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Estimation of the Renewal Function.

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Estimation of the Renewal Function

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NOTATIONS

$h(t)$	Renewal density function
$H(t)$	Renewal function
$r(t)$	Hazard rate function
IHR	Increasing hazard rate
DHR	Decreasing hazard rate
CHR	Constant hazard rate
$\hat{H}_{pf}$	Parametric estimator of the renewal function (by Frees)
$\hat{H}_{pi}$	Parametric integral estimator of the renewal function
$\hat{H}_{n}$	Nonparametric estimator for the renewal function
$\hat{H}_{nf}$	Nonparametric estimator of the renewal function (by Frees)
$\hat{H}_{ne}$	Nonparametric empirical estimator of the renewal function
$\hat{H}_{na}$	Nonparametric asymptotic estimator of the renewal function
$\hat{H}_{nk}$	Nonparametric Kaplan-Meier estimator of the renewal function

ABSTRACT

The computation of the renewal function when the distribution function is completely known has received much attention in the literature. However, in many cases the form of the distribution function is unknown and has to be estimated nonparametrically. Several nonparametric estimators for the renewal function for complete data were suggested by Frees (1986) and Schneider et al. (1988). In many cases, however, censoring of the lifetime might occur. In this study, estimators of the renewal function based on randomly censored data is discussed. We introduce nonparametric estimators of the renewal function and show that the estimators compare well with a parametric estimator. Also, different lifetime distributions with different hazard rates and various censoring distributions were considered in a simulation study.

CHAPTER 1

INTRODUCTION

Renewal theory concerns the study of stochastic processes counting the number of events which take place as a function of time. Here the interoccurrence times between successive events are independent and identically distributed random variables. Although renewal theory originated from the analysis of replacement problems for manufacturers, the theory has many applications to a wide variety of practical probability problems, such as inventory theory (e.g., Karlin 1958; Ross 1970; Schneider 1978; Sahin and Kilari 1984; Sahin and Sinha 1987), where a renewal process is used to model the successive times between demand points; estimation of future demand (e.g., White 1964; Soland 1968a,b, 1969); queueing theory (e.g., Yadin and Naor 1963), and reliability problems (e.g., Barlow and Proschan 1975; Goldberg 1981; Abdel-Hameed 1987; Ait Kadi and Cleroux 1988; Nelson 1988), where a renewal process is used to model the successive repairs or replacements of a failed machine. For example, in a queueing process the embedded events can be the arrival of customers who find the system empty or in an inventory process the embedded events may be the

replenishments of stock when the inventory position drops to the reorder point. In addition, renewal theory has been proven to be a powerful tool in modeling the manpower planning problem (e.g., Bartholomew and Forbes 1979), where a renewal process can be used to model the sequence of resignations from a given assignment job; continuous sampling plans (e.g., Yang 1983, 1985); sampling size determination (e.g., Guttman and Menzefricke 1986); insurance risk analysis (e.g., Buhlmann 1970); and sequential analysis (e.g., Woodroofe 1982). These analyses are usually based on appropriate identification of renewal processes and estimation of renewal function for the specific problem considered. Renewal theory provides powerful and elegant tools for analyzing stochastic processes that replace themselves from time to time. The long-run behavior of a regenerative stochastic process can be modeled in terms of its behavior during a single renewal process.

As an application, we study the problem of estimating the renewal function which occurs in a warranty analysis. Renewal theory provides a natural description of the sequence of product replacement which will give warranty cost. The cost of certain types of warranties is closely tied to functions which arise in the renewal theory. We assume the consumer is purchasing a sequence of products

one at a time where a new product instantaneously replaces a failed one. The assumption of instantaneous replacement allows us to describe the sequence of product lifetimes as a renewal process, and allow the use of the powerful techniques of renewal theory.

A warranty is defined as a contractual obligation under which a manufacturer must agree to repair or replace a commercial or a consumer product should it fail before a specified time. The elapsed "time" may be measured in calendar time units (months, days, etc.) or in usage units (miles, minutes of operations, etc.). Often, manufacturers use warranties as a means of advertising the quality of products to accomplish a variety of goals such as quality assurance, product promotion, and consumer risk reduction. From the point of view of the manufacturers, such warranty cost estimation is a very important consideration since it could drastically affect profitability.

There are two types of warranties which are commonly used for consumer goods: free replacement and pro rata warranties. Under the free replacement warranty, the manufacturer provides as many replacements as necessary to yield service free of charge if the product fails prior to the warranted life. Under the pro rata warranty policy, the item is replaced or repaired on failure, at a cost prorated according to the age of the failed product. In recent years

many published works on warranty analysis use the renewal process to describe product replacements. Beidenweg (1981) and Mamer (1982) discussed the long-run time average cost of warranty for both manufacturer and consumers. Blischke and Scheuer (1975, 1981) estimated the warranty costs to the manufacturer and consumers incurred during a finite period of time for the pro rata and free replacement warranties. Menke (1969) and Glickman and Berger (1976) presented the calculation of the costs incurred during a single warranty period for specific lifetime distributions for the ordinary free replacement warranty. Nguyen and Murthy (1984) used the renewal function to investigate the optimal policy for servicing items sold with warranties. For simplicity, we consider the free replacement warranty in this thesis.

Suppose that upon the purchase of a item subject to stochastical failure, the manufacturer agrees to replace a purchased item at no cost to the consumer, if it fails before the end of the warranty length t. We assume that replacement is immediate at a fixed cost per unit. The manufacturer wants to estimate the warranty cost, an expense of the manufacturer which depends upon the product lifetime of the stochastically failing item. The key random variable in the warranty cost analysis is the number, N, of replacements required. From the manufacturer's point of view, since the warranty costs for a stochastically failed

unit are proportional to the number of replacements, estimating the expected cost of a warranty for a failed item is the same as estimating the expected number of renewals by time t based on the finite number of observed lifetimes. Frees (1986a) suggested several parametric and nonparametric estimators based on a complete sample when all units considered have failed. These failure times may be obtained from life tests prior to introducing the product to the market or they may be actual failure times observed after the product is sold to consumers. However, in both situations, censoring of the lifetime might occur. If censored data are not included, then the information available for estimating is reduced which results in nonsufficient statistics. Those estimators based on failure times only must be strongly biased.

There are several advantages in the estimation of the lifetime based on real failure times rather than on test results. In particular, failure times observed by a customer might be more realistic than those observed in an experiment. To describe an example where censoring occurs we consider the situation where a new product is introduced to the market and we assume that the first item is sold at $t_1=0$. In general t_i, $i=1,2, \ldots,$ give the times at which the ith item is sold. In order to obtain a timely estimate of the warranty cost, the renewal function will be estimated

at time t. Assume that n items have been sold in the interval [0,t]; some of these items might have failed; others are still functioning. Consequently, the failure times of functioning items are censored. Since the starting time, t_i, is random, the censoring time is also random for each item.

If life tests are performed in a laboratory, it is not practical, or economically feasible, to test each device to failure. Therefore, censoring is often used to reduce test time. Basically, Type I (time censoring) and Type II (failure censoring) censoring are used in the literature to distinguish between the two different ways of censoring. There are two reasons why we are concerned with the random censorship in a warranty cost analysis. First, random censorship usually occurs in real selling situations rather than in the laboratory life tests where Type I or Type II censoring is more common. Second, random censorship refers to the situation in which observations are randomly censored. Random censorship is the most general censorship including Type I censoring as a special case (Nelson 1982; Schneider 1986). Due to these reasons, we focus on the random censoring pattern in this study.

The purpose of this study is to propose and to investigate several parametric and nonparametric estimators of the renewal function based on complete and randomly

censored samples. The study is organized in the following way. Chapter 2 discusses definitions and computation of the renewal function for the completely known lifetime distribution. Three lifetime distributions are also discussed. Application of the renewal theory requires knowledge of the renewal function which cannot readily be computed in most situations; an integral equation must be solved or a number of convolutions evaluated. Chapter 3 discusses renewal function estimation for complete samples. Both parametric and nonparametric estimators are discussed. The computation of the renewal function when the distribution function is completely known has received much attention in the literature. However, in many cases the form of the lifetime distribution function is unknown and has to be estimated nonparametrically. The nonparametric estimators of the renewal function for complete data were suggested by Frees (1986a) and Schneider, Lin, and O'Cinneide (1988a). These estimators are basic to the investigations of warranty cost models carried out in the later chapters. Chapter 4 describes the renewal function estimation for randomly censored samples. Several parametric and nonparametric estimators of the renewal function are discussed. In Chapter 5 two examples are presented. First, we use the reliability data in Juran and Gryna (1970), which can also be found in Kolb and Ross (1980), to demonstrate the estimation of the

renewal function for complete samples. Second, we present a numerical example, which was used in Schneider, Lin, and Tang (1988b), to illustrate how the renewal function estimates work for censored samples in a warranty cost analysis. Chapter 6 presents a simulation experiment to evaluate the efficiency of the estimators for both complete and censored samples. The simulation study includes (1) the Weibull, lognormal, and gamma lifetime distributions, (2) various censoring patterns generated by beta distributions, (3) comparison between parametric and nonparametric estimators, (4) comparison among estimators for complete samples and among estimators for censored samples, and (5) the effect of ignoring censored observations and comparing the efficiency of all the estimators for finite samples and for fixed values of t. The bias and mean squared error were used to compare the relative performance of the estimators while the average percent of censoring was included for different censored samples. Finally, in Chapter 7, conclusions and discussion are presented. Since there are several parametric and nonparametric estimators of the renewal function in this study, we shall discuss the advantages of each in various circumstances. Efficiency of the renewal function estimators, particularly for the analysis of warranty costs, should be compared.

CHAPTER 2

RENEWAL FUNCTION

2.1 Definitions

To define the renewal function, let X_1, X_2, ... be independently and identically distributed random variables with the distribution function $F(x)$, $x \geq 0$, which has mean μ and variance σ^2. Consider the sum $S_k = X_1 + \ldots + X_k$, $k \geq 1$ of k failure times. The renewal function $H(t)$ $(t \geq 0)$ is defined as

$$H(t) = \sum_{k=1}^{\infty} F^k(t), \tag{2.1}$$

where $F^k(t) = Pr(S_k \leq t)$ is the k-fold convolution of F for $k \geq 1$. The function $H(t)$ is of interest in renewal theory since $H(t)$ is the expected numbers of renewals in an interval $[0,t]$ for a renewal process with underlying lifetime distribution F. If F is completely known, then $H(t)$ can be computed in principle. However, in most cases F is unknown or the general form of F might be known but the parameters of the distribution are unknown. In those cases $H(t)$ has to be estimated from data. The renewal function might be estimated parametrically if the general form of F is known,

or nonparametrically if the form of F is unknown.

It can be shown that the renewal function H(t) satisfies the so called renewal equation

$$H(t) = F(t) + \int_0^t H(t-y)f(y)dy, \tag{2.2}$$

and the renewal density function h(t) = H'(t) satisfies

$$h(t) = f(t) + \int_0^t h(t-y)f(y)dy, \tag{2.3}$$

where f(y) is the probability density function of the failure times. If h(t) is known, then (2.3) may be used to compute the renewal function H(t) by

$$H(t) = \int_0^t h(u)du. \tag{2.4}$$

An important asymptotic result in renewal theory of interest here is suggested by the relationship

$$\lim_{t \to \infty} H(t) - t/\mu = \sigma^2/(2\mu^2) - 1/2. \tag{2.5}$$

This asymptotic approximation is particularly attractive where F is not completely specifiable, as the renewal function H(t) depends asymptotically only upon the first two moments. This asymptotic result will play an important role in estimating the renewal function H(t). Specifically, when the time interval of interest is large, asymptotic

expressions for the renewal function are used.

In summary, application of renewal theory typically requires knowledge of the renewal function H(t) which cannot readily computed in most situations. An integral renewal equation in (2.2) must be solved or a number of recursively defined convolutions in (2.1) must be evaluated. There is a well-known asymptotic approximation to renewal function H(t) for large t. However, this approximation is not sufficiently accurate for the small-to-moderate values of t encountered in practice.

2.2 Lifetime Distributions

There are three probability distributions most commonly encountered in applications of renewal theory: the Weibull, lognormal, and gamma. Important characteristics of these lifetime distributions are mean, variance, and hazard rate.

We define the mean by $E(x) = \int_{-\infty}^{\infty} xf(x)dx$ and the variance by $V(x)= \int_{-\infty}^{\infty} x^2 f(x)dx-[E(x)]^2$ for continuous lifetime distributions. The hazard rate, an important characteristic, is defined by $r(x) = f(x)/[1-F(x)]$.

The hazard rate has a probabilistic interpretation: $r(t)dt$ represents the probability that an item of age t will fail in the interval $(t,t+dt)$ given the item has "survived" to time t. For this reason, it plays a central role in

lifetime data analysis. Many parametric lifetime models, such as the Weibull and the gamma distributions, have a monotone hazard rate. If r(t) increases monotonically over time, the distribution is said to have increasing hazard rate (IHR). If r(t) decreases monotonically over time, the distribution is said to have decreasing hazard rate (DHR). An IHR indicates that the unit is more likely to fail in the next increment of time than it would be in an interval of the length at an earlier age. This implies the "aging effect", i.e., the unit is wearing out or deteriorating with age. Similarly, a DHR means that the unit is improving with age. A constant hazard rate (CHR) occurs for the exponential distribution and reflects the memoryless property of that distribution. The hazard rate is very useful with lifetime model, since the information about the hazard rate can help in selecting a lifetime distribution or its parameter settings. For example, it is practical to restrict consideration to a model with an IHR or with a hazard rate having some other well-defined properties.

The Weibull Distribution

First, we consider the Weibull distribution which has emerged as the most popular parametric family of lifetime distributions. The reason for the popularity of the Weibull distribution is its flexibility in taking forms and

empirically permitting a good fit to many kinds of data. We consider the Weibull probability density function

$$f(y) = (\beta/\theta^{\beta})y^{\beta-1}\exp[-(y/\theta)^{\beta}],\ y>0, \tag{2.6}$$

with location parameter zero, positive shape parameter β and positive scale parameter θ. Since it is a lifetime distribution, we let the distribution be defined only for positive y.

The mean of the Weibull distribution is given by $\theta\Gamma[1+(1/\beta)]$ and the variance by $\theta^2\{\Gamma[1+(2/\beta)]-(\Gamma[1+(1/\beta)])^2\}$, where $\Gamma(U) = \int_0^\infty Z^{U-1} \exp(-Z)\, dZ$ is the complete gamma function. The Weibull distribution has the hazard rate $r(y) = \beta\theta(\theta y)^{\beta-1}$, which is a power function of time. The hazard rate for the Weibull distribution is increasing for $\beta>1$, decreasing for $\beta<1$, and is independent of y for $\beta=1$. Note that the Weibull lifetime distribution reduces to the exponential distribution if $\beta=1$. Figure 1 gives the hazard-rate curves for the Weibull distribution with various β's.

In many real life tests, the Weibull data are conveniently analyzed in terms of the simple extreme value distribution which has probability density function

$$f(T) = (1/b)\exp[(T-c)/b]\exp\{-\exp[(T-c)/b]\}, \tag{2.7}$$

where $T=\ln\beta$, $c=\ln\theta$, and $b=1/\beta$. The location parameter c may have any value. The scale parameter b must be positive. Then

the maximum likelihood estimates (MLEs) of $\hat{\beta}$ and and $\hat{\theta}$ can be computed. For a detailed discussion of these estimates, see, for instance, Lawless (1982). MLEs might be conveniently obtained by using SAS (1985) or other commercial statistical software.

The Lognormal Distribution

The lognormal distribution, like the Weibull distribution, has been widely used as a lifetime distribution model. The lognormal distribution has density

$$f(y) = (y\sigma)^{-1}(2\pi)^{-1/2}\exp\{-(\log y-\theta)^2/(2\sigma^2)\},\ y>0,\ \sigma>0. \tag{2.8}$$

The mean of the lognormal distribution is given by $\exp(\theta+\sigma^2/2)$ and the variance by $[\exp(\sigma^2)-1][\exp(2\theta+\sigma^2)]$. The hazard rate for the lognormal distribution is

$$r(y) = f(y)/\{1-\Phi[(\log y-\theta)/\sigma]\}.$$

It can be shown that the hazard rate for the lognormal distribution has value 0 at y=0, increases to a maximum, and then decreases, approaching 0 as $y \to \infty$. Figure 2 shows the hazard-rate curves for the lognormal distribution with various σ's.

The Gamma Distribution

The gamma distributed lifetimes have the density

$$f(y) = \beta^{-\alpha} y^{\alpha-1} \exp(-y/\beta)/\Gamma(\alpha), \; y>0, \tag{2.9}$$

where $\alpha>0$ and $\beta>0$ are parameters; β is a scale parameter and α is a shape parameter. The mean is $\alpha\beta$ and the variance is $\alpha\beta^2$. The gamma distribution, like the Weibull distribution, includes the exponential as a special case ($\alpha=1$). The gamma distribution fits a wide variety of lifetime data adequately. The hazard rate of the gamma distribution is not expressible in a simple closed form and hence is difficult to work with. The hazard rate for the gamma distribution is

$$r(y) = f(y)/[1-(1/\Gamma(\alpha))\int_0^y U^{-\alpha} \exp(-U)dU].$$

It can be shown that $r(y)$ is increasing for $\alpha>1$, with $r(0)=0$ and $\lim_{y\to\infty} r(y)=\beta$, decreasing for $\alpha<1$ with $\lim_{y\to\infty} r(y)=\beta$, and is a constant for $\alpha=1$. Figure 3 shows the hazard-rate curves for the gamma distribution with various α's. Notice that the gamma distribution is frequently cited as a lifetime model for repair items: Barnett and Ross (1965), for example, examine some data on computer failures for which the repair time distribution is "reasonably approximated by a gamma distribution with [shape] parameter 1/2".

2.3 Computation of the Renewal Function for Completely Known F

In many of the applications involving renewal theory, it

is necessary to evaluate the renewal function. Although considerable effort has been directed to the theoretical study of the renewal function, the renewal function is quite difficult to compute, even when the lifetime distribution function is known. Equations (2.1) and (2.2) do not have explicit solutions for most lifetime distributions. Therefore, in order to compute the renewal function $H(t)$, one has to resort to numerical techniques either on (2.2) or directly on (2.1).

Several suggestions were made in the literature regarding how to approach this problem (see Soland 1968a,b, 1969; Jaquette 1972; Cleroux and McConalogue 1976; McConalogue 1978, 1981; Baxter 1981; Baxter et al. 1981, 1982; Carlsson 1983; Deligonul 1985; Sahin 1986). Smith and Leadbetter (1963) and Lomnicki (1966) investigated a series-expression method for computing the renewal function for the Weibull model, but did not provide any tables.

The basic renewal function and some quantities have been evaluated by Soland (1968a, 1969). Soland (1968b) has presented the renewal tables for the Weibull and the gamma distributions with increasing hazard rate only. For example, the tables give true values of the renewal function for distributions with shape parameter at least 2. Soland's algorithm first calculates the renewal density function $h(t)$ in (2.3) by approximating the integral with a finite sum of

h(y) for y<t and then computes the renewal function H(t) by numerical integration of the expression (2.4). Therefore, given the estimators for the parameters of f(y), one can compute the renewal function H(t) numerically.

The Cleroux-McConalogue's algorithm (1976) generates highly accurate piecewise polynomial approximations to convolutions of probability distributions that are twice differentiable. The essence of this algorithm is a cubic-spline representation of the form $F^k(t)$ in (2.1) where F is bounded. The major limitation of the Cleroux-McConalogue's algorithm is that it cannot be applied to the Weibull and the gamma distributions with decreasing hazard rate (Baxter 1981). McConalogue (1978, 1981) generalized this algorithm, permitting its application to a subclass of those distribution functions F with a shape parameter no less than 0.5. It has been used in Baxter et al. (1981, 1982) to compute extensive renewal tables for a number of distributions with a wide range of values for the shape parameter, with a reported accuracy to four to six decimals.

Soland's algorithm is much simpler, and an extensive numerical study and comparison with tables by Baxter et al. showed that Soland's algorithm performs well as long as the hazard rate is not rapidly decreasing. When the hazard rate decreases too fast, Soland's algorithm does not perform well. For instance, for the gamma distribution with the shape

parameter α=0.55, Soland's algorithm is not efficient (Schneider et al. 1988b). Under these situations, Cleroux-McConalogue's algorithm should be used.

CHAPTER 3

RENEWAL FUNCTION ESTIMATION FOR COMPLETE SAMPLES

In Chapter 2, the computation of the renewal function for completely known F was discussed. However, in most cases F is not completely known or the general form of F might be known but the parameters of the distribution are unknown. In those cases the renewal function must be estimated from a sample. The renewal function might be estimated either parametrically or nonparametrically.

3.1 Parametric Estimator

If the samples are not censored, the estimation of the renewal function when the distribution function is completely known has received much attention in the literature. Recently, Frees (1986) suggested a parametric estimator $\hat{H}_{pf}(t)$ for the renewal function based on equation (2.2) for complete data. Let $\hat{F}$ be the resulting estimator of the distribution function F. This estimator is defined by

$$\hat{H}_{pf} = \sum_{k\geq 1} \hat{F}^k(t;\hat{p}_1,\hat{p}_2,\ldots,\hat{p}_h) \tag{3.1}$$

where $\hat{F}^k$ is the k-fold convolution of $F(t;p_1,p_2,\ldots,p_h)$ and

$\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_h$ are parameter estimates. Thus, estimating the renewal function in (3.1) of a parametric family is a two-stage procedure. The first stage is to construct consistent estimates of the parameters $p_1, p_2, \ldots, p_h$, for example, the maximum likelihood estimators. Second, a numerical computation of $\hat{H}_{pf}(t)$ based on (3.1), or (3.2) and (3.3) is performed.

$$\hat{h}_{pf}(t) = \hat{f}(t) + \int_o^t \hat{h}_{pf}(t-y)\hat{f}(y)dy, \tag{3.2}$$

$$\hat{H}_{pf}(t) = \int_o^t \hat{h}_{pf}(u)du. \tag{3.3}$$

The parametric estimator $\hat{H}_{pf}$, in equation (3.1), is a consistent estimator of H(t). To evaluate (3.2) and (3.3), an algorithm developed by Cleroux and McConalogue (1976) may be used and is generally more effective. However, Soland's algorithm is simpler and seems to be adequate for situations where the hazard rate is not decreasing rapidly.

3.2 Nonparametric Estimators

Since the parametric approach is dependent upon the assumption of known lifetimes, questions of the robustness of the estimators $\hat{H}_{pf}$ under this assumption arise. Based on these robustness considerations, nonparametric estimators

of the renewal function are considered.

Estimator For Large t

The approximation, based on (2.5), is particularly attractive where F(t) is not completely known and t is large enough, as it depends only upon the first two moments.

If t is large enough, then an important estimator of H(t) suggested by Frees(1986a,b) is the following

$$\hat{H}_n(t) = t/\hat{\mu}_n + \hat{\sigma}_n^2/(2\hat{\mu}_n^2) - 1/2, \tag{3.4}$$

where $\hat{\mu}_n$ and $\hat{\sigma}_n$ are estimators of the mean and variance of F based on a sample of failure times recorded up to time t. For complete samples, the mean μ and variance σ^2 can be conveniently estimated by the sample mean and sample variance. This asymptotic estimator, in terms of distributional measures, depends only upon the sample mean and sample variance of F. Conceptually, it will be accurate if t is large enough. However, any estimator based on (3.4) is not a pointwise consistent estimator of H(t) since $\hat{H}_n(t)$ is asymptotically biased for any fixed t, even if $n \to \infty$. Notice that $o(1) = H(t) - t/\mu - \sigma^2/2\mu^2 + 0.5$ changes sign several times as t increases and, thus, the bias of the nonparametric estimator $\hat{H}_n(t)$, based on equation (3.4), is not monotonically decreasing as t increases. Therefore, as

expected, even though estimators of the type in (3.4) are based on recorded observations, they do not perform well for small t. This was pointed out by Frees (1986a).

Frees' Estimator

The second nonparametric estimator for a complete sample introduced by Frees (1986a,b) is based on the sum of the "convolutions without replacement" of the empirical distribution function in equation (3.5). Consider a random sample $X_1, X_2, \ldots, X_n$ of size n. The nonparametric Frees estimator is defined as

$$\hat{H}_{nf}(t) = \sum_{k=1}^{m} \hat{F}_n^{(k)}(t), \tag{3.5}$$

where

$$\hat{F}_n^{(k)}(t) = {}_nC_k^{-1} \sum_i I(X_{i1}+\ldots+X_{ik} \le t), \tag{3.6}$$

and the sum extends over all sub-samples without replacement of size k from $X_1, X_2, \ldots, X_n$. Here, I(A), the indicator function, is one if the event A occurs, and is zero otherwise. The design parameter $m<n$ was introduced by Frees (1986a,b) to reduce the amount of computation, since, in the case of $m=n$, one has to evaluate 2^n-1 indicators. Frees (1986a,b) used the algorithm due to John and Robinson (1983) and Yandell and Lindahl (1985) to compute $\hat{F}_n^{(k)}(t)$. However,

this algorithm is not of polynomial time in m and may require up to n^m operations to compute (3.5). Schneider et al. (1988a) suggested a polynomial time algorithm due to Pagano and Tritchler (1983). This algorithm uses the fast Fourier transform to compute $\hat{F}_n^{(k)}(t)$ for t in the interval $(0,t^*)$, where $t^* = k_{i=1,...,n}\max(X_i)$. To implement this algorithm, the range of $(0,t^*)$ has to be discretized. This amounts to rescaling the x_i values appropriately and rounding them to the nearest integer to obtain sufficient accuracy. To compute the m convolutions in (3.5), the algorithm needs on the order of $m^2\max(X_i)n$ operations. Thus, the main advantage of this algorithm is that as sample size increases, the CPU time it requires is bounded above by a polynomial as shown by Schneider et al. (1988a).

Empirical Estimator

The third nonparametric estimator of the renewal function is based on the renewal equation (2.4). We replace F(t) with $\hat{F}_n(t)$, the empirical distribution function, and obtain a nonparametric estimator $\hat{H}_{ne}(t)$ as the solution to

$$\hat{H}_{ne}(t) = \hat{F}_n(t) + \int_o^t \hat{H}_{ne}(t-x)d\hat{F}_n(x), \tag{3.7}$$

which is denoted as a nonparametric empirical renewal function. As Frees points out, this is the nonparametric

maximum likelihood estimator of H(t). The integral operation in (3.7) can be solved recursively, similar to the approach Soland (1969) used in the parametric case. This method needs only t^2 operations to compute $\hat{H}_{ne}(t)$. Thus, the computation time does not increase with the sample size.

As is shown by O'Cinneide and Schneider (1988), the two renewal function estimators $\hat{H}_{nf}$ and $\hat{H}_{ne}$ (with a suitable choice of design parameter m in Frees' estimator) have the same asymptotic behavior as random functions; this is described by the following theorem: $\sqrt{n}(\hat{H}_n - H)$ converges (for $\hat{H}_{nf}$ and $\hat{H}_{ne}$) in distribution to $(W^{\circ}oF)*G$ as $n \to \infty$ (in the Skorohod topology) on any finite interval [0,t], where * denotes convolution, and G is given by

$$G(t) = \sum_{k=0}^{\infty}(k+1)\ P(S_k \leq t).$$

Frees (1986b) has already established this asymptotic behavior for each fixed t.

CHAPTER 4

RENEWAL FUNCTION ESTIMATION FOR CENSORED SAMPLES

4.1 Censoring Model

The random censoring model is often used to achieve theoretical results in reliability analyses and in survival analyses. For example, in a warranty analysis, some manufacturers offer warranties on every new item. In this situation, Y_i is the random variable of interest (e.g., the lifetime of an item, the age at death, etc.), and C_i is the censoring variable (e.g., the warranty period, the period of an observational study, etc.). The random censorship which is a result of the random starting times can be described by the following model. The sample is randomly censored , i.e., a random sample $(Y_1, \ldots, Y_n)$ is drawn from a population with cumulative distribution function (cdf) $F(y)$, and a random sample $(C_1, \ldots, C_n)$ is drawn from cdf. $G(C)$, where C_i $i= 1, \ldots, n$ are the random censoring times. The observed times are then

$$X_i = G_i Y_i + (1 - G_i) C_i, \tag{4.1}$$

where

$$G_i = \begin{cases} 1 & \text{if } Y_i \le C_i \\ 0 & \text{if } Y_i > C_i. \end{cases}$$

Therefore, we observe the pairs of (X_i, G_i), where $G_i = 0$ means that Y_i is censored and $G_i = 1$ means that Y_i is a failure time. Here we also assume that F has a positive mean μ and variance σ^2.

4.2 Parametric Estimator

As mentioned in Section 3.1, estimating the renewal function in (3.1) is a two-stage procedure. The first stage is to construct consistent estimates of the parameters, for example, the MLEs of p_1, p_2, ..., and p_h. Second, the mean and variance of y can be estimated.

In order to illustrate the procedure, we consider the Weibull lifetime distribution. Maximum likelihood estimates of parameters β and θ of the Weibull distribution in (2.6) have desirable asymptotic properties (Mann et al. 1974). However, because there is no closed-form expression for the maximum likelihood estimates, the use of a numerical iterative procedure is necessary to solve them for a given sample. Since information about G(C) is seldom available, we consider here a conditional maximum likelihood estimator, i.e., conditioned on the censoring times C_i.

The natural logarithm of the likelihood function for

the pairs of (X_i, G_i) is given by

$$\ln L_i = G_i[\ln\beta - \ln\theta + (\beta-1)(\ln X_i - \ln\theta) - X_i^{\beta}\theta^{-\beta}] - (1-G_i)(X_i/\theta)^{\beta}. \quad (4.2)$$

The first partial derivatives of $\ln L_i$ with respect to the parameters are given by

$$d\ \ln L_i/\ d\theta = G_i\{-\theta^{-1}[\beta - \beta(X_i/\theta)^{\theta}]\} + (1-G_i)\beta(X_i/\theta)^{\beta}/\theta, \quad (4.3)$$

$$d\ \ln L_i/\ d\beta = G_i\{\beta^{-1} + \ln X_i - \ln\theta - (X_i/\theta)^{\beta}\ln(X_i/\beta)\} - (1-G_i) (X_i/\theta)^{\beta}\ln(X_i/\theta). \quad (4.4)$$

Let $\overline{\ln X}$ be the mean of the uncensored $\ln X_i$, N_u be the number of uncensored observations, and Σ_u, Σ_c and Σ_a be the sum over the uncensored, censored and all data, respectively. One can easily obtain the two equations

$$\ln\theta = \beta^{-1}[\ln(\Sigma_a X_i^{\beta}) - \ln(N_u)], \quad (4.5)$$

$$\Sigma_a X_i^{\beta}\ln X_i^{\beta} - \Sigma_a X_i^{\beta}(1+\beta\overline{\ln X}) = 0. \quad (4.6)$$

Thus, the maximum likelihood estimates can be obtained by solving equation (4.6) for β and then computing $\ln\theta$ using (4.5). After estimates of β and θ have been found, one can use Soland's algorithm to compute the parametric estimator $\hat{H}_{pi}(t)$.

4.3 Nonparametric Estimators

In this section we consider nonparametric estimates of the renewal function for censored samples. The first estimator is an asymptotical estimator for large t. The second estimator is an empirical estimator.

Estimator For Large t

Based on (2.5) if the data are not censored, the mean μ and variance σ^2 can be conveniently estimated by the sample mean and sample variance. However, if data are randomly censored, then a nonparametric estimation of μ and σ^2 becomes more complicated. The mean μ can be obtained via the Buckley-James (1978) estimator which is identical to the Susarla-Tsai-Van Ryzin estimator (1984). Note that estimates for μ can conveniently be obtained by SAS (1985), while the variance σ^2 is estimated by using the Schneider-Weissfeld estimator (1986) which is not yet available on commercial statistical software.

The Buckley-James estimator of the mean μ based on the vector

$$Y_j^* = G_j Y_j + (1 - G_j)\, E(Y_j \mid Y_j > C_j),\ j=1,\ldots,n \tag{4.7}$$

is defined by

$$\hat{\mu} = \frac{1}{n} \sum_{j=1}^{n} Y_j^*. \tag{4.8}$$

In order to apply this technique to equation (4.7) for the estimation of μ, one has to establish appropriate estimators for $E(Y_j \mid Y_j > C_j)$. In the parametric case it is easy to find explicit expressions for this expectation and replace it by an appropriate estimate. Buckley and James, who consider the case where the underlying distribution function F is unknown, estimate F using the Kaplan-Meier estimator (Kaplan and Meier 1958)

$$1 - \hat{F}(Y) = \pi \, (1-d_{(i)}/n_{(i)})^{G_i}, \tag{4.9}$$

where the product is over $Y_{(i)} \leq Y$ and $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ are the ordered values, $n_{(i)}$ is the number of $Y_{(j)}$ not less than $Y_{(i)}$ and $d_{(i)}$ denotes the number of failures at $Y_{(i)}$. The Kaplan-Meier estimator is then used to estimate $E(Y_j \mid Y_j > C_j)$ in the following manner. Censored observations are replaced by

$$E(Y_j \mid Y_j > C_j) = \sum_{k=1}^{n} W_{jk} \, Y_k, \tag{4.10}$$

for j=1, ... , n, where the sum is over the set of uncensored values and

$$W_{jk} = \begin{cases} V_k / (1 - \hat{F}(C_j)) & Y_j < Y_k \\ 0 & \text{otherwise,} \end{cases} \tag{4.11}$$

where V_k is the mass of the Kaplan-Meier estimator at the uncensored points. Notice that the Kaplan-Meier estimator $\hat{F}(Y)$ assigns the remaining mass to the largest residual if it is censored.

Schneider and Weissfeld (1986) proposed an estimator of the variance σ^2 by applying the above reasoning to the computation of

$$\sum_{j=1}^{n} (Y_j^* - \hat{\mu})^2 . \tag{4.12}$$

Replacing the Y_j^* by their estimated expectations, we have

$$\hat{\sigma}^2 = \frac{1}{n} \{ \sum_{j=1}^{n} [G_j (Y_j - \hat{\mu})^2 + (1 - G_j) \sum_{k=1}^{n} W_{jk} (Y_k - \hat{\mu})^2] \} . \tag{4.13}$$

This estimator uses the information in both the censored and uncensored observations for the estimation of σ^2. To reduce bias in equation (4.12), we apply the bias correction (see Schneider and Weissfeld 1986)

$$\hat{\sigma}_c^2 = \hat{\sigma}^2 \, n_u / (n_u - 1) , \tag{4.14}$$

where n_u is the number of uncensored observations.

The nonparametric estimates $\hat{\mu}$ and $\hat{\sigma}_c^2$ are used in the equation (2.5) to obtain the nonparametric asymptotic estimator $\hat{H}_{na}(t)$ of the renewal function.

Empirical Estimator

The second alternative nonparametric estimator $\hat{H}_{nk}(t)$, based on the Kaplan-Meier estimates, is similar to the nonparametric empirical estimator given by equation (3.7) which is solved by Soland's (1969) algorithm.

Some of the structure of the nonparametric methods rests on a simple property of order statistics: the distribution of the area under the form of the density function between any two ordered observations is independent of the form of the density function. The expected area under pdf f(y) between two successive observations is 1/n. Thus, on the average, the n ordered observations divide the area under f(y) into n equal parts of area 1/n each. If there are no censored observations in a sample of size n, the empirical survivor function is defined as

$$\hat{S}(t) = (\text{Numbers of observations} \geq t)/n, \ t \geq 0. \qquad (4.15)$$

This is a step function that decreases by 1/n just after each observed lifetime. If there are k lifetimes equal to t,

the $\hat{S}(t)$ drops by k/n just past t. In the randomly censoring situations, it is important to be able to obtain nonparametric estimates of various characteristics of the survival function S. For the randomly censored data, Kaplan and Meier (1958) give the product limit method, which is the actuarial method in the limit as all intervals go to zero length. They also show that the method provides the nonparametric maximum likelihood estimator of S itself.

Now, when dealing with censored samples, the Kaplan-Meier estimator (1958) in (4.9) can be used here. The method used to compute $\hat{H}_{nk}(t)$ is similarily to the approach used to compute in the nonparametric empirical estimator $\hat{H}_{ne}(t)$ in (3.7). Therefore,
we have

$$\hat{H}_{nk}(t) = \hat{F}_k(t) + \int_o^t \hat{H}_{nk}(t-x)d\hat{F}_k(x). \tag{4.16}$$

Here, we replace F with the Kaplan-Meier estimator $\hat{F}_k$ to obtain $\hat{H}_{nk}(t)$; while in computing $\hat{H}_{ne}(t)$, F is replaced by the empirical distribution function $\hat{F}_n$.

Because of the relative computational simplicity of the algorithm, we propose that both $\hat{H}_{ne}$ and $\hat{H}_{nk}$ may provide efficient methods to estimate the renewal function. One advantage of the nonparametric estimator over the parametric estimator is its simplicity. While the nonparametric

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Because of the relative computational simplicity of the algorithm, we propose that both $\hat{H}_{ne}$ and $\hat{H}_{nk}$ may provide efficient methods to estimate the renewal function. One advantage of the nonparametric estimator over the parametric estimator is its simplicity. While the nonparametric

estimators for randomly censored samples are available in closed forms, the maximum likelihood estimates have to be found by an iterative procedure such as the Newton-Raphson method.

CHAPTER 5

EXAMPLES

5.1 Example for Complete Samples

In order to illustrate how to calculate the parametric and nonparametric estimators of the renewal function for complete samples, 107 observed lifetimes of units of electronic ground support equipment in Juran and Gryna (1979) were used. These were previously used in Kolb and Ross (1980). In this typical reliability example it can be seen, in Figure 4, that there are three periods: infant mortality period, constant hazard rate period, and wear-out period. During the first 20 hours of operation, the observed hazard rate decreases. The hazard rate between 20 and 100 hours is very constant and was indicated as the normal operating period. Finally, after 100 hours of operation, the hazard rate seems to increase steadily, reflecting wearout. Such hazard rates are usually termed "bathtub shaped".

Consider the situation where the manufacturer of equipment agrees to replace the equipment for a certain length of time, say, T. This type of agreement is assumed in a free replacement warranty and T is the duration of the

warranty. Suppose that we would like to estimate H(20) and H(100). Distributions like the Weibull or the gamma distribution impose very strong restrictions on the data. This is illustrated by their inability to produce the bathtub curves and to interpret data adequately. Thus, in this example, we calculate nonparametric estimates $\hat{H}_{ne}$ and $\hat{H}_{nf}$ for t=20 and t=100 assuming that the probability distribution of the lifetimes is not known.

The results are summarized in Table 1. All computations were performed on an IBM 3084 at Louisiana State University. Compared with the estimates of Frees (1986b), the nonparametric estimate $\hat{H}_{ne}$ gives a very similar result. However, the estimator $\hat{H}_{ne}$ uses much less computation time when compared to Frees' estimator. Another advantage is that the design parameter, m, used in calculating $\hat{H}_{ne}$ does not need to be specified. A simulation result, shown in Schneider et al. (1988a), indicated that $\hat{H}_{ne}$ has a slightly larger bias than the estimator $\hat{H}_{nf}$, but the differences in the MSEs are not large. Therefore, these results suggest using the nonparamatric estimator $\hat{H}_{ne}$ for situations where the data are not censored and the lifetime distributions are unknown. However, a disadvantage of $\hat{H}_{ne}$ is that it does not seem to perform well in some extreme cases such as when the hazard rate is rapidly decreasing (Schneider et al. 1988a).

5.2 Example for Censored Samples: Warranty Data

To demonstrate an example where random censoring occurs we consider the situation where a new product is sold to customers at different times, where t_i, i=1,2,... denotes the time at which the product is sold. These times are random. After 12 months the manufacturer wants to estimate the expected number of failures given by H(t) in order to estimate his warranty costs. In other words, the manufacturer wants to estimate the expected number of failures which will occur within 24 months. Some of the items have failed within 12 months; some are still running. Table 2 gives a summary of the relevant data. Fourteen items have been sold within 12 months, the first four of which have failed during that time and were replaced by new items. We assume that these four items were still running after 12 months. Using the two-parameter Weibull distribution with density in (2.6), we obtain the maximum likelihood estimates $\hat{\beta}=7.856$ and $\hat{\theta}=7.8116$. Based on these estimates, we used Soland's algorithm to compute an estimate of the expected number of failures for 24 months

$$\hat{H}_{pi}(24) = 2.71.$$

The nonparametric estimate of the renewal function is based on the nonparametric estimates of the mean and variance,

which are

$$\hat{\mu} = 7.2725 \text{ and } \hat{\sigma}^2_c = 1.2906,$$

respectively. Equation (3.4) then gives

$$\hat{H}_{na}(24) = 24/7.2725 + 1.2906/(2*7.2725^2) - 0.5 = 2.81.$$

Also, we calculate an estimate based on the lognormal distribution with density in (2.8). The maximum likelihood estimates of the parameters are $\hat{\theta}$=2.0355 and $\hat{\sigma}$=0.2359. Soland's algorithm thus gives

$$\hat{H}_{pi}(24) = 2.58.$$

Finally, we calculate the nonparametric estimates which are based on the Kaplan-Meier estimates. Equation (4.16) gives then

$$\hat{H}_{nk}(24) = 3.18.$$

Since the hazard rate plot for this example is highly irregular, no parametric estimates can fit. The parametric estimator, $\hat{H}_{pi}$, computed using the Weibull or the lognormal distribution, is quite suspect in this example.

CHAPTER 6

SIMULATION STUDY

To study the difference between parametric and nonparametric estimates, relative performance of the estimators, $\hat{H}_{pi}$, $\hat{H}_{na}$, and $\hat{H}_{nk}$, was investigated by a simulation study for finite samples and fixed values of t (t=0.25, 0.50, 0.75, 1.00, and 1.25) when random censoring is presented. Four important factors were considered in the simulation study: (1) sample size, (2) lifetime distribution, (3) censoring distribution, and (4) hazard rate.

In each case 1000 samples of size 10, 20, and 30 were drawn. Since the application of renewal function estimation to a warranty analysis is of interest, the simulation comparisons provided here emphasize a small t relative to the mean.

Baxter et al. (1981) provided true values of the renewal function for several lifetime distributions, and, therefore, the simulation study was limited to those distributions. Three probability distributions were considered: the Weibull, lognormal, and gamma. Note that the curves of the hazard rate for different lifetime

distributions have different shapes. The shape of the hazard rate is dependent upon the parameter settings. To illustrate the effect of different hazard rates, we use different parameter settings to examine the estimation of the renewal function. Namely, there are three types of shapes of the hazard rates most commonly encountered in applications of renewal theory: IHR, DHR, and CHR. The values of shape parameter for which the renewal functions are estimated are as follows:

Weibull:	β=.55, 1.0, and 3.0.
Lognormal:	σ=.5, 1.0, and 2.0.
Gamma:	α=.55, 1.0, and 2.0.

The lower limit of the shape parameter for the Weibull and the gamma distributions is due to the inability of the Cleroux-McConalogue's algorithm to evaluate true values of the renewal function in the renewal tables with a shape parameter <0.5 (Baxter et al. 1981).

As mentioned, more attention will be paid to renewal function estimation for the randomly censored sample. In order to obtain a variety of censoring patterns, the beta distribution with density

$$f(y) = \frac{1}{B(p,q)} (y/a)^{p-1}(1-y/a)^{q-1}\, I_{(0,a)}(y). \qquad (6.1)$$

is chosen. This distribution has a wide range of shapes for different parameter settings, including the uniform distribution as well as symmetrical, left-, and right-skewed distributions. The beta distribution has been used successfully (Schneider and Weissfeld 1986) as a censoring distribution since it creates various degrees of random censoring patterns by varying the parameters, p and q, of the distribution. Three cases for each lifetime are considered: equal censoring, increasing censoring, and decreasing censoring. Based on the shapes of the beta distributions, three parameters settings for the beta distributions were selected: (p=1.0, q=1.0), (p=1.0, q=2.0), and (p=2.0,q=1.0). Figure 5 shows probability density function for the beta distribution. Note that the distribution with parameter settings (p=1.0, q=2.0) is right-skewed; while the distribution with parameters settings (p=2.0, q=1.0) is left-skewed. The average percent of censoring, bias, and mean squared error based on 1000 convergent samples were used to compare the relative performance of the estimators: $\hat{H}_{pi}$, $\hat{H}_{na}$, and $\hat{H}_{nk}$.

The bias and mean squared error were used to compare the relative performance of the estimators. The IMSL Fortran subroutines, GGWIB, GGNLG, GGAMR, and GGBTR, were used to produce all of the random deviates.

Ignoring censoring values results in heavily biased

estimates due to the reduction of the information available for use in the estimates. To demonstrate the importance of censored data estimation, we performed a simulation study where censored values are simply ignored. Table 3 presents the simulation results for parametric estimators $\hat{H}_{pi}(t)$ where estimation is based only upon uncensored values. When we compare these results with censored data estimates $\hat{H}_{pi}(t)$ in Tables 10-12, we can see the effect which censored data have. Ignoring censoring values results in heavily biased estimates due to the reduction of the information available for estimation.

The Weibull Distribution

Tables 4-12 show the simulation results for the Weibull distribution based on 1000 samples and censored with the beta distribution where we have different parameter settings.

First, both IHR and CHR cases were considered. We see that when t is small, relative to the mean, the asymptotic estimator $\hat{H}_{na}$, based on equation (2.5), is the most biased estimator. However, the mean squared errors of $\hat{H}_{na}$ are not much higher than those of $\hat{H}_{pi}$ and $\hat{H}_{nk}$ unless t is small. The estimator $\hat{H}_{nk}$, based on the Kaplan-Meier estimator, is more biased only when t=1.25. But the mean squared errors are not quite different. The right-skewed parameter

settings produce more censorship than the left-skewed parameter settings. When the beta distribution has right-skewed parameter settings (p=1.0, q=2.0), both bias and mean squared error are relatively large due to the decreasing censoring effect which is thought have a great impact for estimators $\hat{H}_{na}$ and $\hat{H}_{na}$. Note that the percent of the censorship is dependent upon the shapes of the beta distribution.

Second, the DHR case was considered. Tables 4-6 show that no estimator dominates others. But, both $\hat{H}_{na}$ and $\hat{H}_{nk}$ under increasing censoring perform better than those under decreasing censoring. Notice that one disadvantage of $\hat{H}_{nk}$ is that both bias and mean squared error of $\hat{H}_{nk}$ are monotonically increasing as t increases.

The Lognormal Distribution

Tables 13-21 present simulation results for lognormal lifetimes. We can see that $\hat{H}_{na}$ is inferior to the parametric estimators $\hat{H}_{pi}$ only when $t \le 0.50$. This result is due to the large bias of the estimators based on the asymptotic equation (2.5). We also notice that $o(1) = H(t) - t/\mu - \sigma^2/(2\mu^2) + 1/2$ changes the sign several times as t increases and, thus, the bias of the estimator $\hat{H}_{na}$, based on (2.5), is not monotonically decreasing as t increases. But, this convergence depends upon the mean and the variance. In

case of σ=2.0 both $\hat{H}_{na}$ and $\hat{H}_{nk}$ performed poorly due to the large mean (μ=7.4) of the lognormal lifetime distribution. In this situation, $\hat{H}_{pi}$ should be used. Again, when t increases, both $\hat{H}_{na}$ and $\hat{H}_{nk}$ behave better under increasing censoring than under decreasing censoring.

Due to the shape of r(t) for the lognormal distribution (in Figure 2), there is no clear effect of the hazard rate on all the estimators. The situation is further complicated by the sometimes oscillatory behavior of H(t).

The Gamma Distribution

Tables 22-27 show the simulation results for the gamma distribution based on 1000 samples and censored with the beta distribution where we have different parameter settings. The Weibull and lognormal maximum likelihood estimates are compared to the nonparametric estimators, $\hat{H}_{na}$ and $\hat{H}_{nk}$. Since the gamma distribution is reduced to the exponential distribution with CHR when α=1, only the IHR and DHR cases were considered. For the DHR case, no estimator dominates others. Both bias and mean squared error of all the estimators are relatively large. This is most likely due to inability of Soland's algorithm to handle DHR situations. However, the parametric estimator, $\hat{H}_{pi}$ performs surprisingly well relative to the nonparametric estimators for the IHR

situation. The nonparametric estimators $\hat{H}_{na}$ are more biased than the parametric estimators $\hat{H}_{pi}$, even though they are based on the wrong distribution models. However, the mean squared error of the nonparametric estimators is similar to that of the parametric estimators for $t \geq 1$. Again, for the IHR case, it was found that both $\hat{H}_{na}$ and $\hat{H}_{nk}$ behaved better under increasing censoring than decreasing censoring.

CHAPTER 7

CONCLUSION

There are three approaches to estimating the renewal function: (1) the lifetime distribution F is known and all the parameters are known, (2) the lifetime distribution F is known but the parameters are unknown, and (3) the lifetime distribution F is unknown. Figure 6 shows these approaches. We are dealing with different approaches for each situation. Under the first situation, both Soland's algorithm and Cleroux-McConalogue's algorithm can be used to solve the renewal function numerically. Under the second situation, both Soland's algorithm and Cleroux-McConalogue's algorithm can be used to estimate the renewal function parametrically for either censored data or uncensored data. Also, the asymptotic formula of the renewal function can be used to estimate H(t) via the estimates of mean and variance. The study suggests that if the lifetime distribution is known, then the parametric estimator $\hat{H}_{pi}$ should be used. Under the third situation, the nonparametric empirical estimator can be used for the uncensored case as long as the hazard rate is not rapidly decreasing, while the nonparametric Kaplan-Meier estimator or the nonparametric asymptotic

estimator can be used for the censored case. Even though neither the nonparametric Kaplan-Meier nor the nonparametric asymptotic estimators is not consistent, the mean squared errors are not higher than the MSEs of the parametric estimator $\hat{H}_{pi}$. Another advantage of nonparametric estimators is that parametric estimators have to involve an iterative procedure for finding the MLEs.

Since, for a warranty cost analysis, t is small relatively to the mean, there are some limitations in using $\hat{H}_{na}$. When t is very small and the lifetime distribution is unknown, the nonparametric Kaplan-Meier estimator, $\hat{H}_{nk}$, is suggested.

As shown in the simulation results, both the hazard rate and the censoring distribution are crucial factors in determining estimators of the renewal function. Unfortunately, both Soland's and Cleroux-McConalogue's algorithms have some limitations to their use under the DHR case. When the lifetime distribution has a fast DHR, it is very difficult and sometimes impossible to solve (2.2) or (2.3). Under such circumstances, an asymptotic approximation of (2.5) will be quite useful if t is not too small.

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Table 1. Comparison of estimates $\hat{H}_{ne}$ and $\hat{H}_{nf}$ for electronic ground support equipment.

t	$\hat{H}_{ne}$	$\hat{H}_{nf}$
20	0.4554	0.4587
100	1.4388	1.4624
CPU for t=20	0.00*	10.88
CPU for t=100	0.01	10.91

* less than 0.01

Table 2. Warranty data for example with censoring time t_c=12 months.

Item #	Time item is sold in months	Life or censoring time in months		Item #	Time item is replaced in months	Censoring time in months	
i	t_i	X_i	G_i*	i	t_i	X_i	G_i
1	0.00	7.92	1	11	7.92	4.08	0
2	1.09	4.55	1	12	4.55	7.45	0
3	2.45	6.77	1	13	6.77	5.23	0
4	3.12	6.77	1	14	6.77	5.23	0
5	4.49	7.51	0				
6	4.92	7.08	0				
7	5.41	6.59	0				
8	5.92	6.08	0				
9	6.85	5.15	0				
10	10.05	1.95	0				

* G_i = 1: Item has failed before t_c
G_i = 0: Item is still running at time t_c

Table 3. Bias and mean squared error of parametric estimator in 500 samples when censored values are ignored.

(Lifetime distribution: Weibull with β=3,θ=1,μ=0.89)
Sample size n = 10.

	$\hat{H}_{pi}$	
t	BIAS	MSE
0.25	0.0133	0.0023
0.50	0.0510	0.0241
0.75	0.1157	0.0724
1.00	0.1782	0.0855
1.25	0.2318	0.1611

Table 4. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=.55,θ=1.0,μ=1.7)
(Censoring distribution: Beta with p=1.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.522		0.013	0.064	-0.144	0.139	0.082	0.136
	0.50	0.806		-0.030	0.109	-0.022	0.269	0.126	0.297
	0.75	1.047	41	-0.098	0.164	0.143	0.516	0.170	0.539
	1.00	1.267		-0.181	0.236	0.328	0.909	0.261	0.926
	1.25	1.473		-0.276	0.331	0.528	1.462	0.384	1.479
20	0.25	0.522		0.012	0.030	-0.217	0.100	0.045	0.058
	0.50	0.806		-0.056	0.055	-0.147	0.129	0.063	0.119
	0.75	1.047	41	-0.150	0.098	-0.036	0.187	0.085	0.198
	1.00	1.267		-0.257	0.167	0.097	0.298	0.115	0.314
	1.25	1.473		-0.374	0.266	0.243	0.474	0.148	0.480
30	0.25	0.522		0.013	0.018	-0.239	0.091	0.030	0.034
	0.50	0.806		-0.065	0.034	-0.188	0.100	0.040	0.072
	0.75	1.047	41	-0.166	0.069	-0.094	0.116	0.052	0.118
	1.00	1.267		-0.281	0.133	0.021	0.163	0.064	0.184
	1.25	1.473		-0.404	0.229	0.149	0.252	0.085	0.265

Table 5. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=.55,θ=1.0,μ=1.7)
(Censoring distribution: Beta with p=1.0 and q=2.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.522		0.008	0.069	-0.122	0.146	0.084	0.143
	0.50	0.806		-0.033	0.128	0.083	0.344	0.136	0.365
	0.75	1.047	49	-0.097	0.208	0.329	0.767	0.253	0.777
	1.00	1.267		-0.176	0.315	0.597	1.451	0.464	1.464
	1.25	1.473		-0.268	0.443	0.878	2.418	0.750	2.441
20	0.25	0.522		-0.001	0.031	-0.222	0.103	0.039	0.061
	0.50	0.806		-0.078	0.059	-0.093	0.133	0.053	0.126
	0.75	1.047	50	-0.177	0.108	0.077	0.235	0.083	0.238
	1.00	1.267		-0.288	0.184	0.269	0.443	0.165	0.450
	1.25	1.473		-0.408	0.292	0.474	0.772	0.357	0.808
30	0.25	0.522		-0.001	0.017	-0.243	0.090	0.032	0.036
	0.50	0.806		-0.085	0.038	-0.141	0.086	0.042	0.073
	0.75	1.047	50	-0.192	0.081	-0.002	0.119	0.055	0.119
	1.00	1.267		-0.311	0.155	0.167	0.218	0.100	0.228
	1.25	1.473		-0.437	0.263	0.345	0.399	0.210	0.423

Table 6. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=.55,θ=1.0,μ=1.7)
(Censoring distribution: Beta with p=2.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.522		0.019	0.061	-0.152	0.138	0.081	0.130
	0.50	0.806		-0.025	0.101	-0.062	0.234	0.115	0.266
	0.75	1.047	32	-0.094	0.147	0.070	0.402	0.149	0.441
	1.00	1.267		-0.180	0.210	0.222	0.664	0.202	0.674
	1.25	1.473		-0.276	0.296	0.389	1.034	0.267	1.012
20	0.25	0.522		0.020	0.029	-0.218	0.097	0.043	0.056
	0.50	0.806		-0.044	0.045	-0.168	0.119	0.060	0.110
	0.75	1.047	33	-0.136	0.077	-0.075	0.154	0.076	0.168
	1.00	1.267		-0.242	0.132	0.039	0.223	0.096	0.247
	1.25	1.473		-0.358	0.217	0.167	0.338	0.114	0.342
30	0.25	0.522		0.023	0.018	-0.237	0.088	0.032	0.033
	0.50	0.806		-0.048	0.029	-0.198	0.096	0.042	0.066
	0.75	1.047	33	-0.146	0.058	-0.116	0.105	0.052	0.104
	1.00	1.267		-0.258	0.113	0.013	0.135	0.063	0.158
	1.25	1.473		-0.379	0.199	0.103	0.197	0.078	0.218

Table 7. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=1.0,θ=1.0,μ=1.0)
(Censoring distribution: Beta with p=1.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.250		0.027	0.033	-0.123	0.051	0.029	0.037
	0.50	0.500		0.062	0.077	-0.024	0.082	0.056	0.090
	0.75	0.750	43	0.073	0.130	0.075	0.159	0.084	0.170
	1.00	1.000		0.063	0.193	0.174	0.282	0.132	0.300
	1.25	1.250		0.034	0.270	0.273	0.451	0.200	0.486
20	0.25	0.250		0.033	0.017	-0.156	0.042	0.016	0.017
	0.50	0.500		0.057	0.038	-0.085	0.046	0.028	0.041
	0.75	0.750	43	0.049	0.061	-0.015	0.070	0.043	0.075
	1.00	1.000		0.018	0.089	0.056	0.116	0.059	0.123
	1.25	1.250		-0.034	0.128	0.126	0.182	0.075	0.196
30	0.25	0.250		0.038	0.013	-0.161	0.038	0.016	0.010
	0.50	0.500		0.059	0.026	-0.100	0.034	0.024	0.025
	0.75	0.750	43	0.045	0.038	-0.039	0.044	0.031	0.047
	1.00	1.000		0.007	0.055	0.022	0.069	0.039	0.077
	1.25	1.250		-0.052	0.080	0.083	0.107	0.051	0.115

Table 8. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=1.0,θ=1.0,μ=1.0)
(Censoring distribution: Beta with p=1.0 and q=2.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.250		0.028	0.033	-0.108	0.049	0.039	0.040
	0.50	0.500		0.065	0.081	0.041	0.100	0.075	0.108
	0.75	0.750	55	0.078	0.148	0.191	0.235	0.129	0.251
	1.00	1.000		0.069	0.235	0.340	0.452	0.256	0.481
	1.25	1.250		0.041	0.346	0.489	0.752	0.422	0.786
20	0.25	0.250		0.029	0.017	-0.162	0.044	0.018	0.018
	0.50	0.500		0.043	0.039	-0.054	0.047	0.030	0.047
	0.75	0.750	57	0.026	0.068	0.053	0.089	0.045	0.093
	1.00	1.000		-0.015	0.107	0.161	0.169	0.097	0.186
	1.25	1.250		-0.074	0.163	0.268	0.288	0.216	0.324
30	0.25	0.250		0.033	0.011	-0.172	0.041	0.016	0.011
	0.50	0.500		0.041	0.024	-0.081	0.033	0.025	0.028
	0.75	0.750	57	0.016	0.039	0.009	0.050	0.030	0.053
	1.00	1.000		-0.034	0.064	0.100	0.093	0.057	0.102
	1.25	1.250		-0.103	0.104	0.191	0.161	0.128	0.192

Table 9. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=1.0,θ=1.0,μ=1.0)
(Censoring distribution: Beta with p=2.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.250		0.028	0.032	-0.126	0.050	0.030	0.035
	0.50	0.500		0.064	0.070	-0.046	0.073	0.054	0.079
	0.75	0.750	29	0.075	0.110	0.035	0.126	0.071	0.139
	1.00	1.000		0.063	0.156	0.115	0.210	0.099	0.222
	1.25	1.250		0.033	0.210	0.196	0.326	0.138	0.342
20	0.25	0.250		0.035	0.018	-0.152	0.040	0.018	0.016
	0.50	0.500		0.067	0.037	-0.091	0.041	0.030	0.037
	0.75	0.750	29	0.067	0.054	-0.030	0.058	0.040	0.064
	1.00	1.000		0.042	0.073	0.031	0.089	0.054	0.099
	1.25	1.250		-0.002	0.098	0.092	0.136	0.064	0.143
30	0.25	0.250		0.039	0.013	-0.158	0.036	0.016	0.010
	0.50	0.500		0.067	0.026	-0.104	0.032	0.023	0.022
	0.75	0.750	30	0.062	0.036	-0.050	0.038	0.030	0.039
	1.00	1.000		0.031	0.046	0.004	0.055	0.037	0.063
	1.25	1.250		-0.020	0.063	0.058	0.083	0.047	0.089

Table 10. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=3,θ=1,μ=0.89)
(Censoring distribution: Beta with p=1.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.016		-0.004	0.001	-0.158	0.028	0.002	0.002
	0.50	0.118		0.003	0.012	0.031	0.010	0.009	0.014
	0.75	0.352	43	0.048	0.032	0.088	0.026	0.029	0.037
	1.00	0.672		0.058	0.035	0.059	0.034	0.045	0.058
	1.25	0.985		0.014	0.042	0.038	0.048	0.057	0.064
20	0.25	0.016		-0.007	0.000	-0.167	0.030	0.001	0.001
	0.50	0.118		0.002	0.007	0.015	0.005	0.005	0.007
	0.75	0.352	45	0.045	0.017	0.066	0.013	0.014	0.019
	1.00	0.672		0.032	0.017	0.030	0.016	0.024	0.027
	1.25	0.985		-0.013	0.021	0.003	0.023	0.030	0.030
30	0.25	0.016		-0.007	0.000	-0.168	0.029	0.001	0.001
	0.50	0.118		0.008	0.005	0.013	0.003	0.009	0.005
	0.75	0.352	44	0.050	0.012	0.062	0.009	0.015	0.012
	1.00	0.672		0.029	0.011	0.025	0.010	0.016	0.019
	1.25	0.985		-0.017	0.014	-0.004	0.015	0.019	0.020

Table 11. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=3,θ=1,μ=0.89)
(Censoring distribution: Beta with p=1.0 and q=2.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.016		-0.001	0.001	-0.142	0.025	0.004	0.002
	0.50	0.118		0.018	0.016	0.064	0.020	0.026	0.022
	0.75	0.352	61	0.075	0.049	0.139	0.052	0.072	0.068
	1.00	0.672		0.085	0.119	0.127	0.073	0.110	0.117
	1.25	0.985		0.047	0.134	0.124	0.103	0.145	0.118
20	0.25	0.016		-0.004	0.000	-0.162	0.028	0.002	0.001
	0.50	0.118		0.011	0.008	0.029	0.007	0.013	0.009
	0.75	0.352	66	0.048	0.021	0.088	0.021	0.022	0.026
	1.00	0.672		0.026	0.027	0.062	0.026	0.047	0.056
	1.25	0.985		-0.027	0.038	0.043	0.036	0.082	0.053
30	0.25	0.016		-0.004	0.000	-0.166	0.029	0.003	0.001
	0.50	0.118		0.011	0.006	0.019	0.004	0.007	0.006
	0.75	0.352	68	0.043	0.014	0.074	0.014	0.014	0.018
	1.00	0.672		0.012	0.016	0.042	0.016	0.032	0.033
	1.25	0.985		-0.044	0.026	0.018	0.022	0.051	0.037

Table 12. Bias and mean squared error of estimators in 1000 samples for the Weibull lifetimes.

(Lifetime distribution: Weibull with β=3,θ=1,μ=0.89)
(Censoring distribution: Beta with p=2.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.016		-0.005	0.000	-0.162	0.029	0.003	0.002
	0.50	0.118		0.000	0.010	0.022	0.008	0.007	0.012
	0.75	0.352	22	0.043	0.027	0.075	0.019	0.023	0.028
	1.00	0.672		0.051	0.028	0.042	0.025	0.033	0.040
	1.25	0.985		0.010	0.031	0.016	0.035	0.040	0.044
20	0.25	0.016		-0.008	0.000	-0.167	0.029	0.002	0.001
	0.50	0.118		0.001	0.006	0.014	0.004	0.006	0.006
	0.75	0.352	22	0.048	0.015	0.063	0.011	0.016	0.015
	1.00	0.672		0.038	0.014	0.026	0.012	0.021	0.019
	1.25	0.985		-0.004	0.016	-0.003	0.017	0.026	0.021
30	0.25	0.016		-0.009	0.000	-0.168	0.029	0.002	0.001
	0.50	0.118		0.005	0.004	0.012	0.002	0.008	0.004
	0.75	0.352	22	0.053	0.011	0.060	0.008	0.016	0.009
	1.00	0.672		0.034	0.009	0.023	0.008	0.018	0.014
	1.25	0.985		-0.009	0.011	-0.007	0.012	0.019	0.014

Table 13. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with θ=0.0,σ=0.5,μ=1.1)
(Censoring distribution: Beta with p=1.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.025		-0.007	0.001	-0.196	0.042	-0.022	0.001
	0.50	0.165		-0.074	0.010	-0.086	0.017	-0.074	0.016
	0.75	0.359	53	-0.100	0.031	-0.029	0.020	-0.053	0.038
	1.00	0.561		-0.053	0.050	0.019	0.034	-0.015	0.056
	1.25	0.762		0.004	0.066	0.068	0.057	0.033	0.085
20	0.25	0.025		0.008	0.000	-0.206	0.044	-0.022	0.001
	0.50	0.165		-0.074	0.008	-0.106	0.016	-0.079	0.011
	0.75	0.359	54	-0.110	0.020	-0.060	0.013	-0.075	0.020
	1.00	0.561		-0.077	0.026	0.022	0.016	-0.034	0.025
	1.25	0.762		-0.021	0.032	0.016	0.024	-0.006	0.038
30	0.25	0.025		-0.007	0.000	-0.203	0.043	-0.021	0.001
	0.50	0.165		-0.068	0.006	-0.106	0.014	-0.075	0.009
	0.75	0.359	54	-0.101	0.016	-0.062	0.010	-0.064	0.014
	1.00	0.561		-0.074	0.019	-0.026	0.012	-0.030	0.018
	1.25	0.762		-0.021	0.022	0.011	0.017	-0.005	0.025

Table 14. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with θ=0.0, σ=0.5, μ=1.1)
(Censoring distribution: Beta with p=1.0 and q=2.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.025		-0.006	0.001	-0.174	0.034	-0.021	0.001
	0.50	0.165		-0.066	0.010	-0.031	0.012	-0.057	0.019
	0.75	0.359	53	-0.041	0.033	0.058	0.029	-0.001	0.047
	1.00	0.561		0.062	0.069	0.139	0.065	-0.110	0.107
	1.25	0.762		0.124	0.097	0.221	0.120	0.205	0.152
20	0.25	0.025		-0.012	0.000	-0.202	0.043	-0.022	0.001
	0.50	0.165		-0.082	0.009	-0.085	0.013	-0.075	0.012
	0.75	0.359	54	-0.104	0.024	-0.021	0.014	-0.059	0.026
	1.00	0.561		-0.040	0.038	0.035	0.025	-0.019	0.051
	1.25	0.762		0.029	0.053	0.092	0.046	0.084	0.089
30	0.25	0.025		-0.011	0.000	-0.207	0.044	-0.021	0.001
	0.50	0.165		-0.080	0.008	-0.096	0.013	-0.077	0.010
	0.75	0.359	54	-0.107	0.020	-0.039	0.011	-0.066	0.020
	1.00	0.561		-0.056	0.027	0.011	0.016	-0.028	0.031
	1.25	0.762		0.013	0.036	0.061	0.029	0.037	0.058

Table 15. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with θ=0.0,σ=0.5,μ=1.1)
(Censoring distribution: Beta with p=2.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.025		-0.005	0.000	-0.200	0.043	-0.022	0.001
	0.50	0.165		-0.067	0.009	-0.098	0.017	-0.076	0.014
	0.75	0.359	34	-0.100	0.049	-0.049	0.017	-0.063	0.029
	1.00	0.561		-0.025	0.017	-0.009	0.025	-0.032	0.039
	1.25	0.762		0.066	0.037	0.032	0.039	0.009	0.055
20	0.25	0.025		0.005	0.000	-0.204	0.043	-0.022	0.001
	0.50	0.165		-0.064	0.006	-0.107	0.015	-0.078	0.010
	0.75	0.359	35	-0.099	0.016	-0.065	0.011	-0.070	0.016
	1.00	0.561		-0.076	0.020	-0.030	0.013	-0.033	0.019
	1.25	0.762		-0.026	0.023	0.005	0.018	-0.006	0.024
30	0.25	0.025		-0.005	0.000	-0.202	0.042	-0.021	0.001
	0.50	0.165		-0.061	0.005	-0.107	0.014	-0.076	0.008
	0.75	0.359	35	-0.094	0.013	-0.066	0.009	-0.064	0.012
	1.00	0.561		-0.073	0.014	-0.033	0.009	-0.030	0.013
	1.25	0.762		-0.025	0.015	0.001	0.012	-0.007	0.016

Table 16. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with θ=0.0,σ=1.0,μ=1.6)
(Censoring distribution: Beta with p=1.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.084		0.008	0.005	-0.147	0.036	0.012	0.011
	0.50	0.260		-0.022	0.020	-0.058	0.036	0.020	0.034
	0.75	0.445	55	-0.023	0.048	0.022	0.063	0.035	0.070
	1.00	0.627		-0.001	0.088	0.105	0.114	0.061	0.128
	1.25	0.804		0.032	0.140	0.192	0.191	0.109	0.216
20	0.25	0.084		0.005	0.002	-0.175	0.037	0.004	0.005
	0.50	0.260		-0.029	0.009	-0.108	0.026	0.004	0.014
	0.75	0.445	56	-0.041	0.021	-0.050	0.028	0.010	0.029
	1.00	0.627		-0.035	0.038	0.012	0.041	0.013	0.047
	1.25	0.804		-0.017	0.061	0.077	0.066	0.025	0.079
30	0.25	0.084		0.009	0.002	-0.175	0.035	0.008	0.003
	0.50	0.260		-0.022	0.006	-0.113	0.022	0.012	0.009
	0.75	0.445	56	-0.036	0.014	-0.059	0.020	0.014	0.019
	1.00	0.627		-0.033	0.025	-0.003	0.027	0.019	0.032
	1.25	0.804		-0.019	0.040	0.057	0.042	0.022	0.049

Table 17. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with θ=0.0,σ=1.0,μ=1.6)
(Censoring distribution: Beta with p=1.0 and q=2.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.084		0.006	0.005	-0.118	0.027	0.019	0.012
	0.50	0.260		0.004	0.026	0.024	0.039	0.049	0.044
	0.75	0.445	65	0.054	0.073	0.157	0.105	0.102	0.113
	1.00	0.627		0.127	0.150	0.293	0.226	0.212	0.249
	1.25	0.804		0.202	0.257	0.433	0.406	0.373	0.433
20	0.25	0.084		-0.001	0.002	-0.172	0.036	0.006	0.005
	0.50	0.260		-0.029	0.012	-0.074	0.022	0.009	0.017
	0.75	0.445	70	-0.026	0.029	0.015	0.034	0.012	0.037
	1.00	0.627		0.002	0.056	0.107	0.069	0.032	0.081
	1.25	0.804		0.040	0.096	0.204	0.128	0.133	0.167
30	0.25	0.084		0.001	0.002	-0.180	0.037	0.006	0.003
	0.50	0.260		-0.026	0.008	-0.093	0.021	0.009	0.012
	0.75	0.445	70	-0.026	0.019	-0.014	0.024	0.015	0.026
	1.00	0.627		-0.005	0.038	0.068	0.044	0.021	0.048
	1.25	0.804		0.028	0.064	0.154	0.084	0.069	0.103

Table 18. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with θ=0.0,σ=1.0,μ=1.6)
(Censoring distribution: Beta with p=2.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.084		0.012	0.005	-0.157	0.038	0.008	0.009
	0.50	0.260		-0.021	0.017	-0.084	0.034	0.015	0.029
	0.75	0.445	42	-0.033	0.038	-0.019	0.048	0.027	0.055
	1.00	0.627		-0.026	0.067	0.048	0.078	0.035	0.089
	1.25	0.804		-0.008	0.103	0.119	0.125	0.055	0.140
20	0.25	0.084		0.011	0.002	-0.174	0.036	0.005	0.004
	0.50	0.260		-0.023	0.008	-0.114	0.025	0.006	0.012
	0.75	0.445	42	-0.040	0.018	-0.063	0.025	0.011	0.024
	1.00	0.627		-0.042	0.030	-0.008	0.033	0.015	0.038
	1.25	0.804		-0.033	0.047	0.050	0.050	0.021	0.058
30	0.25	0.084		0.014	0.002	-0.175	0.035	0.007	0.003
	0.50	0.260		-0.018	0.005	-0.118	0.022	0.011	0.008
	0.75	0.445	42	-0.037	0.012	-0.070	0.019	0.014	0.016
	1.00	0.627		-0.040	0.020	-0.020	0.021	0.018	0.026
	1.25	0.804		-0.035	0.030	0.035	0.031	0.019	0.037

Table 19. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with θ=0.0,σ=2.0,μ=7.4)
(Censoring distribution: Beta with p=1.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.174		0.076	0.027	-0.072	0.056	0.150	0.073
	0.50	0.367		0.065	0.058	0.027	0.108	0.171	0.140
	0.75	0.542	54	0.053	0.100	0.143	0.213	0.185	0.235
	1.00	0.705		0.042	0.149	0.271	0.379	0.209	0.383
	1.25	0.860		0.030	0.203	0.408	0.612	0.251	0.590
20	0.25	0.174		0.070	0.014	-0.136	0.039	0.124	0.035
	0.50	0.367		0.048	0.025	-0.071	0.046	0.131	0.063
	0.75	0.542	55	0.024	0.041	0.010	0.069	0.132	0.094
	1.00	0.705		-0.001	0.062	0.104	0.117	0.120	0.133
	1.25	0.860		-0.027	0.087	0.207	0.194	0.111	0.185
30	0.25	0.174		0.075	0.012	-0.144	0.035	0.129	0.030
	0.50	0.367		0.051	0.017	-0.089	0.034	0.137	0.047
	0.75	0.542	55	0.022	0.025	-0.017	0.041	0.128	0.065
	1.00	0.705		-0.007	0.038	0.067	0.064	0.113	0.085
	1.25	0.860		-0.038	0.055	0.160	0.108	0.092	0.109

Table 20. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with θ=0.0,σ=2.0,μ=7.4)
(Censoring distribution: Beta with p=1.0 and q=2.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.174		0.091	0.033	-0.031	0.050	0.161	0.079
	0.50	0.367		0.114	0.084	0.147	0.151	0.197	0.170
	0.75	0.542	54	0.141	0.158	0.341	0.373	0.276	0.372
	1.00	0.705		0.168	0.250	0.548	0.731	0.409	0.711
	1.25	0.860		0.190	0.353	0.763	1.235	0.651	1.250
20	0.25	0.174		0.072	0.016	-0.126	0.038	0.125	0.036
	0.50	0.367		0.062	0.032	-0.016	0.050	0.133	0.068
	0.75	0.542	55	0.053	0.055	0.112	0.104	0.131	0.111
	1.00	0.705		0.044	0.086	0.251	0.211	0.144	0.195
	1.25	0.860		0.033	0.122	0.400	0.378	0.255	0.391
30	0.25	0.174		0.076	0.013	-0.141	0.035	0.127	0.030
	0.50	0.367		0.064	0.023	-0.048	0.035	0.136	0.052
	0.75	0.542	55	0.050	0.039	0.063	0.064	0.130	0.079
	1.00	0.705		0.035	0.059	0.186	0.129	0.122	0.116
	1.25	0.860		0.019	0.084	0.317	0.240	0.164	0.221

Table 21. Bias and mean squared error of estimators in 1000 samples for the lognormal lifetimes.

(Lifetime distribution: Lognormal with $\theta=0.0,\sigma=2.0,\mu=7.4$)
(Censoring distribution: Beta with p=2.0 and q=1.0)

				$\hat{H}_{pi}$		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.174		0.073	0.024	-0.086	0.057	0.145	0.070
	0.50	0.367		0.051	0.048	-0.010	0.094	0.167	0.132
	0.75	0.542	46	0.027	0.079	0.083	0.161	0.174	0.202
	1.00	0.705		0.003	0.115	0.188	0.266	0.172	0.280
	1.25	0.860		-0.021	0.155	0.302	0.416	0.176	0.397
20	0.25	0.174		0.071	0.013	-0.139	0.039	0.126	0.035
	0.50	0.367		0.042	0.021	-0.085	0.044	0.132	0.060
	0.75	0.542	46	0.010	0.032	-0.014	0.059	0.129	0.085
	1.00	0.705		-0.024	0.048	0.069	0.092	0.117	0.115
	1.25	0.860		-0.058	0.068	0.161	0.147	0.102	0.149
30	0.25	0.174		0.076	0.011	-0.147	0.035	0.129	0.029
	0.50	0.367		0.045	0.014	-0.099	0.032	0.136	0.045
	0.75	0.542	46	0.010	0.020	-0.034	0.036	0.128	0.061
	1.00	0.705		-0.027	0.030	0.043	0.051	0.114	0.075
	1.25	0.860		-0.065	0.045	0.129	0.083	0.091	0.091

Table 22. Bias and mean squared error of estimators in 1000 samples for the gamma lifetimes.

(Lifetime distribution: Gamma with α=.55, β=1.0, μ=.55)
(Censoring distribution: Beta with p=1.0 and q=1.0)

				$\hat{H}_{pi}$ (Weibull)		$\hat{H}_{pi}$ (lognormal)		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.74		0.06	0.11	-0.06	0.09	-0.02	0.24	0.10	0.25
	0.50	1.25		-0.04	0.17	-0.17	0.17	0.08	0.62	0.17	0.67
	0.75	1.73	18	-0.25	0.30	-0.33	0.32	0.22	1.28	0.25	1.35
	1.00	2.20		-0.51	0.58	-0.54	0.57	0.37	2.21	0.36	2.31
	1.25	2.66		-0.81	1.09	-0.80	0.98	0.52	3.42	0.50	0.57
20	0.25	0.74		0.08	0.05	-0.05	0.04	-0.04	0.11	0.07	0.10
	0.50	1.25		-0.07	0.06	-0.18	0.09	-0.00	0.22	0.11	0.23
	0.75	1.73	17	-0.31	0.18	-0.37	0.21	0.06	0.40	0.15	0.43
	1.00	2.20		-0.61	0.49	-0.60	0.47	0.14	0.65	0.19	0.68
	1.25	2.66		-0.95	1.05	-0.88	0.90	0.22	0.98	0.24	1.02
30	0.25	0.74		-0.74	1.24	-0.74	1.03	-0.74	1.09	-0.74	1.22
	0.50	1.25		-1.25	2.95	-1.25	2.70	-1.25	3.14	-1.25	3.41
	0.75	1.73	18	-1.73	4.97	-1.73	4.82	-1.73	6.17	-1.73	6.49
	1.00	2.20		-0.65	0.51	-0.63	0.47	0.04	0.39	0.10	0.41
	1.25	2.66		-1.00	1.10	-0.92	0.92	0.09	0.58	0.13	0.61

Table 23. Bias and mean squared error of estimators in 1000 samples for the gamma lifetimes.

(Lifetime distribution: Gamma with α=.55,β=1.0,μ=.55)
(Censoring distribution: Beta with p=1.0 and q=2.0)

				$\hat{H}_{pi}$ (Weibull)		$\hat{H}_{pi}$ (lognormal)		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.74		0.04	0.11	-0.07	0.09	-0.04	0.24	0.10	0.25
	0.50	1.25		-0.09	0.19	-0.18	0.19	0.11	0.68	0.18	0.71
	0.75	1.73	28	-0.31	0.37	-0.35	0.35	0.29	1.43	0.29	1.49
	1.00	2.20		-0.58	0.71	-0.57	0.63	0.48	2.51	0.44	2.66
	1.25	2.66		-0.89	1.28	-0.83	1.08	0.68	3.93	0.63	4.15
20	0.25	0.74		0.05	0.05	-0.06	0.04	-0.08	0.11	0.07	0.10
	0.50	1.25		-0.12	0.09	-0.19	0.10	-0.01	0.24	0.11	0.25
	0.75	1.73	28	-0.38	0.25	-0.38	0.24	0.09	0.46	0.15	0.49
	1.00	2.20		-0.68	0.61	-0.61	0.51	0.21	0.79	0.22	0.82
	1.25	2.66		-1.03	1.23	-0.89	0.96	0.32	1.23	0.31	1.28
30	0.25	0.74		-0.74	1.19	-0.74	1.02	-0.74	1.02	-0.74	1.21
	0.50	1.25		-1.25	2.84	-1.25	2.69	-1.25	3.09	-1.25	3.42
	0.75	1.73	28	-1.73	4.81	-1.73	4.80	-1.73	6.22	-1.73	6.50
	1.00	2.20		-0.73	0.62	-0.65	0.50	0.08	0.41	0.11	0.43
	1.25	2.66		-1.08	1.28	-0.93	0.97	0.17	0.63	0.17	0.66

Table 24. Bias and mean squared error of estimators in 1000 samples for the gamma lifetimes.

(Lifetime distribution: Gamma with α=.55,β=1.0,μ=.55)
(Censoring distribution: Beta with p=2.0 and q=1.0)

				$\hat{H}_{pi}$ (Weibull)		$\hat{H}_{pi}$ (lognormal)		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.74		0.08	0.11	-0.06	0.08	-0.00	0.23	0.10	0.24
	0.50	1.25		-0.02	0.15	-0.17	0.16	0.07	0.56	0.17	0.60
	0.75	1.73	8	-0.23	0.25	-0.34	0.29	0.17	1.08	0.23	1.15
	1.00	2.20		-0.48	0.49	-0.56	0.53	0.28	1.82	0.31	1.88
	1.25	2.66		-0.78	0.94	-0.82	2.76	0.40	2.84	0.40	2.84
20	0.25	0.74		0.10	0.05	-0.06	0.04	-0.01	0.12	0.07	0.10
	0.50	1.25		-0.03	0.05	-0.18	0.08	-0.02	0.22	0.10	0.23
	0.75	1.73	8	-0.27	0.15	-0.37	0.24	0.05	0.37	0.14	0.41
	1.00	2.20		-0.56	0.42	-0.60	0.45	0.11	0.59	0.18	0.64
	1.25	2.66		-0.90	0.93	-0.88	0.87	0.17	0.86	0.22	0.92
30	0.25	0.74		-0.74	1.28	-0.74	1.03	-0.74	1.13	-0.74	1.22
	0.50	1.25		-1.25	3.04	-1.25	2.71	-1.25	3.17	-1.25	3.41
	0.75	1.73	8	-1.73	5.10	-1.73	4.84	-1.73	6.16	-1.73	6.47
	1.00	2.20		-0.59	0.42	-0.62	0.44	0.03	0.33	0.11	0.35
	1.25	2.66		-0.93	0.96	-0.90	0.88	0.07	0.49	0.13	0.51

Table 25. Bias and mean squared error of estimators in 1000 samples for the gamma lifetimes.

(Lifetime distribution: Gamma with α=2.0,β=1.0,μ=2.0)
(Censoring distribution: Beta with p=1.0 and q=1.0)

				$\hat{H}_{pi}$ (Weibull)		$\hat{H}_{pi}$ (lognormal)		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.02		-0.00	0.00	0.01	0.00	-0.22	0.06	0.00	0.00
	0.50	0.09		0.00	0.01	0.00	0.01	-0.12	0.03	0.01	0.01
	0.75	0.18	56	0.01	0.02	-0.00	0.01	-0.05	0.02	0.03	0.03
	1.00	0.28		0.03	0.03	-0.00	0.03	0.01	0.03	0.03	0.04
	1.25	0.39		0.05	0.05	0.01	0.04	0.07	0.05	0.04	0.06
20	0.25	0.02		-0.00	0.00	0.01	0.00	-0.24	0.06	0.00	0.00
	0.50	0.09		0.00	0.00	0.00	0.00	-0.16	0.03	0.00	0.01
	0.75	0.18	58	0.01	0.01	-0.01	0.01	-0.09	0.02	0.01	0.01
	1.00	0.28		0.02	0.02	-0.01	0.01	-0.04	0.02	0.01	0.02
	1.25	0.39		0.03	0.02	-0.01	0.02	0.00	0.02	0.02	0.03
30	0.25	0.02		-0.00	0.00	0.01	0.00	-0.25	0.06	0.00	0.00
	0.50	0.09		0.00	0.00	0.00	0.00	-0.16	0.03	0.01	0.00
	0.75	0.18	58	0.01	0.01	-0.01	0.01	-0.10	0.02	0.01	0.01
	1.00	0.28		0.02	0.01	-0.01	0.01	-0.05	0.01	0.01	0.01
	1.25	0.39		0.03	0.02	-0.01	0.01	-0.01	0.01	0.01	0.02

Table 26. Bias and mean squared error of estimators in 1000 samples for the gamma lifetimes.

(Lifetime distribution: Gamma with α=2.0,β=1.0,μ=2.0)
(Censoring distribution: Beta with p=1.0 and q=2.0)

				$\hat{H}_{pi}$ (Weibull)		$\hat{H}_{pi}$ (lognormal)		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.02		0.01	0.00	0.01	0.00	-0.21	0.05	0.01	0.00
	0.50	0.09		0.02	0.01	0.02	0.01	-0.07	0.02	0.03	0.02
	0.75	0.18	67	0.05	0.03	0.03	0.02	0.04	0.03	0.07	0.03
	1.00	0.28		0.09	0.05	0.06	0.04	0.13	0.06	0.09	0.06
	1.25	0.39		0.13	0.08	0.10	0.07	0.22	0.12	0.14	0.11
20	0.25	0.02		0.00	0.00	0.01	0.00	-0.25	0.06	0.00	0.00
	0.50	0.09		0.00	0.00	-0.00	0.00	-0.14	0.03	0.01	0.01
	0.75	0.18	73	0.01	0.01	-0.01	0.01	-0.06	0.02	0.01	0.01
	1.00	0.28		0.02	0.02	-0.01	0.02	0.01	0.02	0.01	0.02
	1.25	0.39		0.02	0.03	0.00	0.03	0.06	0.03	0.02	0.04
30	0.25	0.02		-0.00	0.00	0.00	0.00	-0.25	0.07	0.00	0.00
	0.50	0.09		0.00	0.00	-0.00	0.00	-0.15	0.03	0.01	0.00
	0.75	0.18	73	0.01	0.01	-0.01	0.01	-0.08	0.01	0.01	0.01
	1.00	0.28		0.02	0.01	-0.01	0.01	-0.02	0.01	0.01	0.01
	1.25	0.39		0.02	0.01	-0.00	0.02	0.04	0.02	0.01	0.03

Table 27. Bias and mean squared error of estimators in 1000 samples for the gamma lifetimes.

(Lifetime distribution: Gamma with α=2.0,β=1.0,μ=2.0)
(Censoring distribution: Beta with p=2.0 and q=1.0)

				$\hat{H}_{pi}$ (Weibull)		$\hat{H}_{pi}$ (lognormal)		$\hat{H}_{na}$		$\hat{H}_{nk}$	
n	t	H (true)	% Censorship	BIAS	MSE	BIAS	MSE	BIAS	MSE	BIAS	MSE
10	0.25	0.02		-0.00	0.00	0.01	0.00	-0.23	0.06	0.00	0.00
	0.50	0.09		-0.00	0.01	0.00	0.01	-0.14	0.03	0.01	0.01
	0.75	0.18	42	0.01	0.02	-0.00	0.01	-0.07	0.02	0.02	0.02
	1.00	0.28		0.02	0.03	-0.01	0.02	-0.02	0.03	0.02	0.03
	1.25	0.39		0.03	0.04	-0.01	0.03	0.02	0.04	0.02	0.05
20	0.25	0.02		-0.00	0.00	0.01	0.00	-0.24	0.06	0.00	0.00
	0.50	0.09		0.00	0.00	0.00	0.00	-0.16	0.03	0.01	0.01
	0.75	0.18	43	0.01	0.01	-0.00	0.01	-0.10	0.02	0.01	0.01
	1.00	0.28		0.02	0.01	-0.01	0.01	-0.05	0.02	0.01	0.02
	1.25	0.39		0.03	0.02	-0.01	0.02	-0.01	0.02	0.01	0.02
30	0.25	0.02		-0.00	0.00	0.01	0.00	-0.24	0.06	0.00	0.00
	0.50	0.09		0.00	0.00	0.01	0.00	-0.16	0.03	0.01	0.00
	0.75	0.18	44	0.01	0.01	-0.00	0.00	-0.10	0.02	0.00	0.01
	1.00	0.28		0.02	0.01	-0.01	0.01	-0.06	0.01	0.01	0.01
	1.25	0.39		0.03	0.01	-0.01	0.01	-0.02	0.01	0.01	0.01

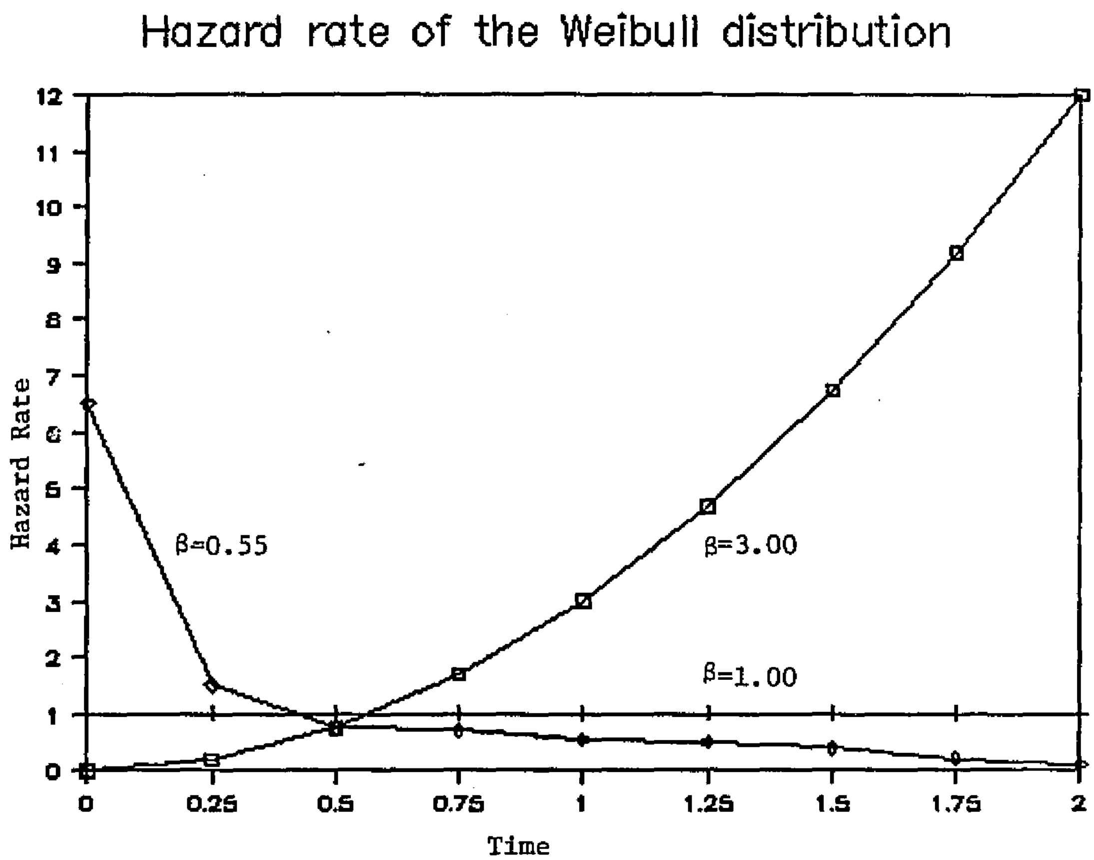


Figure 1. The hazard-rate curves for the Weibull distribution.

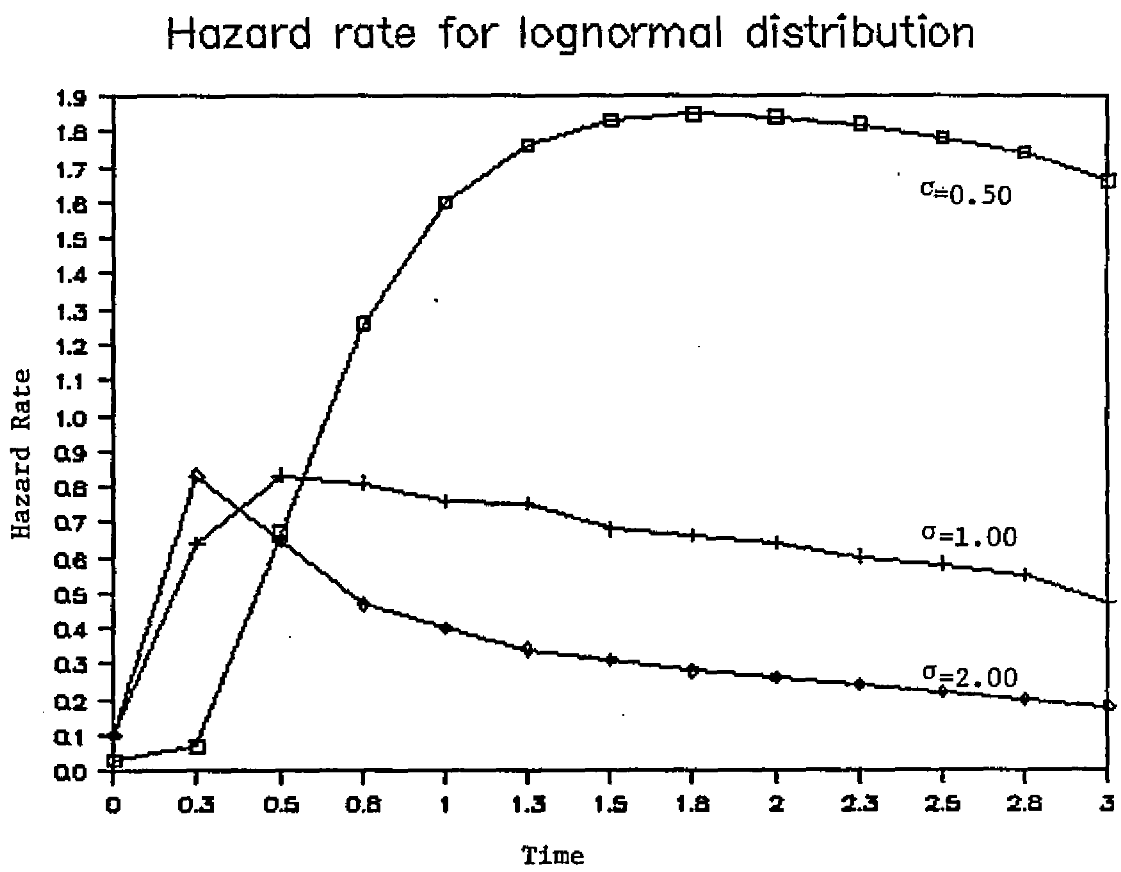


Figure 2. The hazard-race curves for the lognormal distribution.

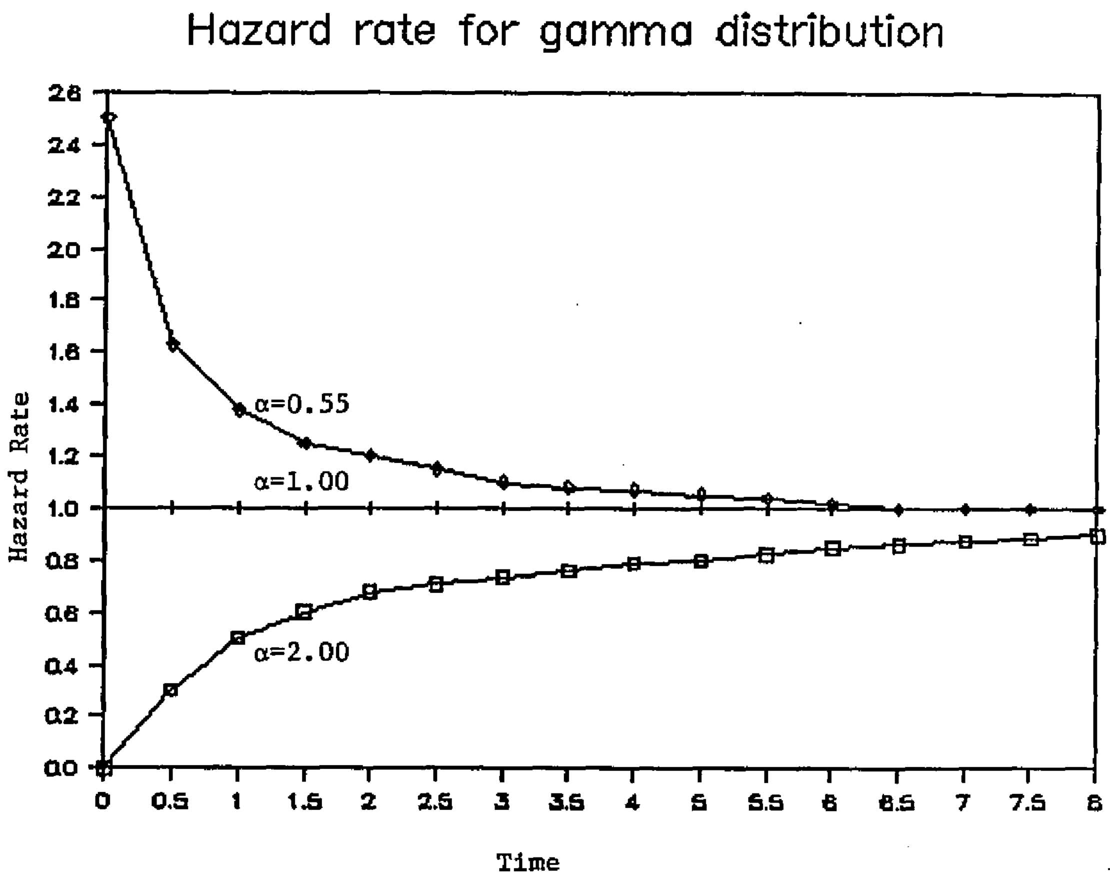


Figure 3. The hazard-rate curves for the gamma distribution.

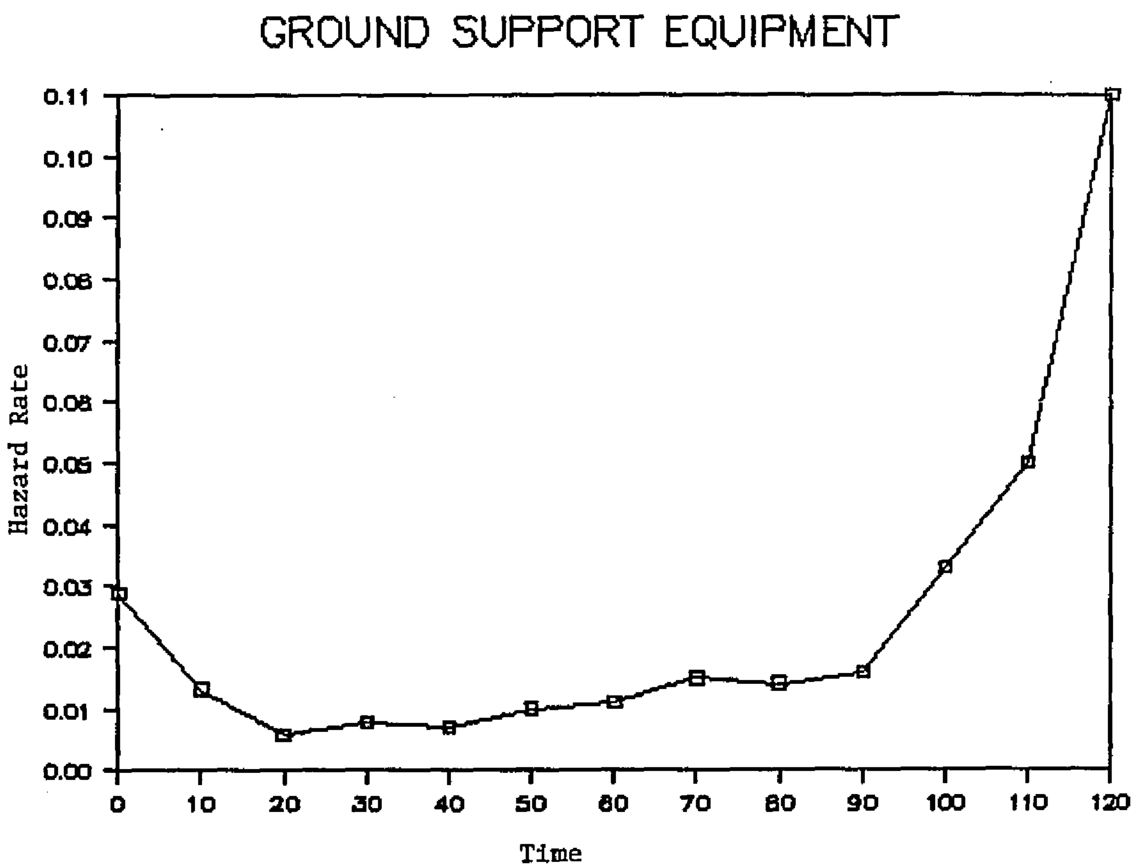


Figure 4. The hazard-rate curve for electronic ground support equipment.

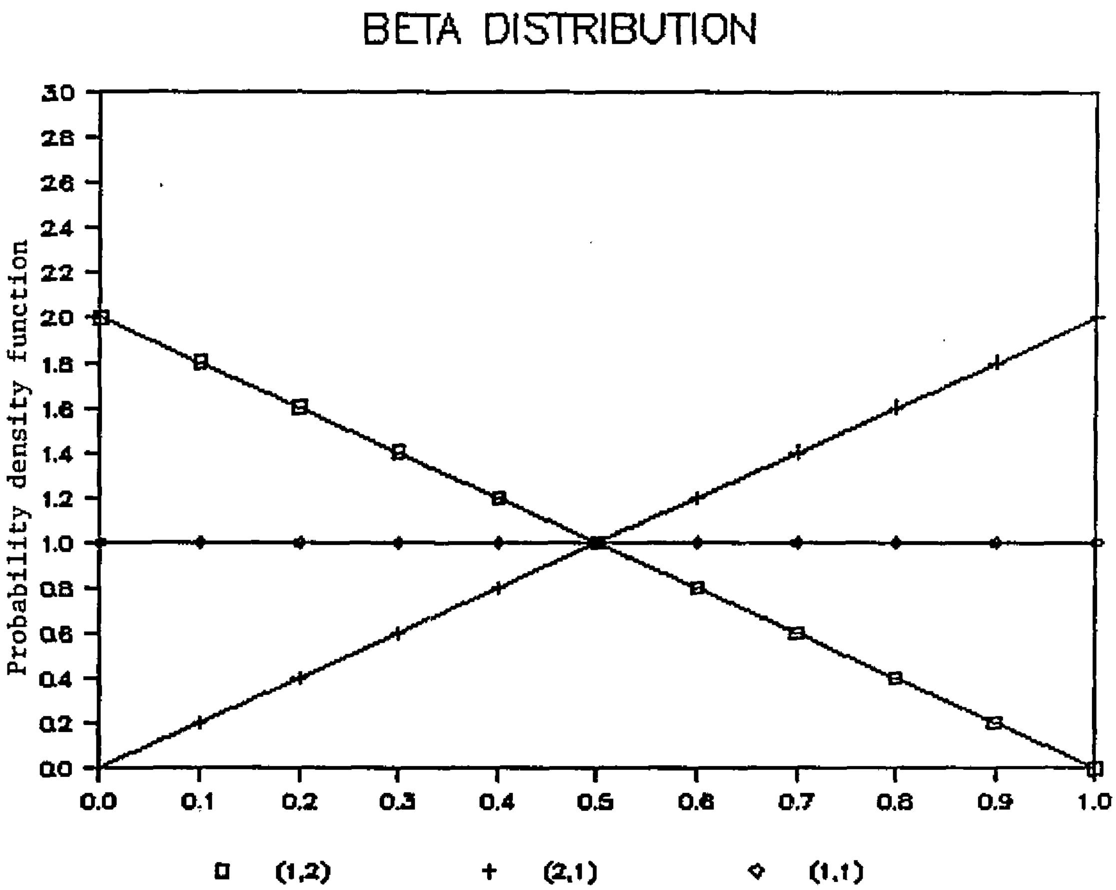
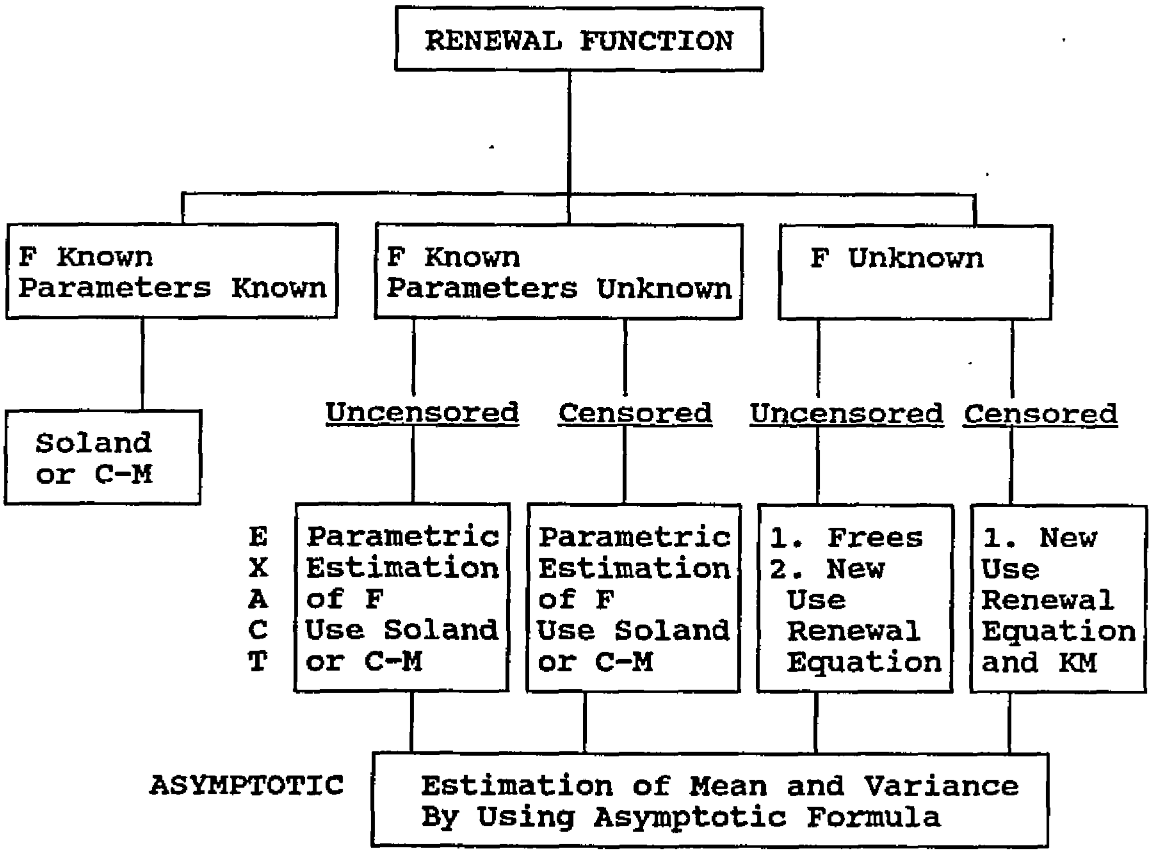


Figure 5. Probability density function for the beta distribution.



EXACT: $$H(t) = \sum_{k=1}^{\infty} F^k(t) = F(t) + \int_0^t H(t-y)f(y)\,dy$$

ASYMPTOTIC: $$\lim_{t \to \infty} H(t) = t/\mu + \sigma^2/(2\mu^2) - 1/2$$

Figure 6. Diagram of the renewal function estimation.

VITA

Binshan Lin was born June 23, 1953 in Chai-Yi, Taiwan. He received his Bachelor of Science in Psychology from National Chengchi University in Taiwan. His academic life in the U.S. started in Louisiana State University, Baton Rouge, Louisiana in 1983. In 1985, he recieved his Master of Applied Statistics from Louisiana State University. He started his doctoral program in the College of Business Administration at Louisiana State University in 1985. During the course of his studies he majored in Quantitative Business Analysis and minored in Experimental Statistics. From June 1986 to August 1988 he served as a teaching assistant, teaching Management Information System, Statistics, and Operations Research in the Department of Quantitative Business Analysis.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Binshan Lin

Major Field: Business Administration (Quantitative Business Analysis)

Title of Dissertation: Estimation of the Renewal Function

Approved: .

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Major Professor and Chairman

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Dean of the Graduate School

EXAMINING COMMITTEE:

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Date of Examination:

July 15, 1988