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Backstepping and Sequential Predictors for Control Systems

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BACKSTEPPING AND SEQUENTIAL PREDICTORS FOR CONTROL SYSTEMS

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Abstract

We provide new methods in mathematical control theory for two significant classes of control systems with time delays, based on backstepping and sequential prediction. Our bounded backstepping results ensure global asymptotic stability for partially linear systems with an arbitrarily large number of integrators. We also build sequential predictors for time-varying linear systems with time-varying delays in the control, sampling in the control, and time-varying measurement delays. Our bounded backstepping results are novel because of their use of converging-input-converging-state conditions, which make it possible to solve feedback stabilization problems under input delays and under boundedness conditions on the feedback control. Our sequential predictors work is novel in its ability to cover time-varying measurement delays and sampling which were beyond the scope of existing sequential predictor methods for time-varying linear systems, and in the fact that the feedback controls that we obtain from our sequential predictors do not contain any distributed terms.

Chapter 1. Background

In this chapter, we introduce the basic concepts required for this dissertation. This dissertation is in the area of control theory, which is a research area at the interface of applied mathematics and engineering that studies classes of dynamical systems that contain forcing functions. Dynamical systems of this type are called control systems. For dynamical systems that are represented as systems of ordinary differential equations, these forcing functions are usually represented as nonconstant parameters in the right side. The forcing functions can depend on the state of the dynamical system and on time, and are called controls. Much of control theory is devoted to control design, which entails finding formulas for the controls that ensure that all solutions of the dynamical system enjoy some desired prescribed behavior, such as asymptotic convergence towards an equilibrium point, or the requirement that all solutions of the system remain in some region of interest in the state space at all nonnegative times. Control systems are often used in biological and engineering models, where the controls can represent forces that can be applied to a physical system or the effects of feeding organisms in their habitat.

There is now a large well known literature on control design for cases where the current state of the dynamical system is available for use in the control; see for instance the standard text [26] by Khalil that is often used in graduate nonlinear control courses in engineering departments, and the more recent shorter version [27]. When the controls only depend on time, they are called open loop controls. Controls that depend on states of the dynamical systems are called feedback controls; see [3] for a laypersons' introduction to feedback control. However, in many engineering applications, measurements of the current state of the system may not be available for use in the control. In engineering applications, this can arise from

time lags in the process that measures the state of the system, or from time delays in the communication from the sensors (which measure the state of the system) to the actuators (which apply the control forces to the system), and then the control system becomes a forced delay system.

There is a large literature on delay systems, which can be viewed as special cases of functional differential equations; see for instance [19] for functional differential equations without control designs, and the books [17, 18, 28, 68] on delayed control systems. One natural approach to solving feedback control design problems under delays is to solve the prescribed stabilization problem with the delays set equal to zero to obtain a feedback control that depends on current state values, and then to replace the current state values in the control by time delayed state values, which is called the emulation approach. However, the emulation approach can lead to bounds on the allowable delays that are too small for the application under consideration. This led to a large literature on control designs where the delay values are used in the control design process, which are called delay compensating controllers. The delay compensating control literature began with the pioneering work of Smith [56] in the 1950s on the Smith predictor, and the reduction model approach that was first developed by Artstein and Mayne in [4, 32], and that was further developed by many others, leading to many advances including the prediction approaches that were studied by Krstic and his collaborators in [25], [28], and other works.

However, delay compensating controls can be challenging to implement in engineering applications, since they usually contain distributed terms where the control formulas contain integrals of the control on certain intervals, or the controls may be only implicitly defined by an integral equation that does not admit an explicit closed form solution; see [43]. This was one motivation for the sequential predictor

approach from [8], where instead of distributed terms, the feedback control depends on the state of a dynamic extension. The dynamic extensions in sequential prediction consist of stacks of ordinary differential equations, which include copies of the original system running on different time scales combined with additional stabilizing terms. On the other hand, even if a control system has no delays, it may be helpful to introduce delays into the feedback, by having the feedback depend on several time lagged state values of the state, which is the artificial delay approach that was used in [36]. The artificial delay approach can be useful for satisfying requirements that the feedback control take all of its values in a suitable bounded set, which are called input constraints.

This dissertation will present two new classes of feedback designs, with the first using artificial delays in a new way to obtain bounded controls, and the second providing sequential predictors under input delays (which are also called feedback delays), measurement delays, and sampling in the controller. Measurement delays result in additional delays in the sequential predictors, and these additional delays are used to more faithfully model time lags in the communication from the physical plant to the controller. Sampling in controls is used to model cases where the state values may only be available at discrete time instants instead of being available for continuous measurement. Although sampling usually occurs when control designs are applied in engineering systems, we believe that our sequential predictors work is the first to analyze the effects of sampling in controls for time-varying systems in a theorem, without imposing the delay bounds that would arise from the emulation approach. Before presenting our new developments, we use the rest of this chapter to provide basic definitions that we use in much of the sequel.

Throughout the dissertation, we confine our analysis to control systems that are governed by systems of ordinary differential equations of the form

$$\dot{x}(t) = \mathcal{F}_0(t, x_t, \delta(t)) \tag{1.1}$$

having a subset of Euclidean space as its state space, where δ is an unknown measurable locally essentially bounded function that is also valued in a subset of Euclidean space, and where the functions x_t are defined by $x_t(s) = x(t+s)$ for all values of $t \geq 0$ and $s \leq 0$ such that $t+s$ is in the domain of x. Here and in the sequel, the dimensions of our Euclidean spaces are arbitrary unless otherwise noted. Additional conditions on the vector field $\mathcal{F}_0$ ensure that $x(t)$ is uniquely defined for all $t \geq t_0$ for all choices of the initial function x_{t_0}, all choices of the initial times $t_0 \geq 0$, and all choices of δ, which is the forward completeness property. Later in the dissertation, we discuss these additional conditions in those places where forward completeness conditions are relevant, and we assume for simplicity in the sequel that the initial time t_0 is always zero. The functions x_t can represent the effects of delayed values in feedback controls, in cases where

$$\mathcal{F}_0(t, x_t, \delta(t)) = G(t, x_t, u(t, x_t), \delta(t)) \tag{1.2}$$

for some control u and some function G, in which case (1.1) is called a closed loop system, and then we say that

$$\dot{x}(t) = G(t, x_t, u(t, x_t), \delta(t)) \tag{1.3}$$

is a control system in closed loop with the control u. To define a solution of (1.1), one needs the initial function x_0 to be defined on $[-T, 0]$ where T is an upper bound on the delays. One desirable property for systems of the form (1.1) is a delayed version of the input-to-state stability (or ISS) property that was introduced by

Sontag in his well known paper [57]. The ISS property generalizes uniform global asymptotic stability, in a way that ensures boundedness of all solutions of the system over $[0, +\infty)$ when δ is bounded; see [59] for more motivation for ISS. In order to define ISS for (1.1), we need the following preliminary definitions.

We use $|\cdot|$ to denote the usual Euclidean norm and the induced matrix norm, and $|\phi|_\infty$ (resp., $|\phi|_\mathcal{I}$) is the essential supremum (resp., supremum over any interval $\mathcal{I}$) for any bounded (resp., locally bounded) $\mathbb{R}^n$ valued measurable function ϕ. We use C^0 to mean continuous. We say that a C^0 function $\gamma : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}$ and write $\gamma \in \mathcal{K}$ provided it is strictly increasing and $\gamma(0) = 0$. We say that it is of class $\mathcal{K}_\infty$ if, in addition, $\gamma(r) \to +\infty$ as $r \to +\infty$. We say that a C^0 function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ belongs to class $\mathcal{KL}$ provided for each fixed $s \geq 0$, the function $\beta(\cdot, s)$ belongs to class $\mathcal{K}$, and for each fixed $r \geq 0$, the function $\beta(r, \cdot)$ is non-increasing and $\beta(r, s) \to 0$ as $s \to +\infty$. We then say that (1.1) is input-to-state stable (also written as ISS) with respect to δ provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial functions x_0 and all locally essentially bounded choices of the function δ, the corresponding solution of (1.1) satisfies

$$|x(t)| \leq \beta(|x_0|_\infty, t) + \gamma(|\delta|_{[0,t]}) \tag{1.4}$$

for all $t \geq 0$. The preceding ISS estimate includes uniform global asymptotic stability, as the special case where $\mathcal{F}_0$ does not depend on δ, in which case the estimate takes the form

$$|x(t)| \leq \beta(|x_0|_\infty, t) \tag{1.5}$$

for all $t \geq 0$. Since (1.1) is a nonlinear system, it is not generally possible to express the solutions $x(t)$ in explicit closed form, even if the system does not contain any uncertainties δ. This can complicate the task of determining whether (1.1) is ISS

with respect to δ. Instead, one can often prove ISS properties indirectly, by constructing special kinds of Lyapunov functions called ISS Lyapunov functions, and then one uses the fact that the existence of the ISS Lyapunov function implies ISS; see [60] for necessary and sufficient conditions for ISS in terms of ISS Lyapunov functions for systems without delays, which have analogs for systems with delays. Although Lyapunov functions have been used frequently for many years in the dynamical systems literature (starting from Lyapunov's use of them in his 1892 dissertation [30]), the construction of Lyapunov functions remains a formidable challenge for more complicated control systems that can contain delays, uncertainties, and unknown parameters [31] that is still an area of ongoing research interest. Some of the work in this dissertation will entail constructing Lyapunov functions.

We also use the following basic definitions. A C^0 function $W : \mathbb{R}^n \to [0, +\infty)$ is called positive definite provided $W(0) = 0$ and $W(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. A C^0 function $V : [0, +\infty) \times \mathbb{R}^n \to [0, +\infty)$ is called uniformly proper and positive definite provided there are functions $\alpha_0 \in \mathcal{K}_\infty$ and $\alpha_1 \in \mathcal{K}_\infty$ such that the inequalities $\alpha_0(|x|) \le V(t, x) \le \alpha_1(|x|)$ hold for all $t \ge 0$ and $x \in \mathbb{R}^n$.

Chapter 2.
Bounded Backstepping

2.1 Introduction

Backstepping is a standard method for building globally asymptotically stabilizing feedback controls, by recursively building feedback controls for subsystems of the original systems and then combining the feedbacks for the subsystems to produce the final feedback control for the original system. However, traditional backstepping does not in general provide bounded controls and may not always be suited for systems with input delays. Therefore, this chapter continues work from [33], [34], [35], [36], [44], and [45] on novel backstepping results that help overcome the obstacles to using classical backstepping; see [26] and [29] for traditional backstepping. There are significant applications that call for backstepping but where the existing backstepping literature does not apply, e.g., systems with general nonlinear subsystems where there are bounds on the allowable sup norms of the delay compensating controls, which produce challenges that we overcome in this work.

In this chapter, we focus on systems of the form

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \mathcal{F}(t, x(t), z(t), \eta(t)) \\ \dot{z}_i(t) & = & z_{i+1}(t), \quad i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) & = & u(t) + \sum_{j=1}^{k} v_j z_j(t) \end{array}\right. \tag{2.1}$$

with a scalar valued control u and any number $k \geq 2$ of integrators, where x is valued in $\mathbb{R}^n$ for any n, $\mathcal{F}$ is known, the v_j's are known real constants, the unknown measurable essentially bounded function η represents model uncertainty, and the nonlinear x subsystem will satisfy a converging-input-converging-state condition that we specify below. Our converging-input-converging-state condition is a variant of the CICS condition from [53], but our condition is a requirement on an auxiliary system that we believe had not been considered in earlier studies of

the CICS condition, and we believe that the CICS condition had not be used in backstepping-based feedback design prior to our group's research. Many nonlinear systems admit changes of variables that produce systems of the form (2.1); see [20, Section 9.1], and Section 2.6 below for examples that illustrate the value of our theory. We write our controls as $u(t)$, but they will be feedbacks that depend on t through their dependence on states of (2.1) and of a dynamic extension.

In most of what follows, we assume that the current values of the state are available for measurement, but our main result will still use a delay in the state values in our feedback control since this so-called artificial delay is needed to design a bounded control; see Section 2.5 for an extension to cases where there are also delays in the measurements $x(t)$ of the state of the nonlinear subsystem of (2.1). The work [36] also used both a converging-input-converging-state assumption on a suitably transformed version of the nonlinear system and artificial delays, but a notable improvement in the present work as compared with [36] is that here we allow an arbitrarily large number k of integrators, while [35, 36] only allowed one integrator. Although our bounded backstepping work [44] also allowed an arbitrarily large number of integrators and cases where current values of the states were not available for use in the control, a notable advantage of the present work over [44] is that we produce a globally bounded control for (2.1) while the controls for the original systems in [44] were not globally bounded. Also, whereas [44] required k artificial delays in the control and did not use dynamic extensions, here we only require one artificial delay, so in this sense we obtain a simpler feedback.

The works [33], [34], [35] and [45] did not use converging-input-converging-state conditions or artificial delays. Moreover, our work differs from the backstepping works [34] (which uses a forwarding method to cover the one integrator case), [33] (which also only covers one integrator, under persistency of excitation con-

ditions that we do not use here), [45] (which produces unbounded controls), and [62] and [63] (which use Lie derivatives or measurements of the full state without satisfying the input constraints that we satisfy here). Therefore, our novel combination of converging-input-converging-state conditions with artificial delays and bounded controllers for (2.1) is a valuable contribution with the potential for many applications. The work to follow improves on our conference version [39] by also incorporating delays in the available state measurements and input-to-state stability with respect to the uncertainties η, and allowing the nonlinear subsystem to depend on all components of the vector z. These three features were not present in [39], which was confined to cases where $\mathcal{F}$ was a function of only (t, x, z_1).

2.2 Lemmas and Main Result

To state our lemmas and main result, we require the following additional notation. Let $f^{(i)}$ denote the ith derivative of an i times differentiaable function $f : [0, +\infty) \to \mathbb{R}$ with $f^{(0)} = f$, and $\sigma_r : \mathbb{R} \to [-r, r]$ is the saturation that is defined for all constants $r > 0$ by $\sigma_r(s) = s$ for all $s \in [-r, r]$ and $\sigma_r(s) = r\text{sign}(s)$ otherwise. Let I_n is the $n \times n$ identity matrix. An integral $\int_a J(\ell)\mathrm{d}\ell$ of a continuous column vector valued function $J = (J_1, \dots J_L)^\top$ on an interval a is defined to be the column vector whose ith entry is $\int_a J_i(\ell)\mathrm{d}\ell$ for all i. We require the following two lemmas:

Lemma 2.1. *Let $T > 0$ be a constant, and $\mu_0 : [-T, +\infty) \to \mathbb{R}$ be any continuous function, and set*

$$\begin{aligned} &\zeta(t) = \int_{t-T}^{t} e^{\ell-t} Q(t, \ell, \ell + T)\mu_0(\ell)\mathrm{d}\ell, \;\; \Omega_j(t) = \zeta^{(j-1)}(t) \\ &\textit{and } \mu_i(t) = \tfrac{1}{T}\int_{t-T}^{t} e^{\ell-t}\tfrac{(t-\ell)^{i-1}}{(i-1)!}\mu_0(\ell)\mathrm{d}\ell \end{aligned} \tag{2.2}$$

for all $j \in \{1, ..., k + 1\}$ and $i \in \mathbb{N}$, where $Q(t, a, b) = (t - a)^{k-1}(t - b)^{k-1}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k \geq 2$. Then there are constants $c_{i,j}(T) \in \mathbb{R}$

for all $i \in \{1, ..., 2k-1\}$ *and all* $j \in \{1, ..., k\}$*, and constants* $g_i(T) \in \mathbb{R}$ *for all* $i \in \{-1, 0, ..., 2k-1\}$*, such that the following hold for all* $t \geq 0$ *:*

$$\begin{aligned} \Omega_j(t) &= \sum_{i=1}^{2k-1} c_{i,j}(T)\mu_i(t) \quad \textit{and} \\ \Omega_{k+1}(t) &= \sum_{i=0}^{2k-1} g_i(T)\mu_i(t) + g_{-1}(T)\mu_0(t-T). \end{aligned} \tag{2.3}$$

□

Proof. For each $j \in \{1, 2, \ldots, k\}$, Ω_j will be a linear combination of integrals, each of which having an integrand of the form $e^{\ell - t}(t-\ell)^\alpha(t-\ell-T)^\beta$ with integers $\alpha \in \{0, 1, \ldots, k-1\}$ and $\beta \in \{0, 1, \ldots, k-1\}$, so the required constants $c_{i,j}$ can be obtained by applying the binomial formula

$$(a+b)^\beta = \sum_{j=0}^{\beta} \tfrac{\beta!}{j!(\beta-j)!} a^j b^{\beta-j} \tag{2.4}$$

with the choices $a = t - \ell$ and $b = -T$ for those integrals in the sums having positive β values. If $j < k$, then all of the α's and β's in the sums will be positive integers. On the other hand, if $j = k$, then the linear combination of integrals in the formula for Ω_k will include constant multiples of the integrals

$$\textstyle\int_{t-T}^{t} e^{\ell-t}(t-\ell)^{k-1}\mu_0(\ell)\mathrm{d}\ell \;\text{ and }\; \int_{t-T}^{t} e^{\ell-t}(t-\ell-T)^{k-1}\mu_0(\ell)\mathrm{d}\ell \tag{2.5}$$

and the derivatives of (2.5) in the formula for $\Omega_{k+1} = \Omega_k'$ will be linear combinations of terms that include $-e^{-T}T^{k-1}\mu_0(t-T)$ and $(-T)^{k-1}\mu_0(t)$, which will provide the constants g_{-1} and g_0 in the lemma. The remaining terms $T_i(t)$ in the linear combination in the formula for Ω_k will only have positive powers α and β, and computing their derivatives $T_i'(t)$ will produce the g_i's in the formula for Ω_{k+1} for $i = 1, 2, \ldots, 2k-1$, by again applying the binominal formula (2.4) to the integrand factors $(t-\ell-T)^\beta$ with positive integers β. □

In the next lemma (which was shown in [61]), we say that a linear system is *not exponentially unstable* provided its poles are all in the closed left-half plane:

Lemma 2.2. *Let $k \geq 2$ be an integer and $v = (v_1, \dots, v_k)$ be any vector of k real constants such that*

$$\left\{ \begin{array}{rcl} \dot{z}_i(t) & = & z_{i+1}(t), \quad i \in \{1, \dots, k-1\} \\ \dot{z}_k(t) & = & u + \sum\limits_{i=1}^{k} v_i z_i \end{array} \right. \tag{2.6}$$

is not exponentially unstable when $u = 0$. Then there is a bounded locally Lipschitz function $\vartheta : \mathbb{R}^k \to \mathbb{R}$ such that (2.6), in closed loop with $u = \vartheta(Z)$ where $Z = (z_1, \dots, z_k)^\top$, is globally asymptotically and locally exponentially stable to 0. $\square$

We now fix functions Λ_j such that the Ω_j's from Lemma 2.1 can be written as

$$\Omega_j(t) = \int_{t-T}^{t} \Lambda_j(\ell, t)\mu_0(\ell)\mathrm{d}\ell \ \text{ for } \ 1 \leq j \leq k \ \text{ and all } \ t \geq 0 \tag{2.7}$$

for all choices of the continuous function $\mu_0 : [-T, +\infty) \to \mathbb{R}$, where we omit the dependence of the Λ_j's on T for brevity. By a simple induction argument on the index j that we omit here, we can prove that each function $\Lambda_j(\ell, t)$ can be written as a function $\mathcal{D}_j(t - \ell, t - \ell - T)$ of the differences $t - \ell$ and $t - \ell - T$. For instance, we have

$$\begin{array}{rcl} \Lambda_1(\ell, t) & = & \mathcal{D}_1(\ell - t, \ell - t - T) \quad = \quad e^{\ell - t}(t - \ell)^{k-1}(t - \ell - T)^{k-1} \text{ and} \\ \Lambda_2(\ell, t) & = & \mathcal{D}_2(\ell - t, \ell - t - T) \quad = \quad e^{\ell - t}\left[-(t - \ell)^{k-1}(t - \ell - T)^{k-1}\right. \\ & & \qquad\qquad\qquad\qquad\qquad +(k-1)(t - \ell)^{k-2}(t - \ell - T)^{k-1} \\ & & \qquad\qquad\qquad\qquad\qquad \left. +(k-1)(t - \ell)^{k-1}(t - \ell - T)^{k-2}\right] \end{array} \tag{2.8}$$

and the formulas for the other Λ_j's and $\mathcal{D}_j$'s can be computed from Lemma 2.1. Notice for later use that the Λ_i's are all bounded on $[t - T, t]$ for each $t \geq 0$ and $T > 0$. For instance, when $k = 2$, we have

$$\max\{|\Lambda_j(\ell, t)| : 1 \leq j \leq 2, \ell \in [t - T, t], t \geq 0\} \leq \max\{T^2, T(T + 2)\}. \tag{2.9}$$

We will assume the following, where $\Lambda = (\Lambda_1, \dots, \Lambda_k)^\top$:

Assumption 1. *(i) The function $\mathcal{F}$ in (2.1) is continuous in t and η, globally Lipschitz in (x, z), and satisfies*

$$\mathcal{F}(t, 0, 0, 0) = 0 \text{ for all } t \geq 0. \tag{2.10}$$

(ii) There are a globally Lipschitz bounded function $\omega : \mathbb{R}^n \to [-\bar{\omega}, \bar{\omega}]$ having some bound $\bar{\omega} > 0$ such that $\omega(0) = 0$ and a constant $T > 0$ such that for each continuous function $\delta : [0, +\infty) \to \mathbb{R}^k$ that exponentially converges to 0, the following is true: All solutions $\xi : [0, +\infty) \to \mathbb{R}^n$ of the system

$$\dot{\xi}(t) = \mathcal{F}\left(t, \xi(t), \int_{t-T}^{t} \Lambda(\ell, t)\omega(\xi(\ell))\mathrm{d}\ell + \delta(t), 0\right) \tag{2.11}$$

satisfy $\lim_{t\to+\infty} \xi(t) = 0$. □

We refer to part (ii) of Assumption 1 as our *converging-input-converging-state assumption*; see Section 2.5 for a generalization involving measurement delays in the ξ measurements in the function ω. An important special case is where $\mathcal{F}$ has the form $\mathcal{F}(t, x, z, \eta) = \mathcal{F}_d(t, x) + \mathcal{F}_c(t, x)[z + \eta]$ for some drift term $\mathcal{F}_d$ and some control term $\mathcal{F}_c$, i.e., affinenss with respect to z and η. In this special case, our condition (2.10) is the requirement that $\mathcal{F}_d(t, 0) = 0$ for all $t \geq 0$, and (2.11) has the form

$$\dot{\xi}(t) = \mathcal{F}_d(t, \xi(t)) + \mathcal{F}_c(t, \xi(t)) \left[\int_{t-T}^{t} \Lambda(\ell, t)\omega(\xi(\ell))\mathrm{d}\ell + \delta(t)\right].$$

See Section 2.3 for readily checked sufficient conditions for the required converging-input-converging-state condition in the preceding affine case. The system (2.11) differs from the nonlinear subsystem of (2.1) because the third argument of $\mathcal{F}$ in (2.1) has been replaced by the sum of an integral term and $\delta(t)$, and because η has

been set to 0. In terms of the Jordan matrix

$$J_{2k-1} = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 & 1 \\ 0 & \dots & \dots & 0 & -1 \end{bmatrix} \in \mathbb{R}^{(2k-1)\times(2k-1)}, \tag{2.12}$$

our main result is as follows, where the global Lipschitzness properties of $\mathcal{F}$ and ω, and sufficient conditions for ISS of (2.17) are also provided in Section 2.3. See Section 2.4 for a proof of the following theorem.

Theorem 2.3. *Let $k \geq 2$, and $T > 0$ and $\mathcal{F}$ and ω be such that Assumption 1 holds, where $k \in \mathbb{N}$. Let ϑ and v satisfy the requirements from Lemma 2.2. Consider the augmented (x, Z, Y) system, consisting of (2.1) and*

$$\dot{Y}(t) = J_{2k-1}Y(t) + \tfrac{e_{2k-1}}{T}\omega(x(t)) \tag{2.13}$$

where $e_{2k-1} = (0, 0, \dots, 1)^\top \in \mathbb{R}^{2k-1}$ is the $(2k-1)$-st standard basis vector, in closed loop with the control

$$\begin{aligned} &u(Z(t), Y_t, x_t) = \\ &\sigma_{\bar{c}}\left(\mathcal{M}(Y_t)\right) + g_0(T)\omega(x(t)) + g_{-1}(T)\omega(x(t-T)) + \vartheta(Z_\star(t)) \end{aligned} \tag{2.14}$$

with the saturation level

$$\bar{c} = \left| \sum_{j=1}^{k} v_j \mathcal{C}_j(T) - \mathcal{G}(T) \right| e^{|J_{2k-1}|T}\bar{\omega} \tag{2.15}$$

where $Z_\star(t) = (z_1(t) - \mathcal{C}_1(T)\Psi(Y_t), \dots, z_k(t) - \mathcal{C}_k(T)\Psi(Y_t))^\top$, $\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T)$, $\mathcal{G}(T) = [g_{2k-1}(T) \;\dots\dots\; g_1(T)]$, and

$$\begin{aligned} &\mathcal{M}(Y_t) = \left(\mathcal{G}(T) - \sum_{j=1}^{k} v_j \mathcal{C}_j(T)\right)\Psi(Y_t) \quad \textit{and} \\ &\mathcal{C}_j(T) = [c_{2k-1,j}(T) \;\dots\dots\; c_{1,j}(T)],\ 1 \leq j \leq k \end{aligned} \tag{2.16}$$

and where the constants $c_{i,j}$ *and* g_i *satisfy the requirements from Lemma 2.1 for the function* $\mu_0(t) = \omega(x(t))$. *Then all maximal solutions* $(x, Z, Y)(t)$ *of the augmented* (x, Z, Y) *system, consisting of (2.1) and (2.13) and with (2.14) as the control, satisfy* $\lim_{t\to+\infty}(x, Z, Y)(t) = 0$ *when* $\eta = 0$. *If, in addition, the system*

$$\dot{\xi}(t) = \mathcal{F}\left(t, \xi(t), \int_{t-T}^{t} \Lambda(\ell, t)\omega(\xi(\ell))\mathrm{d}\ell + \delta(t), \eta(t)\right) \tag{2.17}$$

is ISS with respect to (δ, η), *then the* (x, Z) *system (2.1) in closed loop with (2.14) is ISS with respect to* η. □

Remark 2.4. As in [44], we can extend Theorem 2.3 to cases where in addition to the artificial delay T, there is a delay in the available measurements of $x(t)$ from the original system (2.1), which can represent cases where the current state $x(t)$ may not be available for use in the feedback control; see Section 2.5. However, as we noted above, [44] does not provide a bounded control for (2.1) even if the v_i's are all zero, and the converging-input-converging-state assumption in [44] has a k-fold integral instead of the simpler single integral we have in (2.11). □

We next provide sufficient conditions for our converging-input-converging-state assumption to hold.

2.3 Checking Assumption 1

To state our Lyapunov function based sufficient conditions for our converging-input-converging-state conditions on (2.11) to hold, and for the ISS property of (2.17) from Theorem 2.3 to hold, we use the following well known result, called Barbalat's Lemma [26]:

Lemma 2.5. *If* $\phi : \mathbb{R} \to \mathbb{R}$ *is uniformly continuous on* $[0, +\infty)$ *and*

$$\lim_{t\to+\infty} \int_0^t \phi(m)\,\mathrm{d}m \tag{2.18}$$

exists and is finite, then $\lim_{t\to+\infty} \phi(t) = 0$.

We also use the system

$$\dot{x}(t) = \mathcal{F}(t, x(t), \Lambda_*(T)\omega(x(t)), \eta(t)), \tag{2.19}$$

where $\mathcal{F}$ is from (2.1), $\Lambda_* : [0, +\infty) \to \mathbb{R}^p$ is defined by

$$\Lambda_*(T) = \int_{t-T}^{t} (\Lambda_1(\ell, t), \ldots, \Lambda_p(\ell, t))^\top \mathrm{d}\ell, \tag{2.20}$$

the constant $T > 0$ will be specified, the functions Λ_i's satisfy the requirements from (2.7), and $p \in [1, k]$ is such that $\mathcal{F}$ is a function of $(t, x, z_1, \ldots, z_p, \eta)$, where $z_1, \ldots, z_p$ are the first p components of the state z of the linear subsystem of (2.1). The definition (2.20) is justified by the fact that each function $\Lambda_i(\ell, t)$ for $i = 1, 2, \ldots, p$ can be written as a function of $t - \ell - T$ and $t - \ell$, so the right side of (2.20) can be written as

$$\begin{aligned}
&\int_{t-T}^{t} (\Lambda_1(\ell, t), \ldots, \Lambda_p(\ell, t))^\top \mathrm{d}\ell \\
&= \int_{t-T}^{t} (\mathcal{D}_1(t-\ell, t-\ell-T), \ldots, \mathcal{D}_p(t-\ell, t-\ell-T))^\top \mathrm{d}\ell \\
&= \int_{-T}^{0} (\mathcal{D}_1(-\ell, -\ell-T), \ldots, \mathcal{D}_p(-\ell, -\ell-T))^\top \mathrm{d}\ell
\end{aligned}$$

and so does not depend on t. In the next assumption, V_t and V_x are the partial derivative with respect to t and the gradient with respect to x, respectively, and the uniform global Lipschitzness in x means that the global Lipschitz constants can be chosen independently of the other variable t:

Assumption 2. *There are functions $f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times p}$ that are uniformly globally Lipschitz in x and continuous on $[0, +\infty) \times \mathbb{R}^n$, such that $\mathcal{F}(t, x, q, \eta) = f(t, x) + g(t, x)(q + \eta)$ holds for all $t \geq 0$, $x \in \mathbb{R}^n$, $q \in \mathbb{R}^p$, and $\eta \in \mathbb{R}^p$. Also, there exist a C^1 uniformly proper and positive definite function $V : [0, +\infty) \times \mathbb{R}^n \to [0, +\infty)$; a uniformly continuous positive definite function $W : \mathbb{R}^n \to [0, +\infty)$; positive constants T, r_1, and r_3; and a constant*

$r_2 \geq 0$ *such that for all* $(t,x) \in [0,+\infty) \times \mathbb{R}^n$*, we have*

$$V_t(t,x) + V_x(t,x)\big(f(t,x) + g(t,x)\Lambda_*(T)\omega(x)\big) \leq -W(x), \tag{2.21a}$$

$$|V_x(t,x)g(t,x)| \leq \sqrt{W(x)}, \tag{2.21b}$$

$$|\omega(x)| \leq r_1\sqrt{W(x)}, \tag{2.21c}$$

$$|f(t,x)| \leq r_2\sqrt{W(x)}, \tag{2.21d}$$

$$\text{and } \ |g(t,x)| \leq r_3, \tag{2.21e}$$

where $\omega : \mathbb{R}^n \to \mathbb{R}$ *is bounded, satisfies* $\omega(0) = 0$*, and admits a global Lipschitz constant* $C > 0$ *on* $\mathbb{R}^n$*.* □

We emphasize that the linearity of $\mathcal{F}$ in the q and η variables will play a key role in this section. See [42] for conditions under which (2.21) can be satisfied. Set

$$\Lambda_a(T) = \textstyle\int_{t-T}^{t} |(\Lambda_1, \ldots, \Lambda_p)(\ell, t)| \mathrm{d}\ell, \tag{2.22}$$

which is independent of t because the Λ_i's can be written as functions of $t - \ell$ and $t - \ell - T$; the proof that the right side of (2.22) is independent of t is the same as the argument we used to show that the right side of (2.20) is independent of t except with a norm on the p tuples in the earlier argument. We also set $\Lambda_+(T) = \sup_{t\geq 0}\{|(\Lambda_1(\ell,t), \ldots, \Lambda_p(\ell,t))| : t - T \leq \ell \leq t\}$, which is finite because of our choice of Λ.

Proposition 2.6. *If Assumption 2 holds, then for all integers* $k \geq 2$*, and for all constants* $T > 0$ *such that*

$$4\big(T\Lambda_a(T)C\big)^2 \left[2r_2^2 + \tfrac{5}{2}\big(r_1 r_3 T\Lambda_+(T)\big)^2\right] < 1, \tag{2.23}$$

Assumption 1 is satisfied. If, in addition, W *is proper, then (2.17) is ISS with respect to* (δ, η)*.* □

Proof. We first prove the first assertion of the proposition (where $\eta = 0$), and then we indicate the additional arguments needed to prove the second assertion. Fix any continuous function $\delta : [0, +\infty) \to \mathbb{R}^p$ that exponentially converges to 0. Along all solutions $x(t)$ of (2.11), the control affine structure of $\mathcal{F}$ gives

$$\begin{aligned} \dot{x}(t) &= \{f(t, x(t)) + g(t, x(t))\Lambda_*(T)\omega(x(t))\} \\ &+ g(t, x(t))\left[\delta(t) + \textstyle\int_{t-T}^{t} \Lambda^\flat(\ell, t)(\omega(x(\ell)) - \omega(x(t)))\mathrm{d}\ell\right] \end{aligned} \tag{2.24}$$

where $\Lambda^\flat = (\Lambda_1, \ldots, \Lambda_p)^\top$. Combining (2.24) with (2.21a)-(2.21b) now gives

$$\begin{aligned} \dot{V}(t) &\leq -W(x(t)) + |V_x(t, x(t))g(t, x(t))| \\ &\quad \times \left[\left|\textstyle\int_{t-T}^{t} \Lambda^\flat(\ell, t)(\omega(x(\ell)) - \omega(x(t)))\mathrm{d}\ell\right| + |\delta(t)|\right] \\ &\leq -W(x(t)) \\ &\quad + p\sqrt{W(x(t))}\left(\left\{\Lambda_a(T) \sup_{\ell \in [t-T, t]} |\omega(x(\ell)) - \omega(x(t))|\right\} + |\delta(t)|\right) \end{aligned} \tag{2.25}$$

for all $t \geq 0$, where the first inequality used the fact that the portion of the dynamics (2.24) that is contained in the curly braces agrees with the dynamics from (2.21a) combined with the triangle inequality and the fact that (2.21a) holds for all $t \geq 0$ and $x \in \mathbb{R}^n$, and where the second inequality in (2.25) used our bound on the function $|V_x(t, x)g(t, x)|$ from (2.21b) and our formula (2.22) for $\Lambda_a(T)$ after moving the norm inside the integral.

We next use the global Lipschitz constant C on ω and apply the Fundamental Theorem of Calculus to find a useful upper bound on the supremum that is contained in (2.25). To this end, we first use inequalities (2.21d)-(2.21e) from Assumption 2 to obtain the upper bound

$$|\dot{x}(t)| \leq r_2\sqrt{W(x(t))} + r_3\left\{|\Lambda_+(T)| \textstyle\int_{t-T}^{t} |\omega(x(\ell))|\mathrm{d}\ell + |\delta(t)|\right\} \tag{2.26}$$

along all solutions of (2.11). Applying $(a + b)^2 \leq 2(a^2 + b^2)$ for suitable $a \geq 0$ and $b \geq 0$, and then applying $(a + b)^2 \leq (5/4)a^2 + 5b^2$ where a and b are the

terms being added together in curly braces in (2.26) and then Jensen's inequality, it follows that along all solutions of (2.24), we have

$$|\dot{x}(t)|^2 \;\leq\; 2r_2^2 W(x(t)) + 2r_3^2\left(5|\delta(t)|^2 + \tfrac{5}{4}Tr_1^2\Lambda_+^2\int_{t-T}^{t} W(x(\ell))\mathrm{d}\ell\right), \qquad (2.27)$$

where $W(x(\ell))$ in the integrand is present because of our condition (2.21c) relating ω to W.

We can now combine (2.25) and (2.27) and then use Jensen's and Young's inequalities $\sqrt{W(x(t))}b \leq \frac{1}{4}W(x(t)) + b^2$, with $b = |\delta(t)|$ and then b being the quantity in curly braces in (2.25), to get

$$\begin{aligned}\dot{V}(t) &\leq -\tfrac{1}{2}W(x(t)) + |\delta(t)|^2 + \Lambda_a^2(T)C^2\sup_{\ell\in[t-T,t]}|x(\ell)-x(t)|^2\\ &\leq -\tfrac{1}{2}W(x(t)) + |\delta(t)|^2 + \Lambda_a^2(T)C^2T\int_{t-T}^{t}|\dot{x}(\ell)|^2\mathrm{d}\ell\\ &\leq -\tfrac{1}{2}W(x(t))\\ &\quad + |\delta(t)|^2 + \Lambda_a^2(T)C^2T\bigg(2r_2^2\int_{t-T}^{t}W(x(\ell))\mathrm{d}\ell + 10r_3^2T|\delta|^2_{[t-T,t]}\\ &\quad + \tfrac{5}{2}r_3^2T^2\Lambda_+^2(T)r_1^2\int_{t-2T}^{t}W(x(\ell))\mathrm{d}\ell\bigg)\\ &\leq -\tfrac{1}{2}W(x(t)) + \mathcal{N}_1\int_{t-2T}^{t}W(x(\ell))\mathrm{d}\ell + \mathcal{N}_2|\delta|^2_{[t-T,t]}\end{aligned} \qquad (2.28)$$

along all solutions of (2.11) for all $t \geq 0$, where

$$\begin{aligned}&\mathcal{N}_1 = T(\Lambda_a(T)C)^2\left(2r_2^2 + \tfrac{5}{2}\left(r_1r_3\Lambda_+(T)T\right)^2\right) \text{ and}\\ &\mathcal{N}_2 = 10(T\Lambda_a(T)r_3C)^2 + 1,\end{aligned} \qquad (2.29)$$

by using the bound $\int_{s-T}^{s}W(x(\ell))\mathrm{d}\ell \leq \int_{t-2T}^{t}W(x(\ell))\mathrm{d}\ell$ for all $s \in [t-T,t]$. Then our condition (2.23) implies that $4T\mathcal{N}_1 < 1$, so we can find a constant $\lambda > 1$ that is close enough to 1 so that $2T\mathcal{N}_1\lambda < 1/2$. Then

$$V_1(t,x_t) = V(t,x(t)) + \lambda\mathcal{N}_1\int_{t-2T}^{t}\int_{s}^{t}W(x(\ell))\mathrm{d}\ell\mathrm{d}s \qquad (2.30)$$

satisfies

$$\begin{aligned}\dot{V}_1 \;&\leq\; -\left\{\tfrac{1}{2} - 2T\mathcal{N}_1\lambda\right\}W(x(t))\\ &\quad - (\lambda-1)\mathcal{N}_1\int_{t-2T}^{t}W(x(\ell))\mathrm{d}\ell + \mathcal{N}_2|\delta|^2_{[t-T,t]}\end{aligned} \qquad (2.31)$$

for all $t \geq 0$ along all solutions of (2.11), since for all $t \geq 0$, we have

$$\tfrac{d}{dt}\int_{t-2T}^{t}\int_{s}^{t} W(x(\ell))\mathrm{d}\ell\mathrm{d}s = 2TW(x(t)) - \int_{t-2T}^{t} W(x(\ell))\mathrm{d}\ell. \tag{2.32}$$

Using our assumption that δ converges to 0 exponentially as $t \to +\infty$, we can find positive constants $\bar{\delta}_1$ and $\bar{\delta}_2$ such that $|\delta(t)| \leq \bar{\delta}_1 e^{-\bar{\delta}_2 t}$ for all $t \geq 0$. Therefore, since the quantity in curly braces in (2.31) is positive and $\lambda > 1$, we can integrate (2.31) on $[0, M]$ for any constant $M > 0$ to get

$$\begin{aligned} &\sup_{t\geq 0} V_1(t, x_t) \\ &\leq\ V_1(0, x_0) + \mathcal{N}_2 \int_0^{+\infty} |\delta|^2_{[0,\ell]}\mathrm{d}\ell\ \leq\ V_1(0, x_0) + \mathcal{N}_2\bar{\delta}_1^2 \int_0^{+\infty} e^{-2\bar{\delta}_2(\ell-T)}\mathrm{d}\ell \\ &\leq V_1(0, x_0) + \mathcal{N}_2\tfrac{\bar{\delta}_1^2}{2\bar{\delta}_2}\ <\ +\infty. \end{aligned}$$

Since V is uniformly proper and positive definite, we conclude that $|x(t)|$ is bounded, so $x(t)$ is uniformly continuous, by the structure of the dynamics (2.11) when $\eta = 0$. Since W is uniformly continuous, it follows that $W(x(t))$ is a uniformly continuous function of t, and integrating (2.31) gives

$$\int_0^{+\infty} W(x(\ell))\mathrm{d}\ell < +\infty. \tag{2.33}$$

Therefore, Barbalat's Lemma implies that $\lim_{t\to+\infty} W(x(t)) = 0$, so since W is positive definite, we conclude that $\lim_{t\to+\infty} x(t) = 0$. This proves the first assertion of the proposition.

To prove the second assertion of the proposition, fix a choice of the measurable essentially bounded function η. Then the preceding analysis applies to the corresponding system (2.17), save for the fact that we must add the additional term $V_x(t, x(t))g(t, x(t))\eta(t)$ to the right sides of the decay estimates on V. We can use Jensen's inequality to check that this additional term is bounded above by

$$\sqrt{W(x(t))}|\eta(t)| \leq \tfrac{c_*}{2}W(x(t)) + \tfrac{1}{2c_*}|\eta(t)|^2 \tag{2.34}$$

where $c_* > 0$ is the constant in curly braces in (2.31). If we add the right side of (2.34) to the right side of (2.31) and use the fact that

$$\int_{t-2T}^{t} \int_{s}^{t} W(x(\ell))\mathrm{d}\ell\mathrm{d}s \le 2T \int_{t-2T}^{t} W(x(\ell))\mathrm{d}\ell \tag{2.35}$$

for all $t \ge 0$, then we can find a function $\gamma_0 \in \mathcal{K}_\infty$ and a constant $k_* > 0$ such that

$$\dot{V}_1 \le -\gamma_0(V_1(t, x_t)) + k_*|(\delta, \eta)|_{[0,t]}^2 \tag{2.36}$$

along all solutions of (2.11), using the properness of V and W to find a $\gamma_1 \in \mathcal{K}_\infty$ such that $\gamma_1(V(t,x)) \le (c_*/2)W(x)$ for all $t \ge 0$ and $x \in \mathbb{R}^n$, then choosing $\gamma_0(\ell) = \min\{\gamma_1(\ell/2), r_*\ell\}$ with $r_* = (\lambda - 1)/(4T\lambda)$ (by the relation $\gamma_0(a+b) \le \gamma_0(2a) + \gamma_0(2b)$ where a and b are the terms being added in the formula (2.30)). Hence, V_1 is an ISS Lyapunov function for (2.11), so the ISS properties follow by standard arguments [26]. □

Remark 2.7. Proposition 2.7 requires $T > 0$ to be small enough, but due to the structure of our controller in Theorem 2.3, we cannot pick $T = 0$. In Section 2.5, we will see how picking T small enough can ensure that the ISS property is maintained even when there are measurement delays D in the x values in our feedbacks. □

Remark 2.8. Conditions (2.21) agree with the sufficient conditions in [44], save for the fact that instead of (2.21b), [44] required a constant $r_0 > 0$ such that $|V_x(t,x)g(t,x)| \le r_0\sqrt{W(x)}$ for all $t \ge 0$ and $x \in \mathbb{R}^n$. However, one may assume that $r_0 = 1$, by replacing g, ω, r_1, and r_3 by g/r_0, $r_1\omega$, and r_3/r_0 respectively without relabelling, so there is no loss of generality in assuming that $r_0 = 1$. □

2.4 Proof of Theorem 2.3

The forward completeness of the closed loop systems defined in the statement of the theorem follow from the boundedness of ω and of the control. Theorem 2.3

will now follow from three more lemmas, which we state next. The first of these lemmas follows from [58, Lemma A.3.2] (applied to the entire function $\mathcal{E}(x) = e^{xt}$ for any $t \in \mathbb{R}$ to compute $\mathcal{E}(J_{2k-1})$):

Lemma 2.9. *For the Jordan matrix J_{2k-1} defined in (2.12), the equality*

$$e^{J_{2k-1}t} = e^{-t} \begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{2(k-1)}}{(2(k-1))!} \\ 0 & 1 & t & \dots & \frac{t^{2k-3}}{(2k-3)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & t \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{2.37}$$

holds for all $t \in \mathbb{R}$ and integers $k \geq 2$. □

Later in the proof of Theorem 2.3, we specialize the following lemma to the case where $\mu_0(t) = \omega(x(t))$:

Lemma 2.10. *Let $\mu_0 : [-T, +\infty) \to [-\bar{\mu}, \bar{\mu}]$ be any continuous function having a bound $\bar{\mu}$. Then the functions μ_i from (2.2) in Lemma 2.1, and the functions $\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T)$ for all solutions Y of*

$$\dot{Y}(t) = J_{2k-1}Y(t) + \frac{e_{2k-1}}{T}\mu_0(t), \tag{2.38}$$

are such that for all $t \geq 0$, we have

$$\nu_{2k-1}(t) = \Psi(Y_t) \ \text{ and } \ |\Psi(Y_t)| \leq e^{|J_{2k-1}|T}\bar{\mu}, \tag{2.39}$$

where $\nu_{2k-1}(t) = (\mu_{2k-1}(t), \dots, \mu_1(t))^\top$ for all $t \geq 0$. □

Proof. By integrating (2.38) over $[t-T, t]$ for any $t \geq 0$, we deduce that

$$\begin{aligned} &\Psi(Y_t) = Y(t) - e^{TJ_{2k-1}}Y(t-T) = \varrho(t), \text{ where} \\ &\varrho(t) = \textstyle\int_{t-T}^{t} e^{J_{2k-1}(t-\ell)}\frac{e_{2k-1}}{T}\mu_0(\ell)\mathrm{d}\ell \end{aligned} \tag{2.40}$$

for all $t \geq 0$. On the other hand, using (2.37), we obtain

$$\varrho(t) = \tfrac{1}{T}\int_{t-T}^{t} e^{-(t-\ell)} \begin{bmatrix} \frac{(t-\ell)^{2(k-1)}}{(2(k-1))!} \\ \vdots \\ t-\ell \\ 1 \end{bmatrix} \mu_0(\ell)d\ell = \nu_{2k-1}(t), \tag{2.41}$$

which proves the first conclusion of the lemma. The second conclusion of the lemma follows since (2.40) gives

$$\begin{aligned} \left|Y(t) - e^{TJ_{2k-1}}Y(t-T)\right| = |\varrho(t)| \ &\leq \ \int_{t-T}^{t} \left|e^{J_{2k-1}(t-\ell)}\frac{e_{2k-1}}{T}\right| |\mu_0(\ell)|\, \mathrm{d}\ell \\ \leq \int_{t-T}^{t} \left|e^{J_{2k-1}(t-\ell)}\frac{e_{2k-1}}{T}\right| \bar{\mu}\mathrm{d}\ell & \end{aligned} \tag{2.42}$$

for all $t \geq 0$, because of the bound $\bar{\mu}$ on μ_0, which proves the lemma. □

Lemma 2.11. *Let $\mu_0 : [-T, +\infty) \to [-\bar{\mu}, \bar{\mu}]$ be any continuous function having a bound $\bar{\mu}$, and let the constants v_i and the function ϑ satisfy the requirements from Lemma 2.2. Consider the linear system*

$$\begin{cases} \dot{z}_i(t) &= z_{i+1}(t), \ \ i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) &= u(t) + \sum_{j=1}^{k} v_j z_j(t) \end{cases} \tag{2.43}$$

in closed loop with the control

$$u(Z(t), Y_t, x_t) = \sigma_{\bar{c}}\left(\mathcal{M}(Y_t)\right) + g_0(T)\mu_0(t) + g_{-1}(T)\mu_0(t-T) + \vartheta(Z_\star(t)) \tag{2.44}$$

with the saturation level $\bar{c}$ for $\sigma_{\bar{c}}$ defined by

$$\bar{c} = \left|\sum_{j=1}^{k} v_j \mathcal{C}_j(T) - \mathcal{G}(T)\right| e^{|J_{2k-1}|T}\bar{\mu} \tag{2.45}$$

and where Y satisfies (2.38) and $\mathcal{M}$, $Z_\star$, $\mathcal{G}$, and the $\mathcal{C}_j$'s and g_j's are defined as in Theorem 2.3. Then the dynamics for the vector $\tilde{Z}(t) = (\tilde{z}_1(t), ..., \tilde{z}_k(t))$ are globally asymptotically and locally exponentially stable to the origin, where $\tilde{z}_i(t) = z_i(t) - \Omega_i(t)$ for $i = 1, 2, \ldots, k$ and the Ω_i's are defined in (2.2) in Lemma 2.1. □

Proof. The fact that $\dot{\Omega}_i = \Omega_{i+1}$ for all $i \in \{1, 2, \ldots, k\}$ and the structure of the dynamics (2.43) allow us to conclude that the dynamics for the functions $\tilde{z}_i(t) = z_i(t) - \Omega_i(t)$ are

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}_i(t) & = & \tilde{z}_{i+1}(t), \quad i \in \{1, \ldots, k-1\} \\ \dot{\tilde{z}}_k(t) & = & u(t) - \Omega_{k+1}(t) + \sum_{j=1}^{k} v_j[\tilde{z}_j(t) + \Omega_j(t)] \end{array} \right. . \tag{2.46}$$

Using our conclusion from Lemma 2.10 that

$$\nu_{2k-1}(t) = \Psi(Y_t) \tag{2.47}$$

where $\nu_{2k-1}(t) = (\mu_{2k-1}(t), \ldots, \mu_1(t))^\top$ as before, it follows from (2.3) that

$$\begin{array}{rcl} \dot{\tilde{z}}_k(t) & = & u(t) - \mathcal{G}(T)\nu_{2k-1}(t) - g_0(T)\mu_0(t) - g_{-1}(T)\mu_0(t-T) \\ & & + \sum_{j=1}^{k} v_j \tilde{z}_j(t) + \sum_{j=1}^{k} v_j \mathcal{C}_j(T)\nu_{2k-1}. \end{array} \tag{2.48}$$

Hence, (2.47) gives

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}_i(t) & = & \tilde{z}_{i+1}(t), \quad i \in \{1, \ldots, k-1\} \\ \dot{\tilde{z}}_k(t) & = & u(t) + \displaystyle\sum_{j=1}^{k} v_j \tilde{z}_j(t) - g_0(T)\mu_0(t) - g_{-1}(T)\mu_0(t-T) + \bar{g}\Psi(Y_t) \end{array} \right. \tag{2.49}$$

where

$$\bar{g} = \sum_{j=1}^{k} v_j \mathcal{C}_j(T) - \mathcal{G}(T). \tag{2.50}$$

Next note that since Lemma 2.1 gives $\Omega_j = \mathcal{C}_j(T)\nu_{2k-1}$ for $j = 1, \ldots, k$, it follows that

$$\tilde{z}_i(t) = z_i(t) - \Omega_i(t) = z_i(t) - \mathcal{C}_i(T)\nu_{2k-1}(t) \text{ for } i \in \{1, \ldots, k\}. \tag{2.51}$$

Thus, (2.47) gives $\tilde{z}_i(t) = z_i(t) - \mathcal{C}_i(T)\Psi(Y_t)$ for all $t \geq 0$ and all $i \in \{1, \ldots, k\}$, so $Z_\star(t) = \tilde{Z}(t) = (\tilde{z}_1(t), \ldots, \tilde{z}_k(t))$ for all $t \geq 0$. Also, $\mathcal{M}(Y_t) = -\bar{g}\Psi(Y_t)$. Therefore, our choice (2.44) of the control gives

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}_i(t) & = & \tilde{z}_{i+1}(t), \quad i \in \{1, \ldots, k-1\} \\ \dot{\tilde{z}}_k(t) & = & \displaystyle\sum_{j=1}^{k} v_j \tilde{z}_j(t) + \sigma_{\bar{c}}\left(-\bar{g}\Psi(Y_t)\right) + \bar{g}\Psi(Y_t) + \vartheta(Z_\star(t)) . \end{array} \right. \tag{2.52}$$

According to (2.39), we have

$$|\bar{g}\Psi(Y_t)| \leq |\bar{g}|\, e^{|J_{2k-1}|T}\overline{\mu} = \bar{c} \tag{2.53}$$

for all $t \geq 0$. From the definition of the saturation level $\bar{c}$ of $\sigma_{\bar{c}}$, it follows that for all $t \geq 0$, we have

$$\left\{ \begin{array}{rcl} \dot{\tilde{z}}_i(t) & = & \tilde{z}_{i+1}(t), \;\; i \in \{1, \ldots, k-1\} \\ \dot{\tilde{z}}_k(t) & = & \vartheta(\tilde{Z}(t)) + \sum\limits_{i=1}^{k} v_i \tilde{z}_i(t) \end{array} \right. \tag{2.54}$$

so the lemma follows from our choice of ϑ in Lemma 2.2. $\square$

We now combine the preceding lemmas to prove Theorem 2.3. We begin by proving the first conclusion of the theorem, in which $\eta = 0$. In this case, the closed loop system is

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \mathcal{F}(t, x(t), z(t), 0) \\ \dot{z}_i(t) & = & z_{i+1}(t), \;\; i \in \{1, \ldots, k-1\} \\ \dot{z}_k(t) & = & u(Z(t), Y_t, x_t) + \displaystyle\sum_{j=1}^{k} v_j z_j(t) \\ \dot{Y}(t) & = & J_{2k-1}Y(t) + \frac{e_{2k-1}}{T}\omega(x(t)). \end{array} \right. \tag{2.55}$$

Using the fact that the control (2.44) from Lemma 2.11 agrees with our control (2.14) from Theorem 2.3 when we select $\mu_0(t) = \omega(x(t))$, it follows from using Lemma 2.11 with the choice $\mu_0(t) = \omega(x(t))$ that

$$\lim_{t \to +\infty} |z_i(t) - \Omega_i(t)| = 0 \tag{2.56}$$

for all $i = 1$ to k, and $\tilde{z}_i = z_i - \Omega_i$ exponentially converges to 0 for all i.

Next notice that the x subsystem of (2.55) can be written as

$$\dot{x}(t) \;\; = \;\; \mathcal{F}(t, x(t), \Omega(t) + \tilde{z}(t), 0) \tag{2.57}$$

where $\Omega = (\Omega_1, \ldots, \Omega_k)^\top$, when we choose the bounded function $\mu_0(t) = \omega(x(t))$. Hence, we can use the converging-input-converging-state portion of our Assumption 1 (with the choices $\delta = \tilde{z}$ and $\eta - 0$) to conclude that $\lim_{t\to+\infty} |x(t)| = 0$ and therefore that for all $i \in \{1, 2, \ldots, k\}$, we have $\lim_{t\to+\infty} \Omega_i(t) = 0$, since $\omega(0) = 0$ and ω is continuous at 0. It follows that

$$\lim_{t\to+\infty} z(t) = \lim_{t\to+\infty} (z(t) - \Omega(t)) + \lim_{t\to+\infty} \Omega(t) = 0. \tag{2.58}$$

On the other hand,

$$\dot{Y} = J_{2k-1} Y + \varepsilon \tag{2.59}$$

is ISS with respect to ε, by the Hurwitzness of J_{2k-1} as we defined this matrix in (2.12), which makes it possible to use a Riccati equation to find a quadratic Lyapunov function for $\dot{Y} = J_{2k-1}Y$ of the form $Y^\top PY$ for some positive definite matrix P which is then an ISS Lyapunov function for (2.59) with ε playing the role of the uncertainty. This provides positive constants c_a and c_b such that $|Y(t)| \leq c_a(|Y(t/2)|e^{-c_b t} + \sup\{|\varepsilon(\ell)| : t/2 \leq \ell \leq t\})$ along all solutions of (2.59) for all $t \geq 0$. Specializing the preceding argument to the function $\varepsilon(t) = e_{2k-1}\omega(x(t))/T$ which converges to 0 as $t \to +\infty$ now gives the first conclusion of Theorem 2.3. This follows because all solutions of (2.59) for bounded choices of ε are bounded, so for each constant $\delta_0 > 0$, we can find a constant $T_0 > 0$ that is large enough so that $\max\{|Y(t/2)|e^{-c_b t}, \sup\{|\varepsilon(\ell)| : t/2 \leq \ell \leq t\}\} < \delta_0/(2c_a)$ for all $t > T_0$, which gives $|Y(t)| \leq \delta_0$ for all $t \geq T_0$.

It remains to prove the second conclusion of the theorem. To this end, first note that with the notation from our proof of the first conclusion of the theorem, the dynamics for $\tilde{z}$ are globally asymptotically stable to 0, so the interconnection of the perturbed dynamics $\dot{x}(t) = \mathcal{F}(t, x(t), \Omega(t) + \tilde{z}(t), \eta(t))$ with $\mu_0(t) = \omega(x(t))$ and the $\tilde{z}$ dynamics will be ISS with respect to η, by standard small gain arguments.

Then the structure of the function Ω implies that the $(x, z) = (x, \tilde{z} + \Omega)$ dynamics are ISS with respect to η. This completes the proof of our theorem.

2.5 Extension to Systems with Measurement Delays

This section is connected with, and provides a nontrivial extension of, Section 2.2, by explaining how the framework of Theorem 2.3 is general enough to allow cases where current values $x(t)$ are not available for measurement or for use in the control. Such cases occur in engineering applications where the control must be computed on a computer that is far from the actual plant, which was the case for instance in the work [48] which used small marine robots to search for oil pollution. Our strategy in this section is to find values of T that ensure that the required converging-input-converging-state assumption is satisfied for cases where current values $x(t)$ are not available for use in the control. See Remark 2.13 for a detailed description of how our work in this section adds value relative to the existing delay compensation literature.

Although [44] did not provide a bounded backstepping controller for the original system (2.1), it allowed cases where current values of the x components of the state of the original system were not available for use in the control, leading to feedback controls in which $x(t)$ must be replaced by time delayed values $x(t - D)$ of x for a constant delay $D > 0$. In the same way, we can extend Theorem 2.3 above to allow cases where one must use time lagged values of x instead of current ones. This is done by replacing $\omega(x(\ell))$ in the preceding analysis by $\omega(x(\ell - D))$ for constant values of the delay D, so instead of placing a converging-input-converging-state assumption on (2.11) in Assumption 1, we must replace (2.11) by the delayed version

$$\dot{\xi}(t) = \mathcal{F}\left(t, \xi(t), \int_{t-T}^{t} \Lambda(\ell, t)\omega(\xi(\ell - D))\mathrm{d}\ell + \delta(t), 0\right), \tag{2.60}$$

and then the conclusions of the theorem remain true with $x(\ell)$ replaced by $x(\ell - D)$ in the feedback control. However, our sufficient conditions from Proposition 2.6 do not apply in cases such as (2.60) with measurement delays. This motivates the following analog of Proposition 2.6 that provides sufficient conditions for our delayed version of the converging-input-converging-state condition to hold, and which can therefore facilitate checking the requirements of our theorem when constant measurement delays D are introduced in the x measurements. In what follows, we use the same choices of $\Lambda_a(T)$ from (2.22) and $\Lambda_+(T) = \sup_{t \geq 0}\{|(\Lambda_1(\ell, t), \dots, \Lambda_p(\ell, t))| : t - T \leq \ell \leq t\}$ as in Section 2.3, which are still independent of t, and which also do not depend on D.

Proposition 2.12. *If Assumption 2 holds, and if the constants $T > 0$ and $D > 0$ are such that*

$$\mathcal{R}(T) < 1 \text{ and} \tag{2.61a}$$

$$C\Lambda_a(T)\big(r_2 + r_3\Lambda_+(T)r_1(D+T)\big)(D+T) < \tfrac{1-\mathcal{R}(T)}{8\sqrt{2}}, \text{ where} \tag{2.61b}$$

$$\mathcal{R}(T) = 4\big(T\Lambda_a(T)C\big)^2\left[2r_2^2 + \tfrac{5}{2}\big(r_1 r_3 T\Lambda_+(T)\big)^2\right] \tag{2.61c}$$

then the following is true: For each continuous function $\delta : [0, +\infty) \to \mathbb{R}^p$ that exponentially converges to zero, all solutions of (2.60) converge to 0 as $t \to +\infty$. If, in addition, the function W from Assumption 2 is proper, then the system

$$\dot{\xi}(t) = \mathcal{F}\left(t, \xi(t), \textstyle\int_{t-T}^{t} \Lambda(\ell, t)\omega(\xi(\ell - D))\mathrm{d}\ell + \delta(t), \eta(t)\right) \tag{2.62}$$

is ISS with respect to (δ, η).

Proof. We indicate the changes needed in the proof of Proposition 2.6. We let $c_* > 0$ be the constant in curly braces in (2.31) as before, and where λ is chosen as in the proof of Proposition 2.6. We may assume that $\lambda > 1$ is close enough to 1

so that the requirements from (2.61) are still true if we replace $\mathcal{R}(T)$ by $\mathcal{R}(T) = 4\lambda(T\Lambda_a(T)C)^2[2r_2^2 + 2.5(r_1 r_3 T\Lambda_+(T))^2]$ (by the strictness of the inequalities in (2.61)), and we make this replacement in the rest of the proof. Then, using our notation from the proof of Proposition 2.6, we have

$$c_* = 0.5(1 - \mathcal{R}(T)) = 0.5(1 - 4T\mathcal{N}_1\lambda). \tag{2.63}$$

In what follows, we use Λ_a and Λ_+ to mean $\Lambda_a(T)$ and $\Lambda_+(T)$, respectively, to keep our notation simple. Using the function V from Assumption 2 and Young's Inequality, the additional term that must be added to the decay estimate on V can be bounded above as follows:

$$\begin{aligned}
&V_x(t, x(t))g(t, x) \textstyle\int_{t-T}^{t} \Lambda^\flat(\ell, t)[\omega(x(\ell - D)) - \omega(x(\ell))]\mathrm{d}\ell \\
&\le C\Lambda_a\sqrt{W(x(t))} \textstyle\int_{t-T-D}^{t} |\dot{x}(s)|\mathrm{d}s \\
&\le C\Lambda_a\sqrt{W(x(t))} \\
&\quad \times \textstyle\int_{t-T-D}^{t} \left\{|f(s, x(s))| + |g(s, x(s))| \left[\Lambda_+ \textstyle\int_{s-T}^{s} |\omega(x(\ell - D))|\mathrm{d}\ell + |\eta(s)|\right]\right\} \mathrm{d}s \\
&\le C\Lambda_a\sqrt{W(x(t))} \\
&\quad \times \textstyle\int_{t-T-D}^{t} \left(r_2\sqrt{W(x(s))} + r_3\Lambda_+ \textstyle\int_{t-D-2T}^{t} |\omega(x(\ell - D))|\mathrm{d}\ell + r_3|\eta(s)|\right) \mathrm{d}s \\
&\le C\Lambda_a\sqrt{W(x(t))} \\
&\quad \times \left[(r_2 + r_3\Lambda_+ r_1(D + T)) \textstyle\int_{t-2D-2T}^{t} \sqrt{W(x(\ell))}\mathrm{d}\ell + r_3(D + T)|\eta|_{[0,t]}\right] \\
&\le \tfrac{c_*}{4}W(x(t)) + \tfrac{1}{c_*}\left(pC\Lambda_a\left[(r_2 + r_3\Lambda_+ r_1(D + T)) \textstyle\int_{t-2(D+T)}^{t} \sqrt{W(x(\ell))}\mathrm{d}\ell\right.\right. \\
&\quad \left.\left. + r_3(D + T)|\eta|_{[0,t]}\right]\right)^2 \\
&\le \tfrac{c_*}{4}W(x(t)) + \tfrac{4}{c_*}\left\{C\Lambda_a\left(r_2 + r_3\Lambda_+ r_1(D + T)\right)\right\}^2 (D + T) \textstyle\int_{t-2D-2T}^{t} W(x(\ell))\mathrm{d}\ell \\
&\quad + \tfrac{2}{c_*}(Cr_3\Lambda_a(D + T))^2|\eta|_{[0,t]}^2,
\end{aligned}$$

where the last inequality also used Young's inequality, the relation $(a + b)^2 \le 2a^2 + 2b^2$ for suitable nonnegative values of a and b, and then Jensen's inequality. Using the inequality (2.61b) and choosing $\lambda > 1$ close enough to 1, it follows that

we can find a constant $\lambda_* > 1$ that is close enough to 1 and which is such that

$$\tfrac{8\lambda_*}{c_*}\left\{C\Lambda_a(D+T)\left(r_2+r_3\Lambda_+ r_1(D+T)\right)\right\}^2 < \tfrac{c_*}{4}, \tag{2.64}$$

since $c_* = 0.5(1-\mathcal{R}(T))$. Then reasoning analogously to the argument that produced (2.31) shows that the time derivative of

$$\begin{aligned} &V_2(t,x_t) = V_1(t,x_t) \\ &+\tfrac{4\lambda_*}{c_*}\left\{C\Lambda_a\left(r_2+r_3\Lambda_+ r_1(D+T)\right)\right\}^2(D+T)\int_{t-2(D+T)}^{t}\int_{\ell}^{t} W(x(s))\mathrm{d}s\mathrm{d}\ell \end{aligned} \tag{2.65}$$

along all solutions of (2.62) admits positive constants c_{**} and c_{***} such that

$$\dot{V}_2 \leq -c_{**}W(x(t)) + c_{***}|(\delta,\eta)|^2_{[0,t]}. \tag{2.66}$$

If, in addition, W is proper, then we can argue as in the proof of Proposition 2.6 to find a function $\gamma_a \in \mathcal{K}_\infty$ and a positive constant k_a such that

$$\dot{V}_2 \leq -\gamma_a(V_2(t,x_t)) + k_a|(\delta,\eta)|^2_{[0,t]} \tag{2.67}$$

(by using the bound (2.35) except with T in (2.35) replaced by $D+T$). Then the rest of the proof is the same as in the last part of the proof of Proposition 2.6 except with V_1 replaced by V_2. $\square$

Remark 2.13. There is a large recent literature on delay compensating control design for nonlinear systems, largely involving prediction, which replaces time lagged state values in controls by predicted state values [6, 7, 22, 23, 24, 25, 47, 50, 51, 67]. However, as we noted in Chapter 1 above, prediction generally leads to dynamic controls that contain distributed terms (i.e., terms that use all values of the control or the state along certain time intervals), which can be difficult to implement in practice [25]. See also the reduction model controls [28] which are only expressed implicitly as solutions of integral equations. Hence, potential advantages of the

controls that can be obtained using our approach from this section include (a) the lack of distributed terms in our controls, (b) our ability to satisfy control bounds, (c) our ability to prove global asymptotic stability of the closed loop system from Theorem 2.3 under any measurement delay $D > 0$ for which (2.60) satisfies the required converging-input-converging-state condition (with no other restriction on the size of D), and (d) the robust performance of our controls in terms of ISS.

2.6 Illustrations

Our Lyapunov function based sufficient conditions are convenient for checking our converging-input-converging-state assumptions from Theorem 2.3. We illustrate this point in this section, in two examples. In our first example, we apply our Lyapunov sufficient conditions directly. In our second example, our Lyapunov sufficient conditions do not apply directly, but we use a mixture of our Lyapunov and trajectory based methods to check our converging-input-converging-state conditions.

Our second example illustrates the point that it may only be necessary to check our sufficient conditions locally in a neighborhood of the equilibrium, instead of globally, which eliminates the need to find a global Lyapunov function as required in Assumption 2. For simplicity, this section only considers cases where there are no measurement delays D, but we can apply the methods from the preceding section to cover measurement delays as well.

For the first illustration, consider the three-dimensional system

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \frac{|x(t)|}{1+|x(t)|} + z_1(t) \\ \dot{z}_1(t) & = & z_2(t) \\ \dot{z}_2(t) & = & u(t) \end{array}\right. \tag{2.68}$$

which is not amenable to classical backstepping, because the right side of $\dot{x}(t)$ in the dynamics is not differentiable. In terms of our notation from Section 2.3, we

choose $k=2$, $n=1$, $p=1$, and

$$\mathcal{F}(t,x,z_1)=\tfrac{|x|}{1+|x|}+z_1 \ \text{ and } \ \omega(x)=-\tfrac{1}{\Lambda_*(T)}\left(\tfrac{|x|}{1+|x|}+2\tfrac{x}{1+|x|}\right)\ , \tag{2.69}$$

where

$$\Lambda_*(T)=\int_{-T}^{0}\Lambda_1(\ell+t,t)\mathrm{d}\ell=\int_{-T}^{0}e^{\ell}\ell^{k-1}(\ell+T)^{k-1}\mathrm{d}\ell=2-T-e^{-T}(2+T). \tag{2.70}$$

We compute a constant $T>0$ such that Assumption 1 is satisfied. First note that since $p=1$, and since $\Lambda_1(\ell,t)\leq 0$ for all $t\geq 0$ and $\ell\in[t-T,t]$, we have $\Lambda_a(T)=-\Lambda_*(T)=|\Lambda_*(T)|$. Since (2.69) are globally Lipschitz functions and $\mathcal{F}$ is an affine function of z_1 and ω is bounded, it suffices to find constants r_i for $i=0,1,2,3$ and functions V and W such that Assumption 2 is satisfied with

$$f(t,x)=\tfrac{|x|}{1+|x|} \ \text{and} \ g(t,x)=1 \tag{2.71}$$

and then to choose T such that our condition (2.23) holds.

To this end, we check that Assumption 2 is satisfied using the functions

$$V(t,x)=\textstyle\int_0^x\sigma_1(\ell)\mathrm{d}\ell \ \text{ and } \ W(x)=\tfrac{2\sigma_1(x)x}{1+|x|}. \tag{2.72}$$

Since (2.71) give

$$f(t,x)+g(t,x)\Lambda_*(T)\omega(x)=-\tfrac{2x}{1+|x|} \tag{2.73}$$

our conditions (2.21) on the r_i's from Assumption 2 for the preceding choices of f, g, V, and W will be satisfied if

$$\begin{aligned}&|\sigma_1(x)|\leq\sqrt{\tfrac{2\sigma_1(x)x}{1+|x|}},\ \tfrac{1}{|\Lambda_*(T)|}\tfrac{3|x|}{1+|x|}\leq r_1\sqrt{\tfrac{2\sigma_1(x)x}{1+|x|}},\ \tfrac{|x|}{1+|x|}\leq r_2\sqrt{\tfrac{2\sigma_1(x)x}{1+|x|}},\\ &\text{and} \ \ 1\leq r_3.\end{aligned} \tag{2.74}$$

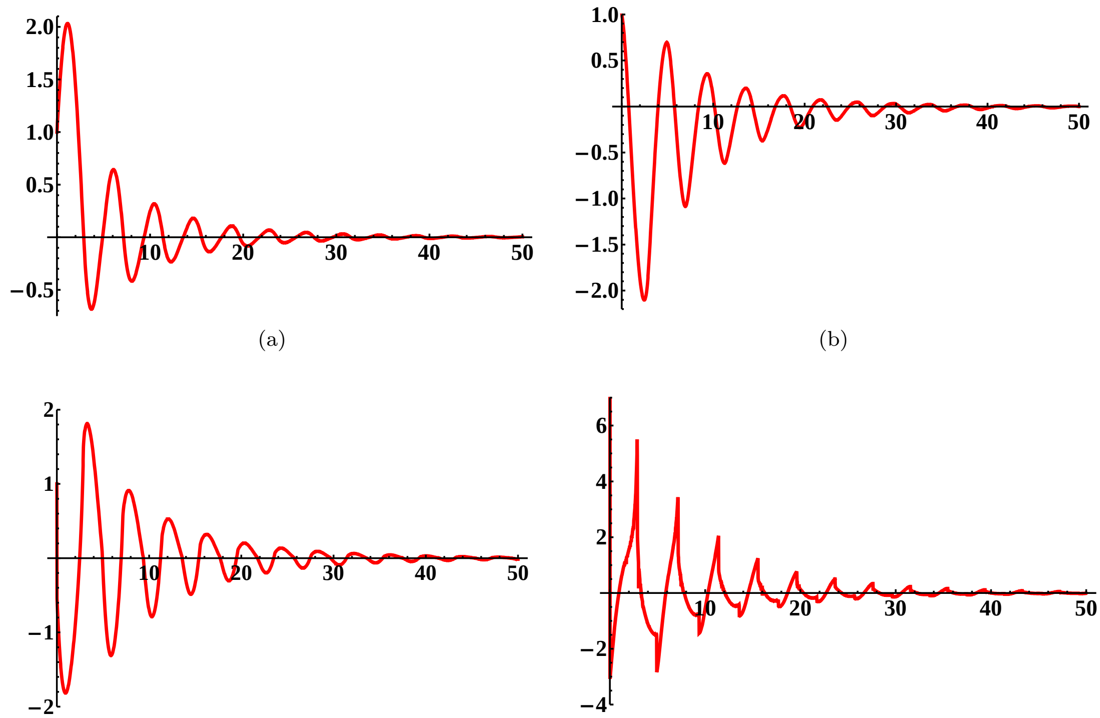
By separately considering points $x\in[-1,1]$ and points $x\notin[-1,1]$, it follows easily that Assumption 2 is satisfied with the choices

$$C=\tfrac{3}{|\Lambda_*(T)|},\ r_1=\tfrac{3}{\sqrt{2}|\Lambda_*(T)|},\ r_2=1,\ \text{and}\ r_3=1. \tag{2.75}$$

Hence, our requirement (2.23) on $T > 0$ from Proposition 2.6 holds if

$$1 > 4(3T)^2 \left[2 + \tfrac{5}{2} \left(\tfrac{3T^3}{\sqrt{2}|2-T-e^{-T}(2+T)|} \right)^2 \right] \tag{2.76}$$

and we can use Mathematica [66] to check that the right side of (2.76) takes the value 0.912536 at $T = 0.11$. Hence, Assumption 1 is satisfied with $T = 0.11$, and then the desired controller is provided by Theorem (2.3). See Figure 2.1 simulations of the closed loop system with the preceding values, which were done using the `NDSolve` command in Mathematica.



(a) (b) (c) (d)

Figure 2.1. Convergence of (x, z) States and Control u for First Illustration: x in Upper Left, z_1 in Upper Right, z_2 in Lower Left, u in Lower Right

We can sometimes apply Theorem 2.3 by checking Assumption 1 through a mixture of Lyapunov and direct trajectory analyses. For instance, in the second

illustration, consider the three dimensional system

$$\begin{cases} \dot{x} &= x^2 - x^3 + z_1 \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= u. \end{cases} \tag{2.77}$$

As noted in [26, pages 593-594], the system (2.77) is globally asymptotically stabilized to 0 by the control

$$u(x,z) = -\tfrac{\partial V_0}{\partial z_1}(x,z_1) + \tfrac{\partial \phi}{\partial z_1}(x,z_1)z_2 - z_2 + \tfrac{\partial \phi}{\partial x}(x,z_1)(x^2 - x^3 + z_1) + \phi(x,z_1),$$
$$\text{where } \ V_0(x,z_1) = \tfrac{1}{2}x^2 + \tfrac{1}{2}(z_1 + x + x^2)^2 \ \text{ and}$$
$$\phi(x,z_1) = -2x - (1+2x)(x^2 - x^3 + z_1) - z_1 - x^2,$$

which is unbounded since it satisfies $\lim_{x\to+\infty} u(x,0) = -\infty$. Our work [44] provided the unbounded control

$$\begin{aligned} u(t) = \tfrac{1}{(1-e^{-\tau})^2}\big\{ J(x(t)) - 2e^{-\tau}J(x(t-\tau)) + e^{-2\tau}J(x(t-2\tau))\big\} \\ -2z_2(t) - z_1(t), \quad \text{where } \ J(x) = -\sin\left(\tfrac{\pi x}{2}\right)\mathbf{1}_{[-2,2]}(x) \end{aligned} \tag{2.78}$$

that rendered (2.77) globally asymptotically stable to 0, where the indicator function $\mathbf{1}_{[-2,2]}$ is defined to be 1 on $[-2,2]$, and 0 on $\mathbb{R}\setminus[-2,2]$. Here we show how our new Theorem 2.3 provides a globally bounded globally asymptotically stabilizing controller for (2.77), using the choice of $\omega = J/\Lambda_*(T)$ with J as defined in (2.78), and with $p = 1$, and $k = 2$ and with the artificial $T > 0$ to be specified.

To verify Assumption 1 with the preceding choices, first note that for each continuous function $\delta : \mathbb{R} \to \mathbb{R}$ that exponentially converges to 0 and each initial state $x_0 \in \mathbb{R}$, we can find a value $T_*(x_0,\delta) \in [0,+\infty)$ such that the corresponding solution of

$$\dot{x}(t) = x^2(t) - x^3(t) + \int_{t-T}^{t} \Lambda_1(\ell,t)\omega(x(\ell))\mathrm{d}\ell + \delta(t) \tag{2.79}$$

satisfies $x(t) \in [-0.8, 3/2]$ for all $t \geq T_*(x_0,\delta)$. This can be done by noting that the integral in (2.79) is bounded by 1 (since $\Lambda_a(T) = |\Lambda_*(T)|$), that $x^2 - x^3 \leq -1.125$

for all $x \geq 3/2$, and that $x^2 - x^3 \geq 1.152$ for all $x \leq -0.8$, so the right side terms $x^2(t) - x^3(t)$ in (2.79) dominate the other right side terms, since we may assume that t is large enough so that $|\delta(t)| \leq 0.12$. Hence, it suffices to check the inequalities (2.21) from Assumption 2 for all $x \in [-0.8, 3/2]$, by only considering time values $t \geq T_*(x_0, \delta)$.

We now check the estimates from (2.21) for all $x \in [-0.8, 3/2]$ using $V(x) = \frac{1}{2}x^2$, $W(x) = x^2$, $f(x) = x^2 - x^3$, and $g(x) = 1$. First note that simple calculations (e.g., using Mathematica [66]) give $x^2 - x^3 - \sin(\pi x/2) \leq -x$ (resp., $\geq -x$) for all $x \in [0, 3/2]$ (resp., $x \in [-0.8, 0]$) which gives $\nabla V(x)(f(x) + \Lambda_*(T)\omega(x)) \leq -W(x)$, $|x^2 - x^3| \leq 1.44|x|$, and $|\sin(\pi x/2)| \leq (\pi/2)|x|$ when $x \in [-0.8, 3/2]$, so we can choose $r_1 = \pi/(2|\Lambda_*(T)|)$, $r_2 = 1.44$, $r_3 = 1$, and $C = \pi/(2|\Lambda_*(T)|)$. Hence, we can use our formula (2.70) for $\Lambda_*(T)$ to check that the sufficient condition (2.23) from Proposition 2.6 (for $\lim_{t\to+\infty} x(t) = 0$ to hold) is satisfied if

$$1 \;>\; (T\pi)^2 \left(2(1.44)^2 + \frac{5\pi^2}{8} \left(\frac{T^3}{2 - T - e^{-T}(2+T)} \right)^2 \right) \tag{2.80}$$

which is satisfied for all $T \in (0, 0.0209]$. Therefore, we can satisfy our requirements with $T = 0.0209$, and then the desired bounded control is provided by Theorem 2.3. See Figure 2.2 for simulations of the closed loop system with the preceding values, which were also done using the `NDSolve` command in Mathematica.

2.7 Conclusions

We provided a new bounded backstepping technique for a large class of cascaded partially linear systems with arbitrarily large numbers of integrators, under a converging-input-converging-state assumption involving the nonlinear subsystems. For many cases where the nonlinear part of the system is control affine, we used Lyapunov functions to provide sufficient conditions for our converging-input-converging-state assumption to be satisfied. Although our controller involves a

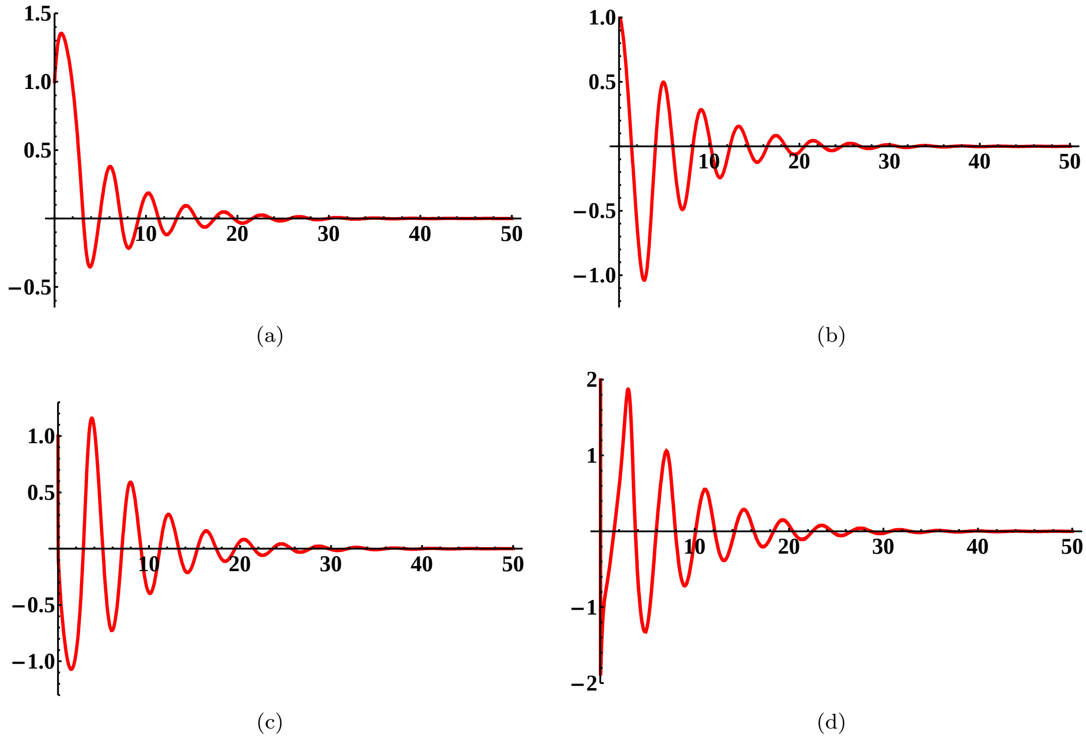


Figure 2.2. Convergence of (x, z) States and Control u for Second Illustration: x in Upper Left, z_1 in Upper Right, z_2 in Lower Left, u in Lower Right

dynamic extension, it has an advantage that it provides bounded controllers for the original system, which would not have been possible under our assumptions if we had instead relied on previous results.

Chapter 3.
Sequential Predictors

3.1 Introduction

This chapter continues our group's search (begun in [37, 38]) for sequential predictor based delay compensation methods for systems where the sup norms of the feedback delays are allowed to be arbitrarily large. Whereas [38] was confined to nonlinear time-invariant systems with constant feedback delays, and [37] was confined to time-varying linear systems with time-varying C^1 delays, and neither [37] nor [38] allowed measurement delays or outputs or sampling, the novelty and value of this work is that we cover time-varying linear systems under arbitrarily long feedback delays, sampling in the feedback control, measurement delays, and outputs. Our novel synergy of Lyapunov and trajectory-based methods makes it possible to prove global exponential stability under sampling without any distributed terms in the controls, using a new set of sampling sequential predictors.

A common feature among predictor methods is that they replace the state of the system in the feedback control by a new state variable that eliminates the effects of the feedback delay. Sequential predictors [8] (also known as chain predictors) are useful for compensating for arbitrarily long delays and for overcoming the obstacles that one may encounter when using other delay compensation methods (such as the reduction model approach) that would lead to controls whose distributed or non-explicit terms may be difficult to compute in practice [2, 49], or emulation approaches that produce upper bounds on the allowable feedback delays (but see, e.g., [1], [10], [11], [12], and [69] for predictive controls without distributed terms for certain special cases of time invariant systems, and [13] for chain observers that do not cover the problems we address here). Since distributed terms use values of the state or of the control on an interval of past times, they may not always

lend themselves to applications. See [22] for feedback controls under sampling and measurement delays for time invariant systems, using distributed terms, and Remark 3.2 below for more on the connections between our work and [22]. Instead of distributed terms or emulation, sequential predictors use dynamic extensions that consist of copies of the original system running on different time scales and additional stabilizing terms. The delayed state of the last dynamic extension is used in the control formula. While sequential predictor methods are advantageous because of their lack of distributed terms, the existing sequential predictor methods for time-varying linear systems [37] use the current state $x(t)$ of the original systems in the first sequential predictor, and their C^1 requirement on the feedback delay excludes the sawtooth shaped delays that would arise from sampling. However, there may be delays in the transmission of the state measurements from the physical system, e.g., in the marine robotic work [48] whose controls were computed on a computer that was far from the physical system but where the delays are known. There are many other cases where the delays are known or can be estimated; see, e.g., [14]. Therefore, this chapter studies the combined effects of (i) feedback delays $h(t)$ in the original system, (ii) sampling in the feedback in the original system (allowing nonperiodic sampling), (iii) outputs (meaning only a function $y = Cx$ of the state of the original system is available for measurement for some constant matrix C, instead of having state measurements available for use in the control), and (iv) measurement delays $\tau(t)$ in the sequential predictors where the output of the original system $y(t)$ in the first sequential predictor is replaced by $y(t - \tau(t))$.

While non-C^1 time-varying delays can be used to model sampling, we find it convenient to use C^1 time-varying feedback delays $h(t)$ (as we did in [37]) and then to apply a sampling operator to the C^1 delay, which sets our work apart from [37, 38, 65]where no sampling was allowed. Sampling is usually done in engineering

implementations, but the effects had previously been ignored in the sequential predictors literature under time-varying delays. It is important to consider the effects of sampling in feedbacks, since sampling that is too infrequent may make it impossible to solve feedback control problems.

We illustrate how our new methods can lead to a smaller required number of sequential predictor than were required in [37]. This work can benefit engineering systems that are prone to measurement delays and sampling, and reduce the computational burden that would come from computing too large a number of sequential predictors. This work adds value compared with our preliminary conference paper [65] which did not analyze the effects of sampling and where the sequential predictor method was only applied to the pendulum dynamics without sampling (while here we cover the pendulum system with sampling). To cover sampling, we derive a new set of sequential predictors in which the predictors also contain sampling and output measurements.

3.2 Main Result for Sequential Predictors

Before stating our main result, we require the following additional notation. For a strictly increasing unbounded sequence of nonnegative values t_j with $t_0 = 0$, and each strictly increasing function $p : \mathbb{R} \to \mathbb{R}$, the sampler σ is defined by the composition $(\sigma \circ p)(t) = p(t_j)$ for all $j \geq 0$ and $t \in [t_j, t_{j+1})$. Hence, if p is nonnegative valued, then $\sigma \circ p : [0, \infty) \to [0, \infty)$. When p is the identity function, $(\sigma \circ p)(t)$ is written as $\sigma(t)$. We use the notation $f_t(s) = f(t + s)$ for any function f and any $t \geq 0$ and $s \leq 0$ for which $t + s$ is in f's domain. The dimensions of our Euclidean spaces are arbitrary unless otherwise indicated, I_n is the $n \times n$ identity matrix, $|\cdot|$ is the usual Euclidean norm, and $|\cdot|_\infty$ (resp., $|\cdot|_\mathcal{I}$) denotes the essential supremum over $[0, \infty)$ (resp., any interval $\mathcal{I} \subseteq [0, \infty)$) in the Euclidean norm.

We study systems with outputs of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u((\sigma \circ \Omega_m)(t)) + \delta(t), \quad y(t) = Cx(t) \tag{3.1}$$

whose state x, control u, and output y are valued in $\mathbb{R}^n$, $\mathbb{R}^\ell$, $\mathbb{R}^r$ respectively, $\Omega_m(t) = t - h(t)$ for a known time-varying delay $h : \mathbb{R} \to [0, \infty)$, the matrix valued functions A and B and $C \in \mathbb{R}^{r \times n}$ are known, the measurable locally essentially bounded function δ represents uncertainty, and the composition $\sigma \circ \Omega_m$ is defined as above with $p(t) = \Omega_m$ where $\{t_j\}$ is a given sequence of sample times such that $t_0 = 0$ and that admit positive constants ε_1 and ε_2 such that

$$\varepsilon_1 \leq t_{j+1} - t_j \leq \varepsilon_2 \tag{3.2}$$

for all $j \geq 0$. The control u will be specified. We will interconnect (3.1) with a dynamic control whose right side depends on delayed output values $y(t - \tau(t))$, and τ may differ from h. Assume:

Assumption 3. *The nonnegative valued functions h and τ are C^1 and bounded from above by constants $c_h \geq 0$ and $c_\tau \geq 0$ respectively, $\dot{h}$ and $\dot{\tau}$ have finite lower bounds, $\dot{h}$ and $\dot{\tau}$ are bounded from above by constants $l_h \in [0, 1)$ and $l_\tau \in [0, 1)$ respectively, and $\dot{h}$ has a global Lipschitz constant $n_h > 0$.* □

Assumption 4. *The functions A and B in (3.1) are bounded and continuous on $\mathbb{R}$, and there are known bounded continuous functions $K : [0, \infty) \to \mathbb{R}^{\ell \times n}$ and $L : [0, \infty) \to \mathbb{R}^{n \times r}$ such that the systems*

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) \;\; \textit{and} \;\; \dot{z}(t) = [A(t) + L(t)C]z(t) \tag{3.3}$$

are uniformly globally exponentially stable on $\mathbb{R}^n$ to 0. □

Assumption 5. *The function K from Assumption 4 admits a global Lipschitz constant $l_K > 0$.* □

Assumption 4 is a time-varying analog of the usual conditions in the constant coefficients case that (A, B) is controllable and (A, C) is observable. By Assumption 4 and standard results (e.g., [26, Theorem 4.14]), we can find bounded C^1 functions $P_i : [0, \infty) \to \mathbb{R}^{n\times n}$ and positive constants $\bar{c}_i$ such that the functions $V_i(t, x) = x^\top P_i(t)x$ satisfy (i) $V_i(t, x) \geq \bar{c}_i|x|^2$ for all $t \geq 0$ and $x \in \mathbb{R}^n$ and $i = 1, 2$, (ii) $\frac{d}{dt}V_1(t, x(t)) \leq -|x(t)|^2$ holds along all solutions $x : [0, \infty) \to \mathbb{R}^n$ of $\dot{x}(t) = [A(t) + B(t)K(t)]x(t)$, and (iii) $\frac{d}{dt}V_2(t, z(t)) \leq -|z(t)|^2$ holds along all solutions $z : [0, \infty) \to \mathbb{R}^n$ of $\dot{z}(t) = [A(t) + L(t)C]z(t)$. For the rest of this work, we fix functions P_i and positive constants $\bar{c}_i$ satisfying the preceding requirements, and we assume that the initial functions at time 0 are constant, e.g., $x(\ell)$ is constant on $(-\infty, 0]$.

In terms of an integer $m > 1$ that we specify later, we also set

$$\Omega_i(t) = t - \frac{i}{m}h(t) \text{ and } \theta_j(t) = \Omega^{-1}_{m-j+1}(\Omega_{m-j}(t)) \tag{3.4}$$

for all $i \in \{0, ..., m\}$ and $j \in \{1, \dots, m\}$, $R_1 = \dot{\theta}_1$, and $R_i(t) = \dot{\theta}_i(t)R_{i-1}(\theta_i(t))$ for $i = 2, \dots, m$, where the Ω_i's are invertible because Assumption 3 implies that $\Omega_i'(t) \geq 1 - l_h > 0$ for all i and t. We also set

$$u_c = n_h \frac{c_h}{(1 - l_h)^2} + \frac{l_h}{1 - l_h}, \tag{3.5}$$

$\phi(t) = t - \tau(t)$, and $G_i(t) = \Omega_m^{-1}(\Omega_{m-i}(t))$ for all $t \geq 0$ and $i \in \{0, 1, \dots, m\}$. Note for later use that $G_1 = \theta_1$. Also, for all $i \in \{1, 2, \dots, m\}$, we have $G_{i-1} \circ \theta_i = G_i$, $\dot{G}_i = R_i$, and (by [37, Lemma 1]) $|R_i|_\infty \leq \left(1 + \frac{u_c}{m}\right)^m$. Moreover, Assumption 3 provides a constant $g_0 > 0$ such that $\dot{G}_i(t) \geq g_0$ for all $t \geq 0$ and $i \in \{1, 2, \dots, m\}$. Therefore, we can construct bounded sequences $L_i : [0, \infty) \to \mathbb{R}^{n\times r}$ and $P_{2i} : [0, \infty) \to \mathbb{R}^{n\times n}$ of continuous functions and constants $\bar{c}_{2i} > 0$ such that for each $i \in \{1, 2, \dots, m\}$, the function $V_{2i}(t, x) = x^\top P_{2i}(t)x$ satisfies $V_{2i}(t, x) \geq \bar{c}_{2i}|x|^2$ for

all $t \geq 0$ and $x \in \mathbb{R}^n$ and $\frac{d}{dt}V_{2i}(t, x(t)) \leq -|x(t)|^2$ along all solutions of

$$\dot{x}(t) = [R_i(t)A(G_i(t)) + L_i(t)C]x(t) \tag{3.6}$$

for all $t \geq 0$. For instance, we can choose

$$P_{2i}(t) = \frac{1}{g_0}P_2(G_i(t)),\ L_i(t) = R_i(t)L(G_i(t)),\ \text{and } \bar{c}_{2i} = \frac{\bar{c}_2}{g_0} \tag{3.7}$$

for all i and $t \geq 0$ (but see Remark 3.2 for other choices). We prove the following:

Theorem 3.1. *Let Assumptions 3-5 hold and the sequences* $\{t_j\}$, $\{P_{2i}\}$, *and* $\{L_i\}$ *satisfy the requirements above. Set* $\bar{P} = \max_i |P_{2i}|_\infty$. *Assume that* $\max_{1\leq i\leq m} |L_iC|_\infty > 0$ *and that*

$$2|P_1B|_\infty\varepsilon_2\left(\sqrt{2}|K|_\infty e^{\varepsilon_2^2|W|_\infty^2}|A+W|_\infty + l_K\right) < 1 \tag{3.8}$$

where $W(t) = B(t)K(\sigma(t))$ *and that* $m \geq 2$ *is an integer such that*

$$\begin{gathered} 4\bar{P}\max_{1\leq i\leq m}|L_iC|_\infty\left(c_\tau + \frac{c_h}{m(1-l_h)}\right)\max\{\mathcal{M}_1, \mathcal{M}_2\} < 1, \\ \text{where } \mathcal{M}_1 = \left(1+\frac{u_c}{m}\right)^m|A|_\infty \text{ and } \mathcal{M}_2 = \max_{1\leq i\leq m}|L_iC|_\infty\sqrt{\frac{1+\frac{u_c}{m}}{1-l_\tau}}\ . \end{gathered} \tag{3.9}$$

Then we can construct positive constants μ_1 *and* μ_2 *such that for all solutions* $x(t)$ *of (3.1) in closed loop with* $u(t) = K(\Omega_m^{-1}(t))z_m(t)$, *where* z_m *is the last* n *components of*

$$\begin{aligned} \dot{z}_i(t) &= R_i(t)\big[A(G_i(t))z_i(t) + B(G_i(t))u((\sigma\circ\Omega_m)((\Omega_m^{-1}\circ\Omega_{m-i})(t)))\big] \\ &\quad + L_i(t)C\mathcal{E}_i(\theta_i^{-1}(\phi(t))),\ 1 \leq i \leq m \end{aligned} \tag{3.10}$$

having the state space $\mathbb{R}^{mn}$, *we have*

$$|(x(t), \mathcal{E}(t))| \ \leq\ \mu_1\left(|(x,\mathcal{E})|_{[-c_h-c_\tau,0]}e^{-\mu_2 t} + |\delta|_\infty\right) \tag{3.11}$$

for all $t \geq 0$, *where*

$$\mathcal{E}(t) = (\mathcal{E}_1, \ldots, \mathcal{E}_m)(t) = (z_1(t) - x(\theta_1(t)), z_2(t) - z_1(\theta_2(t)), \ldots, z_m(t) - z_{m-1}(\theta_m(t)))$$

. □

3.3 Novelty and Value of Sequential Predictors Theorem

Remark 3.2. When $C = I_n$, we satisfy the requirements from our theorem with

$$L_i(t) = -I_n - R_i(t)A(G_i(t)) \text{ and } P_{2i}(t) = \frac{1}{2}I_n, \tag{3.12}$$

in which case (3.8)-(3.10) agree with the predictors and conditions in [65] if we replace σ in (3.10) by the identity operator (but [65] does not allow sampling or outputs). The z_i dynamics in (3.10) is called the ith sequential (sub)predictor. Constantness of C was used to ensure that (3.10) depends on the delayed output $y(t-\tau(t))$ but not the state. The method from the notable work [22] for constant delays of combining the input and measurement delays into one delay $h+\tau$ in u does not apply here, because for time-varying delays, it would produce the condition $\dot{h} + \dot{\tau} < 1$, and because [22] leads to distributed terms and requires time-invariant systems with periodic sampling. □

Remark 3.3. When $\tau = 0$ and $L_i = R_i L(G_i)$ for all i, we can choose m large enough to satisfy (3.9), using the limit $\lim_{m\to\infty}\left(1 + \frac{u_c}{m}\right)^{m/u_c} = e$. The intuition of a large m in (3.9) is that larger c_h's allow larger prediction horizons $|h|_\infty$ and so need more sequential predictors to produce predicted values for the control. Condition (3.9) is needed even in the absence of measurement delays and sampling, but when $\tau = 0$, we can pick $c_\tau = 0$. If m is large enough and $C = I_n$, and if we make the choices $L_i(t) = -I_n + R_i(t)A(G_i(t))$ and $P_{2i}(t) = \frac{1}{2}I_n$ from Remark 3.2, then we can again use the limit definition of e and (3.26) to check that (3.9) holds if

$$\frac{c_\tau}{\sqrt{1-l_\tau}} \in \left[0, \frac{1}{2(1+e^{u_c}|A|_\infty)^2}\right). \tag{3.13}$$

Condition (3.13) holds in cases of interest, since $\tau(t)$ only comes from measurement delays, whereas $h(t)$ comes from both delays in the actuation in (3.1) and from measurement delays from the mth sequential predictor to the plant. Therefore, h

would be much larger than τ in practice. However, (3.8) can be satisfied if ε_2 is sufficiently small, and we can allow cases where $c_\tau > c_h$. Theorem 3.1 improves on [37] for cases with no measurement delays τ, $C = I_n$, and no sampling. When $\tau = 0$ and when there is no sampling and $C = I_n$, [37] required

$$\max\left\{2, 4u_c^\sharp\left(\frac{1}{\sqrt{2}}\left(1+\frac{u_c}{m}\right)^m |A|_\infty + u_c^\sharp\left(1+\frac{u_c}{m}\right)\right)\frac{c_h}{1-l_h}\right\} < m, \tag{3.14}$$

where, in terms of our notation from (3.9), $u_c^\sharp = 1 + \left(1+\frac{u_c}{m}\right)^m |A|_\infty$. Section 3.6 below illustrates how (3.9) can allow a smaller m compared with (3.14) when $\tau = 0$ and $C = I_n$. $\square$

3.4 Lemmas to Prove Sequential Predictors Theorem

We first provide four lemmas, which will use the fact that for each constant $c_* > 0$ and each continuous function $q : [-c_*, \infty) \to [0, \infty)$ and $t \geq 0$, the conditions

$$\begin{aligned} &\int_{t-c_*}^t \int_\ell^t q(r)\mathrm{d}r\mathrm{d}\ell \leq c_* \int_{t-c_*}^t q(r)\mathrm{d}r \ \text{ and} \\ &\tfrac{d}{dt}\int_{t-c_*}^t \int_\ell^t q(r)\mathrm{d}r\mathrm{d}\ell = c_* q(t) - \int_{t-c_*}^t q(r)\mathrm{d}r \end{aligned} \tag{3.15}$$

hold. Throughout this section, we maintain our notation and assumptions from Theorem 3.1.

Lemma 3.4. *We can find positive constants v_1 and v_2 such that the time derivative of $V_s(t, x_t) = V_1(t, x(t)) + v_1 \int_{t-\varepsilon_2}^t \int_s^t |x(\ell)|^2 \mathrm{d}\ell\mathrm{d}s$ along all solutions of*

$$\dot{x}(t) = A(t)x(t) + B(t)K(\sigma(t))x(\sigma(t)) \tag{3.16}$$

satisfies $\dot{V}_s \leq -v_2 V_s(t, x_t)$ for all $t \geq 0$, where $V_1(t, x) = x^\top P_1(t)x$ as before. $\square$

Proof: We can rewrite the system (3.16) as $\dot{x}(t) = W^\sharp(t)x(t) + W(t)(\Delta x)(t)$, where $(\Delta x)(t) = x(\sigma(t)) - x(t)$, $W(t) = B(t)K(\sigma(t))$ and $W^\sharp(t) = A(t) + W(t)$. Hence, we can use Jensen's Inequality and the estimate $|p + q|^2 \leq 2|p|^2 + 2|q|^2$ for

suitable choices of p and q to get

$$\begin{aligned}
|(\Delta x)(t)|^2 &\le \left|\int_{\sigma(t)}^t \dot{x}(\ell)\mathrm{d}\ell\right|^2 \\
&\le \varepsilon_2 \int_{\sigma(t)}^t |\dot{x}(\ell)|^2 \mathrm{d}\ell \\
&\le \varepsilon_2 \int_{\sigma(t)}^t \big(|W^\sharp|_\infty |x(\ell)| + |W|_\infty |(\Delta x)(\ell)|\big)^2 \mathrm{d}\ell \\
&\le 2\varepsilon_2 \int_{\sigma(t)}^t |W^\sharp|_\infty^2 |x(\ell)|^2 \mathrm{d}\ell + 2\varepsilon_2 |W|_\infty^2 \int_{\sigma(t)}^t |(\Delta x)(\ell)|^2 \mathrm{d}\ell,
\end{aligned}$$

since $t-\sigma(t) = t - t_j \le \varepsilon_2$ for all $t \in [t_j, t_{j+1})$ and $j \ge 0$, so Gronwall's inequality from [5, p.218] applied to the integrand $w(\ell) = |\Delta x(\ell)|^2$ on $[\sigma(t), t]$ gives

$$|\Delta x(t)|^2 \le 2\varepsilon_2 e^{2\varepsilon_2^2 |W|_\infty^2} \int_{\sigma(t)}^t |W^\sharp|_\infty^2 |x(\ell)|^2 \mathrm{d}\ell \tag{3.17}$$

for all $t \in [t_j, t_{j+1})$ and $j \ge 0$, because σ is constant on each interval $[t_j, t_{j+1})$. Since $\sigma(t) \ge t - \varepsilon_2$ for all $t \ge 0$, and since (3.16) can also be written as

$$\begin{aligned}
\dot{x}(t) &= (A(t) + B(t)K(t))x(t) + B(t)(K(\sigma(t)) \\
&\quad -K(t))x(t) + B(t)K(\sigma(t))(\Delta x)(t),
\end{aligned} \tag{3.18}$$

it follows from Assumption 4 and the relation $rs \le \Delta_0 r^2/2 + s^2/(2\Delta_0)$ with the choices $\Delta_0 = 1 - 2|P_1 B|_\infty l_K \varepsilon_2$ (which is positive because of (3.8)), $r = |x(t)|$, and $s = 2|P_1 W|_\infty |\Delta x(t)|$ that along all solutions of (3.16) for all $t \ge 0$, we have

$$\begin{aligned}
\dot{V}_1 &\le -|x(t)|^2 + 2|x(t)||P_1 B|_\infty l_K \varepsilon_2 |x(t)| \\
&\quad +2x^\top(t)P_1(t)W(t)(x(\sigma(t)) - x(t)) \\
&= -\Delta_0 |x(t)|^2 + \{x^\top(t)\}\{2P_1(t)W(t)(x(\sigma(t)) - x(t))\} \\
&\le -\frac{\Delta_0}{2}|x(t)|^2 + \frac{2}{\Delta_0}|P_1 W|_\infty^2 |x(\sigma(t)) - x(t)|^2 \\
&\le -\frac{\Delta_0}{2}|x(t)|^2 + v_* \int_{t-\varepsilon_2}^t |x(\ell)|^2 \mathrm{d}\ell,
\end{aligned} \tag{3.19}$$

where $v_* = (4\varepsilon_2/\Delta_0)|P_1 W|_\infty^2 |W^\sharp|_\infty^2 e^{2\varepsilon_2^2 |W|_\infty^2}$, because $|K(\sigma(t)) - K(t)| \le l_K \varepsilon_2$ for all $t \ge 0$. Since $W^\sharp = A + W$, (3.8) also implies that $\varepsilon_2 v_* < \Delta_0/2$. Fix a constant $\lambda > 1$ that is close enough to 1 so that $\lambda \varepsilon_2 v_* < \Delta_0/2$, and set $v_1 = \lambda v_*$. Then

we can use (3.15) with $c_* = \varepsilon_2$ and $q(\ell) = |x(\ell)|^2$ to check the conclusions of the lemma with $v_2 = \min\{(0.5\Delta_0 - \lambda\varepsilon_2 v_*)/|P_1|_\infty, (\lambda - 1)/(\lambda\varepsilon_2)\}$, by lower bounding the single integral in (3.19) using the double integral from the V_s formula. $\square$

Lemma 3.5. *The dynamics for $\mathcal{E}$ in Theorem 3.1 are*

$$\begin{cases} \dot{\mathcal{E}}_1(t) = R_1(t)A(G_1(t))\mathcal{E}_1(t) + L_1(t)C\mathcal{E}_1(\theta_1^{-1}(\phi(t))) - \dot{\theta}_1(t)\delta(\theta_1(t)) \\ \dot{\mathcal{E}}_i(t) = R_i(t)A(G_i(t))\mathcal{E}_i(t) + L_i(t)C\mathcal{E}_i(\theta_i^{-1}(\phi(t))) \\ \qquad\qquad - \dot{\theta}_i(t)L_{i-1}(\theta_i(t))C\mathcal{E}_{i-1}(\theta_{i-1}^{-1}(\phi(\theta_i(t)))),\ 2 \le i \le m. \end{cases} \tag{3.20}$$

Also, the system (3.1), in closed loop with $u(t) = K(\Omega_m^{-1}(t))z_m(t)$ and (3.10), can be written as

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)K(\sigma(t))x(\sigma(t)) \\ &+\{B(t)[K(\sigma(t))\mathcal{E}_1(\theta_2(...\theta_m((\sigma \circ \Omega_m)(t))))+ \\ &... + K(\sigma(t))\mathcal{E}_m((\sigma \circ \Omega_m)(t))] + \delta(t)\}, \end{aligned} \tag{3.21}$$

and for all $t \ge 0$, we have $z_m(t) = x(\Omega_m^{-1}(t)) + \mathcal{E}_1(\theta_2(...\theta_m(t)...)) + ... + \mathcal{E}_m(t)$. $\square$

Proof: Setting $z_0 = x$ and $\xi_i(t) = z_{i-1}(\theta_i(t))$ gives $\mathcal{E}_i(t) = z_i(t) - \xi_i(t) = z_i(t) - z_{i-1}(\theta_i(t))$ for all $i \in \{1, ..., m\}$, and

$$\begin{aligned} \dot{\xi}_1(t) &= R_1(t)A(\theta_1(t))\xi_1(t) + \dot{\theta}_1(t)\delta(\theta_1(t)) \\ &\quad +R_1(t)B(\theta_1(t))u\left((\sigma \circ \Omega_m)(\theta_1(t))\right). \end{aligned} \tag{3.22}$$

Our choices of $G_1 = \theta_1$ and z_1 in (3.10) then give the $\dot{\mathcal{E}}_1$ formula from (3.20). If $2 \le i \le m$, then $G_{i-1} \circ \theta_i = G_i$ and $\dot{\theta}_i(t)R_{i-1}(\theta_i(t)) = R_i(t)$ and so also

$$\begin{aligned} \dot{\xi}_i(t) &= R_i(t)B(G_i(t))u((\sigma \circ \Omega_m)((\Omega_m^{-1} \circ \Omega_{m-i+1})(\theta_i(t))) \\ &\quad + \dot{\theta}_i(t)L_{i-1}(\theta_i(t))C[z_{i-1}(\theta_{i-1}^{-1} \circ \phi(\theta_i(t))) - z_{i-2}(\phi \circ \theta_i(t))] \\ &\quad +R_i(t)A(G_i(t))\xi_i(t)\ . \end{aligned} \tag{3.23}$$

Combining the formula for $\dot{z}_i$ from (3.10) with (3.23) and using the fact that $\Omega_{m-i+1}(\theta_i(t)) = \Omega_{m-i}(t)$ implies that for each $i \in \{2, \ldots, m\}$, the variable $\mathcal{E}_i =$

$z_i - \xi_i$ satisfies the formula for $\dot{\mathcal{E}}_i$ in (3.20). The z_m formula in the conclusion follows from the formulas

$$z_m(t) = \mathcal{E}_m(t) + z_{m-1}(\theta_m(t)) \tag{3.24}$$

and $\theta_1 \circ \ldots \circ \theta_m = \Omega_m^{-1}$, and repeated applications of the relations

$$z_i(\theta_{i+1}) = \mathcal{E}_i(\theta_{i+1}) + z_{i-1}(\theta_i \circ \theta_{i+1}).[\tag{3.25}$$

Then (3.21) follows from replacing t by $\sigma \circ \Omega_m(t)$ in the z_m formula, and then substituting the final z_m formula into $u(t) = K(\Omega_m^{-1}(t))z_m(t)$. $\square$

We also use the following lemma that was shown in [37]:

Lemma 3.6. *The θ_i's in (3.4) and the constant (3.5) satisfy*

$$|\dot{\theta}_i(\ell) - 1| \leq \frac{u_c}{m} \quad \text{and} \quad |\theta_i^{-1}(\ell) - \ell| \leq \frac{c_h}{1 - l_h} \frac{1}{m} \tag{3.26}$$

for all $\ell \in \mathbb{R}$ and $i \in \{1, 2, \ldots, m\}$. $\square$

See the appendix below for a proof of the preceding lemma. Our final lemma is:

Lemma 3.7. *Let $\lambda > 1$ be a constant such that $\lambda \mathcal{H} < 1$, where $\mathcal{H}$ is the left side of the inequality in (3.9), and $i \in \{1, 2, \ldots, m\}$. Set*

$$c_m^\sharp = c_\tau + \frac{c_h}{m(1 - l_h)} \quad \textit{and} \quad \bar{L} = \max_{1 \leq i \leq m} |L_i C|_\infty, \tag{3.27}$$

and set

$$\begin{aligned}
&Q_i^\sharp(t, s_t) = V_{2i}(t, s(t)) + \lambda^2 \bar{\alpha} \mathcal{I}(s_t), \ \ \textit{where } \mathcal{I}(s_t) = \textstyle\int_{t-2c_m^\sharp}^t \int_\ell^t |s(r)|^2 \mathrm{d}r \mathrm{d}\ell, \\
&V_{2i}(t, s) = s^\top P_{2i}(t) s, \quad \textit{and} \\
&\bar{\alpha} = 4\bar{P}^2 c_m^\sharp \bar{L}^2 \max\left\{ \left(1 + \frac{u_c}{m}\right)^{2m} |A|_\infty^2, \bar{L}^2 \left(1 + \frac{u_c}{m}\right) \frac{1}{1 - l_\tau} \right\}.
\end{aligned} \tag{3.28}$$

Then we can find a constant $c_0 > 0$ such that the time derivative of (3.28) along all solutions of

$$\dot{s}(t) = R_i(t) A(G_i(t)) s(t) + L_i(t) C s(\theta_i^{-1}(\phi(t))) \tag{3.29}$$

satisfies $\frac{d}{dt} Q_i^\sharp(t, s_t) \leq -c_0 Q_i^\sharp(t, s_t)$ for all $t \geq 0$. $\square$

Proof: Fix an index i. We rewrite (3.29) as

$$\dot{s}(t) \quad = \quad H_i(t)s(t) - L_i(t)C\mathcal{D}_i(t), \tag{3.30}$$

where

$$H_i(t) = R_i(t)A(G_i(t)) + L_i(t)C \ \text{ and } \ \mathcal{D}_i(t) = s(t) - s(\theta_i^{-1}(\phi(t))), \tag{3.31}$$

and where for all $t \geq 0$,

$$\begin{aligned} &|\mathcal{D}_i(t)| \leq \\ &\textstyle\int_{\theta_i^{-1}(\phi(t))}^{t} |R_i(\ell)A(G_i(\ell))s(\ell)|\mathrm{d}\ell + \int_{\theta_i^{-1}(\phi(t))}^{t} |L_i(\ell)Cs(\theta_i^{-1}(\phi(\ell)))|\mathrm{d}\ell. \end{aligned} \tag{3.32}$$

Since $\theta_i^{-1}(t) \leq t$, the second inequality in (3.26) gives $0 \leq t - \theta_i^{-1}(\phi(t)) = \tau(t) + \phi(t) - \theta_i^{-1}(\phi(t)) \leq c_m^\sharp$ for all $t \geq 0$. Therefore, Jensen's inequality gives

$$\left(\int_{\theta_i^{-1}(\phi(t))}^{t} |s(\ell)|\mathrm{d}\ell\right)^2 \leq c_m^\sharp \int_{\theta_i^{-1}(\phi(t))}^{t} |s(\ell)|^2\mathrm{d}\ell, \tag{3.33}$$

and (using $q = \theta_i^{-1}(\phi(\ell))$ and the second inequality in (3.26))

$$\begin{aligned} \left(\textstyle\int_{\theta_i^{-1}(\phi(t))}^{t} |s(\theta_i^{-1}(\phi(\ell)))|\mathrm{d}\ell\right)^2 &\leq \left(\textstyle\int_{\tau^\sharp(t)}^{t} |s(\theta_i^{-1}(\phi(\ell)))|\mathrm{d}\ell\right)^2 \\ &\leq c_m^\sharp \textstyle\int_{\theta_i^{-1}\left(\phi(t)-c_m^\sharp\right)}^{\theta_i^{-1}(\phi(t))} |s(q)|^2 \dfrac{\dot{\theta}_i(q)}{1-l_\tau}\mathrm{d}q \end{aligned}$$

for all $t \geq 0$, where $\tau^\sharp(t) = t - \tau(t) - \dfrac{c_h}{m(1-l_h)}$, since each θ_i^{-1} is increasing.

We next use the functions

$$\begin{aligned} &\mathcal{P}_i(t) = \bar{L} \textstyle\int_{\theta_i^{-1}(\phi(t))}^{t} |R_i(\ell)||A(G_i(\ell))||s(\ell)|\mathrm{d}\ell \ \text{ and} \\ &\mathcal{J}_i(t) = \bar{L}^2 \textstyle\int_{\theta_i^{-1}(\phi(t))}^{t} |s(\theta_i^{-1}(\phi(\ell)))|\mathrm{d}\ell \end{aligned}$$

and Young's Inequality $ab \leq 0.25a^2 + b^2$ twice with $a = |s(t)|$ to conclude that

$$\begin{aligned} &\frac{d}{dt}V_{2i}(t, s(t)) \leq -|s(t)|^2 + 2\bar{P}|s(t)|\mathcal{P}_i^\sharp(t) \\ &\leq -\frac{1}{2}|s(t)|^2 + \bar{\alpha}\left[\int_{\theta_i^{-1}(\phi(t))}^{t} |s(\ell)|^2\mathrm{d}\ell + \int_{t-2c_m^\sharp}^{\theta_i^{-1}(\phi(t))} |s(q)|^2\mathrm{d}q\right] \end{aligned} \tag{3.34}$$

along all solutions of (3.30), where

$$\mathcal{P}_i^\sharp(t) = \mathcal{P}_i(t) + \mathcal{J}_i(t), \tag{3.35}$$

by also using the fact that (3.26) gives $\max_{1\le i\le m}|R_i|_\infty \le (1+(u_c/m))^m$ and the upper bound $\bar{P}$ for all of the $|P_{2i}|_\infty$'s and

$$\theta_i^{-1}\left(\phi(t) - c_m^\sharp\right) \ge \theta_i^{-1}\left(t - c_m^\sharp - c_\tau\right) \ge t - 2c_m^\sharp, \tag{3.36}$$

to upper bound $\mathcal{P}_i^2$ and $\mathcal{J}_i^2$, respectively. It follows from (3.34) that along all solutions of (3.30) for all $t \ge 0$, we have

$$\frac{d}{dt}Q_i^\sharp(t, s(t)) \le \left[-\frac{1}{2} + 2\lambda^2 c_m^\sharp \bar{\alpha}\right]|s(t)|^2 + \frac{\bar{\alpha}}{2c_m^\sharp}(1-\lambda^2)\mathcal{I}(s_t), \tag{3.37}$$

where the $1/(2c_m^\sharp)$ factor followed from applying the inequality from (3.15) with $c_* = 2c_m^\sharp$ and $q(r) = |s(r)|^2$. By (3.9) and our choice of $\lambda > 1$ and the bound $V_{2i}(t,s)/\bar{P} \le |s|^2$, the coefficients of $|s(t)|^2$ and $\mathcal{I}(s_t)$ in (3.37) are negative and this gives the required positive constant c_0. □

3.5 Proof of Sequential Predictors Theorem

We next use the preceding four lemmas to prove the theorem. This requires a different argument from [37], e.g., because the functions $\sigma \circ p$ for continuous functions p will not be differentiable, so the approach from [37] would not apply. First note that the dynamics (3.21) agrees with (3.16), save for the fact that the terms in curly braces in (3.21) have been added. Therefore, along all solutions of the closed loop system (3.21), the inequality $ab \le \frac{v_2\bar{c}_1}{2}a^2 + \frac{1}{2v_2\bar{c}_1}b^2$ with the choices $a = |x(t)|$ and $b = 2|P_1|_\infty|\delta^\sharp(t)|$ combined with the relation

$$V_s(t, x_t) \ge V_1(t, x(t)) \ge \bar{c}_1|x(t)|^2 \tag{3.38}$$

implies that the function V_s from Lemma 3.4 satisfies

$$\begin{aligned} \dot{V}_s &\leq -\tfrac{v_2}{2}V_s(t,x_t) - \tfrac{\bar{c}_1 v_2 |x(t)|^2}{2} + \{|x(t)|\}\{2|P_1|_\infty|\delta^\sharp(t)|\} \\ &\leq -\tfrac{v_2}{2}V_s(t,x_t) + \tfrac{2|P_1|_\infty^2|\delta^\sharp(t)|^2}{\bar{c}_1 v_2} \\ &\leq -\tfrac{v_2}{2}V_s(t,x_t) + \tfrac{2(m+1)}{\bar{c}_1 v_2}|P_1|_\infty^2 \\ &\quad \times\Big(|BK(\sigma)|_\infty^2\Big[|\mathcal{E}_1(\theta_2(...\theta_m((\sigma\circ\Omega_m)(t))...))|^2 + \quad \dots \\ &\quad +|\mathcal{E}_m((\sigma\circ\Omega_m)(t))|^2\Big] + |\delta(t)|^2\Big) \end{aligned} \tag{3.39}$$

where $\delta^\sharp$ is the quantity in curly braces in (3.21) and W is from Theorem 3.4, by using the Cauchy inequality for squaring a sum of $m+1$ nonnegative terms to produce the $m+1$ factor.

We next use the functions V_{2i} from Lemma 3.7 to build an ISS Lyapunov-Krasovskii functional for the dynamics (3.20) for the error variable $\mathcal{E}$. By applying the inequality $ab \leq \frac{c_0\bar{c}_{21}}{2}a^2 + \frac{1}{2c_0\bar{c}_{21}}b^2$ with $a = |\mathcal{E}_1(t)|$ and $b = 2\bar{P}|\dot{\theta}_1|_\infty|\delta|_\infty$, we can find a constant $c_1 > 0$ such that along all solutions of the $\mathcal{E}_1$ subsystem of (3.20), we have

$$\begin{aligned} \tfrac{d}{dt}Q_1^\sharp(t,(\mathcal{E}_1)_t) &\leq -c_0 Q_1^\sharp(t,(\mathcal{E}_1)_t) + \{|\mathcal{E}_1(t)|\}\{2\bar{P}|\dot{\theta}_1|_\infty|\delta|_\infty\} \\ &\leq -\tfrac{c_0 Q_1^\sharp(t,(\mathcal{E}_1)_t)}{2} + c_1|\delta|_\infty^2, \end{aligned} \tag{3.40}$$

by also using the positive definite quadratic lower bound on V_{21} and a bound on $|\dot{\theta}_1(t)|$. We can use the second inequality in (3.26) multiple times to find positive constants c_2 and c_3 such that

$$|\mathcal{E}_1(\theta_1^{-1}(\phi(\theta_2(t))))|^2 \leq c_2\left(|\mathcal{E}_1(t)|^2 + \int_{t-c_3}^t |\mathcal{E}_1(\ell)|^2 \mathrm{d}\ell + |\delta|_\infty^2\right) \text{ for all } t \geq 0 \tag{3.41}$$

along all solutions of (3.20). This is done by first upper bounding the left side of (3.41) by $2|\mathcal{E}_1(t)|^2 + 2|\mathcal{E}_1(\theta_1^{-1}(\phi(\theta_2(t)))) - \mathcal{E}_1(t)|^2$, then applying the Mean Value Theorem to $\mathcal{E}_1(t)$ to upper bound $|\mathcal{E}_1(\theta_1^{-1}(\phi(\theta_2(t)))) - \mathcal{E}_1(t)|^2$, then using Jensen's inequality and the change of variables $q = \theta_1^{-1}(\phi(\ell))$ to transform an integrand

containing $\mathcal{E}_1(\theta_1^{-1}(\phi(\ell)))$ (in the formula for the upper bound for $|\dot{\mathcal{E}}_1(\ell)|^2$) into an integral with the integrand term $\mathcal{E}_1(q)$, and finally using the relations

$$\theta_1^{-1}(\phi(\theta_2(t))) \geq \phi(\theta_2(t)) - \tfrac{c_h}{m(1-l_h)} \geq \theta_2(t) - c_m^\sharp \geq t - \tfrac{c_h}{m(1-l_h)} - c_m^\sharp \tag{3.42}$$

and $\theta_1^{-1}(\phi(t)) \geq t - c_m^\sharp$, which follow from (3.26), if $\theta_1^{-1}(\phi(\theta_2(t))) \leq t$ (or using the ISS decay estimate (3.40) if $\theta_1^{-1}(\phi(\theta_2(t))) \geq t$). For instance, we can choose $c_3 = \frac{c_h}{m(1-l_h)} + 2c_m^\sharp$.

Since Young's inequality provides an upper bound

$$|P_{22}|_\infty|\mathcal{E}_2(t)||\dot{\theta}_2|_\infty|L_1C|_\infty|\mathcal{E}_1(\theta_1^{-1}\circ\phi\circ\theta_2(t))| \leq \frac{c_0}{2}Q_2^\sharp(t,(\mathcal{E}_2)_t)+L_*|\mathcal{E}_1(\theta_1^{-1}\circ\phi\circ\theta_2(t))|^2$$

for some constant $L_* > 0$, we can therefore use (3.41) and (3.15) to find positive constants c_4, c_5, c_6, and c_7 such that the time derivative of

$$Q_2^{\sharp\sharp}(t,(\mathcal{E}_1,\mathcal{E}_2)_t) = c_4Q_1^\sharp(t,(\mathcal{E}_1)_t) + Q_2^\sharp(t,(\mathcal{E}_2)_t) + c_5\textstyle\int_{t-c_3}^t\int_\ell^t|\mathcal{E}_1(r)|^2\mathrm{d}r\mathrm{d}\ell \tag{3.43}$$

along all solutions of the $(\mathcal{E}_1,\mathcal{E}_2)$ subsystem of (3.20) satisfies

$$\tfrac{d}{dt}Q_2^{\sharp\sharp}(t,(\mathcal{E}_1,\mathcal{E}_2)_t) \leq -c_6Q_2^{\sharp\sharp}(t,(\mathcal{E}_1,\mathcal{E}_2)_t) + c_7|\delta|_\infty^2 \tag{3.44}$$

for all $t \geq 0$, where c_5 is chosen to cancel the effects of the integral in (3.41), and then c_4 is chosen to cancel the additional $|\mathcal{E}_1(t)|^2$ terms that are produced. Repeating this process inductively for $i = 2,3,\ldots,m$ provides the desired ISS Lyapunov-Krasovskii functional $Q_m^{\sharp\sharp}(t,(\mathcal{E})_t)$ that admits positive constants c_8 and c_9 that satisfies

$$\tfrac{d}{dt}Q_m^{\sharp\sharp}(t,(\mathcal{E})_t) \leq -c_8Q_m^{\sharp\sharp}(t,(\mathcal{E})_t) + c_9|\delta|_\infty^2 \tag{3.45}$$

along all solutions of the $\mathcal{E}$ dynamics in (3.20) for all $t \geq 0$.

We now use the decay estimate (3.45) to cancel the effects of the terms in squared brackets in (3.39). The structure of (3.20) allows us to find positive constants c_{10}

and c_{11} such that the quantity in squared brackets in (3.39) is bounded above by $\mathcal{B}_*(t) = c_{10}\left(\int_{t-c_{11}}^{t} |\mathcal{E}(\ell)|^2 \mathrm{d}\ell + |\mathcal{E}(t)|^2 + |\delta|_\infty^2\right)$, using the ISS decay estimate (3.45) to produce an ISS decay estimate on $|\mathcal{E}(t)|^2$ to express the squared quantities in squared brackets in (3.39) in terms of integrals having the upper limit of integration t and the integrand $|\mathcal{E}(\ell)|^2$ as we did for $\mathcal{E}_1$ above. Hence, we can use the quadratic structure of V_s and $Q_m^{\sharp\sharp}$ and the decay estimates (3.39) and (3.45) to find positive constants c_{12}, c_{13}, c_{14}, and c_{15} such that the time derivative of

$$V^\sharp(t,(x,\mathcal{E})_t) = V_s(t,x_t) + c_{12} Q_m^{\sharp\sharp}(t,\mathcal{E}_t) + c_{13} \int_{t-c_{11}}^{t} \int_{\ell}^{t} |\mathcal{E}(p)|^2 \mathrm{d}p\mathrm{d}\ell \qquad (3.46)$$

along all solutions of the $(x,\mathcal{E})$ dynamics for all $t \geq 0$ satisfies

$$\tfrac{d}{dt} V^\sharp(t,(x,\mathcal{E})_t) \leq -c_{14} V^\sharp(t,(x,\mathcal{E})_t) + c_{15}|\delta|_\infty^2, \qquad (3.47)$$

where c_{13} and then c_{12} were chosen to cancel the $\mathcal{B}_*(t)$ in the upper bound for $\dot{V}_s$ (by an analog of the argument that produced c_5 and c_4). Hence $V^\sharp$ is an ISS Lyapunov-Krasovskii functional for the closed loop $(x,\mathcal{E})$ from the statement of the theorem, which gives the conclusion. □

3.6 Illustration: Pendulum Example

We can apply Theorem 3.1 to many systems and allow sampling and measurement delays where current output values $y(t)$ are not available to use in the sequential predictors. When $A = 0$, we can allow any constant measurement delay $\tau \in [0, 1/2)$ (by (3.13)), but we can also allow drift. For instance, this section illustrates Theorem 3.1 using a benchmark pendulum example from [37] where we show why Theorem 3.1 allows a smaller number of predictors than [37] when $\tau = 0$.

The simple pendulum model is [37]

$$\dot{r}_1(t) = r_2(t), \quad \dot{r}_2(t) = -\frac{g}{l}\sin(r_1(t)) + \frac{1}{\bar{m}l^2} v(t - h(t)) \qquad (3.48)$$

where l is the pendulum length in meters, $h(t)$ is the time varying delay in the input v, $\bar{m}$ is the pendulum mass, and $g = 9.8$ m/s is the gravitational constant. The goal in [37] was to track a prescribed C^1 reference trajectory $(r_{1,s}(t), r_{2,s}(t))$ for which $\dot{r}_{1,s}(t) = r_{2,s}(t)$ for all $t \geq 0$. The change of feedback

$$u(t - h(t)) = \frac{1}{w_0}\left[\frac{1}{\bar{m}l^2}v(t - h(t)) - \dot{r}_{2,s}(t) - \frac{g}{l}\sin(r_{1,s}(t))\right] \tag{3.49}$$

for any constant $w_0 > 0$ then gives the tracking dynamics

$$\dot{\tilde{r}}_1(t) = w_0\tilde{r}_2(t), \;\; \dot{\tilde{r}}_2(t) = \frac{g}{w_0 l}[\sin(r_{1,s}(t)) - \sin(\tilde{r}_1(t) + r_{1,s}(t))] + u(t - h(t)) \tag{3.50}$$

for the error variables

$$\tilde{r}_1 = r_1 - r_{1,s}(t) \;\text{ and }\; \tilde{r}_2 = (r_2 - r_{2,s}(t))/w_0, \tag{3.51}$$

where the change of feedback used the fact that $\Omega_m(t) = t - h(t)$ is invertible, which follows from our condition $l_h \in (0, 1)$.

The work [43] proved that under the constant delay $h = 1$ and with $w_0 = 1$ and $r_{1,s}(t) = \omega t$ where $\omega > 0$ is a large enough constant, the origin of the linearization

$$\dot{x}_1(t) = w_0 x_2(t), \;\; \dot{x}_2(t) = -\frac{g}{w_0 l}\cos(\omega t)x_1(t) + u(t - h(t)) \tag{3.52}$$

of (3.50) around 0, in closed loop with a distributed control is globally exponentially stable, and [36] constructed a globally asymptotically stabilizing sequential predictor control for the original nonlinear system (3.50) for constant h. However, [36] did not apply to time-varying delays or outputs.

The work [37] constructed sequential predictors for the linearized error dynamics (3.52) having time-varying delays h when $w_0 = 1$ and $\tau = 0$ (without allowing sampling or measurement delays). The assumptions in [37] were the same as Assumptions 3-4 in the special case $\tau = 0$ and $C = I_n$. Given any constant $\omega > 0$,

the assumptions from [37] are satisfied using

$$A(t) = \begin{bmatrix} 0 & w_0 \\ -\frac{g}{w_0 l}\cos(\omega t) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and}$$
$$K(t) = \begin{bmatrix} \frac{g}{w_0 l}\cos(\omega t) - w_0 & -w_0 \end{bmatrix}$$

which satisfies $|A|_\infty = \max\{g/(w_0 l), w_0\}$. Using the preceding choices and $h(t) = 1 + \alpha \sin(t)$, $l_h = n_h = \alpha = 1/7$, $c_h = 1 + \alpha$, $w_0 = 1$, and any constants $\omega > 0$ and $l > g$, the smallest m such that (3.14) is satisfied is $m = 47$. On the other hand, when $\tau = 0$ and with the preceding choices of the parameters and

$$L_i(t) = -[I_n + R_i(t)A(G_i(t))] \;\text{ and }\; P_{2i}(t) = \frac{1}{2}I_2, \tag{3.53}$$

condition (3.9) holds with $m = 17$. Hence, in the special case of (3.52) with $w_0 = 1$, Theorem 3.1 provided a 63% reduction in the number m of required sequential predictors compared to [37], so when $\tau = 0$, Theorem 3.1 does not reduce to the theorem in [37]. To see how we can also cover nonzero τ's, note that when τ is constant, we can use the preceding parameter choices with $l > g/w_0^2$ to check that (3.13) holds if $\tau \in [0, 1/(2(1 + e^{u_c} w_0)^2))$, since $g/(w_0 l) \le w_0$ gives $|A|_\infty = w_0$. Hence, we can allow any constant $\tau \in [0, 0.5)$ by picking m large enough and $w_0 > 0$ small enough. For example, if we choose

$$\tau = 0.15,\; h(t) = 1 + \alpha \sin(t),\; \alpha = 1/7,\; w_0 = 0.1,\; C = I_2, \tag{3.54}$$

and the choices (3.53), then (3.9) from Theorem 2.3 can be satisfied with $m = 7$.

To see how we can also cover sampling, notice that the preceding choices of A, B, and K imply that the following matrices $H = A + BK$ and P_1 satisfy $H^\top P_1 + P_1 H = -I_2$:

$$H = w_0 \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \text{ and } P_1 = \frac{1}{w_0}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \tag{3.55}$$

This allows us to choose P_1 from (3.55) when computing the left side of (3.8), as follows. We continue our assumption that $l > g/w_0^2$, and we fix sampling times $\{t_j\}$ that satisfy our requirements for some ε_i's as defined in Section 3.2. Then, with the notation of Theorem 3.1,

$$|P_1 B| = \frac{\sqrt{5}}{2w_0} \text{ and } |W|_\infty = |K|_\infty = w_0\sqrt{\left(1+\tfrac{g}{w_0^2 l}\right)^2 + 1}.$$

Also, we can upper bound the norm of $A + W$ using the fact that for all unit vectors $v \in \mathbb{R}^2$,

$$\begin{aligned}
&|(A(t)+W(t))v| \\
&= \left|\left(w_0 v_2, \left(\frac{g}{w_0 l}\left(\cos(\omega\sigma(t)) - \cos(\omega t)\right) - w_0\right) v_1 - w_0 v_2\right)\right| \\
&\leq 3w_0 + \frac{g}{w_0 l}\left|\left(\cos(\omega\sigma(t)) - \cos(\omega t)\right)\right| \leq \bar{B}, \quad \text{where } \bar{B} = 3w_0 + \frac{g}{w_0 l}\omega\varepsilon_2,
\end{aligned} \tag{3.56}$$

by the subadditivity of the square root and the bound $\sup_{t\geq 0}|\sigma(t) - t| \leq \varepsilon_2$. Hence, (3.8) holds if

$$\frac{\sqrt{5}}{w_0}\varepsilon_2\left(\sqrt{2}|K|_\infty \bar{B} e^{\varepsilon_2^2 |K|_\infty^2} + \frac{g}{w_0 l}\omega\right) < 1. \tag{3.57}$$

For fixed values of g, l, $\omega > 0$, and ε_2, the left side of (3.57) converges to ∞ as $w_0 \to 0^+$. This illustrates the tradeoff that reducing $w_0 > 0$ can increase the range of allowable upper bounds c_τ on τ, but also reduce the allowable values of the maximum dwell time ε_2.

3.7 Conclusions

We advanced the theory of sequential predictors for linear time-varying systems with time-varying delays, by proving a general theorem, and illustrating how we can reduce the number of required sequential predictors compared with the literature, to reduce the computational burden from applying the method. We allowed sampling, outputs, and measurement delays that model more realistic cases where the current state of the physical system may not be available for measurement.

Such measurement delays and sampling had not been considered in previous treatments of sequential predictors for time-varying systems with time-varying delays, which also did not allow outputs. We required significant changes in the earlier formulas for sequential predictors, as well as new conditions on the required numbers of sequential predictors. We hope to develop analogs for PDE with feedback and measurement delays, outputs, and sampling.

Chapter 4.
Further Research

Although we advanced the theory of bounded backstepping and sequential predictors, there remain interesting related open problems that we hope to address in our future research. For bounded backstepping, we hope to study analogs for parabolic or other types of partial differential equations, which can provide bounded analogs of the backstepping results in [28]. We also hope to study bounded backstepping problems where there may be uncertain vector fields that define the systems, and where there may be prescribed input bounds that need to be satisfied. Although we provided sufficient conditions for our converging-input-converging-state condition to hold in terms of the existence of suitable strict Lyapunov functions, it would be interesting to determine whether our sufficient conditions are also necessary conditions in control affine cases.

For our sequential predictors work, we hope to find analogs for nonlinear systems that would allow time-varying delays and sampling in the controls, i.e., for systems that are nonlinear in the state. This could build on our group's work [37], which provided sequential predictors for nonlinear systems with constant delays but that did not allow sampling in the control. We also hope to develop analogs of our sequential predictor work for partial differential equations with feedback and measurement delays, and for sampling with uncertain coefficient matrices, which would extend the partial differential equations results from [54] by allowing time-varying systems with nonconstant delays and sampling. We will also look for ways to minimize the number of sequential predictors in our dynamical extensions, and to use numerical analysis methods to compare the computational burdens for implementing sequential predictor methods versus other predictor methods that produce distributed terms in controls.

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Appendix. Proof of Lemma 3.6

We provide a proof (from [37]) of Lemma 3.6 (from p.46) that we used to prove our theorem about sequential predictors.

We continue to use our notation

$$\Omega_i(t) = t - \frac{i}{m}h(t) \ \text{ and } \ \theta_j(t) = \Omega_{m-j+1}^{-1}(\Omega_{m-j}(t)) \tag{A.1}$$

$$u_c = n_h \frac{c_h}{(1-l_h)^2} + \frac{l_h}{1-l_h}, \tag{A.2}$$

as before, where the positive constants c_h, $l_h \in (0,1)$, and n_h are from Assumption 3. Fix any $i \in \{1, 2, \ldots, m\}$ and $t \geq 0$. Using (A.1), we have $\theta_i(t) = \Omega_{m-i+1}^{-1}(\Omega_{m-i}(t))$ and $\Omega_{m-i}(t) = \Omega_{m-i+1}(t) + \dfrac{1}{m}h(t)$, so

$$\theta_i(t) \ = \ t + \Omega_{m-i+1}^{-1}\left(\Omega_{m-i+1}(t) + \frac{1}{m}h(t)\right) - \Omega_{m-i+1}^{-1}\left(\Omega_{m-i+1}(t)\right). \tag{A.3}$$

Therefore, we can use the Mean Value Theorem to obtain a $w \in \mathbb{R}$ (depending on t and i in general) for which

$$\theta_i(t) \ = \ t + \frac{1}{\dot{\Omega}_{m-i+1}(w)}\frac{1}{m}h(t). \tag{A.4}$$

Hence, Assumption 3 implies

$$|\theta_i(t) - t| \ \leq \ \frac{1}{1-l_h}\frac{1}{m}h(t) \ \leq \ \frac{c_h}{1-l_h}\frac{1}{m}\,. \tag{A.5}$$

Then the second inequality in (3.26) follows after we replace t by $\theta_i^{-1}(t)$ in (A.5).

To verify the first inequality in (3.26), we first note that

$$\begin{aligned}\dot{\theta}_i(t) \ &= \ \frac{1-\frac{m-i}{m}\dot{h}(t)}{1-\frac{m-i+1}{m}\dot{h}\left(\Omega_{m-i+1}^{-1}(\Omega_{m-i}(t))\right)} \\ &= \ \frac{1-\frac{m-i+1}{m}\dot{h}(t)}{1-\frac{m-i+1}{m}\dot{h}\left(\Omega_{m-i+1}^{-1}(\Omega_{m-i}(t))\right)} + \frac{\frac{1}{m}\dot{h}(t)}{1-\frac{m-i+1}{m}\dot{h}\left(\Omega_{m-i+1}^{-1}(\Omega_{m-i}(t))\right)}\,.\end{aligned} \tag{A.6}$$

By the relation $\Omega_{m-i}(t) = \Omega_{m-i+1}(t) + \frac{1}{m}h(t)$, the function

$$\mathcal{G}_i(t) = \Omega_{m-i+1}^{-1}\left(\Omega_{m-i+1}(t) + \tfrac{1}{m}h(t)\right) \tag{A.7}$$

satisfies the following relation

$$\begin{aligned}
\dot{\theta}_i(t) &= \frac{1-\frac{m-i+1}{m}\dot{h}(t)}{1-\frac{m-i+1}{m}\dot{h}(\mathcal{G}_i(t))} + \frac{\frac{1}{m}\dot{h}(t)}{1-\frac{m-i+1}{m}\dot{h}\big(\Omega^{-1}_{m-i+1}(\Omega_{m-i}(t))\big)} \\
&= \frac{m-i+1}{m}\frac{\dot{h}\big(t+\mathcal{G}_i(t)-\Omega^{-1}_{m-i+1}(\Omega_{m-i+1}(t))\big)-\dot{h}(t)}{1-\frac{m-i+1}{m}\dot{h}(\mathcal{G}_i(t))} \qquad \text{(A.8)} \\
&\quad +\frac{1}{m}\frac{\dot{h}(t)}{1-\frac{m-i+1}{m}\dot{h}\big(\Omega^{-1}_{m-i+1}(\Omega_{m-i}(t))\big)} + 1\ ,
\end{aligned}$$

which can be checked by rewriting $\dot{h}(t)$ in the first numerator in (A.8) as $\dot{h}(t) = (\dot{h}(t) - \dot{h}(\mathcal{G}_i(t)) + \dot{h}(\mathcal{G}_i(t))$. We conclude that

$$|\dot{\theta}_i(t) - 1| \leq \frac{m-i+1}{m}n_h\frac{\big|\mathcal{G}_i(t)-\Omega^{-1}_{m-i+1}(\Omega_{m-i+1}(t))\big|}{1-\frac{m-i+1}{m}l_h} + \frac{1}{m}\frac{l_h}{1-\frac{m-i+1}{m}l_h}, \qquad \text{(A.9)}$$

using the Lipschitz constant n_h for $\dot{h}$. Then our choice of $\mathcal{G}_i$ and the Mean Value Theorem applied to the function Ω^{-1}_{m-i+1} give

$$\begin{aligned}
|\dot{\theta}_i(t) - 1| &\leq \frac{m-i+1}{m}n_h\frac{\big|\Omega_{m-i+1}(t)+\frac{1}{m}h(t)-\Omega_{m-i+1}(t)\big|}{\big(1-\frac{m-i+1}{m}l_h\big)^2} + \frac{1}{m}\frac{l_h}{1-\frac{m-i+1}{m}l_h} \\
&\leq \frac{1}{m}\frac{m-i+1}{m}n_h\frac{c_h}{\big(1-\frac{m-i+1}{m}l_h\big)^2} + \frac{1}{m}\frac{l_h}{1-\frac{m-i+1}{m}l_h}.
\end{aligned}$$

Now the lemma follows from our formula (A.2) for u_c.

Vita

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