

6-25-2018

Invariant of Noncommutative Algebras and Poisson Geometry

Bach Van Nguyen

Louisiana State University and Agricultural and Mechanical College

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INVARIANT OF NONCOMMUTATIVE ALGEBRAS AND POISSON GEOMETRY

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Bach Van Nguyen

B.S., University of California at Davis, 2012

M.S., Louisiana State University, 2014

August 2018

Acknowledgments

First, the author gratefully acknowledge his advisor, Professor Milen Yakimov, for being an excellent mathematician, a great advisor, and for the guidance and support through out the years. His encouragements and constant inspirations have helped shape the author's mathematical career.

In addition, the math department at Louisiana State University and its faculty members have been very supportive of their graduate students. Especially, the author thanks Professor Pramod Achar, William Adkins, Richard Litherland, Bogdan Oporowski, and also Professor Juana Moreno for spending their valuable time serving on this thesis committee.

Also, without the unconditional love and support from his family, the author would has never made it this far in life, so he would like to wholeheartedly thank his parents and sister.

Further, the author likes to acknowledge his teachers from grade school, undergraduate, and graduate school for their contributions to the development of his education and career.

Lastly, many wonderful friendships have been formed through out his life and so the author is thankful for his friends.

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Abstract

In this dissertation, we describe the structure of discriminant of noncommutative algebras using the theory of Poisson quantization and ring theoretic properties of Poisson algebra. In particular, under appropriate conditions, we express the discriminant of specialization of $\mathbb{K}[q^{\pm 1}]$ -algebras as product of Poisson prime elements in some Poisson central subalgebra. In addition, we provide methods for computing noncommutative discriminant in various settings using results obtained for specialization of $\mathbb{K}[q^{\pm 1}]$ -algebras. Further, to demonstrate, we explicitly compute the discriminant of algebra of quantum matrices and quantum Schubert cell algebras specializing at roots of unity. This dissertation is part of the collaboration with Trampel and Yakimov in [26].

Chapter 1

Introduction

Discriminant has long been a key in the study of number theory, algebraic geometry and combinatorics. In noncommutative algebra, it is used to study orders and lattices; more details are in Reiner's book [27]. Beside the classical applications of discriminant in the study of noncommutative algebras, new and exciting applications have been found recently by Bell, Brown, Ceken, Palmieri, Wang, Yakimov, and Zhang. In [5, 6], discriminant is used as an invariant to describe the automorphism group of noncommutative algebras, which is a difficult problem in algebra, see [30, 31]. Relating to the automorphism problem, the Zariski cancellation problem also can be studied using discriminant, [1]. One of the most studied subject in mathematics is representation theory of algebras. On that note, Brown and Yakimov in [4] have successfully used discriminant to describe representations of prime PI algebras.

With these wonderful new activities involving discriminant, it is one's interest to study and find better and more efficient methods to compute discriminant of noncommutative algebras, since it is a challenging problem to calculate the discriminant in general. One of the difficult steps is computing the determinant of the defining matrix in definition of discriminant. Before the results in [26], people have only been using various techniques from noncommutative algebra to calculate discriminant, but still involve finding the determinant at the end. For instance, the case quasipolynomial ring and some quantum Weyl algebras in [5, 6, 7]. Further, the factors of discriminant have been observed in [5, 7] but not fully understood. Our results in [26] provides the geometric and algebraic meanings for such factors of discriminant. In addition, ours new description of the noncommutative

discriminant opens up many interesting directions to tackle the computational problem of discriminant and other new application problems associate to discriminant. Among those interesting directions are the connections to (quantum) cluster algebras [23, 26].

This thesis is part of the work with Trampel and Yakimov in [26]. In that paper, we obtain new approach for tackling the computation of discriminant of algebras which are coming from specializations. We draw connection between Poisson geometry associate to specialization of algebras and noncommutative discriminant. The discriminant is an element of the center of our algebra. More precisely, in our context, the center will be a Poisson algebra and the discriminant is a Poisson normal element. When this Poisson algebra is "nice", the discriminant is decomposed into product of Poisson prime elements. These Poisson prime elements carry geometric meanings which in turn is translated onto the discriminant. In other word, the discriminant can be studied via the symplectic foliation of our Poisson variety associate to the Poisson central algebra in the picture. Our results apply to many important classes of quantum algebras specialized at roots of unity.

1.1 Discriminant and Poisson Primes

Let R be an $\mathbb{K}[q^{\pm 1}]$ -algebra where \mathbb{K} is a field of characteristic zero. Let $\epsilon \in \mathbb{K}^\times$ such that $q - \epsilon$ is not a zero divisor. We define the specialization of R at ϵ to be the \mathbb{K} -algebra $R_\epsilon := R/(q - \epsilon)R$. We let σ be the quotient map. The center Z_ϵ of R_ϵ is a Poisson algebra with the Poisson bracket defined to be

$$\{\sigma(x_1), \sigma(x_2)\} := \sigma\left(\frac{x_1x_2 - x_2x_1}{q - \epsilon}\right)$$

for $x_i \in \sigma^{-1}(Z_\epsilon)$. Suppose that R_ϵ is free module of finite rank over some Poisson subalgebra C_ϵ of Z_ϵ . Let $Y = \{y_j | 1 \leq j \leq N\}$ be a C_ϵ -basis of R_ϵ . This give rises to a map $\text{tr} : R_\epsilon \longrightarrow C_\epsilon$ which is independent of the basis Y in the sense that if

one use a different basis then it is differ by a constant. Then the discriminant of R_ϵ over C_ϵ is

$$d(R_\epsilon/C_\epsilon) := \det([\text{tr}(y_i y_j)]) \in C_\epsilon.$$

More on the discriminant of algebra in general and its application to orders can be found in Chapter 2. The discussion of discriminant and R_ϵ is in Section 4.1 and 4.2.

We call an element of a Poisson algebra A , Poisson normal if the principal ideal generated by that element is closed under Poisson bracket. Similarly, a Poisson prime element is one such that the associated principal ideal is a prime ideal and is also closed under the Poisson bracket. In addition, the zero locus of a Poisson prime element is a union of the symplectic leaves of the Poisson variety $\text{Spec} A$. The converse also true. We describe in more detail of this topic in Section 3.3 and 3.4.

Here we state the theorem which relates discriminant and Poisson geometry. The details are described in Section 4.2.

Theorem 1.1.1. (N.–Trampel–Yakimov, [26, Theorem 3.2])

Let R be a $\mathbb{K}[q^{\pm 1}]$ -algebra. Let $\epsilon \in \mathbb{K}^\times$ be such that $q - \epsilon$ is regular in R . Suppose R_ϵ is a free module of finite rank over some Poisson subalgebra C_ϵ of $Z(R_\epsilon)$.

1. *The discriminant $d(R_\epsilon/C_\epsilon)$ is a Poisson normal element of $(C_\epsilon, \{.,.\})$.*
2. *When C_ϵ is a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain, $d(R_\epsilon/C_\epsilon) = 0$ or*

$$d(R_\epsilon/C_\epsilon) =_{C_\epsilon^\times} \prod_{i=1}^m p_i$$

for some (not necessarily distinct) Poisson prime elements $p_i \in C_\epsilon$.

Theorem 1.1.1 provides new methods and effective techniques to attack the computational problem of discriminant. We describe in details these techniques in various setting in Section 4.3. Moreover, we use our new-founded methods to explicitly calculate the discriminant of the quantum Weyl algebra in two generators, which previously has been done with more complicate techniques by Chan, Young, and Zhang in [7]. This can also be found in Section 4.3.

1.2 Result on Algebra of Quantum Matrices

For applications of our Theorem 1.1.1 and new computational techniques, we consider the algebra of square quantum matrices $R_q[M_n]$. We take the specialization $R_\epsilon[M_n]$ at an odd primitive ℓ^{th} root of unity $\epsilon \in \mathbb{K}$. Then $R_\epsilon[M_n]$ is free over its central subalgebra $C_\epsilon[M_n]$ generated by the ℓ^{th} power of the generators $\sigma(x_{ij})$ of $R_\epsilon[M_n]$. The Poisson central subalgebra $C_\epsilon[M_n]$ is isomorphic to a polynomial ring, so it is a UFD. Moreover, the Poisson primes in $C_\epsilon[M_n]$ are minors $\Delta_{I;J}$ for $I, J \subset [1, n]$.

Theorem 1.2.1. (N.–Trampel–Yakimov)[26, Theorem 4.1]

Let \mathbb{K} be a field of characteristic 0, $\ell > 2$ an odd integer and $\epsilon \in \mathbb{K}$ a primitive $\ell - th$ root of unity. Then

$$d(R_\epsilon[M_n]/C_\epsilon[M_n]) =_{\mathbb{K}^\times} \prod_{k=1}^n \Delta_k^L \prod_{j=1}^{n-1} \bar{\Delta}_j^L$$

where $L = \ell^{n^2-1}(\ell - 1)$.

The definition of algebra of square quantum matrices and its specializations together with Poisson structures are mentioned in Chapter 6. Theorem 1.2.1 is discussed in Section 6.3.

1.3 Result on Quantum Schubert Cell Algebra

Let \mathfrak{g} be a simple Lie algebra and W be its corresponding Weyl group. Let G be the split connected, simply connected algebraic group with Lie algebra \mathfrak{g} . For any

element $w \in W$, Lusztig [25] and De Concini, Kac, and Procesi [13] introduced the quantum Schubert cell algebra $U^-[w]$ which is a subalgebra of the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Let $U_\epsilon^-[w]$ be the specialization at a primitive ℓ -th root of unity $\epsilon \in \mathbb{K}$. Let $C_\epsilon^-[w]$ be the central subalgebra generated by the ℓ -th power of the Lusztig's root vectors. Then $U_\epsilon^-[w]$ is a free module over $C_\epsilon^-[w]$. Moreover, $C_\epsilon^-[w]$ is isomorphic to the coordinate ring of the Schubert cell $B_+w.B_+$ which is subvariety of the full flag variety G/B_+ , where B_+ is the positive Borel subgroup of G . In addition, $C_\epsilon^-[w]$ is a polynomial ring in term of its generators and its Poisson primes are identified with the generalized minors $\Delta_{\lambda, w\lambda}$ on G , where λ is a dominant weight.

Theorem 1.3.1. (N.–Trampel–Yakimov)[26, Theorem 5.3]

Let \mathfrak{g} be a simple Lie algebra, w a Weyl group element, and $l > 2$ an odd integer which is $\neq 3$ in the case of G_2 . Let \mathbb{K} be a field of characteristic zero containing a primitive l -th root of unity. Then

$$d(U_\epsilon^-[w]/C_\epsilon^-[w]) =_{\mathbb{K}^\times} \prod_{i \in S(w)} \Delta_{\omega_i, w\omega_i}^L$$

where $L := \ell^{N-1}(\ell - 1)$ and ω_i are the fundamental weights of \mathfrak{g} and $S(w)$ is the support of w .

Details on quantum group, quantum Schubert cell algebra and its Poisson structure, and necessarily Lie theoretic background are in Chapter 7.

Chapter 2

Discriminant of Noncommutative Algebras

In this chapter, we recall the definition of discriminant of an algebra which is mentioned in various places such as [27, 1, 4, 5, 6, 7]. We also discuss some of its important properties both old and new, and cover some examples. We also include some historical application of discriminant in the study of order of algebras, as found in the book of Reiner, [27].

2.1 Discriminant: Definition and Properties

Let R be an algebra, and $\text{tr} : R \longrightarrow R$ be a linear map.

Definition 2.1.1. *We call (R, tr) an algebra with trace if for any $x, y \in R$, we have that*

- $\text{tr}(xy) = \text{tr}(yx)$
- $\text{tr}(x)y = y \text{tr}(x)$
- $\text{tr}(\text{tr}(x)y) = \text{tr}(x) \text{tr}(y)$.

We also call tr a trace map of R .

Observe that, the image of tr is a central subalgebra of R , and tr is linear over this subalgebra. Now we give examples of trace maps.

Example 2.1.2. 1. *Let R be the ring of n by n matrices over some commutative domain F , which means $R = M_n(F)$. Then the map of taking trace of a matrix in R to F makes R into an algebra with trace. Here, $x \in F$ is viewed as an element in R by multiplication with the identity matrix.*

2. Let C be any subalgebra of $Z(R)$, the center of R . Suppose that R is free of rank n as a C -module. Then for a chosen C -basis of R , we consider the map

$$\text{tr} : R \longrightarrow M_n(C) \hookrightarrow C,$$

where the first arrow is coming from left multiplication on R , and the second arrow is taking trace of a matrix. It is straight forward to see that tr is a trace map, thus makes (R, tr) into an algebra with trace. We call such a trace map the canonical trace map. Our main results will take place in scenario as in this example of trace map.

3. Next, we consider another interesting example which can be viewed as more general than the first two. Let R be a finite dimensional algebra over some algebraically closed field K . Let V be a finite dimensional representation of R . Then the map

$$R \longrightarrow \text{End}_K(V) \longrightarrow K$$

is a trace map, where the last arrow is again taking trace of a matrix, and provide that the composition is not zero.

Now we are ready to define the discriminant of an algebra R .

Definition 2.1.3. [27, 26]

Let (R, tr) be an algebra with trace. Denote C to be the central subalgebra of R which is the image of tr . We define the following:

1. Let $Y = \{y_1, \dots, y_n\}$ be a set of n elements in R . Then the discriminant of Y is defined as

$$d_n(Y : \text{tr}) := \det([\text{tr}(y_i y_j)]) \in C.$$

2. The n -discriminantal ideal $D_n(R/C)$ is defined to be the ideal of C generated by $d_n(Y : \text{tr})$ for all set of n elements $Y \subset R$.

3. Let R be a free of rank n C -module, and tr be the canonical trace map. Let $Y = \{y_i\}_{i=1}^n$ be a C -basis of R . Then we define the discriminant of R over C to be

$$d(R/C) :=_{C^\times} d_n(Y : \text{tr}).$$

Remark 2.1.4. Note that the discriminant of R over C is well defined and is independent of the choice of C -basis for R . Let $X = \{x_i\}_{i=1}^n$ be another C -basis for R . Denote $B = (b_{ij})$ be the change of basis matrix, where $x_i = \sum_{j=1}^n b_{ij}y_j$. Then we have

$$\text{tr}(x_i x_j) = \text{tr}\left(\sum_r b_{ir}y_r \sum_s b_{js}y_s\right) = \sum_r \sum_s b_{ir}b_{js} \text{tr}(y_r y_s) = \sum_r \sum_s b_{ir}b_{js} \text{tr}(y_s y_r).$$

Thus $\det([\text{tr}(x_i x_j)]) = \det(B[\text{tr}(y_s y_r)]^T B^T) = \det(B)^2 \det[\text{tr}(y_s y_r)]$ which give us the equation

$$d_n(X : \text{tr}) = \det(B)^2 d_n(Y : \text{tr}). \quad (2.1.1)$$

Therefore, discriminant of R over C , $d(R/C)$ is well defined up to a unit in C . Moreover, the discriminantal ideal in this situation is a principal ideal generated by $d_n(Y : \text{tr})$.

Let us give a small example of how to compute discriminant for a specific algebra.

Example 2.1.5. [5, Example 1.7]

Consider the Weyl algebra in two generators, $R = K\langle x, y \rangle / (xy + yx - 1)$, and K is a field of characteristic zero. Then it is easy to check that $X := x^2$ and $Y := y^2$ are central elements in R . Moreover, center of R is the polynomial ring $C = K[X, Y]$. View R as a C -module, then R is free with basis $V = \{1, x, y, xy\}$. Here, we take the canonical trace map tr . With a quick calculation, one obtains

$$\begin{aligned} \text{tr}(1) = \text{trace} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} &= 4, & \text{tr}(x) = \text{trace} \begin{bmatrix} 0 & x^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^2 \\ 0 & 0 & 1 & 0 \end{bmatrix} &= 0 \\ \\ \text{tr}(y) = \text{trace} \begin{bmatrix} 0 & 1 & y^2 & 0 \\ 0 & 0 & 0 & -y^2 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} &= 0, & \text{tr}(xy) = \text{trace} \begin{bmatrix} 0 & 0 & 0 & -x^2y^2 \\ 0 & 1 & y^2 & 0 \\ 0 & -x^2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} &= 2. \end{aligned}$$

Hence, the discriminant of R over C is

$$d(R/C) = d_4(V : \text{tr}) = \det \begin{bmatrix} 4 & 0 & 0 & 2 \\ 0 & 4x^2 & 2 & 0 \\ 0 & 2 & 4y^2 & 0 \\ 2 & 0 & 0 & 2 - 4x^2y^2 \end{bmatrix} = -2^4(4x^2y^2 - 1)^2.$$

Let F be a commutative domain. Recall that derivation of a F -algebra R is a F -linear map $\partial : R \longrightarrow R$ which satisfies the Leibniz's rule, $\partial(ab) = a\partial(b) + \partial(a)b$, for any $a, b \in R$. In the next proposition, we show how derivation and discriminant of an algebra are related to each other. This relationship will play a key role in our main theorems in the later chapters.

Proposition 2.1.6. (N.–Trampel–Yakimov [26, Proposition 2.2])

Let R be an algebra with trace $\text{tr} : R \longrightarrow C \subset Z(R)$. Also suppose that R is a free of rank n module over C .

1. If ∂ is a derivation of R such that $\partial \text{tr}(x) = \text{tr}(\partial(x))$ for all $x \in R$, then

$$\partial(d_n(Y : \text{tr})) = 2 \text{trace}(B) d_n(Y : \text{tr}),$$

for any C -basis $Y := \{y_i\}_{i=1}^n$ of R , where $B = (b_{ij}) \in M_n(C)$ is given by $\partial(y_i) = \sum_j b_{ij} y_j$.

2. Suppose tr is the canonical trace map as in Definition 2.1.3 part 3. Then every derivation ∂ of R , which preserves C , satisfies the property $\partial \text{tr}(x) = \text{tr}(\partial(x))$ for all $x \in R$.

Proof. For the first statement, we use the fact that discriminants of (R, C) and $(R[t]/(t^2), C[t]/(t^2))$ are the same. Now let ∂ be such that $\partial \text{tr}(x) = \text{tr}(\partial(x))$ for all $x \in R$. Then it is clear that the derivation $(1 + t\partial)$ also satisfies the same condition as ∂ . Denote $(1 + t\partial)Y := \{(1 + t\partial)y_i\}_{i=1}^n$. Since t^2 acts by zero, we have that $(1 + t\partial) \text{tr}(y_i y_j) = \text{tr}((1 + t\partial)y_i (1 + t\partial)y_j)$. Thus, equation 2.1.1 gives us the following

$$(1 + t\partial)d_n(Y : \text{tr}) = d_n((1 + t\partial)Y : \text{tr}) = \det(I_n + tB)^2 d_n(Y : \text{tr}).$$

Using the property of determinant and the fact that t^2 is zero, the right hand side of equation above can be expressed as $(1 + \text{trace}(B)t)^2 d_n(Y : \text{tr})$. On the other hand, $(1 + t\partial)d_n(Y : \text{tr}) = d_n(Y : \text{tr}) + t\partial d_n(Y : \text{tr})$. So by comparing coefficient of t on both side, one obtains

$$\partial(d_n(Y : \text{tr})) = 2 \text{trace}(B) d_n(Y : \text{tr}).$$

So statement 1 is proved.

For the second statement, let ∂ be a derivation of R such that $\partial(C) \subset C$. Choose a C -basis for R , then the embedding $R[t]/(t^2) \hookrightarrow M_n(C[t]/(t^2))$ implies that $(1 + t\partial) \text{tr}(x) = \text{tr}((1 + t\partial)x)$ for all $x \in R$, since we are working with the canonical trace map over $R[t]/(t^2)$. Hence the statement follows from the fact that $\text{tr}(x)$ and $\text{tr}(\partial x)$ are in C . \square

2.2 Discriminant and Orders of Algebras

In this section, we provide some connection of discriminant to the study of orders of algebra. These notes can be found in the book of I. Reiner, [27] and in [28].

The theory of maximal order arises from the work of Dedekind on factorization problems over ring of algebraic integers and algebraic number field. Since then, it has played important role in the study of integral representations of groups and algebraic number theory [28]. Moreover, the theory itself “is of interest in its own right, and is essentially the study of noncommutative arithmetic. The beauty of the subject stems from the fascinating interplay between the arithmetical properties of orders, and the algebraic properties of the algebras containing them”, [27].

Let R be a noetherian integral domain and K be its quotient field. For an finite dimensional K -algebra A , an R -order in A is a subring $\Lambda \subset A$ such that:

- The center of Λ contains R ,
- Λ is finitely generated as R -module and,
- $K\Lambda = A$.

For example, we take G be a finite group, then RG is an R -order in KG . Also, the matrix ring $M_n(R)$ is an R -order in $M_n(K)$. Note that, Λ is called *maximal order* if it is not contained in any R -order of A .

Next, we describe how discriminant fits into the setting of orders. First we need some set up. Let V be a finite dimensional vector space over K . Then for $\phi \in \text{End}_K(V)$, one obtains an action of $K[X]$ on V where X acts on V via ϕ . This turns V into a finitely generated $K[X]$ -module. By the fundamental theorem for modules over principal ideal domain,

$$V \cong \sum_{i=1}^r K[x]/(h_i(X)), \quad h_i(X) \in K[X].$$

We define the *characteristic polynomial of ϕ over K* to be

$$\text{char}_K \phi = \prod_{i=1}^r h_i(X).$$

Now, when A is a finite dimensional K -algebra, $\alpha \in A$ can be viewed as an element in $\text{End}_K(A)$ via left multiplication. Suppose $A = \sum_{i=1}^m K u_i$, and we write $\alpha u_i = \sum_j^m a_{ij} u_j$. Then the characteristic polynomial of α is

$$\text{char}_K \alpha = \det(\delta_{ij} X - a_{ij}) = X^m - (T_{A/K} \alpha) X^{m-1} + \cdots + (-1)^m N_{A/K} \alpha$$

where the trace map $T_{A/K} \alpha = \text{trace}(\alpha)$ and norm map $N_{A/K} \alpha = \det(\alpha)$. Further, we suppose that A is separable over K , which means that there is a finite separable field extension $K \subset E$ such that $E \otimes_K A \cong \prod_{i=1}^s M_{n_i}(E)$. Hence, for each $\alpha \in A$, we can associate to α the matrix

$$\bar{\alpha} := \begin{bmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_s \end{bmatrix}$$

where $\alpha_i \in M_{n_i}(E)$. The *reduced characteristic polynomial of α over K* is defined to be

$$\text{red. char}_{A/K} \alpha = \text{char}(\bar{\alpha}) = X^n - (\text{tr}(\alpha)) X^{n-1} + \cdots + (-1)^n \text{nr}(\alpha)$$

where $\text{tr}(\alpha)$ and $\text{nr}(\alpha)$ are the reduced trace of α and reduced norm of α , respectively. It turns out that $\text{red. char}_{A/K} \alpha \in K[X]$ and doesn't depend on the choice of E . Moreover, if α is integral over R , then $\text{red. char}_{A/K} \alpha \in R[X]$.

Theorem 2.2.1. [27, Theorem 10.1]

Let Λ be an R -order in A . Then for each $\alpha \in \Lambda$, $\text{red. char}_{A/K} \alpha \in R[X]$, and so are $\text{tr}(\alpha)$ and $\text{nr}(\alpha)$.

Now we set $d(\Lambda)$ to be the m -discriminant ideal of Λ in R with respect to the reduced trace map, where $m = \dim_K(A)$. Using discriminant, one obtains the proof of the next theorem.

Theorem 2.2.2. [27, Theorem 10.3]

Let Γ be a subring of A containing R , such that $K\Gamma = A$, and suppose that each $\alpha \in \Gamma$ is integral over R . Then Γ is an R -order in A .

As a consequences, every R -order in A is contained in a maximal R -order in A . Moreover, there exists at least one maximal R -order in A , [27, 10.4].

Chapter 3

Poisson Geometry and Related Topics

With its roots in physics, Poisson geometry holds a special place in mathematics for its vast influence on many other subjects such as the theory of Lie groups and Lie algebras, noncommutative algebras and deformation/quantization theory, and mathematical physics. In this chapter, we review definition of Poisson algebra, Poisson manifold, and describe some of their algebraic and geometric properties. Also, we discuss the standard Poisson structure on Lie groups and connections to Lie bialgebras. These notes can be found in standard textbooks such as [8, 10, 15]. Also, background materials on differential geometry are from [22].

Through out this chapter, we denote k be a field of characteristic zero and algebra over k is associative with unity.

3.1 Poisson Algebra

Definition 3.1.1. *Let A be a commutative k -algebra. We call $(A, \{.,.\})$ a Poisson algebra if $\{.,.\} : A \otimes A \longrightarrow A$ is a k -bilinear Lie bracket (i.e. antisymmetric and obeys the Jacobi identity) of A which satisfies the Leibniz identity:*

$$\{ab, c\} = a\{b, c\} + \{a, c\}b, \quad \forall a, b, c \in A.$$

We call $\{.,.\}$ the Poisson bracket of A .

A morphism between Poisson algebras is an algebra morphism which preserves the Poisson bracket, that is $f(\{a, b\}_A) = \{f(a), f(b)\}_{A'}$ for an k -algebra map $f : (A, \{.,.\}_A) \longrightarrow (A', \{.,.\}_{A'})$ and $a, b \in A$. Also, note that $\{a, .\}$ is a derivation on A by definition, for any $a \in A$. Moreover, this derivation is related to the Hamiltonian vector field, which will be discussed in the next section.

An example of Poisson algebra is the ring of polynomials, $A = k[x, y]$, where the Poisson bracket is defined by taking partial derivative:

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}, \quad \forall f, g \in A.$$

This also can be extended to the case when $A = k[x_1, \dots, x_n, y_1, \dots, y_n]$.

Another example of Poisson algebra is the ring of smooth functions on a symplectic manifold. Let M be a smooth manifold over $k = \mathbb{R}$, and let $\mathcal{O}(M)$ be its ring of smooth functions on M . We call M a *symplectic manifold* if it is equipped with a non-degenerate regular 2-form $\omega \in \Gamma(\wedge^2 TM)$ such that $d\omega = 0$. We call such ω a *symplectic form* on M . Note that (M, ω) has even dimension, since ω is nondegenerate and skew-symmetric. Now, for each function $f \in \mathcal{O}(M)$, we can associate to it a vector field $(V_f : \mathcal{O}(M) \longrightarrow \mathcal{O}(M))$ such that $\omega(W, V_f) = df(W)$, for any vector field W on M . Then the Poisson bracket on $\mathcal{O}(M)$ is defined to be

$$\{f, g\} := \omega(V_f, V_g) = V_f(g), \quad \forall f, g \in \mathcal{O}(M).$$

A detailed proof of this can be found in [10]. When the manifold is $M = \mathbb{R}^{2n}$, we can take the symplectic form to be $\omega = \sum_i dx_i \wedge dy_i$, where the coordinates in M are $(x_i, y_i | i = 1, \dots, n)$. Then $V_f = \sum_i (\frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i})$, and the Poisson bracket is

$$\{f, g\} = \sum_i (\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i}).$$

This generalizes the Poisson structure of polynomial ring as described in previous example.

3.2 Poisson Manifold

Let M be a smooth manifold. We call M a *Poisson manifold* if $\mathcal{O}(M)$ admits a Poisson bracket $\{.,.\}$. So symplectic manifolds are Poisson manifolds.

Suppose $(M, \{.,.\})$ is a Poisson manifold. We will see how the Poisson bracket and bivector field on M are related to each other. As mentioned earlier in section

3.1, for $f \in \mathcal{O}(M)$, $\{f, \cdot\} : \mathcal{O}(M) \longrightarrow \mathcal{O}(M)$ is a derivation, which gives rise to a unique vector field V_f , due to the fact that there is an isomorphism between space of derivations on $\mathcal{O}(M)$ and smooth vector fields on M . We call V_f the *Hamiltonian vector field* associated to f . Note that the bracket $\{\cdot, \cdot\}$ defines a derivation in either argument, so $\{f, g\}$ only depends on the derivatives df and dg . Then one obtains a map between cotangent and tangent bundle $T^*M \longrightarrow TM$ such that $df \mapsto V_f$, for all $f \in \mathcal{O}(M)$. In particular, there exists $\pi \in \Gamma(\wedge^2 TM)$, which is viewed as a skew-symmetric bilinear form on the cotangent bundle of M , uniquely defined by the bracket

$$\{f, g\} = \pi(df, dg).$$

We call π the *Poisson bivector field*. Moreover, this Poisson bivector field π satisfies the condition $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket, a graded Lie bracket for multivector fields. The vanishing of π under the Schouten bracket is equivalent to Jacobi identity for $\{\cdot, \cdot\}$. Therefore, a bivector field $\pi \in \Gamma(\wedge^2 TM)$ determines a Poisson bracket if and only if $[\pi, \pi] = 0$.

When the Poisson manifold is a symplectic manifold, then the symplectic form ω gets identified with the Poisson bivector field π . On the other hand, a nondegenerate Poisson bivector field would induces a symplectic structure on the manifold.

A morphism between Poisson manifolds M and N is a smooth map $F : M \longrightarrow N$ such that it preserves the Poisson structures of $\mathcal{O}(M)$ and $\mathcal{O}(N)$:

$$\{f, g\}_M \circ F = \{f \circ F, g \circ F\}_N, \quad \forall f, g \in \mathcal{O}(M).$$

Also, we call N a Poisson submanifold of a Poisson manifold M if N is a submanifold of M and if the restriction of Poisson bivector field of M to N lies in $\Gamma(\wedge^2 TN)$. In other word, $\mathcal{O}(N)$ is a Poisson subalgebra of $\mathcal{O}(M)$ with $\{\cdot, \cdot\}_N$ is the restriction of $\{\cdot, \cdot\}_M$ on N .

The product of two Poisson manifolds M and N , $M \times N$ is again a Poisson manifold, in which for any $f, g \in \mathcal{O}(M \times N)$, the Poisson bracket is given by

$$\{f, g\}_{M \times N}(x, y) = \{f(\cdot, y), g(\cdot, y)\}_M(x) + \{f(x, \cdot), g(x, \cdot)\}_N(y),$$

for $x \in M$ and $y \in N$.

3.3 Symplectic Leaf

Without being technical, a partition of a manifold M into disjoint, connected, immersed submanifolds is called a *foliation* of M . We call these submanifolds the *leaves* of the foliation. In this section, we construct a natural foliation of Poisson manifold, where the leaves are symplectic manifolds; hence, the name *symplectic leaves*.

Let M be a Poisson manifold and π be its Poisson bivector field. Arises from π is the bundle map $\tilde{\pi} : T^*M \longrightarrow TM$ such that $\tilde{\pi}(v)(f) = \pi(v, df)$, for $v \in T^*M$ and $f \in \mathcal{O}(M)$. For each $x \in M$, the *rank of π at x* is the rank of $\tilde{\pi}_x : T_x^*M \longrightarrow T_xM$. Due to the skew-symmetry of π , this rank is an even number for any $x \in M$.

Suppose S is a submanifold of M . We call S an *integral submanifold* of $\text{Im}\tilde{\pi}$ when S is path connected and

$$T_x S = \text{Im}\tilde{\pi}_x, \quad \forall x \in S.$$

Observe that, points in S all have same rank. Moreover, integral submanifolds of $\text{Im}\tilde{\pi}$ are immersed symplectic manifolds. Now we define a *symplectic leaf* of M to be the maximal integral submanifold of $\text{Im}\tilde{\pi}$. Thus the symplectic leaves form a symplectic foliation of our Poisson manifold M .

Recall that an integral curve of a smooth vector field V is a smooth curve γ , such that the tangent vector at each point along γ is equal to the tangent vector of V at that point. Then, for any two points in an integral submanifold, they are

connected by a piecewise smooth path whose each piece is an integral curve of some Hamiltonian vector field. Thus one can also define the symplectic leaf as an equivalent class in M , where x and y are equivalent if they are connected by such a curve.

Let \mathfrak{g} be a finite dimensional Lie algebra with the Lie bracket $[\cdot, \cdot]$ and \mathfrak{g}^* be the dual of \mathfrak{g} . As an example, we discuss the Poisson structure on \mathfrak{g}^* and its symplectic leaves. First, we construct a Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$. For any $\alpha \in \mathfrak{g}^*$, the tangent space of \mathfrak{g}^* at α is identified with \mathfrak{g}^* . Thus we have $T_\alpha^* \mathfrak{g}^* \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$. When $f \in \mathcal{O}(\mathfrak{g}^*)$, the differential one form df gives us a map $T_\alpha \mathfrak{g}^* \cong \mathfrak{g}^* \longrightarrow T_\alpha^* \mathfrak{g}^* \cong \mathfrak{g}$. Now we define the Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$ to be

$$\{f, g\}(\alpha) := \langle [df(\alpha), dg(\alpha)] , \alpha \rangle, \quad \forall \alpha \in \mathfrak{g}^*.$$

Let G be a connected Lie group with the Lie algebra \mathfrak{g} . The action of G on itself by conjugation induces an *adjoint* action of G on \mathfrak{g} denoted by Ad , where $\text{Ad}(g)(x) = gxg^{-1}$, for $g \in G$ and $x \in \mathfrak{g}$. When view \mathfrak{g} as the tangent space of G at the identity, $\text{Ad}(g)(x)(f) = x(gfg^{-1})$, for $f \in \mathcal{O}(G)$. Differentiate Ad at the identity, we obtain an action of \mathfrak{g} on \mathfrak{g} , denoted by ad , where $\text{ad}(x)(y) = [x, y]$, for $x, y \in \mathfrak{g}$. There is also the *coadjoint* action Ad^* of G on \mathfrak{g}^* , given by $\langle \text{Ad}^*(g)(\alpha), x \rangle = \langle \alpha, \text{Ad}(g^{-1})(x) \rangle$, for $\alpha \in \mathfrak{g}^*$. Again, differentiating Ad^* at the identity, we get ad^* .

Now we show that coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ has a symplectic structure, and thus makes it into a symplectic leaf of \mathfrak{g}^* . Using the Poisson structure on \mathfrak{g}^* , we obtain the bivector field $\pi_\alpha : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$, for $\alpha \in \mathfrak{g}^*$, where $\pi_\alpha(x, y) = \alpha([x, y])$, $\forall x, y \in \mathfrak{g}$. The symplectic structure on \mathbb{O} , will come from the restriction of π to \mathbb{O} . From now, we take $\alpha \in \mathbb{O}$. Note that $\mathbb{O} \cong G/G^\alpha$, where G^α is the stabilizer of α in G under Ad^* . Then the tangent space of \mathbb{O} at α is $T_\alpha \mathbb{O} \cong T_\alpha(G/G^\alpha) \cong \mathfrak{g}/\mathfrak{g}^\alpha$,

with $\mathfrak{g}^\alpha = \text{Lie}G^\alpha$. We need to show that $\pi_\alpha(x, \cdot)$ is identically zero on \mathfrak{g} for all $x \in \mathfrak{g}^\alpha$. Since $g \in G^\alpha$ if and only if $\text{Ad}^*(g)(\alpha) = \alpha$, at the Lie algebra level, $\text{ad}^*(x)(\alpha) = 0$ if and only if $x \in \mathfrak{g}^\alpha$. Then $\pi_\alpha(x, \cdot)$ vanishes on \mathfrak{g} if and only if $x \in \mathfrak{g}^\alpha$ because $\pi_\alpha(x, \cdot) = \text{ad}^*(x)(\alpha)$. Next, $d\pi_\alpha = 0$ is due to the construction of π and using the Jacobi identity. Hence $\pi_\alpha : \mathfrak{g}/\mathfrak{g}^\alpha \times \mathfrak{g}/\mathfrak{g}^\alpha \longrightarrow \mathbb{C}$, for $\alpha \in \mathbb{O}$, gives us a symplectic structure on \mathbb{O} . The fact that \mathbb{O} is an integral submanifold of $\text{Im}\pi$ is clear. Therefore \mathbb{O} is a symplectic leaf of \mathfrak{g}^* .

3.4 Poisson Normal and Poisson Prime Elements

Now we are back to the algebra side. In order to relate noncommutative discriminant to Poisson geometry, we need to discuss the notion of Poisson normal and prime elements of a Poisson algebra.

Let $(A, \{.,.\})$ be a Poisson algebra over a field k of characteristic zero.

Definition 3.4.1. *We call $a \in A$ Poisson normal if for every $x \in A$, $\{a, x\} = ay$, for some $y \in A$. When A is a domain as a commutative algebra, this is equivalent to $\{a, x\} = a\partial(x)$, for some Poisson derivation ∂ of A .*

Suppose A is a domain. We call $p \in A$ Poisson prime if it is a prime element of the algebra A and is Poisson normal.

Remark 3.4.2. [26, Remark 2.4]

Consider working over $k = \mathbb{C}$ and $\text{Spec}A$ is a smooth Poisson variety. Then an prime element $p \in A$ is Poisson prime if and only if its zero locus $\mathcal{V}(p)$ is a union of symplectic leaves.

If $\mathcal{V}(p)$ is a union of symplectic leaves, then $\{p, g\}$ vanishes on the smooth locus of p for all $g \in A$. Therefore (p) is a Poisson ideal. On the other hand, when (p) is Poisson ideal, and \mathcal{L} be a leaf such that $\mathcal{L} \cap \mathcal{V}(p) \neq \emptyset$ and $\mathcal{L} \not\subset \mathcal{V}(p)$, then for

any smooth point $m \in \mathcal{L} \cap \mathcal{V}(p)$ there exist $\mathfrak{g} \in A$ so that $\{p, g\}(m) \neq 0$. Thus contradicts the condition that (p) is Poisson ideal.

We call a noetherian Poisson algebra A a *Poisson unique factorization domain* if it is an integral domain as an algebra and that every nonzero Poisson prime ideal of A contains a Poisson prime element. [26, Definition 2.5]. With this definition, we state the follow proposition which motivates our main result in Chapter 4.

Proposition 3.4.3. [26, Proposition 2.6]

Let A be a Poisson algebra. Suppose A is either a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain. Then every nonzero, nonunit Poisson normal element $a \in A$ has a unique factorization of the form

$$a = \prod_{i=1}^m p_i$$

where $p_i \in A$ are Poisson prime elements, not necessarily distinct. The uniqueness is up to taking associates and permutations.

Proof. When A is a noetherian Poisson UFD, the proof is similar to [9, Proposition 2.1]. Suppose A is a UFD as commutative algebra. The proposition follows from the fact that if $p|a$ then p is a Poisson prime whenever a is Poisson normal. If $p|a$ then, $a = p^r b$ for some $b \in A$ such that $p \nmid b$. Since a is Poisson normal, $\{a, x\} = ay$ for all $x \in A$ and some $y \in A$. Replacing a with $p^r b$ we get

$$\begin{aligned} \{p^r b, x\} &= \{p^r, x\}b + p^r \{b, x\} \\ &= r\{p, x\}p^{r-1}b + p^r \{b, x\} = p^r b. \end{aligned}$$

Thus $p^r | \{p, x\}p^{r-1}b$, and hence $p | \{p, x\}$ for all $x \in A$.

□

3.5 Poisson Lie Group

Similar to the relationship between Lie groups and Lie algebras, we have the Poisson Lie groups and Lie bialgebras. In this section, we give a quick introduction to the Poisson Lie group theory which we will use in later chapter. These notes are taken from [15] and [8].

Let G be a Lie group. We call G a *Poisson Lie group* if G is endowed with a Poisson structure which is compatible with the Lie structure in the sense that the multiplication map

$$m : G \times G \longrightarrow G$$

is a Poisson map. More precisely, for any x_0 and $y_0 \in G$ and any $f, g \in \mathcal{O}(G)$,

$$\{f, g\}(x_0 y_0) = \{f(x y_0), g(x y_0)\}(x_0) + \{f(x_0 y), g(x_0 y)\}(y_0)$$

where $f(x y_0)$ and $g(x y_0)$ are functions in terms of x (similarly for y). A morphism between Poisson Lie groups is a morphism of Lie groups and also of Poisson manifolds. Further, a Lie subgroup of G is a *Poisson Lie subgroup* if it is a Poisson submanifold of the Poisson manifold G . Let H be a Poisson Lie subgroup of G . Then the quotient G/H is called a *Poisson homogeneous space*, and it admits a unique Poisson manifold structure such that the projection map $G \longrightarrow G/H$ is a Poisson morphism.

Let $\pi \in \Gamma(\wedge^2 TG)$ be the corresponding bivector field for G . We can also describe the compatibility between the Poisson structure and Lie structure on G in term of this bivector field:

$$\pi(xy) = (d_x(R_y) \otimes d_x(R_y))\pi(x) + (d_y(L_x) \otimes d_y(L_x))\pi(y) \quad (3.5.1)$$

where L_x and R_y are the left and right multiplication map by x and y , respectively. Therefore, a Poisson structure on a Lie group G is a Poisson Lie group structure

if and only if $\pi(xy)$ is the sum of the left translate by x of its value at y and the right translate by y of its value at x , [8, Proposition 1.2.2]. As a consequence, the rank of π is zero at the identity element in G , and thus the Poisson structure of a Poisson Lie group G is never symplectic [8, Corollary 1.2.3].

It is worth to mention that the left and right multiplication by elements of G are not Poisson maps in general. However, G acts on itself by left or right multiplication is Poisson, that is $G \times G \longrightarrow G$ is a Poisson map, where $G \times G$ is product of Poisson manifolds. In addition, the inverse map $i(x) = x^{-1}$ for $x \in G$, is an anti-Poisson map, that is $\{f \circ i, g \circ i\}(x) = -\{f, g\}(x^{-1})$.

3.6 Lie Bialgebra

Let G be a Poisson Lie group with the Poisson bivector π . Since the rank of π at the identity element $e \in G$ is zero, $\{e\}$ is a symplectic leaf of G . We consider the Lie algebra associate to G which is $\mathfrak{g} = T_e G$ and its dual $\mathfrak{g}^* = T_e^* G$. The dual \mathfrak{g}^* admits a natural Lie algebra structure. Let $\mathcal{O}(G)_e$ be the ring of germs of smooth functions defined over a neighborhood of e , and I be the ideal of functions vanishing at e , which is a unique maximal ideal of $\mathcal{O}(G)_e$. Then the Poisson bracket on $\mathcal{O}(G)$ descends to $\mathcal{O}(G)_e$ and is given by $\{f, g\} = \pi(df \otimes dg)$, for $f, g \in \mathcal{O}(G)_e$. Since $\pi_e = 0$, the image of $\{., .\}$ on $\mathcal{O}(G)_e$ is contained in I . Hence $\{., .\}$ makes I into a Lie algebra. Moreover, I^2 is a Lie ideal of I under this Lie bracket $\{., .\}$. Therefore the quotient $I/I^2 \cong T_e^* G$ is a Lie algebra. Note that, this construction works in general for any Poisson manifold (M, π) and $x \in M$ such that $\pi_x = 0$.

Denote $[\cdot, \cdot]_{\mathfrak{g}^*} : \wedge^2 \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ the Lie algebra structure on \mathfrak{g}^* , constructed above. The dual of this map is $\delta : \mathfrak{g} \longrightarrow \wedge^2 \mathfrak{g}$, given by $\langle X, [\xi_1, \xi_2]_{\mathfrak{g}^*} \rangle = \langle \delta(X), \xi_1 \otimes \xi_2 \rangle$ for $X \in \mathfrak{g}$. In terms of the Poisson bivector π , we have $\delta = d\pi$, where we view π as a map from G to $\wedge^2 \mathfrak{g}$. Then the map δ satisfies the *co-Jacobi identity*

$$Alt(\delta \otimes \text{id})\delta(X) = 0, \quad \forall X \in \mathfrak{g},$$

where $Alt(a \otimes b \otimes c) = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b$. A vector space V with a linear map $\delta : V \longrightarrow \wedge^2 V$ satisfying the co-Jacobi identity is called a *Lie coalgebra*. Hence (\mathfrak{g}, δ) is a Lie coalgebra. Moreover, the Lie algebra and Lie coalgebra structure on \mathfrak{g} are compatible in the sense that: for $X, Y \in \mathfrak{g}$,

$$\delta([X, Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [X \otimes 1 + 1 \otimes X, \delta(Y)]. \quad (3.6.2)$$

Equation 3.6.2 follows from equation 3.5.1 and the fact that $\delta = d\pi$. Thus δ is a 1-cocycle for \mathfrak{g} with values in $\wedge^2 \mathfrak{g}$.

Definition 3.6.1. *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra. Then $(\mathfrak{g}, [\cdot, \cdot], \delta)$ is called a Lie bialgebra if the skew symmetric linear map $\delta : \mathfrak{g} \longrightarrow \wedge^2 \mathfrak{g}$, cobracket, satisfies the co-Jacobi identity and the 1-cocycle condition 3.6.2.*

A morphism between Lie bialgebras $\phi : (\mathfrak{g}, \delta_{\mathfrak{g}}) \longrightarrow (\mathfrak{h}, \delta_{\mathfrak{h}})$ is a Lie algebra morphism such that $(\phi \otimes \phi) \circ \delta_{\mathfrak{g}} = \delta_{\mathfrak{h}} \circ \phi$. Lie subalgebra of \mathfrak{g} is called a *Lie subbialgebra* if it is closed under the cobracket. Also, for $\mathfrak{h} \subset \mathfrak{g}$ a Lie ideal, then the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$ is a quotient Lie bialgebra if and only if \mathfrak{h} is a *Lie coideal*, that is $\delta(\mathfrak{h}) \subset \mathfrak{g} \otimes \mathfrak{h} + \mathfrak{h} \otimes \mathfrak{g}$.

To summarize we have the following proposition:

Proposition 3.6.2. [15, Proposition 2.1]

Let G be a Poisson Lie group. Then the Lie algebra $\mathfrak{g} = Lie(G)$ is naturally a Lie bialgebra.

Example 3.6.3. [15, Example 2.3] *Consider the Lie algebra $sl_2(\mathbb{C})$ with basis*

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

We define the Lie bialgebra structure on $sl_2(\mathbb{C})$ to be

$$\delta(e) = \frac{1}{2}e \wedge h, \quad \delta(f) = \frac{1}{2}f \wedge h, \quad \delta(h) = 0.$$

It is straight forward to check that δ satisfies the co-Jacobi identity and the 1-cocycle condition. We call this the standard Lie bialgebra structure on $sl_2(\mathbb{C})$.

Similar to the correspondence between Lie groups and Lie algebra, we have the correspondence between Poisson Lie groups and Lie bialgebra. We now state the main theorem of Poisson Lie theory:

Theorem 3.6.4. (Drinfeld)[15, Theorem 2.2] *The functor $G \mapsto Lie(G)$ between the category of connected, simply connected Poisson Lie groups and the category of finite dimensional Lie bialgebras is an equivalence of categories.*

3.7 Manin Triple, Dual, Double, and Poisson Action

For a Lie bialgebra \mathfrak{g} , the relationship between \mathfrak{g} and its dual \mathfrak{g}^* is better explained through the notion of Manin triple. Moreover, it is easier to construct a Lie bialgebra structure on \mathfrak{g} by using a Manin triple.

Definition 3.7.1. *A triple of Lie algebras $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ is a Manin triple if \mathfrak{p} is equipped with a nondegenerate symmetric bilinear form (\cdot, \cdot) which is invariant under the adjoint action of \mathfrak{p} , and such that*

- \mathfrak{p}_+ and \mathfrak{p}_- are Lie subalgebras of \mathfrak{p} ,
- $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ as vector space, and
- \mathfrak{p}_+ and \mathfrak{p}_- are isotropic with respect to (\cdot, \cdot) .

Proposition 3.7.2. [8, Proposition 1.3.4]

For any finite dimensional Lie algebra \mathfrak{g} , there is a one-to-one correspondence between Lie bialgebra structures on \mathfrak{g} and Manin triples $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ such that $\mathfrak{p}_+ = \mathfrak{g}$.

As a result, if \mathfrak{g} is a Lie bialgebra, then its dual \mathfrak{g}^* is also a Lie bialgebra. This is so because swapping \mathfrak{p}_+ and \mathfrak{p}_- creates a new Manin triple. More details are in the proof of Proposition 1.3.4 in [8]. Since Lie bialgebra is self dual, by the main theorem of Poisson Lie theory, we also have the notion of dual Poisson Lie group. Let G be a connected, simply connected Poisson Lie group, and \mathfrak{g} be its Lie algebra with canonical Lie bialgebra structure. Then the connected, simply connected Poisson Lie group G^* corresponds to the Lie bialgebra \mathfrak{g}^* is called the *dual Poisson Lie group* of G . Moreover, $G^{**} \cong G$.

Example 3.7.3. (Manin triple for the standard Lie bialgebra structure on $sl_2(\mathbb{C})$)

Recall the standard Lie bialgebra structure on $\mathfrak{g} = sl_2(\mathbb{C})$ in example 3.5.3, we now give the corresponding Manin triple to that. Let $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}$ be the direct sum of Lie algebras. Define the pairing on \mathfrak{p} to be

$$((x, y), (x', y'))_{\mathfrak{p}} = (x, x')_{\mathfrak{g}} - (y, y')_{\mathfrak{g}}$$

for $x, x', y, y' \in \mathfrak{g}$ and $(\cdot, \cdot)_{\mathfrak{g}}$ is the standard inner product on \mathfrak{g} . Then we take \mathfrak{p}_+ to be the diagonal subalgebra of \mathfrak{p} and

$$\mathfrak{p}_- = \{(x, y) \in \mathfrak{b}_- \oplus \mathfrak{b}_+ \mid \mathfrak{h} - \text{component of } x + y \text{ is zero}\}$$

where \mathfrak{b}_- and \mathfrak{b}_+ are Lie subalgebras of \mathfrak{g} generated by f, h and e, h respectively. Then $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ is a Manin triple that corresponds to the standard Lie bialgebra structure on $\mathfrak{g} = sl_2(\mathbb{C})$. Note that, this construction can be extended to any symmetrizable Kac–Moody algebra, [8, Example 1.3.8].

For the dual \mathfrak{g}^* , its Lie bialgebra structure is given by

$$\delta(e^*) = \frac{1}{2}h^* \wedge e^*, \quad \delta(f^*) = \frac{1}{2}h^* \wedge f^*, \quad \delta(h^*) = e^* \wedge f^*.$$

We denote here e^*, f^* , and h^* the generators of \mathfrak{g}^* with relations

$$[h^*, e^*] = \frac{-1}{2}e^*, \quad [h^*, f^*] = \frac{-1}{2}f^*, \quad [e^*, f^*] = 0.$$

Example 3.7.4. Dual Poisson Lie group of $SL_2(\mathbb{C})$, [15]

Again, we will be working with $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. The connected Lie group corresponding to \mathfrak{g} is $G = SL_2(\mathbb{C})$. The standard Poisson structure on G is given in terms of bivector field as follow:

$$\pi\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) = \frac{t}{2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \pi\left(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\right) = \frac{t}{2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\pi\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\right) = 0.$$

Then the connected, simply connected dual Poisson Lie group is

$$G^* = \left\{ \left(\begin{bmatrix} p & q \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p & q' \\ 0 & 1 \end{bmatrix} \right) \mid p > 0 \right\},$$

and its Poisson structure is given by

$$\pi\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{t}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \wedge \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

$$\pi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) = \frac{t}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \wedge \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right),$$

$$\pi\left(\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}\right) = t \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \wedge \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right),$$

Given a finite dimensional Lie bialgebra \mathfrak{g} , its dual \mathfrak{g}^* is also a Lie bialgebra via the Manin triple. In addition, there is another construction in which one can also obtain a new Lie bialgebra from \mathfrak{g} .

Proposition 3.7.5. [8]

There is a canonical Lie bialgebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ so that \mathfrak{g} and $(\mathfrak{g}^*)^{op}$ are Lie sub-bialgebras.

We call $\mathfrak{g} \oplus \mathfrak{g}^*$ the *double* of \mathfrak{g} and is denoted by $D(\mathfrak{g})$. Thus we also have the notion of double of a Poisson Lie group, $D(G)$, where G and G^* are Poisson Lie subgroups of $D(G)$.

A Poisson Lie group G is said to act on a Poisson manifold M on the left if $G \times M \longrightarrow M$ is a Poisson map where $G \times M$ is a product of Poisson manifolds. Such an action is called *Poisson action*. Now suppose that G is a connected Poisson Lie group, and G^* , $D(G)$ be its connected, simply connected dual and respectively, double Poisson Lie group. Then, there is an Poisson action of G^* on G , called the "dressing action", given in terms of the double $D(G)$, see [8]. It turns out that the G^* -orbits under dressing action are precisely the symplectic leaves of G , [8, Proposition 1.5.3]. Moreover, any closed Lie subgroup H of G is a Poisson Lie subgroup if and only if H is invariant under the dressing action of G^* , [8, Proposition 1.5.5]. In particular, the dressing action of G^* on G induces an Poisson action of G^* on G/H and the orbits are also the symplectic leaves of G/H , whenever H is a Poisson Lie subgroup of G , [8, Theorem 1.5.6].

Chapter 4

Discriminant of Algebras coming from Specialization

Continue with the spirit of Poisson geometry as in previous chapter but focus more on the algebra side, we will discuss the theory of quantum deformation of Poisson algebras, [14, Chapter 11.6]. We then develop theory and methods for calculating discriminant of algebra coming from specialization, using its Poisson structure. We finish the chapter with results for n -discriminant ideal also in the setting of specialization of algebras.

4.1 Quantum Deformation of Poisson Algebras

Let R be a commutative algebra and A be a R -algebra. For $h \in R$ which is not a zero divisor of A , we consider the quotient A/hA . Suppose A/hA is commutative, and let $\sigma : A \rightarrow A/hA$ be the canonical projection. We define a bracket on A/hA as follow: for any $x, y \in A/hA$,

$$\{x, y\} = \sigma([a, b]/h), \quad (4.1.1)$$

where $a, b \in A$ such that $\sigma(a) = x$ and $\sigma(b) = y$. Since A/hA is commutative, $\sigma([a, b]) = 0$, so $[a, b]/h$ makes sense. Take a' be another representative of x . Then $a - a' = hc$ for some $c \in A$. We have

$$\sigma([a, b]/h) - \sigma([a', b]/h) = \sigma([a - a', b]/h) = \sigma([c, b]) = 0.$$

Thus, the bracket $\{.,.\}$ is independent of the choice of representative for x and y . It is easy to check that this bracket makes A/hA into a Poisson algebra. We then call A a *quantization* of the Poisson algebra A/hA , [14].

One can make the Poisson structure from 4.1.1 become more interesting by letting A/hA be a noncommutative algebra. Let $a \in A$ such that $\sigma(a)$ is in the center of A/hA .

Let $y \in A/hA$ and let $b \in A$ be its representative. Once again, $[a, b]/h \in A$ is well defined. Consider the map $D_a : A/hA \longrightarrow A/hA$, where $D_a(y) = \sigma([a, b]/h)$. As before, D_a is a well defined map of A/hA . We have the following theorem.

Theorem 4.1.1. [14]

The map D_a is a derivation on A/hA . For a different representative a' of x , $D_a - D_{a'}$ is an inner derivation, which corresponds to $\sigma((a - a')/h)$. Moreover, $\phi \circ D_a \circ \phi^{-1} = D_{\phi(a)}$, for any automorphism ϕ of A .

As a result we have:

Corollary 4.1.2. [14]

The center of A/hA has a natural Poisson structure induced by:

$$\{x, y\} := D_a(y), \quad \text{where } x = \sigma(a).$$

Moreover, the automorphism of A induces a Poisson automorphism of the center of A/hA .

We note that, one can also consider a Hopf algebra structure on A , then investigate the relationship between the Hopf structure on A/hA or the center $Z(A/hA)$ with the defined Poisson bracket. This leads to notion of quantum deformation of Poisson algebraic group. Unfortunately, we will not discuss this topic in detail in this manuscript.

4.2 Specialization of Algebras and Noncommutative Discriminant

Following the setting above, we let R be an $\mathbb{K}[q^{\pm 1}]$ -algebra, with \mathbb{K} is a field of characteristic zero and q is an indeterminate, and let $\epsilon \in \mathbb{K}^\times$. We define the specialization of R at ϵ to be the \mathbb{K} -algebra $R_\epsilon := R/(q - \epsilon)R$. As before, σ is the canonical projection of R onto R_ϵ . Suppose that $q - \epsilon$ is a regular element, which means that it is not a left or right zero divisor in R . Then by Corollary 4.1.2, the

center $Z(R_\epsilon)$ is a Poisson algebra with the Poisson bracket:

$$\{x, y\} := \sigma([a, b]/(q - \epsilon))$$

where $x, y \in Z(R_\epsilon)$ and $a, b \in R$ are their representatives, respectively.

We have the following result which is a consequence of Theorem 4.1.1

Proposition 4.2.1. [26]

For any $z \in Z(R_\epsilon)$, the Hamiltonian derivation $\{z, \cdot\}$ of $Z(R_\epsilon)$ has a lift to an algebra derivation of R_ϵ , which is given by

$$\partial_a(\sigma(b)) := \sigma([a, b]/(q - \epsilon)), \quad a \in \sigma^{-1}(z), \quad b \in R.$$

Note that ∂_a is the same as D_a in Theorem 4.1.1.

Now we are ready to state our main theorem of this chapter, which describes the property and structure of discriminant of R_ϵ using the defined Poisson bracket.

Theorem 4.2.2. (N.–Trampel–Yakimov, [26, Theorem 3.2])

Let R be a $\mathbb{K}[q^{\pm 1}]$ -algebra. Let $\epsilon \in \mathbb{K}^\times$ be such that $q - \epsilon$ is regular in R . Suppose R_ϵ is a free module of finite rank over some Poisson subalgebra C_ϵ of $Z(R_\epsilon)$.

1. *The discriminant $d(R_\epsilon/C_\epsilon)$ is a Poisson normal element of $(C_\epsilon, \{\cdot, \cdot\})$.*
2. *When C_ϵ is a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain, $d(R_\epsilon/C_\epsilon) = 0$ or*

$$d(R_\epsilon/C_\epsilon) =_{C_\epsilon^\times} \prod_{i=1}^m p_i$$

for some (not necessarily distinct) Poisson prime elements $p_i \in C_\epsilon$.

Remark 4.2.3. [26]

Note that product of 0 primes is 1 by convention. Also $d(R_\epsilon/C_\epsilon)$ is defined up to a unit of C_ϵ because of the fact that x is a Poisson normal element if and only

if ux is a Poisson normal, for any unit element u . In addition, by [21, Example 5.12] there are examples of Poisson structures on polynomial algebras which are not Poisson UFDs. On the other hand, one can construct Poisson UFDs which are not UFDs as commutative algebras. Hence, the two classes of algebras in Theorem 4.2.2 (2) are not properly contained in each other.

Proof. Let $Y := \{y_1, \dots, y_N\}$ be a C_ϵ -basis for R_ϵ . Let $z \in C_\epsilon$ and $a \in \sigma^{-1}(z)$, then by Proposition 4.2.1, the Hamiltonian derivation $\{z, \cdot\}$ of $Z(R_\epsilon)$ extends to a derivation of R_ϵ , denoted by ∂_a . Also ∂_a preserves the Poisson subalgebra C_ϵ by definition. Thus by Proposition 2.1.6, we have the following equation:

$$\{z, d_N(Y : \text{tr})\} = 2 \text{trace}(B(a)) d_N(Y : \text{tr})$$

where $B(a) = (b_{ij})$ is a matrix with entries in C_ϵ , given by $\partial_a(y_i) = \sum_j b_{ij} y_j$. Note that the trace map tr here is the canonical trace map with respect to the basis Y of R_ϵ . Therefore statement (1) of the theorem follows.

The second statement follows from the first one and Proposition 3.4.3. □

4.3 Methods for Computing Discriminant

Using Theorem 4.2.2, we describe here methods for computing the discriminant $d(R_\epsilon/C_\epsilon)$ in various situations, [26].

1. When R_ϵ admits a \mathbb{Z}^n grading and C_ϵ is a homogeneous subalgebra, we can work with a homogeneous C_ϵ -basis Y of R_ϵ . Then the trace map tr will be homogeneous and $d_N(Y : \text{tr})$ is graded and

$$\deg d_N(Y : \text{tr}) = 2 \sum_{y \in Y} \deg y.$$

Also, the units in C_ϵ will have degree zero, so the discriminant is homogeneous. Therefore, the Poisson prime factors p_i of $d_N(Y : \text{tr})$ need to be also

homogeneous and

$$\sum_{i=1}^m \deg p_i = \deg d(R_\epsilon/C_\epsilon) = 2 \sum_{y \in Y} \deg y.$$

2. Next, we need some description of the Poisson prime factors of $d(R_\epsilon/C_\epsilon)$. In that regard, the geometry for symplectic leaves of Poisson manifolds appear in theory of quantum groups are well studied. For examples, we have the Belavin–Drinfeld Poisson structures in [32], the varieties of Lagrangian subalgebras in [16, 17], and the nonstandard Poisson structures on simple Lie groups in [24]. Using Remark 3.4.2, we can infer results about Poisson primes of the corresponding Poisson algebras. In addition, the Poisson primes of algebras which are Poisson–CGL extensions are described in [19].
3. Theorem 4.2.2 (1) says that the discriminant $d_N(Y : \text{tr})$ is a Poisson normal element of C_ϵ . Thus, when C_ϵ is a domain, by Definition 3.4.1, $d_N(Y : \text{tr})$ gives rise to a derivation ∂_{discr} of C_ϵ , so that for any $z \in C_\epsilon$, we have

$$\{d_N(Y : \text{tr}), z\} = d_N(Y : \text{tr})\partial_{discr}(z).$$

This derivation is described in the proof of Theorem 4.2.2. Similarly, one obtains the derivations ∂_{p_i} for each of the Poisson prime factor p_i , since being Poisson prime is also being Poisson normal. Hence the equation

$$\sum_{i=1}^m \partial_{p_i} = \partial_{discr}.$$

When the conditions in part two of Theorem 4.2.2 are not met, we have the following approach. Since each element in C_ϵ gives rise to a Hamiltonian derivation and thus a Hamiltonian vector field on $\text{Spec}C_\epsilon$, and since $d_N(Y : \text{tr})$ is Poisson normal, the value of all these Hamiltonian vector fields at $d_N(Y : \text{tr})$ is completely determined via ∂_{discr} . In other word, one has a

description of the evolution of $d_N(Y : \text{tr})$ under all Hamiltonian flows on $\text{Spec}C_\epsilon$.

4. In the case that R is a filtered algebra, we can use the leading term argument, [6, Proposition 4.10], to describe leading term of $d_N(Y : \text{tr})$, thus provide further restrictions on which Poisson prime factors p_i can appear in the decomposition.

Utilze Theorem 4.2.2 and these methods, we now provide the discriminant formula for a more general algebra than one in Example 2.1.5. Notice that, this formula has been proven in [7] but with more involved methods.

We consider the quantum Weyl algebra A_q over $\mathbb{K}[q^{\pm 1}]$ with two generators x_1, x_2 and relation

$$x_1x_2 = qx_2x_1 + 1.$$

Let $\epsilon \in \mathbb{K}$ be a primitive ℓ -th root of unity, $\ell > 1$. Let $A_\epsilon := A_q/(q - \epsilon)A_q$ and denote $\sigma : A_q \longrightarrow A_\epsilon$ be the projection. Then the center Z_ϵ of A_ϵ is isomorphic to $\mathbb{K}[z_1, z_2]$, where $z_i := \sigma(x_i)^\ell$ for $i \in \{1, 2\}$. Also, the algebra A_ϵ is free over Z_ϵ with basis

$$Y := \{\sigma(x_1)^{n_1}\sigma(x_2)^{n_2} \mid n_1, n_2 \in [0, \ell - 1]\}.$$

Theorem 4.3.1. [26, 7, Theorem 3.4, Theorem 2.4]

Let \mathbb{K} be a field of characteristic zero, and let ℓ, ϵ be as above. Then the discriminant of A_ϵ over Z_ϵ is given by

$$d(A_\epsilon/Z_\epsilon) =_{\mathbb{K}^\times} (z_1z_2 - t)^{\ell(\ell-1)}$$

where $t := (1 - \epsilon)^{-\ell}$.

Proof. [26, page 281-282]

We will be working over \mathbb{C} for the sake of simplicity, but one can always do a base change to get back to \mathbb{K} .

First, we calculate the Poisson bracket between z_1 and z_2 . Note that moving the ℓ power of x_1 passes that of x_2 , we obtains the relation:

$$x_1^\ell x_2^\ell - q^{\ell^2} x_2^\ell x_1^\ell = \sum_{i=0}^{\ell-1} t_i x_2^i x_1^i$$

where $t_0 = \prod_{m=1}^{\ell} (1 + q + \cdots + q^{m-1})$ and $t_i \in \mathbb{K}[q^{\pm 1}]$. Using this, one has

$$\begin{aligned} \{z_1, z_2\} &= \sigma\left(\frac{(q^{\ell^2} - 1)x_2^\ell x_1^\ell + \sum_{i=0}^{\ell-1} t_i x_2^i x_1^i}{q - \epsilon}\right) \\ &= \sigma\left(\frac{(q^{\ell^2} - 1)x_2^\ell x_1^\ell + t_0}{q - \epsilon}\right) \\ &= \ell^2 \epsilon^1 (z_1 z_2 - t). \end{aligned}$$

Using the Poisson bracket of z_1 and z_2 , we can describe the symplectic leaves of $\text{Spec} Z_\epsilon \cong \mathbb{C}^2$. For a point in \mathbb{C}^2 , the integral curve through that point for the Hamiltonian vector field $\{z_i, \cdot\}$ is determined by the hyperplane $z_1 z_2 = t$. Therefore, the symplectic leaves are points on this hyperplane and the complement of the hyperplane.

Since $z_1 z_2 - t$ is a prime element in Z_ϵ , applying Remark 3.4.2, it is also a Poisson prime element. For any other prime element of Z_ϵ , its zero locus must has a nontrivial intersection with the 2-dimensional symplectic leaf, and thus not a Poisson prime. Hence $z_1 z_2 - t$ is the only Poisson prime in Z_ϵ . Also, since Z_ϵ is a UFD, so applying Theorem 4.2.2, we have

$$d(A_\epsilon/Z_\epsilon) =_{\mathbb{K}^\times} (z_1 z_2 - t)^m$$

for some positive integer m .

To determine m , we employ the third method, described above. We first calculate the following:

$$\{z_1, d_{\ell^2}(Y : \text{tr})\} = m \ell^2 \epsilon^{-1} z_1 d_{\ell^2}(Y : \text{tr}).$$

Consider the derivation $\partial_{x_1^\ell}$, as in Proposition 4.2.1, then by Proposition 2.1.6, $m\ell^2\epsilon^{-1}z_1 = 2\text{trace}(B)$, where B is the matrix obtained by applying $\partial_{x_1^\ell}$ to Z_ϵ -basis Y of A_ϵ . Trace of B is the sum of Z_ϵ -coefficient of y in $\partial_{x_1^\ell}(y)$, when written in term of basis Y . Since $x_1x_2 = qx_2x_1 + 1$, the coefficient of y is easily calculated and is equal to $\ell n_2\epsilon^{-1}z_1$. Hence, we have

$$m\ell^2\epsilon^{-1} = 2 \sum_{y \in Y} \ell n_2 \epsilon^{-1} = 2\ell^2\epsilon^{-1} \sum_{n_2=0}^{\ell-1} n_2 = \ell^3(\ell-1)\epsilon^{-1}.$$

This implies $m = \ell(\ell-1)$. The theorem is proven. □

4.4 Discriminant Ideal

We recall the definition of n -discriminant ideal to be the ideal generated by $d_n(Y : \text{tr})$ for all the set of n elements Y . In this section, we state results on this ideal in the setting of specialization of algebras without the freeness condition like before in Theorem 4.2.2.

Let R be a $\mathbb{K}[q^{\pm 1}]$ -algebra for an infinite field \mathbb{K} . Let $\epsilon \in \mathbb{K}^\times$ be such that $q - \epsilon$ is not a zero divisor in R . Suppose C_ϵ is a Poisson subalgebra of $Z(R_\epsilon)$ and that R_ϵ is equipped with a trace map $\text{tr} : R_\epsilon \longrightarrow C_\epsilon$. We also suppose further that $\text{tr} \circ \partial = \partial \circ \text{tr}$ for all derivation ∂ of R_ϵ which preserves C_ϵ .

Theorem 4.4.1. (N-Trampel–Yakimov, [26, Theorem 3.6])

For all positive integers n , the discriminant ideal $D_n(R_\epsilon/C_\epsilon)$ is a Poisson ideal of C_ϵ . Moreover, for any derivation ∂ of R_ϵ such that $\partial(C_\epsilon) \subset C_\epsilon$, we have $\partial(D_n(R_\epsilon/C_\epsilon)) \subset D_n(R_\epsilon/C_\epsilon)$.

The first statement in the theorem follows from the second by applying Proposition 4.2.1. The second follows from [26, Proposition 3.7], which is stated for any algebra with a trace map and derivations satisfying conditions mentioned above. We state the proposition here.

Proposition 4.4.2. [26, Proposition 3.7]

Let S be a \mathbb{K} -algebra with \mathbb{K} as before. Suppose that $\text{tr} : S \longrightarrow C \subset Z(S)$ commutes with all derivation ∂ of S which preserves C . Then $\partial(D_n(S/C)) \subset D_n(S/C)$.

Let S and C as above. For a positive integer n , we define a symmetric C -multilinear form on S^n

$$\langle \cdot, \cdot \rangle : S^n \times S^n \longrightarrow C, \quad \langle X, Y \rangle := \det([\text{tr}(x_i y_j)]_{i,j=1}^n)$$

where $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n) \in S^n$. Then the *modified discriminant ideal* $MD_n(S/C)$, [6], is defined to be the ideal of C generated by $\langle X, Y \rangle$ for all $X, Y \in S^n$. Observe that, the discriminant ideal is contained in the modified discriminant ideal. When the freeness condition holds, they are equal and is generated by the discriminant of the chosen set of basis elements.

Theorem 4.4.3. [26, Theorem 3.8]

Let \mathbb{K} , n , S , C , tr , and ∂ be as before. Then $\partial(MD_n(S/C)) \subset MD_n(S/C)$.

In the setting of Theorem 4.4.1, the modified discriminant ideal $MD_n(R_\epsilon/C_\epsilon)$ is a Poisson ideal of C_ϵ . Moreover, $\partial(MD_n(R_\epsilon/C_\epsilon)) \subset MD_n(R_\epsilon/C_\epsilon)$ for all derivations ∂ of R_ϵ such that $\partial(C_\epsilon) \subset C_\epsilon$.

Chapter 5

Generalization to Poisson Orders

K. Brown and I. Gordon, in [3], introduced a new important class of algebras called Poisson orders. They were used by them, in the same paper, to study the geometry and representation theory of symplectic reflection algebras. In addition, the theory of Poisson orders also include many other important algebras such as quantum groups at roots of unity and restricted Lie algebra and its universal enveloping algebra. In this chapter, we will generalize our results in Chapter 4 to the setting of Poisson orders.

5.1 Definition of Poisson Orders

Let \mathbb{K} be a field of characteristic 0. Suppose S is an affine \mathbb{K} -algebra and is finitely generated as C -module for some central subalgebra C . We call S is a *Poisson C -order*, [3, 26, 2.1, 3.6], if there exists a \mathbb{K} -linear map

$$D : C \longrightarrow \text{Der}_{\mathbb{K}}(S) : z \mapsto D_z ,$$

such that

1. C is stable under $D(C)$, and
2. the induced bracket $\{.,.\}$ on C given by $\{z_1, z_2\} = D_{z_1}(z_2)$ turns C into a Poisson algebra over \mathbb{K} .

In the setting of specialization of algebras, R_{ϵ} is a Poisson Z_{ϵ} -order, provided it is a finite Z_{ϵ} -module. Here, the map D is coming from Proposition 4.2.1.

5.2 Discriminant Theorem for Poisson Orders

We now state the generalization of theorems in Section 4.2 and 4.4.

Theorem 5.2.1. [26, Theorem 3.10] *Let S be a \mathbb{K} -algebra with \mathbb{K} a field of characteristic zero. Suppose S is a Poisson C -order. Let $\text{tr} : S \rightarrow C$ be a trace map commuting with all derivations of S which preserving C .*

1. *If S is a free of finite rank C -module, then $d(S/C)$ is a Poisson normal element of C . Moreover, if C is a UFD as a commutative algebra or a Noetherian Poisson UFD, then either $d(S/C) = 0$ or*

$$d(S/C) =_{C^\times} \prod_{i=1}^m p_i$$

for some (not necessarily distinct) Poisson prime elements $p_i \in C$.

2. *For all positive integers n , the ideal $D_n(S/C)$ and $MD_n(S/C)$ are Poisson ideal of C . In addition, for any derivation ∂ of S that preserves C , we have $\partial(D_n(S/C)) \subset D_n(S/C)$ and $\partial(MD_n(S/C)) \subset MD_n(S/C)$.*

Chapter 6

Discriminant of Quantum Matrices

The algebra of quantum $m \times n$ matrices is a quantization of the coordinate ring of space of all $m \times n$ matrices (over a field \mathbb{K} of characteristic 0). They have been one of the main objects in the study of noncommutative algebras and noncommutative algebraic geometry. Here, we will explicitly compute the discriminant formula for the specializations of algebra of square quantum matrices at odd roots of unity. The rectangular case will be in the next chapter and is an application of theory from quantum Schubert cell algebras, since it is more involved and required Lie theoretic approach.

Notations through out for this chapter: Let \mathbb{K} be a field of characteristic 0. Let $\epsilon \in \mathbb{K}$ be an odd primitive ℓ -th root of unity with $\ell > 2$. Also, n is a positive integer greater than 1. Also we follow the setting in Section 4 of [26].

6.1 Algebra of Square Quantum Matrices and Its Specialization

We define here the algebra of square quantum matrices $R_q[M_n]$ as a $\mathbb{K}[q^{\pm 1}]$ -algebra with generators x_{ij} for $i, j \in [1, n]$ satisfying following relations:

$$x_{ij}x_{kj} = qx_{kj}x_{ij} \quad \text{for } i < k,$$

$$x_{ij}x_{ir} = qx_{ir}x_{ij} \quad \text{for } j < r,$$

$$x_{ij}x_{kr} = x_{kr}x_{ij} \quad \text{for } i < k, \ j > r,$$

$$x_{ij}x_{kr} - x_{kr}x_{ij} = (q - q^{-1})x_{ir}x_{kj} \quad \text{for } i < k, \ j < r.$$

Let we give an example when $n = 2$. The algebra $R_q[M_2]$ is generated by x_{11} , x_{12} , x_{21} , and x_{22} with relations:

$$x_{11}x_{12} = qx_{12}x_{11}, \quad x_{11}x_{21} = qx_{21}x_{11}, \quad x_{12}x_{21} = x_{21}x_{12}$$

$$x_{22}x_{12} = q^{-1}x_{12}x_{22}, \quad x_{22}x_{21} = q^{-1}x_{21}x_{22}$$

$$x_{11}x_{22} - x_{22}x_{11} = (q - q^{-1})x_{12}x_{21}.$$

Note that there is an obvious embedding of $R_q[M_2]$ into $R_q[M_n]$ for any fixed $i < k$ and $j < m$, where

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} x_{ij} & x_{im} \\ x_{kj} & x_{km} \end{bmatrix}$$

Now we consider the specialization at ϵ , $R_\epsilon[M_n] := R_q[M_n]/(q - \epsilon)R_q[M_n]$ and denote $\sigma : R_q[M_n] \longrightarrow R_\epsilon[M_n]$, as before, the projection. It is natural to see that ℓ^{th} power of the generators x_{ij} , under σ , are central elements of $R_\epsilon[M_n]$. We denote these central elements by $z_{ij} := \sigma(x_{ij})^\ell \in Z(R_\epsilon[M_n])$. Then we let $C_\epsilon[M_n]$ be the central subalgebra generated by the z_{ij} 's. We have the identification

$$C_\epsilon[M_n] \cong \mathbb{K}[z_{ij}, 1 \leq i, j \leq n].$$

Viewing the generators z_{ij} of $C_\epsilon[M_n]$ as entries of a $n \times n$ matrix, we consider the minors from the lower left corner and from top right corner and denote them as follow:

$$\Delta_k := \Delta_{[n-k+1, n]; [1, k]}, \quad \bar{\Delta}_j := \Delta_{[1, j]; [n-j+1, n]} \in C_\epsilon[M_n],$$

for each $k \in [1, n]$. One can see that $\Delta_n = \bar{\Delta}_n$. Let $\Delta_{I;J} \in C_\epsilon[M_n]$ be the minor over the rows in $I \subset [1, n]$ and the columns in $J \subset [1, n]$.

Since the algebra $R_q[M_n]$ has a PBW basis over $\mathbb{K}[q^{\pm 1}]$, in terms of its generators x_{ij} , its specialization at ϵ , $R_\epsilon[M_n]$, also has a BPW basis, but over $C_\epsilon[M_n]$. In other word, $R_\epsilon[M_n]$ is a free $C_\epsilon[M_n]$ -module with the following basis:

$$Y := \{\sigma(x_{11})^{k_{11}} \cdots \sigma(x_{nn})^{k_{nn}} \mid 0 \leq k_{11}, \dots, k_{nn} \leq \ell - 1\}$$

where the ordering of $\sigma(x_{ij})$ in each monomial is by column, starting with the first one, and from top down. So the rank of $R_\epsilon[M_n]$ over $C_\epsilon[M_n]$ is ℓ^{n^2} . In addition, we have the identification

$$C_\epsilon[M_n] \cong \mathbb{K}[z_{ij}, 1 \leq i, j \leq n].$$

6.2 Poisson Structures and $C_\epsilon[M_n]$

As discussed in Chapter 5, the center $Z(R_\epsilon[M_n])$ admits a canonical Poisson bracket $\{.,.\}$ due to Corollary 4.1.2. In order to apply our Theorem 4.2.2 to $R_\epsilon[M_n]$, we need the central subalgebra $C_\epsilon[M_n]$ to be a Poisson subalgebra of $Z(R_\epsilon[M_n])$ under the Poisson bracket $\{.,.\}$. We have the following lemma, which addresses this issue. Note that this lemma can be derived from [12, Theorem 7.6], in which one relates the Poisson structure $\{.,.\}$ of $Z(R_\epsilon[M_n])$ coming from specialization to the standard Poisson structure on simple Lie group and Poisson dual Lie group.

Lemma 6.2.1. [26, Lemma 4.2] *For an odd integer $\ell > 2$, $C_\epsilon[M_n]$ is a Poisson subalgebra of $(Z(R_\epsilon[M_n]), \{.,.\})$ and the Poisson bracket on the generators is given by :*

$$\{z_{ij}, z_{km}\} = \ell^2 \epsilon^{-1} (\text{sign}(k - i) + \text{sign}(m - j)) z_{im} z_{kj}.$$

To prove this lemma directly, one use the embedding of $R_q[M_2]$ into $R_q[M_n]$ to restrict to just the case of $n = 2$. Then the only nontrivial bracket calculation is the one between z_{11} and z_{22} . For this, one uses the Proposition 4.2.1 and the fact that ℓ is odd to guarantee the non-vanishing and vanishing of certain coefficients arising from applying the derivation $\partial_{x_{11}^\ell}$.

If ℓ is even, we have an example, [26, Remark 4.3], where $C_\epsilon[M_n]$ is not a Poisson subalgebra of $Z(R_\epsilon[M_n])$. Take $\ell = 4$ and $\epsilon = \pm i$, then

$$\{z_{11}, z_{22}\} = -32\epsilon\sigma(x_{22})^2\sigma(x_{12})^2\sigma(x_{21})^2\sigma(x_{11})^2.$$

Next, we discuss a natural grading on $R_q[M_n]$ and the homogeneous Poisson prime elements of $C_\epsilon[M_n]$.

We assign to each x_{ij} a degree $\deg x_{ij} := e_i + e_{n+j}$, where $\{e_1, \dots, e_{2n}\}$ the standard basis of \mathbb{Z}^{2n} . This defines a \mathbb{Z}^{2n} -grading on $R_q[M_n]$ and thus on $R_\epsilon[M_n]$. Since $\deg z_{ij} = \ell(e_i + e_{n+j})$, the central subalgebra $C_\epsilon[M_n]$ is homogeneous and the canonical trace map tr is a graded.

Proposition 6.2.2. [26, Proposition 4.4]

The minors

$$\Delta_1, \dots, \Delta_n, \bar{\Delta}_1, \dots, \bar{\Delta}_{n-1}$$

are homogeneous Poisson prime elements of $C_\epsilon[M_n]$. In addition, for any $i, j \in [1, n]$, we have following identities:

$$\{\Delta_k, z_{ij}\} = \ell^2 \epsilon^{-1} (\langle e_{[n-k+1, n]}, e_i \rangle - \langle e_{[n+1, n+k]}, e_{n+j} \rangle) \Delta_k z_{ij}$$

$$\{\bar{\Delta}_k, z_{ij}\} = -\ell^2 \epsilon^{-1} (\langle e_{[1, k]}, e_i \rangle - \langle e_{[2n-k+1, 2n]}, e_{n+j} \rangle) \bar{\Delta}_k z_{ij}.$$

Here, the pairing appears in Proposition 6.2.2 is defined to be $\langle e_i, e_j \rangle = \delta_{ij}$. Also for $i \leq j$, we write $e_{[i, j]}$ in place of $e_i + \dots + e_j$.

Proof. (sketch)[26, page 289]

We work over \mathbb{C} for sake of simplicity. Then $C_\epsilon[M_n]$ is realized as the coordinate ring of space of $n \times n$ complex matrices, M_n , which admits a rational torus action of $H := (\mathbb{C}^\times)^{2n}$ where

$$(t_1, \dots, t_{2n}) \cdot [z_{ij}]_{i,j=1}^n := [t_i t_{n+j} z_{ij}]_{i,j=1}^n.$$

Moreover, this action of H preserves the Poisson structure on $C_\epsilon[M_n]$.

Being a homogeneous Poisson prime element is equivalent to having a zero locus which is a union of H -orbit of symplectic leaves. Brown, Goodearl, and Yakimov, in

[2], have classified possible type of H -orbits of symplectic leaves of M_n , and they are: a Zariski dense H -orbit of leaves, $2n - 1$ H -orbits of leaves of codimension 1, and finitely many H -orbits of leaves with codimension higher than 1. Further, Yakimov has calculated the zero loci of the minors $\Delta_1, \dots, \Delta_n, \bar{\Delta}_1, \dots, \bar{\Delta}_{n-1}$ in [33, Theorem 5.3]; they are the Zariski closure of those $2n - 1$ leaves of codimension 1. Therefore the minors Δ_i and $\bar{\Delta}_j$ are Poisson primes.

On the other hand, if p is any other homogeneous Poisson prime in $C_\epsilon[M_n]$, then its zero locus cannot intersect the single Zariski dense H -orbit. Since this H -orbit is dense and the action of H is compatible with the Poisson structure on $C_\epsilon[M_n]$, a nontrivial intersection would implies the zero locus of p contains M_n . With similar reasoning for the H -orbit of codimension 1 and higher, f must be one of those minors Δ_i or $\bar{\Delta}_j$.

For the identities involving Poisson bracket of Δ_i and $\bar{\Delta}_j$ with the generators of $C_\epsilon[M_n]$, they are arisen from the result of Gekhtman, Shapiro, and Vainshtein in [18, Lemma 3.2].

□

6.3 Discriminant Formula for $R_\epsilon[M_n]$

Now we state the main theorem of this chapter.

Theorem 6.3.1. (N.–Trampel–Yakimov)[26, Theorem 4.1]

Let \mathbb{K} be a field of characteristic 0, $\ell > 2$ an odd integer and $\epsilon \in \mathbb{K}$ a primitive ℓ – th root of unity. Then

$$d(R_\epsilon[M_n]/C_\epsilon[M_n]) =_{\mathbb{K}^\times} \prod_{k=1}^n \Delta_k^L \prod_{j=1}^{n-1} \bar{\Delta}_j^L$$

where $L = \ell^{n^2-1}(\ell - 1)$.

Denote $\delta := d_{\ell^{n^2}}(Y : \text{tr})$ be the discriminant of the $C_\epsilon[M_n]$ -basis Y , mentioned in Section 6.2, of $R_\epsilon[M_n]$. Since $C_\epsilon[M_n]$ is a polynomial, it is a UFD. Then Theorem

4.2.2 and Proposition 6.2.2 imply that

$$\delta = t \prod_{i=1}^n \Delta_i^{m_i} \prod_{j=1}^{n-1} \bar{\Delta}_j^{\bar{m}_j}$$

with $t \in C_\epsilon[M_n]^\times = \mathbb{K}^\times$, and some $m_i, \bar{m}_j \in \mathbb{N}$. Then using methods as described in Chapter 4.6, one shows that $m_i = \bar{m}_j = \ell^{n^2-1}(\ell - 1)$. Next, we elaborate more on the details of calculations for m_i and \bar{m}_j .

Recall that the algebra $R_\epsilon[M_n]$ is naturally graded by \mathbb{Z}^{2n} , with $\deg x_{ij} = e_i + e_{n+j}$ where e_i are the standard basis vectors of \mathbb{Z}^{2n} . Also the trace map tr is graded, and $C_\epsilon[M_n]$ -basis Y of $R_\epsilon[M_n]$ is homogeneous, thus we have

$$\begin{aligned} \deg \delta &= 2 \sum_{k_{11}, \dots, k_{nn}=0}^{\ell-1} \deg x_{11}^{k_{11}} \cdots x_{nn}^{k_{nn}} \\ &= 2 \sum_{i,j \in [1,n]} \left(\frac{\ell(\ell-1)}{2} \ell^{n^2-1} \right) (e_i + e_{n+j}) = n(\ell-1) \ell^{n^2} e_{[1,2n]}. \end{aligned}$$

Since we know δ is written as product of the minors, so we can compare the degree of the product of minors with right hand side of above equation. Note that the degree of each minor is

$$\deg \Delta_i = \ell(e_{[n-i+1,n]} + e_{[n+1,n+i]}),$$

$$\deg \bar{\Delta}_j = \ell(e_{[1,j]} + e_{[2n-j+1,2n]}),$$

so $\deg \Delta_i \bar{\Delta}_j = \ell e_{[1,2n]}$ if and only if $i + j = n$. In particular, one has

$$\begin{aligned} \deg \prod_{i=1}^n \Delta_i^{m_i} \prod_{j=1}^{n-1} \bar{\Delta}_j^{\bar{m}_j} &= \sum_{i=1}^n m_i \ell(e_{[n-i+1,n]} + e_{[n+1,n+i]}) + \sum_{j=1}^{n-1} \bar{m}_j \ell(e_{[1,j]} + e_{[2n-j+1,2n]}) \\ &= n \ell^{n^2} (\ell - 1) e_{[1,2n]}. \end{aligned}$$

This implies that $\bar{m}_j = m_{n-j}$ and thus

$$\sum_{i=1}^n m_i = n \ell^{n^2-1} (\ell - 1).$$

Next, we will use the derivation of $R_\epsilon[M_n]$ given by

$$\partial_{x_{ij}^\ell}(\sigma(a)) = \sigma([x_{ij}^\ell, a]/(q - \epsilon)),$$

to determine m_i , because of the identity from Proposition 2.1.6

$$\partial_{x_{ij}^\ell}(\delta) = 2 \operatorname{trace}(B(i, j)) z_{ij} \delta,$$

where $B(i, j)$ is the matrix obtained from applying the derivation to $C_\epsilon[M_n]$ -basis Y . First we set up some notation. For any element $r \in R_\epsilon[M_n]$, it can be expressed as

$$r = \sum_{y \in Y} \sum_{\mu} t_{\mu, y} \mu y$$

for some $t_{\mu y} \in \mathbb{K}$ and all monomials μ of $C_\epsilon[M_n] \cong \mathbb{K}[z_{ij} | 1 \leq i, j \leq n]$. Denote the coefficient of the summand μy in expression of r to be $\operatorname{coeff}_{\mu, y}(r) := t_{\mu, y}$.

We have the following lemma:

Lemma 6.3.2. [26, Lemma 4.5]

For $y = \sigma(x_{11})^{k_{11}} \cdots \sigma(x_{nn})^{k_{nn}} \in Y$,

$$\operatorname{coeff}_{z_{ij}, y}(\partial_{x_{ij}^\ell}(y)) = \left(\sum_{a=1}^n \operatorname{sign}(a - j) k_{ia} + \sum_{a=1}^n \operatorname{sign}(a - i) k_{aj} \right) \ell \epsilon^{-1}.$$

Let $x^{\mathbf{k}} = x_{11}^{k_{11}} \cdots x_{nn}^{k_{nn}}$ for $\mathbf{k} = (k_{11}, \dots, k_{nn}) \in \mathbb{N}^{n^2}$. Then the lemma follows by using the defining relations of $R_q[M_n]$:

$$x_{ij}^\ell x^{\mathbf{k}} = q^{-\sum_{a=1}^{j-1} k_{ia} \ell - \sum_{a=1}^{i-1} k_{aj} \ell} x^{\mathbf{k}'} + \text{lower order terms},$$

$$x^{\mathbf{k}} x_{ij}^\ell = q^{-\sum_{a=j+1}^n k_{ia} \ell - \sum_{a=i+1}^n k_{aj} \ell} x^{\mathbf{k}'} + \text{lower order terms},$$

where $\mathbf{k}' = (k'_{11}, \dots, k'_{nn})$ with $k'_{ab} := k_{ab} + \ell \delta_{ai} \delta_{bj}$.

By Lemma 6.3.2, the trace of matrix $B(i, j)$ is:

$$\begin{aligned}
\text{trace}(B(i, j)) &= \sum_{y \in Y} \text{coeff}_{z_{ij}, y}(\partial_{x_{ij}^\ell}(y)) \\
&= \sum_{y \in Y} \left(\sum_{a=1}^n \text{sign}(a - j) k_{ia} + \sum_{a=1}^n \text{sign}(a - i) k_{aj} \right) \ell \epsilon^{-1}. \\
&= \left(\ell^{n^2} \frac{\ell - 1}{2} (n - 2j + 1) + \ell^{n^2} \frac{\ell - 1}{2} (n - 2i + 1) \right) \ell \epsilon^{-1} \\
&= (n - i - j + 1)(\ell - 1) \ell^{n^2+1} \epsilon^{-1}.
\end{aligned}$$

Now view δ as product of the Poisson primes and by the identities in Proposition 6.2.2,

$$\partial_{x_{ij}^\ell}(\delta) = \{z_{ij}, \delta\} = M_{ij} z_{ij} \delta$$

where

$$M_{ij} = \ell^2 \epsilon^{-1} \left(- \sum_{k=n-i+1}^n m_k + \sum_{k=i}^{n-1} \bar{m}_k + \sum_{k=j}^n m_k - \sum_{k=n-j+1}^{n-1} \bar{m}_k \right).$$

Since $\bar{m}_i = m_{n-i}$ for $i \in [1, n-1]$, we get

$$M_{ij} = \ell^2 \epsilon^{-1} \left(- \sum_{k=n-i+1}^n m_k + \sum_{k=1}^{n-i} m_k + \sum_{k=j}^n m_k - \sum_{k=1}^{j-1} m_k \right).$$

Hence, for $j \in [1, n-1]$,

$$2m_j \ell^2 \epsilon^{-1} = M_{i,j} - M_{i,j+1} = 2 \text{trace}(B(i, j)) - 2 \text{trace}(B(i, j+1)) = 2\ell^{n^2+1}(\ell - 1) \epsilon^{-1}.$$

This implies $m_j = (\ell - 1) \ell^{n^2-1}$ for $j \in [1, n-1]$. Since we already know $\sum_{i=1}^n m_i = n \ell^{n^2-1}(\ell - 1)$, so m_n is immediate. Therefore Theorem 6.1.3 is proved.

Chapter 7

Discriminant of Quantum Schubert Cell Algebras

In this chapter, we explicitly calculate the formula for discriminant of quantum Schubert cell algebras specialized at roots of unity, for all simple Lie algebras.

7.1 Quantized Universal Enveloping Algebra of Lie Algebras

Our note on definition of quantum group here follows from [20].

Let \mathfrak{g} be a complex semisimple Lie algebra. Denote Φ be the root system with respect to a fixed Cartan subalgebra, and W be the Weyl group of Φ . Let $(.,.)$ be the W -invariant bilinear form on \mathbb{Q} -span of Φ . Let $[c_{ij}] \in M_r(\mathbb{Z})$ be the Cartan matrix. Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots, and $\{s_1, \dots, s_r\} \subset W$ be the corresponding set of simple reflections.

Let \mathbb{K} be a field of characteristic zero, and q be an indeterminate. Denote $q_i = q^{\|\alpha_i\|/2}$. Now for all $a \in \mathbb{Z}$,

$$[a]_i := \frac{q_i^a - q_i^{-a}}{q_i - q_i^{-1}}$$

and we also define $[n]_i!$ and $\binom{a}{n}_i$ in the similar way, for $n \geq 0$.

Then the quantized universal enveloping algebra $U_q(\mathfrak{g})$ is a $\mathbb{K}(q)$ -algebra with generators E_i , F_i , and $K_i^{\pm 1}$ for $i \in [1, r]$ and following relations:

$$K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i,$$

$$K_i E_j K_i^{-1} = q^{(\alpha_i, \alpha_j)} E_j,$$

$$K_i F_j K_i^{-1} = q^{-(\alpha_i, \alpha_j)} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{s=0}^{1-c_{ij}} (-1)^s \binom{1-c_{ij}}{s}_i E_i^{1-c_{ij}-s} E_j E_i^s = 0 \quad \text{for } i \neq j,$$

$$\sum_{s=0}^{1-c_{ij}} (-1)^s \binom{1-c_{ij}}{s}_i F_i^{1-c_{ij}-s} F_j F_i^s = 0 \quad \text{for } i \neq j.$$

We denote U_q^+ , U_q^- , and U_q^0 be the subalgebras of $U_q(\mathfrak{g})$ generated by E_i , respectively F_i , and $K_i^{\pm 1}$ for $i \in [1, r]$. Then by the defining relation for K_i 's, U_q^0 is a commutative algebra. Moreover, for each root γ in the root lattice $\mathcal{Q} := \mathbb{Z}\Phi$, we write

$$K_\lambda = \prod_i K_i^{m_i}, \quad \text{where } \lambda = \sum m_i \alpha_i.$$

Thus $K_\lambda K_\mu = K_{\lambda+\mu}$ for all $\lambda, \mu \in \mathbb{Z}\Phi$, and also

$$K_\lambda E_i K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_i, \quad K_\lambda F_i K_\lambda^{-1} = q^{-(\lambda, \alpha_i)} F_i.$$

In addition, $U_q(\mathfrak{g})$ is graded by the root lattice \mathcal{Q} where $\deg E_i = \alpha_i$, $\deg F_i = -\alpha_i$, and $\deg K_i^{\pm 1} = 0$. Hence the subalgebras U_q^+ , U_q^- , and U_q^0 are graded, since their respected generators are homogeneous elements.

The algebra $U_q(\mathfrak{g})$ is also a Hopf algebra with the coproduct Δ , counit ε , and antipode S defined as:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \varepsilon(E_i) = 0, \quad S(E_i) = -K_i^{-1} E_i,$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \varepsilon(F_i) = 0, \quad S(F_i) = -F_i K_i,$$

$$\Delta(K_i) = K_i \otimes K_i, \quad \varepsilon(K_i) = 1, \quad S(K_i) = K_i^{-1}.$$

One of the nice property of $U_q(\mathfrak{g})$ is that it admits a PBW basis in terms of its generators. Let I and J be finite sequences of simple roots. Then the set of monomials $F_I K_\mu E_J$, with $\mu \in \mathcal{Q}$, form a basis for $U_q(\mathfrak{g})$. This implies that $U_q(\mathfrak{g})$ has a triangular decomposition as vector space:

$$U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^0 \otimes U_q^+.$$

For $i \in [1, r]$, we define the following elements in $U_q(\mathfrak{g})$:

$$E_i^{(a)} := \frac{E_i^a}{[a]_i!} \quad \text{and} \quad F_i^{(a)} := \frac{F_i^a}{[a]_i!},$$

for all $a \geq 0$. Then for each simple reflection s_i , we associate to it a $\mathbb{K}(q)$ -algebra automorphism T_i of $U_q(\mathfrak{g})$ defined by:

$$\begin{aligned} T_i(K_\mu) &= K_{s_i \mu} = T_i^{-1}(K_\mu) \\ T_i(E_i) &= -F_i K_i, \quad T_i^{-1}(E_i) = -K_i^{-1} F_i \\ T_i(F_i) &= -K_i^{-1} E_i, \quad T_i^{-1}(F_i) = -E_i K_i \end{aligned}$$

and for $i \neq j$,

$$\begin{aligned} T_i(E_j) &= \sum_{s=0}^{-c_{ij}} (-1)^s q_i^{-s} E_i^{(-c_{ij}-s)} E_j E_i^{(s)} \\ T_i^{-1}(E_j) &= \sum_{s=0}^{-c_{ij}} (-1)^s q_i^{-s} E_i^{(s)} E_j E_i^{(-c_{ij}-s)} \\ T_i(F_j) &= \sum_{s=0}^{-c_{ij}} (-1)^s q_i^s F_i^{(s)} F_j F_i^{(-c_{ij}-s)} \\ T_i^{-1}(F_j) &= \sum_{s=0}^{-c_{ij}} (-1)^s q_i^s F_i^{(-c_{ij}-s)} F_j F_i^{(s)} \end{aligned}$$

The subgroup of $\text{Aut}_{\mathbb{K}(q)}(U_q(\mathfrak{g}))$ generated by T_i for all $i \in [1, r]$ is called the braid group of W , because T_i satisfies the braid relation on modules of $U_q(\mathfrak{g})$, as showed by Lusztig. For any word $w \in W$, choose a reduced expression $w = s_{i_1} \cdots s_{i_N}$, then we write $T_w = T_{i_1} \cdots T_{i_N}$.

7.2 Quantum Schubert Cell Algebras

Quantum Schubert cell algebras are subalgebras of $U_q(\mathfrak{g})$, defined for each element of the Weyl group.

Let $w \in W$, and fix a reduced expression for w , say $w = s_{i_1} \cdots s_{i_N}$. We define the root $\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$, for each $k \in [1, N]$. Then the quantum Schubert

cell algebra $U^-[w]$ is a $\mathbb{K}[q^{\pm 1}]$ -subalgebra of $U_q(\mathfrak{g})$ generated by the quantum root vectors

$$F_{\beta_j} := T_{i_1} \dots T_{i_{j-1}}(F_{i_j}),$$

for $j \in [1, N]$, and they satisfy the Levendorskii–Soibelman straightening relation: for $1 \leq j < m \leq N$,

$$F_{\beta_m} F_{\beta_j} - q^{-(\beta_m, \beta_j)} F_{\beta_j} F_{\beta_m} = \sum_{k_{j+1}, \dots, k_{m-1} \in \mathbb{N}} t_{k_{j+1}, \dots, k_{m-1}} F_{\beta_{j+1}}^{k_{j+1}} \dots F_{\beta_{m-1}}^{k_{m-1}}$$

for some $t_{k_{j+1}, \dots, k_{m-1}} \in \mathbb{Q}[q^{\pm 1}]$. The algebra $U^-[w]$ does not depend on the choice of reduced expression of w , [25, 11]. When w is the longest word, we get back the algebra U_q^+ . In addition, the algebra $U^-[w]$ has a PBW basis

$$\{F_{\beta_1}^{k_1} \dots F_{\beta_N}^{k_N} \mid k_1, \dots, k_N \in \mathbb{N}\}.$$

Let $\epsilon \in \mathbb{K}$ be a primitive ℓ -th root of unity. The specialization at ϵ of the quantum Schubert cell algebra is $U_\epsilon^-[w] := U^-[w]/(q - \epsilon)U^-[w]$. Denote $\sigma : U^-[w] \longrightarrow U_\epsilon^-[w]$ be the canonical projection map, and we define the following elements:

$$z_{\beta_j} := (\epsilon^{\|\alpha_{i_j}\|/2} - \epsilon^{-\|\alpha_{i_j}\|/2})^\ell \sigma(F_{\beta_j})^\ell \in U_\epsilon^-[w], \quad j \in [1, N].$$

Let $C_\epsilon^-[w]$ be the \mathbb{K} -subalgebra of $U_\epsilon^-[w]$ generated by these z_{β_j} for all $j \in [1, N]$. A result of De Concini, Kac, and Procesi in [12] says that $C_\epsilon^-[w]$ is a central subalgebra of $U_\epsilon^-[w]$.

Theorem 7.2.1. [12][26, Theorem 5.2]

For all integers $\ell > 1$, $C_\epsilon^-[w]$ is a subalgebra of $Z(U_\epsilon^-[w])$. It is isomorphic to the polynomial algebra in terms of the generators z_{β_j} , $j \in [1, N]$ and is independent of the choice of reduced expression of w .

Moreover, $U_\epsilon^-[w]$ is a free $C_\epsilon^-[w]$ -module with basis

$$Y := \{\sigma(F_{\beta_1})^{k_1} \dots \sigma(F_{\beta_N})^{k_N} \mid k_1, \dots, k_N \in [0, \ell - 1]\}.$$

7.3 Poisson Structures and $U_\epsilon^-[w]$

In this section, we will be working over $\mathbb{K} = \mathbb{C}$ for the sake of simplicity. As mentioned before, we can do a direct base change from \mathbb{C} to any field of characteristic zero and our result also follows.

The goals in this section are to understand the canonical Poisson structure on $C_\epsilon^-[w]$ obtained from specialization and its Poisson primes.

First, We set up some notations. Let G be the split, connected, simply connected Lie group over \mathbb{K} with Lie algebra \mathfrak{g} . Let B_+ and B_- be the opposite Borel subgroups of G and U_+ , U_- be their corresponding unipotent radicals. The maximal torus in G is $H = B_+ \cap B_-$. Then the Weyl group W is identified with the normalizer of H in G , $N_G(H)$. In addition, we denote $\{e_i, f_i\}$ for $i \in [1, r]$ be the Chevalley generators of \mathfrak{g} , where e_i 's generate $\text{Lie}(U_+)$ and f_i 's generate $\text{Lie}(U_-)$. Let β be a positive root of \mathfrak{g} . We define the root vectors

$$e_\beta := \text{Ad}_u(e_{\alpha_i}) \quad \text{and} \quad f_\beta := \text{Ad}_u(f_{\alpha_i})$$

where $u \in W$ such that $\beta = u(\alpha_i)$. Definition of e_β and f_β are independent of the choice of $u \in W$ and α_i .

For $w \in W$, we consider the Schubert cell $B_+w \cdot B_+$, a subvariety of the full flag variety G/B_+ . To be clear, $w \cdot B_+$ is an element in G/B_+ , and the Schubert cell is an B_+ -orbit in G/B_+ . Next, for any $v \leq w \in W$, We define a smooth irreducible subvariety of G/B_+ , the Richardson variety $R_{v,w} := B_-v \cdot B_+ \cap B_+w \cdot B_+$.

Recall the standard Poisson structure on G , as discussed in Chapter 3, Section 5, and we write (G, π_{st}) to emphasize that. The Lie subgroups B_\pm are also Poisson submanifolds of G under the standard Poisson structure π_{st} , by construction. On top of that, their corresponding Lie algebra \mathfrak{b}_\pm are co-Lie ideal of the Lie bialgebra \mathfrak{g} . Thus, the quotient (full flag variety) G/B_+ inherits a unique Poisson structure

such that the map $G \longrightarrow G/B_+$ is a Poisson map. We denote the π to be the Poisson structure on G/B_+ .

Let H acts on G/B_+ by multiplication. Then by the results of Evans and Lu in [17], the H -orbits of symplectic leaves of $(G/B_+, \pi)$ are precisely the Richardson varieties $R_{v,w}$; moreover, intersection of the Zariski closure of Richardson variety with Schubert cell has the following decomposition

$$\overline{R_{v,w}} \cap B_+w \cdot B_+ = \sqcup_{u \in W, v \leq u \leq w} R_{u,w},$$

by Richardson in [29].

Since the action of G on itself by left multiplication is Poisson and since (B_+, π_{st}) is a Poisson manifold, $(G/B_+, \pi)$ is a Poisson (B_+, π_{st}) -space. Thus the Schubert cell $(B_+w \cdot B_+, \pi)$ is a Poisson homogeneous space for (B_+, π_{st}) .

Back to our algebra $C_\epsilon^-[w]$, we relate to the Schubert cell via the following algebra isomorphisms

$$C_\epsilon^-[w] \cong \mathbb{K}[U_+ \cap w(U_-)] \cong \mathbb{K}[B_+w \cdot B_+]. \quad (7.3.1)$$

The first isomorphism is given by

$$f \in \mathbb{K}[U_+ \cap w(U_-)] \mapsto f(\exp(z_{\beta_1}e_{\beta_1}) \dots \exp(z_{\beta_N}e_{\beta_N})) \in C_\epsilon^-[w].$$

For the second isomorphism, it is coming from the fact that $U_+ \cap w(U_-) \cong B_+w \cdot B_+$ via $g \mapsto gw \cdot B_+$.

Theorem 7.3.1. [26, Theorem 5.4]

The composition of isomorphisms in equation 7.3.1 is an isomorphism of Poisson algebras:

$$(C_\epsilon^-[w], \{.,.\}) \longrightarrow (\mathbb{K}[B_+w \cdot B_+], \ell^2\epsilon^{-1}\{.,.\}_\pi).$$

Thus $C_\epsilon^-[w]$ is a Poisson subalgebra of $Z(U_\epsilon^-[w])$ under the canonical Poisson bracket coming from specialization at ϵ . Next we discuss the Poisson primes in $C_\epsilon^-[w]$.

Let P^+ be the set of dominant integral weights and ω_i for $i \in [1, r]$ be the fundamental weights. Let ρ be the sum of all fundamental weights. For $\lambda \in P^+$, we consider $L(\lambda)$, the irreducible highest weight \mathfrak{g} -module of weight λ . Let b_λ be the highest weight vector of $L(\lambda)$ and ξ_λ be in the dual representation $L^*(\lambda)$ such that $\langle \xi_\lambda, b_\lambda \rangle = 1$. Then for any $u, v \in W$ we define the generalized minors

$$\Delta_{u\lambda, v\lambda} \in \mathbb{K}[G]$$

where $\Delta_{u\lambda, v\lambda}(g) := \langle \xi_\lambda, u^{-1}gvb_\lambda \rangle$, for $g \in G$.

Note that $U_\epsilon^-[w]$ and $C_\epsilon^-[w]$ are also graded by the root lattice \mathcal{Q} . Moreover, the projection map σ is \mathcal{Q} -graded. Define the support of $w \in W$ to be

$$S(w) := \{i \in [1, r] \mid s_i \text{ occurs in one and hence in any reduced expression of } w\}.$$

Proposition 7.3.2. [26, Proposition 5.5]

Using Theorem 7.3.1, the homogeneous Poisson prime elements of $C_\epsilon^-[w]$ are $\Delta_{\omega_i, w\omega_i}$, for $i \in S(w)$. Moreover, they satisfy

$$\{\Delta_{\omega_i, w\omega_i}, z\} = -\ell\epsilon^{-1}\langle (w+1)\omega_i, \gamma \rangle \Delta_{\omega_i, w\omega_i} z,$$

where $\gamma \in \mathcal{Q}$ and z is any element in the γ -graded component of $C_\epsilon^-[w]$.

The proof of Proposition 7.3.2 relies on a result of Yakimov in [33, Theorem 4.7], which says that the vanishing ideal of $\overline{R_{s_i, w}} \cap B_+ w \cdot B_+$ is generated by $\Delta_{\omega_i, w\omega_i}$ for each $i \in S(w)$. Then applying the results in [17] and [29] as mentioned earlier. For the second statement in the theorem, we use again a result of Yakimov in [34, eq. (5.1)].

7.4 Discriminant Formula for $U_\epsilon^-[w]$

Now we are ready to state the main theorem of this chapter. Using Theorem 4.2.2 and Proposition 7.3.2, we have:

Theorem 7.4.1. [26, Theorem 5.3]

Let \mathfrak{g} be a simple Lie algebra, $w \in W$, and $\ell > 2$ be an odd integer which is not 3 for the case of G_2 . Suppose \mathbb{K} is a field of characteristic zero and $\epsilon \in \mathbb{K}$ is a primitive ℓ -th root of unity. Then

$$d(U_\epsilon^-[w]/C_\epsilon^-[w]) =_{\mathbb{K}^\times} \prod_{i \in S(w)} \Delta_{\omega_i, w\omega_i}^L$$

where $L = \ell^{N-1}(\ell - 1)$.

The idea for proof of Theorem 7.4.1 is similar to that of Theorem 6.3.1.

7.5 Algebra of Quantum Rectangular Matrices $R_q[M_{m,n}]$

Earlier in Chapter 6, we prove the formula for discriminant of algebra of square quantum matrices specialized at roots of unity. Now using our results on quantum Schubert cell algebras, we prove that for the algebra of quantum rectangular matrices.

Let $1 \leq m \leq n \in \mathbb{Z}$. We define the algebra of (rectangular) quantum matrices $R_q[M_{m,n}]$ to be the $\mathbb{K}[q^{\pm 1}]$ -algebra with generators x_{ij} with $i \in [1, m]$ and $j \in [1, n]$ and defining relations as in Chapter 6.1.

Now consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_{m+n}$, and the quantum Schubert cell algebra $U^-[c^m]$, where $c = (12 \dots (m+n)) \in S_{m+n}$ is the Coxeter element of the symmetric group on $m+n$ symbols. Then by [34], $U^-[c^m]$ is isomorphic to $R_q[M_{m,n}]$ via

$$F_{\beta_k} \mapsto (-q)^{i+j-2} x_{ij}$$

for compatible indices i, j , and $k \in [1, mn]$.

Similar to the setting in Chapter 6.1 and 6.2, we specialize $R_q[M_{m,n}]$ to a root of unity $\epsilon \in \mathbb{K}$.

Theorem 7.5.1. [26, Theorem 5.7]

Suppose \mathbb{K} is a field of characteristic zero, and $\ell > 2$ is an odd integer, and $\epsilon \in \mathbb{K}$ is a primitive ℓ -th root of unity. Then we have

$$d(R_\epsilon[M_{m,n}]/C_\epsilon[M_{m,n}]) =_{\mathbb{K}^\times} \prod_{j=1}^{m-1} (\Delta_{[m-j+1,m];[1,j]}^L \Delta_{[1,j];[n-j+1,n]}^L) \prod_{k=1}^{n-m+1} \Delta_{[1,m];[k,m+k-1]}^L$$

where $L = \ell^{mn-1}(\ell - 1)$.

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Vita

Bach V. Nguyen was born in Vietnam in the 1980's. He arrived to the United States of America in mid 2000's and completed his bachelor degree in pure mathematics at University of California at Davis in June 2012. Right after that, he joined the Ph. D. program at Department of Mathematics of Louisiana State University. On June 20, 2018, he defended his doctoral thesis, under the direction of Professor Milen Yakimov. Following that, he will be holding a visiting assistant professor position at the Department of Mathematics of Temple University.