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Unavoidable Immersions and Intertwines of Graphs

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UNAVOIDABLE IMMERSIONS AND INTERTWINES OF GRAPHS

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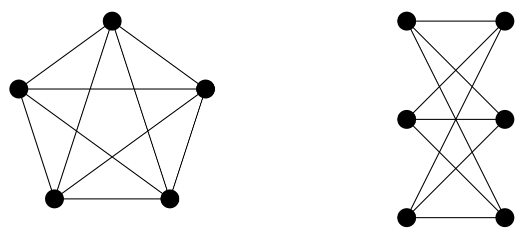
Abstract

The topological minor and the minor relations are well-studied binary relations on the class of graphs. A natural weakening of the topological minor relation is an immersion. An immersion of a graph H into a graph G is a map that injects the vertex set of H into the vertex set of G such that edges between vertices of H are represented by pairwise-edge-disjoint paths of G. In this dissertation, we present two results: the first giving a set of unavoidable immersions of large 3-edge-connected graphs and the second on immersion intertwines of infinite graphs. These results, along with the methods used to prove them, are analogues of results on the graph minor relation. A conjecture for the unavoidable immersions of large 4-edge-connected graphs is also stated with a partial proof.

Introduction

0.1 Graph Minors and Graph Immersions

A graph is a mathematical object often thought of as a abstraction of a network, consisting of vertices and edges with some pairs of vertices connected by edges. In this section, we attempt to give an intuitive understanding of the topics discussed in this dissertation. For formal definitions we refer the reader to Section 0.2. One of the major areas of study in graph theory involves characterizing various classes of graphs in terms of substructures. The best-known examples of this idea are Kuratowski's Theorem [13] and Wagner's theorem [29], which state that a graph may be embedded in the plane so that no pair of its edges cross if and only if it does not admit, respectively, a topological minor or minor of K_5 or $K_{3,3}$.



K_5 $K_{3,3}$

Figure 1. The minor-minimal non-planar graphs.

In this dissertation, we focus on two main graph relations: the minor relation and the immersion relation. Both of these relations are weakened forms of the topological minor relation. A topological minor of a graph G is often visualized in what is called a topological minor model, where the vertex set of the minor is injected into the vertex set of G, and the edges of the topological minor are represented by pairwise-internally-disjoint paths between these vertices. A minor of G can be represented in a similar way, though the vertices of the minor are

injected into a set of pairwise-disjoint, connected subgraphs of G as opposed to the vertex set of G. An immersion model instead weakens the restriction on the paths connecting the vertices which are the image of the immersed graph. That is, in an immersion model, the vertex set of the immersed graph is injected into the vertex set of G, and the edges are represented by a set of pairwise-edge-disjoint paths of G.

The topological minor relation, as well as the minor relation, have been extensively studied in the mid to late 20th century, especially in the Graph Minors Project of Robertson and Seymour. The results of this seminal project span twenty-three papers and 762 pages. The implications of this project are far-reaching and continue to exert a strong influence on the study of structural graph theory. One of the main results of this project asserts that every infinite list of graphs contains two elements one of which is a minor of the other. As a corollary of this result, every minor-closed class of graphs can be characterized by a finite list of forbidden minors. For example, as Wagner's theorem states, there are exactly two minor-minimal graphs, K_5 and $K_{3,3}$, that characterize the class of planar graphs. More generally, for any surface, the class of graphs embeddable on that surface can be characterized by a finite number excluded minors.

Another area to which the Graph Minors Project made a significant contribution is Ramsey Theory, which focuses on characterizing the unavoidable substructures of large graphs. For example, if the order of a simple graph is sufficiently large then it either has an induced subgraph of a large complete graph or a large edgeless graph.

While the main results of the Graph Minors Project lie outside of classical Ramsey Theory, the methods developed as a result of that project are very useful in the study of large graphs. In particular, the results concerning the dichotomy between

the tree-width of graphs and the existence of a large grid minor will be discussed in Chapter 1 and Chapter 4.

Although the minor relation was the main focus of the Graph Minors Project, the immersion relation was also studied in the final paper of the series [26]. The main result of that project on the immersion relation is analogous to that on the minor relation. That is, every infinite list of graphs has two graphs such that one is immersed in the other. This result prompted a similar, though more recent, interest in the study of the immersion relation. Even though the tools and techniques developed in the Graph Minors Project have applications to both minor relation and immersion relation, the study of the latter has received significantly less attention. In fact, several results on graph immersions are analogues of results on graph minors. This includes the two theorems and the conjecture presented in this dissertation, as well as several of the supporting results used to prove those theorems.

Chapter 1 contains a brief survey of relevant work in graph minors and graph immersions. We also discuss the similarities and key differences between the minor and immersion relations, and give proofs of some simple, useful results on graph immersions. Chapter 2 is devoted to the background and notation of the first main result regarding the unavoidable immersions of 3- and 4-edge-connected graphs. The proof of the first main result concerning the unavoidable immersions of 3-edge-connected graphs is presented in Chapter 3, and a conjecture for the unavoidable immersions of 4-edge-connected graphs is stated with a partial proof in Chapter 4. The second main result, which regards immersion intertwines, is discussed and proved in Chapter 5. We close the dissertation in Chapter 6 by stating several possible directions of future research in the study of graph immersions, as well as

some open questions raised by the two main results and the conjecture presented in Chapters 3–5.

The remainder of the introduction introduces the notation and terminology that will be used throughout the document. Any notation not explicitly defined in this dissertation, will follow that of Diestal [8] and West [30].

0.2 Notation and Basic Concepts

A *graph* G is a discrete mathematical object consisting of a set of vertices $V(G)$, a set of edges $E(G)$, and an incidence relation $I(G)$. We define the *order* of a graph G as the cardinality of the vertex set, denoted $|V(G)|$, and the *size* of G as the cardinality of the edge set $|E(G)|$. Graphs are generally presented as topological objects, with vertices represented by distinct points and edges represented by curves connecting a pair of vertices or loops connecting a vertex to itself. If a vertex v and an edge e are incident, we call v an *end vertex* of e. If two distinct vertices u and v are both end-vertices of an edge e, we say that u and v are *adjacent*. The degree of a vertex v, denoted $\deg(v)$, is the number of edges incident with v. For a positive integer k, the notation P_k will be used to denote the graph consisting of a path on k vertices, C_k the graph consisting of a cycle on k vertices, K_k the complete graph on k vertices, and $K_{k,j}$ the complete bipartite graph with bipartitions of order k and j.

We call the collection of edges with the same end-vertices a *parallel class*. If S is a subset of $V(G)$, we denote the graph formed by deleting the vertices in S and all edges of G incident with a vertex of S by $G - S$, and we let $G(S)$ denote the subgraph of G *induced* by S. Similarly, if T is a subset of $E(G)$, we let $G \setminus T$ denote the graph obtained from G be deleting the edges of T, and $G(T)$ be the subgraph of G *induced* by the edges of T. We call two graphs G and H *isomorphic*, denoted $G \cong H$, if there is an incidence preserving bijection

$\iota : V(G) \cup E(G) \to V(H) \cup E(H)$ that maps the vertices of G to the vertices of H and the edges of G to the edges of H.

A graph is *connected* if, for each pair of distinct vertices $u, v \in V(G)$, there is a there is a path between u and v, called a u,v-path. A maximal connected subgraph of G is called a *component*, and a graph with more than one component is called *disconnected*. The *connectivity* of G is the minimum number of vertices whose deletion disconnects G. We say G is *k-connected* if k is less than or equal to the connectivity of G. If G is $k-1$ connected, but for each set S of $k-1$ vertices whose deletion disconnects G the graph $G-S$ has a component of order exactly one, then we say that G is *internally-k-connected*. Similarly, we define the *edge-connectivity* of G as the minimum number of vertices whose deletion disconnects G. The graph G is *l-edge-connected* if l is less than or equal to the edge-connectivity of G. If u and v are distinct vertices of G, a *u, v-cut* is a subset S of $V(G)$ such that there is no u, v-path in $G - S$. More generally, if S is set of vertices of G such that $G - S$ contains more components than G, we refer to S as a *vertex cut*. A *block* of a graph G is a maximally-2-connected subgraph of G. If G is two connected, then G itself is called a block.

Two important results of Menger [15] regarding vertex and edge-connectivity that will be used often in this dissertation are stated below.

Theorem 0.1 (Menger)**.** *If u and v are distinct, non-adjacent vertices of a graph G then the maximum number of pairwise internally vertex disjoint u, v-paths is equal to the minimum size of a u, v-cut.*

Theorem 0.2 (Menger)**.** *If u and v are distinct vertices of a graph G, then the minimum size of a subset of $E(G)$ that seperates u and v is equal to the maximum number of pairwise edge disjoint u, v-paths.*

We note that the two versions of Menger's theorem above immediately imply the following corollaries, respectively.

Corollary 0.3. *Let G be a k-connected graph and let u and v be two distinct vertices of G. Then there are at least k pairwise internally vertex disjoint u, v-paths.*

Corollary 0.4. *Let G be an l-edge-connected graph and let u and v be two distinct vertices of G. Then there are at least k pairwise edge disjoint u, v-paths.*

While we primarily focus on graph immersions in this dissertation, there are several containment relations between graphs that we wish to formally define. Let G and H graphs. We say that H is an *induced subgraph* of G if H is isomorphic to a graph obtained from G by a series of vertex deletions. We call H a *subgraph* of G, denoted $H \subseteq G$, if H is isomorphic to a graph obtained from G by a series of vertex and edge deletions.

Let G be a graph and e be an edge of G. A *contraction* of e, is the operation consisting of deleting the edge e and identifying the vertices u and v into a new vertex w such that each edge of $G \backslash e$ incident to u or v is incident to w. In particular, if e was a member of a parallel class of size at least 2, then the edges of that parallel class are loops adjacent to w. We denote the graph G after contracting the edge e by G/e. If H is a graph isomorphic to a graph obtained from G by a sequence of operations, each of which is either a vertex deletion, edge deletion, or contraction of edges which are incident with a vertex of degree exactly two, we say that H is a *topological minor* of G, or $H \leq_{\mathrm{t}} G$. We refer to contractions of this nature as *series contractions.* A natural weakening of the topological minor relation is minor relation. The primary difference between the minor relation and the topological minor relation is that the operations corresponding to the minor relation allow

the contraction of any edge, regardless of the degree of its end-vertices. If H is isomorphic to a graph that can be obtained from G by deleting vertices, deleting edges, and contracting edges, we say that H is a *minor* of G, or $H \leq_{\mathrm{m}} G$. The minor relation has been well studied in the recent century and continues to be an area of significance in structural graph theory. We discuss the minor relation further as well as some relevant results in Chapter 1.

It is sometimes helpful to define topological minors in terms of a function as opposed to operations. Let $\mathscr{P}(G)$ denote the set of all nontrivial, finite paths of G. We say $H \leq_{\mathrm{t}} G$ if there is a map $\tau : V(H) \cup E(H) \to V(G) \cup \mathscr{P}(G)$, sometimes abbreviated as $\tau : H \to G$, such that:

(1) if $v \in V(H)$, then $\tau(v) \in V(G)$;

(2) if v and u are distinct vertices of H, then $\tau(v) \neq \tau(u)$;

(3) if $e \in E(H)$ has end vertices v and u, then $\tau(e) \in \mathscr{P}(G)$ and the path $\tau(e)$ connects $\tau(v)$ with $\tau(u)$, and;

(4) if e and f are distinct edges of H, then the paths $\tau(e)$ and $\tau(f)$ are internally-vertex-disjoint.

This definition allows us to weaken the relation in a different way. Namely, by mapping the edges of H to edge-disjoint paths of G as opposed to internally-vertex-disjoint paths. We say H is *immersed* in G if there is a map, called an immersion, $\varphi : V(H) \cup E(H) \to V(G) \cup \mathscr{P}(G)$, sometimes abbreviated as $\varphi : H \to G$, such that:

(i) if $v \in V(H)$, then $\varphi(v) \in V(G)$;

(ii) if v and u are distinct vertices of H, then $\varphi(v) \neq \varphi(u)$;

(iii) if $e \in E(H)$ has end vertices v and u, then $\varphi(e) \in \mathscr{P}(G)$ and the path $\varphi(e)$ connects $\varphi(v)$ with $\varphi(u)$, and;

(iv) if e and f are distinct edges of H, then the paths $\varphi(e)$ and $\varphi(f)$ are edge-disjoint.

We use $H \leq_{\mathrm{im}} G$ to denote that H is immersed in G, or, equivalently, G admits an immersion of H. See Figure 2 for an example, where colors are used to differentiate intersecting paths that are images of the edges of K_5. If C is a subgraph of H, then the restriction of φ to $V(C) \cup E(C)$ will be abbreviated by $\varphi|_C$.

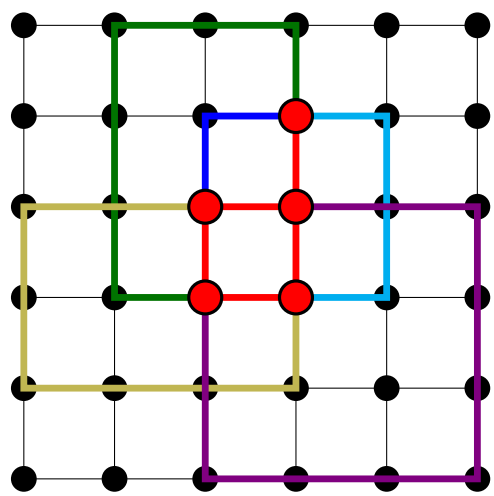


Figure 2. An immersion of K_5 into the 6×6 grid.

The relation defined above is also known as *weak immersion* and sometimes denoted $H \leq_{\mathrm{wim}} G$. An extra condition may also be applied to the definition, resulting in the strong immersion relation. We say H is *strongly immersed* in G or $H \leq_{\mathrm{sim}} G$ if, in addition to satisfying conditions (i)–(iv), φ also satisfies:

(v) $e \in E(H)$ has end vertices v and u and w is a vertex of H other than v and u, then $\varphi(w) \notin V(\varphi(e))$.

The shorthand "immersion" and the notation $\leq_{\text{im}}$ is used in the literature to describe both the strong and weak immersion relations depending on the context of the result. As this dissertation contains results involving both relations, we specify in each chapter which relation the term will apply to.

Like the minor relation, the immersion relation (both strong and weak) $\leq_{\text{im}}$ is transitive. If G, G', and G'' are graphs and $\alpha : G \to G'$ and $\beta : G' \to G''$ are immersions, we define a *composition* $\beta \circ \alpha : G \to G''$ of α and β in the following manner. If $v \in V(G)$ then $(\beta \circ \alpha)(v) = \beta(\alpha(v))$. If $e \in E(G)$ with end-vertices x and y is mapped by α to the path $P \in \mathcal{P}(G')$, then the images of the edges of P under β form a trail T in G'' with end-vertices $(\beta \circ \alpha)(x)$ and $(\beta \circ \alpha)(y)$. As the trail T is a connected subgraph of G, it contains a path, which may not be unique, joining $(\beta \circ \alpha)(x)$ to $(\beta \circ \alpha)(y)$; we let any such path be $(\beta \circ \alpha)(e)$. A composition is itself an immersion of G into G''.

Chapter 1
Some Results in Graph Minors and Graph Immersions

1.1 Path Liftings

While both strong and weak immersions are defined as functions, it is also possible to associate each relation with a corresponding set of operations. To this end, we introduce the concept of the weak and strong lifting operations, which are both weakenings of the series contraction operation. Let G be a graph, let u and v be vertices of G, and let P be a uv-path in G of length at least two. A *weak lifting* of P is the operation which consists of deleting the edges of P and then adding an edge between u and v.

This operation lets us realize an immersed graph as the result of a sequence of path lifts and deletions. If H is a graph weakly immersed in G via the weak immersion φ, we can obtain a graph isomorphic to H by taking the weak lift each path of length greater than one of G which is the image of an edge of H under φ, and then deleting any vertices or edges of G that are not images of vertices or edges of H under φ. If S is the set of paths of G which are lifted, E is the set of edges of G which are deleted, and V is the set of vertices of G which are deleted, then we say that φ is the weak immersion *corresponding to* the lift of S and the deletion of V and E.

Conversely, Let H be a graph obtained from G by a sequence of deletions and path lifts, let V be the set of vertices of G deleted to obtain H, let E be the set of edges of G deleted to obtain H, and let S be the set of paths of G weakly lifted to obtain H. Define the map $\varphi : H \to G$ as follows. If e is an edge of H, then either e is also an edge of G or e is the edge resulting from the lift of a path P_e of G. In the latter case, define $\varphi(e) = P_e$ If e is an edge of both G and H, define $\varphi(e) = e$. Finally, if v is a vertex of H, then it must also be a vertex of G. Thus,

define $\varphi(v) = v$. We similarly say φ is the weak immersion *corresponding to* the lift of S and the deletion of V and E. This idea is summarized in the following theorem.

Theorem 1.1. *Let G and H be graphs. Then $H \leq_{\text{wim}} G$ if and only if H is isomorphic to a graph that can be obtained from G by lifting a set of paths and then deleting a set of vertices and a set of edges.*

In order to provide an analogous set of operations corresponding to the strong immersion relation, we need to modify the weak lifting operation to reflect the addition of the fifth condition (v) to the definition of weak immersion. Let S be a set of pairwise edge-disjoint paths in a graph G. We say that S is *strongly liftable* if no end-vertex of a path in S is an internal vertex of another path in S. The operation of *strongly lifting* the set S consists of deleting all internal vertices of all paths in S, and adding edges joining every pair of vertices of G that are end-vertices of the same path in S. Using similar constructions to above, we obtain a theorem for the strong immersion relation, which is analogous to Theorem 1.1.

Theorem 1.2. *Let G and H be graphs. Then $H \leq_{\text{sim}} G$ if and only if H is isomorphic to a graph obtained from G by strongly lifting a strongly liftable set S of paths and deleting a set of vertices and a set of edges.*

Although the series of operations of the minor relation are commutative, the operations corresponding to the weak immersion relation, namely vertex deletion and path liftings, do not. For example, let G be a graph, let P a path of G, and let v an internal vertex of P. Consider the graph obtained by lifting P and deleting v. If v is deleted before P is lifted, it is impossible to lift $P \setminus v$ as it is no longer a path. However, if P is lifted before the deletion of v, then the operations are valid and a new graph can be obtained. Therefore, it is necessary to lift the path P

before deleting any vertices that are internal vertices of P. However, we also note that this counterexample only applies to the weak immersion relation as strongly lifting a path includes the deletion of the internal vertices of that path.

1.2 The Graph Minors Project

One of the most influencial and best known results in Graph Theory is the proof of Wagner's Conjecture by Robertson and Seymour in the Graph Minors Project [25]. As stated in the introduction, a number of mathematical methods and results were developed as part of this project that will be used to prove the theorems in this dissertation. In order to state one of the main result of the Graph Minors Project, we begin with a few definitions.

A pair $(\mathscr{G}, \preceq)$, where $\mathscr{G}$ is a class of graphs and $\preceq$ is a binary relation on $\mathscr{G}$, is called a *quasi-order* if the relation $\preceq$ is both reflexive and transitive. An antichain of $(\mathscr{G}, \preceq)$ is a sequence S of graphs in $\mathscr{G}$ such that each pair of elements of S is incomparable with respect to $\preceq$. A quasi-order $(\mathscr{G}, \preceq)$ is a *well-quasi-order* if it admits no infinite antichain and no infinite strictly decreasing sequence.

Theorem 1.3 (Robertson And Seymour)**.** *The class of finite graphs is a well-quasi-order with the minor relation*

While the effects of the Graph Minors Project and the methods developed to solve it are numerous, from computational complexity to excluded minor theorems to infinite graph theory, we focus primarily on two concepts. The first is a quantification of the resemblence of a graph to a tree and a decomposition of that graph into a parts which have a tree-like structure, called tree-width and a tree-decomposition respectively. According to Diestel [8], tree-decompositions and tree-width were first introduced by Halin [12] and later reintroduced by Robertson and Seymour [22]. A formal definition of both a tree-decomposition and tree-width

are given in Chapter 4, however, we present an intuitive definition of tree-width here. Let n be a positive integer and let G_0 be isomorphic to K_n. For a nonnegative integer i, the graph G_{i+1} is obtained from G_i by adding a new vertex v_{i+1} to G_i and connecting v_{i+1} to every vertex of a clique of size n in G_i. If a graph G is isomorphic to G_i for any i, then we say that G is an *n-tree*. A graph has tree-width at most n if and only if it is a subgraph of an n tree.

Tree-width and tree-decompositions have many applications in graph minors and structural graph theory. For example, a relationship between the tree-width of a graph and the size of the largest grid graph minor it admits was discovered [23] in the Graph Minors Project.

Theorem 1.4 (Robertson and Seymour)**.** *For every positive integer k, there is an integer n such that every graph of tree width at least n admits the $k \times k$ grid, Γ_k, as a minor.*

If a graph is planar, then it is a minor of a sufficiently large grid. Therefore, the previous theorem also gives a dichotomy between planar graphs and tree-width.

Theorem 1.5 (Robertson and Seymour)**.** *Given a graph H, the graphs without an H minor have bounded tree width if and only if H is planar.*

These relations are very useful in extremal graph theory. In particular, tree-decompositions and Theorem 1.5 were used by Oporowski, Oxley, and Thomas to find the unavoidable minors and the unavoidable topological minors for 3-connected graphs as well as the unavoidable minors and the unavoidable topological minors for internally-4-connected graphs. As both a member of the set of unavoidable topological minors for 3-connected graphs and a member of the set of unavoidable topological minors of internally-4-connected graphs are planar, the proof is divided into two cases: one where a graph has bounded tree-width and

another where a graph has unbounded tree-width. In the latter case, Theorem 1.5 immediately gives a desired topological minor. In the remaining case, an analysis of a tree-decomposition of the graph to produces a desired topological minors. These results are discussed in more detail, along with their immersion counterparts, in Chapters 2–4.

The other aspect of the graph minors project we focus on in this dissertation is the proof of Wagner's Conjecture, Theorem 1.3. A consequence of Theorem 1.3 is any antichain of $(\mathscr{G}, \leq_{\mathrm{m}})$ is finite. This implies that if $\mathscr{H}$ is a minor-closed class of graphs, then there are finitely many graphs that are minor-minimal with that property not they are not contained in $\mathscr{H}$. These minor-minimal graphs, which are called the excluded minors of $\mathscr{H}$, provide a complete characterization for the class $\mathscr{H}$ in terms of the minimal counterexamples to containment in $\mathscr{H}$. As mentioned in the introduction, the excluded minors for embeddability on the plane are K_5 and $K_{3,3}$. The only other complete list of excluded minors for embeddability on a surface is the list of excluded minors for embeddability on the projective plane, which has thirty-five excluded minors [3]. Another minor-closed class is the class of graphs of tree-width less than or equal to a positive integer n. The excluded minors for these classes are known for all values of n which are less than five.

Excluded minor characterizations reduce the question of the inclusion of a graph in a minor-closed class to finitely many tests for a fixed graph as a minor. From a computational complexity standpoint, the Graph Minors Project showed that testing for any fixed minor can be performed in cubic time [24]. This gives a polynomial time algorithm for testing a given graphs inclusion in a minor-closed class for which the excluded minors are known.

Although Theorem 1.3 states that the class of finite graphs is a well-quasi-order with the minor relation, the class of infinite graphs is not well-quasi-ordered with

the minor relation. Thomas [27] gave a construction of an infinite minor antichain of infinite graphs, though members of the antichain are uncountable. It is still unknown whether the class of countable graphs is a well-quasi-order with the minor relation. Later, Oporowski was able to use this antichain to disprove a conjecture of Seymour by constructing an infinite graph which admitted no proper-self-minor; that is, an infinite graph such that no graph isomorphic to it can be obtained by a non-trivial sequence of deleting vertices, deleting edges, and contracting edges. Oporowski was also able to show that the class of infinite graphs with the minor relation does not have the finite intertwine property. The immersion analogue of this results is proved in Chapter 5.

1.3 Immersion Analogues of Graph Minors Results

While the primary goal of the Graph Minors Project was settle Wagner's Conjecture, the final paper in the series proved a conjecture of Nash-Williams. Similar to Wagner's Conjecture, Nash-Williams conjectured that the class of finite graphs is a well-quasi-order with the strong immersion relation, and also a well-quasi-order with the weak immersion relation. Robertson and Seymour [26] showed the following.

Theorem 1.6 (Robertson and Seymour). *The class of finite graphs is a well-quasi-order with the weak immersion relation.*

While the immersion relation has received less attention than the minor relation, some results have been published in the study of immersions that are analogues of results in graph minors. In addition, methods developed in the study of minors have been modified to accommodate the differences between the immersion relation and the minor relation. One of the most apparent differences between the methods in graph minors and their counterparts in graph immersions that many tools used

in graphs immersions are stated in terms of edge-connectivity instead of vertex-connectivity. Because of the nature of the lifting operation, if H and G are two graphs such that $H \leq_{\text{im}} G$, the maximum degree of H is at most the maximum degree of G. The degree of an end-vertex of a lifted path will remain the same, while the degree of an internal vertex will be reduced by two. This distinction is highlighted in a theorem of Chudnovsky, Dvořák, Klimošová, and Seymour, which describes the relationship between the tree-width of a graph and the existence of a large grid immersion in a result analogous to Theorem 1.4.

Theorem 1.7 (Chudnovsky, Dvořák, Klimošová, and Seymour)**.** *Let g be an integer greater than one. Then there is an function $f_{1.7}$ such that every 4-edge-connected graph with tree-width at least $f_{1.7}(g)$ admits an immersed $g \times g$ grid, Γ_g.*

As planarity is not closed under the immersion relation, the existence of a grid immersion does not necessarily imply the existence of a planar graph immersion. For example, the graph K_5, which is not planar, is immersed in Γ_6, which is planar. In addition, as a grid has maximum degree four, no graph with maximum degree greater than four can be immersed in a grid. However, if H is a graph with maximum degree at most four, then it can be immersed in a sufficiently large grid. Therefore, the immersion analogue of 1.5 states a relationship between the tree-width of a 4-edge-connected graph and the existence of an immersion of a fixed graph with maximum degree at most 4.

Theorem 1.8 (Chudnovsky, Dvořák, Klimošová, and Seymour)**.** *Let H be a graph with maximum degree at most four. Then there is an integer n such that every 4-edge-connected graph of tree-width at least n admits an immersion of H.*

Another reason edge-connectivity is often used in place of vertex-connectivity in the study of immersions is the interaction between path liftings and vertex cuts. Unlike the operations corresponding to the minor relation, the lifting operation requires at least two edges, whereas deletion and contraction refer only to a single edge or single vertex. For this reason, the lifting operation often fails to preserve structural properties of vertex cuts. In particular, those in graphs obtained by k-sums.

We formally define the k-sum of two graphs in Chapter 2, however, we give an informal definition here. A k-sum of two graphs H and G is formed by identifying the vertices of a clique of order k in H and the vertices of a clique of order k in G and deleting the edges of both cliques. The vertices of the k-sum resulting from the identification of the cliques form a vertex cut of the k-sum of G and H. With the minor and topological minor relation, the set of vertices in a minor or topological minor corresponding to the identified vertices of the k-sum form a vertex cut of that minor or topological minors. However, the identification of the cliques has the possibility of creating several paths between the non-identified vertices of H and the non-identified vertices of G. A lift of any of these paths would create an edge between a vertex of H and a vertex of G. As a result, the identified clique vertices are not necessarily a vertex cut of a graph immersed in the k-sum.

Because the lifting operation is equivalent to finding edge disjoint paths between pairs of vertices, an edge cut in a graph G will correspond to an edge cut in an immersed graph of at most the same size. Any lifted path containing an edge e of an edge cut replaces e in the corresponding edge cut of the immersed graph. This distinction is useful in the proof of the main result of Chapter 5 regarding the finite intertwine property of infinite graphs under the immersion relation.

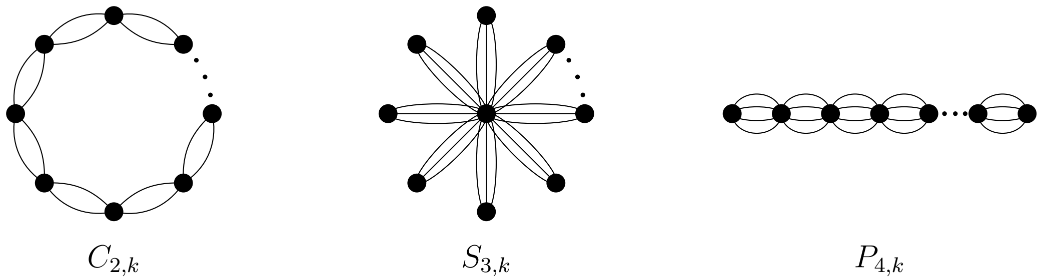


Figure 1.1. The graphs $C_{2,k}$, $S_{3,k}$, and $P_{4,k}$.

Finally, edge-connectivity provides an advantage to vertex-connectivity due to the relationship between the edge-connectivity of a graph and the existence of edge-disjoint paths in that graph. Although the vertex-connectivity of a graph is at most the edge-connectivity of a that graph, the difference between the vertex-connectivity and edge-connectivity of a graph is unbounded. Consider the following example. Let S_k denote the complete bipartite graph $K_{1,k}$, and let $S_{n,k}$ denote the graph obtained from S_k by replacing each edge of S_k by parallel class of size n (see Figure 1.1 for example). The graph $S_{n,k}$ has connectivity one, as the deletion of the vertex of degree nk separates the graph into k components. However, $S_{n,k}$ has edge-connectivity n. Thus, the edge-connectivity of $S_{n,k}$ provides a more detailed description of a graph, particularly in the context of the lifting operation. Namely, that for the minimum of n and k, $S_{n,k}$ admits an immersed K_n or K_k, and therefore, admits an immersion of any graph on at most $\min\{n, k\}$ vertices.

Similarly, let P_k denote the path on k vertices, and let $P_{n,k}$ denote the graph obtained from P_k by replacing each edge of P_k with a parallel class of size n (see figure 1.1). Again, $P_{n,k}$ has vertex-connectivity one and edge-connectivity n. As before, the edge-connectivity of $P_{n,k}$ gives detailed description of the structure of $P_{n,k}$ in the context the immersion relation as $P_{n,k}$ admits an immersion of cycle with every edge replaced by a parallel class of size $\lfloor \frac{n}{2} \rfloor$, denoted $C_{\lfloor \frac{n}{2} \rfloor,k}$ (see Figure 1.1).

Because of the usefulness of viewing immersion problems in terms of edge-connectivity (particularly in Chapter 5), we define an analogue to a block of a graph. A *blob* is a maximal 2-edge-connected subgraph of G or a single vertex which is only incident with cut-edges or loops of G. If a graph is 2-edge-connected or a single vertex, the graph itself is also a blob. We refer the read to Figure 1.2 for an example. The three blobs of the graph in Figure 1.2 are the monochromatic subgraphs represented with the colors red, green, and blue. We also note that the two cut-edges, represented in black, are not members of any blob. In fact, every edge of a graph is either contained in a blob, or a cut-edge.

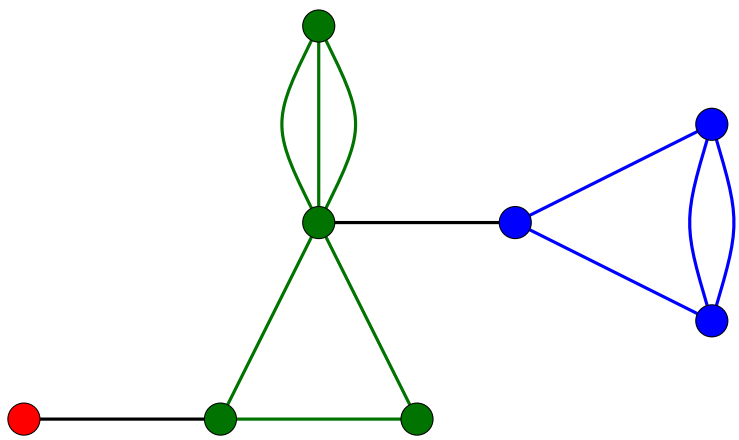


Figure 1.2. The blobs of a 1-edge-connected graph.

A lemma about blobs and the immersion relation can be stated as follows.

Lemma 1.9. *Let $H \leq_{\text{im}} G$ via the immersion φ and let C be a blob of H. Then there is a blob D of G such that $C \leq_{\text{im}} D$ via the immersion $\varphi|_C$.*

Proof. Let x and y be two vertices in $V(C)$. Since C is a blob, x and y are joined by at least two edge-disjoint paths. Let P_1 and P_2 be two such paths. As $\varphi(P_1)$ and $\varphi(P_2)$ are edge disjoint paths in G joining $\varphi(x)$ and $\varphi(y)$, and as blobs are maximal, it follows that $\varphi(P_1)$ and $\varphi(P_2)$ (and their endpoints $\varphi(x)$ and $\varphi(y)$) are contained in some blob, D, of G. Thus $\varphi|_C$ immerses C in D. $\square$

This dissertation presents immersion analogues of two graph minors results, as well as a conjecture, which is given partial proof. The first result is an analogue of Oporowski's Theorem [19] on minor intertwines of infinite graphs. Formally defined in Chapter 5, a minor intertwine of two graphs G_1 and G_2 is a graph G that is minor minimal with the property that it contains both G_1 and G_2 as minors. Extending the work of Andreae ([1] and [2]) we construct two infinite graphs which have infinitely many immersion intertwines. This shows that the class of infinite graphs does not possess the finite immersion intertwine property. In Chapter 3 we give an immersion analogue of the result of Oporowski, Oxley, and Thomas [20] by stating the unavoidable immersions for 3-edge-connected graphs. And finally, in Chapter 4 we present a conjecture for the unavoidable immersions of 4-edge-connected graphs along with a partial proof.

Chapter 2
Unavoidable Subgraphs, Minors, and Immersions

2.1 Unavoidable Substructures

In this chapter, as well as Chapter 3 and Chapter 4, we concern ourselves primarily with the weak immersion relation. As such, we will omit the weak and weak from the phrases weak immersion, weak liftable, and weak lifting and instead specify stron if we are referring to the strong immersion relation.

The study of unavoidable substructures in graphs is usually guided by questions of the following form: Given a graph property $\mathcal{P}$, is there a function f such that every graph with property $\mathcal{P}$ and order at least $f(k)$ "contains" a "highly structured" graph with property $\mathcal{P}$ and order at least k? There are many different results of this type, each with different meanings of the property $\mathcal{P}$ and of the phrases "contains" and "highly structured". The classical result of this type is a theorem of Ramsey [21], in which the property $\mathcal{P}$ is of a graphs being simple, that is, containing no parallel edges or loops, the phrase "contains" is understood as having an induced subgraph, and "highly structured" refers to a complete graph or an edgeless graph. More formally:

Theorem 2.1 (Ramsey)**.** *There is a function $f_{2.1}$ such that, for every positive integer k, every simple graph of order at least $f(k)$ contains an induced, complete subgraph on k vertices, or an independent set of k vertices.*

For example, $f_{2.1}(3)$ may be taken as six, or equivalently, if a simple graph has order at least six, then it contains an induced subgraph of K_3 or an independent set of order three. Using somewhat imprecise language, we say that a large complete graph, or a large edgeless graph is an unavoidable induced subgraph of a sufficiently large simple graph. Alternatively, in graph theory jargon, we say that

complete graphs and edgeless graphs are the unavoidable induced subgraphs of simple graphs.

Many different "Ramsey-like" results, such as unavoidable colorings of hypercubes [10] or the unavoidable parallel-minors of 4-connected graphs [6], have been proved since Ramsey's Theorem. This document is focused on the properties of vertex-connectivity and edge-connectivity, and the subgraph, topological minor, minor, and immersion relations. In particular, we rephrase the previous question as follows: Given a k-connected (l-edge-connected) graph, is there a function f such that every k-connected (l-edge-connected) graph and order at least $f(k)$ "contains" a "highly structured" k-connected (l-edge-connected) graph of order at least k? If we consider connected graphs, it is easy to show (see [8] for example) that the unavoidable subgraphs are a long path or a vertex of high degree.

Theorem 2.2. *There is a function $f_{2.2}$ such that, for every positive integer k, every connected graph of order at least $f_{2.2}(k)$ contains $K_{1,k}$ or P_k as a subgraph.*

Note that the statement in Theorem 2.2 would be false if "subgraph" were replaced by "induced subgraph" since complete graphs would serve as counterexamples. For a similar reason, when considering 2-connected graphs, we weaken the containment relation further by allowing the series contraction operation. (See [8] for the proof of the following theorem.)

Theorem 2.3. *For every integer k greater than or equal to two, there is a function $f_{2.3}$ such that every 2-connected graph of order at least $f_{2.3}(k)$ contains a topological minor isomorphic to C_k or $K_{2,k}$.*

Suppose "topological minor" was replaced in the previous theorem by "subgraph", and consider the graph $C_{f_{2.3}(k)}$. The cycle $C_{f_{2.3}(k)}$ does not contain cycle of length k as a subgraph. Thus, the set of unavoidable subgraphs of large 2-connected

graphs contains the infinite family of cycle graphs. However, by allowing the series contraction operation, we reduce the infinite number of unavoidable subgraphs to the two unavoidable topological minors given in Theorem 2.3.

The set of unavoidable topological minors of large 3-connected graphs is also known, though the proof is substantially more difficult than the proofs of Theorem 2.2 and Theorem 2.3.

Theorem 2.4 (Oporowski, Oxley, and Thomas)**.** *For every integer k greater than or equal to three, there is a function $f_{2.4}$ such that every 3-connected graph with at least $f_{2.4}(k)$ vertices contains a topological minor isomorphic to W_k, L_k^+, or $K_{3,k}$.*

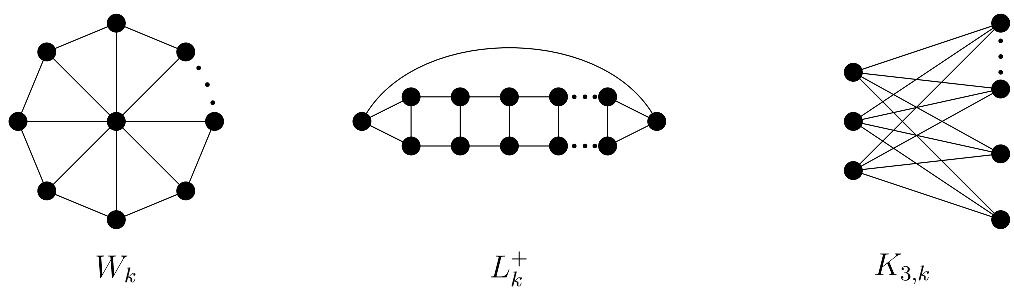


Figure 2.1. The unavoidable topological minors of large 3-connected graphs.

Although the set of unavoidable topological minors of large 3-connected graph is known, by replacing "topological minor" with "minor" in the previous theorem, we can simplify the set of unavoidable substructures. As L_k^+ contains a minor of W_k, the number of unavoidable minors of large 3-connected graphs is one fewer than the number of unavoidable topological minors of 3-connected graphs.

Theorem 2.5 (Oporowski, Oxley, and Thomas)**.** *There is a function $f_{2.5}$ such that, for every integer k greater than or equal to three, every 3-connected graph with at least $f_{2.5}(k)$ vertices contains a minor isomorphic to W_k or $K_{3,k}$.*

As discussed in Chapter 1, immersion results are often stated in terms of edge-connectivity as opposed to vertex-connectivity. This is the primary difference in the statement of the unavoidable immersion theorems compared to the previous results for the unavoidable topological minors and unavoidable minors of large graphs. As 1-edge-connected graphs are also 1-connected and immersion is a weakening of the subgraph relations, the unavoidable immersions of 1-edge-connected graphs are identical to the unavoidable subgraphs of 1-connected graphs. If we consider 2-edge-connected graphs, the decomposition theorems we use later in this chapter combined with Theorem 2.2 and Theorem 2.3 show that the lone unavoidable immersion of large 2-edge-connected graphs is a long cycle.

As was the case in Theorem 2.4, the unavoidable immersion of large 3-edge-connected graphs is significantly more difficult to find. This is the first of two main results of this dissertation.

Theorem 2.6. *There is a function $f_{2.6}$ such that, for every integer $k \geq 3$, every* 3-*edge-connected graph with order at least $f_{2.6}(k)$ admits an immersion of L_k^+ or $P_{2,k}^+$.*

L_k^+ $\qquad$ $P_{2,k}^+$

Figure 2.2. The unavoidable immersions of large 3-edge-connected graphs.

Theorem 2.6 is proved in Chapter 3. However, we first discuss several decomposition tools that will be useful in that proof.

2.2 Decomposition Theorems and k-Sums

Let k be an integer and let H and G be graphs such that both H and G have a clique of order k. A k-sum of H and G, denoted $H \oplus_k G$, is the graph obtained from identifying the vertices of a clique of order k in H with a vertices of a clique of order k in G, then deleting the edges between the vertices of the identified cliques. It is important to note that the k-sum is not a well-defined operation. That is, given two graphs H and G, the graph $H \oplus_k G$ is not necessarily unique. A given graph my have many cliques of order k, and therefore, a k-sum depends on the choice of clique in H and G.

For this reason, we introduce notation to specify the vertices of H and G that are identified in the graph $H \oplus_k G$. As we focus on 1-sums and 2-sums in the proof of Theorem 2.6, we adapt our notation to those choices of k. Let v_G and v_H be the vertices of the cliques of order one identified in the graph $H \oplus_1 G$. We call v_G and v_H the *identified vertices* of the 1-sum $H \oplus_1 G$. Similarly, let the e_G with end vertices v_G and u_G be the edge in a clique of order two in G, and e_H with end vertices v_H and u_H be the edge in a clique of order two in H. Further, suppose that v_H and v_G are identified in the graph $H \oplus_2 G$, and u_H and u_G are similar identified in $H \oplus G$. We say that u_H and u_G are the *identified vertices* of H in the 2-sum $H \oplus_2 G$, v_G and u_G are the *identified vertices* of G in the 2-sum $H \oplus_2 G$, and e_H and e_G are the *marker edges* of H and G, respectively, in the 2-sum $H \oplus_2 G$.

Now suppose G is a connected graph. Construct a tree T such that the vertex set of T consists of the disjoint union of the blocks of G and the cut vertices of G, that is, vertices of G which belong to more than one block. Two vertices of T are adjacent if and only if one is a vertex of G and the other is a block containing that vertex. Since a block is a maximal two connected subgraph, it is easy to see that T must be a tree, as a cycle would imply that the blocks that compose the

cycle would not be maximal. We call T a *block-tree* (see Figure 2.3). The following result is very well-known and can be found, for example, in [8].

Theorem 2.7. *Every connected graph has a unique block-tree.*

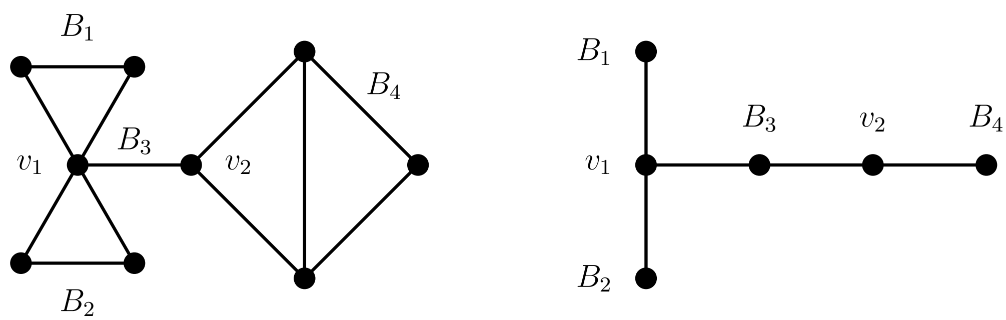


Figure 2.3. A connected graph and its block tree.

It is often convenient to view a connected graph G as a series of 1-sums of the blocks of G, where the 1-sums are informed by the structure of the block tree. If G consists of a single block, then G is vacuously obtained by a trivial number of 1-sums. Proceeding inductively, G can be constructed as a series of 1-sums of it's blocks by choosing the cut vertices of G as the identified vertices. More formally, let B_1 and B_2 be two blocks of G such that $v \in V(G)$ is a vertex of both B_1 and a vertex of B_2. Then the subgraph of G induced by B_1 and B_2 is equivalent to $B_1 \oplus_1 B_2$ where v is the identified vertex of the 1-sum $B_1 \oplus_1 B_2$. This allows us to rephrase Theorem 2.7 in terms of 1-sums.

Theorem 2.8. *Let G be a connected graph. Then G can be constructed by a series of 1-sums of blocks of G, where the identified vertices of the 1-sums correspond to the cut vertices of G.*

Similarly, Cunningham and Edmonds [7] provide a decomposition of a 2-connected graph into 3-connected graphs, cycles, and cocycles. We introduce new notation to formally describe this decomposition.

Let $\{E_1, E_2\}$ denote a partition of the edge set of a 2-connected graph G into the sets E_1 and E_2. A *split* of G is a partition $\{E_1, E_2\}$ such that the cardinality of E_1 is at least two, the cardinality of E_2 is at least two, and $V(G(E_1)) \cap V(G(E_2)) = 2$. Let u and v be the two vertices of $V(G(E_1)) \cap V(G(E_2)) = 2$ in the split $\{E_1, E_2\}$. We construct two graphs G_1 and G_2 by adding an edge e between u and v in $E(G_i)$ for $i \in [2]$. This set of graphs $\{G_1, G_2\}$ is called a *simple decomposition* of G associated with the split $\{E_1, E_2\}$ and the marker edge e. A *decomposition*, D, of G is the set of graphs obtained by a (possibly trivial) series of simple decompositions.

Theorem 2.9 (Cunningham, Edmonds)**.** *Every* 2*-connected graph has a unique, minimal, decomposition into* 3*-connected graphs, cycles, and cocycles.*

We now prove an easy lemma about these decompositions that will be used in the proof of Theorem 2.6 in Chapter 3.

Lemma 2.10. *Let G be a* 2*-connected graph, let D be a decomposition of G, and let H be a member of D. Then $H \leq_{\text{im}} G$.*

Proof. Let G be a 2-connected graph with decomposition $D = \{H_i\}$ for $i \in [|D|]$. In addition, let $M(H_i)$ be the set of marker edges of H_i, and consider an edge $e \in M(H_i)$ with endpoints u and v. As u and v are endpoints of a marker edge e, they are members of some other member of D. Without loss of generality, assume $u, v \in V(H_j)$ for $i \neq j$. As H_j is a cocycle, cycle, or 3-connected graph, by Menger's Theorem (0.1), there must exist a path $P \subseteq H_j$ such that P_j does not contain the edge e. Consider subgraph of G induced by the edges of G that are not marker edges and the paths P_j for all j distinct from i.

Let H be the graph formed by lifting each path P_j and then deleting all edges and vertices that are not members of $V(H_i), E(H_i - M(H_i)$, or the lift of a path P_j for all j such that $V(H_j) \cap V(H_i) \neq \emptyset$. Then H is isomorphic to H_i via the map

$\varphi : H_i \to H$, where $\varphi(v) = v$ for $v \in H_i$, $\varphi(e) = e$ for $e \in E(H_i) - M(H_i)$, and $\varphi(e)$ equals the edge resulting from the lift of P_j for $e \in M(H_i)$ and $e \in H_j$. Thus, as $H \leq_{\mathrm{im}} G$, we have that $H_i \leq_{\mathrm{im}} G$. $\square$

It is often useful to use a decomposition to impose a tree-like structure similar to a block tree on a 2-connected graph G. We can associate a tree T with a decomposition D by letting the vertices of T be the members of D and letting two vertices of T be adjacent if and only if the corresponding members of D share a marker edge. Furthermore, T provides a blueprint for constructing G from 2-sums of the graphs of D. Let f be an edge of T with end vertices G_1 and G_2. Then G_1 and G_2 share a marker edge. Let e denote the marker edge associated with the simple decomposition $\{G_1, G_2\}$. Consider the graph $G_1 \oplus_2 G_2$ where e is the identified edge of the 2-sum. The decomposition $D' = (D - \{G_1, G_2\}) \cup G_1 \oplus_2 G_2$ is a decomposition of G with one fewer member than D, and the tree T' associated with D' is equivalent to T/e.

By repeating this process inductively, this series of 2-sums of the graphs of G produces the graph G. We note that the 2-sum of two cycles is a cycle, and similarly the 2-sum of a two cocycles is a cocycle. Therefore, a series of 2-sums that produce G might not necessarily be unique. However, a there is a unique series of 2-sums if we place the following conditions on the series: no two cocycles are 2-summed, no two cycles are 2-summed, and no cocycles of size two are used in any 2-sum. We summarize this idea in the following theorem.

Theorem 2.11. *Let G be a 2-connected graph. Then G can be obtained by a series of 2-sums of cycles, cocycles, and 3-connected graphs. Further, if no two cycles share a marker edge, no two cocycles share a marker edge, and no cocycle of size two is used in any 2-sum, then the series of 2-sums is unique.*

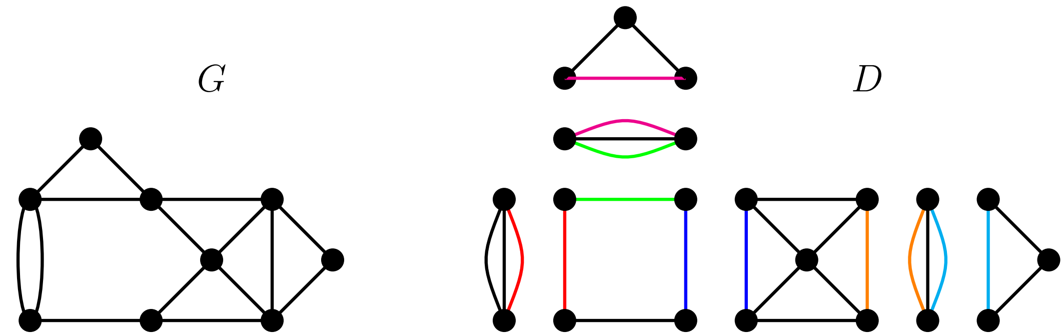


Figure 2.4. A 2-connected graph decomposed into cycles, cocycles, and 3-connected graphs.

Theorem 2.9 and Theorem 2.11 are illustrated in Figure 2.4. Each component of the graph on the right is a member of the decomposition D of the graph G. The non-black edges correspond to the marker edges of the decomposition. The non-black edges also denote the identified cliques in the series of 2-sums of graphs of D that yield the graph G. Furthermore, as no pair of cycles shares a marker edge, no pair of cocycles shares a marker edge, and a cocycle of size two is not a member of D, the decomposition is unique. Both Theorem 2.8 and Theorem 2.11 will be useful in the proof of Theorem 2.6 in the following chapter.

Chapter 3
Unavoidable Immersions of 3-Edge-Connected Graphs

3.1 Proof of Theorem 2.6

We proceed with the proof of Theorem 2.6 by the use of several lemmas, three of which corresponded to the cases of a 3-edge-connected graph G being connectivity one, connectivity two, or 3-connected. In the proofs of Lemma 3.5 and 3.3 we make use of the decomposition theorems in Chapter 2 as well as Theorem 2.2 to investigate the tree-like structure of G. For the proof of Lemma 3.2 we use the result of Oporowski, Oxley, and Thomas [20], Theorem 2.4, to find the required immersions.

We first prove an easy lemma that will be useful in the proof of Theorem 2.6

Lemma 3.1. *The graph L_k^+ is immersed in both of the graphs $S_{3,2k}$ and W_{2k}.*

Proof. We prove this lemma in four parts. First, we show that $S_{3,2k}$ admits an immersion of W_{2k}. In turn W_{2k} admits an immersion of the k-rung möbius ladder, and finally, the k-rung möbius ladder admits an immersion of L_k^+.

Consider $S_{3,2k}$. Label the vertex of with degree $3k$ by v, and label the remaining vertices by v_i for $i \in \{0, \ldots, 2k-1\}$. For every $i \in \{0, \ldots, 2k-1\}$, there is a path P_i of length two from the vertex v_i to the vertex v_{i+1} (where addition in the indices is performed $\pmod{2k}$) through v such that P_i and P_j are edge-disjoint if $i \neq j$. Let G be the graph obtained by lifting the P_i for each $i \in \{0, \ldots, 2k-1\}$. Then G is isomorphic to W_{2k}.

For each $l \in \{0, \ldots, k-1\}$ Let P_l' be the path of length two in G from v_l to v_{l+k} through the vertex v. Let G' denote the graph obtained from G by lifting P_l for each $l \in \{0, \ldots, k-1\}$ and deleting the vertex v. Then G' is isomorphic to the k-rung möbius ladder. (See Figure 3.1.)

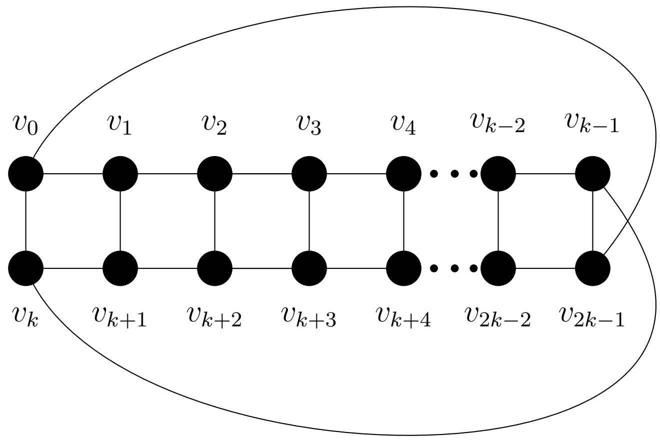


Figure 3.1. The k-rung möbius ladder.

Let G'' be the graph obtained from G' by lifting the path $\{v_1, v_0, v_k\}$, lifting the path $\{v_{2k-2}, v_{2k-1}, v_{k-1}\}$, and then deleting the vertices v_0 and v_{2k-1}. Then G'' is isomorphic to L_k^+, and by transitivity, $L_k^+ \leq_{\mathrm{im}} S_{3,2k}$ and $L_k^+ \leq_{\mathrm{im}} W_{2k}$.

$\square$

The following class of graphs will be useful in Chapter 3 and Chapter 4. Given a set of non-negative integers $\{a_1, \ldots, a_n\}$ and a non-negative integer k, define the graph $K_{(a_0,a_1,\ldots,a_n),k}$ to be the bipartite graph with bipartitions $A = \bigcup_{i\in[n]} A_i$ and B constructed in the following way. Let B be a set of k vertices, and for each $i \in [n]$ add a set A_i of a_i vertices joined to each vertex of B by parallel classes of size i (see Figure 3.2 for an example).

We now proceed with the proof of Theorem 2.6.

Lemma 3.2. *Let k be an integer greater than or equal to three and let G be a 3-connected graph. Then there is a function $f_{3.2}$ such that if G has order at least $f_{3.2}(k)$ then G admits an immersed L_k^+ (see Figure 2.2).*

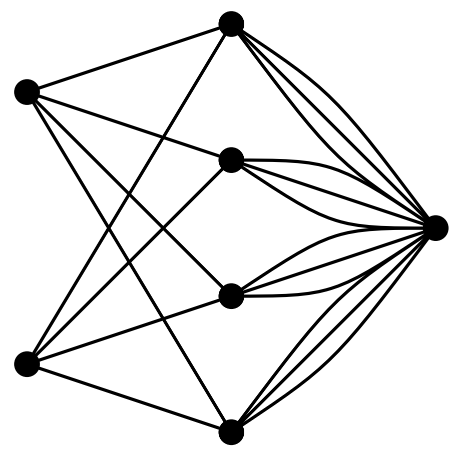


Figure 3.2. The graph $K_{(2,0,1),4}$.

Proof. Let $f_{3.2}(k) = f_{2.4}(6k)$ and let G be a 3-connected graph with order $f_{3.2}(k)$. Then by Theorem 2.4, as the order of G is at least $f_{2.4}(6k)$, G admits a topological minor isomorphic to L^+_{6k}, W_{6k}, or $K_{3,6k,}$. If G admits a topological minor isomorphic to L^+_{6k} then it also admits a topological minor isomorphic to L^+_k, and therefore, an immersed L^+_k.

Suppose, instead, that G admits a topological minor isomorphic to W_{6k}. Then by Lemma 3.1, G admits an immersion of L^+_k.

Finally, suppose that G admits a topological minor isomorphic to $K_{3,6k}$. We show that $S_{3,2k} \leq_{\text{im}} K_{3,6k}$. Label the vertices of the partition of size 3 with the labels b_0, b_1, and b_2, and label the vertices of the partition of size $6k$ with the labels c_0, c_1, …, and c_{6k}. For each $i \in \{0, 1, \ldots, 6k-3\}$ such that $i \equiv 0 \pmod 3$, lift the paths $\{b_0, c_{i+1}, b_1, c_i\}$ and $\{b_0, c_{i+2}, b_2, c_i\}$. After all the specified paths have been lifted, delete the vertices b_1, b_2, c_{i+1} and c_{i+2}. The resulting graph is isomorphic to $S_{3,2k}$ and, therefore, G admits an immersed $S_{3,2k}$. Thus, by Lemma 3.1, G admits an immersed L^+_k.

□

Lemma 3.3. *Let k be an integer greater than or equal to three and let G be a 3-edge-connected graph with vertex-connectivity two. Then there is a function $f_{3.3}$ such that if G has order at least $f_{3.3}(k)$ then G admits an immersed L_k^+ or $P_{2,k}^+$ (see Figure 2.2).*

Proof. Let $f_{3.3}(k) = (f_{3.2}(k) - 1)f_{2.2}(8k\binom{f_{3.2}(k)}{2})$ and let G be a 3-edge-connected graph with vertex-connectivity two and order $f_{3.3}(k)$. By theorem 2.9, G can be decomposed into two sums of 3-connected graphs, cycles, and cocycles. Let D denote the unique decomposition and let T be the tree associated with D. As the order of G is at least $f_{3.3}(k)$, the decomposition D must either contain a cycle or 3-connected graph of order at least $f_{3.2}(k)$, or the number of graphs in D must be at least $f_{2.2}(8k\binom{f_{3.2}(k)}{2})$.

First, suppose D contains a 3-connected graph, H, of order at least $f_{3.2}(k)$. Then by Lemma 2.10, $H \leq_{\mathrm{im}} G$. Furthermore, by Lemma 3.2, H admits an immersed L_k^+. By transitivity, we have that G also admits and immersed L_k^+.

Suppose instead that D contains a cycle of order at least $f_{3.2}(k)$. Denote this cycle C, let m be the order of C, and label the vertices of C as $c_1, c_2, \ldots, c_m$ in cyclic order. We show that all but one pair of adjacent vertices of C are a marker edge for a 2-sum with either a parallel class of size three or greater, a cycle, or a 3-connected graph.

Assume the contrary. In particular, suppose without loss of generality that the edge f, with end-vertices c_i and c_{i+1} for $i \in [m]$, and the edge e, with end-vertices c_j and c_{j+1} for $j \in [m]\backslash\{i\}$, are not marker edges. Consider the graph $G\backslash\{e, f\}$ and the vertices c_i and c_{i+1}. There is no member of D other than C that contains both c_i and c_{i+1} since e is not a marker edge. Furthermore, as T is acyclic, every path P between c_i and c_{i+1} must contain either the edge e or the edge f. Therefore, there is

no path between c_i and c_{i+1} in $G \setminus \{e, f\}$. This contradicts the 3-edge-connectivity of G. Hence all but one pair of adjacent vertices of C are marker edges.

Thus we can assume there is at most one edge that is not a marker edge. Without loss of generality, let e with end vertices c_m and c_1 be this edge.

Consider c_i and c_{i+1} for $i \in [m-1]$, and let e_i denote the edge incident both c_i and c_{i+1}. As e_i is a marker edge, c_i and c_{i+1} are the identified vertices in a 2-sum with a member O of D that is not a parallel class of size less than three. Thus, by Menger's Theorem (Theorem 0.2), there are two edge disjoint paths joining c_i and c_{i+1}contained in O. Lift these paths if they are of length long than two, and delete the vertices and remaining edges of O that are not the aforementioned paths or lifts of the paths.

Repeat this process for all $i \in [m-1]$. Then delete all vertices of $V(G) - V(C)$. The resulting graph is isomorphic to $P^+_{2,m}$ and therefore $P^+_{2,m} \leq_{\text{im}} G$. As $m \geq f_{3.2}(k) \geq k$, this implies that $P^+_{2,k} \leq_{\text{im}} G$.

Instead, suppose that the cardinality of D is at least $f_{2.2}(8k\binom{f_{3.2}(k)}{2})$ and each member of D has order less than $f_{3.2}(k)$. That is, the order of the tree T is at least $f_{2.2}(8k\binom{f_{3.2}(k)}{2})$. By Theorem 2.2, the graph T contains a subgraph of a path of length $8k\binom{f_{3.2}(k)}{2}$ or a vertex of degree $8k\binom{f_{3.2}(k)}{2}$.

Assume first that T has a vertex v of degree $8k\binom{f_{3.2}(k)}{2}$. The vertex v corresponds to a graph of D which is either a cocycle, cycle, or 3-connected graph. In order to address a commonly occurring situation in each case, we now prove a short lemma.

Lemma 3.4. *Let k be an integer greater than or equal to three, let H be a cocycle, cycle, or 3-connected graph in a unique decomposition D of a 2-connected, 3-edge-connected graph G, and let x and y two vertices of H. Further, suppose that x and y are the identified vertices in at least $8k$ of the 2-sums of H with graphs in D. Then H admits an immersed L^+_k.*

Proof. Let n be the number of 2-sums for which x and y are the identified vertices. Let H, x, and y be defined as in the lemma statement, and let $\{H_i\}_{i\in[n]}$ be the set of graphs of D which are two summed to with H at $\{x, y\}$. First note that either H is a cocycle, or at most one of the graphs in $\{H_i\}_{i\in[n]}$ is a cocycle by the definition of D. Suppose, without loss of generality, that H_n is a cocycle and that each other H_i for $i \in [n-1]$ is either a cycle or a 3-connected graph. Then for each $i \in [n-1]$ there is a vertex v_i of H_i such that v_i is not an end-vertex of the marker edge corresponding to the 2-sum $H \oplus_2 H_i$.

By Menger's theorem (Theorem 0.2), as G is a 3-edge-connected graph, there are three edge-disjoint paths W_{1_i}, W_{2_i}, and W_{3_i} from the vertex v_i to the vertices $\{x, y\}$. Furthermore, each path W_{j_i} for $j \in [3]$ is a subgraph of H_i by the definition of D. By the pigeon hole principle, as the $n \geq 8k$ and $\{x, y\}$ are joined to each v_i by three edge disjoint paths, there must be $4k$ graphs H_i for which either x is the endpoint of (at least) two paths of W_{j_i} or y is the endpoint of (at least) two paths of W_{j_i}.

Suppose, again without loss of generality, that H_i for $i \in [4k]$ are graphs such that W_{1_i} and W_{2_i} have endpoints v_i and x and W_{3_i} has endpoints v_i and y. We first show that G admits an immersed $K_{\{1,2\},4k}$ (see Figure 3.2) and then show that $K_{\{1,2\},4k}$ admits an immersed $S_{3,2k}$.

Let G' be the graph obtained from G by lifting the paths W_{j_i} for $i \in [4k]$ and $j \in [3]$ of length greater than one, reducing each remaining parallel class to size two if x is an end vertex of the parallel class or one if y is an end vertex of the parallel class, and deleting all vertices that are not x, y, or v_i. Then G' is isomorphic to $K_{\{1,2\},4k}$ and $K_{\{1,2\},4k} \leq_{\text{im}} G$.

Now let G'' be the graph obtained from G' by lifting the path x, v_{i+1}, y, v_i for all odd $i \in [4k]$, then deleting y and v_i for all even $i \in [4k]$. Then G'' is isomorphic to

$S_{3,2k}$ and, therefore, $S_{3,2k} \leq_{\text{im}} G$. Hence, $L_k^+ \leq_{\text{im}} G$ by transitivity and Lemma 3.1.

$\square$

First suppose the vertex v of T corresponds to a cocycle with vertices x and y. As the degree of v is at least $8k$, x and y are the identified vertices in at least $8k$ 2-sums. Therefore, by Lemma 3.4, $L_k^+ \leq_{\text{im}} G$.

Next, suppose that v corresponds to a cycle or 3 connected graph, H, in D. By the pigeonhole principle, as the degree of v is at least $8k\binom{f_{3.2}(k)}{2}$ and the order of H is less than $f_{3.2}(k)$, there must exist two adjacent vertices, x' and y', of H so that x' and y' are the identified vertices of at least $8k$ 2-sums. Then by Lemma 3.4, we have $L_k^+ \leq_{\text{im}} G$.

Lastly, suppose T contains a path of length at least $8k\binom{f_{3.2}(k)}{2}$. Then T also contains a path S of length $6k$. Label the vertices of S as $s_0, s_1, \ldots, s_{6k}$. The path S corresponds to a series of 2-sums of graphs of D. For $i \in \{0, \ldots, 6k\}$, let H_{s_i} be the member of D corresponding to the vertex s_i of T, and denote the two vertices forming the intersection of H_{s_i} and $H_{s_{i+1}}$ as x_i and y_i. That is, x_i and y_i are the two vertices of the identified K_2 in the 2-sum of the graphs H_{s_i} and $H_{s_{i+1}}$.

Consider x_0 and x_{6k}. By Menger's Theorem (Theorem 0.2), there are three edge disjoint paths, P_1, P_2, and P_3 from x_0 to x_{6k}. Furthermore, as T is acyclic, the path P_l for $l \in [3]$ must contain either x_i or y_i for each $i \in \{1, \ldots, 6k-1\}$. Hence, as S has length at least $6k$, by construction, both H_{s_i} and $H_{s_{i+1}}$ cannot be cocycles, and there are three edge disjoint paths from x_0 to x_{6k}, two of the paths must intersect at least k times. Without loss of generality, suppose that P_1 and P_2 share at least k vertices, and denote first $k-2$ of these vertices distinct from x_0 and x_{6k} $\{v_1, v_2, \ldots, v_{k-2}\}$

Let F be the graph that is obtained from G by the following operations: lift the path P_3, lift the segments of the path P_1 between the vertices x_0 and v_1, v_{k-2} and x_{6k}, and v_m and v_{m+1} for $m \in [k-3]$ if the segments have length greater than one, lift the segments of the path P_2 between the vertices x_0 and v_1, v_{k-2} and x_{6k}, and v_m and v_{m+1} for $m \in [k-3]$ if the segments have length greater than one, delete all vertices except x_0, x_{6k}, and v_m, and delete any remaining edges that so that every parallel class has size at most 2. Then F is isomorphic to $P^+_{2,k}$ and therefore, $P^+_{2,k} \leq_{\text{im}} G$.

$\square$

Lemma 3.5. *Let k be an integer greater than or equal to three and let G be a 3-edge-connected graph with vertex-connectivity one. Then there is a function $f_{3.5}$ such that if G has order at least $f_{3.5}(k)$ then G admits an immersed $S_{3,k}$, L^+_k, or $P^+_{2,k}$ (see Figure 2.2).*

Proof. Assume G is 3-edge-connected, the vertex-connectivity of G is one, and G has order at least $f_{3.5}(k) = (f_{3.3}(k) - 1)(f_{2.2}(f_{3.3}(k)) + 1$. Then G has a block-tree decomposition T, where the vertices of T correspond to either blocks of G or to cut vertices of G. As the order of G is $(f_{3.3}(k) - 1)(f_{2.2}(f_{3.3}(k)) + 1$, either a block of G has order at least $f_{3.3}(2k)$ or the T has order at least $f_{2.2}(f_{3.3}(k))$.

First, assume that T has order at least $f_{2.2}(f_{3.3}(k))$ and no block of G has order $f_{3.3}(k)$. Then by Theorem 2.2, T contains a path of length at least $(f_{3.3}(k))$ or a vertex of degree at least $f_{3.3}(k)$. Consider the case where T contains a path P of length $m \geq f_{3.3}(k)$. Label the vertices of P as $t_0, t_1, \ldots, t_{m-1}$. As path P corresponds to a series of one sums of blocks in the graph G, the vertex t_i for $i \in \{1, \ldots, m-1\}$ corresponds to a cut-vertex of G if i is odd, and t_i corresponds to a block of G if i is even. Label one vertex of the block of G corresponding to t_0

as x_{t_0}. In a similar manner, label a vertex of the block of G corresponding to t_{m-1} as $x_{t_{m-1}}$.

By Menger's Theorem (Theorem 0.2), as G is 3-edge-connected, there are three edge disjoint paths in G, P_1, P_2, and P_3, with endpoints x_{t_0} and $x_{t_{m-1}}$. As T is an acyclic graph, the paths P_1, P_2, and P_3 must intersect at the cut vertices of G corresponding to the vertices t_i for which i is odd. Consider the graph H_1 formed by lifting the path P_1, lifting the segments of the paths P_2 and P_3 with endpoints v_{t_i} and $t_{v_{i+2}}$ for $i \in \{1, \dots, m-1\}$ such that i is even, and then deleting the vertices and edges of G with the exception of the edges formed by the the described liftings and the vertices v_{t_i} for $i \in \{1, \dots, m-1\}$ such that i is even. The resulting graph H_1 is $P^+_{2,\frac{m-1}{2}}$, which, as $\frac{m-1}{2} \geq k$, implies that G admits an immersion of $P^+_{2,k}$.

Next, suppose that T contains a vertex, labeled s_0, of degree at least $f_{3.3}(k)$. Then s_0 corresponds to either a cut-vertex of G, labeled v_{s_0}, or a block of G, labeled B_{s_0}. First assume that s_0 corresponds to a block B_{s_0} of G. Then each neighbor of s_0 in T corresponds to a distinct vertex of the block B_{s_0} which is also a cut-vertex in G. As the degree of s_0 is at least $f_{3.3}(k)$, the order of B_{s_0} must also be at least $f_{3.3}(k)$, which contradicts our assumed bound on the order of the blocks of G, and therefore, s_0 must correspond to a cut-vertex of G.

Suppose then that s_0 corresponds to v_{s_0}, a cut-vertex of G. Then the neighbors of s_0 in T correspond to blocks of G each containing v_{s_0}. As $f_{3.3}(k) \geq 2k$, label $2k$ of these blocks $B_0, B_1, \dots, B_{2k}$. Each of these blocks must contain a vertex distinct from v_{s_0} by the definition of the block-tree and cut-vertex. So for each $i \in \{0, \dots, 2k\}$ label a vertex of the block B_i distinct from v_{s_0} as v_i. As G is 3-edge-connected, by Menger's Theorem (Theorem 0.2), there exist three edge disjoint paths, Q_{i_1}, Q_{i_2}, and Q_{i_3}, between v_i and v_{s_0}. Furthermore, as T is a tree

, these paths lie completely within B_i. This implies that, for $j \in \{0, \ldots, 2k\}$ and $n, m \in [3]$, if Q_{j_m} is a path in B_j, then Q_{j_m} and Q_{i_n} share no edges.

Consider the graph H_2 formed by lifting the paths Q_{i_n} for each $i \in \{0, \ldots, 2k\}$ and each $n \in [3]$, and then deleting all edges and vertices of G that are not v_i, v_{s_0}, and the edges formed by the described liftings. Then H_2 is isomorphic to $S_{3,2k}$, and therefore, G admits an immersed $S_{3,2k}$. Hence, by Lemma 3.1, G also admits an immersed L_k^+.

Finally, assume G contains a block, labeled B, of order at least $f_{3.3}(k)$. By Lemma 3.3, as blocks are 2-connected subgraphs and B has order at least $f_{3.3}(k)$, B admits an immersed L_k^+ or $P_{2,k}^+$. Therefore, G also admits an immersed L_k^+ or $P_{2,k}^+$. □

We now proceed with the proof of Theorem 2.6.

Proof of Theorem 2.6. Let $k \geq 3$ be an integer and G be a 3-edge-connected graph with order $f_{2.6}(k) = f_{3.5}(k)$. We consider three cases: the connectivity of G is at least 3, the connectivity of G is 2, or the connectivity of G is 1. If the connectivity of G is three, then by Lemma 3.2, as the order of G is $f_{2.6}(k)$, G admits an immersed L_k^+ or $P_{2,k}^+$. Similarly, if G has connectivity two, then by Lemma 3.3, the graph G admits an immersed L_k^+ or $P_{2,k}^+$., and if G is connectivity one, then G admits an immersed L_k^+ or $P_{2,k}^+$ by Lemma 3.5. □

Chapter 4
Unavoidable Immersions of 4-edge-connected graphs

4.1 Unavoidable Topological Minors and Minors

In addition to finding the unavoidable minors of large 3-connected graphs, Oporowski, Oxley, and Thomas also used a similar method to find the unavoidable minors and the unavoidable topological minors of large internally-4-connected graphs.

Theorem 4.1 (Oporowski, Oxley, and Thomas)**.** *For every integer k greater than or equal to four, there is a function f such that every internally-4-connected graph with at least $f(k)$ vertices contains a topological minor isomorphic to the 2k-spoke alternating double wheel, $K_{4,k}$, the k-rung möbius ladder, the k-rung circular ladder, or $K'_{4,k}$.*

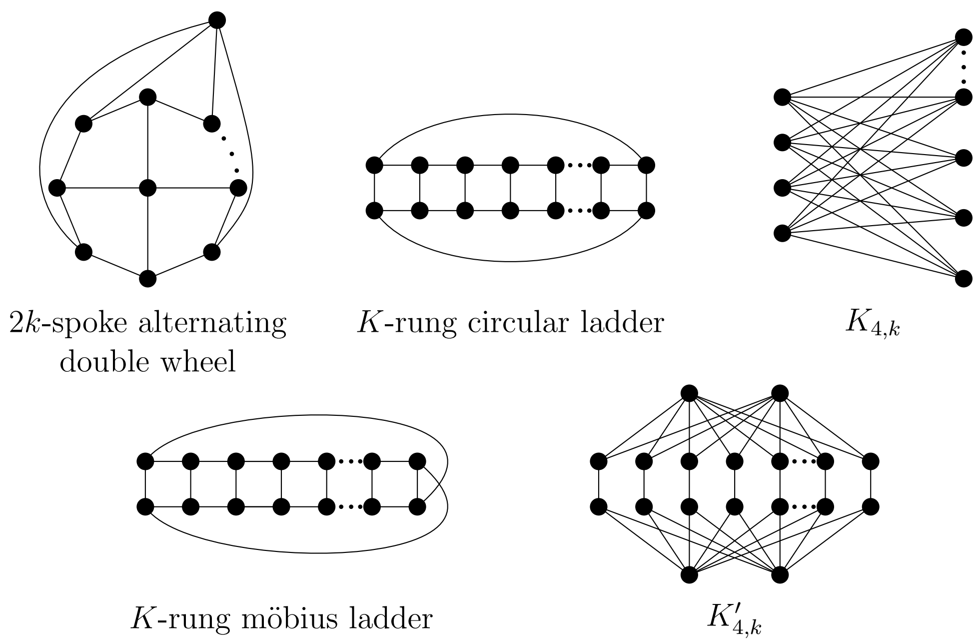


Figure 4.1. The unavoidable topological minors of large internally-4-connected graphs.

Similar to Theorem 2.4 and Theorem 2.5, the number of unavoidable graphs is reduced “topological minor” is replaced with “minor” in the previous theorem. This gives the following result.

Theorem 4.2 (Oporowski, Oxley, and Thomas)**.** *For every integer k greater than or equal to four, there is a function f such that every internally-4-connected graph with at least $f(k)$ vertices contains a minor isomorphic to $2k$-spoke double wheel, the k-rung circular ladder, the k-rung möbius ladder, or $K_{4,k}$.*

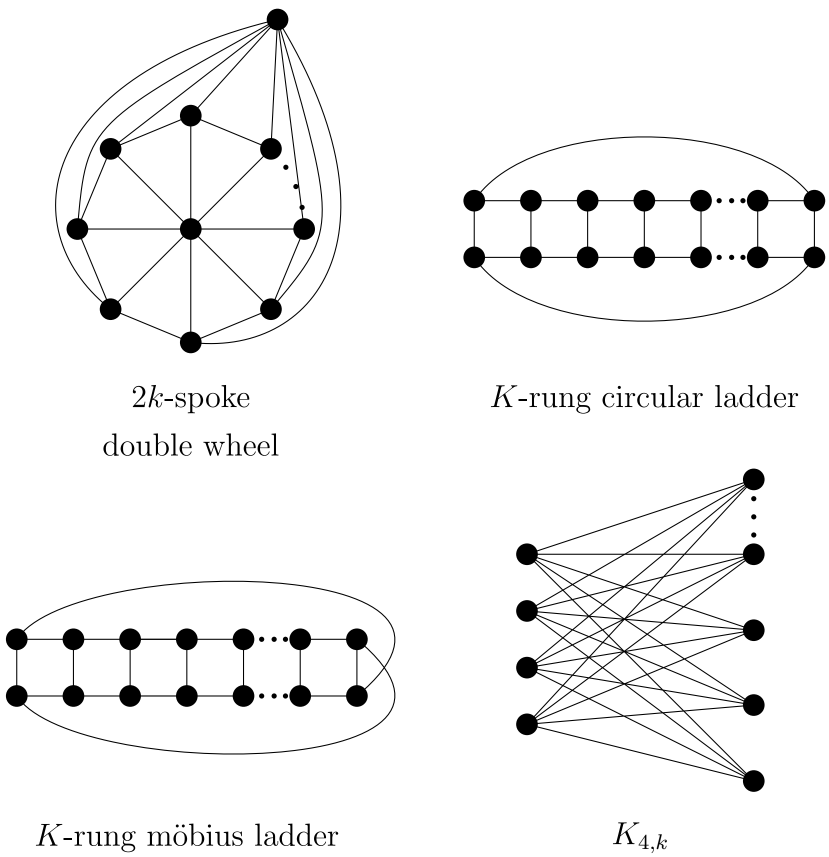


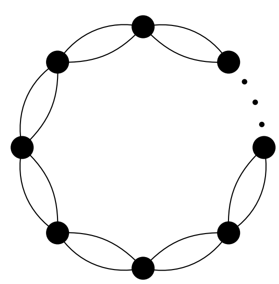
Figure 4.2. The unavoidable minors of large internally-4-connected graphs.

While the block-tree of a graph, as well as the similar decomposition of 2-connected graphs [7] proved useful in the proof of Theorem 2.6, a method the more closely resembles the proof given by Oporowski, Oxley, and Thomas in [20]

is needed for the partial proof of Conjecture 4.3. Despite the existence of a decomposition theorem for 3-connected graphs analogous to Theorem 2.9 and the analogous result in graph minors, both of these results involve internally-4-connected graphs. This makes dividing the proof of Conjecture 4.3 into cases depending on the vertex-connectivity of a graph problematic. In particular because, unlike the minor relation, the degree of an immersed graph will always be at most the degree of the graph it is immersed in. Likewise, the connectivity and edge-connectivity cannot increase from an path lift or a deletion. As a large 4-edge-connected graph may contain many vertices of degree three, attempting to prove Conjecture 4.3 with a method similar to the proof of Theorem 2.6 deteriorates into analysis on a complex series of many, highly dependent cases. Therefore, a different approach is needed to find the unavoidable immersions of 4-edge-connected graphs .

Although a complete proof eludes us at the moment, we present a conjecture for the unavoidable immersion for 4-edge-connected graphs.

Conjecture 4.3. *For every integer k greater than or equal to four, there is a function $f_{4.3}$ such that every 4-edge-connected graph of order at least $f_{4.3}(k)$ admits an immersed double cycle of length k, $C_{2,k}$. (See Figure 4.3.)*



$C_{2,k}$

Figure 4.3. The conjectured unavoidable immersion for large 4-edge-connected graphs.

The next section contains the definition of a tree-decomposition, which will be used in the partial proof of Conjecture 4.3.

4.2 Tree-Width and Tree-Decompositions

As we discussed in Chapter 1, the concept of tree-width is important in both the Graph Minors Project as well as other areas of structural graph theory. Let G be a graph, T be a tree, and $\mathcal{Y} = \{Y_t\}_{t \in V(T)}$ be a family of subsets of $V(G)$ indexed by the vertices of T. The pair $(T, \mathcal{Y})$ is called a *tree-decomposition* of G if:

(TD1) $V(G) = \bigcup_{t \in T} Y_t$;

(TD2) if e is an edge in $E(G)$, then there is a vertex $t \in V(T)$ such that both end-vertices of e are in Y_t, and;

(TD3) if t, t', and t'' are vertices of T such that t' lies on the path between t and t'', then $V_t \cap V_{t''} \subset V(t')$.

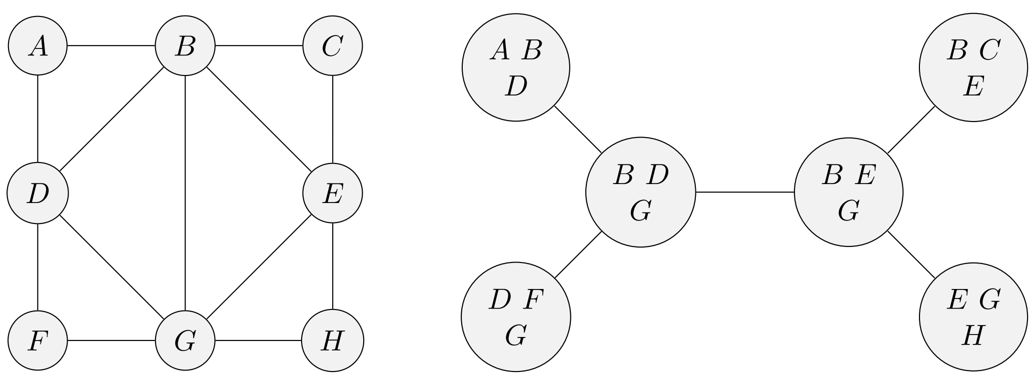


Figure 4.4. A tree-decomposition of a graph.

See Figure 4.4 for an example of a tree-decomposition of a graph. We call the subgraphs $G(Y_t)$ and the sets Y_t themselves the parts of the decomposition $(T, \mathcal{Y})$. The *width* of a tree-decomposition $(T, \mathcal{Y})$ is defined to be the number $\max\{|Y_t| - 1 : t \in V(T)\}$. The *tree-width* of a graph G, often denoted $\mathrm{tw}(G)$, is the least

width of any tree-decomposition of G. As mentioned previously, tree-width is a quantification of the resemblance of a graph G to a tree. The smaller the $\mathrm{tw}(G)$, the closer the structure of G to a tree. If $\mathrm{tw}(G) = 1$, then G itself is a tree. We call a tree decomposition *linked* if, in addition to (TD1)-(TD3), $(T, \mathcal{Y})$ satisfies the following condition:

(TD4) if t and t' are two vertices of T and k is a positive integer then either there are k disjoint paths in G between Y_t and $Y_{t'}$ or there is a vertex t'' of T on the path between t and t' such that $|Y_{t''}| < k$

Thomas showed in [28] that if a graph admits a tree-decomposition of width w, then it also admits a linked tree-decomposition of width w. This allows us to impose a Menger-like property on the paths between vertices in different parts of the tree-decomposition. Oporowski, Oxley, and Thomas further refined the definition of tree-decomposition using the following definitions. Let $(T, \mathcal{Y})$ be a tree-decomposition of a graph G, t' be a vertex of T and B be a component of $T - t'$. We say that a vertex v in Y'_t is *B-tied* if v is in Y_t for some $t \in V(B)$. We say a path P in G is *B-confined* if the path has length at least two and every internal vertex of P belongs to the set $\bigcup_{t \in V(B)} (Y_t - Y_{t'})$. A tree-decomposition is called *lean* if, in addition to satisfying the conditions (TD1)-(TD4), $(T, \mathcal{Y})$ also satisfies the following three conditions:

(TD5) if t and t' are distinct vertices of T, then $Y_t \neq Y_{t'}$;

(TD6) if t' is a vertex of T and B is a component of $T - t'$, then $\bigcup_{t \in V(B)} (Y_t - Y_{t'}) \neq \emptyset$, and;

(TD7) if t is a vertex of T, B is a component of $T - t$, and u and v are B-tied vertices in Y_t, then there is a B-confined path in G between u and v.

The conditions (TD4)–(TD7) impose a more rigorous structure on tree-decompositions, which will be essential to the proof of Conjecture 4.3. We are able to consider lean tree-decompositions instead of tree-decompositions thanks to the following result of Oporowski, Oxley, and Thomas [20].

Theorem 4.4. *If a graph has a tree-decomposition of width at most w, where w is integer, then that graph has a lean tree-decomposition of width at most w.*

Lean tree-decompositions are important in the proof of Theorem 4.1 as well as the partial proof of Conjecture 4.3.

4.3 Partial proof of Conjecture 4.3

We proceed with the proof of Conjecture 4.3 by investigating the tree-width of a large 4-edge-connected graph G. In case G has bounded tree-width, we are able to use Theorem 2.2 to show that tree-decomposition of G either has a high degree vertex or a long path. A lemma and a conjectured lemma are presented the former and latter cases, respectively. The proof of the conjecture lemma would complete the proof of Conjecture 4.3. We begin with a corollary of Theorem 1.8 which will be used the the proof.

Corollary 4.5. *Let k be a positive integer. Then there is a function $f_{4.5}$ such that every graph of tree-width at least $f_{4.5}(k)$ admits an immersion of $C_{2,k}$.*

We now proceed with a lemma and conjectured lemma, and proof of the former.

Lemma 4.6. *Let G be a 4-edge-connected graph such that the tree width of G is less than $f_{4.5}(k)$, let $(T, \mathcal{Y})$ be a lean tree-decomposition of G, and let t be a vertex of T with degree at least:*

$$n = \left(\binom{f_{4.5}(k)}{4} + 3\binom{f_{4.5}(k)}{3} + 3\binom{f_{4.5}(k)}{2} + f_{4.5}(k) \right) (9k-1) + 1.$$

Then G admits an immersed $C_{2,k}$.

Proof. Let G, $(T, \mathcal{Y})$, n, and t be as decribed in the theorem statement. Label the neighbors of t by $t_1, t_2, \ldots, t_n$. Consider the parts Y_t and Y_{t_i} for some integer $i \in [n]$. As $(T, \mathcal{Y})$ is lean, there is a vertex v_i in $Y_{t_i} - Y_t$. By Menger's Theorem (Theorem 0.2), there are four edge-disjoint paths, $P_{i,j}$ for $j \in [4]$, between v_i and Y_{t_i}. Let B_{t_i} be the component of $T - t$ containing t_i. Then by the definition of lean tree-decomposition, $P_{i,j}$ is contained, for all i and j, in B_{t_i} with the possible exception of its end-vertex in Y_t.

Denote the set of end-vertices of the paths $P_{i,j}$ by S_i, and note that the cardinality of S_i is either one, two, three, or four. Then for some integer n, as the degree of t is at least n and $|Y_t| \leq f_{4.5}$, there is a set of vertices S_n such that for at least $9k$ many parts of G, the end vertices of the set of paths $P_{i,j}$ is equal to S_n. That is, $S_i = S_n$ for at least $9k$ integers i. Without loss of generality, suppose that $S_i = S_n$ for all $i \in [9k]$, and let $S = \{S_i\}_{i \in 9k}$. We consider four cases where the cardinality of S_n is either one, two, three, or four.

First suppose $|S_n| = 1$. Consider the subgraph of G induced by the paths $P_{i,j}$ for $i \in [4k]$ and $j \in [4]$. Let G' be the graph obtained from this subgraph by lifting any paths $P_{i,j}$ for which the length of $P_{i,j} \geq 2$. Then G' contains a subgraph isomorphic to $S_{4,4k}$, and by extension, a subgraph isomorphic to $S_{4,k}$. Let v denote the vertex of $S_{4,k}$ with degree $4k$, and $v_1, v_2, \ldots, v_k$ denote the neighbors of v. Then by lifting two of the paths $v_i, v, v_i + 1$ for each i and then deleting v, we find that $C_{2,k} \leq_{\text{im}} S_{4,k}$. Therefore, $C_{2,k} \leq_{\text{im}} G$.

Next we suppose that $|S_n| = 2$. Label the vertices of S_n as s_1 and s_2. As there are four edge-disjoint paths from v_i to S_n, and S_n has cardinality two, there either must be a vertex of S_n that is an end-vertex of three of the paths $P_{i,j}$ or each vertex of S_n is an end-vertex of two paths $P_{i,j}$. Since $|S| \geq 9k$, by the pigeonhole principle, there are at least $2k$ parts Y_{t_i} such that there are three paths $P_{i,j}$ which

all share a common end-vertex in S_n, or $2k$ parts Y_{t_i} for which exactly two sets of two paths $P_{i,j}$ share a common end-vertex in S_n.

Consider the former case. Again by the pigeonhole principle, as there are $2k$ parts Y_{t_i} such that there are three paths $P_{i,j}$ which all share a common end-vertex in S_n, there are at least $2k$ parts Y_{t_i} such that v_i is joined by three of the paths $P_{i,j}$ to, without loss of generality, s_1. Without loss of generality, suppose that the parts Y_{t_i} for $i \in [2k]$ are such that v_i is joined to s_1 by the paths $P_{i,1}$, $P_{i,2}$, and $P_{i,3}$, and to s_2 by the path $P_{i,4}$. Consider the subgraph of G induced by the paths $P_{i,j}$ and let G' be the graph obtained from this subgraph, by lifting any paths $P_{i,j}$ of length at least two. Then G' contains a subgraph isomorphic to $K_{(1,0,1),2k}$. We claim that $S_{4,k}$ is immersed in $K_{(1,0,1),2k}$.

Let s_1 be the vertex of $K_{(1,0,1),2k}$ joined to each vertex in the partition of order $2k$ by parallel classes of size 3, and s_2 the other vertex of the bipartition of order two. Let $v_1, \ldots, v_{2k}$ denote the vertices of the bipartition of order $3k$. Let G'' be the graph obtained from $K_{(1,0,1),2k}$ by lifting the paths s_1, v_{i+1}, s_2, v_i for which $i \in [2k]$ is odd, then deleting the vertices s_2 and v_j for which $j \in [2k]$ is even. Then G'' is isomorphic to $S_{4,k}$. By the reasoning given in the first case, $C_{2,k} \leq_{\text{im}} S_{4,k}$, and hence by transitivity, $C_{2,k} \leq_{\text{im}} G$.

Now suppose, instead, that there are $2k$ parts Y_{t_i} for which exactly two sets of two paths $P_{i,j}$ share a common end-vertex in S_n. By the pigeonhole principle and without loss of generality, we can assume that there are at least $2k$ parts, Y_{t_i} for which v_i is joined to s_1 by the paths $P_{i,1}$ and $P_{i,2}$, and joined to s_2 by the paths $P_{i,3}$ and $P_{i,4}$. Again without loss of generality, suppose that Y_{t_i} for $i \in [2k]$ are parts with the this property. Consider the subgraph of G induced by the paths $P[i, k]$ for $i \in [2k]$,

and let G' be the graph obtained from this subgraph by lifting any paths $P_{i,j}$ of length at least two. Then G' contains a subgraph isomorphic to $K_{(0,2),2k}$. We claim that $S_{4,k}$ is immersed in $K_{(0,2),2k}$. Let s_1 and s_2 be the vertices of the bipartition of order two. Let $v_1, \dots, v_{2k}$ denote the vertices of the bipartition of order $2k$. Note that for each odd integer $i \in [2k]$ there are two edge-disjoint paths from s_1 to v_i through the vertices v_{i+1} and s_2. Let G'' be the graph obtained from $K_{(0,2),2k}$ by lifting these paths, then deleting the vertices s_2 and v_j for which $j \in [2k]$ is even. Then G'' is isomorphic to $S_{4,k}$. By the reasoning given above, $C_{2,k} \leq_{\text{im}} S_{4,k}$, and hence by transitivity, $C_{2,k} \leq_{\text{im}} G$.

Instead, consider the case where $|S_n| = 3$. As there are four edge-disjoint paths from v_i to S_n, and S_n has cardinality three, there must be a vertex of S_n that is an end-vertex of two of the paths $P_{i,j}$. Let s_1, s_2 and s_3 be the vertices of S_n. Since $|S| \geq 9k$, by the pigeonhole principle and without loss of generality, there are at least $3k$ members of S such that the vertex s_1 is an end-vertex of the paths $P_{i,1}$ and $P_{i,2}$, s_2 is an end-vertex of the path $P_{i,3}$, and s_3 is an end-vertex of $P_{i,4}$.

Consider the subgraph of G induced by the paths $P_{i,j}$ and let G' be the graph obtained from this subgraph, by lifting any paths $P_{i,j}$ of length at least two. Then G' contains a subgraph isomorphic to $K_{(2,1),3k}$. We claim that $S_{4,k}$ is immersed in $K_{(2,1),3k}$. Let s_1 be the vertex of $K_{(2,1),3k}$ joined to each vertex in the partition of order $3k$ by parallel classes of size 2, and s_2 and s_3 be the other vertices of the bipartition of order three. Let $v_1, \dots, v_{3k}$ denote the vertices of the bipartition of order $3k$. Let G'' be the graph obtained from $K_{(2,1),3k}$ by lifting the paths s_1, v_{i+1}, s_2, v_i and s_1, v_{i+2}, s_3, v_i for $i \in [3k], i \equiv 1 \pmod 3$, then deleting the vertices s_2, s_3, and v_j for $j \in [3k], j \not\equiv 1 \pmod 3$. Then G'' is isomorphic to $S_{4,k}$. By the reasoning above, $C_{2,k} \leq_{\text{im}} S_{4,k}$, and hence by transitivity, $C_{2,k} \leq_{\text{im}} G$.

Finally, suppose $|S_n| = 4$. Label the vertices of S_n as s_1, s_2, s_3, and s_4. Consider the subgraph of G induced by the paths $P_{i,j}$ and let G' be the graph obtained from this subgraph, by lifting any paths $P_{i,j}$ of length at least two. Then G' contains a subgraph isomorphic to the graph $K_{4,9k}$, and therefore a subgraph isomorphic to $K_{4,4k}$. We claim that $S_{4,k} \leq_{\text{im}} K_{4,4k}$. Let the vertices of the bipartition of order $4k$ be labeled $v_1, v_2, \ldots, v_{4k}$ and the vertices of the remaining bipartition labeled $s_1, \ldots, s_4$. For $i \in [4k], i \equiv 1 \pmod 4$, lift the path s_1, v_{i+1}, s_2, v_i, the path s_1, v_{i+2}, s_3, v_i, and the path s_1, v_{i+3}, s_4, v_i, then delete s_2, s_3, s_4 and v_i for $i \not\equiv 1 \pmod 4$. The resulting graph is isomorphic to $S_{4,k}$, and therefore $S_{4,k} \leq_{\text{im}} K_{4,4k}$. By the reasoning above, we also have that $C_{2,k} \leq_{\text{im}} K_{4,4k}$ and hence, by transitivity, $C_{2,k} \leq_{\text{im}} G$. □

Conjecture 4.7. *Let G be a 4-edge-connected graph such that the tree width of G is less than $f_{4.5}(k)$ and let $(T, \mathcal{Y})$ be a lean tree-decomposition of G. Then there is an integer m such that if P is a path of T such that P has length at least m, then G admits an immersed $C_{2,k}$.*

We now use Lemma 4.6 and Conjecture 4.7 to give a partial proof of Conjectured 4.3.

Proof of Conjecture 4.3. Let $k \geq 4$ be an integer, n and m be as described in Lemma 4.6 and Conjectured Lemma 4.7 respectively, and G a 4-edge-connected graph of order at least $f_{4.3}(k) = \big(f_{2.2}(\max\{m,n\})\big)\big(f_{4.5}(k) - 1\big)$. First, suppose that G has tree-width at least $f_{4.5}(k)$. Then as the maximum degree of $C_{2,k}$ is four, by Theorem 4.5, $C_{2,k}$ is immersed in G.

Suppose then, that the tree width of G is at most $f_{4.5}(k) - 1$. This implies that G admits a tree-decomposition of width at most $f_{4.5}(k) - 1$. By Theorem 4.4, G also admits a lean tree-decompositions $(T, \mathcal{Y})$ of width $f_{4.5}(k) - 1$. Since G has

order at least $f_{4.3}(k)$ and $\mathrm{tw}(G)$ is at most $f_{4.5}(k) - 1$, the order of T must be at least $f_{2.2}(\max\{m, n\})$. Thus, T either contains a path of length $\max\{m, n\}$ or has a vertex of degree at least $\max\{m, n\}$. Then, by Lemma 4.6 and Conjectured Lemma 4.7, G admits an immersed $C_{2,k}$. $\square$

Chapter 5
Immersion Intertwines

5.1 Intertwines

In this chapter, we concern ourselves primarily with the strong immersion relation. As such, we will omit the strong and strongly from the phrases strong immersion, strongly liftable, and strong lifting and instead specify weak if we are referring to the weak immersion relation. As we also consider infinite graphs in this chapter, we begin with a few definitions. If $\varphi : H \to G$ is an immersion and $\varphi|_{V(H)}$ is a bijection such that two vertices, v and v', of H are adjacent if and only if their images, $\varphi(v)$ and $\varphi(v')$, are adjacent in G, then we say that φ induces an isomorphism between H and G; otherwise φ is *proper*. If $H = G$, then φ is a *self-immersion*, and, if additionally, it induces the identity map, then it is *trivial*. Furthermore, a self-immersion of G is proper if and only if at least at least one vertex or edge has been deleted, or at least one path has been lifted. That is, if V, E, and S are the set of deleted vertices, deleted edges, and lifted paths, respectively, of G corresponding to a self immersion of G, then that self-immersion of G is proper if and only if at least one of the sets V, E, and S is nonempty.

Suppose $(\mathscr{G}, \leq)$ is a quasi-order and G_1 and G_2 are two elements of $\mathscr{G}$. An *intertwine* of G_1 and G_2 is an element G of $\mathscr{G}$ satisfying the following conditions:

- $G_1 \leq G$ and $G_2 \leq G$, and
- if $G' \leq G$ and $G \nleq G'$, then $G_1 \nleq G$ or $G_2 \nleq G$.

The class of all intertwines of G_1 and G_2 is denoted by $\mathscr{I}_{\leq}(G_1, G_2)$. A quasi-order $(\mathscr{G}, \leq)$ satisfies the *finite intertwine property* if for every pair G_1 and G_2 of elements of $\mathscr{G}$, the class of intertwines $\mathscr{I}_{\leq}(G_1, G_2)$ has no infinite antichains. It is clear that if $(\mathscr{G}, \leq)$ is a well-quasi-order, then it also satisfies the finite intertwine

property. However, it is well known that the converse is not true; for example, see [19].

As mentioned in Chapter 1, Thomas proved [27] that $(\mathscr{G}_\infty, \leq_{\mathrm{m}})$, where $\leq_{\mathrm{m}}$ denotes the minor relation, is not a well-quasi-order. Oporowski proved [18] the existence of an infinite graph which contains no proper minor isomorphic to itself, and later showed [19] that $(\mathscr{G}_\infty, \leq_{\mathrm{m}})$ does not satisfy the finite intertwine property. Andreae showed [1] that $\mathscr{G}_\infty$ is a well-quasi-order under neither the strong nor weak immersion relations and gave a construction [2] for an infinite graph admitting only the trivial self-strong-immersion.

In a result analogous to [19], we strengthen Andreae's result by showing that $(\mathscr{G}_\infty, \leq_{\mathrm{im}})$ does not satisfy the finite intertwine property. In particular, we construct two graphs G_1 and G_2, and an infinite class $\mathscr{F}$ in $\mathscr{G}_\infty$ such that:

(IT1) $\mathscr{F}$ is an immersion antichain;

(IT2) every graph in $\mathscr{F}$ is connected;

(IT3) both G_1 and G_2 are subgraphs of each graph in $\mathscr{F}$;

(IT4) if G' is properly immersed in a graph G in $\mathscr{F}$, then $G_1 \not\leq_{\mathrm{im}} G'$ or $G_2 \not\leq_{\mathrm{im}} G'$.

Note that (IT3) implies that G_1 and G_2 are immersed in G. Hence, the existence of graphs G_1, G_2 and a class of graphs $\mathscr{F}$ satisfying (IT1)–(IT4) implies the following statement, which is the main result of this chapter.

Theorem 5.1. *The quasi-order $(\mathscr{G}_\infty, \leq_{\mathrm{im}})$ does not satisfy the finite intertwine property.*

5.2 The Construction of an Infinite Family of Intertwines

We will exhibit two graphs G_1 and G_2 in $\mathscr{G}_\infty$ such that $\mathscr{I}_{\leq_{\mathrm{im}}}(G_1, G_2)$ is infinite. The construction of G_1 and G_2 begins with the following results, which are immediate consequences of, respectively, Lemma 4, and Lemmas 1 and 2 of [2].

Theorem 5.2. *There is an infinite set $\mathscr{H}$ of pairwise-disjoint infinite blobs such that $|H| \leq |\mathscr{H}|$ for all $H \in \mathscr{H}$, and $\mathscr{H}$ forms an immersion antichain.*

Theorem 5.3. *Given an immersion antichain $\mathscr{H}$ of pairwise-disjoint infinite blobs such that $|H| \leq |\mathscr{H}|$ for all $H \in \mathscr{H}$, there is a connected graph G such that the set of blobs of G is $\mathscr{H}$ and G admits no self-immersion except for the trivial one.*

Let $\mathscr{H}$ be an antichain as described in Theorem 5.2. For the exact construction of the graph G in Theorem 5.3 we refer the reader to Lemma 1 of [2]. However, we note that it is important for the proof of Theorem 5.1 that every edge not contained in a blob of G is a cut-edge with the end-vertices in different members of $\mathscr{H}$. Partition $\mathscr{H}$ into countably many sets $\{\mathscr{H}_i\}_{i\in\mathbb{Z}}$ with the cardinality of each $\mathscr{H}_i$ equal to $|\mathscr{H}|$. Then, by Theorem 5.3, for each $i \in \mathbb{Z}$, there is a connected graph B_i whose set of blobs is $\mathscr{H}_i$, and that admits no proper self-immersion. Furthermore, Lemma 1.9 implies that if i and j are distinct integers, then $B_i \not\leq_{\mathrm{im}} B_j$, as no blob of B_i is immersed in a blob of B_j. Therefore, the set of graphs $\{B_i\}_{i\in\mathbb{Z}}$ is an immersion antichain.

For each graph B_i, label one vertex u_i. Let P be a two-way infinite path with vertices labeled $\{v_i\}_{i\in\mathbb{Z}}$ such that, for each integer i, the vertex v_i is adjacent to v_{i+1} and v_{i-1}. We construct the graph G_1 by taking the disjoint union of P and the graphs B_i for which i is odd, and then identifying the vertices u_i and v_j for $i = j$. Similarly, we construct the graph G_2 by taking the disjoint union of P and

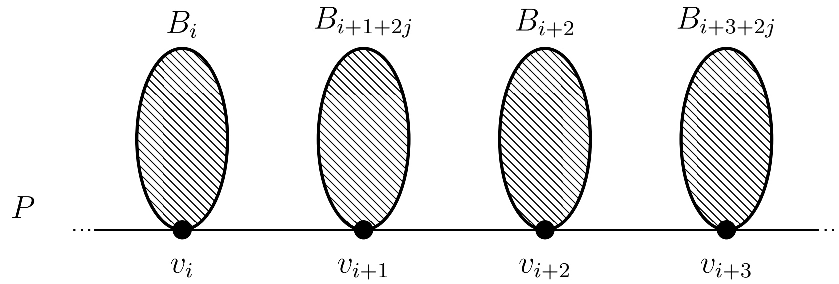


Figure 5.1. The graph F_j, an intertwine of G_1 and G_2.

the graphs B_i for which i is even, and then identifying the vertices u_i and v_j for $i = j$.

Now let j be an integer. Take the disjoint union of G_1 and all the graphs B_i for which i is even. Then, for each even integer i, identify the vertex v_i of G_1 with the vertex u_{i+2j} of the graph B_{i+2j}. Let F_j be the resulting graph (see Figure 5.1) and define $\mathscr{F}$ as the set $\{F_j\}_{j\in\mathbb{Z}}$.

The following lemma immediately implies the main result, Theorem 5.1.

Lemma 5.4. *The set of graphs $\mathscr{F} = \{F_j\}_{j\in\mathbb{Z}}$ is an immersion antichain. Furthermore, each $F_j \in \mathscr{F}$ is an immersion intertwine of the graphs G_1 and G_2.*

Proof. Let j be an integer. It is easy to see that F_j satisfies (IT2) and (IT3). Therefore, in order to show that F_j is an immersion intertwine of G_1 and G_2, it suffices to prove that it also satisfies (IT4).

Suppose, for contradiction, that F_j' is a graph that is properly immersed in F_j via a map φ, and both G_1 and G_2 are immersed in F_j'. Then we can obtain F_j' from F_j by deleting a set of vertices V, deleting a set of edges E, and then lifting a liftable set of paths S, with at least one of these sets being nonempty. We consider two cases depending on whether there is an integer i for which B_i meets $V \cup E \cup S$.

First, assume that no B_i meets $V \cup E \cup S$. Then the sets V and S are empty, as all the vertices of F_j are contained in the subgraphs $\{B_n\}_{n\in\mathbb{Z}}$, and E consists of some edges of P.

Suppose the edge $e = \{v_k, v_{k+1}\}$ is in E where k is odd; the argument is symmetric when k is even. The graph $F_j \setminus e$ has exactly two components, with the subgraphs B_k and B_{k+2} in distinct components. Label the component containing B_k as C_1 and the component containing B_{k+2} as C_2.

Let A be a blob of B_k. As A and each blob of C_2 are members of the antichain $\mathscr{H}$, by Lemma 1.9, we have $A \not\leq_{\mathrm{im}} C_2$. Hence, by transitivity, $B_k \not\leq_{\mathrm{im}} C_2$. It follows similarly that $B_{k+2} \not\leq_{\mathrm{im}} C_1$. But as G_1 is connected and the only components of $F_j \setminus e$ are C_1 and C_2, we have that $G_1 \not\leq_{\mathrm{im}} F_j \setminus e$. Furthermore, as $F'_j \leq_{\mathrm{im}} F_j \setminus e$, by transitivity, $G_1 \not\leq_{\mathrm{im}} F'_j$; a contradiction.

Now suppose that, for some odd integer i, the graph B_i meets $V \cup E \cup S$; again, the argument is symmetric if i is even. Let the set V_{B_i} denote the vertices of V contained in B_i. Similarly, define E_{B_i} to be the set of edges contained in both B_i and E, and S_{B_i} to be the set of subgraphs of the paths of S which are entirely contained in B_i. We note that, due to the construction of F_j, the graphs of S are paths themselves. Let F''_j be the graph obtained from F_j by lifting S_{B_i}, deleting V_{B_i}, and deleting E_{B_i}, and let B'_i be the graph obtained from B_i by lifting S_{B_i}, deleting V_{B_i}, and deleting E_{B_i}. Then by our assumption, we have that $B_i \subset G_1 \leq_{\mathrm{im}} F'_j \leq_{\mathrm{im}} F''_j \lneq_{\mathrm{im}} F_j$. In particular, let $\tau : B_i \to F''_j$ be the immersion of B_i into F''_j and let $\gamma : B'_i \to B_i$ be the immersion of B'_i into B_i corresponding to V_{B_i}, E_{B_i}, and S_{B_i}.

Consider the image of B_i under τ. We claim that $\tau(B_i) \subset (B'_i)$. Suppose not. Then either there is a vertex $v \in V(B_i)$ such that $\tau(v) \not\subset V(B'_i)$ or edge $e \in E(B_i)$ such that $\tau(e) \not\subset \mathscr{P}(B'_i)$. Assume the former. As every vertex of B_i is contained in a blob of B_i by its construction, v is contained in some blob $A \subset B_i$. By Lemma

1.9, the image of A under τ must be contained in some blob A' of F_j''. As each blob of F_j'' not contained in B_i is a member of the antichain $\mathscr{H}$ distinct from A, we have that A' and A are distinct members of $\mathscr{H}$ and $A \leq_{\text{im}} A'$; a contradiction.

Then it must be that there is some edge $e \in E(B_i')$ such that $\tau(e) \not\subset \mathscr{P}(B_i')$. If e is contained in some blob of B_i we arrive at a similar contradiction. Therefore, e must be an edge of B_i not contained in any blob of B_i. Thus e must be a cut-edge of B_i joining two blobs A_1 and A_2. Let $a_1 \in V(A_1)$ and $a_2 \in V(A_2)$ be the end-vertices of e. Then $\tau(e)$ a path in F_j'' joining $\tau(a_1)$ and $\tau(a_2)$. By the argument above, both $\tau(a_1)$ and $\tau(a_2)$ are contained in B_i'. The path $\tau(e)$ must contain an edge not contained in B_i'. However, the only edges not contained in B_i' with an end-vertex in B_i' are edges of P, which are cut-edges; a contradiction as both end-vertices of $\tau(e)$ are contained in B_i'. Therefore, $\tau(e)$ must be entirely contained in B_i.

Thus, $\tau(B_i) \subset B_i'$, and by extension, $B_i \leq_{\text{im}} B_i'$ via τ. As $V_{B_i} \cup E_{B_i} \cup S_{B_i}$ is non-empty by assumption, the immersion $\gamma : B_i' \to B_i$ is proper. Therefore, since B_i is mapped by τ to some, not necessarily proper, subset of B_i', we have $B_i \lneq_{\text{im}} B_i$ via the composition $\gamma \circ \tau$; a contradiction. Hence, $G_1 \nleq_{\text{im}} F_j''$ and, by transitivity, $G_1 \nleq_{\text{im}} F_j'$.

Hence, $\mathscr{F}$ satisfies the condition (IT4).

To show that $\mathscr{F}$ is an antichain in $(\mathscr{G}_\infty, \leq_{\text{im}})$, suppose that F_i is immersed in F_j for some distinct integers i and j. By construction, F_i and F_j are not isomorphic. Therefore, F_i is properly immersed in the intertwine F_j and so either $G_1 \nleq_{\text{im}} F_i$ or $G_2 \nleq_{\text{im}} F_i$. But both G_1 and G_2 are immersed in F_i by construction; a contradiction. The conclusion follows. $\square$

Chapter 6
Conclusion

The study of graph immersion is among the newer ares of study in graph theory. Compared to much of mathematics it has barely started its development. However, the results of the last century in graph minors have laid significant groundwork for productive study of the immersion relation. The methods developed by Robertson, Seymour, and the many others who have studied structural graph theory are being adapted to fit the nature of the immersion relation. While some results in graph immersions have followed closely to their minor counterparts, such as Theorem 1.3 and Theorem 1.6, other questions have taken longer to resolve or still remain unanswered. Here we present some open areas of study in graph immersions, including questions raised by the main results of this dissertation.

6.1 Unavoidable Immersions of 4-Edge-Connected Graphs

The first open question considered is Conjecture 4.3 regarding the unavoidable immersion of 4-edge-connected graphs. As stated in Chapter 4, Conjectured Lemma 4.7 is the sole remaining piece needed to complete the proof of Conjecture 4.3. While the proof of Theorem 2.6 seems to offer a suitable method, the decomposition theorem for 3-connected graphs and the internal-4-connectivity condition of Theorem 4.1 make a proof using this method too complex. The analysis of the long path in the tree-decomposition also becomes more complex when considering edge-connectivity as opposed to vertex-connectivity,. However, the structure given by the tree decomposition seems to be the a promising approach once the proof method of Oporowski, Oxley, and Thomas [20] can be modified to accommodate edge-connectivity as opposed to vertex-connectivity. This adaptation is what I believe to be the only barrier to a proof of 4.7, and therefore a complete proof of Conjecture 4.3.

6.2 Well-Quasi-Orderings

The second question we pose arises from the work of Andreae and the main result of Chapter 5, Theorem 5.1. While Andreae showed that the class of infinite graphs is not a well-quasi-order with the strong immersion relation, the antichain provided as a counterexample has an uncountable vertex set. As Theorem 5.3 and Theorem 5.1 rely on the existence of this antichain, these results both refer specifically to the class of infinite graphs containing uncountably large graphs. It is still unknown whether the class of countable graphs is a well-quasi-order with the strong immersion relation. It is also not known whether there is a countably infinite graph that does not admit a proper strong self-immersion, or whether the class of countable graphs with the strong immersion relation satisfies the finite intertwine property. If the existence of a such an antichain and a countably infinite graph with no proper self immersion constructed from that antichain similar to the construction in [2], the methods present in Chapter 5 would show that the class of countable graphs with the strong immersion property does not have the finite intertwine property. In addition, the question of the well-quasi-ordering of infinite graphs (both countable and uncountable) remains open for the weak immersion relation. The questions of an infinite graph admitting no proper weak self-immersion and the class of infinite graphs with the weak immersion having finite intertwine property also remain open, with little progress having been made towards a solution.

Another open problem is the question of whether the class of finite graphs is a well-quasi-order with the strong immersion relation. Despite the success of the Graph Minors Project and the construction of an strong immersion antichain of infinite graphs, Nash-Williams' conjecture for the strong immersion relation remains open. Robertson and Seymour state in [26] "it seemed to us at one time that we had a proof of the stronger, but even if it was correct it was very much more

complicated, and it is unlikely that we will write it down". Despite this, there has been a renewed interest in the last year in answering Nash-Williams' conjecture for strong immersions.

6.3 Excluded Immersions and Excluding Fixed Graphs

There are also a number of results in graph minors that have unproven analogues in graph immersions. Theorem 1.6 states that there are a finite number of excluded immersions for any immersion-closed class of graphs. While several characterizations for minor-closed classes exist (such as planar embeddability, projective embeddability, and bounded tree-width), little has been done to find the excluded immersions for immersion-closed classes of graphs. This is partially due to the stark differences between minor-closed properties of graphs and immersion-closed properties. For example, as mentioned before, embeddability is not closed under the strong or weak immersion operation. For the planar case this is easy to see as K_5 can be immersed in an 6×6 grid (see Figure 2). In the case of bounded tree-width, the graph $S_{n,k}$ has tree-width one. However, $S_{n,k}$ admits an immersion of the complete graph $K_{\min\{n,k\}}$, which has tree-width $\min\{n,k\} - 1$. Therefore, the class of graphs with bounded tree-width is not immersion closed either.

A class of graphs which is closed under the immersion relation is the class of graphs with pathwidth less than or equal to a positive integer n. The pathwidth of a path embedding of a graph G is obtained by injecting the vertex set of G onto the vertex set of a path P, and taking the maximum number of edges of G between components of $P - e$ for all $e \in E(G)$. The pathwidth of G is the minimum pathwidth of an path embedding of G onto P over all possible path embeddings. Another immersion-closed class is the class of graphs with congestion less than or equal to n. Congestion is defined similarly to path width, though the vertices of G are mapped to the leaves of a subcubic tree instead of a path. For very small values

of n, some results are known or trivial. For example, the excluded immersion for the class of graphs with congestions at most two is K_4, however, little else is known about the exluded immersions of pathwidth or congestion for great values of n.

Similarly, the classification of graphs that exclude a fixed graph as an immersion has not yet been studied in great depth, although there has been more progress than in characterizing immersion-closed classes with excluded immersions. In a theorem closely related to Theorem 1.5, Marx and Wollen [14] showed that for a fixed graph H, a large enough graph with maximum degree at least the maximum degree of H either admits an immersion of H or has bounded tree-cut-width. In a more constructive result, Belmonte, Giannopoulou, Lokshtanov, and Thilikos [4] were able to determine that every graph not admitting an immersion W_4, the wheel on five vertices, can be construction via 1-, 2- and 3-edge-sums of subcubic graphs and graphs of bounded tree width. Similarly, Giannopoulou, Kamiński, and Thilikos [11] gave a constructive characterization for graphs not admitting immersions of K_5 or $K_{3,3}$ as 1-, 2-, and 3-edge sums of planar subcubic graphs and graphs of branchwidth at most 10. While these results constitute a significant number of the results in graph immersions, the number of similar graph minors results is far greater, and much work remains to be done in classifying graphs excluding an immersed fixed graph.

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