6-29-2018

Non-local Methods in Fracture Dynamics

Eyad Said
Louisiana State University and Agricultural and Mechanical College, eyadsaid@gmail.com

NON-LOCAL METHODS IN FRACTURE DYNAMICS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Eyad Said
B.S., Damascus University, 2004
M.S., Damascus University, 2006
August 2018

Acknowledgments

This dissertation would not be possible without many contributions. I would like first to express my deep gratitude to my advisor, Professor Robert Lipton, for his advice, patience and guidance. Thank you for always believing in me and always pushing me to achieve more.

I am grateful to Professor Stephen Shipman for many discussions and insight. Thank you for always being there and always challenging me with questions that were designed to make me think more and do more.

I am very thankful to Professor James Oxley. Thank you for all the support and all the help you tirelessly provide. You made me a better teacher with all your attention to details. You always told me what I needed to hear, not what I wanted to hear.

I would also like to thank Prashant Kumar Jha for his valuable help.

I offer my thanks to everyone at the LSU Mathematics department for providing me with a community and an excellent environment to study and work.

A special thank you is offered to Charlotte Senn for being a great friend and offering continuous support.

I dedicate this dissertation to my wife, Afraa. I would have never made it without your love, support and understanding. I will never be able to express my gratitude. To my parents, who never doubted me and were always next to me even though we are thousands of miles apart. And to my kids, Joury and Adam, you are the reason for everything I do.

Table of Contents

Acknowledgments ... ii

Abstract ... iv

Chapter 1: Introduction ... 1

Chapter 2: Dynamic Brittle Fracture from Non-Local Potentials ... 5
- 2.1 Nonlocal Dynamics ... 5
- 2.2 Existence of Solutions ... 9
- 2.3 Stability Analysis ... 12
- 2.4 Control of the Softening Zone ... 16
- 2.5 Calibration of the Model ... 21
- 2.6 Linear Elastic Operators in the Limit of Vanishing Horizon ... 28
- 2.7 Conclusions ... 31

Chapter 3: State Based Dynamic Damage Propagation with Memory ... 32
- 3.1 Formulation ... 32
- 3.2 Existence of Solutions ... 36
- 3.3 Energy Balance ... 43
- 3.4 Explicit Damage Models, Cyclic Loading and Strain to Failure ... 47
- 3.5 Numerical Results ... 52
- 3.6 Linear Elastic Operators in the Small Horizon Limit ... 55
- 3.7 Conclusions ... 58

References ... 60

Vita ... 63

Abstract

We first introduce a regularized model for free fracture propagation based on non-local potentials. We work within the small deformation setting and the model is developed within a state based peridynamic formulation. At each instant of the evolution we identify the softening zone where strains lie above the strength of the material. We show that deformation discontinuities associated with flaws larger than the length scale of non-locality δ can become unstable and grow. An explicit inequality is found that shows that the volume of the softening zone goes to zero linearly with the length scale of non-local interaction. This scaling is consistent with the notion that a softening zone of width proportional to δ converges to a sharp fracture set as the length scale of nonlocal interaction goes to zero. Here the softening zone is interpreted as a regularization of the crack network. Inside quiescent regions with no cracks or softening the nonlocal operator converges to the local elastic operator at a rate proportional to the radius of nonlocal interaction. This model is designed to be calibrated to measured values of critical energy release rate, shear modulus, and bulk modulus of material samples. For this model one is not restricted to Poission ratios of $1/4$ and can choose the potentials so that small strain behavior is specified by the isotropic elasticity tensor for any material with prescribed shear and Lamé moduli.

Then a model for dynamic damage propagation is developed using non-local potentials. The model is posed using a state based peridynamic formulation. The resulting evolution is seen to be well posed. At each instant of the evolution we identify a damage set. On this set the local strain has exceeded critical values either for tensile or hydrostatic strain and damage has occurred. The damage set is nondecreasing with time and is associated with damage state variables defined

at each point in the body. We show that a rate form of energy balance holds at each time during the evolution. Away from the damage set we show that the nonlocal model converges to the linear elastic model in the limit of vanishing nonlocal interaction.

Chapter 1
Introduction

We address the problem of free crack propagation in homogeneous materials. The crack path is not known a-priori and is found as part of the problem solution. Our approach is to use a nonlocal formulation. We will work within the small deformation setting and the model is developed within a state based peridynamic formulation. Peridynamics [17], [20] is a nonlocal formulation of continuum mechanics expressed in terms of displacement differences as opposed to spatial derivatives of the displacement field. These features provide the ability to simultaneously simulate both smooth displacements and defect evolution. The net force acting on a point x is due to the strain between x and neighboring points y. The neighborhood of nonlocal interaction between x and its neighbors y is confined to ball of radius δ centered at x denoted by $B_\delta(x)$. The radius of the ball is called the horizon. Numerical implementations based on nonlocal peridynamic models exhibit formation and localization of features associated with phase transformation and fracture see for example [3], [21],[16], [8],[1], [14], [2], [10], [19],[22], [9]. A recent review can be found in [7].

In the second chapter we are motivated by the recent models proposed and studied in [12], [13], and [14]. Calibration has been investigated in [4]. These models are defined by double well two point strain potentials. Here one potential well is centered at the origin and associated with elastic response while the other well is at infinity and associated with surface energy. The rational for studying these models is that they are shown to be well posed over the class of square integrable non-smooth displacements and in the limit of vanishing non-locality the dynamics localize and recovers features of sharp fracture propagation see, [12] and [13]. In this work we extend this modeling approach to the state based formulation. Our work is further motivated by the recent numerical-experimental study carried out in [4] demonstrating that the bond based model is unable to capture the Poission ratio for a sample of PMMA at room temperature. Here we develop a double well state based potential for which the Poission ratio is no longer constrained to be 1/4. We show that for this model we can choose the potentials so that the small strain behavior is specified by the isotropic elasticity tensor for any material with prescribed shear and Lamé moduli.

In the third chapter we address the problem of damage propagation in materials. Here the damage evolution is not known a-priori and is found as part of the problem solution. Our approach is to use a nonlocal formulation with the purpose of using the least number of parameters to describe the model. We will work within the small deformation setting and the model is developed within a state based peridynamic formulation. Here strains are expressed in terms of displacement differences as opposed to spatial derivatives. For the problem at hand the non-locality provides the flexibility to simultaneously model non-differentiable displacements and damage evolution.

The recent model studied in [12], [13],[14], [26] is defined by double well two point strain potentials. Here one potential well is centered at the origin and associated with elastic response while the other well is at infinity and associated with surface energy. The rational for studying these models is that they are shown to be well posed and, in the limit of vanishing non-locality, the dynamics recovers features associated with sharp fracture propagation see, [12] and [13]. While memory is not incorporated in this model it is seen that the inertia of the evolution keeps the forces in a softened state over time as evidenced in simulations [14]. This modeling approach is promising for fast cracks but for cyclic loading and slowly propagating fractures an explicit damage-fracture modeling with memory is needed. In this work we develop this approach for more general models that allow for three point nonlocal interactions and irreversible damage. The use of three point potentials allows one to model a larger variety of elastic properties. In the lexicon of peridynamics we adopt an ordinary state based formulation [17], [20]. We introduce non-local forces that soften irreversibly as the shear strain or dilatational strain increases beyond critical values. The fracture set for this model is defined to be the set of points x where the maximum of the ratio of tensile strain to the critical strain over a neighborhood $B_\delta(x)$ exceeds a threshold value. Both the fracture set and damage set increase monotonically in time. This model is shown to deliver a mathematically well posed evolution. Our proof of this is motivated by recent work [24] where existence of solution for bond based peridynamic models with damage is established. Recently another well posed bond based model with damage has been proposed in [23] where fracture simulations are carried out.

In addition to being state based, our modeling approach differs from [24] and earlier bond based work [18] and uses differentiable damage variables. This feature allows us to establish an energy balance equation relating kinetic energy, potential energy, and energy dissipation at each instant during the evolution. At each instant we identify the set undergoing damage where the local energy dissipation rate is positive. On this set the local strain has exceeded a critical value and damage has occurred. Damage is irreversible and the damage set is monotonically increasing with time. Explicit damage models are illustrated and stress strain curves for both cyclic loading and strain to failure are provided. These models are illustrated in two numerical examples. In the first example, we consider a square domain and apply a time periodic y-directed displacement along the top edge while fixing the bottom, left and right edges. We track the strain and force over 3 loading periods. The simulations show that bonds suffer damage and the strain vs force plot is similar to the one predicted by the damage law. In the second example, we apply a shear load to the top edge while fixing the bottom edge and leaving left and right edges free. As expected we find that damage appears along the diagonal of square, see Figure 3.14.

We conclude by noting that for this model the forces scale inversely with the length of the horizon. With this in mind we consider undamaged regions and we are able to show that the nonlocal operator converges to a linear local operator associated with the elastic wave equation. In this limit the elastic tensor can have any combination of Poison ratio and Young's modulus. The Poison ratio and Young modulus are determined uniquely by explicit formulas in terms of the nonlocal potentials used to define the model. This result is consistent with small horizon convergence results for convex energies, see [25], [28], [21]. Further reading and complete derivations can be found in the recent monograph [27].

Chapter 2
Dynamic Brittle Fracture from Non-Local Potentials

2.1 Nonlocal Dynamics

We formulate the nonlocal dynamics. Here we will assume displacements u are small (infinitesimal) relative to the size of the three dimensional body D. The tensile strain is written as $S = S(y, x, t; u)$ and given by

$$S(y, x, t; u) = \frac{u(t, y) - u(t, x)}{|y - x|} \cdot e_{y-x}, \qquad e_{y-x} = \frac{y - x}{|y - x|}, \tag{2.1}$$

where e_{y-x} is a unit direction vector and $\cdot$ is the dot product. It is evident that $S(y, x, t; u)$ is the tensile strain along the direction e_{y-x}. We introduce the influence function $\omega^\delta(|y - x|)$ such that ω^δ is nonzero for $|y - x| < \delta$, zero outside. Here we will take $\omega^\delta(|y - x|) = \omega(|y - x|/\delta)$ with $\omega(r) = 0$ for $r > 1$ nonnegative for $r < 1$ and ω is bounded.

The spherical or hydrostatic strain at x is given by

$$\theta(x, t; u) = \frac{1}{V_\delta} \int_{D \cap B_\delta(x)} \omega^\delta(|y - x|) S(y, x, t; u) |y - x| \, dy, \tag{2.2}$$

where V_δ is the volume of the ball $B_\delta(x)$ of radius δ centered at x. Here we have employed the normalization $|y - x|/\delta$ so that this factor takes values in the interval from 0 to 1.

Motivated by potentials of Lennard-Jones type, we define the force potential for tensile strain given by

$$\mathcal{W}^\delta(S(y, x, t; u)) = \alpha \omega^\delta(|y - x|) \frac{1}{\delta |y - x|} f(\sqrt{|y - x|} S(y, x, t; u)) \tag{2.3}$$

and the potential for hydrostatic strain

$$\mathcal{V}^\delta(\theta(x, t; u)) = \frac{\beta g(\theta(x, t; u))}{\delta^2} \tag{2.4}$$

where $\mathcal{W}^\delta(S(y, x, t; u))$ is the pairwise force potential per unit length between two points x and y and $\mathcal{V}^\delta(\theta(x, t; u))$ is the hydrostatic force potential density at x. They are described in terms of their potential functions f and g see Figure 2.1. These two potentials are double well potentials are chosen so that the associated forces acting between material points x and y are initially elastic and then soften and decay to zero as the strain between points increases, see Figure 2.2 for the tensile force. This force is negative for compression and a similar force hystatic strain law follows from the potential for hydrostatic strain. The first well for $\mathcal{W}^\delta(S(y,x,t;u))$ and $\mathcal{V}^\delta(\theta(x,t;u))$ is at zero tensile and hydrostatic strain respectively. With this in mind we make the choice

$$f(0) = f'(0) = g(0) = g'(0). \tag{2.5}$$

The second well is at infinite tensile and hydrostatic strain and is characterized by the horizontal asymptotes $\lim_{S\to\infty} f(S) = f_\infty$ and $\lim_{\theta\to\infty} g(\theta) = g_\infty$ respectively, see Figure 2.1.

The critical tensile strain $S_c^+ > 0$ for which the force begins to soften is given by the inflection point $r_1^+ > 0$ of f and is

$$S_c^+ = \frac{r_1^+}{\sqrt{|y-x|}}. \tag{2.6}$$

The critical negative tensile strain is chosen much larger in magnitude than S_c^+ and is

$$S_c^- = \frac{r_1^-}{\sqrt{|y-x|}}, \tag{2.7}$$

with $r_1^- < 0$ and $r_1^+ << |r_1^-|$. The critical value $0 < \theta_c^+$ where the force begins to soften under positive hydrostatic strain for $\theta(x,t;u) > \theta_c^+$ is given by the inflection point r_2^+ of g and is

$$\theta_c^+ = r_2^+. \tag{2.8}$$

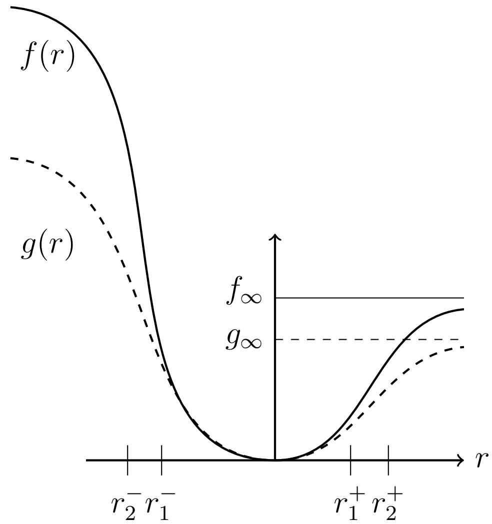

Figure 2.1. Potential function $f(r)$ for tensile force and potential function $g(r)$ for hydrostatic force .

The critical compressive hydrostatic strain where the force begins to soften for negative hydrostatic strain is chosen much larger in magnitude than θ_c^+ and is

$$\theta_c^- = r_2^-, \tag{2.9}$$

with $r_2^- < 0$ and $r_2^+ < |r_2^-|$. For this model we suppose the inflection points for g and f satisfy the ordering

$$r_2^- < r_1^- < 0 < r_1^+ < r_2^+. \tag{2.10}$$

This ordering is chosen to illustrate ideas for a material that is weaker in shear strain than hydrostatic strain. With this choice and the appropriate influence function ω^δ, if the hydrostatic stress is positive at x and is above the critical value θ_c^+ then there are points y in the peridynamic neighborhood for which the tensile stress between x and y is above S_c^+. This aspect of the model is established and addressed in section 2.4.

The potential energy is given by

$$\begin{aligned} PD^\delta(u) = \frac{1}{V_\delta} \int_D \int_{D \cap B_\delta(x)} |y-x| \mathcal{W}^\delta(S(y,x,t;u))\, dydx \\ + \int_D \mathcal{V}^\delta(\theta(x,t;u))\, dx. \end{aligned} \tag{2.11}$$

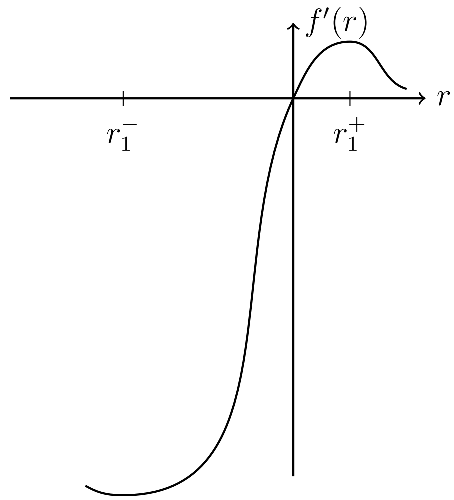

Figure 2.2. Cohesive tensile force.

The material is assumed homogeneous and the density is given by ρ and the applied body force is denoted by $b(x,t)$. We define the Lagrangian

$$\mathrm{L}(u,\partial_t u,t)=\frac{\rho}{2}||\dot{u}||^2_{L^2(D;\mathbb{R}^3)}-PD^\delta(u)+\int_D b\cdot u dx,$$

here $\dot{u}$ is the velocity given by the time derivative of u and $\|\dot{u}\|_{L^2(D;\mathbb{R}^3)}$ denotes the L^2 norm of the vector field $\dot{u}:D\to\mathbb{R}^3$. Applying the principal of least action together with a straight forward calculation gives the nonlocal dynamics

$$\rho\ddot{u}(x,t)=\mathcal{L}^T(u)(x,t)+\mathcal{L}^D(u)(x,t)+b(x,t),\text{ for }x\in D, \tag{2.12}$$

where

$$\mathcal{L}^T(u)(x,t)=\frac{2}{V_\delta}\int_{D\cap B_\delta(x)}\frac{\omega^\delta(|y-x|)}{\delta|y-x|}\partial_S f(\sqrt{|y-x|}S(y,x,t;u))e_{y-x}\,dy, \tag{2.13}$$

and

$$\mathcal{L}^D(u)(x,t)=\frac{1}{V_\delta}\int_{D\cap B_\delta(x)}\frac{\omega^\delta(|y-x|)}{\delta^2}\left[\partial_\theta g(\theta(y,t;u))+\partial_\theta g(\theta(x,t;u))\right]e_{y-x}\,dy. \tag{2.14}$$

The dynamics is complemented with the with initial data

$$u(x,0)=u_0(x),\qquad \partial_t u(x,0)=v_0(x). \tag{2.15}$$

It is readily verified that this is an ordinary state based peridynamic model. The forces are defined by the derivatives of the potential functions and the derivative associated with the tensile strain potential is sketched in Figure 2.2. We show in the next section that this initial value problem is well posed.

2.2 Existence of Solutions

The regularity and existence of the solution depends on the regularity of the initial data and body force. In this work we choose a general class of body forces and initial conditions. The initial displacement u_0 and velocity v_0 are chosen to be integrable and belonging to $L^\infty(D;\mathbb{R}^3)$. The body force $b(x,t)$ is chosen such that for every $t \in [0,T_0]$, b takes values in $L^\infty(D,\mathbb{R}^3)$ and is continuous in time. The associated norm is defined to be $\|b\|_{C([0,T_0];L^\infty(D,\mathbb{R}^3))} = max_{t\in[0,T_0]}\|b(x,t)\|_{L^\infty(D,\mathbb{R}^3)}$. The space of continuous functions in time taking values in $L^\infty(D;\mathbb{R}^3)$ for which this norm is finite is denoted by $C([0,T_0];L^\infty(D,\mathbb{R}^3))$. The space of functions twice differentiable in time taking values in $L^\infty(D,\mathbb{R}^3)$ such that both derivatives belong to $C([0,T_0];L^\infty(D,\mathbb{R}^3))$ is written as $C^2([0,T_0];L^\infty(D,\mathbb{R}^3))$.

We will establish existence and uniqueness for the evolution by writing the second order ode as an equivalent first order system. The nonlocal dynamics (2.12) can be written as a first order system. Set $y = (y_1, y_2)$ where $y_1 = u$ and $y_2 = u_t$. Now, set $F^\delta(y,t) = \Big(F_1(y,t), F_2(y,t)\Big)^T$ where:

$$\begin{aligned} F_1(y,t) &= y_2 \\ F_2(y,t) &= \mathcal{L}^T(y_1)(t) + \mathcal{L}^D(y_1)(t) + b(t) \end{aligned} \tag{2.16}$$

And the initial value problem is given by the equivalent first order system

$$\begin{aligned} \frac{d}{dt}y &= F^\delta(y,t) \\ y(0) = (y_1(0), y_2(0)) &= (u_0, v_0) \end{aligned} \tag{2.17}$$

The existence of a unique solution to the initial value problem is asserted in the following theorem.

Theorem 2.1. For a body force $b(t,x)$ in $C^1\Big([0,T];L^\infty(D,\mathbb{R}^3)\Big)$ and initial data $y_1(0)$ and $y_2(0)$ in $L^\infty((D;\Re^d)\times L_0^\infty(D;\mathbb{R}^3)$ there exists a unique solution $y(t)$ such that $y_1=u$ is in
$C^2\Big([0,T];L^\infty(D,\mathbb{R}^3)\Big)$ for the dynamics described by 2.17 with initial data in $L^\infty((D;\mathbb{R}^3)\times L^\infty(D;\mathbb{R}^3)$ and body force $b(t,x)$ in $C^1\Big([0,T];L^\infty(D,\mathbb{R}^3)\Big)$

Proof. We will show that the model is *Lipschitz continuous* and then apply the theory of ODE in Banach spaces, e.g. [5], to guarantee the existence of a unique solution. It is sufficent to show that

$$||\mathcal{L}^T(u)(x,t)+\mathcal{L}^D(u)(x,t)-(\mathcal{L}^T(v)(x,t)+\mathcal{L}^D(v)(x,t))||_{L^\infty(D)}\leq C||u-v||_{L^\infty(D)} \tag{2.18}$$

For ease of notation, we introduce the following vectors

$$\vec{U}=u(y)-u(x),$$

$$\vec{V}=v(y)-v(x).$$

We write

$$\mathcal{L}^T(u)(x,t)+\mathcal{L}^D(u)(x,t)-(\mathcal{L}^T(v)(x,t)+\mathcal{L}^D(v)(x,t))=\mathcal{I}_1+\mathcal{I}_2. \tag{2.19}$$

Here

$$\begin{aligned}\mathcal{I}_1=\frac{2\alpha}{\delta V_\delta}\int_{D\cap B_\delta(x)}\frac{\omega^\delta(|y-x|)}{\sqrt{|y-x|}}\Big\{&f'(\sqrt{|y-x|}S(y,x,t;u))\\&-f'(\sqrt{|y-x|}S(y,x,t;v))\Big\}e_{(y-x)}dy\end{aligned} \tag{2.20}$$

And

$$\begin{aligned}\mathcal{I}_2=\frac{\beta}{\delta^2 V_\delta}\int_{D\cap B_\delta(x)}\omega^\delta(|y-x|)\Big(&g'(\theta(y,t;u))+g'(\theta(x,t;u))\\&-(g'(\theta(y,t;v))+g'(\theta(x,t;v)))\Big)e_{(y-x)}dy\end{aligned} \tag{2.21}$$

Since f'' is bounded a straight forward calculation gives:

$$|f'(\sqrt{|y-x|}S(y,x,t;u)) - f'(\sqrt{|y-x|}S(y,x,t;v))$$
$$\leq \sqrt{|y-x|}\sup_{s\in\mathbb{R}}\{|f''(s)|\}|S(y,x,t;u) - S(y,x,t;v)|$$

and and $|e_{y-x}| = 1$ so we can bound $\mathcal{I}_1$ by

$$|\mathcal{I}_1| \leq \frac{2\alpha}{\delta V_\delta}\int_{D\cap B_\delta(x)} \omega^\delta(|y-x|)\sup_{x\in D}\{|f''(x)|\}|S(y,x,t;u) - S(y,x,t;v)|\,dy \quad (2.22)$$

In what follows $C_1 = \sup_{s\in\mathbb{R}}\{|f''(s)|\} < \infty$ and we make the change of variable

$$y = x + \delta\xi$$
$$|y-x| = \sigma|\xi|$$
$$dy = \delta^3 d\xi,$$

and a straight forward calculation shows

$$\mathcal{I}_1 \leq \frac{2\alpha C_1}{\delta^2}\int_{H_1(0)\cap\{x+\delta\xi\in D\}} |\omega(\xi)|\frac{|u(x+\delta\xi) - u(x) - (v(x+\delta\xi) - v(x))|}{|\xi|}\,d\xi \quad (2.23)$$

Which leads to the inequality

$$||\mathcal{I}_1||_{L^\infty(D;\mathbb{R}^3)} \leq \frac{4\alpha C_1 C_2}{\delta^2}||u-v||_{L^\infty(D;\mathbb{R}^3)}, \quad (2.24)$$

with $C_2 = \int_{H_1(0)}|\xi|^{-1}\omega(|\xi|)\,d\xi$.

Now we can work on the second part, where we follow a similar approach. Noting that g'' is bounded we let $C_3 = \sup_{\theta\in\mathbb{R}}\{|g''(\theta)|\} < \infty$ and $C_4 = \int_{H_1(0)}|\xi|\omega(|\xi|)d\xi$ to find that

$$|g'(\theta(y,t;u)) - g'(\theta(y,t;v))| \leq C_3|\theta(y,t;u) - \theta(y,t;v)| \leq \frac{2C_3C_4}{\delta^2}\|u-v\|_{L^\infty(D;\mathbb{R}^3)}$$

and

$$|g'(\theta(x,t;u)) - g'(\theta(x,t;v))| \leq C_3|\theta(x,t;u) - \theta(x,t;v)| \leq \frac{2C_3C_4}{\delta^2}\|u-v\|_{L^\infty(D;\mathbb{R}^3)}$$

so

$$\|\mathcal{I}_2\|_{L^\infty(D;\mathbb{R}^3)} \leq \frac{4\beta C_3 C_4}{\delta^2}\|u-v\|_{L^\infty(D;\mathbb{R}^3)}. \tag{2.25}$$

Adding (2.24) and (2.25) gives the desired result

$$\begin{aligned}||\mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t) - (\mathcal{L}^T(v)(x,t) + \mathcal{L}^D(v)(x,t))||_{L^\infty(D;\mathbb{R}^3)} \\ \leq \frac{4(\alpha C_1 C_2 + \beta C_3 C_4)}{\delta^2}\|u-v\|_{L^\infty(D;\mathbb{R}^3)}\end{aligned} \tag{2.26}$$

□

2.3 Stability Analysis

In this section we identify a source for crack nucleation as a material defect represented by a jump discontinuity in the displacement field. To illustrate the ideas we assume the defect is in the interior of the body and at least δ away from the boundary. This jump discontinuity can become unstable and grow in time. We proceed with a perturbation analysis and consider a time independent body force density b and a smooth equilibrium solution u. Now assume that the defect perturbs u in the neighborhood of a point x by a piece-wise constant vector field s that represents a jump in displacement across a planar surface with normal vector ν. We assume that this jump occurs along a defect of length 2δ on the planar surface.

The smooth equilibrium solution $u(x,t)$ is a solution of

$$0 = \mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t) + b(x) \tag{2.27}$$

Now consider a $u^P(x,t)$ perturbed solution that differs from equilibrium solution $u(x,t)$ by the jump across the planar surface with specified by unit normal vector ν. We suppose the surface passes through x extends across the peridynamic neighborhood centered at x. Points y for which $(y-x)\cdot\nu < 0$ are denoted by $\mathcal{E}_\nu^-$ and points for which $(y-x)\cdot\nu \geq 0$ are denoted by $\mathcal{E}_\nu^+$, see Figure 2.3.

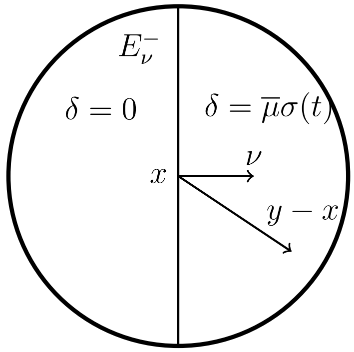

Figure 2.3. Jump discontinuity.

The perturbed solution u^P satisfies

$$\rho \ddot{u^P} = \mathcal{L}^T(u^P)(x,t) + \mathcal{L}^D(u^P)(x,t) + b(x) \tag{2.28}$$

Here the perturbed solution $u^P(x,t)$ is given by the equilibrium solution plus a piecewise constant perturbation and is written

$$u^P(y,t) = u(y,t) + s(y,t) \tag{2.29}$$

Where

$$s(y,t) = \begin{cases} 0 & y \in \mathcal{E}_\nu^- \\ \bar{\mu}\sigma(t) & y \in \mathcal{E}_\nu^+ \end{cases} \tag{2.30}$$

Subtracting (2.27) from (2.28) gives

$$\rho \ddot{u^P} = \mathcal{L}^T(u^P)(x,t) + \mathcal{L}^D(u^P)(x,t) - \mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t) \tag{2.31}$$

Here the second term in $\mathcal{L}^D(u)$ vanishes as we are away from the boundary and the integrand is odd in the y variable with respect to the domain $B_\delta(x)$. Since $u^P = u + s$ and s is small we expand $f'(\sqrt{|y-x|}(S(y,x,t;u+s)$ in Taylor series in s. Noting that $\theta(x,t;u+s) = \theta(x,t;u) + \theta(x,t;s)$ and $\theta(x,t;s)$ is initially infinitesimal we also expand $g'(\theta(x,t;u+s)$ in a Taylor series in $\theta(x,t;s)$. Applying the expansions to 2.31 shows that to leading order

$$\rho\ddot{u^P} = \rho\ddot{\sigma}\overline{\mu} = \frac{2\alpha}{\delta V_\delta}\int_{B_\delta(x)}\frac{\omega^\delta(|y-x|)}{\sqrt{|y-x|}}f''(\sqrt{|y-x|}S)(s(y,t)-s(x,t))\cdot e_{(y-x)}e_{(y-x)}dy$$
$$+\left\{\frac{\beta}{V_\delta\delta^2}\int_{B_\delta(x)}\omega^\delta(|y-x|)g''(\theta(y,t;u))\frac{1}{V_\delta}\right.$$
$$\left.\times\int_{B_\delta(y)}\omega^\delta(|z-y|)(s(z,t)-s(y,t))\cdot e_{z-y}\,dz e_{y-x}\,dy\right\}$$
$$= I_1 + I_2, \tag{2.32}$$

where I_1 and I_2 are the first, second terms on the right hand side of (2.32). A straight forward calculation using (2.30) shows that

$$I_1 = -\frac{2\alpha}{\delta V_\delta}\int_{B_\delta(x)\cap\mathcal{E}_\nu^-}\frac{J^\delta(|y-x|)}{|y-x|}f''(\sqrt{|y-x|}S)e_{(y-x)}\cdot\bar{\mu}\sigma(t)e_{(y-x)}dy \tag{2.33}$$

We next calculate I_2. A straight forward but delicate calculation gives

$$\frac{1}{V_\delta}\int_{B_\delta(y)}\frac{\omega^\delta(|z-y|)}{\delta}(s(z,t)-s(y,t))\cdot e_{z-y}\,dz = b(y)\cdot\overline{\mu}\sigma(t) \tag{2.34}$$

where

$$b(y) = \frac{1}{V_\delta}\int_0^{2\pi}\int_a^\delta\int_0^{\overline{\phi}}\omega(|z-y|)e(\theta,\phi)|z-y|^2\sin\phi\,d\phi\,d\theta\,d|z-y| \tag{2.35}$$

and the limits of the iterated integral are

$$a = |(y-x)\cdot\nu| \quad \overline{\phi} = \arccos\left(\frac{|(y-x)\cdot\nu|}{|z-y|}\right) \tag{2.36}$$

and $e(\theta,\phi)$ is the vector on the unit sphere with direction specified by the angles θ and ϕ. Calculation now gives

$$I_2 = \frac{\beta}{V_\delta\delta^2}\left(\int_{B_\delta(x)\cap\mathcal{E}_\nu^-}\omega^\delta(|y-x|)g''(\theta(y,t;u))b(y)\cdot\overline{\mu}\sigma(t)e_{y-x}dy\right.$$
$$\left.-\int_{B_\delta(x)\cap\mathcal{E}_\nu^+}\omega^\delta(|y-x|)g''(\theta(y,t;u))b(y)\cdot\overline{\mu}\sigma(t)e_{y-x}dy\right) \tag{2.37}$$

where

$$\int_{B_\delta(x)\cap\mathcal{E}_\nu^-}\frac{\omega^\delta(|y-x|)}{\delta}b(y)\cdot\overline{\mu}s(t)e_{y-x}dy=\int_{B_\delta(x)\cap\mathcal{E}_\nu^+}\frac{\omega^\delta(|y-x|)}{\delta}b(y)\cdot\overline{\mu}s(t)e_{y-x}dy \tag{2.38}$$

We now take the dot product of both sides of (2.32) with $\overline{\mu}$ to get

$$\rho\ddot{\sigma}=\frac{(\mathsf{A}+\mathsf{B}^{sym})\overline{\mu}\cdot\overline{\mu}}{|\overline{\mu}|^2}\sigma(t) \tag{2.39}$$

where

$$\mathsf{A}=-\frac{2\alpha}{\delta V_\delta}\int_{B_\delta(x)\cap\mathcal{E}_\nu^-}\frac{J^\delta(|y-x|)}{|y-x|}f''(\sqrt{|y-x|}S)e_{(y-x)}\otimes e_{(y-x)}dy \tag{2.40}$$

and $\mathsf{B}^{sym}=(\mathsf{B}+\mathsf{B}^T)/2$ with

$$\begin{aligned}\mathsf{B}=&\frac{\beta}{V_\delta\delta^2}\left(\int_{B_\delta(x)\cap\mathcal{E}_\nu^-}\omega^\delta(|y-x|)g''(\theta(y,t;u))b(y)\otimes e_{y-x}dy\right.\\&\left.-\int_{B_\delta(x)\cap\mathcal{E}_\nu^+}\omega^\delta(|y-x|)g''(\theta(y,t;u))b(y)\otimes e_{y-x}dy\right)\end{aligned} \tag{2.41}$$

Inspection shows that

$$f''(\sqrt{|y-x|}S)<0,\ \text{ when } S>S_c^+ \tag{2.42}$$

Thus the eigenvalues of A can be non-negative whenever the tensile strain is positive and greater than S_c^+ so that the force is in the softening regime for a preponderance of points y inside $B_\delta(x)$. In general the defect will be stable if all eigenvalues of the stability matrix $\mathsf{A}+\mathsf{B}^{sym}$ are negative. On the other hand the defect will be unstable if at least one eigenvalue of the stability matrix is positive.

We collect results in the following proposition.

Proposition 2.2. Fracture nucleation condition about a defect

A condition for crack nucleation at a defect passing through a point x is that the associated stability matrix $\mathsf{A}+\mathsf{B}^{sym}$ has at least one positive eigenvalue.

If the equilibrium solution is constant then $\theta(y,t;u) =$ constant and $\mathsf{B}^{sym} = 0$. For this case the fracture nucleation condition simplifies and depends only on the eigenvalues of the matrix A. In the next section we analyze the size of the set where the tensile strain is greater than S_c^+ so that the tensile force is in the softening regime for points y inside $B_\delta(x)$.

2.4 Control of the Softening Zone

We define the softening zone in terms of the collection of centers of peridynamic neighborhoods with tensile strain exceeding S_c^+. In what follows we probe the dynamics to obtain mathematically rigorous and explicit estimates on the size of the softening zone in terms of the radius of the peridynamic horizon. In this section we assume $\omega^\delta = 1$, $\delta < 1$, and from the definition of the hydrostatic strain $\theta(x,t;u)$ we have the following lemma.

Lemma 2.3. Hydrostatic softening implies tensile softening.

If $\theta_c^+ < \theta(x,t;u)$ then $S_c^+ < S(y,x,t;u)$ for some subset of points y inside the peridynamic neighborhood centered at x and

$$\{x \in D :\, \theta(x) > \theta_c^+\} \subset \{x \in D :\, S(x,y,t;u) > S_c^+, \textit{ for some } y \textit{ in } B_\delta(x)\} \tag{2.43}$$

Proof. Suppose $\theta_c^+ < \theta(x,t;u)$ then there are points y in $B_\delta(x)$ for which

$$\theta_c^+ < |y-x|S(y,x,t;u) < \sqrt{|y-x|}S(y,x,t;u) \tag{2.44}$$

so

$$S_c^+ < \frac{\theta_c^+}{\sqrt{|y-x|}} < S(y,x,t;u) \tag{2.45}$$

since $r_1^+ < r_2^+ = \theta_c^+$. This directly implies

$$\{x \in D :\, \theta(x) > \theta_c^+\} \subset \{x \in D :\, S(x,y,t;u) > S_c^+, \text{ for some } y \text{ in } B_\delta(x)\} \tag{2.46}$$

and the lemma is proved. $\square$

This inequality shows that the collection of neighborhoods where softening is due to the hydrostatic force is also subset of the neighborhoods where there is softening due to tensile force. Motivated by this observation we focus on peridynamic neighborhoods where the tensile strain is above critical. We start by defining the *softening zone*. The set of points y in $B_\delta(x)$ with tensile strain larger than critical can be written as

$$A_\delta^+(x) = \{y \in B_\delta(x) : \, S(y, x, t; u) > \mathcal{S}_c^+\}$$

From the monotonicity of the force potential f we can also express this set as

$$A_\delta^+(x) = \{y \in B_\delta(x); f(\sqrt{|y-x|}S(y, x, t; u)) \geq f(r_1^+)\}$$

We define the weighted volume of the set A_δ^+ in terms of its characteristic function $\chi_{A_\delta^+}(y)$ taking the value one for $y \in A_\delta^+$ and zero outside. The weighted volume of A_δ^+ is given by $\int_{B_\delta(x)} \chi_{A_\delta^+}(y)|y-x|\,dy$ and the weighted volume of $B_\delta(x)$ is $m = \int_{B_\delta(x)} |y-x|\,dy$. The weighted volume fraction $P_\delta(x)$ of $y \in B_\delta(x)$ with tensile strain larger than critical is given by the ratio

$$P_\delta(x) = \frac{\int_{B_\delta(x)} \chi_{A_\delta^+}(y)|y-x|\,dy}{m}$$

Definition 2.4. Softening Zone. *Fix any volume fraction $0 < \gamma \leq 1$, and with each time t in the interval $0 \leq t \leq T$, define the softening zone $SZ^\delta(\gamma, t)$ to be the collection of centers of peridynamic neighborhoods for which the weighted volume fraction of points y with strain $S(y, x, t; u)$ exceeding the threshold $\mathcal{S}_c$ is greater than γ, i.e,*

$$SZ^\delta(\gamma, t) = \left\{x \in D; P_\delta(x) > \gamma\right\} \tag{2.47}$$

We now show that the volume of $SZ^\delta(\gamma, t)$ goes to zero linearly with the horizon δ for properly chosen initial data and body force. This scaling is consistent with the notion that a softening zone of width proportional to δ converges to a sharp fracture as the length scale δ of nonlocal interaction goes to zero. We define the sum of kinetic and potential energy as

$$W(t) = \frac{\rho}{2}||\dot{u}||^2_{L^2(D,\mathbb{R}^d)} + PD^\delta(u(t)) \tag{2.48}$$

and set

$$C(t) = \Big(\frac{1}{\sqrt{\rho}}\int_0^t ||b||_{L^2(D,\mathbb{R}^d)} d\tau + \sqrt{W(0)}\Big)^2 \tag{2.49}$$

Here $C(t)$ is a measure of the total energy delivered to the body from initial conditions and body force up to time t. The tensile toughness is defined to be the energy of tensile tension between x and y per unit length necessary for softening and is given by $f(r_1^+)/\delta$. We now state the geometric dependence of the softening zone on horizon.

Theorem 2.5. *The volume of the softening zone SZ^δ is controlled by the horizon δ according to the following relation expressed in terms of the total energy delivered to the system, the tensile toughness, and the weighted volume fraction of points y where the tensile strain exceeds S_c^+,*

$$\text{Volume}(SZ^\delta(\gamma, t)) \leq \frac{\delta C(t)}{\gamma m f(r_1^+)} \tag{2.50}$$

Remark It is clear that for zero initial data such that $u(0, x) = 0$ that $C(t)$ depends only on the body force $b(t, x)$ and initial velocity. For this choice we see that the softening zone goes to zero linearly with the horizon δ.

We now establish the theorem using Gronwall's inequality and Tchebychev's inequality. The peridynamic energy density at a point x is

$$E^\delta(x) = \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} |y - x|\mathcal{W}^\delta(S(y, x, t; u))\, dy + \mathcal{V}^\delta(\theta(x, t; u)) \tag{2.51}$$

Which can also be rewritten with the following change of variable $y - x = \delta\xi$

$$E^\delta(x) = \frac{\alpha}{\delta V_1}\int_{D\cap B_1(0)} \omega(|\xi|) f(\sqrt{\delta|\xi|}S(x + \delta\xi, x, t; u))d\xi + \frac{\beta g(\theta(x, t; u))}{\delta^2} \tag{2.52}$$

Recall from the monotonicity of $f(r)$ that $r_1^+ < r$ implies $f(r_1^+) < f(r)$. Now define the set where the strain exceeds the threshold S_c^+

$$S^{+,\delta} = \Big\{(\xi, x) \in B_1(0) \times D;\ x + \delta\xi \in D \ \text{ and } \ f(r_1^+) < f(\sqrt{\delta|\xi|}S(x + \delta\xi, x, t; u))\Big\} \tag{2.53}$$

A straight forward calculation with $\omega(|\xi|) = 1$ shows that

$$\begin{aligned} \frac{f(r_1^+)}{\delta} \int_{S^{+,\delta}} |\xi| d\xi dx &\leq \int_{S^{+,\delta}} \frac{1}{\delta} f(\sqrt{\delta|\xi|}S(x + \delta\xi, x, t; u) d\xi dx \\ &\leq \int_D E^\delta(x)\, dx = PD^\delta(u(t)) \end{aligned} \tag{2.54}$$

We define the weighted volume of the set $S^{+,\delta}$ to be

$$\tilde{V}(S^{+,\delta}) = \int_{S^{+,\delta}} |\xi| d\xi dx \tag{2.55}$$

and inequality (2.54) becomes

$$\frac{f(r_1^+)}{\delta} \tilde{V}(S^{+,\delta}) \leq \int_{S^{+,\delta}} \frac{1}{\delta} f(\sqrt{\delta|\xi|}S(x + \delta\xi, x, t, u)) d\xi dx \leq PD^\delta(u(t)) \tag{2.56}$$

Next we use Gronwall's inequality to prove the following theorem that shows that the kinetic and peridynamic energies of the solution $u(x, t)$ are bounded by the energy put into the system.

Theorem 2.6.

$$C(t) \geq \frac{\rho}{2} \|\dot{u}\|^2_{L^2(D;\mathbb{R}^2)} + PD^\delta(u(t)). \tag{2.57}$$

Proof. We start by multiplying both sides of (2.12) by $\dot{u}$ to get

$$\rho\ddot{u}(t) \cdot \dot{u}(t) = \big(\mathcal{L}^T(u)(x, t) + \mathcal{L}^D(u)(x, t)\big) \cdot \dot{u}(t) + b(t) \cdot \dot{u}(t)$$

Applying the product rule in the first term and integration by parts in the second term gives

$$\frac{1}{2} \frac{d}{dt} \Big[\rho||\dot{u}||^2_{L^2(D,\mathbb{R}^d)} + 2PD^\delta(u(t))\Big] = \int_D b(t) \cdot \dot{u}(t)\, dx$$

Application of Cauchy's inequality to the right hand side gives

$$\frac{1}{2}\frac{d}{dt}\Big[\rho||\dot{u}||^2_{L^2(D,\mathbb{R}^d)} + 2PD^\delta(u(t))\Big] = \int_D b(t)\cdot\dot{u}(t)dx \leq ||b(t)||_{L^2(D,\mathbb{R}^d)}||\dot{u}(t)||_{L^2(D,\mathbb{R}^d)} \tag{2.58}$$

Now set $\tilde{W}(t) = \rho||\dot{u}||^2_{L^2(D,\mathbb{R}^d)} + 2PD^\delta(u(t)) + \zeta$ where ζ is a positive number and can be taken arbitrarily small and (2.58) becomes,

$$\frac{1}{2}\tilde{W}'(t) =\leq ||b(t)||_{L^2(D,\mathbb{R}^d)}||\dot{u}(t)||_{L^2(D,\mathbb{R}^d)} \leq ||b(t)||_{L^2(D,\mathbb{R}^d)}\frac{\sqrt{\tilde{W}(t)}}{\sqrt{\rho}}$$

Now we can write

$$\frac{1}{2}\int_0^t \frac{\tilde{W}'(\tau)}{\sqrt{\tilde{W}(\tau)}}d\tau \leq \frac{1}{\sqrt{\rho}}\int_0^t ||b||_{L^2(D,\mathbb{R}^d)}d\tau$$

Which simplifies to

$$\sqrt{\tilde{W}(t)} - \sqrt{\tilde{W}(0)} \leq \frac{1}{\sqrt{\rho}}\int_0^t ||b||_{L^2(D,\mathbb{R}^d)}d\tau \tag{2.59}$$

since ζ can be made arbitrarily small we find that

$$\sqrt{W(t)} - \sqrt{W(0)} \leq \frac{1}{\sqrt{\rho}}\int_0^t ||b||_{L^2(D,\mathbb{R}^d)}d\tau \tag{2.60}$$

and (2.57) follows. □

We apply inequality 2.56 and Theorem 2.6 to get the fundamental inequality.

$$\tilde{V}(S^{+,\delta}) \leq \frac{C(t)\delta}{f(\bar{r})} \tag{2.61}$$

The fundamental inequality above is defined on $B_1(0) \times D$ and we now use it to bound the volume of the softening zone on D. Introducing the characteristic function $\chi^{S^{+,\delta}}(\xi, x)$ taking the value 1 when $(\xi, x) \in S^{+,\delta}$ and 0 otherwise. We immediately have

$$mP_\delta(x) = \int_{B_1(0)} \chi^{S^{+,\delta}}(\xi, x)|\xi|d\xi$$

So we can rewrite equation (2.55) as

$$
\begin{aligned}
\tilde{V}(S^{+,\delta}) &= \int_D \int_{B_1(0)} \chi^{S^{+,\delta}}(\xi,x)|\xi| d\xi dx \\
&= m \int_D P_\delta(x)\, dx
\end{aligned}
\tag{2.62}
$$

Now applying Tchebyshev's inequality to (2.62) with $SZ^\delta(\gamma,t)$ defined by (2.47) gives the desired result

$$
\mathrm{Volume}(SZ^\delta(\gamma,t)) \leq \frac{1}{\gamma}\int_D P_\delta(x)\, dx = \frac{\tilde{V}(S^{+,\delta})}{m\gamma} \leq \frac{C(t)\delta}{m\gamma f(r_1^+)} \tag{2.63}
$$

2.5 Calibration of the Model

In this section we show how to calibrate this model using the known elastic properties and energy release rate of fracture associated with a given material.

Calibrating The Peridynamic Energy To Elastic Properties

We start by considering a body D for which the strain S is small. Here *small* means for a fixed $|y-x|$ we have $|S| << |S_c^\pm|$, $|\theta| << |\theta_c^\pm|$. Now we proceed to calculate the peridynamic energy density inside the material due to the presence of a small deformation $u(x)$. Suppose that the *strain* at the length scale of a neighborhood of *horizon* δ is a linear function, *i.e.*

$$
\begin{aligned}
S(u,y,x) &= \frac{u(y)-u(x)}{|y-x|}\cdot\frac{y-x}{|y-x|} \\
&= F\frac{y-x}{|y-x|}\cdot\frac{y-x}{|y-x|} = Fe\cdot e
\end{aligned}
\tag{2.64}
$$

here F is a 3 by 3 matrix. We expand the first potential with respect to S and the second in θ keeping in mind that

$$
f(0) = f'(0) = g(0) = g'(0) = 0
$$

to get

$$
\begin{aligned}
f\Big(\sqrt{|y-x|}S\Big) &= \frac{|y-x|}{2} f''(0)S^2 + O(S^3) \\
g\Big(\theta(x,t;S)\Big) &= \frac{1}{2} g''(0)\theta^2 + O(\theta^3)
\end{aligned}
\tag{2.65}
$$

So we write the *energy density* which was defined in equation 2.51 for points x of distance δ away from the boundary ∂D to leading order

$$\begin{aligned} E^\delta =& \frac{1}{V_\delta}\frac{\alpha f''(0)}{2\delta}\int_{H_\delta(x)} \omega^\delta(|y-x|)|y-x|(Fe\cdot e)^2 dy \\ &+ \frac{\beta g''(0)}{2\delta^2}\left(\frac{1}{V_\delta}\int_{H_\delta(x)} \omega^\delta(|y-x|)|y-x|Fe\cdot e\, dy\right)^2 \end{aligned} \tag{2.66}$$

The change of variable $\delta\xi = y - x$ gives to leading order

$$\begin{aligned} E^\delta =& \frac{\alpha f''(0)}{2V_1}\int_{H_1(0)} \omega(|\xi|)|\xi|(Fe\cdot e)^2 d\xi \\ &+ \frac{\beta g''(0)}{2V_1^2}\left(\int_{H_1(0)} \omega(|\xi|)|\xi|Fe\cdot e\, d\xi\right)^2 \end{aligned} \tag{2.67}$$

Observe that $(Fe\cdot e)^2 = \sum_{ijkl} F_{ij}F_{kl}e_ie_je_ke_l$ and the first term in (2.67) is given by

$$\sum_{ijkl} \mathbb{M}_{ijkl}F_{ij}F_{kl} \tag{2.68}$$

where

$$\begin{aligned} \mathbb{M}_{ijkl} &= \frac{\alpha f''(0)}{2V_1}\int_{\mathcal{H}_1(0)} |\xi|\omega(|\xi|)\, e_ie_je_ke_l\, d\xi \\ &= \frac{\alpha f''(0)}{2V_1}\int_0^1 |\xi|^3\omega(|\xi|)\, d|\xi| \int_{S^2} e_ie_je_ke_l\, de \end{aligned} \tag{2.69}$$

where de is an element of surface measure on the unit sphere. Next observe $Fe\cdot e = \sum_{kj} F_{kj}e_ke_j$ and the second term in (2.67) is given by

$$\frac{\beta g''(0)}{2V_1^2}\left(\sum_{ij}\Lambda_{ij}F_{ij}\right)^2 = \frac{\beta g''(0)}{2V_1^2}\sum_{ijkl}\Lambda_{ij}\Lambda_{kl}F_{ij}F_{kl} \tag{2.70}$$

where

$$\Lambda_{jk} = \int_{\mathcal{H}_1(0)} |\xi|\omega(|\xi|)\, e_je_k\, d\xi = \int_0^1 |\xi|^3\omega(|\xi|)d|\xi| \int_{S^2} e_je_k\, de \tag{2.71}$$

Focusing on the first term we show that

$$\mathbb{M}_{ijkl} = 2\mu\left(\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2}\right) + \lambda\delta_{ij}\delta_{kl} \tag{2.72}$$

where μ and λ are given by

$$\lambda = \mu = \frac{\alpha f''(0)}{10} \int_0^1 |\xi|^3 \omega(|\xi|) \, d|\xi| \tag{2.73}$$

To see this we write

$$\Gamma_{ijkl}(e) = e_i e_j e_k e_l \tag{2.74}$$

to observe that $\Gamma(e)$ is a totally symmetric tensor valued function defined for $e \in S^2$ with the property

$$\Gamma_{ijkl}(Qe) = Q_{im} e_m Q_{jn} e_n Q_{ko} e_o Q_{lp} e_p = Q_{im} Q_{jn} Q_{ko} Q_{lp} \Gamma_{mnop}(e) \tag{2.75}$$

for every rotation Q in SO^3. Here repeated indices indicate summation. We write

$$\int_{\mathcal{H}_1(0)} |\xi|^3 \omega(|\xi|) \, e_i e_j e_k e_l \, d\xi = \int_0^1 |\xi|^3 \omega(|\xi|) d|\xi| \int_{S^2} \Gamma_{ijkl}(e) \, de \tag{2.76}$$

to see that for every Q in SO^3

$$Q_{im} Q_{jn} Q_{ko} Q_{lp} \int_{S^2} \Gamma_{ijkl}(e) \, de = \int_{S^2} \Gamma_{mnop}(Qe) \, de = \int_{S^2} \Gamma_{mnop}(e) \, de \tag{2.77}$$

Therefore we conclude that $\int_{S^2} \Gamma_{ijkl}(e) \, de$ is invariant under SO^3 and is therefore an isotropic symmetric 4^{th} order tensor and necessairily of the form

$$\int_{S^2} \Gamma_{ijkl}(e) \, de = a \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) + b \delta_{ij} \delta_{kl} \tag{2.78}$$

So $\mathbb{M}$ can be written in the form

$$\mathbb{M}_{ijkl} = 2\mu \left(\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2} \right) + \lambda \delta_{ij} \delta_{kl} \tag{2.79}$$

with suitable choices of μ and λ. To evaluate μ and λ we note the following relations between μ and λ for isotropic 4^{th} order tensors of the form above and their contractions

$$\mathbb{M}_{iijj} = 3(2\mu + 3\lambda) \tag{2.80}$$

$$\mathbb{M}_{ijij} = 3(4\mu + \lambda) \tag{2.81}$$

These relations can be readily verified by direct calculation.

On the other hand from the definition of $\mathbb{M}$ given by (2.69) we have

$$\mathbb{M}_{iijj} = \frac{\alpha f''(0)}{2V_1} \int_0^1 |\xi|^3 \omega(|\xi|)\, d|\xi| \int_{S^2} e_i^2 e_j^2 \, de = \frac{4\pi\alpha f''(0)}{2V_1} \int_0^1 |\xi|^3 \omega(|\xi|)\, d|\xi| \quad (2.82)$$

$$\mathbb{M}_{ijij} = \frac{\alpha f''(0)}{2V_1} \int_0^1 |\xi|^3 \omega(|\xi|)\, d|\xi| \int_{S^2} e_i^2 e_j^2 \, de = \frac{4\pi\alpha f''(0)}{2V_1} \int_0^1 |\xi|^3 \omega(|\xi|)\, d|\xi| \quad (2.83)$$

since $e_i^2 = \sum_i e_i^2 = 1$. Equation (2.73) now follows on recalling that $V_1 = \frac{4}{3}\pi$ and solving the system given by (2.80) and (2.81).

Focusing on the second term of (2.67) given by (2.70) we show that

$$\Lambda_{ij} = \frac{4\pi}{3} \int_0^1 |\xi|^3 \omega(|\xi|)\, d|\xi| \delta_{ij} \quad (2.84)$$

To see this we write

$$\Lambda_{ij}(e) = e_i e_j \quad (2.85)$$

to observe that $\Lambda(e)$ is a totally symmetric tensor valued function defined for $e \in S^2$ with the property

$$\Lambda_{ij}(Qe) = Q_{im} e_m Q_{jn} e_n = Q_{im} Q_{jn} \Lambda_{mn}(e) \quad (2.86)$$

for every rotation Q in SO^3. As before repeated indices indicate summation. We consider

$$\int_{S^2} \Lambda_{ij}(e)\, de \quad (2.87)$$

to see that for every Q in SO^3

$$Q_{im} Q_{jn} \int_{S^2} \Lambda_{ij}(e)\, de = \int_{S^2} \Lambda_{mn}(Qe)\, de = \int_{S^2} \Lambda_{mn}(e)\, de \quad (2.88)$$

Therefore we conclude that $\int_{S^2} \Lambda_{ij}(e)\, de$ is an isotropic symmetric 2^{nd} order tensor and of the form

$$\int_{S^2} \Lambda_{ij}(e)\, de = a\delta_{ij} \quad (2.89)$$

i.e., a multiple of the identity. So from (2.71) Λ is of the form

$$\Lambda_{ij} = \gamma \delta_{ij} \tag{2.90}$$

To evaluate γ we take the trace of (2.71) and (2.84) follows.

Now the second term is given by

$$\frac{\beta g''(0)}{2V_1^2}\left(\frac{4\pi}{3}\right)^2\left(\int_0^1 |\xi|^3\omega(|\xi|)d|\xi|\right)^2\sum_{ijkl}\delta_{ij}\delta_{kl}F_{ij}F_{kl} =$$

$$= \beta g''(0)\frac{1}{2}\left(\int_0^1 |\xi|^3\omega(|\xi|)d|\xi|\right)^2\sum_{ijkl}\delta_{ij}\delta_{kl}F_{ij}F_{kl} = \mathbb{K}_{ijkl}F_{ij}F_{kl} \tag{2.91}$$

Collecting results we see that the leading order the energy is given by

$$E^\delta = \sum_{ijkl}(\mathbb{M}_{ijkl} + \mathbb{K}_{ijkl})F_{ij}F_{kl} = \sum_{ijkl}\left(2\overline{\mu}\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2} + \overline{\lambda}\delta_{ij}\delta_{kl}\right)F_{ij}F_{kl} \tag{2.92}$$

where the shear modulus is given by

$$\overline{\mu} = \frac{\alpha f''(0)}{10}\int_0^1 |\xi|^3\omega(|\xi|)\,d|\xi| \tag{2.93}$$

and the Lame constant is given by

$$\overline{\lambda} = \frac{\alpha f''(0)}{10}\int_0^1 |\xi|^3\omega(|\xi|)\,d|\xi| + \frac{\beta g''(0)}{2}\left(\int_0^1 |\xi|^3\omega(|\xi|)d|\xi|\right)^2 \tag{2.94}$$

One is free to choose α and β provided that the resulting elastic tensor satisfies the constraints of ellipticity. Here one is no longer restricted to Poisson ratios of 1/4 as in the bond based formulation.

An identical calculation shows that for two dimensional problems the elastic constants are given by

$$\overline{\mu} = \frac{\alpha f''(0)}{8}\int_0^1 |\xi|^2\omega(|\xi|)\,d|\xi| \tag{2.95}$$

and

$$\overline{\lambda} = \frac{\alpha f''(0)}{8}\int_0^1 |\xi|^2\omega(|\xi|)\,d|\xi| + \frac{\beta g''(0)}{2}\left(\int_0^1 |\xi|^2\omega(|\xi|)d|\xi|\right)^2 \tag{2.96}$$

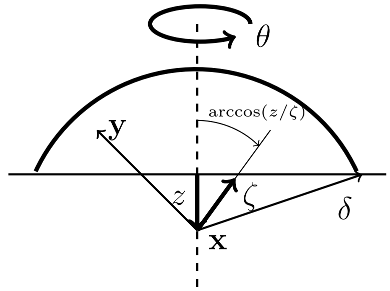

Figure 2.4. Evaluation of energy release rate $\mathcal{G}_s$. For each point x along the dashed line, $0 \leq z \leq \delta$, the work required to break the interaction between x and y in the spherical cap is summed up in (2.97) using spherical coordinates centered at x.

and one is no longer restricted to Poisson ratio 1/3 materials.

We note here that the two dimensional moduli $\bar{\mu}$ and $\bar{\lambda}$ are directly related to the well known moduli appearing in the plane strain or plane stress solutions for isotropic materials. This relationship is now well known and can be found in see [11] also [15].

Calibrating Energy Release Rate

In regions of large strain the same force potentials (2.3) and (2.4) are used to calculate the amount of energy consumed by a crack per unit area of growth, i.e., the energy release rate. The energy release rate equals the work necessary to eliminate force interaction on either side of a fracture surface per unit fracture area. In this model the energy release rate has two components one associated with the force potential for tensile strain (2.3) and the other associated with the force potential for hydrostatic strain (2.4). The critical energy release rate $\mathcal{G}s$ associated with fracture under tensile forces is found to be the same for all choices of horizon δ. However the critical energy release rate for hydrostatic fracture $\mathcal{G}_h$ increases with decreasing horizon and becomes infinite as $\delta \to 0$ at the rate $1/\delta$.

For tensile forces we use (2.3) and calculate the work required to eliminate interaction between two points x and y, this is given by $\mathcal{W}^\delta(\infty) = \lim_{S\to\infty} \mathcal{W}^\delta(S)$ where $\mathcal{W}^\delta(\infty) = \omega^\delta(|y-x|)f_\infty/\delta$. We suppose x gives the center of the peridynamic neighborhood located a distance z away from the planar interface separating upper and lower half spaces. We suppose x lies in the lower half space and the points y lie in the upper half space inside the peridynamic neighborhood of x, see Figure 2.4. The critical energy release rate $\mathcal{G}_s$ associated with tensile forces equals the work necessary to eliminate force interaction on either side of a fracture surface per unit fracture area. It is given in three dimensions by integration of $\mathcal{W}^\delta(\infty)$ over the intersection of the neighborhood of x and the upper half space given by the spherical cap, see Figure 2.4,

$$\mathcal{G}_s = \frac{4\pi}{V_\delta} \int_0^\delta \int_z^\delta \int_0^{\cos^{-1}(z/\zeta)} \mathcal{W}^\delta(\infty, \zeta)\zeta^2 \sin\phi \, d\phi \, d\zeta \, dz \tag{2.97}$$

where $\zeta = |y - x|$. This integral is calculated and for d dimensions $d = 1, 2, 3$, the result is

$$\mathcal{G}_s = M \frac{2\omega_{d-1}}{\omega_d} f_\infty \tag{2.98}$$

where $M = \int_0^1 r^d \omega(r) dr$ and ω_d is the volume of the d dimensional unit ball, $\omega_1 = 2, \omega_2 = \pi, \omega_3 = 4\pi/3$. We see from this calculation that the critical energy release rate is independent of δ.

For hydrostatic forces we use (2.4) and calculate the work required to eliminate interaction between x and the upper half plane. As before we suppose x gives the center of the peridynamic neighborhood located a distance z away from the planar interface separating upper and lower half spaces. We suppose x lies in the lower half space and the peridynamic neighborhood of x intersects the upper half space, see Figure 2.5.

The critical energy release rate $\mathcal{G}_h$ associated with hydrostatic forces equals the work necessary to eliminate force interaction on either side of a fracture surface per unit fracture area. The work per unit volume needed to eliminate hydrostatic

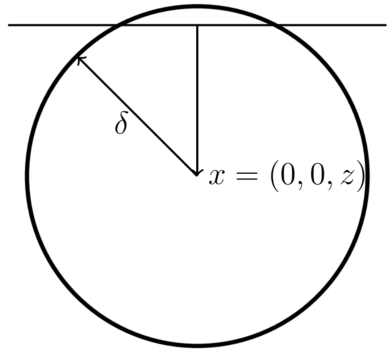

Figure 2.5. Hydrostatic energy release rate $\mathcal{G}_h$.

interaction between a point x and its neighbors is

$$\mathcal{V}^\delta(\infty)(x) = \lim_{\theta\to\infty} \frac{\beta g(\theta)}{\delta^2} = \frac{\beta g_\infty}{\delta^2} \tag{2.99}$$

For points $x = (0, 0, z)$, with $0 < |z| < \delta$ above and below the $z = 0$ plane the work per unit area to eliminate hydrostatic interaction between the lower half space $z < 0$ and upper half space $z > 0$ is

$$\mathcal{G}_h = 2\int_0^\delta \frac{\beta g_\infty}{\delta^2}\, dz = \frac{2\beta g_\infty}{\delta} \tag{2.100}$$

For d dimensions $d = 1, 2, 3$, the result is the same and

$$\mathcal{G}_s = \frac{2\beta g_\infty}{\delta} \tag{2.101}$$

We see from this calculation that the energy release rate for hydrostatic fracture is increasing at the rate $1/\delta$.

2.6 Linear Elastic Operators in the Limit of Vanishing Horizon

In this section we consider smooth evolutions u in space and show that away from fracture set the operators $\mathcal{L}^T + \mathcal{L}^D$ acting on u converge to the operator of linear elasticity in the limit of vanishing nonlocality. We denote the fracture set by $\tilde{D}$ and consider any open un-fractured set D' interior to D with its boundary a finite distance away from the boundary of D and the fracture set $\tilde{D}$. In what follows we suppose that the nonlocal horizon δ is smaller than the distance separating the boundary of D' from the boundaries of D and $\tilde{D}$.

Theorem 2.7. *Convergence to linear elastic operators. Suppose that $u(x,t) \in C^2([0,T_0], C^3(D,\mathbb{R}^3))$ and for every $x \in D' \subset D \setminus \tilde{D}$, then there is a constant $C > 0$ independent of nonlocal horizon δ such that for every (x,t) in $D' \times [0,T_0]$, one has*

$$|\mathcal{L}^T(u(t)) + \mathcal{L}^D(u(t)) - \nabla \cdot \mathbb{C}\,\mathcal{E}(u(t))| < C\delta \tag{2.102}$$

where the the elastic strain is $\mathcal{E}(u) = (\nabla u + (\nabla u)^T)/2$ and the elastic tensor is isotropic and given by

$$\mathbb{C}_{ijkl} = 2\bar{\mu}\left(\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2}\right) + \bar{\lambda}\delta_{ij}\delta_{kl} \tag{2.103}$$

with shear modulus $\bar{\mu}$ and Lamé coefficient $\bar{\lambda}$ given by (2.93) and (2.94). The numbers α and β can be chosen independently and can be any pair of real numbers such that $\mathbb{C}$ is positive definite.

Proof. We start by showing

$$|\mathcal{L}^T(u(t)) - \frac{f''(0)}{2\omega_3}\int_{B_1(0)} e|\xi|J(|\xi|)e_ie_je_k\,d\xi\partial^2_{jk}u_i(x)| < C\delta \tag{2.104}$$

where $\omega_3 = 4\pi/3$ and $e = e_{y-x}$ are unit vectors on the sphere, here repeated indices indicate summation. To see this recall the formula for $\mathcal{L}^T(u)$ and write $\partial_S f(\sqrt{|y-x|}S) = f'(\sqrt{|y-x|}S)\sqrt{|y-x|}$. Now Taylor expand $f'((\sqrt{|y-x|}S)$ in $\sqrt{|y-x|}S$ and Taylor expand $u(y)$ about x, denoting e_{y-x} by e to find that all odd terms in e integrate to zero and

$$|\mathcal{L}^T(u(t))_l - \frac{2}{V_\delta}\int_{B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|}\frac{f''(0)}{4}|y-x|^2\partial^2_{jk}u_i(x)e_ie_je_ke_l, dy| < C\delta$$
$$l = 1,2,3. \tag{2.105}$$

On changing variables $\xi = (y-x)/\delta$ we recover (3.71). Now we show

$$|\mathcal{L}^D(u(t))_k - \frac{1}{\omega_3}\int_{B_1(0)} |\xi|\omega(|\xi|)e_ie_j\,d\xi\frac{\beta g''(0)}{2\omega_3}\int_{B_1(0)} |\xi|\omega(|\xi|)e_ke_l\,d\xi\partial^2_{lj}u_i(x)| < C\delta$$
$$k = 1,2,3. \tag{2.106}$$

We note for $x \in D'$ that $D \cap B_\delta(x) = B_\delta(x)$ and the integrand in the second term of (3.6) is odd and the integral vanishes. For the first term in (3.6) we Taylor expand $\partial_\theta g(\theta)$ about $\theta = 0$ and Taylor expand $u(z)$ about y inside $\theta(y,t)$ noting that terms odd in $e = e_{z-y}$ integrate to zero to get

$$|\partial_\theta g(\theta(y,t)) - g''(0)\frac{1}{V_\delta}\int_{B_\delta(y)} \omega^\delta(|z-y|)|z-y|\partial_j u_i(y) e_i e_j \, dz| < C\delta^3 \tag{2.107}$$

Now substitution for the approximation to $\partial_\theta g(\theta(y,t)$ in the definition of $\mathcal{L}^D$ gives

$$\begin{aligned} |\mathcal{L}^D(u) - &\left\{ \frac{1}{V_\delta} \int_{B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2} e_{y-x} \frac{1}{2V_\delta} \right. \\ &\times \left. \int_{B_\delta(y)} \omega^\delta(|z-y|)|z-y|\beta g''(0)\partial_j u_i(y) e_i e_j \, dz \, dy \right\}| < C\delta \end{aligned} \tag{2.108}$$

We Taylor expand $\partial_j u_i(y)$ about x, note that odd terms involving tensor products of e_{y-x} vanish when integrated with respect to y in $B_\delta(x)$ and we obtain (3.73).

We now calculate as in ([13] equation (6.64)) or in Section 2.5 to find that

$$\begin{aligned} &\frac{f''(0)}{2\omega_3} \int_{B_1(0)} |\xi| J(|\xi|) e_i e_j e_k e_l \, d\xi \partial^2_{jk} u_i(x) = \\ &(2\mu_1 \left(\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2}) + \lambda_1 \delta_{ij}\delta_{kl} \right) \partial^2_{jk} u_i(x) \end{aligned} \tag{2.109}$$

where

$$\mu_1 = \lambda_1 = \frac{f''(0)}{10} \int_0^1 r^3 \omega(r) \, dr \tag{2.110}$$

Next observe that a straight forward calculation gives

$$\frac{1}{\omega_3} \int_{B_1(0)} |\xi| \omega(|\xi|) e_i e_j \, d\xi = \delta_{ij} \int_0^1 r^3 \omega(r) \, dr \tag{2.111}$$

and we deduce that

$$\begin{aligned} &\frac{1}{\omega_3} \int_{B_1(0)} |\xi| \omega(|\xi|) e_i e_j \, d\xi \frac{\beta g''(0)}{2\omega_3} \int_{B_1(0)} |\xi| \omega(|\xi|) e_k e_l \, d\xi \partial^2_{lj} u_i(x) \\ &= \frac{\beta g''(0)}{2} \left(\int_0^1 r^3 \omega(r) \, dr \right)^2 \delta_{ij}\delta_{kl} \partial^2_{lj} u_i(x) \end{aligned} \tag{2.112}$$

Theorem 3.6 follows on adding (3.76) and (3.79)

□

2.7 Conclusions

We have introduced a regularized model for free fracture propagation based on non-local potentials. At each instant of the evolution we identify the softening zone where strains lie above the strength of the material. We have shown that discontinuities associated with flaws larger than the length scale of non-locality δ can become unstable and grow. An explicit inequality is found that shows that the volume of the softening zone goes to zero linearly with the length scale of non-local interaction. This scaling is consistent with the notion that a softening zone of width proportional to δ converges to a sharp fracture as the length scale of nonlocal interaction goes to zero. Inside quiescent regions with no cracks the nonlocal operator converges to the local elastic operator at a rate proportional to the radius of nonlocal interaction. We that the model can be calibrated to measured values of critical energy release rate, shear modulus, and bulk modulus of material samples. The double well state based potential developed here no longer has Poisson ratio constrained to be $1/4$. For this model we can choose the potentials so that the small strain behavior is specified by the isotropic elasticity tensor for any material with prescribed shear and Lamé moduli.

The energy release rate necessary for tensile forces to create fractures is constant in δ where as the forces necessary to create a fracture using hydrostatic forces grows as $1/\delta$. Thus creation of fracture surfaces by hydrostatic forces will not be seen when

$$\delta < \frac{2\beta g_\infty}{G_s} \tag{2.113}$$

On the other hand the elastic properties for small strains can be made to correspond to any positive definite isotropic elastic tensor.

Chapter 3
State Based Dynamic Damage Propagation with Memory

3.1 Formulation

In this work we assume the displacements u are small (infinitesimal) relative to the size of the three dimensional body D. The tensile strain is denoted $S = S(y, x, t; u)$ and given by

$$S(y, x, t; u) = \frac{u(t, y) - u(t, x)}{|y - x|} \cdot e_{y-x}, \qquad e_{y-x} = \frac{y - x}{|y - x|} \tag{3.1}$$

where e_{y-x} is a unit direction vector and $\cdot$ is the dot product. It is evident that $S(y, x, t; u)$ is the tensile strain along the direction e_{y-x}. We introduce the nonnegative weight $\omega^\delta(|y-x|)$ such that $\omega^\delta = 0$ for $|y-x| > \delta$ and the hydrostatic strain at x is defined by

$$\theta(x, t; u) = \frac{1}{V_\delta} \int_{D \cap B_\delta(x)} \omega^\delta(|y - x|)|y - x| S(y, x, t; u)\, dy \tag{3.2}$$

where V_δ is the volume of the ball $B_\delta(x)$ of radius δ centered at x. The weight is chosen such that $\omega^\delta(|y - x|) = \omega(|y - x|/\delta)$, and

$$\ell_1 = \frac{1}{V_\delta} \int_{B_\delta(x)} \omega^\delta(|y - x|)\, dy < \infty \tag{3.3}$$

We follow [17] and [24] and introduce a nonnegative damage factor taking the value one in the undamaged region and zero in the fully damaged region. The damage factor for the force associated with tensile strains is written $H^T(u)(y, x, t)$, the corresponding factor for hydrostatic strains is written $H^D(u)(x, t)$. Here we assume no damage and $H^T(u)(y, x, t) = 1$ until a critical tensile strain S_c is reached. For tensile strains greater than S_c damage is initiated and $H^T(y, x, t)$ drops below 1. The fully damaged state is $H^T(y, x, t) = 0$. For hydrostatic strains we assume no damage until a critical positive dilatational strain θ_c^+ or a negative compressive strain (θ_c^-) is reached. Again $H^D(x, t) = 1$ until a critical hydrostatic strain is reached and then drops below 1 with the fully damaged state being $H^D(x,t) = 0$. We postpone description of the specific form of the history dependent damage factors until after we have defined the nonlocal forces.

The force at a point x due to tensile strain is given by

$$\mathcal{L}^T(u)(x,t) = \frac{2}{V_\delta}\int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|} H^T(u)(y,x,t)\partial_S f(\sqrt{|y-x|}S(y,x,t;u))e_{y-x}\,dy \tag{3.4}$$

Here $J^\delta(|y-x|)$ is a nonnegative bounded function such that $J^\delta = 0$ for $|y-x| > \delta$ and $M = \sup\{y \in B_\delta(x); J^\delta(|y-x|)\}$ and

$$\ell_2 = \frac{1}{V_\delta}\int_{B_\delta(x)} \frac{J^\delta(|y-x|)}{|y-x|^2}\,dy < \infty \text{ and } \ell_3 = \frac{1}{V_\delta}\int_{B_\delta(x)} \frac{J^\delta(|y-x|)}{|y-x|^{3/2}}\,dy < \infty \tag{3.5}$$

Both J^δ and ω^δ are prescribed and characterize the influence of nonlocal forces on x by neighboring points y. Here ∂_S is the partial derivative with respect to strain. The function $f = f(r)$ is twice differentiable for all arguments r on the real line and f', f'' are bounded. Here we take $f(r) = \alpha r^2/2$ for $r < r_1$ and $f = r$ for $r_2 < r$, with $r_1 < r_2$, see Figure 3.1. The factor $\sqrt{|y-x|}$ appearing in the argument of $\partial_S f$ ensures that the nonlocal operator $\mathcal{L}^T$ converges to the divergence of a stress tensor in the small horizon limit when its known a-priori that displacements are smooth, see section 3.6.

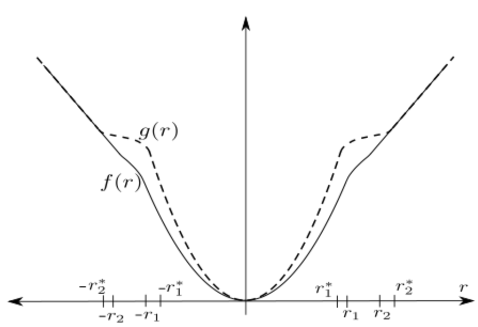

Figure 3.1. Generic plot of $f(r)$ (Solid line) and $g(r)$ (Dashed line).

The force at a point x due to the hydrostatic strain is given by

$$\mathcal{L}^D(u)(x,t) = \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}\left[H^D(u)(y,t)\partial_\theta g(\theta(y,t;u)) + \right. \\ \left. H^D(u)(x,t)\partial_\theta g(\theta(x,t;u))\right] e_{y-x}\, dy \tag{3.6}$$

where the function $g(r) = \beta r^2/2$ for $r < r_1^*$, $g = r$ for $r_2^* < r$, with $r_1^* < r_2^*$ and g is twice differentiable and g', g'' are bounded, see Figure 3.1. It is readily verified that the force $\mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t)$ satisfies balance of linear and angular momentum.

The damage factor for tensile strain $H^T(u)(y,x,t)$ is given in terms of the functions $h(x)$ and $j_S(x)$. Here h is nonnegative, has bounded derivatives (hence Lipschitz continuous), takes the value one for negative x and for $x \geq 0$ decreases and is zero for $x > x_c$, see Figure 3.2. Here we are free to choose x_c to be any small and positive number. The function $j_S(x)$ is nonnegative, has bounded derivatives (hence Lipschitz continuous), takes the value zero up to a positive critical strain S_C and then takes on positive values. We will suppose $j_S(x) \leq \gamma|x|$ for some $\gamma > 0$, see Figure 3.3. The damage factor is now defined to be

$$H^T(u)(y,x,t) = h\left(\int_0^t j_S(S(y,x,\tau;u))\, d\tau\right) \tag{3.7}$$

It is clear from this definition that damage occurs when the stress exceeds S_c for some period of time and the bond force decreases irrevocably from its undamaged value. The damage function defined here is symmetric, i.e., $H^T(u)(y,x,t) = H^T(u)(x,y,t)$. For hydrostatic strain we introduce the nonnegative function j_θ with bounded derivatives (hence Lipschitz continuous). We suppose $j_\theta = 0$ for an interval containing the origin given by (θ_c^-, θ_c^+) and takes positive values outside this interval, see Figure 3.4. As before we will suppose $j_\theta(x) \leq \gamma|x|$ for some $\gamma > 0$, The damage factor for hydrostatic strain is given by

$$H^D(u)(x,t) = h\left(\int_0^t j_\theta(\theta(x,\tau;u))\,d\tau\right) \tag{3.8}$$

For this model it is clear that damage can occur irreversibly for compressive or dilatational strain when the possibly different critical values θ_c^- or θ_c^+ are exceeded.

The *damage set* at time t is defined to be the collection of all points x for which $H^T(y,x,t)$ or $H^D(u)(x,t)$ is less than one. This set is monotonically increasing in time. The *process zone* at time t are the collection of points x undergoing damage such that $\partial_t H^T(y,x,t) < 0$ or $\partial_t H^D(x,t) < 0$. Explicit examples of $H^T(u)(y,x,t)$ and $H^D(u)(x,t)$ are given in section 3.4.

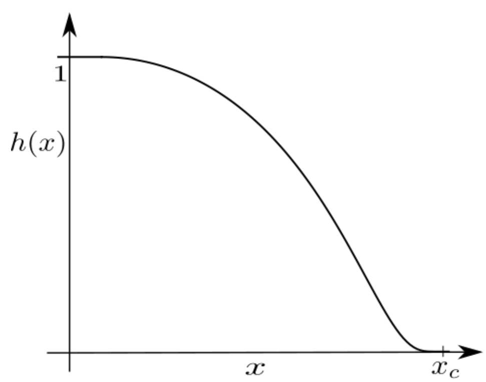

Figure 3.2. Generic plot of $h(x)$.

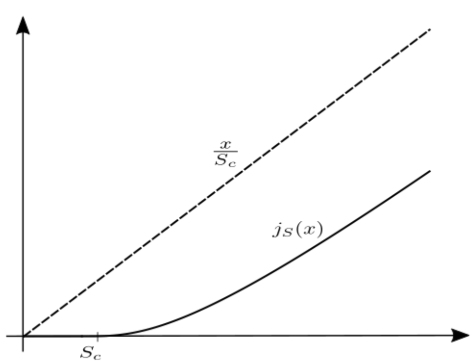

Figure 3.3. Generic plot of $j_S(x)$ with S_c.

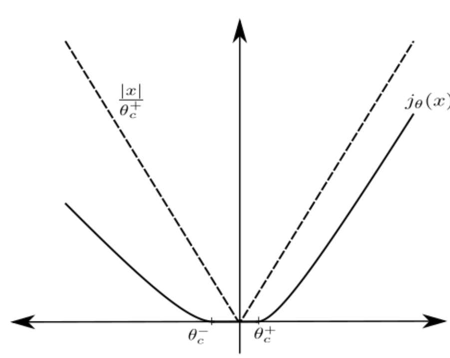

Figure 3.4. Generic plot of $j_\theta(x)$ with θ_c^+, and θ_c^-.

We define the body force $b(x,t)$ and the displacement $u(x,t)$ is the solution of the initial value problem given by

$$\rho\partial_t^2 u(x,t) = \mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t) + b(x,t) \text{ for } x \in D \text{ and } t \in (0,T) \tag{3.9}$$

with initial data

$$u(x,0) = u_0(x), \qquad \partial_t u(x,0) = v_0(x) \tag{3.10}$$

It is easily verified that this is an ordinary state based peridynamic model. We show in the next section that this initial value problem is well posed.

3.2 Existence of Solutions

The regularity and existence of the solution depends on the regularity of the initial data and body force. In this work we choose a general class of body forces and initial conditions. The initial displacement u_0 and velocity v_0 are chosen to be integrable and bounded and belonging to $L^\infty(D;\mathbb{R}^3)$. The space of such functions is denoted by $L^\infty(D;\mathbb{R}^3)$ The body force $b(x,t)$ is chosen such that for every $t \in [0,T_0]$, b takes values in $L^\infty(D,\mathbb{R}^3)$ and is continuous in time. The associated norm is defined to be $\|b\|_{C([0,T_0];L^\infty(D,\mathbb{R}^3))} = max_{t\in[0,T_0]}\|b(x,t)\|_{L^\infty(D,\mathbb{R}^3)}$. The associated space of continuous functions in time taking values in $L^\infty(D;\mathbb{R}^3)$ for which this norm is finite is denoted by $C([0,T_0];L^\infty(D,\mathbb{R}^3))$. The space of functions twice differentiable in time taking values in $L^\infty(D,\mathbb{R}^3)$ such that both derivatives belong to $C([0,T_0];L^\infty(D,\mathbb{R}^3))$ is written as $C^2([0,T_0];L^\infty(D,\mathbb{R}^3))$. We now assert the existence and uniqueness for the solution of the initial value problem.

Theorem 3.1. Existence and uniqueness of the damage evolution. *The initial value problem given by* (3.9) *and* (3.10) *has a solution* $u(x,t)$ *such that for every* $t \in [0,T_0]$, *u takes values in* $L^\infty(D,\mathbb{R}^3)$ *and is the unique solution belonging to the space* $C^2([0,T_0];L^\infty(D,\mathbb{R}^3))$.

To prove the theorem we will show

1. The operator $\mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t)$ is a map from $C([0,T_0];L^\infty(D,\mathbb{R}^3))$ into itself.

2. The operator $\mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t)$ is Lipschitz continuous with respect to the norm of $C([0,T_0];L^\infty(D,\mathbb{R}^3))$.

The theorem then follows from an application of the Banach fixed point theorem.

To establish properties (1) and (2) we state and prove the following lemmas for the damage factors.

Lemma 3.2. *Let $H^T(u)(y,x,t)$ and $H^D(u)(x,t)$ be defined as in (3.7) and (3.8). Then for $u \in C([0,T_0];L^\infty(D,\mathbb{R}^3))$ the mappings*

$$(y,x) \mapsto H^T(u)(y,x,t) : D \times D \to \mathbb{R}, \qquad x \mapsto H^D(u)(x,t) : D \to \mathbb{R} \tag{3.11}$$

are measurable for every $t \in [0,T_0]$, and the mappings

$$t \mapsto H^T(u)(y,x,t) : [0,T_0] \to \mathbb{R}, \qquad t \mapsto H^D(u)(x,t) : [0,T_0] \to \mathbb{R} \tag{3.12}$$

are continuous for almost all (y,x) and x respectively. Moreover for almost all $(y,x) \in D \times D$ and all $t \in [0,T_0]$ the map

$$u \mapsto H^T(u)(y,x,t) : C([0,T_0];L^\infty(D,\mathbb{R}^3)) \to \mathbb{R} \tag{3.13}$$

is Lipschitz continuous, and for almost all $x \in D$ and all $t \in [0,T_0]$ the map

$$u \mapsto H^D(u)(x,t) : C([0,T_0];L^\infty(D,\mathbb{R}^3)) \to \mathbb{R} \tag{3.14}$$

is Lipschitz continuous.

Proof. The measurability properties are immediate. In what follows constants are generic and apply to the context in which they are used. We establish continuity in time for $H^D(u)$. For $\hat{t}$ and t in $[0, T_0]$ we have

$$
\begin{aligned}
&|H^D(u)(x,\hat{t}) - H^D(u)(x,t)| \\
&= |h\left(\int_0^{\hat{t}} j_\theta(\theta(x,\tau;u))\,d\tau\right) - h\left(\int_0^{t} j_\theta(\theta(x,\tau;u))\,d\tau\right)| \\
&\leq C_1 \int_{\min\{\hat{t},t\}}^{\max\{\hat{t},t\}} j_\theta(\theta(x,\tau;u))d\tau \\
&\leq \gamma\, C_1 \int_{\min\{\hat{t},t\}}^{\max\{\hat{t},t\}} |\theta(x,\tau;u)|d\tau \\
&\leq \gamma\, \ell_1 C_1 C_2 |\hat{t} - t| 2 \|u\|_{C([0,T_0];L^\infty(D,\mathbb{R}^3))}
\end{aligned}
\tag{3.15}
$$

The first inequality follows from the Lipschitz continuity of h, the second follows from the growth condition on j_θ, and the third follows from (3.3).

We establish continuity in time for $H^T(u)$. For $\hat{t}$ and t in $[0, T_0]$ we have

$$
\begin{aligned}
&|H^T(u)(x,\hat{t}) - H^T(u)(x,t)| \\
&= |h\left(\int_0^{\hat{t}} j_S(S(y,x,\tau;u))\,d\tau\right) - h\left(\int_0^{t} j_S(S(y,x,\tau;u))\,d\tau\right)| \\
&\leq C_1 \int_{\min\{\hat{t},t\}}^{\max\{\hat{t},t\}} j_S(S(y,x,\tau;u))d\tau \\
&\leq \gamma\, C_1 \int_{\min\{\hat{t},t\}}^{\max\{\hat{t},t\}} |S(y,x,\tau;u)|d\tau \\
&\leq \gamma\, C_1 C_2 \frac{|\hat{t} - t|}{|y - x|} 2 \|u\|_{C([0,T_0];L^\infty(D,\mathbb{R}^3))}
\end{aligned}
\tag{3.16}
$$

The first inequality follows from the Lipschitz continuity of h, the second follows from the growth condition on j_S, and the third follows from the definition of strain (3.1).

To demonstrate Lipschitz continuity for $H^D(u)(x,t)$ we write

$$\begin{aligned}
&|H^D(u)(x,t)) - H^D(v)(x,t)| \\
&= |h\left(\int_0^t j_\theta(\theta(x,\tau;u))\,d\tau\right) - h\left(\int_0^t j_\theta(\theta(x,\tau;v))\,d\tau\right)| \\
&\leq C_1|\int_0^t (j_\theta(\theta(x,\tau;u) - j_\theta(\theta(x,\tau;v))\ d\tau| \\
&\leq C_1 C_2 \int_0^t |\theta(x,\tau;u) - \theta(x,\tau;v)|\,d\tau \\
&\leq 2t\ell_1 C_1 C_2 \|u-v\|_{C([0,t];L^\infty(D,\mathbb{R}^3))}
\end{aligned} \tag{3.17}$$

The first inequality follows from the Lipschitz continuity of h, the second follows from the Lipschitz continuity of j_θ, and the third follows from (3.3). The Lipschitz continuity for $H^S(u)(y,x,t)$ follows from similar arguments using the Lipschitz continuity of h, and j_S, and (3.1), and we get

$$|H^T(u)(y,x,t)) - H^T(v)(y,x,t)| \leq \frac{2tC_1C_2C_3}{|y-x|}\|u-v\|_{C([0,t];L^\infty(D,\mathbb{R}^3))} \tag{3.18}$$

□

Proof of Theorem 3.1. We establish (1) by first noting that

$$|\mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t)| \leq \frac{C}{\delta^2} \tag{3.19}$$

where C is a constant. This estimate follows from the boundedness of f', g', $H^T(u)$, and $H^D(u)$ and the integrability of the ratios $J^\delta(|y-x|)/|y-x|^2$, $J^\delta(|y-x|)/|y-x|^{3/2}$, and $\omega^\delta(|y-x|)$. Thus $\|\mathcal{L}^T(u)(x,t)+\mathcal{L}^D(u)(x,t)\|_{L^\infty(D,\mathbb{R}^3)}$ is uniformly bounded for all $t \in [0, T_0]$.

To complete the demonstration of (1) we point out that the force functions $\partial_S f$ and $\partial_\theta g$ are Lipschitz continuous in their arguments. The key features are given in the following lemma.

Lemma 3.3. *Given two functions v and w in $L^\infty(D,\mathbb{R}^3)$ then*

$$|\partial_S f(\sqrt{|y-x|}S(y,x;v)) - \partial_S f(\sqrt{|y-x|}S(y,x;w))| \leq \frac{2C}{\sqrt{|y-x|}}\|v-w\|_{L^\infty(D,\mathbb{R}^3)} \tag{3.20}$$

and

$$|\partial_\theta g(\theta(x;v)) - \partial_\theta g(\theta(x;w))| \leq 2\ell_1 C\|v-w\|_{L^\infty(D,\mathbb{R}^3)} \tag{3.21}$$

Proof.

$$\begin{aligned}
&|\partial_S f(\sqrt{|y-x|}S(y,x;v)) - \partial_S f(\sqrt{|y-x|}S(y,x;w))| \\
&\leq C\sqrt{|y-x|}|S(y,x;v) - S(y,x;w)| \leq \frac{2C}{\sqrt{|y-x|}}\|v-w\|_{L^\infty(D,\mathbb{R}^3))}
\end{aligned} \tag{3.22}$$

where the first inequality follows from the Lipschitz continuity of $\partial_S f$, and the second follows from the definition of S.

For $\partial_\theta g$ we have

$$|\partial_\theta g(\theta(x;v)) - \partial_\theta g(\theta(x;w))| \leq C|\theta(x;v) - \theta(x;w)| \leq 2\ell_1 C_1\|v-w\|_{L^\infty(D,\mathbb{R}^3))} \tag{3.23}$$

where the first inequality follows from the Lipschitz continuity of $\partial_\theta g$, and the second follows from the definitions of θ and S. □

We have

$$\begin{aligned}
&|\mathcal{L}^T(u)(x,\hat{t}) - \mathcal{L}^T(u)(x,t)| \\
&\leq \frac{2}{V_\delta}\int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|}|\partial_S f(\sqrt{y-x}S(y,x,\hat{t};u)) - \partial_S f(\sqrt{y-x}S(y,x,t;u))|\,dy \\
&+ \frac{2}{V_\delta}\int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|}|H^T(u)(y,x,\hat{t}) - H^T(u)(y,x,t)|\,dy
\end{aligned} \tag{3.24}$$

From the above, (3.18), and Lemma 3.3 we see that

$$\begin{aligned}
&\|\mathcal{L}^T(u)(x,\hat{t}) - \mathcal{L}^T(u)(x,t)\|_{L^\infty(D,\mathbb{R}^3)} \\
&\leq \frac{\ell_3 C_3}{\delta}\|u(x,\hat{t}) - u(x,t)\|_{L^\infty(D,\mathbb{R}^3)} + \frac{\ell_2\gamma\, C_1 C_2}{\delta}|\hat{t}-t|2\|u\|_{C([0,T_0];L^\infty(D,\mathbb{R}^3))}
\end{aligned} \tag{3.25}$$

and we see $\mathcal{L}^T$ is well defied and maps $C([0,T_0];L^\infty(D,\mathbb{R}^3))$ into itself.

We show the continuity in time for $\mathcal{L}^D(u)(x,t)$. Now we have

$$
\begin{aligned}
&|\mathcal{L}^D(u)(x,\hat{t}) - \mathcal{L}^D(u)(x,t)| \\
&\leq \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}|\partial_\theta g(\theta(y,\hat{t};u)) - \partial_\theta g(\theta(y,t;u))|\,dy \\
&+ \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}|H^D(u)(y,\hat{t}) - H^D(u)(y,t)|\,dy \\
&+ \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}|\partial_\theta g(\theta(x,\hat{t};u)) - \partial_\theta g(\theta(x,t;u))|\,dy \\
&+ \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}|H^D(u)(x,\hat{t}) - H^D(u)(x,t)|\,dy
\end{aligned}
\tag{3.26}
$$

and applying Lemma 3.3 and (3.17) to (3.26) we get the continuity

$$
\begin{aligned}
&|\mathcal{L}^D(u)(x,\hat{t}) - \mathcal{L}^D(u)(x,t)| \leq \frac{4\ell_1^2 C_1}{\delta^2}\|u(\hat{t},x) - u(t,x)\|_{L^\infty(D,\mathbb{R}^3)} \\
&+ \frac{\gamma\, 4\ell_1^2 C_1 C_2}{\delta^2}|\hat{t}-t|\|u\|_{C([0,T_0];L^\infty(D,\mathbb{R}^3))}
\end{aligned}
\tag{3.27}
$$

We conclude that $\mathcal{L}^D$ is well-defied and maps $C([0,T_0];L^\infty(D,\mathbb{R}^3))$ into itself and item (1) is proved.

To show Lipschitz continuity consider any two functions u and w belonging to $C([0,T_0];L^\infty(D,\mathbb{R}^3))$, $t\in[0,T_0]$ to write

$$
\begin{aligned}
&|\mathcal{L}^T(u)(x,t) + \mathcal{L}^D(u)(x,t) - [\mathcal{L}^T(w)(x,t) + \mathcal{L}^D(w)(x,t)]| \\
&\leq \frac{2}{V_\delta}\int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|}|\partial_S f(\sqrt{|y-x|}S(y,x,t;u)) - \partial_S f(\sqrt{|y-x|}S(y,x,t;w))|\,dy \\
&+ \frac{2}{V_\delta}\int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|}|H^T(u)(y,x,t) - H^T(w)(y,x,t)|\,dy \\
&+ \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}|\partial_\theta g(\theta(y,t;u)) - \partial_\theta g(\theta(y,t;w))|\,dy \\
&+ \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}|H^D(u)(y,t) - H^D(w)(y,t)|\,dy \\
&+ \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}|\partial_\theta g(\theta(x,t;u)) - \partial_\theta g(\theta(x,t;w))|\,dy \\
&+ \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}|H^D(u)(x,t) - H^D(w)(x,t)|\,dy
\end{aligned}
\tag{3.28}
$$

Applying (3.17) and (3.18) to (3.28) delivers the estimate

$$\begin{aligned}&\|\mathcal{L}^T(u)(x,t)+\mathcal{L}^D(u)(x,t)-[\mathcal{L}^T(w)(x,t)+\mathcal{L}^D(w)(x,t)]\|_{C([0,t];L^\infty(D,\mathbb{R}^3))}\\&\leq\frac{C_1+tC_2}{\delta^2}\|u-w\|_{C([0,t];L^\infty(D,\mathbb{R}^3))}\end{aligned}\tag{3.29}$$

where C_1 and C_2 are constants not depending on time u or w. For $T_0>t$ we can choose a constant $L>(C_1+T_0C_2)/\delta^2$ and

$$\begin{aligned}&\|\mathcal{L}^T(u)(x,t)+\mathcal{L}^D(u)(x,t)-[\mathcal{L}^T(w)(x,t)+\mathcal{L}^D(w)(x,t)]\|_{C([0,t];L^\infty(D,\mathbb{R}^3))}\\&\leq L\|u-w\|_{C([0,t];L^\infty(D,\mathbb{R}^3))},\ \text{for all } t\in[0,T_0]\end{aligned}\tag{3.30}$$

This proves the Lipschitz continuity and item (2) of the theorem is proved. Note that $u(\tau)=w(\tau)$ for all $\tau\in[0,t]$ implies $\mathcal{L}^T(u)(x,t)+\mathcal{L}^D(u)(x,t)=[\mathcal{L}^T(w)(x,t)+\mathcal{L}^D(w)(x,t)]$ and $\mathcal{L}^T(u)(x,t)+\mathcal{L}^D(u)(x,t)$ is a Volterra operator.

We write evolutions $u(x,t)$ belonging to $C([0,t];L^\infty(D,\mathbb{R}^3))$ as $u(t)$ and $(Vu)(t)$ is the sum

$$(Vu)(t)=\mathcal{L}^T(u)(t)+\mathcal{L}^D(u)(t)\tag{3.31}$$

We seek the unique fixed point of $u(t)=(Iu)(t)$ where I maps $C([0,t];L^\infty(D,\mathbb{R}^3))$ into itself and is defined by

$$(Iu)(t)=u_0+tv_0+\int_0^t(t-\tau)(Vu)(\tau)+b(\tau)\,d\tau\tag{3.32}$$

This problem is equivalent to finding the unique solution of the initial value problem given by (3.9) and (3.10). We now show that I is a contraction map and by virtue of the Banach fixed point theorem we can assert the existence of a fixed point in $C([0,t];L^\infty(D,\mathbb{R}^3))$. To see that I is a contraction map on $C([0,t];L^\infty(D,\mathbb{R}^3))$ we introduce the equivalent norm

$$|||u|||_{C([0,t];L^\infty(D,\mathbb{R}^3))}=\max_{t\in[0,T_0]}\{e^{-2LT_0t}\|u\|_{L^\infty(D,\mathbb{R}^3)}\}\tag{3.33}$$

and show I is a contraction map with respect to this norm. We apply (3.30) to find for $t \in [0, T_0]$ that

$$\begin{aligned}\|(Iu)(t) - (Iw)(t)\|_{L^\infty(D,\mathbb{R}^3)} &\leq \int_0^t (t-\tau)\|(Vu)(\tau) - (Vw)(\tau)\|_{L^\infty(D,\mathbb{R}^3)}\, d\tau \\ &\leq LT_0 \int_0^t \|u - w\|_{C([0,\tau];L^\infty(D,\mathbb{R}^3))}\, d\tau \\ &\leq LT_0 \int_0^t \max_{s\in[0,\tau]}\{\|u(s) - w(s)\|_{L^\infty(D,\mathbb{R}^3)} e^{-2LT_0 s}\} e^{2LT_0\tau}\, d\tau \\ &\leq \frac{e^{2LT_0 t} - 1}{2} |||u - w|||_{C([0,T_0];L^\infty(D,\mathbb{R}^3))}\end{aligned} \tag{3.34}$$

and we conclude

$$|||(Iu)(t) - (Iw)(t)|||_{C([0,T_0];L^\infty(D,\mathbb{R}^3))} \leq \frac{1}{2}|||u - w|||_{C([0,T_0];L^\infty(D,\mathbb{R}^3))} \tag{3.35}$$

so I is a contraction map. From the Banach fixed point theorem there is a unique fixed point $u(t)$ belonging to $C([0, T_0]; L^\infty(D, \mathbb{R}^3))$ and it is evident from (3.32) that $u(t)$ also belongs to $C^2([0, T_0]; L^\infty(D, \mathbb{R}^3))$. This concludes the proof of Theorem 3.1.

3.3 Energy Balance

The evolution is shown to exhibit a balance of energy at all times. In this section we describe the potential and the energy dissipation rate and show energy balance in rate form. The potential energy at time t for the evolution is denoted by $U(t)$ and is given by

$$\begin{aligned}U(t) = \frac{2}{V_\delta}\int_D \int_{D\cap B_\delta(x)} &\frac{J^\delta(|y-x|)}{\delta} H^T(u)(y,x,t) f(\sqrt{|y-x|}S(y,x,t;u))\, dydx \\ &+ \int_D \frac{1}{\delta^2} H^D(u)(x,t) g(\theta(x,t;u))\, dx\end{aligned} \tag{3.36}$$

The energy dissipation rate $\partial_t R(t)$ is

$$\partial_t R(t) = -\frac{2}{V_\delta}\int_D \int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta}\partial_t H^T(u)(y,x,t) f(\sqrt{|y-x|}S(y,x,t;u))\,dydx$$
$$- \int_D \frac{1}{\delta^2}\partial_t H^D(u)(x,t) g(\theta(x,t;u))\,dx \tag{3.37}$$

The derivatives $\partial_t H^T(u)(y,x,t)$ and $\partial_t H^D(u)(x,t)$ are easily seen to be non-positive and the dissipation rate satisfies $\partial_t R(t) \geq 0$. The kinetic energy is

$$K(t) = \rho \int_D \frac{|\partial_t u(x,t)|^2}{2} dx \tag{3.38}$$

The energy balance in rate form is given in the following theorem.

Theorem 3.4. *The rate form of energy balance for the damage-fracture evolution is given by*

$$\partial_t K(t) + \partial_t U(t) + \partial_t R(t) = \int_D b(x,t)\cdot \partial_t u(x,t)\,dx \tag{3.39}$$

Proof of Theorem 3.4. We multiply both sides of the evolution equation (3.9) by $\partial_t u(x,t)$ and integrate over D to get

$$\rho \int_D \partial_t^2 u(x,t)\cdot\partial_t u(x,t)\,dx = \int_D \mathcal{L}^T(u)(x,t)\cdot\partial_t u(x,t)\,dx \tag{3.40}$$
$$+ \int_D \mathcal{L}^D(u)(x,t)\cdot\partial_t u(x,t)\,dx + \int_D b(x,t)\cdot\partial_t u(x,t)\,dx \tag{3.41}$$

The term on the left side of the equation is immediately recognized as $\partial_t K(t)$. The first and second terms on the right hand side of the equation are given in the following lemma.

Lemma 3.5. *One has the following integration by parts formulas given by*

$$\int_D \mathcal{L}^T(u)(x,t)\cdot\partial_t u(x,t)\,dx$$
$$= -\frac{2}{V_\delta}\int_D \int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta} H^T(u)(y,x,t)\partial_t f(\sqrt{|y-x|}S(y,x,t;u))\,dydx \tag{3.42}$$

and

$$\int_D \mathcal{L}^D(u)(x,t)\cdot\partial_t u(x,t)\,dx = -\int_D \frac{1}{\delta^2}H^D(u)(x,t)\partial_t g(\theta(x,t;u))\,dx \tag{3.43}$$

Now note that

$$\begin{aligned}
&\partial_t U(t) + \partial_t R(t)\\
&= \frac{2}{V_\delta}\int_D\int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta} H^T(u)(y,x,t)\partial_t f(\sqrt{|y-x|}S(y,x,t;u))\,dydx\\
&+ \int_D \frac{1}{\delta^2}H^D(u)(x,t)\partial_t g(\theta(x,t;u))\,dx
\end{aligned} \tag{3.44}$$

and the energy balance theorem follows from (3.40) and (3.44).

We conclude by proving the integration by parts Lemma 3.5. We start by proving (3.43). We expand $\partial_t g(\theta(x,t))$

$$\begin{aligned}
&\partial_t g(\theta(x,t;u))\\
&= \partial_\theta g(\theta(x,t;u))\frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \omega^\delta(|y-x|)|y-x|\frac{\partial_t u(y)-\partial_t u(x)}{|y-x|}\cdot e_{y-x}\,dy
\end{aligned} \tag{3.45}$$

and write

$$-\int_D \frac{1}{\delta^2}H^D(u)(x,t)\partial_t g(\theta(x,t;u))\,dx = A(t)+B(t) \tag{3.46}$$

where

$$A(t) = -\int_D \frac{1}{\delta^2}H^D(u)(x,t)\partial_\theta g(\theta(x,t;u))\frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \omega^\delta(|y-x|)\partial_t u(y)\cdot e_{y-x}\,dydx \tag{3.47}$$

and

$$B(t) = \int_D \frac{1}{\delta^2}H^D(u)(x,t)\partial_\theta g(\theta(x,t;u))\frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \omega^\delta(|y-x|)\partial_t u(x)\cdot e_{y-x}\,dydx \tag{3.48}$$

Next introduce the characteristic function of D denoted by $\chi_D(x)$ taking the value one inside D and zero outside and together with the properties of $\omega^\delta(|y-x|)$ we rewrite $A(t)$ as

$$A(t) = -\int_{\mathbb{R}^3\times\mathbb{R}^3} \chi_D(x)\chi_D(y)\omega^\delta(|y-x|) \times \frac{1}{\delta^2}H^D(u)(x,t)\partial_\theta g(\theta(x,t;u))\frac{1}{V_\delta}\partial_t u(y)\cdot e_{y-x}\,dydx \tag{3.49}$$

we switch the order of integration and note $-e_{y-x} = e_{x-y}$ to obtain

$$A(t) = \int_D \frac{1}{V_\delta}\int_{D(x)\cap B_\delta(y)} \frac{\omega^\delta(|y-x|)}{\delta^2}H^D(u)(x,t)\partial_\theta g(\theta(x,t;u))e_{x-y}\,dx\cdot\partial_t u(y)dy \tag{3.50}$$

We can move $\partial_t u(x)$ outside the inner integral, regroup factors, and write $B(t)$ as

$$B(t) = \int_D \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}H^D(u)(x,t)\partial_\theta g(\theta(x,t;u))e_{y-x}\,dy\cdot\partial_t u(x)dx \tag{3.51}$$

We rename the inner variable of integration y and the outer variable x in (3.50) and add equations (3.50) and (3.51) to get

$$A(t)+B(t) = \int_D \mathcal{L}^D(u)(x,t)\cdot\partial_t u(x,t)\,dx \tag{3.52}$$

and (3.43) is proved.

The steps used to prove (3.42) are similar to the proof of (3.43) so we provide only the key points of its derivation below. We expand $\partial_t f(\sqrt{|y-x|}S)$ to get

$$\begin{aligned}&\partial_t f(\sqrt{|y-x|}S(y,x,t;u))\\ &= \partial_S f(\sqrt{|y-x|}S(y,x,t;u))\frac{\partial_t u(y)-\partial_t u(x)}{|y-x|}\cdot e_{y-x}\end{aligned} \tag{3.53}$$

and write

$$\begin{aligned}-\frac{2}{V_\delta}\int_D\int_{D\cap B_\delta(x)} &\frac{J^\delta(|y-x|)}{\delta}H^T(u)(y,x,t)\partial_t f(\sqrt{|y-x|}S(y,x,t;u))\,dydx\\ &= A(t)+B(t)\end{aligned} \tag{3.54}$$

where

$$\begin{aligned}A(t) = -\int_D \frac{1}{V_\delta}\int_{D\cap B_\delta(x)} &\frac{J^\delta(|y-x|)}{\delta|y-x|}H^T(u)(y,x,t)\\ &\times\partial_S f(\sqrt{|y-x|}S(y,x,t;u))\partial_t u(y)\cdot e_{y-x}\,dydx\end{aligned} \tag{3.55}$$

and

$$B(t) = \int_D \frac{1}{V_\delta} \int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|} H^T(u)(y,x,t) \times \partial_S f(\sqrt{|y-x|}S(y,x,t;u))\partial_t u(x)\cdot e_{y-x}\,dydx \tag{3.56}$$

We note that $S(y,x,t;u) = S(x,y,t;u)$ and $H^T(u)(y,x,t) = H^T(u)(x,y,t)$ and proceeding as in the proof of (3.43) we change the order of integration in (3.55) noting that $-e_{y-x} = e_{x-y}$ to get

$$A(t) = \int_D \frac{1}{V_\delta} \int_{D\cap B_\delta(y)} \frac{J^\delta(|y-x|)}{\delta|y-x|} H^T(u)(x,y,t) \times \partial_S f(\sqrt{|y-x|}S(x,y,t;u)))e_{x-y}\,dx\cdot \partial_t u(y)\,dy \tag{3.57}$$

Taking $\partial_t u(x)$ outside the inner integral in (3.56) gives

$$B(t) = \int_D \frac{1}{V_\delta} \int_{D\cap B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|} H^T(u)(y,x,t) \times \partial_S f(\sqrt{|y-x|}S(y,x,t;u))e_{y-x}\,dy\cdot \partial_t u(x)\,dx \tag{3.58}$$

We conclude noting that now

$$A(t)+B(t) = \int_D \mathcal{L}^T(u)(x,t)\cdot\partial_t u(x,t)\,dx \tag{3.59}$$

and (3.42) is proved.

3.4 Explicit Damage Models, Cyclic Loading and Strain to Failure

In this section we provide concrete examples of the damage functions $H^T(u)(y,x,t)$ and $H^D(u)(x,t)$. We provide an example of cyclic loading and the associated degradation in the nonlocal force-strain law as well as the strain to failure curve for monotonically increasing strains. In this work both damage functions H^T and H^D are given in terms of the function h. Here we give an example of $h(x):\mathbb{R}\to\mathbb{R}^+$ as follows

$$h(x) = \begin{cases} \bar{h}(x/x_c), & \text{for } x\in(0,x_c), \\ 1, & \text{for } x\le 0, \\ 0, & \text{for } x\ge x_c. \end{cases} \tag{3.60}$$

with $\bar{h} : [0, 1] \to \mathbb{R}^+$ is defined as

$$\bar{h}(x) = \exp[1 - \frac{1}{1 - (x/x_c)^a}] \tag{3.61}$$

where $a > 1$ is fixed. Clearly, $\bar{h}(0) = 1$, $\bar{h}(x_c) = 0$, see Figure 3.4.

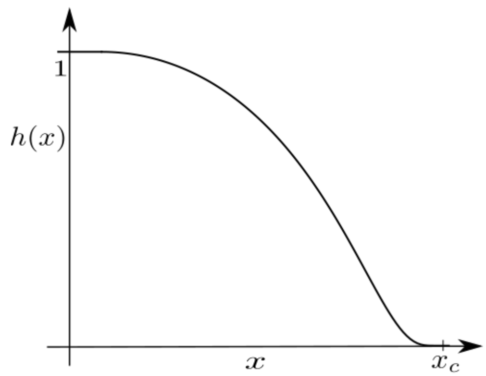

Figure 3.5. Plot of $h(x)$ with $a = 2$.

For a given critical strain $S_c > 0$, we define the threshold function for tensile strain $j_S(x)$ as follows

$$j_S(x) := \begin{cases} \bar{j}(x/S_c), & \forall x \in [S_c, \infty), \\ 0, & \text{otherwise.} \end{cases} \tag{3.62}$$

where $\bar{j} : [1, \infty) \to \mathbb{R}^+$ is given by

$$\bar{j}(x) = \frac{(x-1)^a}{1 + x^b} \tag{3.63}$$

with $a > 1$ and $b \geq a - 1$ fixed. Note that $j_S(1) = 0$. Here the condition $b \geq a - 1$ insures the existence of a constant $\gamma > 0$ for which

$$j_S(x) \leq \gamma |x|, \qquad \forall x \in \mathbb{R}, \tag{3.64}$$

see Figure 3.6.

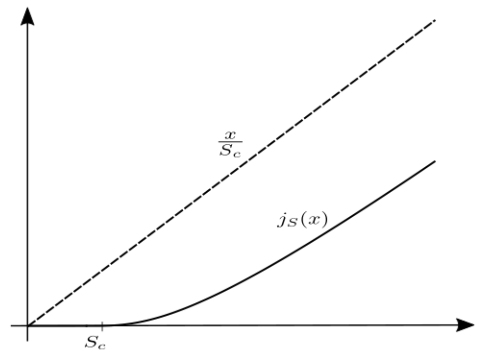

Figure 3.6. Plot of $j_S(x)$ with $a = 4, b = 5$ and $S_c = 2$.

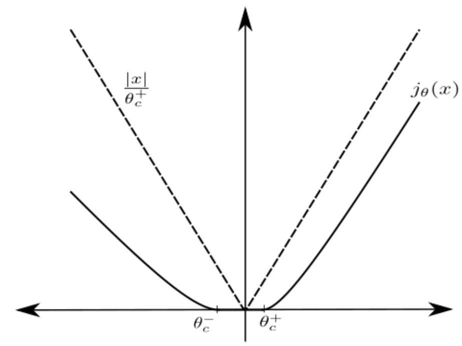

Figure 3.7. Plot of $j_\theta(x)$ with $a = 4, b = 5$, $\theta_c^+ = 2$, and $\theta_C^- = 3$.

For a given critical hydrostatic strains $\theta_c^- < 0 < \theta_c^+$ we define the threshold function $j_\theta(x)$ as

$$j_\theta(x) := \begin{cases} \bar{j}(x/\theta_c^+), & \forall x \in [\theta_c^+, \infty), \\ \bar{j}(-x/\theta_c^-), & \forall x \in (-\infty, -\theta_c^-], \\ 0, & \text{otherwise}, \end{cases} \tag{3.65}$$

where $\bar{j}(x)$ is defined by (3.63) and we plot j_θ in Figure 3.7. We summarize noting that an explicit form for $H^T(u)(y, x, t)$ is obtained by using (3.60) and (3.62) in (3.7) and an explicit form for $H^D(u)(x, t)$ is obtained by using (3.60) and (3.65) in (3.8).

We first provide an example of cyclic damage incurred by a periodically varying tensile strain. Let x, y be two fixed material points with $|y - x| < \delta$ and let $S(y, x, t; u) = S(t)$ correspond to a temporally periodic strain, see 3.8a. Here $S(t)$ periodically takes excursions above the critical strain S_c. During the first period we have

$$S(t) = \begin{cases} t, & \forall t \in [0, S_C + \epsilon], \\ 2(S_C + \epsilon) - t & \forall t \in (S_C + \epsilon, 2(S_c + \epsilon)] \end{cases}$$

and $S(t)$ is extended to $\mathbb{R}^+$ by periodicity, see 3.8a. For this damage model we let η be the area under the curve $j_S(x)$ from $x = S_c$ to $x = S_c + \epsilon$. It is given by

$$\eta = \int_{S_c}^{S_c+\epsilon} j_S(x)dx = \int_{S_c}^{S_c+\epsilon} j_S(S(t))dt$$

From symmetry the area under the curve $j_S(x)$ under unloading from $S_c + \epsilon$ to S_c is also η. The corresponding damage function $H^T(u)(y, x, t)$ is plotted in 3.8b.

In 3.9, we plot the strain-force relation where S is the abscissa and the tensile force given by $H^T((u)(y, x, t))\partial_S f(\sqrt{|y-x|}S(y, x, t; u)))$ is the ordinate. Here the damage factor $H^T(u)(y, x, t)$ drops in value with each cycle of strain loading. After each cycle, the slope (elasticity) in the linear and recoverable part of the force-strain curve decreases due to damage. The force needed to soften the material is the strength and it is clear from the model that the strength decreases after each cycle due to damage.

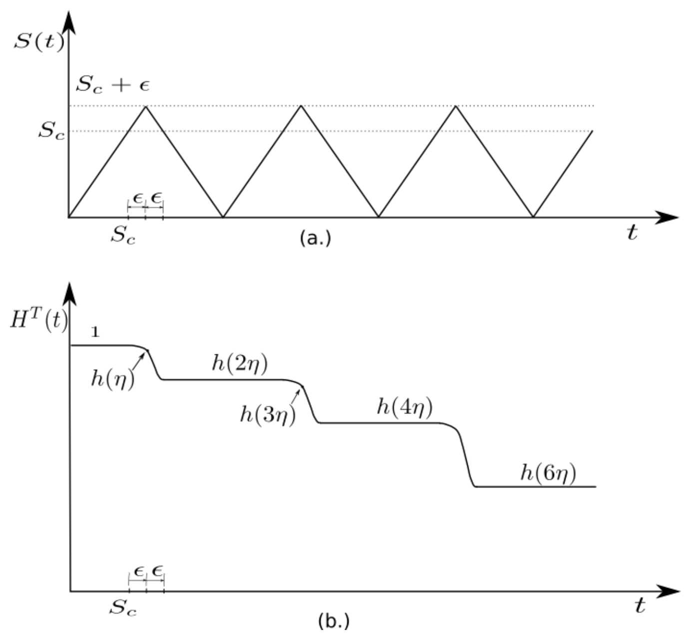

Figure 3.8. (a.) Strain profile. (b.) Damage function plot corresponding to strain profile.

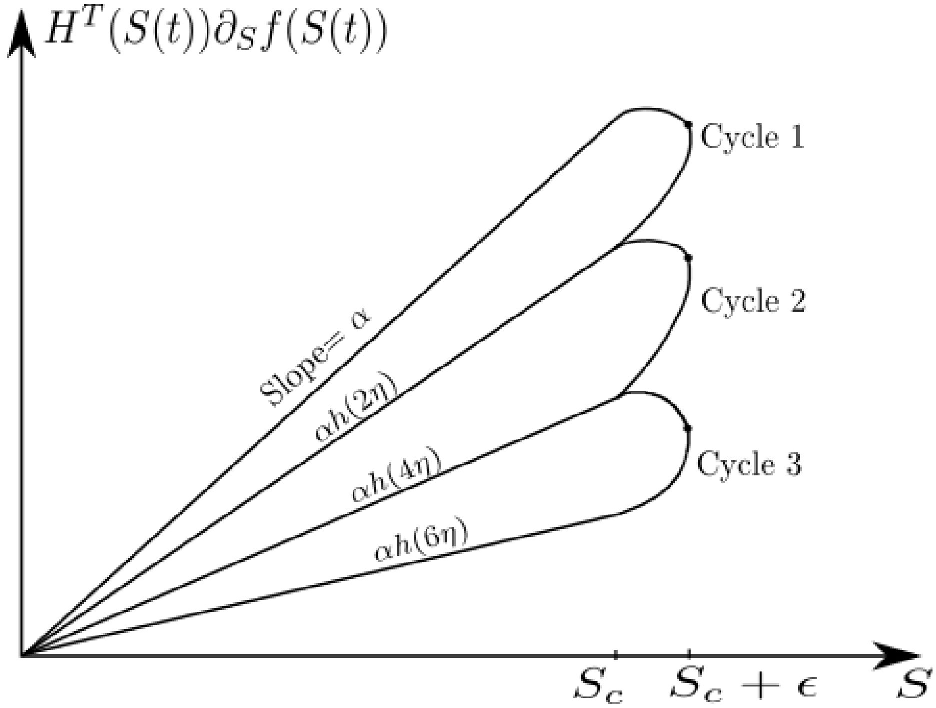

Figure 3.9. Cyclic strain vs Force plot. The initial stiffness is α. Hysteresis is evident in this model.

Application of this rigorously established model to fatigue is a topic of future research but beyond the scope of this article. We note that fatigue models based on perydynamic bond softening are introduced in [29] and with fatigue crack nucleation in the context of the Paris law in [31].

The next example is strain to failure for a monotonically increasing strain. Here we let

$S(y, x, t; u) = S(t) = t$ and plot the corresponding force-strain curve in 3.10. We see that the force-strain relation is initially linear until the strain exceeds S_c, the force then reaches its maximum and subsequently softens to failure. At $S^* \approx 0.55025$, we have $\int_0^{S^*} j_S(t)dt = x_c$, and $H^T = 0$. Here we take $\alpha = 1$.

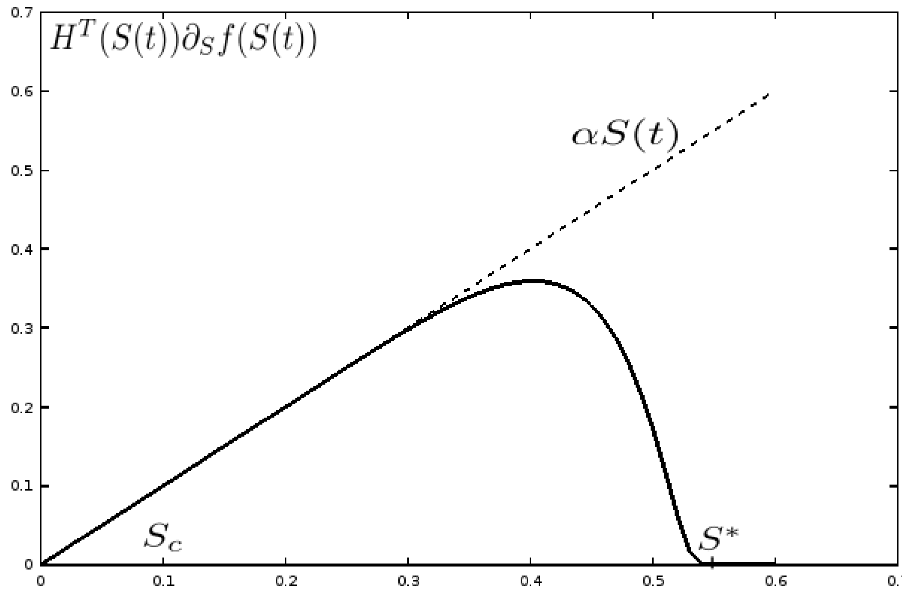

Figure 3.10. Strain vs Force plot where $S(t) = t$. $H^T(S(t))$ begins to drop at $S_c = 0.1$ and $S^* \approx 0.55025$.

3.5 Numerical Results

In this section, we present numerical results. Explicit expressions of the functions described in e the previous section are used in simulating the problem. The damage function h is defined similar to 3.60 with exponent $a = 1.01$ and $x_c = 0.2$. The function j_S is given by 3.62 with $a = 5, b = 5, S_c = 0.01$. The function j_θ is given by 3.65 with $a = 4, b = 5, \theta_c^+ = 0.3, \theta_c^- = 0.4$. Nonlinear potential function f is given by $f(r) = \alpha r^2$ for $r < r_1$ and $f(r) = r$ for $r > r_2$. We let $\alpha = 10$ and let $r_1 = r_2 = 0.05$. Similarly, the nonlinear potential function g is given by $g(r) = \beta r^2$ for $r < r_1^*$ and $g(r) = r$ for $r > r_2^*$. We let $\beta = 1$ and let $r_1^* = r_2^* = 0.05$. The influence function is given by $J^\delta(|y-x|) = \omega^\delta(|y-x|) = 1 - \frac{|y-x|}{\delta}$ for $0 \leq |y-x| \leq \delta$ and $J^\delta(|y-x|) = \omega^\delta(|y-x|) = 0$ otherwise. We consider θ_c^+ and θ_c^- sufficiently high so that we only see damage due to tensile forces and not hydrostatic forces.

In both numerical problems, we consider the material domain $D = [0,1]^2$. We also keep the initial condition fixed to $u_0 = 0$ and $v_0 = 0$. Further, we apply no body force, i.e. $b = 0$. However we will consider boundary loading that is periodic in time. Let $x = (x_1, x_2)$ where x_1 corresponds to the component along horizontal axis and x_2 corresponds to the component along vertical axis.

Periodic Loading

We apply boundary condition $u = 0$ on edge $x_1 = 0$, $x_1 = 1$, and $x_2 = 0$. We consider function $\bar{u}$ of form

$$\bar{u}(t) = \begin{cases} \alpha_{bc} t, & \forall t \in [0, T_{bc}], \\ \alpha_{bc} T_{bc} - t & \forall t \in (T_{bc}, 2T_{bc}] \end{cases} \tag{3.66}$$

and periodically extend the function for any time t. For point x on edge $x_2 = 1$, we apply $u(t, x) = (u_1(t, x), u_2(t, x)) = (0, \bar{u}(t))$. We consider $\alpha_{bc} = 0.01$ and $T_{bc} = 0.216$.

To numerically approximate the evolution equation, we discretize the domain D uniformly with mesh size $h = \delta/5$, where $\delta = 0.15$ in this problem. For time discretization, we consider the velocity verlet scheme for second order in time differential equation and a midpoint quadrature for the spatial discretization. Final time is $T = 1.2$ and size of time step is $\Delta t = 10^{-5}$.

To obtain the hysteresis plot, we chose bonds as shown in 3.11. We track the bond strain $S(y, x, t; u)$ and other relevant quantities. While we track all the bonds shown in 3.11, we only provide plots for the bond which is near to middle top edge. For the bonds in either left and right of the bond at middle top edge, the response is same. For the bond inside the material, the strains are never greater than S_c and therefore it experiences no damage.

3.12 and 3.13 show the strain of the bond and damage H^T of the bond as function of time. It is quite similar to the plots shown in 3.8 and 3.9.

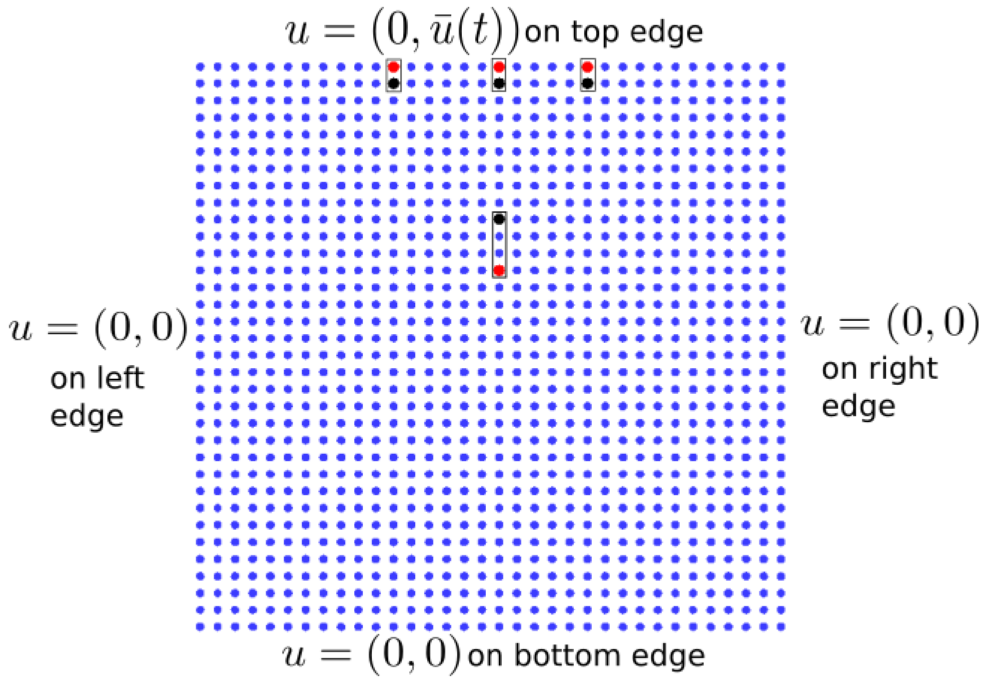

Figure 3.11. Discretization of material domain $D = [0,1]^2$. During simulation bond between red and black material point is tracked to obtain the strain vs stress profile and other information.

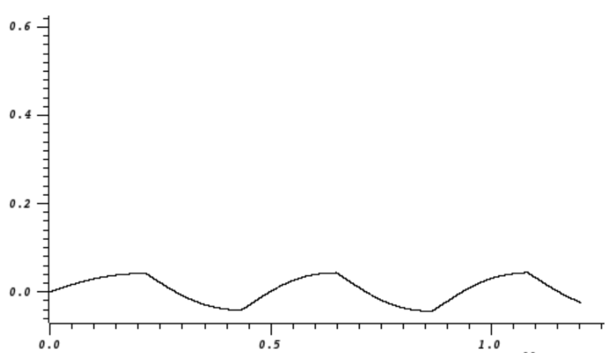

Figure 3.12. Time vs Strain $S(y, x, t; u)$ plot.

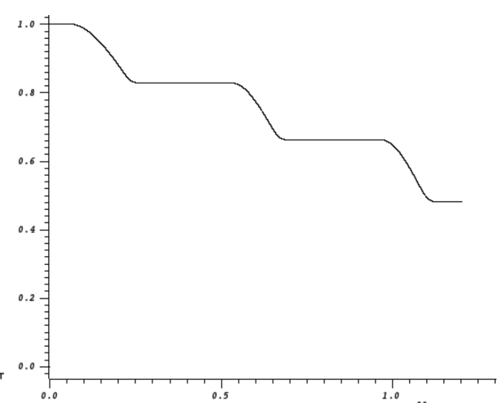

Figure 3.13. Time vs Damage function $H^T((u)(y, x, t))$ plot.

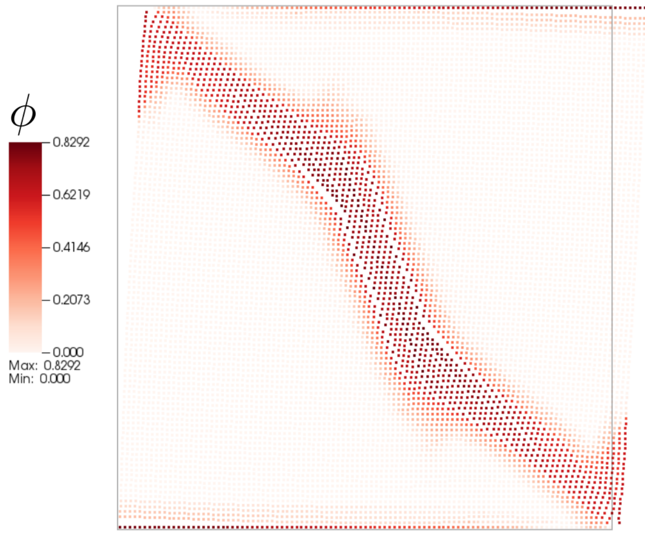

Figure 3.14. Each point in figure shows the discretized mesh node. Strength of color shows the damage ϕ experienced by the mesh node. Box shows reference material domain $[0,1]^2$.

Shear Loading

We apply $u = 0$ on bottom edge, and keep left and right edge free. On top, we apply $u(t,x) = (u_1(t,x), u_2(t,x)) = (\gamma t x_2, 0)$. We chose $\gamma = 0.0001$ and simulate the problem up to time $T = 750$. Time step is $\Delta t = 10^{-5}$.

We choose the size of horizon to be $\delta = 0.05$ and mesh size $h = \delta/5$. As noted in the beginning of the section that we hydrostatic parameters large enough such that the damage is only due to the tensile interaction between material points. For tensile interaction, the extent of damage experienced by a material is defined as

$$\phi(t,x;u) = 1 - \frac{\int_{D\cap B_\delta(x)} H^T(u)(y,x,t)dy}{\int_{D\cap B_\delta(x)} dy} \tag{3.67}$$

Clearly, if all bonds in a horizon of material point x suffer no damage then ϕ will be 0. As the damage of bonds increases ϕ also increases. In 3.14, we show ϕ at final time $t = 750$. As we can see, the damage is along the diagonal of square.

3.6 Linear Elastic Operators in the Small Horizon Limit

In this section we consider smooth evolutions u in space and show that away from damage set the operators $\mathcal{L}^T + \mathcal{L}^D$ acting on u converge to the operator of linear elasticity in the limit of vanishing nonlocality. We denote the damage set by $\tilde{D}$ and consider any open undamaged set D' interior to D with its boundary a finite distance away from the boundary of D and the damage set $\tilde{D}$. In what follows we suppose that the nonlocal horizon δ is smaller than the distance separating the boundary of D' from the boundaries of D and $\tilde{D}$.

Theorem 3.6. *Convergence to linear elastic operators. Suppose that* $u(x,t) \in C^2([0,T_0], C^3(D,\mathbb{R}^3))$ *and no damage, i.e.,* $H^T(y,x,t) = 1$ *and* $H^D(x,t) = 1$*, for every* $x \in D' \subset D \setminus \tilde{D}$*, then there is a constant* $C > 0$ *independent of nonlocal horizon* δ *such that for every* (x,t) *in* $D' \times [0,T_0]$*, one has*

$$|\mathcal{L}^T(u(t)) + \mathcal{L}^D(u(t)) - \nabla\cdot\mathbb{C}\,\mathcal{E}(u(t))| < C\delta \tag{3.68}$$

where the the elastic strain is $\mathcal{E}(u) = (\nabla u + (\nabla u)^T)/2$ *and the elastic tensor is isotropic and given by*

$$\mathbb{C}_{ijkl} = 2\mu\left(\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2}\right) + \lambda\delta_{ij}\delta_{kl} \tag{3.69}$$

with shear modulus μ *and Lamé coefficient* λ *given by*

$$\mu = \frac{f''(0)}{10}\int_0^1 r^3 J(r)\,dr \ \text{ and } \lambda = g''(0)\left(\int_0^1 r^3 J(r)\,dr\right)^2 + \frac{f''(0)}{10}\int_0^1 r^3 J(r)\,dr \tag{3.70}$$

The numbers $f''(0) = \alpha$ *and* $g''(0) = \beta$ *can be chosen independently and can be any pair of real numbers such that* $\mathbb{C}$ *is positive definite.*

Proof. We start by showing

$$|\mathcal{L}^T(u(t)) - \frac{f''(0)}{2\omega_3}\int_{B_1(0)} e|\xi|J(|\xi|)e_ie_je_k\,d\xi\partial^2_{jk}u_i(x)| < C\delta \tag{3.71}$$

where $\omega_3 = 4\pi/3$ and $e = e_{y-x}$ are unit vectors on the sphere, here repeated indices indicate summation. To see this recall the formula for $\mathcal{L}^T(u)$ and write

$\partial_S f(\sqrt{|y-x|}S) = f'(\sqrt{|y-x|}S)\sqrt{|y-x|}$. Now Taylor expand $f'((\sqrt{|y-x|}S)$ in $\sqrt{|y-x|}S$ and Taylor expand $u(y)$ about x, denoting e_{y-x} by e to find that all odd terms in e integrate to zero and

$$|\mathcal{L}^T(u(t))_l - \frac{2}{V_\delta}\int_{B_\delta(x)} \frac{J^\delta(|y-x|)}{\delta|y-x|}\frac{f''(0)}{4}|y-x|^2\partial^2_{jk}u_i(x)e_ie_je_ke_l, dy| < C\delta, \quad l=1,2,3. \tag{3.72}$$

On changing variables $\xi = (y-x)/\delta$ we recover (3.71). Now we show

$$|\mathcal{L}^D(u(t))_k - \frac{1}{\omega_3}\int_{B_1(0)} |\xi|\omega(|\xi|)e_ie_j\,d\xi\frac{g''(0)}{\omega_3}\int_{B_1(0)} |\xi|\omega(|\xi|)e_ke_l\,d\xi\partial^2_{lj}u_i(x)| < C\delta, \quad k=1,2,3. \tag{3.73}$$

We note for $x \in D'$ that $D \cap B_\delta(x) = B_\delta(x)$ and the integrand in the second term of (3.6) is odd and the integral vanishes. For the first term in (3.6) we Taylor expand $\partial_\theta g(\theta)$ about $\theta = 0$ and Taylor expand $u(z)$ about y inside $\theta(y,t)$ noting that terms odd in $e = e_{z-y}$ integrate to zero to get

$$|\partial_\theta g(\theta(y,t)) - g''(0)\frac{1}{V_\delta}\int_{B_\delta(y)} \omega^\delta(|z-y|)|z-y|\partial_j u_i(y)e_ie_j\,dz| < C\delta^3 \tag{3.74}$$

Now substitution for the approximation to $\partial_\theta g(\theta(y,t)$ in the definition of $\mathcal{L}^D$ gives

$$\begin{aligned}|\mathcal{L}^D(u) - \Bigg\{\frac{1}{V_\delta}\int_{B_\delta(x)} \frac{\omega^\delta(|y-x|)}{\delta^2}e_{y-x}\frac{1}{V_\delta}\int_{B_\delta(y)} \omega^\delta(|z-y|)|z-y| \\ \times g''(0)\partial_j u_i(y)e_ie_j\,dz\,dy\Bigg\}| < C\delta\end{aligned} \tag{3.75}$$

We Taylor expand $\partial_j u_i(y)$ about x, note that odd terms involving tensor products of e_{y-x} vanish when integrated with respect to y in $B_\delta(x)$ and we obtain (3.73).

We now calculate as in ([13] equation (6.64)) to find that

$$\begin{aligned}\frac{f''(0)}{2\omega_3}\int_{B_1(0)} |\xi|J(|\xi|)e_ie_je_ke_l\,d\xi\partial^2_{jk}u_i(x) = \\ = (2\mu_1\left(\frac{\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}}{2}) + \lambda_1\delta_{ij}\delta_{kl}\right)\partial^2_{jk}u_i(x)\end{aligned} \tag{3.76}$$

where

$$\mu_1 = \lambda_1 = \frac{f''(0)}{10} \int_0^1 r^3 J(r)\, dr \tag{3.77}$$

Next observe that a straight forward calculation gives

$$\frac{1}{\omega_3} \int_{B_1(0)} |\xi| \omega(|\xi|) e_i e_j \, d\xi = \delta_{ij} \int_0^1 r^3 \omega(r)\, dr \tag{3.78}$$

and we deduce that

$$\begin{aligned} &\frac{1}{\omega_3} \int_{B_1(0)} |\xi| \omega(|\xi|) e_i e_j \, d\xi \frac{g''(0)}{\omega_3} \int_{B_1(0)} |\xi| \omega(|\xi|) e_k e_l \, d\xi \partial^2_{lj} u_i(x) \\ &= g''(0) \left(\int_0^1 r^3 \omega(r)\, dr \right)^2 \delta_{ij} \delta_{kl} \partial^2_{lj} u_i(x) \end{aligned} \tag{3.79}$$

Theorem 3.6 follows on adding (3.76) and (3.79)

□

3.7 Conclusions

We have introduced a simple nonlocal model for free damage propagation in solids. In this model there is only one equation and it describes the dynamics of the displacement using Newton's law $F = ma$. The damage is a consequence of displacement history and diminishes the force strain law as damage accumulates. The modeling allows for both cyclic damage or damage due to abrupt loading. The damage is irreversible and the damage set grows with time. The dissipation energy due to damage together with the kinetic and potential energy satisfies energy balance at every instant of the evolution. Future work will address the question of localization of damage using this model. We believe that if the loading is such that large monotonically increasing strains are generated then damage localization based on material softening and inertia could be anticipated.

In this treatment we have considered dynamic problems only. For this case we have shown uniqueness for the model. The analysis of this model in the absence of inertial forces leads to the quasi-static case where the effects of inertia are absent but memory of the load history is still present. Future work aims to explore this model for this case and understand regimes of body force specimen geometry and boundary loads for which there is loss of uniqueness and associated instability. Such non-uniqueness is well known for quasi-static gradient damage models [30].

References

[1] A. Agwai, I. Guven, and E. Madenci, *Predicting crack propagation with peridynamics: a comparative study*, International Journal of Fracture, 171 (2011), pp. 65–78.

[2] F. Bobaru and W. Hu, *The meaning, selection, and use of the peridynamic horizon and its relation to crack branching in brittle materials*, International Journal of Fracture, 176 (2012), pp. 215–222.

[3] K. Dayal and K. Bhattacharya, *Kinetics of phase transformations in the peridynamic formulation of continuum mechanics*, Journal of the Mechanics and Physics of Solids, 54 (2006), pp. 1811–1842.

[4] P. Diehl, R. Lipton, and M. A. Schweitzer, *Numerical verification of a bond-based softening peridynamic model for small displacements: Deducing material parameters from classical linear theory.* Institut für Numerische Simulation Preprint No. 1630, (2016).

[5] B. K. Driver, *Analysis tools with applications*, E-book, Springer, Berlin, 2003.

[6] Q. Du, M. Gunzburger, R. Lehoucq, and K. Zhou, *Analysis of the volume-constrained peridynamic Navier equation of linear elasticity*, Journal of Elasticity, 113 (2013), pp. 193–217.

[7] F. Bobaru, J. T. Foster, P. H. Geubelle, and S. A. Silling, *Handbook of Peridynamic Modeling*, 2016.

[8] J. T. Foster, S. A. Silling, and W. Chen, *An energy based failure criterion for use with peridynamic states*, International Journal for Multiscale Computational Engineering, 9 (2011).

[9] W. Gerstle, N. Sau, and S. Silling, *Peridynamic modeling of concrete structures*, Nuclear Engineering and Design, 237 (2007), pp. 1250–1258.

[10] Y. D. Ha and F. Bobaru, *Studies of dynamic crack propagation and crack branching with peridynamics*, International Journal of Fracture, 162 (2010), pp. 229–244.

[11] I. Jasiuk, J. Chen, and M. F. Thorpe, *Elastic moduli of two dimensional materials with polygonal and elliptical holes,* Applied Mechanics Rev., **47** (1994), pp. S18–S28.

[12] R. Lipton, *Dynamic brittle fracture as a small horizon limit of peridynamics*, Journal of Elasticity, 117 (2014), pp. 21–50.

[13] R. Lipton, *Cohesive dynamics and brittle fracture*, Journal of Elasticity, 124, Issue 2 (2016), pp. 143-191.

[14] R. Lipton, S. Silling, and R. Lehoucq, *Complex fracture nucleation and evolution with nonlocal elastodynamics*, arXiv preprint arXiv:1602.00247, (2016).

[15] G. W. Milton, The Theory of Composites. Cambridge University Press, Cambridge, 2002.

[16] S. Silling, O. Weckner, E. Askari, and F. Bobaru, *Crack nucleation in a peridynamic solid*, International Journal of Fracture, 162 (2010), pp. 219–227.

[17] S. A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, Journal of the Mechanics and Physics of Solids, 48 (2000), pp. 175–209.

[18] S. A. Silling and E. Askari, *A meshfree method based on the peridynamic model of solid mechanics*, Computers & Structures, 83 (2005), pp. 1526–1535.

[19] S. A. Silling and F. Bobaru, *Peridynamic modeling of membranes and fibers*, International Journal of Non-Linear Mechanics, 40 (2005), pp. 395–409.

[20] S. A. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari, *Peridynamic states and constitutive modeling*, Journal of Elasticity, 88 (2007), pp. 151–184.

[21] S. A. Silling and R. B. Lehoucq, *Convergence of peridynamics to classical elasticity theory*, Journal of Elasticity, 93 (2008), pp. 13–37.

[22] O. Weckner and R. Abeyaratne, *The effect of long-range forces on the dynamics of a bar*, Journal of the Mechanics and Physics of Solids, 53 (2005), pp. 705–728.

[23] Q. Du, Y. Tao, and X. Tian,*A peridynamic model of fracture mechanics with bond-breaking*, Journal of Elasticity, (2017) DOI 10.1007/s10659-017-9661-2.

[24] E. Emmrich and D. Phust, *A short note on modeling damage in peridynamics*, Journal of Elasticity , 123 (2016), pp. 245–252.

[25] E. Emmrich and O. Weckner, *On the well-posedness of the linear peridynamic model and its convergencee towards the Navier equation of linear elasticity*, Commun. Math. Sci., 5 (2007), pp. 851–864.

[26] P. K. Jha and R. Lipton (2017) *Numerical analysis of peridynamic models in Hölder space,* arXiv preprint arXiv:1701.02818.

[27] R. Lipton, E. Said, and P. K. Jha, *Free damage propagation with memory,* Journal of Elasticity *In revision.*

[28] T. Mengesha and Q. Du, *Nonlocal constrained value problems for a linear peridynamic Navier equation,* Journal of Elasticity, 116 (2014), pp. 27–51.

[29] E. Oterkus, I. Guven, and E. Madenci, *Fatigue failure model with peridynamic theory.*,In IEEE Intersociety Conference on Thermal and Thermomechanical Phenomena in Electronic Systems (ITherm), Las Vegas, NV, June 2010, pages 1–6, 2010.

[30] K. Pham and J. J. Marigo, *From the onset of damage to rupture: construction of responses with damage localization for a general class of gradient damage models.* Continuum Mech. Thermodyn., 25 (2013) pp. 147–171.

[31] S. A. Silling and E. Askari, *Peridynamic model for fatigue cracking*, Sandia Report, SAND2014-18590, 2014.

[32] O. Weckner and E. Emmrich, *Numerical simulation of the dynamics of a nonlocal, inhomogeneous, infinite bar*, J. Comput. Appl. Mech, 6 (2005), pp. 311–319.

Vita

Eyad Said was born in Damascus City, Syria. He finished his undergraduate studies at Damascus University in 2004. He earned a master of science degree in banking and finance from Damascus University in 2006. In August 2012 he came to Louisiana State University to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2018