \hat{\Pi}_b^2 \rangle = 1$. Our other measurement scheme is an on-off detection which only discriminates between zero and non-zero photons. This can be mathematically represented by a set of measurement operators:

$$\hat{\Pi}_{\text{off}} = |0\rangle\langle 0|, \quad \hat{\Pi}_{\text{on}} = \hat{I} - |0\rangle\langle 0|,$$

(3.20)

where $\hat{I}$ is an identity operator. For a single mode Gaussian state, the probability of obtaining non-zero photons is given by [92]:

$$P_{\text{on}} = 1 - \frac{2}{\sqrt{\det(\Gamma + \hat{1})}},$$

(3.21)

where $\Gamma$ is the covariance matrix at the output. The phase sensitivity for the on-off scheme
can be calculated using the classical Fisher information. The sensitivity is lower bounded by the classical Crámer-Rao bound:

$$\Delta \phi \geq \frac{1}{\sqrt{F}}, \quad (3.22)$$

and, the Fisher information is given by:

$$F = \sum \frac{1}{P_{on}} \left( \frac{dP_{on}}{d\phi} \right)^2. \quad (3.23)$$

### 3.2.3 Phase sensitivity with thermal and squeezed vacuum state

Let us first look at the phase estimation with a thermal and a squeezed vacuum state. Thermal states have many applications in astronomical, aerospace, defense, etc., and hence supersensitive detection of thermal states are of general interest. We first begin by defining the mean and the covariance matrices for these states, from which we can construct the output Wigner function, and calculate the parity signal for phase estimation. The mean and covariance matrix of a thermal state is:

$$X_{th} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Gamma_{th} = (2n_{th} + 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.24)$$

where $n_{th}$ is the average photon number in the thermal state. Similarly, the mean and the covariance matrix of a squeezed vacuum state is:

$$X_{SV} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Gamma_{SV} = \begin{pmatrix} (\cosh (r) + \sinh (r))^2 & 0 \\ 0 & (\cosh (r) - \sinh (r))^2 \end{pmatrix}. \quad (3.25)$$

Using these, we calculate the phase sensitivity $\Delta \phi$ with parity detection and is given by:

$$\Delta \phi = \sqrt{\frac{2}{n_{OPA} (n_{OPA} + 2) [1 + (1 + 2n_s)(1 + 2n_{th})]}}, \quad (3.26)$$
Here, we have assumed that $\phi = 0$. For an arbitrary value of $\phi$, the phase sensitivity is shown in Appendix B.

Now that we have the phase sensitivity, we would like to compare the performance of our scheme with a standard benchmark, namely, the shot noise limit (SNL) and the Heisenberg limit (HL). To calculate these quantities, we need to find out the mean number of photons $\bar{n}$ inside the interferometer. This can be calculated using Eq. (3.4). As OPAs are nonlinear with a gain factor, they emit spontaneous photons, amplifying the input photon number, and therefore, $\bar{n}$ is not the total input photon number. The mean photon number $\bar{n}$ inside the SU(1,1) interferometer is given by:

$$\bar{n} = \langle \Psi_{\text{in}} | (\hat{a}_1^\dagger \hat{a}_1 + \hat{b}_1^\dagger \hat{b}_1) | \Psi_{\text{in}} \rangle,$$

$$\text{(3.27)}$$

where $| \Psi_{\text{in}} \rangle = | \Psi_{\text{th}} \rangle \otimes | \Psi_\xi \rangle$ and $| \Psi_{\text{th}} \rangle$ is a thermal state, and $| \Psi_\xi \rangle$ is a squeezed vacuum state. Finally, the total photon number inside the SU(1,1) interferometer is:

$$\bar{n} = (n_{\text{OPA}} + 1)(n_{\text{th}} + n_s) + n_{\text{OPA}}.$$  

$$\text{(3.28)}$$

Here, $n_{\text{th}}$ is the input photon number of the thermal state, $n_s = \sinh^2 (r)$ is the photon number of the squeezed vacuum state, $n_{\text{OPA}} = 2 \sinh^2 (g)$ is the number of photons emitted from the first OPA, and $g = g_1 = g_2$ is the parametrical strength of the OPAs. It can be seen that there are two contributions in increasing the mean photon number inside the SU(1,1) interferometer. The amplification of the input photon number of the two states and the amplification of the input vacuum state by spontaneous emission are the contributing processes. Using the total photon number inside the SU(1,1) interferometer, the SNL and the HL for a thermal and a squeezed vacuum states are given by:

$$\Delta \phi_{\text{SNL}} = \frac{1}{\sqrt{\bar{n}}} = \frac{1}{\sqrt{(n_{\text{OPA}} + 1)(n_{\text{th}} + n_s) + n_{\text{OPA}}}}.$$  

$$\text{(3.29)}$$
\[
\Delta \phi_{\text{HL}} = \frac{1}{n} \frac{1}{(n_{\text{OPA}} + 1)(n_{\text{th}} + n_a) + n_{\text{OPA}}}.
\]

(3.30)

Figure 3.2: Phase sensitivity \(\Delta \phi\) with a thermal state and a squeezed vacuum state (black) and HL (blue). It can be seen that the sensitivity gets better with the increase in photon number \(n_{\text{th}}\). Plotted with \(r = 2\).

First, we investigate with a thermal and a squeezed vacuum state as inputs and plot the results in Figure 3.2. By increasing the mean photon number \(n_{\text{th}}\), the sensitivity gets closer to HL.
Figure 3.3: Phase sensitivity $\Delta \phi$ (black) as a function of $r$, along with HL (blue) and SNL (red). The mean photon number is chosen according to Eq. (3.31) for achieving the HL.

We can also state the optimal condition for approaching the HL by comparing equation Eq. (3.30) and Eq. (3.26), and is found to be:

$$n_{th} = \frac{\sinh^2(g) - n_s}{\cosh^2(2g)}.$$  \hspace{1cm} (3.31)

The above expression ensures that the phase sensitivity approaches the HL. It can be proven that regardless of the values of $n_s$ and $g$, the optimal mean photon number ($n_{th}$) of the thermal state is no more than one. Figure 3.3 shows the phase sensitivity as a function of $r$. We see that the sensitivity improves with the increase in the squeezing parameter $r$ of the input squeezed vacuum state.
3.2.4 Phase sensitivity with coherent and displaced-squeezed state

In this section, we look at another phase estimation strategy using a coherent state and a displaced-squeezed-vacuum state. Let us start by writing down the mean and covariance matrix for these states. The mean and covariance matrix of a coherent state is given by:

\[
X_{\text{coh}} = \begin{pmatrix} \sqrt{2} \alpha_1 \cos(\theta_1) \\ \sqrt{2} \alpha_1 \sin(\theta_1) \end{pmatrix}, \quad \Gamma_{\text{coh}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(3.32)

Similarly, the mean and covariance matrix of the displaced-squeezed-vacuum (DSV) state is given by:

\[
X_{\text{DSV}} = \begin{pmatrix} \sqrt{2} \alpha_2 \cos(\theta_2) \\ \sqrt{2} \alpha_2 \sin(\theta_2) \end{pmatrix}, \quad \Gamma_{\text{DSV}} = \begin{pmatrix} (\cosh (r) + \sinh (r))^2 & 0 \\ 0 & (\cosh (r) - \sinh (r))^2 \end{pmatrix}.
\]

(3.33)

As before, we evolve these means and covariance matrices and construct the Wigner function at the output from which we compute the phase sensitivity. As before, we would like to compare the performance of our SU(1,1) interferometer with standard metric, namely, HL and SNL. The HL and the SNL for these particular input states are given by:

\[
\Delta \phi_{\text{HL}} = \frac{1}{\bar{n}_{\text{Total}}} = \frac{1}{\bar{n}_2 + \bar{n}_\xi + \bar{n}_1 + n_{\text{opa}} + 4 \sqrt{\bar{n}_1 \bar{n}_2 \bar{n}_{\text{opa}} (\bar{n}_{\text{opa}} + 2)}},
\]

(3.34)

\[
\Delta \phi_{\text{SNL}} = \frac{1}{\sqrt{\bar{n}_{\text{Total}}}} = \frac{1}{\sqrt{\bar{n}_2 + \bar{n}_\xi + \bar{n}_1 + n_{\text{opa}} + 4 \sqrt{\bar{n}_1 \bar{n}_2 \bar{n}_{\text{opa}} (\bar{n}_{\text{opa}} + 2)}}.
\]

(3.35)

Here, \( \bar{n}_1 \) is the average photon number in the coherent state in the first arm. Similarly, \( \bar{n}_2 \) and \( \bar{n}_\xi = \sinh^2 (r) \) are the average photon number in the displaced and squeezed part of the DSV, with \( r \) being the squeezing parameter. And lastly, \( \bar{n}_{\text{opa}} = 2 \sinh^2 (g) \) is the average photon number of the OPA and \( g \) is the gain parameter.
As before, since parity is a single mode measurement, we only need to calculate the Wigner function at the output mode \( b \). The Wigner function at mode \( b \) can be computed using:

\[
\langle \hat{\Pi}_b \rangle = \exp \left( -\bar{X}_{22}^T \cdot \Gamma_{22}^{-1} \cdot \bar{X}_{22} \right) \frac{1}{\sqrt{|\Gamma_{22}|}},
\]

where \( \bar{X}_{22}^T = (\langle X_{2,3} \rangle, \langle X_{2,4} \rangle) \), \( \Gamma_{22} = \begin{pmatrix} \Gamma_{33}^2 & \Gamma_{34}^2 \\ \Gamma_{43}^2 & \Gamma_{44}^2 \end{pmatrix} \) and \( \bar{X}_{22} = (0, 0)^T \), since the Wigner function is taken at the origin of the phase space.

Figure 3.4: The effect on the phase sensitivity with the increase in the squeezing parameter \( r \). The HL (blue) is given by Eq. (3.34). Plotted with \( \bar{n}_1 = 16, \bar{n}_2 = 4 \) and \( g = 2 \).

The phase sensitivity is calculated using the above expression and Eq. (3.19). The expression for phase sensitivity is long and not illuminating to report here. Upon examining the phase sensitivity as a function of \( \phi \), we find that the minimum is not at zero as in previously reported scheme [71, 77]. Hence, we resort to numerical minimization to find the optimal phase sensitivity. As we resort to a numerical method, we set the phases of
Figure 3.5: Phase sensitivity $\Delta \phi$ as a function of the gain parameter $g$ of the OPA. The phase sensitivity with coherent and DSV (pink) is obtained by numerically optimizing $\phi$. The HL (blue) and SNL (red) are given by Eqs. (3.34, 3.35). Plotted with $\bar{n}_1 = 16, \bar{n}_2 = 4$ and $r = 2$.

the coherent state ($\theta_1$) and DSV state ($\theta_2$) to zero, as they only shift the position of the optimal point.

In Figure 3.4, we show the effect of the increase in the squeezing strength of the OPA on the sensitivity. We see that, with an increase in $r$, the sensitivity of the scheme increases as expected. Similar behavior is observed when increasing the average photon number of the coherent state on the first arm.

Next, we present the sensitivity of our scheme and compare it with SNL and HL (Eq. 3.35, Eq. 3.34). In Figure 3.5, we see that our scheme is sub-shot-noise limited even for a small value of squeezing strength $g$ of the OPA. With higher $g$, the sensitivity of our scheme keeps increasing and approaches the HL for $g \geq 2$. However, the sensitivity of our scheme never goes below the HL.

The parity measurement requires a photon-number-resolving detector (PNRD). These detectors are very costly and difficult to implement in an experimental setup. For Gaussian states, there has been a proposal of obtaining the parity signal without the use of photon
number resolving detector, but it requires post-processing [74]. Here, we present the results using on-off detectors, which only discriminate between zero and non-zero photons, as discussed in the previous section. Figure 3.6 shows the sensitivity of our scheme using on-off detector. We can see that sub-shot-noise sensitivity can be achieved for $g \leq 2$. Thus, if only sub-shot-noise sensitivity is desired for a particular application, a simple measurement setup with an on-off detector is sufficient.

### 3.2.5 Sensitivity with coherent and DSV state with photon loss

Next, we see how photon loss affects the phase sensitivity of our interferometer. Photon loss modeling can be done by adding a fictitious beam splitter with two input vacuum modes in both arms inside the interferometer. The loss after the phase shift is denoted in Figure 3.1 by $L$. Here, $L$ is related to the transmissivity of the beamsplitter $T$ by $L = 1 - T$. In terms of the phase space representation, we need four quadratures to represent the two introduced modes. For simplicity, we assume that the losses are the same in both arms.
The inputs, OPAs, phase shifter and the fictitious BS in the photon loss model are given by:

\[ \Gamma^L_0 = \begin{pmatrix} \Gamma_0 & 0_4 \\ 0_4 & I_2 \oplus I_2 \end{pmatrix}_{8 \times 8}, \]  

(3.37)

\[ S_{\text{OPA}i}^L = \begin{pmatrix} S_{\text{OPA}i} & 0_4 \\ 0_4 & I_4 \end{pmatrix}_{8 \times 8}, \]  

(3.38)

\[ S_\phi^L = \begin{pmatrix} S_\phi & 0_4 \\ 0_4 & I_4 \end{pmatrix}_{8 \times 8}, \]  

(3.39)

\[ S_{\text{BS}}^L = \begin{pmatrix} \sqrt{T}I_4 & \sqrt{1-T}I_4 \\ \sqrt{1-T}I_4 & -\sqrt{T}I_4 \end{pmatrix}_{8 \times 8}, \]  

(3.40)

where 0_4 is a four-by-four zero matrix and I_4 is a four-by-four identity matrix. Similarly, the propagation of \( \bar{X}_L^0 \) and \( \Gamma^L_0 \) through the SU(1,1) interferometer is given by \( S^L = S_{\text{OPA}2}^L \cdot S_{\text{BS}}^L \cdot S_\phi^L \cdot S_{\text{OPA}1}^L \). For completeness, the transformation relation between the inputs and outputs is given by:

\[ \bar{X}^L_2 = S^L \bar{X}^L_0, \]  

(3.41)

\[ \Gamma^L_2 = S^L \Gamma_0 (S^L)^T. \]  

(3.42)

Similar to the ideal case, the parity detection is given by:

\[ \langle \hat{\Pi}_b \rangle^L = \frac{\exp(-\langle \bar{X}^L_{22} \rangle^T \cdot (\Gamma^L_{22})^{-1} \cdot \langle \bar{X}^L_{22} \rangle^T)}{\sqrt{|\Gamma^L_{22}|}}. \]  

(3.43)

where \( \bar{X}^L_{22} = (0,0)^T \), \( \bar{X}^L_{22} = (\langle X^L_{2,3} \rangle, \langle X^L_{2,4} \rangle) \), and \( \Gamma^L_{22} = \begin{pmatrix} \Gamma^L_{33} & \Gamma^L_{34} \\ \Gamma^L_{43} & \Gamma^L_{44} \end{pmatrix} \).
We show the effect of the photon loss on the sensitivity of our scheme in Figure 3.7. The results with 1% and 5% photon loss in each arm are presented. We see that the sensitivity quickly degrades with the increasing loss. If the loss is small, for example, 1%, it is still possible to beat the SNL, but with a loss of 5%, the sensitivity quickly degrades above the shot-noise limit. We also like to point out that other schemes involving parity detection suffer a similar fate in regards to photon loss, as was recently shown in Ref. [80].

3.3 Discussion

In this chapter, we looked at the phase sensitivity of the SU(1,1) interferometer with different input states; a thermal state and a squeezed vacuum state and, a coherent state and a displaced-squeezed-vacuum state.

For a thermal and squeezed vacuum state, we found that our strategy is super sensitive. This should come as surprise to many, considering thermal states does poorly in a Mach-Zehnder interferometer. This should intrigue researchers and open a new avenue towards sensitive detection of thermal light. Our results on super-sensitive phase estimation is an important step toward that goal and also paves a way for practical quantum metrology with
thermal sources, such as photometers, or at different wavelengths where the generation of quantum features, such as coherence, number states, squeezing or entanglement might be challenging. In our work, we employed parity detection as we found out that intensity measurement cannot beat the SNL. It is also worth pointing out that although homodyne detection is simple experimental procedure based on quadrature measurement, it is not applicable in our case because the quadrature means of the thermal state and the squeezed vacuum state are zero.

Similarly, we studied the phase estimation with a coherent and a displaced squeezed state. We showed sub-shot-noise sensitivity which approaches HL with the increase in the gain parameter \((g)\) of the OPA. We also implemented a simple on-off detection strategy and showed that under reasonable value of \(g\), our scheme is sub-shot-noise limited. We also studied the performance of our proposed SU(1,1) interferometer in the presence of photon loss and we found out that the sensitivity degrades quickly. We would like to point out that this is not just the case for our choice of input. It is the same in general for all parity based detection schemes. In the context of SU(1,1), similar conclusion was reached in Ref. [80] with a coherent and a squeezed states.
Chapter 4
Quantum Fisher Information in a SU(1,1) Interferometer

In the last chapter, I discussed in detail the working of a SU(1,1) interferometer and showed how it could be used for phase estimation with different states of light. In this chapter, I discuss the quantum Fisher information for different phase configurations. More precisely, I explain the confusion in the community regarding the use of quantum Fisher information, starting with an example in Mach-Zehnder interferometer, and discuss the methods used in the literature to fix the issue. Then, we apply these tools in SU(1,1) interferometer to obtain a consistent and a tighter bound on the phase sensitivity.

4.1 Overview of QFI analysis in a Mach-Zehnder interferometer

Quantum Fisher information (QFI) is not new to us. We looked at the definition of quantum Fisher information, along with quantum Cramér-Rao bound in the first chapter. In the second chapter, we applied the multivariable quantum Cramér-Rao bound for bounding the phase sensitivity of the multimode interferometer for multiple phase estimation. As we had discussed briefly before, one has to be very careful in claiming a precision bound solely based on the QFI. Without considering a particular measurement, there might be hidden resources that are unaccounted in the QFI calculation, which falsely boosts the precision of estimation. A specific problem in the context of Mach-Zehnder (MZI) was studied in Ref. [93]. Let us look into this problem in detail first, and we apply the same technique in our more complicated SU(1,1) scheme for bounding the phase sensitivity.

The phase estimation scenario considered in Ref. [93] is shown in Figure 4.1. The authors considered a Mach-Zehnder interferometer with arbitrary transmissivity \( \tau \) for estimating the phase generated by the unitary \( U_\phi \). They considered a squeezed vacuum and a coherent state \( (\psi_{in} = |\xi\rangle \otimes |\alpha\rangle) \) as inputs. The unitary \( U_\phi \) takes three different configurations given by: (i) \( \hat{U}_\phi = e^{-i\hat{g}_1 \phi} \), \( \hat{g}_1 = \hat{a}^\dagger \hat{a} \), (ii) \( \hat{U}_\phi = e^{i\hat{g}_2 \phi} \), \( \hat{g}_2 = \frac{1}{2} (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) \), and (iii)
Figure 4.1: A Mach-Zehnder interferometer with different phase configurations. The first two is a single parameter estimation while the third is a two parameter phase estimation scenario.

\[ U_\phi = e^{i\hat{g}_s \phi_s} e^{i\hat{g}_d \phi_d}, \hat{g}_s = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}), \hat{g}_d = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \phi_s = \phi_1 + \phi_2, \text{ and } \phi_d = \phi_1 - \phi_2. \]

In plain language, the situation corresponds to a phase shift in the first arm, equal phase shifts in both arms, and unequal phase shifts in both arms, respectively. Although the first two scenarios have different configurations, they are physically equivalent model with the task of estimating the total phase \( \phi \). The first two scenarios are essentially a single parameter estimation, while the third is a two-parameter estimation problem. To highlight the inconsistent results that are obtained from the QFI, let us just focus on the first two cases. Let us look the QFI for these cases that are given by [93, 94]:

\[ F_Q^{(i)} = 4\tau(1 - \tau) \left(|\alpha|^2 e^{2r} + \sinh^2(r)\right) + 2(1 - \tau)^2 \sinh^2(2r) + 4\tau^2|\alpha|^2, \quad (4.1) \]
\[ F_Q^{(ii)} = 4\tau(1 - \tau)\left(|\alpha|^2 e^{2r} + \sinh^2(r)\right) + (1 - 2\tau)^2 \left(|\alpha|^2 + \frac{1}{2}\sinh^2(2r)\right). \] (4.2)

The above QFIs are clearly different, although they represent physically equivalent systems. The difference is seen more clearly if we assume \( r = 0 \) and \( \tau = 1/2 \), then we get, \( F_Q^{(i)} = 2|\alpha|^2 \) and \( F_Q^{(ii)} = |\alpha|^2 \). Moreover, if we consider even more extreme scenario with \( \tau = 1 \) and \( r = 0 \), where the coherent state is simply transmitted through, and there is no interference at all, we still get positive values for QFI with, \( F_Q^{(i)} = 4|\alpha|^2 \) and \( F_Q^{(ii)} = |\alpha|^2 \)[93, 94]. This surprising result can be explained as follows: the QFI only depends on the change in the probe state with respect to \( \phi \). A coherent state evolves under the action of \( \phi \) and can provide information about the phase \( \phi \). But the main point that is missing is the role of the reference beam. In quantum mechanics, there is no such thing as an absolute phase, and we need an additional reference phase to define \( \phi \). Whenever the inputs are coherent superpositions of different photon number state, such as a coherent state or squeezed vacuum state, etc., this problem arises. To avoid the use of additional phase reference, Jarzyna et al. suggested phase-averaging the input state as [93, 94]:

\[ \rho = \int \frac{d\theta}{2\pi} V_\theta^a V_\theta^b (|r\rangle \otimes |\alpha\rangle) (r) \otimes \langle \alpha| V_\theta^{a\dagger} V_\theta^{b\dagger}. \] (4.3)

where \( V_\theta^y = \exp(-i\theta y^\dagger y) \). Here, both the squeezed and the coherent states are averaged over a common phase \( \theta \). This process makes the input state mixed, making the calculation of the QFI more involved. The QFI \( F_Q^{(\rho)} \) they obtained is different from both \( F_Q^{(i)} \) and \( F_Q^{(ii)} \), and does not depend on the choice of the phase shift generator, \( U^{(i)}_\phi \) or \( U^{(ii)}_\phi \). For \( \tau = 1/2 \), \( F_Q^{(\rho)} \) is given by:

\[ F_Q^{(\rho)} = |\alpha|^2 e^{2r} + \sinh^2(r). \] (4.4)

This suggests that, without discussing the need for a reference beam, \( F_Q^{(i)} \) and \( F_Q^{(ii)} \) cannot be deployed correctly.

For a two-parameter estimation as depicted in (iii), there arises a need for a reference
beam, with respect to which these phase shifts are defined. As before, since this is a two-parameter problem, the multivariable quantum Cramér-Rao bound needs to be calculated. As before, the covariance is bounded by:

\[ C \geq F^{-1} = 4(\langle \partial_i \psi | \partial_j \psi \rangle - \langle \partial_i \psi | \psi \rangle \langle \psi | \partial_j \psi \rangle). \quad (4.5) \]

where \( C_{ij}(i, j = 1, 2) \) is the covariance matrix for parameters \( \phi_1 \) and \( \phi_2 \). If we want to find the precision bound on \( \phi_i \), then the proper bound is given by:

\[ \Delta \phi_i \geq \sqrt{(F^{-1})_{ii}}. \quad (4.6) \]

In general, \( (F^{-1})_{ii} \neq (F_{ii})^{-1} \). To correctly derive a bound, the cross-correlation terms cannot be ignored, and the inverse of the full matrix needs to be taken for a proper bound.

### 4.2 SU(1,1) and phase configurations for single phase estimation

Several QFI-based analysis of the SU(1,1) interferometer have been performed in the last few years. Sparaciari et al. calculated the QFI with both coherent state inputs, and a phase shift in a single arm [95, 96]. Li et al. studied the QFI with a coherent and a squeezed vacuum state as inputs with a phase shift in one arm [77]. They obtained a QFI much higher than that can be obtained with their choice of parity measurement. Gong et al. further extended the study of Li et al. and calculated the QFI for different phase configurations. Specifically, their phase configurations consist of: phase shift \( \phi \) in the first arm, phase shift \( \phi \) in the second arm, and phase shifts \( \phi_1 \) and \( \phi_2 \) in both arms [97]. They found that QFI depends on the position of the phase shift. This implies that the QFI can be different for a physically equivalent setup. This seems bizarre, as one would expect the same QFI if the task is estimating \( \phi \) in all these cases. This is the main motivation of our work. Our goal was to get to the bottom of this discrepancy, and point out the correct way to calculate and interpret the QFI.
Figure 4.2: Schematic of a SU(1,1) interferometer with different phase shifts: phase shift in upper arm with generator $\hat{g}_u$, phase shift in lower arm with generator $\hat{g}_l$, equal phase shift in both arms given by generator $\hat{g}_s$, and phase shifts in both arms with generator $\hat{g}_s - \hat{g}_d$.

Just for completeness, we briefly go over the SU(1,1) model one more time. A schematic of the SU(1,1) interferometer with different phase configurations is shown in Figure 4.1. Two input modes interact via an optical parametric amplifier (OPA), with gain parameter $g$, and then go through a phase shift on one or both of the arms. After the phase shifts, the measurement is done. The second OPA (with a $\pi$ phase difference than the first as before) is not necessary for QFI analysis, as it can be thought of as being part of the measurement process. As before, the relation between the output and input modes of the OPA is given by:

$$\hat{a}_1 = \cosh(g)\hat{a}_0 + e^{i\theta} \sinh(g)\hat{b}_0^\dagger,$$

$$\hat{b}_1 = \cosh(g)\hat{b}_0 + e^{i\theta} \sinh(g)\hat{a}_0^\dagger,$$

(4.7)
where $g$ and $\theta$ are the parametric gain and phase of the OPA. As before, $\hat{a}_i$ ($\hat{b}_i$) and $\hat{a}_i^\dagger$ ($\hat{b}_i^\dagger$) are the annihilation and creation operators in mode A (B) respectively, and the subscripts 0 and 1 represents the input mode and output mode of the first OPA, respectively.

Our model consists of four different phase-shift configurations. The first three configurations, in Figure 4.1, represent single-phase estimation, while the last represents two-phase estimation. The choice of the model depends on the type of application one has in mind [61]. If only the first arm undergoes a phase shift, it is represented by the unitary operator $\hat{U}^u_\phi = e^{i\hat{g}u\phi}$ with generator $\hat{g}_u = \hat{a}_1^\dagger \hat{a}_1$. If the phase shift occurs only in lower arm, then the unitary operator is given by $\hat{U}^l_\phi = e^{i\hat{g}l\phi}$ with generator $\hat{g}_l = \hat{b}_1^\dagger \hat{b}_1$. Similarly, if the phase shift is equally split between the two arms, then the unitary is $\hat{U}^s_\phi = e^{i\hat{g}s\phi}$ with $\hat{g}_s = (\hat{a}_1^\dagger \hat{a}_1 + \hat{b}_1^\dagger \hat{b}_1)/2$. For some applications, such as LIGO, different phase shifts occur in each arms [93]. For this scenario, the unitary operator is given by $\hat{U}_\phi = e^{i\hat{g}_1\phi_1}e^{i\hat{g}_2\phi_2} = e^{i\hat{g}_s\phi_s}e^{i\hat{g}_d\phi_d}$, where $\hat{g}_d = (\hat{a}_1^\dagger \hat{a}_1 - \hat{b}_1^\dagger \hat{b}_1)/2$, and $\phi_1$ and $\phi_2$ are the unknown phases in the two arms. These phases are also described by the phase sum $\phi_s = \phi_1 + \phi_2$ and the phase difference $\phi_d = \phi_1 - \phi_2$.

Now, let us look in detail at the results obtained in Ref. [97]. When the state before the measurement is a pure state given by $|\psi_{\phi}\rangle_{AB} = e^{-i\hat{g}\phi}|\psi\rangle_{AB}$, the QFI is given by [28]:

$$F_Q = 4(\langle \psi | \hat{g}^2 | \psi \rangle - \langle \psi | \hat{g} | \psi \rangle^2). \quad (4.8)$$

First, let us consider vacuum states as inputs, then all single phase configuration results in the same QFI given by:

$$F_Q = n_\kappa(n_\kappa + 2). \quad (4.9)$$

where $n_\kappa = 2\sinh^2(g)$ is the average photon number of the output state due to the contribution from the OPA. This result suggests, that regardless of the position of the phase, the precision bound of the SU(1,1) interferometer is always the same, which makes sense, since all these phase configurations represent an equivalent physical system.
However, if the input states are non-vacuum states, then QFI obtained are different as pointed out in Ref. [97]. The authors calculated the QFI when one of the input is a coherent state, and the other is a coherent or a squeezed vacuum state. For simplicity, let us consider a coherent state $|\beta\rangle$, with $n_{\beta} = |\beta|^2$, and a vacuum state as inputs. The QFIs for these inputs, with generators ($\hat{g}_u$, $\hat{g}_l$, and $\hat{g}_s$) are given by [97]:

$$F_Q(\hat{g}_u) = n_\beta \cosh (4g) + \sinh^2 (2g) + n_\beta (1 - 2 \cosh (2g)),$$

(4.10)

$$F_Q(\hat{g}_l) = n_\beta \cosh (4g) + \sinh^2 (2g) + n_\beta (1 + 2 \cosh (2g)),$$

(4.11)

$$F_Q(\hat{g}_s) = n_\beta \cosh (4g) + \sinh^2 (2g).$$

(4.12)

This situation is similar to the MZI case, where the QFI depends on the position of the phase shift. In the following section, we apply the phase averaging method introduced before, and we find that the QFIs for all these different phase configurations give the same result.

4.3 Single phase estimation with an arbitrary and a vacuum state

To make our results more general, we first consider a single phase estimation in the SU(1,1) interferometer, with an arbitrary state $\hat{\rho}_\chi$ in mode $A$ and a vacuum state in mode $B$.

The calculation of the QFI without an actual implementation of a detection scheme might suggest a false quantum advantage [61, 93]. This is because the optimal measurement that saturates the QFI may include uncounted resources, such as an external strong local oscillator. The possibility of this overestimation is circumvented by eliminating the common reference beam between the input states and the measurement. To apply the phase averaging method, we start by expanding $\hat{\rho}_\chi$ in the photon-number basis as:

$$\hat{\rho}_\chi = \sum_{n,m=0}^{\infty} c_{nm}|n\rangle\langle m|,$$

(4.13)
where $|n\rangle$ is the $n$-photon number state. The reference beam between the inputs and the measurement is removed by phase averaging the input states as in Ref. [93]:

\[
\Psi_{\text{avg}} = \int \frac{d\varphi}{2\pi} \hat{V}_{\varphi}^A \hat{V}_{\varphi}^B \left( \hat{\rho}_\chi \otimes |0\rangle \langle 0|^B \right) \hat{V}_{\varphi}^A \dagger \hat{V}_{\varphi}^B \dagger
\]

\[
= \sum_{n,m=0}^\infty \int \frac{d\varphi}{2\pi} e^{i\varphi(n-m)} c_{nm} |n\rangle \langle m|^A \otimes |0\rangle \langle 0|^B
\]

\[
= \sum_{n=0}^\infty p_n |n\rangle \langle n|^A \otimes |0\rangle \langle 0|^B,
\]

where $\hat{V}_{\varphi}^A = e^{i\varphi \hat{a}^\dagger}$, $\hat{V}_{\varphi}^B = e^{i\varphi \hat{b}^\dagger}$, and $p_n = c_{nn}$ is a real positive number satisfying $\sum_n p_n = 1$ [61, 98]. The state after the first OPA is given by:

\[
\Psi_{\text{OPA avg}} = \hat{T}_{\text{OPA}} |\psi_n\rangle_{AB} \langle \psi_n|_{AB}
\]

\[
\text{where } |\psi_n\rangle_{AB} = \sum_{k=0}^\infty c_{n,k} |k\rangle_A \otimes |k\rangle_B.
\]

This follows from the fact that the photon number difference between two arms is conserved in the SU(1,1) interferometer [72]. Here, we only average the input phases and not the phase of the OPA, and this makes the amplitude factor $c_{n,k}$ a function of the OPA phase $\theta$. Any advantage that can be achieved by using the phase $\theta$ of the OPA is allowed as in the first SU(1,1) scheme [72]. By using the convexity of the QFI, and noticing that $|\psi_n\rangle$ and $|\psi_{n'}\rangle$ are orthogonal for $n \neq n'$, we have [23, 99]:

\[
F_Q (\Psi_{\text{avg}}) = \sum_{n=0}^\infty p_n F_Q (|\psi_n\rangle).
\]

Here, in the left hand side of Eq. (4.16), $\Psi_{\text{avg}}$ is used as an input. If we denote the state
after the phase shift as $\Psi_{\text{avg}}(\phi) = \sum n p_n |\psi_n(\phi)\rangle\langle\psi_n(\phi)|$, then in principle, each $|\psi_{\text{avg}}(\phi)\rangle$ contained in $\Psi_{\text{avg}}(\phi)$ is perfectly and coherently distinguishable by applying projectors

$$P_m = \sum_{j=0}^{\infty} |m + j\rangle\langle m + j| \otimes |j\rangle\langle j|,$$

that is, by the applying quantum non-demolition measurement on that basis. Doing this, one can distinguish $|\psi_n(\phi)\rangle$ in a post-selective way, and then conditionally choose the measurement for phase estimation. This is equivalent to what we have in right hand side of Eq. (4.16), where we can choose $|\psi_n\rangle$ initially as an input with weight $p_n$, and then apply appropriate measurements to each one of them. Hence, the right hand side of Eq. (4.16) is always reachable with $\Psi_{\text{avg}}$.

Now, using the phase-averaged input states, the QFI of $|\psi_n\rangle$, with phase shift in the upper arm with generator $\hat{g}_u$, is given by:

$$F_Q^u(|\psi_n\rangle) = 4(\langle \hat{g}_u^2 \rangle - (\langle \hat{g}_u^2 \rangle^2), \quad (4.17)$$

where $\langle \hat{g}_u \rangle = n \cosh^2(g) + \sinh^2(g)$.

and $\langle \hat{g}_u^2 \rangle = n^2 \cosh^4(g) + (n + 1) \cosh^2(g) \sinh^2(g) + 2n \cosh^2(g) \sinh^2(g) + \sinh^4(g)$. Hence,

$$F_Q^u(|\psi_n\rangle) = 4(n + 1) \sinh^2(g) \cosh^2(g) = (n + 1)n_\kappa(n_\kappa + 2). \quad (4.18)$$

Finally, the QFI of the phase-averaged input state $\Psi_{\text{avg}}$ is given by:

$$F_Q^u(\Psi_{\text{avg}}) = \sum_{n=0}^{\infty} p_n (n + 1)n_\kappa(n_\kappa + 2),$$

$$= (\bar{n}_\chi + 1)n_\kappa(n_\kappa + 2), \quad (4.19)$$

where $\bar{n}_\chi = \sum_n np_n$ is the average photon number of $\hat{\rho}_\chi$.

Similarly, with phase-averaged input states $\Psi_{\text{avg}}$, for the phase shift in lower arm with generator $\hat{g}_l$, the QFI is given by:

$$F_Q^l(\Psi_{\text{avg}}) = (\bar{n}_\chi + 1)n_\kappa(n_\kappa + 2). \quad (4.20)$$
Finally, for the two equal phase shifts with generator $\hat{g}_s$ and phase-averaged input states $\Psi_{\text{avg}}$, the QFI is given by:

$$F^s_Q(\Psi_{\text{avg}}) = (\bar{n}_\chi + 1)n_\kappa(n_\kappa + 2).$$  \hspace{1cm} (4.21)

We see from the results in Eqs. (4.19–4.21) that once we rule out the use of external phase reference at the measurement, all these different phase configurations reduce to the same QFI. Also, if we take $\bar{n}_\chi \rightarrow 0$, meaning we have vacuum input states in both input ports, we get back the result from Eq. (4.9). Moreover, from Eqs. (4.19–4.21), we see that when one of the inputs is a vacuum state, the QFI is proportional to the average photon number $\bar{n}_\chi$ of the arbitrary state $|\chi\rangle$, but does not depend on the state $\hat{\rho}_\chi$. This suggests that if the OPA gain $g$ is fixed, the best strategy is to use a state with higher average photon number. That is, using exotic quantum states as input does not help in boosting the sensitivity, when one of the input states is a vacuum state. This conclusion bears resemblance to the MZI case, in which one cannot beat the shot-noise limit by using a nonclassical state, if one of the inputs is a vacuum state [61]. Also, since the above QFI is proportional to the square of the average photon number of the OPA, the best strategy is to use all the resources to boost the gain $g$ of the OPA.

Now, we compare our QFI with previously known results. If we consider a coherent state $|\alpha\rangle$ as input in mode A, then our QFI is:

$$F^\alpha_Q = (n_\alpha + 1)n_\kappa(n_\kappa + 2).$$  \hspace{1cm} (4.22)

Our QFI is lower than all previous results calculated without phase averaging the input states [Eqs. (4.11–4.12)] as seen in Figure. 4.3. This suggests that to achieve the QFIs in Eqs. (4.11–4.12), external (but uncounted) resources at the measurement is required. Li et al. had calculated the QFI for this inputs, along with the phase sensitivity using parity detection, and had claimed that parity measurement cannot attain this QFI [77]. However,
Figure 4.3: Comparison of the QFI of the phase-averaged state [Eqs. (4.19 – 4.21)] with the results of Gong et al. [Eqs. (4.10 – 4.12)]. Plotted with \( n_\beta = n_\chi = 10 \).

Our result (Eq. 4.22) is exactly saturated by parity detection, suggesting that Li et al. overestimated the QFI by not taking into account hidden (unaccounted) resources.

### 4.4 Two-parameter phase estimation

In this section, we look at the two-parameter phase estimation. First, we input an arbitrary state and a vacuum state, and then a coherent state and a squeezed state. First, just to reiterate again, all the previous models with generator \( \hat{g}_l \), \( \hat{g}_u \), and \( \hat{g}_s \) are single parameter estimation. Although \( \hat{g}_s \) consists of phase \( \phi/2 \) in both arms, the essential task is estimating a single phase \( \phi \). In all these cases, it is also implicitly assumed that the phase difference \( \phi_d \) is known beforehand. There are instances when both arms undergo different phase shifts, and multiparameter estimation becomes necessary. In these scenarios, both phase sum \( \phi_s \) and phase difference \( \phi_d \) are unknown, and even though only \( \phi_s \) is of interest, we have to employ multiparameter estimation, and employ the multiparameter QCRB as in Chapter 2 for bounding the precision limit.
4.4.1 Arbitrary state and vacuum state

Let us consider the estimation of $\phi_s$ and $\phi_d$ in the SU(1,1) interferometer when the inputs are a vacuum state and an arbitrary pure state. For this two-parameter phase estimation scenario, the QCRB is calculated using two-by-two quantum Fisher information matrix (QFIM) given by [26]:

$$F_Q = \begin{bmatrix} F_{dd} & F_{sd} \\ F_{ds} & F_{ss} \end{bmatrix},$$

(4.23)

where $F_{ij} = 4 \langle \hat{g}_i \hat{g}_j \rangle - \langle \hat{g}_i \rangle \langle \hat{g}_j \rangle$, and the subscripts $s$ and $d$ denote $\phi_s$ and $\phi_d$ respectively. Since we are interested in phase sum $\phi_s$, the QCRB is given by:

$$\Delta^2 \phi_s \geq \frac{F_{dd}}{F_{dd}F_{ss} - F_{ds}F_{sd}}.$$

(4.24)

Using our input states $|\chi\rangle \otimes |0\rangle$, each element of $F_Q$ is calculated to be:

$$F_{dd} = V_\chi,$$

$$F_{ds} = F_{sd} = V_\chi \cosh(2g),$$

$$F_{ss} = V_\chi \cosh^2(2g) + (1 + \bar{n}_\chi) \sinh^2(2g),$$

(4.25)

where $\bar{n}_\chi$ is the average photon number of the state $|\chi\rangle$, and $V_\chi = \langle \chi | \hat{n}^2 | \chi \rangle - \langle \chi | \hat{n} | \chi \rangle^2$ is the photon number variance of $|\chi\rangle$. Plugging these values into Eq. 4.24, we get

$$\Delta^2 \phi_s \geq \frac{1}{(\bar{n}_\chi + 1)n_\kappa (n_\kappa + 2)}.$$

(4.26)

Here, it should be noted that if the non-diagonal terms are ignored, i.e. implicitly assume that $\phi_d$ is known a priori, we get a higher QFI than in Eq. (4.26), which misleadingly overestimates the precision limit.
As can be easily seen, if $|\chi\rangle = |\alpha\rangle$, we get:

$$\Delta^2\phi_s \geq \frac{1}{(n_\alpha + 1)n_\kappa(n_\kappa + 2)},$$

which is the QCRB obtained using the phase averaging method. The two methods are equivalent when one of the inputs is a vacuum state. To see this, let $\rho = \hat{\rho}_{\text{in}} \otimes |0\rangle\langle 0|^B$, and we phase average this state as:

$$\Psi_{\text{avg}}^{\text{OPA}} = \int \frac{d\theta}{2\pi} \hat{T}_{\text{OPA}}^{AB} \hat{V}_\theta^A \hat{V}_\theta^B \hat{\rho} \hat{V}_\theta^A \hat{V}_\theta^B \hat{T}_{\text{OPA}}^{AB} = \int \frac{d\theta}{2\pi} \hat{T}_{\text{OPA}}^{AB} \hat{V}_\theta^A \hat{V}_\theta^B \hat{\rho} \hat{T}_{\text{OPA}}^{AB} \hat{V}_\theta^A \hat{V}_\theta^B \hat{T}_{\text{OPA}}^{AB} = \int \frac{d\theta}{2\pi} \hat{\rho} \hat{T}_{\text{OPA}}^{AB} \hat{V}_\theta^A \hat{V}_\theta^B \hat{T}_{\text{OPA}}^{AB} \hat{V}_\theta^A \hat{V}_\theta^B \hat{T}_{\text{OPA}}^{AB} \hat{\rho} \hat{T}_{\text{OPA}}^{AB} \hat{V}_\theta^A \hat{V}_\theta^B \hat{T}_{\text{OPA}}^{AB} \hat{V}_\theta^A \hat{V}_\theta^B \hat{T}_{\text{OPA}}^{AB}.$$

The second equality follows from the fact that the vacuum state is not changed by the phase-shift operator. The third equality holds as the two-mode squeezing (OPA) commutes with the phase shift operation $\hat{V}_\theta^A \hat{V}_\theta^B$. This suggests that the phase averaging is equivalent to adding another unknown phase $\theta$ in the upper arm (mode A) and an unknown phase $-\theta$ in the lower arm (mode B). That is, the interferometer’s phase difference is set to be unknown, and the problem is equivalent to estimating two unknown parameters, $\phi_s$ and $\phi_d$ [94]. Although this results in the same precision bound, the estimation problem has changed. Also, the above results hold only when one of the inputs is a vacuum state, but do not hold in general for both non-vacuum inputs.

### 4.4.2 Coherent and squeezed vacuum state

Now, let us look at both non-vacuum input states. We use a coherent state and a squeezed vacuum state: $|\alpha\rangle_A \otimes |\xi\rangle_B$ as inputs. For simplicity, we assume $\alpha$ is real. By using the QFIM method, we find that the QFI in estimating phase sum $\phi_s$ is given by:

$$F_Q^1 = \sinh^2(2g) \left[ |\alpha|^2 e^{2r} + \cosh^2(r) \right] + \cosh^2(2g) \frac{8|\alpha|^2 \sinh^2(2r)}{4|\alpha|^2 + 2 \sinh^2(2r)}. \quad (4.28)$$
where $r$ is the squeezing strength of the squeezed vacuum state $|\xi\rangle$. We compare our QFI with the results in Ref. [77], which is given by:

$$F_Q^2 = \sinh^2(2g) \left[ |\alpha|^2 e^{2r} + \cosh^2(r) \right] + \cosh^2(2g) \left[ |\alpha|^2 + \frac{1}{2} \sinh^2(2r) \right]. \quad (4.29)$$

The difference between $F_Q^1$ and $F_Q^2$ is:

$$F_Q^1 - F_Q^2 = -\cosh^2(2g) \frac{\left[ -4|\alpha|^2 + \cosh(4r) - 1 \right]^2}{4 \left[ 4|\alpha|^2 + \cosh(4r) - 1 \right]}. \quad (4.30)$$

The difference is always negative, meaning that our QFI is lower, which suggests our QFI $F_Q^1$ gives a tighter QCRB.

Now, let us compare the QFI in Eq. (4.28) with the classical Fisher information (CFI) of parity detection, which is the known best strategy for these particular input states in ideal scenario [77]. The CFI of the parity detection is given by:

$$F_{cl} = \sinh^2(2g) \left[ |\alpha|^2 e^{2r} + \cosh^2(r) \right]. \quad (4.31)$$

It is clear that $F_Q^1$ is larger than $F_{cl}$. This suggests that the parity measurement is not an optimal measurement, and some other type of measurement needs to be employed to saturate the QCRB.

### 4.5 Discussion

In this chapter, I discussed the issue in claiming precision of phase estimation, solely based on the quantum Fisher information (QFI). First, we looked into detail the QFI analysis in MZI, and discussed the role of the reference phase (beam). I also discussed the phase-averaging method, which correctly gives the same QFI for different yet physically equivalent phase configurations. We also looked at the previous QFI results in the SU(1,1) interferometer, which gives different QFI for physically equivalent phase configurations. We employed the phase-averaging method in the SU(1,1) interferometer, and showed that
all these different configurations give the same QFI. We also established that the phase-
averaging method is equivalent to two-parameter estimation for an arbitrary and a vacuum
state. We also showed that our method provides tighter bounds than previously known
results.
Chapter 5
Summary

Over the past few years, several theoretical studies have been done in the field of quantum metrology. The work in this dissertation was a small effort to contribute to the ongoing studies in this giant field. There also have been several breakthroughs in experiments. With technological advancement, it is no doubt that in the near future, quantum metrological techniques will evolve out from a small laboratory to full implementation for practical use in day to day problems, and will positively affect people’s lives. I would like to end by summarizing the work in this dissertation.

In Chapter 1, we looked at the building blocks and tools to carry out the work in this thesis. We looked at different quantum states of light (vacuum state, coherent state, squeezed state, thermal state, Fock state, two-mode squeezed state). We also discussed quantum metrology and the task of phase estimation and introduced relevant quantities used in phase estimation.

We looked into the multimode interferometer in Chapter 2, which uses single photons at the input. We showed that our scheme performs better compared to all previously known schemes. We also implemented a scattershot metrology scheme in which photons enter the interferometer with some probability. We are currently working on extending this scheme for quantum imaging.

Chapter 3 and 4 dealt with a nonlinear interferometer called a SU(1,1) interferometer. In Chapter 3, we looked at the phase estimation with SU(1,1) with different states of light, and showed sub-shot-noise sensitivity. We also showed that advantage with parity measurement is not practical, considering losses in the experimental setup.

In Chapter 4, we clarified the use of the QFI in the SU(1,1) interferometer. We showed that existing results overestimate the precision by not properly taking into account the use of the reference phase. We also showed that phase averaging and QFIM methods are equivalent when one of the inputs is a vacuum state. We also showed that our method
gives a tighter QCRB than previously known.
References


Appendix A
QFI for Multiparameter phase estimation

Here, we calculate the entries of the QFI matrix $F_{\bar{\phi}}$. In general, they are specified by,

$$[F_{\bar{\phi}}]_{l,n} = 4 \left< \hat{b}_l^{\dagger} \hat{b}_l \hat{b}_n^{\dagger} \hat{b}_n \right> - 4 \left< \hat{b}_l^{\dagger} \hat{b}_l \right> \left< \hat{b}_n^{\dagger} \hat{b}_n \right>.$$  \hfill (A.1)

Computing the latter term first,

$$\langle \psi | \hat{b}_l^{\dagger} \hat{b}_l | \psi \rangle = \sum_{q,l=1}^m V_{j,l} V_{j,l} \langle k |^{\otimes m} \hat{a}_q^{\dagger} \hat{a}_l | k \rangle^{\otimes m}$$

$$= \sum_{q=1}^m |V_{j,q}|^2 \langle k |^{\otimes m} \hat{a}_q^{\dagger} \hat{a}_q | k \rangle^{\otimes m}$$

$$= \sum_{q=1}^m \frac{1}{m} \cdot k$$

$$= k.$$

Hence,

$$4 \left< \hat{b}_l^{\dagger} \hat{b}_l \right> \left< \hat{b}_n^{\dagger} \hat{b}_n \right> = 4 k^2.$$  \hfill (A.2)

Meanwhile,

$$\langle \psi | \hat{b}_l^{\dagger} \hat{b}_l \hat{b}_n^{\dagger} \hat{b}_n | \psi \rangle$$

$$= \sum_{i,j,q,p=1}^m V_{i,j} V_{i,j} V_{n,q} V_{n,q} \langle k |^{\otimes m} \hat{a}_i^{\dagger} \hat{a}_j^{\dagger} \hat{a}_q^{\dagger} \hat{a}_p | k \rangle^{\otimes m}$$

$$= \sum_{q,p,q \neq p}^m V_{i,p} V_{i,q} V_{n,q} V_{n,p} \langle k |^{\otimes m} \hat{a}_q^{\dagger} \hat{a}_q \hat{a}_p | k \rangle^{\otimes m}$$

$$+ \sum_{q,p}^m |V_{i,p}|^2 |V_{n,q}|^2 \langle k |^{\otimes m} \hat{a}_q^{\dagger} \hat{a}_p | k \rangle^{\otimes m}.$$  \hfill (A.3)

The second term is essentially just the square of Eq. (A.2), and equal to $k^2$, hence

$$= k^2 + \sum_{q,p,q \neq p}^m (V_{i,p} V_{i,q} V_{n,q} V_{n,p}) k (k + 1).$$  \hfill (A.4)
Rewriting $\omega = e^{2\pi i/m}$ as the first $m^{th}$ root of unity,

$$\omega = \frac{1}{m^2} \sum_{q,p=1,q\neq p}^m \omega^{(l-1)(p-1)}\omega^{-(l-1)(q-1)}\omega^{(n-1)(q-1)}\omega^{-(n-1)(p-1)} ,$$

$$= \frac{1}{m^2} \sum_{q,p=1,q\neq p}^m \omega^{(p-q)(l-1)}\omega^{-(p-q)(n-1)} ,$$

$$= \frac{1}{m^2} \sum_{q,p=1,q\neq p}^m \omega^{(p-q)(l-1)}\omega^{-(p-q)(n-1)} ,$$

$$= \frac{1}{m^2} \sum_{q,p=1,q\neq p}^m \omega^{(p-q)^{l-n}} .$$

(A.5)

If $l - n = 0$, i.e. for the diagonal entries of $\mathcal{F}^\text{quant}_\varphi$, the summand is 1 and hence the sum evaluates to $m^2 - m$. For the off-diagonal terms, let $l - n = k$, so

$$= \frac{1}{m^2} \sum_{q,p=1,q\neq p}^m \omega^{(p-q)}^k ,$$

$$= \frac{1}{m^2} \sum_{q,p=1,q\neq p}^m \omega^{(p-q)}^k .$$

(A.6)

Let $p - q = r$, and note that $r \neq 0$. There are $m$-many $\{p,q\}$ pairs whose difference is $r$ (or congruent to $r$ (mod $m$), since $\omega$ is a $m^{th}$ root of unity). Thus, the sum reduces to,

$$= \frac{1}{m^2} \left[ m \sum_{r=1}^{m-1} \omega^r \right]$$

$$= \frac{1}{m^2} \left[ m[-1] \right]$$

$$= -\frac{1}{m} ,$$

(A.7)

where we have used the fact that the sum over all powers of any $k^{th}$ root of unity is equal to one. Thus, the terms of the QFI matrix simplify to,

$$[\mathcal{F}_\varphi]_{l,n} = \begin{cases} 
4k(k + 1) \cdot \frac{m-1}{m} & l = m \\
4k(k + 1) \cdot -\frac{1}{m} & l \neq m 
\end{cases} .$$

(A.8)
Appendix B
Phase Estimation with SU(1,1) interferometer

B.1 Coherent State and Displaced-Squeezed-Vacuum State

The mean and covariance matrix of coherent state is given by:

\[
X_{\text{coh}} = \begin{pmatrix} \sqrt{2} \alpha_1 \cos(\theta_1) \\ \sqrt{2} \alpha_1 \sin(\theta_1) \end{pmatrix},
\]
\[
\Gamma_{\text{coh}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Similarly, the mean and covariance of the displaced-squeezed-vacuum (DSV) is given by:

\[
X_{\text{DSV}} = \begin{pmatrix} \sqrt{2} \alpha_2 \cos(\theta_2) \\ \sqrt{2} \alpha_2 \sin(\theta_2) \end{pmatrix},
\]
\[
\Gamma_{\text{DSV}} = \begin{pmatrix} (\cosh (r) + \sinh (r))^2 & 0 \\ 0 & (\cosh (r) - \sinh (r))^2 \end{pmatrix}.
\]

The combined input mean and covariance is given by:

\[
X_0 = X_{\text{coh}} \oplus X_{\text{DSV}} = \begin{pmatrix} \sqrt{2} \alpha_1 \cos(\theta_1) \\ \sqrt{2} \alpha_1 \sin(\theta_1) \\ \sqrt{2} \alpha_2 \cos(\theta_2) \\ \sqrt{2} \alpha_2 \sin(\theta_2) \end{pmatrix},
\]
\[
\Gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\cosh (r) + \sinh (r))^2 & 0 \\ 0 & 0 & 0 & (\cosh (r) - \sinh (r))^2 \end{pmatrix}.
\]

The input Wigner function can easily be constructed using this input mean and covariance matrix by plugging them into Eq. (6). We do not need to calculate the input Wigner function, and it just suffices to propagate the mean and covariance matrix through the SU(1,1) interferometer. After propagation through the SU(1,1) interferometer given by Eqs. (10) and (11), the input mean and covariance matrix evolves to output mean and output covariance matrix given by:

\[
X_2 = \begin{pmatrix} \sqrt{2} \alpha_1 \left( \cosh^2(g) \cos (\theta_1 + \phi) - \sinh^2(g) \cos(\theta_1) \right) - \sqrt{2} \alpha_2 \sinh(2g) \sin (\phi/2) \sin (\phi/2 - \theta_2) \\ \sqrt{2} \alpha_1 \left( \cosh^2(g) \sin (\theta_1 + \phi) - \sinh^2(g) \sin(\theta_1) \right) + \sqrt{2} \alpha_2 \sinh(2g) \sin (\phi/2) \cos (\phi/2 - \theta_2) \\ \sqrt{2} \alpha_1 \sinh(2g) \sin (\phi/2) \sin (\theta_1 + \phi/2) + \sqrt{2} \alpha_2 \left( \cosh^2(g) \cos(\theta_2) - \sinh^2(g) \cos(\phi - \theta_2) \right) \\ \sqrt{2} \alpha_1 \sinh(2g) \sin (\phi/2) \cos (\theta_1 + \phi/2) + \sqrt{2} \alpha_2 \left( \sinh^2(g) \sin (\phi - \theta_2) + \cosh^2(g) \sin (\theta_2) \right) \end{pmatrix},
\]

(B.7)
and each element of the covariance matrix is given by:

\[
\Gamma_{2} = \begin{pmatrix}
\Gamma_{11}^{2} & \Gamma_{12}^{2} & \Gamma_{13}^{2} & \Gamma_{14}^{2} \\
\Gamma_{21}^{2} & \Gamma_{22}^{2} & \Gamma_{23}^{2} & \Gamma_{24}^{2} \\
\Gamma_{31}^{2} & \Gamma_{32}^{2} & \Gamma_{33}^{2} & \Gamma_{34}^{2} \\
\Gamma_{41}^{2} & \Gamma_{42}^{2} & \Gamma_{43}^{2} & \Gamma_{44}^{2}
\end{pmatrix},
\]

(B.8)

\[
\Gamma_{11}^{2} = e^{2r} \sinh^2(2g) \sin^4 \left(\frac{\phi}{2}\right) + \sinh^4(g) + \cosh^2(g) \cosh^2(g) \left(e^{-2r} \sin^2(\phi) - 2 \cos(\phi)\right),
\]

\[
\Gamma_{12}^{2} = 4 \sinh^2(g) \cosh^2(g) \sinh(r) \cosh(r) \sin(\phi)(\cos(\phi) - 1) = \Gamma_{21}^{2},
\]

\[
\Gamma_{13}^{2} = -\frac{1}{4} e^{-2r} \left( e^{2r} - 1 \right) \left( e^{2r} + 1 \right) \sinh(g) \cosh(g) \left( 2 \sinh^2(g) \cos(2\phi) + 1 \right)
- \frac{1}{4} e^{-2r} \left( e^{2r} + 1 \right) \sinh(g) \cosh(g) \cosh(2g) \left( -4e^{2r} \cos(\phi) + 3e^{2r} + 1 \right) = \Gamma_{31}^{2},
\]

\[
\Gamma_{14}^{2} = \sinh(g) \cosh(g) \sin(\phi) \left( \sinh^2(r) + \cosh^2(r) + 1 \right) - \sinh(g) \cosh(g) \sinh(2r) \sin(\phi) \left( \cos(2g) - 2 \sinh^2(g) \cos(\phi) \right) = \Gamma_{41}^{2},
\]

\[
\Gamma_{22}^{2} = e^{-2r} \sinh^2(2g) \sin^4 \left(\frac{\phi}{2}\right) - \sinh^2(g) \cosh^2(g) \left( 2 \cos(\phi) - e^{2r} \sin^2(\phi) \right) + \sinh^4(g) + \cosh^4(g),
\]

\[
\Gamma_{23}^{2} = \sinh(g) \cosh(g) \sin(\phi) \left( \sinh(2g) - 2 \sinh^2(g) \cos(\phi) \right)
+ \sinh(g) \cosh(g) \sin(\phi) \left( \sinh^2(r) + \cosh^2(r) + 1 \right) = \Gamma_{32}^{2},
\]

\[
\Gamma_{24}^{2} = \frac{1}{2} e^{-2r} \left( e^{2r} + 1 \right) \sinh(g) \sinh^2(g) \cosh(g) \left( 2e^{2r} \sin^2(\phi) + \cos(2\phi) \right)
+ \frac{1}{4} e^{-2r} \left( e^{2r} + 1 \right) \sinh(g) \cosh(g) \cosh(2g)(3 - 4 \cos(\phi)) + 1 = \Gamma_{42}^{2},
\]

\[
\Gamma_{33}^{2} = e^{-2r} \sinh^4(g) \sin^2(\phi) + e^{2r} \left( \cosh^2(g) - \sinh^2(g) \cos(\phi) \right)^2 + \sinh^2(2g) \sin^2 \left(\frac{\phi}{2}\right),
\]

\[
\Gamma_{34}^{2} = 4 \sinh^2(g) \sinh(r) \cosh(r) \sin(\phi) \left( \cos^2(g) - \sinh^2(g) \cos(\phi) \right) = \Gamma_{43}^{2},
\]
\[ \Gamma_2^{44} = e^{2r} \sinh^4(g) \sin^2(\phi) + e^{-2r} \left( \cosh^2(g) - \sinh^2(g) \cos(\phi) \right)^2 + \sinh^2(2g) \sin^2 \left( \frac{\phi}{2} \right). \]

Using the above mean and covariance matrix, we can easily calculate the sensitivity with the parity and on-off measurement scheme.

\[ \langle \hat{\Pi} \rangle = \frac{8}{v_6} \exp \left[ \frac{128(v_1 \times v_2 - v_3 \times v_4)}{v_5} \right] \]  

(B.9) 

where,

\[ v_1 = \alpha_1 \sinh(2g) \sin \left( \frac{\phi}{2} \right) \cos \left( \theta_1 + \frac{\phi}{2} \right) + \alpha_2 \left( \sinh^2(g) \sin (\phi - \theta_2) + \cosh^2(g) \sin (\theta_2) \right) \]

\[ v_2 = \alpha_1 \sinh(2g) \sin \left( \frac{\phi}{2} \right) \sin \left( \theta_1 + \frac{\phi}{2} \right) + \alpha_2 \left( \cosh^2(g) \cos (\theta_2) - \sinh^2(g) \cos (\phi - \theta_2) \right) \]

\[ \times 4 \sinh^2(g) \sinh(r) \cosh(r) \sin(\phi) \left( \cosh^2(g) - \sinh^2(g) \cos(\phi) \right) \]

\[- \left( \alpha_1 \sinh(2g) \sin \left( \frac{\phi}{2} \right) \cos \left( \theta_1 + \frac{\phi}{2} \right) + \alpha_2 \left( \sinh^2(g) \sin (\phi - \theta_2) + \cosh^2(g) \sin (\theta_2) \right) \right) \]

\[ \times \left( e^{-2r} \sinh^4(g) \sin^2(\phi) + e^{2r} \left( \cosh^2(g) - \sinh^2(g) \cos(\phi) \right)^2 + \sinh^2(2g) \sin^2 \left( \frac{\phi}{2} \right) \right) \]

\[ v_3 = \alpha_1 \sinh(2g) \sin \left( \frac{\phi}{2} \right) \sin \left( \theta_1 + \frac{\phi}{2} \right) + \alpha_2 \left( \cosh^2(g) \cos (\theta_2) - \sinh^2(g) \cos (\phi - \theta_2) \right), \]

\[ v_4 = e^{2r} \sinh^4(g) \sin^2(\phi) + e^{-2r} \left( \cosh^2(g) - \sinh^2(g) \cos(\phi) \right)^2 + \sinh^2(2g) \sin^2 \left( \frac{\phi}{2} \right) \]

\[ \left( \alpha_1 \sinh(2g) \sin \left( \frac{\phi}{2} \right) \sin \left( \theta_1 + \frac{\phi}{2} \right) + \alpha_2 \left( \cosh^2(g) \cos (\theta_2) - \sinh^2(g) \cos (\phi - \theta_2) \right) \right) \]

\[- \left( 4 \sinh^2(g) \sinh(r) \cosh(r) \sin(\phi) \left( \cosh^2(g) - \sinh^2(g) \cos(\phi) \right) \right) \times \]

\[ \left( \alpha_1 \sinh(2g) \sin \left( \frac{\phi}{2} \right) \cos \left( \theta_1 + \frac{\phi}{2} \right) + \alpha_2 \left( \sinh^2(g) \sin (\phi - \theta_2) + \cosh^2(g) \sin (\theta_2) \right) \right), \]

\[ v_5 = 32 \cosh^2(r) \left( \sinh^4(2g) \cos(2\phi) - \sinh^2(4g) \cos(\phi) \right) \left( 4 \cosh(4g - 2r) + 3 \cosh(8g - 2r) + 8 \cosh(4g) + 6 \cosh(8g) + 50 + 4 \cosh(2(2g + r)) + 3 \cosh(2(4g + r)) - 14 \cosh(2r) \right), \]

\[ v_6 = \sqrt{8 \sinh^4(2g) \cos(2\phi) - 8 \sinh^2(4g) \cos(\phi) + 4 \cosh(4g) + 3 \cosh(8g)(4 \cosh(r) - 14 \cosh(2r) + 50)}. \]
B.2 Thermal State and Squeezed-Vacuum State

Similarly as before, the combined input mean and covariance matrix can be constructed. After the evolution, the mean is zero as both thermal and squeezed vacuum has zero means. The terms of the covariance matrix is given by:

\[
\begin{align*}
\gamma_{11} &= e^{-2r} \sin^2\left(\frac{\phi}{2}\right) \sinh^2(2g) + \frac{1}{4} \left[ 3 + \cosh(4g) - 2 \cos \phi \sinh^2(2g) \right] (1 + 2n_{th}), \\
\gamma_{13} &= \gamma_{31} = \sin^2\left(\frac{\phi}{2}\right) \sinh(4g) (- \cosh r + \sinh r) \left( \cosh r + e^n_{th} \right), \\
\gamma_{14} &= \gamma_{41} = e^{-2r} \cosh g \sin \phi \sinh g \left[ 1 + e^{2r} (1 + 2n_{th}) \right], \\
\gamma_{22} &= e^{2r} \sin^2\left(\frac{\phi}{2}\right) \sinh^2(2g) + \frac{1}{4} \left[ 3 + \cosh(4g) - 2 \cos \phi \sinh^2(2g) \right] (1 + 2n_{th}), \\
\gamma_{23} &= \gamma_{32} = \cosh g \sin \phi \sinh g \left( 1 + e^{2r} + 2n_{th} \right), \\
\gamma_{24} &= \gamma_{42} = \frac{1}{2} \sin^2\left(\frac{\phi}{2}\right) \sinh(4g) \left( 1 + e^{2r} + 2n_{th} \right), \\
\gamma_{33} &= \frac{1}{2} e^{2r} \left( 1 + \cos \phi \right) + e^{-2r} \cosh^2(2g) \sin^2\left(\frac{\phi}{2}\right) + \sin^2\left(\frac{\phi}{2}\right) \sinh^2(2g) (1 + 2n_{th}), \\
\gamma_{34} &= \gamma_{43} = \cosh(2g) \sin \phi \sinh(2r), \\
\gamma_{44} &= \frac{1}{2} e^{-2r} \left( 1 + \cos \phi \right) + e^{2r} \cosh^2(2g) \sin^2\left(\frac{\phi}{2}\right) + \sin^2\left(\frac{\phi}{2}\right) \sinh^2(2g) (1 + 2n_{th}).
\end{align*}
\]

Now, the mean value and the covariance matrix of the lower output \(b\) which is given by:

\[
\bar{X}_{22} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Gamma_{22} = \begin{pmatrix} \gamma_{33} & \gamma_{34} \\ \gamma_{43} & \gamma_{44} \end{pmatrix}.
\]

The parity detection signal is given by:

\[
\langle \Pi_b \rangle = \frac{8}{\sqrt{T}}.
\]

where,

\[
T = e^{-2r} \{-7 + 50e^{2r} + 7e^{4r} + (1 + e^{2r})^2[4 \cosh(4g) + 3 \cosh(8g) + 8 \cos(2\phi)\sinh^4(2g) \\
- 8 \cos \phi \sinh^2(4g)]} + 32e^{-2r} \sin^2\left(\frac{\phi}{2}\right) \sinh^2(2g) n_{th} \left( (1 + e^{4r})[3 + \cosh(4g) \\
- 2 \cos \phi \sinh^2(2g)] + 8e^{2r} \sin^2\left(\frac{\phi}{2}\right) \sinh^2(2g)(1 + n_{th}) \right).
\]
and the phase sensitivity is given by:

\[ \Delta \phi = \frac{\Delta \hat{\Pi}_b}{\left| \frac{\partial \langle \hat{\Pi}_b \rangle}{\partial \phi} \right|}, \tag{B1} \]

where,

\[ \Delta \hat{\Pi}_b = \left\{ 1 - 64/\{ e^{-2r} \{-7 + 50e^{2r} - 7e^{4r} + (1 + e^{2r})^2 [4 \cosh(4g) + 3 \cosh(8g) + 8 \cos(2\phi) \sinh^2(4g) \\
- 8 \cos \phi \sinh^2(4g)] + 32e^{-2r} \sin^2(\frac{\phi}{2}) \sinh^2(2g) n_{th} \{(1 + e^{4r})[3 + \cosh(4g) - 2 \cos \phi \sinh^2(2g)] \\
+ 8e^{2r} \sin^2(\frac{\phi}{2}) \sinh^2(2g)(1 + n_{th})}\} \right\}^{1/2}, \tag{B2} \]

\[ \left| \frac{\partial \langle \hat{\Pi}_b \rangle}{\partial \phi} \right| = -\left\{ 128 \sinh^2(2g) \{-2 \sin(2\phi) \sinh^2(2g) \{ \cosh^2 r + n_{th} (1 + \cosh(2r) + n_{th}) \} \} + \sin(\phi) \{4 \cosh(2g) \cosh^2 r + 4n_{th} [\cosh^2(2g) \cosh(2r) + \sinh^2(2g)(1 + n_{th})] \} \right\} / \left\{ e^{-2r} \{-7 \right.
+ (1 + e^{2r})^2 [4 \cosh(4g) + 3 \cosh(8g) + 8 \cos(2\phi) \sinh^4(2g) - 8 \cos \phi \sinh^2(4g)] - 7e^{4r} \\
+ 50e^{2r} + 32 \sin^2(\frac{\phi}{2}) \sinh^2(2g) n_{th} [(1 + e^{4r})(3 + \cosh(4g) - 2 \cos \phi \sinh^2(2g)) \\
+ 8e^{2r} \sin^2(\frac{\phi}{2}) \sinh^2(2g)(1 + n_{th})] \right\}^{3/2}. \tag{B3} \]
Appendix C
QFI for SU(1,1) interferometer

Here, I provide the detailed calculations of Chapter 4. In the calculations, the inputs are vacuum state and an arbitrary pure state.

C.1.1 Phase shift in Upper Arm Only

Our input state is \( |\Psi_in\rangle = |0\rangle \otimes |\chi\rangle \). The Quantum Fisher information (QFI) when the phase shift is only in the upper arm is given by:

\[
F = 4 \left( \langle \Psi_in| (a_1^\dagger a_1)^2 |\Psi_in\rangle - \langle \Psi_in| a_1^\dagger a_1 |\Psi_in\rangle^2 \right) \tag{C.1}
\]

For simplicity, let us break down the calculation step by step. For the calculation of second term, \( a_1^\dagger a_1 \) term can be written as:

\[
a_1^\dagger a_1 = u^2 a_0^\dagger a_0 + |\nu|^2 b_0^\dagger b_0 + u \nu a_0^\dagger b_0^\dagger + u \nu^* a_0^\dagger b_0.
\tag{C.2}
\]

Only one term gives a non-zero element, yielding

\[
\langle \Psi_in| a_1^\dagger a_1 |\Psi_in\rangle = \langle 0 | \otimes \langle \chi | (b_0^\dagger b_0 \sinh^2 (r)) |0\rangle \otimes |\chi\rangle \\
= \sinh^2 (r) \langle \chi | b_0^\dagger b_0 |\chi\rangle \\
= \sinh^2 (r) (\langle b_0^\dagger b_0 \rangle + 1). \tag{C.3}
\]

The \( (a_1^\dagger a_1)^2 \) can be written as:

\[
(a_1^\dagger a_1)^2 = u^4 a_0^\dagger a_0 a_0^\dagger a_0 + u^3 \nu a_0^\dagger a_0 a_0^\dagger b_0^\dagger + u^3 \nu^* a_0^\dagger a_0 b_0^\dagger b_0 + u^2 |\nu|^2 a_0^\dagger a_0 b_0^\dagger b_0 \\
+ u^2 \nu a_0^\dagger b_0^\dagger a_0 b_0^\dagger b_0 + u^2 \nu^* a_0^\dagger b_0^\dagger a_0 b_0^\dagger b_0 + u |\nu|^2 a_0^\dagger b_0^\dagger b_0^\dagger b_0 \\
+ u^3 \nu^* a_0^\dagger b_0^\dagger a_0 b_0^\dagger b_0 + u^2 |\nu|^2 a_0^\dagger b_0^\dagger b_0^\dagger b_0 + u^2 |\nu|^2 \nu a_0^\dagger b_0^\dagger b_0^\dagger b_0 \\
+ u^2 |\nu|^2 a_0^\dagger a_0 b_0^\dagger b_0 + u |\nu|^2 \nu a_0^\dagger b_0^\dagger b_0^\dagger b_0 + u |\nu|^4 b_0^\dagger b_0^\dagger b_0^\dagger b_0. \tag{C.4}
\]

Only two terms survive yielding:

\[
\langle 0 | \otimes \langle \chi | (a_0^\dagger b_0^\dagger b_0^\dagger b_0 \sinh^2 (r) \cosh^2 (r)) |0\rangle \otimes |\chi\rangle \\
= \sinh^2 (r) \cosh^2 (r) \langle 0 | a_0^\dagger a_0 \langle \chi | b_0^\dagger b_0 |\chi\rangle \\
= \sinh^2 (r) \cosh^2 (r) (\langle b_0^\dagger b_0 \rangle + 1). \tag{C.5}
\]
and,
\[
\langle 0 | \otimes \langle \chi | (b_0^\dagger b_0 b_0^\dagger b_0^\dagger \sinh^4 (r)) | 0 \rangle \otimes | \chi \rangle \\
= \sinh^4 (r) \langle \chi | b_0^\dagger b_0 b_0^\dagger b_0^\dagger | \chi \rangle \\
= \sinh^4 (r) \langle \chi | (b_0^\dagger b_0 + 1)(b_0^\dagger b_0 + 1) | \chi \rangle \\
= \sinh^4 (r) \langle \chi | (b_0^\dagger b_0)^2 + 2b_0^\dagger b_0 + 1 | \chi \rangle \\
= \sinh^4 (r) \langle \chi | (b_0^\dagger b_0)^2 + 2(b_0^\dagger b_0) + 1). \tag{C.6}
\]

Putting together the above calculation, the first term of the QFI is given by:
\[
\langle \Psi_{in} | (a_1^\dagger a_1)^2 | \Psi_{in} \rangle = \sinh^4 (r)(\langle (b_0^\dagger b_0)^2 \rangle + 2\langle b_0^\dagger b_0 \rangle + 1) + \sinh^2 (r) \cosh^2 (r)(\langle b_0^\dagger b_0 \rangle + 1). \tag{C.7}
\]

Hence, the Quantum Fisher information (QFI) is:
\[
F = 4 \sinh^2 (r) \cosh^2 (r)(\langle b_0^\dagger b_0 \rangle + 1) + 4 \sinh^4 (r)(\langle (b_0^\dagger b_0)^2 \rangle + 2\langle b_0^\dagger b_0 \rangle + 1) - 4(\sinh^2 (r)(\langle b_0^\dagger b_0 \rangle + 1))^2
\]
When \(| \chi \rangle = | \beta \rangle\), the above QFI becomes Eq. (4.10).

### C.1.2 Phase shift in Lower Arm

Similarly, when the phase shift is in the lower arm, the QFI is given by:
\[
F = 4 \left( \langle \Psi_{in} | (b_1^\dagger b_1)^2 | \Psi_{in} \rangle - \langle \Psi_{in} | b_1^\dagger b_1 | \Psi_{in} \rangle^2 \right) \tag{C.8}
\]

Similarly to the last calculation, let’s break down the calculation step by step. The \(b_1^\dagger b_1\) term can be written as:
\[
b_1^\dagger b_1 = | \nu |^2 a_0^\dagger a_0^\dagger + u^2 b_0^\dagger b_0 + u \nu a_0^\dagger b_0^\dagger + u \nu^* a_0 b_0. \tag{C.9}
\]

Only two of the terms give a non-zero element,
\[
\langle 0 | \otimes \langle \chi | b_0^\dagger b_0 \cosh^2 (r) | 0 \rangle \otimes | \chi \rangle \\
= \cosh^2 (r) \langle \chi | b_0^\dagger b_0 | \chi \rangle \\
= \cosh^2 (r) \langle b_0^\dagger b_0 \rangle, \tag{C.10}
\]

and,
\[
\langle 0 | \otimes \langle \chi | a_0 a_0^\dagger \sinh^2 (r) | 0 \rangle \otimes | \chi \rangle = \sinh^2 (r), \tag{C.11}
\]

Giving,
\[
\langle \Psi_{in} | b_1^\dagger b_1 | \Psi_{in} \rangle = \cosh^2 (r) \langle b_0^\dagger b_0 \rangle + \sinh^2 (r). \tag{C.12}
\]
Similarly, \((b_1^\dagger b_1)^2\) can be written as:

\[
(b_1^\dagger b_1)^2 = |\nu|^4 a_0 a_0^\dagger a_0 a_0^\dagger + u^2|\nu|^2 a_0 a_0^\dagger b_0 b_0^\dagger + u|\nu|^2 \nu a_0 a_0^\dagger a_0^\dagger b_0 b_0^\dagger + u|\nu|^2 \nu^* a_0 a_0^\dagger a_0^\dagger b_0 b_0^\dagger + u^2|\nu|^2 a_0 a_0^\dagger a_0^\dagger b_0 b_0^\dagger + u^3 \nu a_0 b_0^\dagger b_0^\dagger b_0^\dagger + u^3 \nu^* a_0 b_0^\dagger b_0^\dagger b_0^\dagger + u^2|\nu|^2 a_0 a_0^\dagger a_0^\dagger b_0 b_0^\dagger + u^2 \nu^* a_0 a_0^\dagger b_0 b_0^\dagger + u^2 \nu^2 a_0 a_0^\dagger b_0 b_0^\dagger.
\]

Only four terms survive from above expression giving:

\[
\langle 0 | \otimes \langle \chi | \left( a_0 a_0^\dagger \right)^2 | 0 \rangle \otimes | \chi \rangle = \sinh^4 (r),
\]

\[
\langle 0 | \otimes \langle \chi | \left( a_0 a_0^\dagger b_0 b_0^\dagger \right) | 0 \rangle \otimes | \chi \rangle = \langle b_0^\dagger b_0 \rangle \sinh^2 (r) \cosh^2 (r),
\]

\[
\langle 0 | \otimes \langle \chi | \left( b_0^\dagger b_0 \right)^2 | 0 \rangle \otimes | \chi \rangle = \langle (b_0^\dagger b_0)^2 \rangle \cosh^4 (r),
\]

\[
\langle 0 | \otimes \langle \chi | a_0 a_0^\dagger b_0 b_0^\dagger | 0 \rangle \otimes | \chi \rangle = \langle (b_0^\dagger b_0) + 1 \rangle \sinh^2 (r) \cosh^2 (r).
\]

Hence,

\[
\langle \Psi_m | (b_1^\dagger b_1)^2 | \Psi_m \rangle = \sinh^4 (r) + 3 \sinh^2 (r) \cosh^2 (r) \langle b_0^\dagger b_0 \rangle + \cosh^4 (r) \langle (b_0^\dagger b_0)^2 \rangle + \sinh^2 (r) \cosh^2 (r).
\]

Hence, the Fisher information is:

\[
F = 4 (\sinh^4 (r) + 3 \langle b_0^\dagger b_0 \rangle \sinh^2 (r) \cosh^2 (r) + \langle (b_0^\dagger b_0)^2 \rangle \cosh^4 (r) + \sinh^2 (r) \cosh^2 (r)) - 4 (\sinh^2 (r) + \langle b_0^\dagger b_0 \rangle \cosh^2 (r))^2.
\]

Similarly as before, when \(| \chi \rangle = | \beta \rangle\), the QFI becomes to Eq. (4.11).

### C.1.3 Phase in Both Arms

When the phase shifts are in both arm, the QFI is given by:

\[
F = 4 \left( \langle \Psi_m | (a_1^\dagger a_1 + b_1^\dagger b_1)^2 | \Psi_m \rangle - \langle \Psi_m | a_1^\dagger a_1 + b_1^\dagger b_1 | \Psi_m \rangle^2 \right).
\]

Similarly to the previous calculation, the \(a_1^\dagger a_1 + b_1^\dagger b_1\) and \((a_1^\dagger a_1 + b_1^\dagger b_1)^2\) terms can be written as:

\[
a_1^\dagger a_1 + b_1^\dagger b_1 = (a_0 a_0^\dagger + b_0 b_0^\dagger) \cosh (2r) + 2 \sinh^2 (r) + 2a_0^\dagger b_0^\dagger e^{-i\theta} \cosh (r) \sinh (r)
\]

\[
+ 2a_0 b_0 e^{i\theta} \sinh (r) \cosh (r)
\]

Only two of them give a non-zero term:

\[
\langle 0 | \otimes \langle \chi | b_0 b_0^\dagger \cosh (2r) | 0 \rangle \otimes | \chi \rangle = \cosh (2r) \langle b_0^\dagger b_0 \rangle + \cosh (2r),
\]

\[
\langle 0 | \otimes \langle \chi | 2 \sinh (r) | 0 \rangle \otimes | \chi \rangle = 2 \sinh (r).
\]

Hence,

\[
\langle \Psi_m | a_1^\dagger a_1 + b_1^\dagger b_1 | \Psi_m \rangle = \cosh (2r) \langle b_0^\dagger b_0 \rangle + 2 \sinh (r)
\]
Similarly,

\[ \langle \Psi_{in} | (a_1^\dagger a_1 + b_1^\dagger b_1)^2 | \Psi_{in} \rangle = 4\sinh^4(r) + 4\sinh^2(r)cosh(2r)\langle b_0^\dagger b_0 \rangle + \cosh^2(2r)\langle (b_0^\dagger b_0)^2 \rangle \\
+ 4\sinh^2(r)cosh^2(r)\langle b_0^\dagger b_0 \rangle + 4\sinh^2(r)cosh^2(r). \]  

(C.22)

Hence, the QFI is:

\[ F = \cosh^2(2r) \left[ \langle b_0^\dagger b_0 \rangle^2 - \langle b_0^\dagger b_0 \rangle^2 \right] + \sinh^2(2r) \left[ 1 + \langle b_0^\dagger b_0 \rangle \right]. \]  

(C.23)

Similar to before, when \( |\chi\rangle = |\beta\rangle \), the QFI is equal to Eq. (4.12).
Appendix D
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