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Templates for Representable Matroids

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TEMPLATES FOR REPRESENTABLE MATROIDS

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“It is the glory of God to conceal a thing: but the honour of kings is to search out a matter.” Proverbs 25:2

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Abstract

The matroid structure theory of Geelen, Gerards, and Whittle has led to a hypothesis that a highly connected member of a minor-closed class of matroids representable over a finite field is a mild modification (known as a *perturbation*) of a frame matroid, the dual of a frame matroid, or a matroid representable over a proper subfield. They introduced the notion of a *template* to describe these perturbations in more detail. In this dissertation, we determine these templates for various classes and use them to prove results about representability, extremal functions, and excluded minors.

Chapter 1 gives a brief introduction to matroids and matroid structure theory. Chapters 2 and 3 analyze this hypothesis of Geelen, Gerards, and Whittle and propose some refined hypotheses. In Chapter 3, we define frame templates and discuss various notions of template equivalence.

Chapter 4 gives some details on how templates relate to each other. We define a preorder on the set of frame templates over a finite field, and we determine the minimal nontrivial templates with respect to this preorder. We also study in significant depth a specific type of template that is pertinent to many applications. Chapters 5 and 6 apply the results of Chapters 3 and 4 to several subclasses of the binary matroids and the quaternary matroids—those matroids representable over the fields of two and four elements, respectively.

Two of the classes we study in Chapter 5 are the even-cycle matroids and the even-cut matroids. Each of these classes has hundreds of excluded minors. We show that, for highly connected matroids, two or three excluded minors suffice. We also show that Seymour's 1-Flowing Conjecture holds for sufficiently highly connected matroids.

In Chapter 6, we completely characterize the highly connected members of the class of golden-mean matroids and several other closely related classes of quaternary matroids. This leads to a determination of the extremal functions for these classes, verifying a conjecture of Archer for matroids of sufficiently large rank.

Chapter 1: Introduction

In 1935, Whitney [44] introduced the concept of a matroid to unify common ideas of dependence in linear algebra and graph theory. Many fundamental mathematical structures, including error-correcting codes and constraints in an optimization problem, can be modeled by matroids. Matroid theory gives a unique perspective on these structures through the introduction of concepts such as connectivity and minors, both of which generalize concepts from graph theory. In this dissertation, we study problems regarding matroids that are very close to graphic matroids, in a sense that will be made precise in Section 1.4. In particular, we determine all such matroids that are contained in various interesting classes of matroids. Subject to a hypothesis by Geelen, Gerards, and Whittle, this then determines *all* highly connected matroids in such a class. This has consequences for the study of excluded minors and of extremal functions of the class.

Section 1.1 is a brief introduction to matroid theory. However, notation and terminology not explained in this dissertation generally follow Oxley [23]. (One exception is that we denote the vector matroid of a matrix A by $M(A)$ rather than $M[A]$.) Sections 1.2–1.4 give some additional information that goes beyond the basics of matroid theory but is foundational to the rest of this dissertation. In Section 1.5, we present an overview of our main results. The reader familiar with the basics of matroid theory may choose to skip to Section 1.2.

1.1 Introduction to Matroids

This section gives a brief introduction to matroid theory.

Definition

There are several equivalent definitions for the notion of a matroid. The following definition is the most common.

Definition 1.1.1. A *matroid* M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E, called the *ground set*, and a collection $\mathcal{I}$ of subsets of E, called *independent sets*, having the following three properties:

- $\emptyset \in \mathcal{I}$.
- If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

We will often denote E and $\mathcal{I}$ by $E(M)$ and $\mathcal{I}(M)$, respectively. If a subset of E is not a member of $\mathcal{I}$, it is a *dependent* set. If $M = (E, \mathcal{I})$ is a matroid, then M is a matroid *on* E. A minimal dependent set of a matroid is called a *circuit*, and a maximal independent set of a matroid is called a *basis*. The next two theorems, whose proofs can be found in [23], show how a matroid can be defined in terms of circuits and bases.

Theorem 1.1.2 ([23, Corollary 1.1.5]). *Let $\mathcal{C}$ be a collection of subsets of a set E. Then $\mathcal{C}$ is the collection of circuits of a matroid on E if and only if $\mathcal{C}$ has the following properties:*

- $\emptyset \notin \mathcal{C}$.
- *If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*
- *If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.*

Theorem 1.1.3 ([23, Corollary 1.2.5]). *Let $\mathcal{B}$ be a collection of subsets of a set E. Then $\mathcal{B}$ is the collection of bases of a matroid on E if and only if $\mathcal{B}$ has the following properties:*

- $\mathcal{B} \neq \emptyset$.
- *If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 - B_2$, then there is an element y of $B_2 - B_1$ such that $(B_1 - x) \cup \{y\} \in \mathcal{B}$.*

Examples of Matroids

As mentioned above, two areas of mathematics that have been fundamental motivators for the study of matroid theory have been linear algebra and graph theory. In linear algebra, a matrix over a field gives rise to a matroid.

Proposition 1.1.4 ([23, Proposition 1.1.1]). *Let A be an $m \times n$ matrix over a field $\mathbb{F}$, and label its columns $1, 2, \ldots, n$. Let $E = \{1, 2, \ldots, n\}$. Let $\mathcal{I}$ consist of the subsets X of E such that the set of columns labeled by X is linearly independent and such that no pair of elements of X label identical columns. Then $(E, \mathcal{I})$ is a matroid.*

The matroid described in Proposition 1.1.4 is called the *vector matroid* of A; we will denote it by $M(A)$. Two matroids M_1 and M_2 are *isomorphic* if there is a bijection $\psi : E(M_1) \to E(M_2)$ such that $X \in \mathcal{I}(M_1)$ if and only if $\psi(X) \in \mathcal{I}(M_2)$. A matroid M that is isomorphic to the vector matroid of a matrix A is called a *representable* matroid. If A is a matrix over a field $\mathbb{F}$, then M is called $\mathbb{F}$-*representable* or *representable over* $\mathbb{F}$. Performing elementary row operations on a matrix, scaling rows and columns of the matrix by a nonzero element of $\mathbb{F}$, removing or adding zero rows, and performing a field automorphism on the matrix entrywise, all preserve the matrix's vector matroid. Because of this, every representable matroid M is isomorphic to the vector matroid of a matrix of the form $[I_r|A]$, where I is the $r \times r$ identity matrix whose columns are labeled by the elements of a basis for M.

In this dissertation, graphs are allowed to have loops and parallel edges. Every graph gives rise to a matroid.

Proposition 1.1.5 ([23, Proposition 1.1.7]). *Let E be the set of edges of a graph G, and let $\mathcal{C}$ be the collection of edge sets of cycles of G. Then $\mathcal{C}$ is the set of circuits of a matroid on E.*

The matroid described in Proposition 1.1.5 is called the *cycle matroid* of G; we will denote it by $M(G)$. A matroid M that is isomorphic to the cycle matroid of a graph is called a *graphic* matroid.

Duality

Let M be a matroid, with ground set E, whose set of bases is $\mathcal{B}$. Let $\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}$. Then $\mathcal{B}^*$ is the set of bases for a matroid M^*, called the *dual* of M. Let $M = M([I_r|A])$ be an $\mathbb{F}$-representable matroid for some field $\mathbb{F}$, where A is an $r \times n$ matrix. Then $M^* = M([-A^T|I_n])$. Moreover, the vector space spanned by the rows of $[-A^T|I_n]$ is the orthogonal subspace of the vector space spanned by the rows of $[I_r|A]$.

Minors

If $M = (E, \mathcal{I})$ is a matroid, then $M\backslash e$ is the matroid obtained by *deleting* e and is defined by $M\backslash e = (E - e, \{I - e : I \in \mathcal{I}\})$. The matroid obtained from M by *contracting* e is denoted by M/e and is defined as $(M^*\backslash e)^*$. If $T \subseteq E(M)$, then $M\backslash T$ and M/T are, respectively, the matroids obtained by deleting and contracting every element in T. It can be shown that, if C and D are subsets of $E(M)$, then $(M/C)\backslash D = (M\backslash D)/C$. Therefore, we denote this matroid by $M/C\backslash D$. Every matroid of this form is called a *minor* of M. The operations of deletion and contraction and the notion of minors are generalizations of the concepts of the same name in graph theory. The matroid $M\backslash D$ can also be denoted by $M|(E - D)$ and is called the *restriction* of M to $(E - D)$.

A class $\mathcal{M}$ of matroids is *minor-closed* if every minor of a matroid in the class is also in the class. The classes of graphic matroids and $\mathbb{F}$-representable matroids, for each field $\mathbb{F}$, are minor-closed. If N is not a member of $\mathcal{M}$ but every proper minor of N is a minor of $\mathcal{M}$, then N is an *excluded minor* for $\mathcal{M}$.

Frame Matroids

Every graphic matroid is representable over every field. The signed incidence matrix of a graph (where non-loop edges label columns whose two nonzero entries are 1 and -1 and where loops label zero columns) has a vector matroid isomorphic to the cycle matroid of the graph. More generally, a *frame matrix* is a matrix where every column has at most two nonzero entries.

Even more generally, let M be a matroid with ground set E and with a basis B such that, for every element $e \in E - B$, there is a subset $B' \subseteq B$ with $|B'| \leq 2$ such that $B' \cup \{e\}$ is a circuit of M. Then M is a *framed matroid.* A *frame matroid* is a restriction of a framed matroid. The vector matroid of a frame matrix is an example of a frame matroid.

Connectivity

It can be shown that all bases of a matroid M have the same size. This number is called the *rank* of the matroid and is denoted by $r(M)$. If $X \subseteq E(M)$, then the *rank* of X, denoted by $r_M(X)$ (or $r(X)$ if the matroid is known from the context), is the size of the largest independent subset of X. Note that $r(M) = r_M(E(M))$. If $X \subseteq E(M)$ and $r(X) = r(M)$, then X is said to be *spanning*.

Let M be a matroid on ground set E. The *connectivity* of X is denoted by $\lambda_M(X)$ or $\lambda(X)$ and is given by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

This definition for matroid connectivity, also called *Tutte connectivity*, was first given by Tutte [39], one of the pioneers in matroid theory, as a generalization of graph connectivity. If $\lambda(X) < k$ and $\min\{|X|, |E - X|\} \geq k$, then the partition $(X, E - X)$ is called a *k-separation*. If M has no k'-separations for $k' < k$, then M is *k-connected*.

There is a weaker notion of connectivity that we will also use in this dissertation. A matroid M is *vertically k-connected* if, for every set $X \subseteq E(M)$ with $\lambda_M(X) < k$, either X or $E - X$ is spanning. If M is vertically k-connected, then its dual M^* is *cyclically k-connected*.

1.2 Matroid Structure Theory

Robertson and Seymour profoundly transformed graph theory with their Graph Minor Theorem [32]. Geelen, Gerards, and Whittle are on track to do the same for matroid theory with their Matroid Structure Theory for matroids representable over a finite field (see, e.g. [8]). The theorem they intend to prove is the following:

Conjecture 1.2.1 (Matroid Structure Theorem, rough idea)**.** *Let $\mathbb{F}$ be a finite field, and let $\mathcal{M}$ be a proper minor-closed class of $\mathbb{F}$-representable matroids. If $M \in \mathcal{M}$ is sufficiently large and has sufficiently high branch-width, then M has a tree-decomposition, the parts of which correspond to mild modifications of matroids representable over a proper subfield of $\mathbb{F}$, or to mild modifications of frame matroids and their duals.*

The words "tree-decomposition", "parts", "correspond to" and "mild modifications" need (a lot of) elaboration, and hide almost 20 years of very hard work. Whittle [41] described the proof of Rota's Conjecture, which has the Matroid Structure Theorem as a major ingredient, as follows: "It's a little bit like discovering a new mountain—we've crossed many hurdles to reach a new destination and we have returned scratched, bloodied and bruised from the arduous journey—we now need to create a pathway so others can reach it."

This dissertation will only focus on the last part of Conjecture 1.2.1. Geelen, Gerards, and Whittle announced a theorem about that part [7, Theorem 3.1] that we will repeat in Chapter 2 as Conjecture 2.0.1. A *represented matroid* can be

thought of as a particular matrix for a representable matroid. A *rank-($\leq t$) perturbation* of a represented matroid $M(A)$ is obtained by adding a matrix of rank at most t to the matrix A. In Section 1.4, we will discuss represented matroids and perturbations in more detail, but these descriptions are sufficient to state the hypothesis on which much of this dissertation is based.

Hypothesis 1.2.2. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{M}$ be a proper minor-closed class of $\mathbb{F}$-represented matroids. There exist constants $k, t \in \mathbb{Z}_+$ such that each k-connected member of $\mathcal{M}$ is a rank-($\leq t$) perturbation of an $\mathbb{F}$-represented matroid N, such that either*

1. *N is a represented frame matroid,*
2. *N^* is a represented frame matroid, or*
3. *N is confined to a proper subfield of $\mathbb{F}$.*

1.3 Additional Preliminaries

The *extremal function* (also called *growth rate function*) for a minor-closed class $\mathcal{M}$, denoted by $h_{\mathcal{M}}(r)$, is the function whose value at an integer $r \geq 0$ is given by the maximum number of elements in a simple represented matroid in $\mathcal{M}$ of rank at most r. For a matroid M, we denote $|\mathrm{si}(M)|$ by $\varepsilon(M)$; that is, $\varepsilon(M)$ is the number of rank-1 flats of M. We will make use of several results in the literature. The first of these is the Growth Rate Theorem of Geelen, Kung, and Whittle [10, Theorem 1.1].

Theorem 1.3.1 (Growth Rate Theorem)**.** *If $\mathcal{M}$ is a nonempty minor-closed class of matroids, then there exists $c \in \mathbb{R}$ such that either:*

(1) $h_{\mathcal{M}}(r) \leq cr$ for all r,

(2) $\binom{r+1}{2} \leq h_{\mathcal{M}}(r) \leq cr^2$ for all r and $\mathcal{M}$ contains all graphic matroids,

(3) there is a prime power q such that $\frac{q^r-1}{q-1} \leq h_{\mathcal{M}}(r) \leq cq^r$ for all r and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids, or

(4) $\mathcal{M}$ contains all simple rank-2 matroids.

If (2) of the previous theorem holds for $\mathcal{M}$, then $\mathcal{M}$ is *quadratically dense*. If M is a simple rank-r matroid in $\mathcal{M}$ such that $\varepsilon(M) = h_{\mathcal{M}}(r)$, then we call M an *extremal matroid* of $\mathcal{M}$.

The proof of the Growth Rate Theorem was based on work in [13] and [9]. Specifically, [13] contains the following result.

Theorem 1.3.2 ([13, Theorem 1.1])**.** *For any finite field $\mathbb{F}$ and graph G, there exists an integer c such that, if M is an $\mathbb{F}$-represented matroid with no $M(G)$-minor, then $\varepsilon(M) \leq cr(M)$.*

We now clarify some notation and terminology that will be helpful specifically for this dissertation. For a field $\mathbb{F}$ of characteristic $p \neq 0$, we denote the prime subfield of $\mathbb{F}$ by $\mathbb{F}_p$. We denote an empty matrix by $[\emptyset]$. We denote a group of one element by $\{0\}$ or $\{1\}$, if it is an additive or multiplicative group, respectively. If S' is a subset of a set S and G is a subgroup of the additive group of the vector space $\mathbb{F}^S$, we denote by $G|S'$ the projection of G onto $\mathbb{F}^{S'}$. Similarly, if $\bar{x} \in G$, we denote the projection of $\bar{x}$ onto $\mathbb{F}^{S'}$ by $\bar{x}|S'$. Let A be an $m \times n$ matrix. If A' is an $m \times n'$ submatrix of A, with $n' \leq n$, then A' is a *column submatrix* of A. Similarly, if A' is an $m' \times n$ submatrix of A, with $m' \leq m$, then A' is a *row submatrix* of A. Let A and A' be matrices with the same dimensions, and suppose the entry in the i-th row and j-th column of A is nonzero if and only if the entry in the i-th row and j-th column of A' is nonzero. Then A and A' have the same *(zero-nonzero) pattern*. The $n \times \binom{n}{2}$ matrix where every column contains exactly two nonzero entries, the first a 1 and the second a -1, is denoted by D_n. Note that D_n is the signed incidence matrix of the complete graph K_n. If $U \subseteq \mathbb{F}^E$ and $X \subseteq E$, then $U|X = \{u|X : u \in U\}$. If $\Gamma \subseteq \mathbb{F}$, then $\Gamma U = \{\gamma u | \gamma \in \Gamma, u \in U\}$.

We use the following notation and terminology, following [22]. The *weight* of a vector is its number of nonzero entries. If $\mathbb{F}$ is a field and $A \cap B = \emptyset$, then we identify the vector space $\mathbb{F}^A \times \mathbb{F}^B$ with $\mathbb{F}^{A \cup B}$. If U and W are additive subgroups of $\mathbb{F}^E$, then U and and W are *skew* if $U \cap W = \{0\}$.

1.4 Represented Matroids and Perturbations

In some respects, this dissertation is about matrices rather than matroids. However, we use these results about matrices to obtain results about their vector matroids. Because a matroid can have inequivalent representations, it will be useful to have a more formal notion of matroid representations.

Let $\mathbb{F}$ be a field. An $\mathbb{F}$-*represented matroid* (or simply *represented matroid* if the field is understood from the context) is a pair $M = (E, U)$, where U is a subspace of $\mathbb{F}^E$. The *dual* of M is $M^* = (E, U^\perp)$, where $U^\perp$ is the subspace consisting of all vectors orthogonal to every vector in U. A *representation* of $M = (E, U)$ is a matrix A whose row space is U. We write $M = M(A)$. We consider two represented matroids to be *equal* if they have representation matrices that are row equivalent up to column scaling. We denote the vector matroid (in the usual sense) of a representation A of M by $\widetilde{M}$ or $\widetilde{M}(A)$ and call it the *abstract matroid* associated with M. Basic matroid notions such as ground sets, independent sets, bases, circuits, rank, closure, connectivity, etc. are freely carried over from abstract matroids to represented matroids. A *represented frame matroid* is a represented matroid with a representation matrix A that has at most two nonzero entries per column.

The notions of restriction, deletion, contraction, and minors of matroids carry over to represented matroids also. If $M = (E, U)$ and $X \subseteq E$, then we define $M|X = (E - U, U|X)$. We define $M\backslash X = M|(E - X)$ and $M/X = (M^*\backslash X)^*$.

Definition 1.4.1. Let M_1 and M_2 be $\mathbb{F}$-represented matroids on a common ground set. Then M_2 is a *rank-($\leq t$) perturbation* of M_1 if there exist matrices A_1 and P such that $M(A_1) = M_1$, the rank of P is at most t, and $M(A_1 + P) = M_2$.

Definition 1.4.2. Let M_1 and M_2 be $\mathbb{F}$-represented matroids on ground set E. If there is some $\mathbb{F}$-represented matroid M on ground set $E \cup \{e\}$ such that $M_1 = M \backslash e$ and $M_2 = M/e$, then M_1 is an *elementary lift* of M_2, and M_2 is an *elementary projection* of M_1.

Note that an elementary lift of a represented matroid $M(A)$ can be obtained by appending a row to A.

Definition 1.4.3. Let M_1 and M_2 be $\mathbb{F}$-represented matroids on a common ground set. We denote the minimum number of elementary lifts and elementary projections needed to transform M_1 into M_2 by $\mathrm{dist}(M_1, M_2)$, and we denote the smallest integer t such that M_2 is a rank-($\leq t$) perturbation of M_1 by $\mathrm{pert}(M_1, M_2)$.

The following observation will be quite useful; in particular, we use it to prove Lemma 1.4.5 below.

Remark 1.4.4. Suppose that $M_1 = M(A_1)$ is a rank-($\leq t$) perturbation of $M_2 = M(A_2)$. Let P be the matrix of rank at most t such that $A_1 + P = A_2$. Let $\{v_1, v_2, \ldots, v_t\}$ be a basis for the row space of P. Note that neither A_1, P, nor $A_1 + P$ need have full row rank. If $r = r(M_1)$, then we may assume that P has $r + t$ rows. If $a_{i,j} \in \mathbb{F}$ for all i, j, let $a_{i,1}v_1 + a_{i,2}v_2 + \ldots + a_{i,t}v_t$ be the i-th row of P. Then M_1 can be obtained by contracting C from the represented matroid obtained from the following matrix.

$$\begin{array}{c|c}
 & C \\
\hline
\begin{matrix} v_1 \\ \vdots \\ v_t \end{matrix} & -I \\
\hline
A_1 & \begin{matrix} a_{1,1} & \cdots & a_{1,t} \\ \vdots & \ddots & \vdots \\ a_{r+t,1} & \cdots & a_{r+t,t} \end{matrix}
\end{array}$$

Lemma 1.4.5 appears in [7] as Lemma 2.1; however, no proof was given in [7].

Lemma 1.4.5 ([7, Lemma 2.1])**.** *If M_1 and M_2 are $\mathbb{F}$-represented matroids on the same ground set, then*

$$\mathrm{pert}(M_1, M_2) \leq \mathrm{dist}(M_1, M_2) \leq 2\,\mathrm{pert}(M_1, M_2).$$

Proof. A rank-($\leq t$) perturbation of a represented matroid M_1 can be obtained by successively adding t rank-1 matrices to some matrix A_1 with $M(A_1) = M_1$. Therefore, we can prove this result inductively by considering the behavior of elementary lifts, elementary projections, and rank-1 perturbations. An elementary lift of M_1 can be obtained by adding the rank-1 matrix $\left[\begin{array}{c} v \\ \hline 0 \end{array}\right]$, for some vector

v, to the matrix $\left[\begin{array}{ccc} 0 & \cdots & 0 \\ \hline & A_1 & \end{array}\right]$, which represents M_1. Thus, every elementary lift of a represented matroid is also a rank-1 perturbation of the represented matroid. Now, since M_1 is a rank-($\leq t$) perturbation of M_2 if and only if M_2 is a rank-($\leq t$) perturbation of M_1 and since M_1 is an elementary lift of M_2 if and only if M_2 is an elementary projection of M_1, we also have that every elementary projection of a represented matroid is a rank-1 perturbation of the represented matroid. The converse of these statements is not true in general; however, we will now show that every rank-1 perturbation of a represented matroid can be obtained by performing an elementary lift followed by an elementary projection.

Suppose that M_2 is a rank-1 perturbation of M_1. By Remark 1.4.4, there are vectors v and w and a matrix A_1 with $M(A_1) = M_1$ such that M_2 is obtained from the matrix $A = \left[\begin{array}{c|c} v & -1 \\ \hline A_1 & w \end{array}\right]$ by contracting the element represented by the last column. The represented matroid obtained from $A' = \left[\begin{array}{c} v \\ \hline A_1 \end{array}\right]$ is an elementary lift of M_1. Since M_2 is obtained from $M(A)$ by contracting the element represented by the last column, M_2 is an elementary projection of $M(A')$.

The fact that an elementary lift or projection can be obtained by a rank-1 perturbation implies that $\operatorname{pert}(M_1, M_2) \leq \operatorname{dist}(M_1, M_2)$. The fact that a rank-1 perturbation can be obtained by at most two elementary lifts and projections implies that $\operatorname{dist}(M_1, M_2) \leq 2 \operatorname{pert}(M_1, M_2)$. ■

In order to prove some results in Chapter 2, we will need some lemmas regarding duality. The first lemma is an easy corollary of Lemma 1.4.5. In fact, the following lemma still holds if $2t$ is replaced by t, but that best possible result is not necessary for our purposes.

Lemma 1.4.6. *Suppose that M_2 is a rank-($\leq t$) perturbation of M_1. Then M_2^* is a rank-($\leq 2t$) perturbation of M_1^*.*

Proof. By Lemma 1.4.5 and duality of elementary lifts and elementary projections, we have

$$\operatorname{pert}(M_1^*, M_2^*) \leq \operatorname{dist}(M_1^*, M_2^*) = \operatorname{dist}(M_1, M_2) \leq 2 \operatorname{pert}(M_1, M_2) \leq 2t.$$

■

The next lemma, proved by Nelson and Walsh [22], gives a bound on $\varepsilon(M)$, when M is the dual of a represented frame matroid. We use this and Lemma 1.4.8 to prove Lemma 1.4.9 below.

Lemma 1.4.7 ([22, Lemma 6.2])**.** *If M^* is a represented frame matroid, then $\varepsilon(M) \leq 3r(M)$.*

Lemma 1.4.8. *If M is a rank-($\leq t$) perturbation of a $\mathrm{GF}(q)$-represented matroid N, then $\varepsilon(M) \leq q^t \varepsilon(N) + \sum_{i=0}^{t-1} q^i$.*

Proof. We proceed by induction on t. If $t = 0$, then $M = N$, and the result is clear. Now suppose the result holds for rank-($\leq t'$) perturbations for all $t' < t$. Since M is a rank-($\leq t$) perturbation of N there is some represented matroid M' such that M' is a rank-($\leq t-1$) perturbation of N and M is a rank-(≤ 1) perturbation of M'. Thus, there are matrices A and P such that $M' = M(A)$, the rank of P is 1, and $M = M(A+P)$. We will show that the nonloop elements in a rank-1 flat of M' become members of at most q distinct rank-1 flats in M. Let $\{a_1, a_2, \ldots, a_{q-1}, a_q\}$ be the elements of $\mathrm{GF}(q)$, with $a_q = 0$, and let v be a nonzero column in A indexed by an element in a rank-1 flat F of M'. Then the nonloop elements of F are each represented by a column $a_i v$ for some i such that $1 \leq i \leq q-1$. Similarly, let w be a nonzero column of P. Then every column of P is represented by a column $a_i w$ for some i such that $1 \leq i \leq q$. Thus, every element in F that is not a loop in M' becomes represented in $A+P$ by a column of the form $a_i v + a_j w = a_i(v + a_i^{-1} a_j w)$, where $1 \leq i \leq q-1$ and $1 \leq j \leq q$. There are q distinct possible values for $a_i^{-1} a_j$; therefore, the elements of F that are not loops in M' are in at most q distinct rank-1 flats in M. Moreover, after P is added to A, loops in M' become represented by columns of the form $a_j w$. This accounts for one additional rank-1 flat in M. Thus, $\varepsilon(M) \leq q\varepsilon(M') + 1$. By the induction hypothesis, we have $\varepsilon(M) \leq q(q^{t-1}\varepsilon(N) + \sum_{i=0}^{t-2} q^i) + 1 = q^t \varepsilon(N) + \sum_{i=0}^{t-1} q^i$, which proves the result. ∎

Lemma 1.4.9. *Let t be a positive integer, and let $\mathbb{F} = \mathrm{GF}(q)$. Then there are finitely many integers r such that the complete graphic matroid $M(K_{r+1})$ is a rank-($\leq t$) perturbation of the dual of an $\mathbb{F}$-represented frame matroid.*

Proof. Suppose M is a rank-($\leq t$) perturbation of an $\mathbb{F}$-represented matroid N, and let $r = r(M)$. Combining the previous two lemmas, we have $\varepsilon(M) \leq q^t(3r(N)) + \sum_{i=0}^{t-1} q^i$. Since M is a rank-($\leq t$) perturbation of N, we have $r(M) \leq r(N) + t$. Therefore, $\varepsilon(M) \leq 3q^t(r-t) + \sum_{i=0}^{t-1} q^i$. Since q and t are constant, this expression is less than $\binom{r+1}{2} = \varepsilon(M(K_{r+1}))$ for all sufficiently large r. ∎

Notation 1.4.10. Let $g(q,t)$ be the least value n such that for all $n' \geq n$, the complete graphic represented matroid $M(K_{n'})$ is not equal to any represented matroid that is a rank-($\leq t$) perturbation of the dual of a represented frame matroid over $\mathrm{GF}(q)$.

Lemma 1.4.9 can be restated as saying that $g(q,t)$ is finite for every prime power q and positive integer t.

1.5 This Dissertation

In Chapter 2, we construct a family of matroids that are arbitrarily high rank perturbations of graphic and of cographic matroids. The matroids in this family are vertically k-connected, but not k-connected. This shows that the connectivity condition in Hypothesis 1.2.2 cannot be weakened arbitrarily.

In Chapter 3, we introduce frame templates, our main objects of study. A template gives some specifics about what certain perturbations look like. The represented matroids constructed in this way are said to *conform* to the template. We give the definition of a frame template, as found in [7]. We also define certain types of frame templates and show how different frame templates are equivalent in various senses. Templates are used to study highly connected matroids. Therefore, if every matroid conforming to a template has a k'-separation for some $k' \leq k$, where k is a fixed positive integer, then that template need not be considered in any applications. We make this idea precise with the notion of *refined* templates.

Chapter 4 delves deeper into the study of templates. We define a notion of minors on the template level. Based on this minor relationship, we define a preorder on the set of templates over a fixed finite field and determine the set of minimal nontrivial templates with respect to this preorder. We also study, in significant depth, a specific type of template called a Y-template. These Y-templates are simpler than frame templates in general and are used often in the later chapters of the dissertation. We also show in Chapter 4 how templates can be used to determine the extremal functions of many classes of matroids.

The results in Chapters 5 and 6 are based on more detailed versions of Hypothesis 1.2.2. Chapter 5 applies the results of Chapters 3 and 4 to certain classes of binary matroids—those matroids representable over the field of two elements. A significant portion of Chapter 5 is devoted to the study of even-cycle and even-cut matroids. Both of these classes have hundreds of excluded minors, but we show that, in the case of highly connected matroids, three and two matroids suffice for the classes of even-cycle and even-cut matroids, respectively. We prove the following results, where $\mathrm{PG}(3,2)\backslash e$, L_{19}, L_{11}, and H_{12} are defined in Chapter 5.

Theorem 1.5.1. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that a k-connected binary matroid with at least $2k$ elements is an even-cycle matroid if and only if it contains no minor isomorphic to $\mathrm{PG}(3,2)\backslash e$, L_{19}, or L_{11}.*

Theorem 1.5.2. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that a k-connected binary matroid with at least $2k$ elements is an even-cut matroid if and only if it is contains no minor isomorphic to $M(K_6)$ or H_{12}^*.*

In Chapter 5, we also give the extremal function for the class of binary matroids of sufficiently large rank that have no minor isomorphic to $\mathrm{PG}(3,2)$. Moreover, we answer the question of which classes of binary matroids have the same extremal function as the class of graphic matroids. One of these classes is the class of 1-flowing matroids, which generalizes the max-flow min-cut property of graphs. We show that a conjecture that Seymour made in 1981 holds for highly connected matroids.

In Chapter 6, we study certain classes of quaternary matroids—those matroids representable over the field of four elements. In particular, we study the class of golden-mean matroids and some other classes closely related to it. Let $\mathcal{AC}_4$ denote the class of quaternary matroids representable over some field of each characteristic. Let $\mathcal{AF}_4$ denote the class of matroids representable over all fields of size at least

4, and let $\mathcal{SL}_4$ denote the class of quaternary matroids M for which there exists a prime power q' such that M is representable over all fields of size at least q'. We show that the extremal function for the class of golden-mean matroids, as well as for $\mathcal{AC}_4$, for $\mathcal{AF}_4$, and for $\mathcal{SL}_4$, is $\binom{r+3}{2}-5$. This verifies, for matroids of sufficiently large rank, a conjecture of Archer [1]. Moreover, we show the following, where $\mathcal{GM}$ denotes the class of golden-mean matroids.

Theorem 1.5.3. *Suppose Hypothesis 3.2.2 holds. There exists $k \in \mathbb{Z}_+$ such that every k-connected member of $\mathcal{AC}_4$ with at least $2k$ elements is contained in exactly one of $\mathcal{AF}_4$, $\mathcal{GM} - \mathcal{AF}_4$, and $\mathcal{SL} - \mathcal{AF}_4$ and such that every k-connected member of $\mathcal{SL}_4$ with at least $2k$ elements is representable over all fields of size at least 7.*

Chapter 2: A Problematic Family of Dyadic Matroids

Geelen, Gerards, and Whittle did not introduce Hypothesis 1.2.2 as we stated it. They announced (without proof) the following conjecture as a theorem.

Conjecture 2.0.1 ([7, Theorem 3.1])**.** *Let $\mathbb{F}$ be a finite field and let $\mathcal{M}$ be a proper minor-closed class of $\mathbb{F}$-represented matroids. Then there exist $k, t \in \mathbb{Z}_+$ such that each vertically k-connected member of $\mathcal{M}$ is a rank-($\leq t$) perturbation of an $\mathbb{F}$-represented matroid N, such that either*

(i) N is a represented frame matroid,

(ii) N^* is a represented frame matroid, or

(iii) N is confined to a proper subfield of $\mathbb{F}$.

The difference between Hypothesis 1.2.2 and Conjecture 2.0.1 lies in the notion of connectivity that was used. Unfortunately, vertical k-connectivity is insufficient for the conclusion to hold, which we will demonstrate in this chapter. Our examples arise only in very specific situations. For this reason, the proof of the Matroid Structure Theorem itself is not jeopardized. The reader whose main interest is in the later chapters can safely skip to Section 2.3, where we give two hypotheses, in addition to Hypothesis 1.2.2, that can serve as replacements for Conjecture 2.0.1.

2.1 Background

The matroids we will be working with in this chapter are dyadic. The following characterization of the dyadic matroids was shown by Whittle in [43].

Theorem 2.1.1. *A matroid is* dyadic *if and only if it is representable over both* GF(3) *and* GF(5)*.*

A matroid is *ternary* if it is GF(3)-representable. In this chapter, since ternary matroids are uniquely GF(3)-representable [4], we will not make any distinction between matroids and represented matroids. We also extend this convention to binary matroids, particularly complete graphic matroids, since every binary matroid that is representable over some field $\mathbb{F}$ is uniquely $\mathbb{F}$-representable.

In particular, we build a family of dyadic matroids that are vertically k-connected for any desired k, and not a bounded-rank perturbation of either a represented frame matroid or the dual of a represented frame matroid. The construction starts with a cyclically k-connected graph G, modifies it at a number of vertices that grows with $|V(G)|$, and dualizes the resulting matroid.

Since our construction makes use of highly cyclically connected graphs, the next two results allow us to specify some additional details about these graphs. The following lemma seems to be fairly well-known; however, we were unable to find an explicit proof in the literature. For the sake of completeness, we state the result and provide a proof, obtained by combining some older results.

Lemma 2.1.2. *For every positive integer k, there is a cyclically k-connected cubic graph.*

Proof. There is a cubic Cayley graph of girth $g \geq k$. (See, for example, Biggs [3] or Jajcay and Širáň [16, Theorem 2.1]. In particular, [16] contains a nice summary of various related results.) Since Cayley graphs are vertex-transitive, a result of Nedela and Škoviera [21, Theorem 17] states that such a graph has cyclic connectivity g. ■

Thomassen [38, Corollary 3.2] showed the following.

Theorem 2.1.3. *There is a function ξ such that a graph G with minimum degree at least 3 and girth at least $\xi(n)$ has a minor isomorphic to K_n.*

2.2 The Construction

Our construction involves repeated use of the generalized parallel connection of a matroid with copies of $M(K_5)$ over a copy of $M(K_4)$ represented in a specific way. The next two results specify that representation. Both results are easily checked, so we state them without proof.

Lemma 2.2.1. *The following matrix represents $M(K_5)$ over all fields of characteristic other than 2:*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Lemma 2.2.2. *The signed graph shown in Figure 2.1, with negative edges printed in bold, represents $M(K_4)$.*

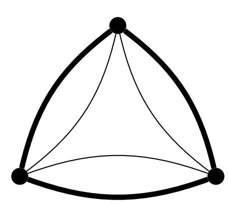

FIGURE 2.1. A Signed-Graphic Representation of $M(K_4)$

This representation of $M(K_4)$ has been encountered before, for example in [46, 14, 37].

Definition 2.2.3. Let G be a cubic graph, and $R \subseteq V(G)$. For each vertex v_i of R, perform the operation of altering the graph on the left in Figure 2.2 to become the signed graph on the right, with negative edges printed in bold. Let G' be the signed graph that results from performing this operation on every vertex in R. Note that G' contains $|R|$ copies of the signed-graphic representation of $M(K_4)$ described

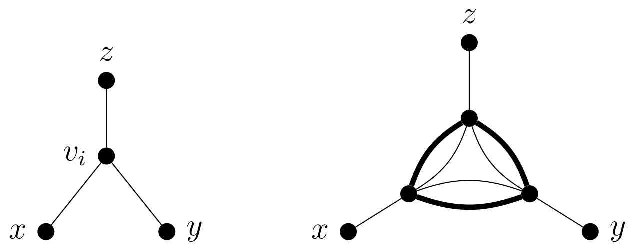

FIGURE 2.2. Changing G to G'

in Lemma 2.2.2. Let $X_1, X_2, \ldots, X_{|R|}$ be the edge sets of these representations of $M(K_4)$. For each X_i, take the generalized parallel connection[1] of $M(G')$ with a copy of $M(K_5)$ over X_i. Delete the X_i, and call the resulting matroid the *ornamentation* of (G, R), denoted by $Or(G, R)$.

Lemma 2.2.4. *For any cubic graph G, with $R \subseteq V(G)$, the ornamentation $Or(G, R)$ is dyadic and has $M(G)$ as a minor.*

Proof. It is well-known that signed-graphic matroids are dyadic (see, for example, [45, Lemma 8A.3]). The construction of $Or(G, R)$ involves generalized parallel connections of a signed graph with copies of $M(K_5)$ over a common representation of $M(K_4)$. Thus, a result of Mayhew, Whittle, and Van Zwam [20, Theorem 3.1] implies that $Or(G, R)$ is dyadic.

Note that $Or(G, R)$ is the result of taking $|R|$ copies of the submatrix $\left[\begin{array}{ccc}1 & 1 & 1\\ \hline & -I & \end{array}\right]$ of the signed incidence matrix of G, with columns indexed by the set $\{1_i, 2_i, 3_i\}$, and altering each of them to become a copy of the following matrix:

$$\begin{array}{c}
\begin{array}{ccccccc} 1_i & 2_i & 3_i & d_i & e_i & f_i & g_i \end{array}\\
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 1\\
-1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}.$$

One can check that in the vector matroid of the above submatrix, deleting g_i and contracting $\{d_i, e_i, f_i\}$ results in the vector matroid of $\left[\begin{array}{ccc}1 & 1 & 1\\ \hline & -I & \end{array}\right]$. Therefore, $M(G)$ is a minor of $Or(G, R)$. ∎

Definition 2.2.5. In the matrix given in the proof of the previous lemma, let $F_i = \{d_i, e_i, f_i, g_i\}$. We will call each F_i a *gadget*.

We are now ready to prove the main result of this chapter.

[1] Each X_i is a modular flat of $M(K_5)$, which is uniquely representable over any field. Therefore, these generalized parallel connections are well-defined.

Theorem 2.2.6. *For every $k, t \in \mathbb{Z}_+$, there exists a vertically k-connected dyadic matroid that is not a rank-($\leq t$) perturbation of either a frame matroid or the dual of a frame matroid.*

Proof. Let g and ξ be the functions given in Notation 1.4.10 and Theorem 2.1.3, respectively. We must define several constants that will be used throughout this proof. First, let $d > 3(2^{\lfloor \log_2(k) \rfloor + 1} - 1)$, and let $c \geq 2t + 20(3^{16t})$. We define

$$m = \max\{c(3^d + 3) + 7, g(3, 2t)\}.$$

Finally, let

$$h = \max\{k + 1, \xi(m)\}.$$

Let G be a cyclically h-connected cubic graph. Such a graph exists by Lemma 2.1.2. This implies that G has girth at least $h \geq \xi(m)$. By Theorem 2.1.3, G has a minor H isomorphic to K_m. Let C and D be the sets of edges such that $G/C \backslash D = H$. Each vertex of H is obtained by contracting all edges in a subtree of $G \backslash D$. Thus, there is a function $\phi : V(G) \to V(H)$ such that $\phi(w) = v$ if w is a vertex in the subtree of $G \backslash D$ that is contracted to result in v.

Claim 2.2.6.1. *There is a set $R = \{v_1, v_2, \ldots, v_c\} \subseteq V(G)$ of size c and $c + 1$ pairwise disjoint sets $\{a_0, b_0, c_0\}$,$\{a_1, b_1, c_1\}$,…,$\{a_c, b_c, c_c\} \subseteq V(H)$ that are also disjoint from $\phi(R)$ such that*

1. *the members of R are pairwise a distance of at least d from each other in G, and*

2. *for each integer i with $1 \leq i \leq c$, if T_i is the subtree of $G \backslash D$ that is contracted to obtain $\phi(v_i)$, then there is a set of vertices $\{a'_i, b'_i, c'_i\} \subseteq V(T_i)$ (possibly some or all of a'_i, b'_i, and c'_i are equal to v_i) and three internally disjoint subpaths of T_i from v_i to a'_i, b'_i, and c'_i such that a'_i, b'_i, and c'_i are neighbors in $G \backslash D$ of some vertex in $\phi^{-1}(a_i)$, $\phi^{-1}(b_i)$, and $\phi^{-1}(c_i)$, respectively.*

Proof. Suppose $\{v_1, a_1, b_1, c_1\}, \ldots, \{v_{k-1}, a_{k-1}, b_{k-1}, c_{k-1}\}$ and $\{a_0, b_0, c_0\}$ were chosen to satisfy (1) and (2), with k maximal. Also suppose, for a contradiction, that $k - 1 < c$. Since G is cubic, there are at most $\sum_{i=0}^{d-1} 3^i < 3^d$ vertices in G whose distance form some v_i is less than d. Thus, after choosing $\{v_1, \ldots v_{k-1}\} \subseteq V(G)$ and $\{a_0, b_0, c_0\}$,$\{a_1, b_1, c_1\}$, …, $\{a_{k-1}, b_{k-1}, c_{k-1}\} \subseteq V(H)$, there are at least $m - (k-1)(3^d + 3) - 3 > m - c(3^d + 3) - 3 \geq 4$ vertices w in $V(H) - (\{a_0, b_0, c_0\} \cup \{\phi(v_1), a_1, b_1, c_1\} \cup \ldots \cup \{\phi(v_1), a_1, b_1, c_1\})$ such that every vertex in $\phi^{-1}(w)$ is at a distance of at least d from each member of $\{v_1, \ldots, v_{k-1}\}$. (In the expression $m - (k-1)(3^d + 3) - 3$, the $+3$ comes from the sets $\{a_i, b_i, c_i\}$ for $i > 0$, and the -3 comes from $\{a_0, b_0, c_0\}$.) Choose one of these vertices w to be $\phi(v_k)$, and let three of the others be $\{a_k, b_k, c_k\}$.

Since each of a_k, b_k, and c_k is a neighbor of $\phi(v_k)$ in H, there must be vertices $\{a'_k, b'_k, c'_k\} \subseteq V(T_k)$ that are neighbors in $G \backslash D$ of some vertex in $\phi^{-1}(a_k)$, $\phi^{-1}(b_k)$, and $\phi^{-1}(c_k)$, respectively. If $a'_k = b'_k = c'_k$, then let $v_k = a'_k = b'_k = c'_k$. If two of

$\{a'_k, b'_k, c'_k\}$ are equal, say $a'_k = b'_k$, then let $v_k = a'_k = b'_k$. Since T_k is a tree, there must be a path in T_k that joins v_k to c'_k. Now suppose $a'_k \neq b'_k \neq c'_k$. Since T_k is a tree, there must be a path P in T_k from a'_k to b'_k. Similarly, there must be a path P' that joins c'_k to some vertex in P. Let v_k be the vertex where these two paths meet. In each of these cases, we have three subpaths of T_k that satisfy (2). Moreover, $R = \{v_1, \ldots, v_k\}$ also satisfies (1) since every vertex in T_k is at a distance of at least d from each member of $\{v_1, \ldots, v_{k-1}\}$. This contradicts the maximality of k and proves the claim. □

Claim 2.2.6.2. *Every circuit of $Or(G, R)$ contains either the edge set of a cycle of G or the edge set of a path in G between two vertices in R.*

Proof. Suppose for a contradiction that C is a circuit of $Or(G, R)$ that contains neither the edge set of a cycle of G nor the edge set of a path in G joining vertices in R. Then $C \cap E(G)$ must consist of the edge sets of vertex-disjoint subtrees $S_1, S_2, \ldots, S_n$ of G such that no S_i contains more than one vertex in R. Thus, $C \subseteq (\cup_{i=1}^c F_i) \cup (\cup_{i=1}^n E(S_i))$. However, we will show by induction on $|\cup_{i=1}^n E(S_i)|$, that $(\cup_{i=1}^c F_i) \cup (\cup_{i=1}^n E(S_i))$ is an independent set. Since no pair of gadgets is represented by submatrices whose sets of rows intersect, $\cup_{i=1}^c F_i$ is an independent set in $Or(G, R)$. Thus, the result holds when $|\cup_{i=1}^n E(S_i)| = 0$. Now, consider $(\cup_{i=1}^c F_i) \cup (\cup_{i=1}^n E(S_i))$ where $|\cup_{i=1}^n E(S_i)| = k > 0$ and the result holds for $|\cup_{i=1}^n E(S_i)| < k$. Delete a pendant edge e in some S_i. By the induction hypothesis, $(\cup_{i=1}^c F_i) \cup (\cup_{i=1}^n E(S_i)) - \{e\}$ is an independent set in $Or(G, R)$. Since e is a pendant edge in some S_i, it must be a coloop in $(Or(G, R))|((\cup_{i=1}^c F_i) \cup (\cup_{i=1}^n E(S_i)))$. Thus, $(\cup_{i=1}^c F_i) \cup (\cup_{i=1}^n E(S_i)$ is an independent set in $Or(G, R)$. By contradiction, this proves the claim. □

Let M be the dual matroid of $Or(G, R)$, and let λ_M and λ_G be the connectivity functions of M and $M(G)$, respectively. Then, by duality, $\lambda_{Or(G,R)} = \lambda_M$.

Claim 2.2.6.3. *The matroid $Or(G, R)$ is cyclically k-connected.*

Proof. Suppose for a contradiction that (X, Y) is a cyclic k'-separation of $Or(G, R)$, where $k' < k$. Let $A \cup B = E(G)$, with $A \subseteq X$ and $B \subseteq Y$. Since $M(G)$ has cyclic connectivity $k > k'$, it has no cyclic k'-separation. Therefore, one of A or B, say A, has no cycles. However, since (X, Y) is a cyclic k'-separation, X and Y each contain a circuit of $Or(G, R)$. Since A, and therefore X, contain no edge set of a cycle of G, we see from Claim 2.2.6.2 that X, and therefore A, contain the edge set of a path in G joining vertices in R. By Claim 2.2.6.1, this path has length at least d. This path must be contained in some component of $G[A]$ with edge set A_1. If a cubic graph either is disconnected or has a cut vertex, then both sides of the separation must contain cycles. Therefore, G is a connected graph with no cut vertices. Let $B_1 = E(G) - A_1$, and let $A_2 = A - A_1$. Suppose $G[B_1]$ is not connected. Then, since $G[A_1]$ is a tree, there is a unique path in $G[A_1]$ from one component of $G[B_1]$ to another. This implies that G has a cut vertex. Thus, we deduce that $G[B_1]$ is connected.

Let r_G be the rank function of $M(G)$. Since B_1 is the disjoint union of B and A_2, we have $r_G(B_1) \leq r_G(B) + r_G(A_2)$. Moreover, since $G[A_1]$ is a component of $G[A]$, we have $r_G(A_1) = r_G(A) - r_G(A_2)$. Therefore, $\lambda_G(A_1) \leq r_G(A) - r_G(A_2) + r_G(B) + r_G(A_2) - r_G(E(G)) = \lambda_G(A)$. Let W be the set of vertices of the vertex boundary between A_1 and B_1. We have $\lambda_G(A_1) = r_G(A_1) + r_G(B_1) - r_G(E(G)) = |V(G[A_1])| - 1 + |V(G[B_1])| - 1 - (|V(G)| - 1) = |W| - 1$. Thus, we have $|W| - 1 = \lambda_G(A_1) \leq \lambda_G(A) \leq \lambda_M(X) < k'$. Therefore, $|W| < k' + 1$.

Note that, since G is cubic and $G[A_1]$ contains no cycle, $G[A_1]$ is a cubic tree whose set of leaves is W. We now claim that no vertex of $G[A_1]$ is at a distance greater than $\lfloor \log_2(k') \rfloor + 1$ from W. Suppose for a contradiction that v is such a vertex. Therefore, there are $3(2^{\lfloor \log_2(k') \rfloor + 1}) > 3(2^{\log_2(k')}) = 3h$ vertices at distance $\lfloor \log_2(k') \rfloor + 2$ from v in $G[A_1]$. This implies that $|W| > 3k'$, contradicting the facts that $|W| < k' + 1$ and k' is a positive integer.

Therefore, each vertex of $G[A_1]$ is a distance of at most $\lfloor \log_2(k') \rfloor + 1$ from W. Thus, since G is cubic, an upper bound for $|A_1|$ is $3(\sum_{i=0}^{\lfloor \log_2(k') \rfloor} 2^i) = 3(2^{\lfloor \log_2(k') \rfloor + 1} - 1) \leq 3(2^{\lfloor \log_2(k) \rfloor + 1} - 1) < d$, contradicting the fact that $G[A_1]$ must have a path of length at least d. Thus, $Or(G, R)$ has no cyclic k'-separation for $k' < k$ and is therefore cyclically k-connected. □

For each $v_i \in R$, let L_i consist of the three edges in H that join $\phi(v_i)$ to the vertices in $\{a_i, b_i, c_i\}$. Let D' consist of all the edges in H incident with a vertex in $\phi(R)$ other than the edges in some L_i.

Claim 2.2.6.4. *Consider c copies of the submatrix $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$ of the signed incidence matrix of K_{m-c}, where each of these submatrices has rows indexed by some $\{a_i, b_i, c_i\}$. Then $Or(G, R)$ has a minor N that is the vector matroid of the matrix obtained from the signed incidence matrix of K_{m-c} by altering each of these submatrices to become the following matrix, where the bottom row is a new row added to the original matrix. Here, $F_i = \{d_i, e_i, f_i, g_i\}$ is a gadget.*

$$
\begin{array}{c}
\\ a_i \\ b_i \\ c_i \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & & d_i & e_i & f_i & g_i \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
-1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & -1 & -1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

Proof. Recall that C and D are the sets of edges such that $G/C \backslash D = H \cong K_m$. Then $N = (Or(G, R))/C \backslash (D \cup D')/\cup_{i=1}^{c} L_i$. Informally, N is the result of "gluing" each F_i onto the set $\{a_i, b_i, c_i\}$ of vertices in K_{m-c}. □

Call the resulting matrix J so that $N = M(J)$. Note that J has m rows and $r(N) = m - 1$. Let N^+ be a rank-$(\leq 2t)$ perturbation of N.

Claim 2.2.6.5. *For some $s \le 2t$, there are vectors $w_1'', \dots, w_s''$ and a submatrix J' of J such that N^+ has a minor isomorphic to the vector matroid of*

$$\begin{array}{|c|}
\hline
w_1'' \\
\vdots \\
w_s'' \\
\hline
J' \\
\hline
\end{array}$$

and such that J' contains at least $20(3^{16t})$ copies of the submatrix

$$\begin{array}{c}
\\ a_i \\ b_i \\ c_i \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccc} d_i & e_i & f_i & g_i \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\end{array}$$

that represents a gadget.

Proof. By Remark 1.4.4, N^+ is the result of contracting C' from the vector matroid of the matrix below, where Δ is some arbitrary ternary matrix.

$$\begin{array}{|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{C'} \\
\hline
w_1 & \\
\vdots & -I \\
w_{2t} & \\
\hline
J & \\
\cline{1-1}
 & \Delta \\
0 & \\
\hline
\end{array}$$

Let V_Δ be the set of row indices of a basis for the rowspace of Δ. Delete from N all elements represented by columns with nonzero entries in V_Δ, along with all gadgets containing such an element. This is equivalent to deleting vertices from the complete graph K_{m-c} that was used to construct N, as well as any gadgets glued onto these vertices. Thus, we are still left with a complete graph with gadgets glued onto it. Moreover, since $|V_\Delta| \le 2t$, we have at least $c - 2t \ge 20(3^{16t})$ gadgets remaining. Since V_Δ is a basis for the rowspace of Δ, we may perform row operations to obtain the following matrix, where J' is a submatrix of J, where each w_i' is a coordinate projection of w_i, and where $\Delta' = \Delta[V_\Delta, C']$.

$$
\begin{array}{c|c|}
 & C' \\
\hline
\begin{array}{c} w'_1 \\ \vdots \\ w'_{2t} \end{array} & -I \\
\hline
J' & 0 \\
\hline
0 & \Delta' \\
\hline
\end{array}
$$

For each element of V_Δ, we contract one element of C', pivoting on an entry in the row of Δ'. We obtain the following matrix.

$$
\begin{array}{c|c|}
 & C'' \\
\hline
\begin{array}{c} w'_1 \\ \vdots \\ w'_{2t} \end{array} & Q \\
\hline
J' & 0 \\
\hline
\end{array}
$$

By contracting C'', we obtain the desired matrix, with $s = 2t - |C''|$. $\square$

Claim 2.2.6.6. *The matrix J' has a submatrix J'', with the same number of rows as J', such that $M(J'')$ is represented by Figure 2.3, where each shaded triangle represents a gadget F_i with vertices a_i, b_i, and c_i positioned at the top, left, and bottom respectively.*

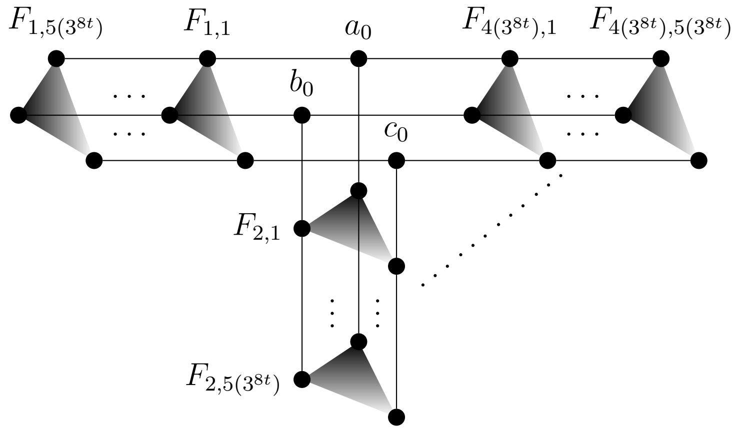

FIGURE 2.3. $M(J'')$

Proof. Partition the set of gadgets into $4(3^{8t})$ subsets $\mathcal{F}_1, \ldots, \mathcal{F}_{4(3^{8t})}$, each of size at least $5(3^{8t})$. This is possible since $20(3^{16t}) = (5)(3^{8t})(4)(3^{8t})$. Let the j-th gadget in $\mathcal{F}_i$ be $F_{i,j} = \{d_{i,j}, e_{i,j}, f_{i,j}, g_{i,j}\}$, and let it be glued onto the vertices $\{a_{i,j}, b_{i,j}, c_{i,j}\}$. Consider the subtree of K_{m-c} shown in Figure 2.4, as well as the two isomorphic

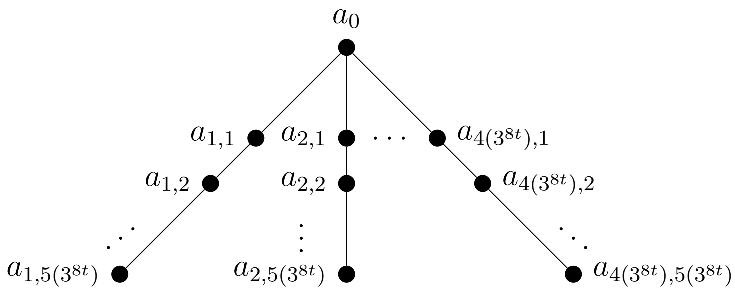

FIGURE 2.4. T_a

subtrees where the $a_{i,j}$ are replaced by $b_{i,j}$ and $c_{i,j}$, respectively. Call these trees T_a, T_b, and T_c, respectively.

Consider the submatrix J'' of J' consisting of the columns indexed by the union of $E(T_a)$, $E(T_b)$, and $E(T_c)$ with the union F of all of the gadgets. One can see that $M(J'')$ can be represented by Figure 2.3. □

Claim 2.2.6.7. *There are ternary matrices U and J''' such that $M(J''')$ can be represented by Figure 2.5 and such that N^+ has a minor N' represented by the following matrix.*

$$\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{F_1} & \multicolumn{1}{c}{F_2} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{F_8} \\ \hline
0 & U & U & \cdots & U \\ \hline
\multicolumn{5}{|c|}{J'''} \\ \hline
\end{array}$$

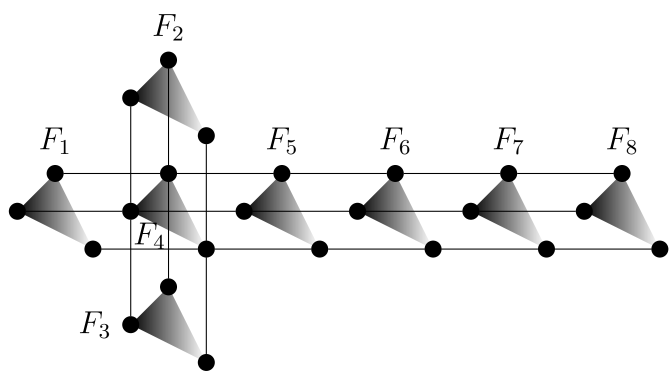

FIGURE 2.5. $M(J''')$

Proof. Let $W = \begin{bmatrix} w''_1 \\ \vdots \\ w''_s \end{bmatrix}$. Since $E(T_a) \cup E(T_b) \cup E(T_c)$ is an independent set in $M(J'')$, we may perform row operations so that the portion of W with columns indexed by $E(T_a) \cup E(T_b) \cup E(T_c)$ becomes the zero matrix. Thus, we have the following matrix.

$E(T_a)$ $E(T_b)$ $E(T_c)$	F
0	W'
J''	

The portion of W' whose columns are indexed by the elements of a gadget is an $s \times 4$ ternary matrix; therefore, there are $3^{4s} \leq 3^{8t}$ possible such matrices. Since each $\mathcal{F}_i$ contains at least $5(3^{8t})$ gadgets, the pigeonhole principle implies that each $\mathcal{F}_i$ contains five gadgets whose corresponding submatrices of W' are equal. Again by the pigeonhole principle, since there are $4(3^{8t})$ sets $\mathcal{F}_i$, there is a set of four $\mathcal{F}_i$ such that each contains at least five gadgets such that all 20 of the gadgets correspond to equal submatrices of W'.

Delete all of the other $c - 2t - 20$ gadgets. All of the remaining gadgets come from four $\mathcal{F}_i$ which we can relabel as $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$. In addition, delete all but one gadget from each of $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$, cosimplify the resulting matroid, and contract the remaining edges incident with either a_0, b_0, or c_0. In the resulting matroid, there are eight remaining gadgets which we relabel as $F_1, F_2, \ldots, F_8$. The elements of each F_i we relabel as $\{d_i, e_i, f_i, g_i\}$, and we relabel the vertices onto which F_i is glued as a_i, b_i, and c_i. This matroid is the desired matroid N'. In Figure 2.5, again each shaded triangle represents a gadget F_i with vertices a_i, b_i, and c_i positioned at the top, left, and bottom respectively. $\square$

Claim 2.2.6.8. *Regardless of U, the matroid N' is not a frame matroid.*

Proof. Let P consist of all edges joining a gadget F_i to a gadget F_{i+1}, for $4 \leq i \leq 7$, and for $1 \leq i \leq 3$, let α_i, β_i, and γ_i be the edges that join a_i, b_i, and c_i to a_4, b_4, and c_4, respectively. Now let N'' be the simplification of

$$N'/P/\{\alpha_1, \beta_2, \gamma_3\}/(\cup_{i=1}^{i=3}\{e_i, f_i, g_i\})/\{d_5, e_6, f_7, g_8\}.$$

In the case where U is the zero matrix, N'' is the generalized parallel connection of $M(K_5)$ with the ternary Dowling geometry of rank 3; that is, N'' is the vector matroid of the following matrix, where the last six columns come from the gadgets $F_5, \ldots, F_8$.

$$\begin{array}{c}
\begin{matrix} d_1 & d_2 & d_3 & d_4 & e_4 & f_4 & g_4 & & & & & & \end{matrix} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

In Secion A.1, we show how the mathematics software system SageMath was used to show that, regardless of U, the matroid N'', and therefore N', are not signed-graphic matroids. The computations were carried out in Version 8.0 of SageMath [34], in particular making use of the *matroids* component [28]. We used the CoCalc (formerly SageMathCloud) online interface.

Indeed, since U has four columns, its rank is at most 4. Therefore, we may assume that U has at most four rows. For $M(U)$, there are 16 possible bases—one of size 0, four of size 1, six of size 2, four of size 3, and one of size 4. For each of these bases, we checked all possible matrices U where the basis indexed an identity matrix, unless the resulting matroid $M(U)$ contained a basis that was already checked. In each case, N'' was found not to be signed-graphic. A ternary matroid is a frame matroid if and only if it is a signed-graphic matroid. Therefore, N' is not a frame matroid. □

Recall that $M^* = Or(G, R)$.

Claim 2.2.6.9. *The matroid M is not a rank-($\leq t$) perturbation of the dual of a frame matroid.*

Proof. Suppose otherwise. Then by Lemma 1.4.6, $Or(G, R)$ is a rank-($\leq 2t$) perturbation of a frame matroid. The class of matroids that are rank-($\leq 2t$) perturbations of a frame matroid is minor-closed. Therefore, by Claim 2.2.6.4, N is a rank-($\leq 2t$) perturbation of a frame matroid. However, by Claims 2.2.6.7 and 2.2.6.8, this is impossible. □

Claim 2.2.6.10. *The matroid M is not a rank-($\leq t$) perturbation of a frame matroid.*

Proof. Suppose otherwise. Then by Lemma 1.4.6, $Or(G, R)$ is a rank-($\leq 2t$) perturbation of the dual of a frame matroid. Recall that, by Theorem 2.1.3, G contains a minor isomorphic to K_m. Therefore, by Lemma 2.2.4, $Or(G, R)$ has a minor isomorphic to $M(K_m)$. But, since $m \geq g(3, 2t)$, Lemma 1.4.9 implies that $M(K_m)$, and therefore $Or(G, R)$, are not rank-($\leq 2t$) perturbations of the dual of a frame matroid. □

By Lemma 2.2.4, $Or(G, R)$, is dyadic. Since the class of dyadic matroids is closed under duality, M is dyadic also. By Claim 2.2.6.3 and duality, M is vertically k-connected. Claims 2.2.6.9 and 2.2.6.10 show that M is not a rank-($\leq t$) perturbation of either a frame matroid or the dual of a frame matroid. This completes the proof of the theorem. ■

Corollary 2.2.7. *The family of matroids given in Theorem 2.2.6 is a counterexample to Conjecture 2.0.1.*

Proof. By Theorem 2.2.6, for every $k, t \in \mathbb{Z}_+$, there exists a vertically k-connected dyadic matroid that is not a rank-($\leq t$) perturbation of either a frame matroid or the dual of a frame matroid. Thus, neither (i) nor (ii) of Conjecture 2.0.1 is satisfied. Moreover, since the matroids given by Theorem 2.2.6 are dyadic, they are representable over GF(3) which has no proper subfield. Therefore, if $\mathbb{F} = \mathrm{GF}(3)$ (or any prime field of odd order, for that matter), then the matroids given by Theorem 2.2.6 do not satisfy (iii) of Conjecture 2.0.1 either. ■

Remark 2.2.8. Our construction relies heavily on a non-standard frame matroid representation of $M(K_4)$, and involves a notion of 4-sums. Each gadget is 4-separating in our construction. The following result by Zaslavsky [46] shows that 5-sums and higher cannot be encountered in an analogous way.

Theorem 2.2.9 ([46, Proposition 5A])**.** *Let Ω be a biased graph such that the frame matroid of Ω is isomorphic to $M(K_m)$ for $m \geq 5$. Then Ω is isomorphic to either $(K_m, \emptyset)$ or Φ'_{m-1}, where the latter is the biased graph obtained by adding an edge e in parallel with an edge of K_m, taking the unbalanced cycles to be the collection of cycles through e, and contracting e in the resulting biased graph.*

This makes us cautiously optimistic that our construction cannot be generalized to have "gadgets" with arbitrary connectivity.

We believe that the subfield case, as stated by Geelen, Gerards, and Whittle [7], remains true.

2.3 Updated Conjectures

We offer several updated conjectures to replace Conjecture 2.0.1. We will state them as hypotheses. First, we restate Hypothesis 1.2.2, which replaces the requirement of vertical connectivity with the stronger requirement of Tutte connectivity.

Hypothesis 2.3.1. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{M}$ be a proper minor-closed class of $\mathbb{F}$-represented matroids. There exist constants $k, t \in \mathbb{Z}_+$ such that each k-connected member of $\mathcal{M}$ is a rank-$(\leq t)$ perturbation of an $\mathbb{F}$-represented matroid N, such that either*

1. *N is a represented frame matroid,*
2. *N^* is a represented frame matroid, or*
3. *N is confined to a proper subfield of $\mathbb{F}$.*

Taken together, the next two hypotheses revise Conjecture 2.0.1 by pairing the condition of vertical connectivity with its natural match of having a large clique minor and by pairing the dual condition of cyclic connectivity with the property of having a large coclique minor.

Hypothesis 2.3.2. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{M}$ be a proper minor-closed class of $\mathbb{F}$-represented matroids. There exist constants $k, t, n \in \mathbb{Z}_+$ such that each vertically k-connected member of $\mathcal{M}$ containing a minor isomorphic to $M(K_n)$ is a rank-$(\leq t)$ perturbation of an $\mathbb{F}$-represented matroid N, such that either*

1. *N is a represented frame matroid or*
2. *N is confined to a proper subfield of $\mathbb{F}$.*

Hypothesis 2.3.3. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{M}$ be a proper minor-closed class of $\mathbb{F}$-represented matroids. There exist constants $k, t, n \in \mathbb{Z}_+$ such that each cyclically k-connected member of $\mathcal{M}$ containing a minor isomorphic to $M^*(K_n)$ is a rank-$(\leq t)$ perturbation of an $\mathbb{F}$-represented matroid N, such that either*

1. *N^* is a represented frame matroid or*
2. *N is confined to a proper subfield of $\mathbb{F}$.*

Chapter 3: Frame Templates

Conjecture 2.0.1 was actually a simplified version of a much more detailed conjecture (which was also stated in [7] as a theorem without proof). In order to state this more detailed conjecture, Geelen, Gerards, and Whittle introduced the notions of *subfield templates* and *frame templates* in [7]. A template is a concise description of certain perturbations of represented matroids. As we will see in Chapters 5 and 6, the study of templates will lead to many applications in the study of the highly connected members of minor-closed classes of representable matroids.

In Section 3.1, we will recall several definitions concerning frame templates which can essentially be found in [7] as well as [15] and [22]. In Section 3.2, we give more detailed versions of Hypotheses 2.3.1–2.3.3 in the language of frame templates. In Section 3.3, we introduce various notions of template equivalence and show that, for all practical purposes of frame templates, it suffices to consider only templates of a certain type—called refined templates.

3.1 Definitions

Let $\mathbb{F}^\times$ denote the multiplicative group of $\mathbb{F}$, and let Γ be a subgroup of $\mathbb{F}^\times$. A Γ-frame matrix is a frame matrix A such that:

- Each column of A with a nonzero entry contains a 1.
- If a column of A has a second nonzero entry, then that entry is $-\gamma$ for some $\gamma \in \Gamma$.

If $\Gamma = \{1\}$, then the vector matroid of a Γ-frame matrix is a graphic matroid. For this reason, we will call the columns of a $\{1\}$-frame matrix *graphic columns*.

A *frame template* over $\mathbb{F}$ is a tuple $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ such that the following hold[1]:

(i) Γ is a subgroup of $\mathbb{F}^\times$.

(ii) C, X, Y_0 and Y_1 are disjoint finite sets.

(iii) $A_1 \in \mathbb{F}^{X \times (C \cup Y_0 \cup Y_1)}$.

(iv) Λ is a subgroup of the additive group of $\mathbb{F}^X$ and is closed under scaling by elements of Γ.

(v) Δ is a subgroup of the additive group of $\mathbb{F}^{C \cup Y_0 \cup Y_1}$ and is closed under scaling by elements of Γ.

Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template. Let B and E be finite sets, and let $A' \in \mathbb{F}^{B \times E}$. We say that A' *respects* Φ if the following hold:

[1]The authors of [7] divided our set X into two separate sets which they called X and D. Their set X can be absorbed into Y_0, therefore we omit it.

(i) $X \subseteq B$ and $C, Y_0, Y_1 \subseteq E$.

(ii) $A'[X, C \cup Y_0 \cup Y_1] = A_1$.

(iii) There exists a set $Z \subseteq E-(C \cup Y_0 \cup Y_1)$ such that $A'[X, Z] = 0$, each column of $A'[B-X, Z]$ is a unit vector, and $A'[B-X, E-(C \cup Y_0 \cup Y_1 \cup Z)]$ is a Γ-frame matrix.

(iv) Each column of $A'[X, E-(C \cup Y_0 \cup Y_1 \cup Z)]$ is contained in Λ.

(v) Each row of $A'[B-X, C \cup Y_0 \cup Y_1]$ is contained in Δ.

The structure of A' is shown below.

		Z	Y_0 Y_1 C
X	columns from Λ	0	A_1
	Γ-frame matrix	unit columns	rows from Δ

Now, suppose that A' respects Φ and that $A \in \mathbb{F}^{B \times E}$ satisfies the following conditions:

(i) $A[B, E-Z] = A'[B, E-Z]$

(ii) For each $i \in Z$ there exists $j \in Y_1$ such that the i-th column of A is the sum of the i-th and the j-th columns of A'.

We say that such a matrix A *conforms* to Φ.

Let M be an $\mathbb{F}$-represented matroid. We say that M *conforms* to Φ if there is a matrix A conforming to Φ such that M is isomorphic to $M(A)/C\backslash Y_1$. We denote by $\mathcal{M}(\Phi)$ the set of $\mathbb{F}$-represented matroids that conform to Φ.

Although the term *coconform* does not appear in [7], we define it in the following obvious way.

Definition 3.1.1. A represented matroid M *coconforms* to a template Φ if its dual M^* conforms to Φ. We denote by $\mathcal{M}^*(\Phi)$ the set of represented matroids that coconform to Φ.

To simplify the proofs in this dissertation, it will be helpful to expand the concept of conforming slightly.

Definition 3.1.2. Let A' be a matrix that respects Φ, as defined above, except that we allow columns of $A'[B-X, Z]$ to be either unit columns or zero columns. Let A be a matrix that is constructed from A' as described above. Thus, $A[B, E-Z] = A'[B, E-Z]$, and for each $i \in Z$ there exists $j \in Y_1$ such that the i-th column of A is the sum of the i-th and the j-th columns of A'. Let M be isomorphic to $M(A)/C\backslash Y_1$. We say that A and M *virtually conform* to Φ and that A' *virtually respects* Φ. If M^* virtually conforms to Φ, we say that M *virtually coconforms* to Φ.

We will denote the set of $\mathbb{F}$-represented matroids that virtually conform to Φ by $\mathcal{M}_v(\Phi)$ and the set of $\mathbb{F}$-represented matroids that virtually coconform to Φ by $\mathcal{M}_v^*(\Phi)$.

3.2 Updated Conjectures

The "template version" of Conjecture 2.0.1 follows as Conjecture 3.2.1. Since a template gives a description of certain types of perturbations, Conjecture 3.2.1 is false, just as Conjecture 2.0.1.

Conjecture 3.2.1 ([7, Theorem 4.2])**.** *Let $\mathbb{F}$ be a finite field, let m be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-represented matroids. Then there exist $k \in \mathbb{Z}_+$ and frame templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ such that*

- *$\mathcal{M}$ contains each of the classes $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$,*
- *$\mathcal{M}$ contains the duals of the represented matroids in each of the classes $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$, and*
- *if M is a simple vertically k-connected member of $\mathcal{M}$ and $\widetilde{M}$ has no $\mathrm{PG}(m-1, \mathbb{F}_p)$-minor, then either M is a member of at least one of the classes $\mathcal{M}(\Phi_1)$, $\dots$, $\mathcal{M}(\Phi_s)$, or M^* is a member of at least one of the classes $\mathcal{M}(\Psi_1)$, $\dots$, $\mathcal{M}(\Psi_t)$.*

To replace Conjecture 3.2.1, we now give "template versions" of the hypotheses given in Section 2.3. The template version of Hypothesis 2.3.1 follows.

Hypothesis 3.2.2. *Let $\mathbb{F}$ be a finite field, let m be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-represented matroids. Then there exist $k \in \mathbb{Z}_+$ and frame templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ such that*

1. *$\mathcal{M}$ contains each of the classes $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$,*
2. *$\mathcal{M}$ contains the duals of the represented matroids in each of the classes $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$, and*
3. *if M is a simple k-connected member of $\mathcal{M}$ with at least $2k$ elements and $\widetilde{M}$ has no $\mathrm{PG}(m-1, \mathbb{F}_p)$-minor, then either M is a member of at least one of the classes $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$, or M^* is a member of at least one of the classes $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$.*

Now we give the template version of Hypotheses 2.3.2 and 2.3.3.

Hypothesis 3.2.3. *Let $\mathbb{F}$ be a finite field, let m be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-represented matroids. Then there exist $k, n \in \mathbb{Z}_+$ and frame templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ such that*

1. *$\mathcal{M}$ contains each of the classes $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$,*

2. $\mathcal{M}$ *contains the duals of the represented matroids in each of the classes* $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$*,*

3. *if* M *is a simple vertically* k*-connected member of* $\mathcal{M}$ *with an* $M(K_n)$*-minor but no* $\mathrm{PG}(m-1, \mathbb{F}_p)$*-minor, then* M *is a member of at least one of the classes* $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$*, and*

4. *if* M *is a cosimple cyclically* k*-connected member of* $\mathcal{M}$ *with an* $M^*(K_n)$*-minor but no* $\mathrm{PG}(m-1, \mathbb{F}_p)$*-minor, then* M^* *is a member of at least one of the classes* $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$*.*

3.3 Template Equivalence and Refinement

Since a template is a rich structure, it may happen that $\mathcal{M}(\Phi) = \mathcal{M}(\Phi')$ even though Φ and Φ' look very different. In this section, we build some tools to deal with such situations.

Definition 3.3.1. Let Φ and Φ' be frame templates over the fields $\mathbb{F}$ and $\mathbb{F}'$, respectively. The pair Φ, Φ' are *strongly equivalent* if $\mathbb{F} = \mathbb{F}'$ and if $\mathcal{M}(\Phi) = \mathcal{M}(\Phi')$. The pair Φ, Φ' are *minor equivalent* if $\mathbb{F} = \mathbb{F}'$ and if the closures of $\mathcal{M}(\Phi)$ and $\mathcal{M}(\Phi')$ under the taking of minors are equal. If there is a one-to-one correspondence between $\mathbb{F}$-represented matroids $M \in \mathcal{M}(\Phi)$ and $\mathbb{F}'$-represented matroids $N \in \mathcal{M}(\Phi')$ with $\widetilde{M} = \widetilde{N}$, then Φ and Φ' are *algebraically equivalent*.

What we call strong equivalence, Nelson and Walsh [22] simply call equivalence. We will reserve the term *equivalent* for a different notion that we will introduce in the next chapter. Nelson and Walsh gave Definitions 3.3.2 and 3.3.3 and proved Lemmas 3.3.4 and 3.3.5 below.

Definition 3.3.2. A frame template $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is *reduced* if there is a partition (X_0, X_1) of X such that

- $\Delta = \Gamma(\mathbb{F}_p^C \times \Delta')$ for some additive subgroup Δ' of $\mathbb{F}^{Y_0 \cup Y_1}$,
- $\mathbb{F}_p^{X_0} \subseteq \Lambda | X_0$ while $\Lambda | X_1 = \{0\}$ and $A_1[X_1, C] = 0$, and
- the rows of $A_1[X_1, C \cup Y_0 \cup Y_1]$ form a basis for a subspace whose additive group is skew to Δ.

We will refer to the partition $X = X_0 \cup X_1$ given in Definition 3.3.2 as the *reduction partition* of Φ.

Definition 3.3.3. A frame template $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ over $\mathbb{F}$ is Y-*reduced* if $\Delta | C = \Gamma(\mathbb{F}_p^C)$ and $\Delta | (Y_0 \cup Y_1) = \{0\}$, and there is a partition (X_0, X_1) of X for which $\mathbb{F}_p^{X_0} \subseteq \Lambda | X_0$ and $\Lambda | X_1 = \{0\}$. We will call the partition $X = X_0 \cup X_1$ the *reduction partition* of Φ.

Lemma 3.3.4 ([22, Lemma 5.6])**.** *Every frame template is strongly equivalent to a reduced frame template.*

Lemma 3.3.5 ([22, Lemma 5.5])**.** *Every frame template is strongly equivalent to a Y-reduced frame template.*

We introduce the following definition.

Definition 3.3.6. A frame template $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is *refined* if it is reduced, with reduction partition $X = X_0 \cup X_1$, and if Y_1 spans the matroid $M(A_1[X_1, Y_0 \cup Y_1])$.

There is also a template form called *standard form* that we will introduce in the next chapter.

In the remainder of this section, we prove a result that will be useful for later chapters of this dissertation and will also be of interest for future work. We wish to show that, for the purposes of using the Hypotheses given in Sections 2.3 and 3.2, only refined frame templates must be considered.

Lemma 3.3.7. *Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a reduced frame template that is not refined. If $M \in \mathcal{M}(\Phi)$, then $E(M) - Y_0$ is not spanning in M.*

Proof. Let A be the matrix conforming to Φ such that $M = M(A)/C\backslash Y_1$. Since Φ is not refined, Y_1 does not span $M(A[Y_0 \cup Y_1])$. Therefore, Y_0 contains a cocircuit in $M(A[Y_0 \cup Y_1])$. In fact, since the definition of reduced implies that $A[X_1, E - (Y_0 \cup Y_1 \cup Z)]$ is the zero matrix, and since every column of $A[X_1, Z]$ is a copy of a column of $A[X_1, Y_1]$, we see that Y_0 contains a cocircuit in $M(A)$. This implies that Y_0 also contains a cocircuit in $M = M(A)/C\backslash Y_1$. Thus, $E(M) - Y_0$ is not spanning in M. ■

Theorem 3.3.8. *If Hypothesis 3.2.2 holds for a class $\mathcal{M}$, then the constant k and the templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ can be chosen so that the templates are refined. Moreover, if Hypothesis 3.2.3 holds for a class $\mathcal{M}$, then the constants k, n, and the templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ can be chosen so that the templates are refined.*

Proof. Suppose that Hypothesis 3.2.2 holds, and let $\Phi \in \{\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t\}$. By Lemma 3.3.4, we may assume that Φ is reduced with reduction partition $X = X_0 \cup X_1$. Suppose for a contradiction that Φ is not refined. Choose $k \geq |Y_0|$. If M is a k-connected represented matroid conforming to Φ, then Lemma 3.3.7 implies that $\lambda_M(Y_0) < r_M(Y_0) \leq |Y_0|$. Therefore, by k-connectivity, we must have $|E(M) - Y_0| < |Y_0|$. Thus, since $2k \geq 2|Y_0|$, we obtain a contradiction and conclude that the constant k and the templates $\Phi_1, \dots, \Phi_s$ can be chosen so that the templates are refined. Moreover, since k-connectivity is closed under duality, the templates $\Psi_1, \dots, \Psi_t$ can be chosen to be refined as well.

Now suppose Hypothesis 3.2.3 holds, and choose $k \geq |Y_0|$. Let M be a simple, vertically k-connected member of some minor-closed class $\mathcal{M}$, and let M have an $M(K_n)$-minor but no $\mathrm{PG}(m-1, \mathbb{F}_p)$-minor for some positive integer m. Part (3) of Hypothesis 3.2.3 implies that M conforms to a template $\Phi \in \{\Phi_1, \dots, \Phi_s\}$. By Lemma 3.3.4, we may assume that Φ is reduced with reduction partition $X = X_0 \cup X_1$. Suppose for a contradiction that Φ is not refined. By Lemma 3.3.7, $E(M) - Y_0$ is not spanning in M. This also implies that $\lambda_M(Y_0) < r_M(Y_0) \leq |Y_0|$.

By vertical k-connectivity, Y_0 is spanning in M. Thus, M is an $\mathbb{F}$-represented matroid of rank $r_M(Y_0) \leq |Y_0|$. Since M is simple, we must have $|E(M)| \leq \frac{|\mathbb{F}|^{|Y_0|}-1}{|\mathbb{F}|-1}$; therefore, we set n sufficiently large so that $|E(K_n)| = \binom{n}{2} > \frac{|\mathbb{F}|^{|Y_0|}-1}{|\mathbb{F}|-1}$. Thus, we see that the constants k, n, and the templates $\Phi_1, \ldots, \Phi_s$ can be chosen so that those templates are refined.

In the case where M is a cosimple, cyclically k-connected member of $\mathcal{M}$ with an $M(K_n)$-minor but no $\mathrm{PG}(m-1, \mathbb{F}_p)$-minor for some positive integer m, we dualize the argument in the previous paragraph to conclude that the constants k, n, and the templates $\Psi_1, \ldots, \Psi_t$ can be chosen so that those templates are refined. ■

Chapter 4: Working with Templates

In many ways, this chapter is the heart of this dissertation. We will give many results that relate frame templates to each other and simplify their use. Chapters 5 and 6 demonstrate the power of templates through applications to various minor-closed classes of matroids. In this chapter, the term *matroid* will mean represented matroid, unless otherwise specified as an abstract matroid.

In Section 4.1, we introduce a minor relation on the set of frame templates over a field. Many problems involving templates can be reduced to problems involving templates where the groups Γ, Λ, and Δ are all trivial and where $C = \emptyset$. We call these simpler templates *Y-templates*. Sections 4.2 and 4.3 give several results about Y-templates that will be used in Chapters 5 and 6. Section 4.4 introduces a preorder on the set of frame templates over a field and determines the set of minimal nontrivial templates with respect to this preorder. Finally, Section 4.5 shows how, subject to Hypothesis 3.2.3, frame templates can be used to determine the extremal functions of quadratically dense classes of representable matroids.

4.1 Reducing a Template

In this section, we will introduce reductions and show that every template reduces to one of several basic templates.

Since templates are used to study minor-closed classes of matroids, a natural question to ask is whether the set of matroids conforming to a particular template is minor-closed. The answer is no, in general. For example, if $|Y_0| = 1$, then a matroid conforms to the following binary frame template if and only if it is a graphic matroid with a loop:

$$(\{1\}, \emptyset, \emptyset, Y_0, \emptyset, [\emptyset], \{0\}, \{0\}).$$

Clearly, this is not a minor-closed class.

Another question to ask is whether there might be some sort of minor relationship between a pair of templates, where every matroid conforming to one template is a minor of a matroid conforming to the other. These questions motivate the following discussion.

Definition 4.1.1. A *reduction* is an operation on a frame template Φ that produces a frame template Φ' such that $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi)$.

Proposition 4.1.2. *The following operations are reductions on a frame template Φ:*

(1) Replace Γ with a proper subgroup.

(2) Replace Λ with a proper subgroup closed under multiplication by elements from Γ.

The content of Sections 4.1 and 4.4 generalize results that were published in Kevin Grace and Stefan H.M. van Zwam, Templates for binary matroids, *SIAM Journal on Discrete Mathematics* **31** (2017), 254–282.

(3) Replace Δ with a proper subgroup closed under multiplication by elements from Γ.

(4) Remove an element y from Y_1. (More precisely, replace A_1 with $A_1[X, Y_0 \cup (Y_1 - y) \cup C]$ and replace Δ with $\Delta|(Y_0 \cup (Y_1 - y) \cup C)$.)

(5) For all matrices A' respecting Φ, perform an elementary row operation on $A'[X, E]$. (Note that this alters the matrix A_1 and performs a change of basis on Λ.)

(6) If there is some element $x \in X$ such that, for every matrix A' respecting Φ, we have that $A'[\{x\}, E]$ is a zero row vector, remove x from X. (This simply has the effect of removing a zero row from every matrix conforming to Φ.)

(7) Let $c \in C$ be such that $A_1[X, \{c\}]$ is a vector of weight one whose nonzero entry is in the row indexed by $x \in X$, and let either $\lambda_x = 0$ for each $\lambda \in \Lambda$ or $\delta_c = 0$ for each $\delta \in \Delta$. We contract c from every matroid conforming to Φ as follows. Let Δ' be the result of adding $-\delta_c A_1[\{x\}, Y_0 \cup Y_1 \cup C]$ to each element $\delta \in \Delta$. Replace Δ with Δ', and then remove c from C and x from X. (More precisely, replace A_1 with $A_1[X - x, Y_0 \cup Y_1 \cup (C - c)]$, replace Λ with $\Lambda|(X - x)$, and replace Δ with $\Delta'|(Y_0 \cup Y_1 \cup (C - c))$.)

(8) Let $c \in C$ be such that $A_1[X, \{c\}]$ is a zero column and $\delta_c = 0$ for all $\delta \in \Delta$. Then remove c from C. (More precisely, replace A_1 with $A_1[X, Y_0 \cup Y_1 \cup (C - c)]$, and replace Δ with $\Delta|(Y_0 \cup Y_1 \cup (C - c))$.)

Proof. Let Φ' be the template that results from performing one of operations (1)-(8) on Φ.

For (1)-(3), every matrix A' respecting Φ' also respects Φ.

For (4), let A' be a matrix respecting Φ', and let M be the matroid $M(A)/C\backslash Y_1$, where A is a matrix conforming to Φ' that has been constructed from A' respecting Φ' as described in Section 1.3. Since Y_1 is deleted to produce M, the only effect of Y_1 on M is that for each $i \in Z$ there exists $j \in Y_1$ such that the i-th column of A is the sum of the i-th and the j-th columns of A'. But each $j \in Y_1$ in the template Φ' is also contained in Y_1 in the template Φ. Therefore, A conforms to Φ, as does M.

For (5) and (6), elementary row operations and removing zero rows produce isomorphic matroids.

Operations (7) and (8) have the effect of contracting c from $M(A)\backslash Y_1$ for every matrix A conforming to Φ. Since all of C is contracted to produce a matroid M conforming to Φ, the matroids we produce by performing either of these operations still conform to Φ. ■

Since we always have $Y_0 \subseteq E(M)$ for every matroid M conforming to Φ, operations (10)-(12) listed in the definition below are not reductions as defined above, but we continue the numbering from Proposition 4.1.2 for ease of reference.

Definition 4.1.3. A template Φ' is a *template minor* of Φ if Φ' is obtained from Φ by repeatedly performing the following operations:

(9) Performing a reduction of type 1–8 on Φ.

(10) Removing an element y from Y_0, replacing A_1 with $A_1[X, (Y_0 - y) \cup Y_1 \cup C]$, and replacing Δ with $\Delta|((Y_0 - y) \cup Y_1 \cup C)$. (This has the effect of deleting y from every matroid conforming to Φ.)

(11) Let $x \in X$ with $\lambda_x = 0$ for every $\lambda \in \Lambda$, and let $y \in Y_0$ be such that $(A_1)_{x,y} \neq 0$. Then contract y from every matroid conforming to Φ. (More precisely, perform row operations on A_1 so that $A_1[X, \{y\}]$ is a unit column with $(A_1)_{x,y} = 1$. Then replace every element $\delta \in \Delta$ with the row vector $-\delta_y A_1[\{x\}, Y_0 \cup Y_1 \cup C] + \delta$. This induces a group homomorphism $\Delta \to \Delta'$, where Δ' is also a subgroup of the additive group of $\mathbb{F}^{C \cup Y_0 \cup Y_1}$ and is closed under scaling by elements of Γ. Finally, replace A_1 with $A_1[X - x, (Y_0 - y) \cup Y_1 \cup C]$, project Λ into $\mathbb{F}^{X-x}$, and project Δ' into $\mathbb{F}^{(Y_0 - y) \cup Y_1 \cup C}$. The resulting groups play the roles of Λ and Δ, respectively in Φ'.)

(12) Let $y \in Y_0$ with $\delta_y = 0$ for every $\delta \in \Delta$. Then contract y from every matroid conforming to Φ. (More precisely, if $A_1[X, \{y\}]$ is a zero vector, this is the same as simply removing y from Y_0. Otherwise, choose some $x \in X$ such that $(A_1)_{x,y} \neq 0$. Then for every matrix A' that respects Φ, perform row operations so that $A_1[X, \{y\}]$ is a unit column with $(A_1)_{x,y} = 1$. This induces a group isomorphism $\Lambda \to \Lambda'$ where Λ' is also a subgroup of the additive group of $\mathbb{F}^X$ and is closed under scaling by elements of Γ. Finally, replace A_1 with $A_1[X - x, (Y_0 - y) \cup Y_1 \cup C]$, project Λ' into $\mathbb{F}^{X-x}$, and project Δ into $\mathbb{F}^{(Y_0 - y) \cup Y_1 \cup C}$. The resulting groups play the roles of Λ and Δ, respectively in Φ'.)

Recall the definitions of virtual respecting and conforming from Definition 3.1.2. Let Φ' be a template minor of Φ, and let A' be a matrix that virtually respects Φ'. Let A be a matrix that virtually conforms to Φ', and let M be a matroid that virtually conforms to Φ'. We say that A' *weakly respects* Φ and that A and M *weakly conform* to Φ. Let $\mathcal{M}_w(\Phi)$ denote the set of represented matroids that weakly conform to Φ, and let $\mathcal{M}^*_w(\Phi)$ denote the set of represented matroids whose duals weakly conform to Φ. If $M \in \mathcal{M}^*_w(\Phi)$, we say that M *weakly coconforms* to Φ.

Lemma 4.1.4. *If a matroid M weakly conforms to a template Φ, then M is a minor of a matroid that conforms to Φ.*

Proof. Let Φ' be a template minor of Φ. As we can see from Definition 4.1.3, every matroid M weakly conforming to Φ' is a minor of a matroid virtually conforming to Φ. It remains to analyze the case where M virtually conforms to Φ; so M is isomorphic to $M(K)/C \backslash Y_1$, where K is built from a matrix K' that virtually respects Φ. Consider the following matrix A' obtained from K' by adding a row r and a column c.

<table>
<tr><td></td><td>c</td><td></td><td colspan="2">Z</td><td>Y_0 Y_1 C</td></tr>
<tr><td>X</td><td>0</td><td>columns from Λ</td><td colspan="2">0</td><td>A_1</td></tr>
<tr><td></td><td>0</td><td>Γ-frame matrix</td><td>0</td><td>unit columns</td><td>rows from Δ</td></tr>
<tr><td>r</td><td>1</td><td>0</td><td>$1 \cdots 1$</td><td>0</td><td>0</td></tr>
</table>

From A', we can obtain a matrix A conforming to Φ such that M is isomorphic to $M(A)/C\backslash Y_1/c$. So M is a minor of a matroid conforming to Φ. ■

An easy consequence of Lemma 4.1.4 is that Hypotheses 3.2.2 and 3.2.3, which deal with minor-closed classes, can be restated in terms of weak conforming.

Corollary 4.1.5. *Suppose Hypothesis 3.2.2 holds. Let $\mathbb{F}$ be a finite field, let m be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-represented matroids. Then there exist $k \in \mathbb{Z}_+$ and refined frame templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ such that*

1. *$\mathcal{M}$ contains each of the classes $\mathcal{M}_w(\Phi_1), \ldots, \mathcal{M}_w(\Phi_s)$,*
2. *$\mathcal{M}$ contains the duals of the represented matroids in each of the classes $\mathcal{M}_w(\Psi_1),\ldots,\mathcal{M}_w(\Psi_t)$, and*
3. *if M is a simple k-connected member of $\mathcal{M}$ with at least $2k$ elements and $\widetilde{M}$ has no $\mathrm{PG}(m-1, \mathbb{F}_p)$-minor, then either M is a member of at least one of the classes $\mathcal{M}_w(\Phi_1), \ldots, \mathcal{M}_w(\Phi_s)$, or M^* is a member of at least one of the classes $\mathcal{M}_w(\Psi_1), \ldots, \mathcal{M}_w(\Psi_t)$.*

Proof. Let $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ be the templates whose existence is implied by Hypothesis 3.2.2. By Theorem 3.3.8, the constant k and the templates can be chosen so that the templates are refined. For $\Phi \in \{\Phi_1, \ldots, \Phi_s\}$, Lemma 4.1.4 implies that every matroid $M \in \mathcal{M}_w(\Phi)$ is a minor of a matroid $N \in \mathcal{M}(\Phi)$. Since $\mathcal{M}$ contains $\mathcal{M}(\Phi)$ and is minor-closed, $\mathcal{M}$ contains $\mathcal{M}_w(\Phi)$ as well. Similarly, $\mathcal{M}$ contains the duals of the matroids in each of the classes $\mathcal{M}_w(\Psi_1), \ldots, \mathcal{M}_w(\Psi_t)$. The third condition above holds since every matroid conforming to a template also weakly conforms to it. ■

The proof of the next corollary is omitted since it is similar to the previous proof.

Corollary 4.1.6. *Suppose Hypothesis 3.2.3 holds. Let $\mathbb{F}$ be a finite field, let m be a positive integer, and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-represented matroids. Then there exist $k, n \in \mathbb{Z}_+$ and refined frame templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ such that*

1. *$\mathcal{M}$ contains each of the classes $\mathcal{M}_w(\Phi_1), \ldots, \mathcal{M}_w(\Phi_s)$,*

2. *$\mathcal{M}$ contains the duals of the represented matroids in each of the classes $\mathcal{M}_w(\Psi_1), \ldots, \mathcal{M}_w(\Psi_t)$,*

3. *if M is a simple vertically k-connected member of $\mathcal{M}$ with an $M(K_n)$-minor but no $\mathrm{PG}(m-1, \mathbb{F}_p)$-minor, then M is a member of at least one of the classes $\mathcal{M}_w(\Phi_1), \ldots, \mathcal{M}_w(\Phi_s)$, and*

4. *if M is a cosimple cyclically k-connected member of $\mathcal{M}$ with an $M^*(K_n)$-minor but no $\mathrm{PG}(m-1, \mathbb{F}_p)$-minor, then M^* is a member of at least one of the classes $\mathcal{M}_w(\Psi_1), \ldots, \mathcal{M}_w(\Psi_t)$.*

If $\mathcal{M}_w(\Phi) = \mathcal{M}_w(\Phi')$, we say that Φ is *equivalent* to Φ' and write $\Phi \sim \Phi'$. It is clear that $\sim$ is indeed an equivalence relation.

4.2 Y-Templates

Many of the applications of frame templates in the upcoming chapters reduce to cases involving templates where the set $C = \emptyset$ and where the groups Γ, Λ, and Δ are trivial. These simpler templates are the topic of this section.

Definition 4.2.1. A *Y-template* over a field $\mathbb{F}$ is a refined template with all groups trivial (so $C = X_0 = \emptyset$). We do not require $\mathbb{F}$ to be finite.

Suppose $\Phi = (\{1\}, \emptyset, X, Y_0, Y_1, A_1, \{0\}, \{0\})$ is a Y-template. Since Φ is refined, Y_1 spans $M(A_1)$. Therefore, by elementary row operations, we may assume that A_1 is of the following form:

$$\begin{array}{ccc} & Y_1 & Y_0 \\ \hline \multicolumn{1}{|c|}{I_{|X|}} & \multicolumn{1}{c|}{P_1} & \multicolumn{1}{c|}{P_0} \\ \hline \end{array}$$

Definition 4.2.2. If A_1 has the form above, then $\mathrm{YT}(P_0, P_1)$ is defined to be the Y-template $(\{1\}, \emptyset, X, Y_0, Y_1, A_1, \{0\}, \{0\})$.

Recall from Section 1.3 that D_n denotes the $n \times \binom{n}{2}$ matrix where every column contains exactly two nonzero entries, the first a 1 and the second a -1. Note that every simple matroid virtually conforming to $\mathrm{YT}(P_0, P_1)$ is a restriction of a matroid of the following form.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
 & 0 & I_{|X|} & \cdots & I_{|X|} & P_1 & \cdots & P_1 & P_1 & P_0 \\
\cline{2-10}
 & & 1\cdots 1 & & & 1\cdots 1 & & & & \\
I_r & D_{r-|X|} & & \ddots & & & \ddots & & 0 & 0 \\
 & & & & 1\cdots 1 & & & 1\cdots 1 & & \\
\hline
\end{array}$$

Definition 4.2.3. The matroid with the representation matrix given above is the *rank-r universal matroid* for $\mathrm{YT}(P_0, P_1)$.

Note that the rank-r universal matroid of $\mathrm{YT}(P_0, P_1)$ need not be simple. For example, columns in $[I|P_1|P_0]$ can be scalar multiples of each other. If a pair of Y-templates have isomorphic universal matroids, one might expect for them to

be strongly equivalent. (Recall from Definition 3.3.1 that Φ and Φ' are strongly equivalent if $\mathcal{M}(\Phi) = \mathcal{M}(\Phi')$.) However, if a matroid M conforms to a template $(\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$, then we always have $Y_0 \subseteq E(M)$. Thus, some elements of the common universal matroid might be contained in Y_0 when thought of as conforming to one template but not in the other template. The next definition is motivated by this technicality.

Definition 4.2.4. Let $\Phi = \mathrm{YT}(P_0, P_1)$ and $\Phi' = \mathrm{YT}(P_0', P_1')$ be Y-templates over a field $\mathbb{F}$. Let X and X' be the sets of row indices for $[P_0|P_1]$ and $[P_0'|P_1']$, respectively. If, for every $r \geq \max\{|X|, |X'|\}$, the rank-r universal matroids of Φ and Φ' are isomorphic, then Φ and Φ' are *semi-strongly equivalent.*

Note that, since every matroid conforming to a Y-template is a restriction of a universal matroid for the template, semi-strong equivalence implies minor equivalence.

Definition 4.2.5. A refined template Φ with reduction partition $X = X_0 \cup X_1$, with $A_1[X_0, Y_1]$ a zero matrix, and with $A_1[X_1, Y_1]$ an identity matrix is a *lifted* template.

Remark 4.2.6. A lifted Y-template is of the form $\mathrm{YT}(P_0, [\emptyset])$.

Remark 4.2.7. The next lemma will mainly be used in the situation where Φ is a Y-template. Keeping that special case in mind may perhaps help to give the reader some intuition of the result. However, we prove the more general result since it may be of interest for future work. In the special case of Y-templates, the next lemma says that the Y-template $\mathrm{YT}(P_0, P_1)$ is equivalent to $\mathrm{YT}\left(\left[\begin{array}{c|c} -P_1 & P_0 \\ \hline I & 0 \end{array}\right], [\emptyset]\right)$.

Lemma 4.2.8. *Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a refined frame template with reduction partition $X = X_0 \cup X_1$. Then Φ is equivalent to a lifted template Φ'.*

Proof. Recall that in a reduced template, $A_1[X_1, C]$ is a zero matrix and that in a refined template, Y_1 spans $M(A_1[X_1, Y_0 \cup Y_1])$. Therefore, by performing elementary row operations, we may assume that A_1 is of the following form, with $Y_1 = R \cup V$ and with each P_i and each Q_i an arbitrary matrix.

$$\begin{array}{c|c|c|c|c|} & C & R & V & Y_0 \\ \hline X_0 & Q_2 & 0 & Q_1 & Q_0 \\ \hline X_1 & 0 & I & P_1 & P_0 \\ \hline \end{array}$$

We now construct Φ'. Let A_1' be of the following form, where S, T, and U are pairwise disjoint sets each also disjoint from R, where $Y_1' = R \cup S$, where $Y_0' = T \cup U$, and where $X' = X_0 \cup X_1 \cup X_2$.

$$\begin{array}{c|c|c|c|c|c|} & C & R & S & T & U \\ \hline X_0 & Q_2 & 0 & 0 & -Q_1 & Q_0 \\ \hline X_1 & 0 & I & 0 & -P_1 & P_0 \\ \hline X_2 & 0 & 0 & I & I & 0 \\ \hline \end{array}$$

Let Δ' be a subgroup of the additive group of $\mathbb{F}^{C\cup Y_1'\cup Y_0'}$ that is isomorphic to Δ with the isomorphism that maps $\begin{array}{cccc} C & R & V & Y_0 \\ [\delta_1 & \delta_2 & \delta_3 & \delta_4] \end{array}$ to $\begin{array}{ccccc} C & R & S & T & U \\ [\delta_1 & \delta_2 & 0 & -\delta_3 & \delta_4] \end{array}$, and let Λ' be a subgroup of the additive group of $\mathbb{F}^{X'}$ that is isomorphic to Λ with the isomorphism that maps $\begin{array}{c} X_0 \\ X_1 \end{array}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ to $\begin{array}{c} X_0 \\ X_1 \\ X_2 \end{array}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{bmatrix}$. Then $\Phi' = (\Gamma, C, X', Y_0', Y_1', A_1', \Delta', \Lambda')$ is a reduced template with reduction partition $X' = X_0 \cup (X_1 \cup X_2)$. Figure 4.1 shows

		Z	C	R	S	T	U
X_0	columns from $\Lambda\|X_0$	0	Q_2	0	0	$-Q_1$	Q_0
X_1	0	0	0	I	0	$-P_1$	P_0
X_2	0	0	0	0	I	I	0
	Γ-frame matrix	unit or zero columns	rows from Δ'				

FIGURE 4.1. A Matrix Virtually Respecting Φ'

the structure of a matrix virtually respecting Φ'. We claim that $\Phi \sim \Phi'$. Since the reduction partition for Φ' is $X' = X_0 \cup (X_1 \cup X_2)$, the set X_0 will still play the role of X_0 in Φ', and $X_1 \cup X_2$ will play the role of X_1 in Φ'. Since $A_1'[X_0, Y_1']$ is a zero matrix, and since $A_1'[X_1 \cup X_2, Y_1']$ is an identity matrix, Φ' is a lifted template. Thus, the lemma follows from the claim that $\Phi \sim \Phi'$.

First, note that $\mathcal{M}_w(\Phi) \subseteq \mathcal{M}_w(\Phi')$ because if M conforms to Φ', then by contracting T (pivoting on the nonzero entries in $A_1[X_2, T]$) we obtain a matroid conforming to Φ. In other words, we repeatedly perform operation (12) on Φ' to obtain Φ. To see that $\mathcal{M}_w(\Phi') \subseteq \mathcal{M}_w(\Phi)$, we will show that $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi)$. Recall that, in a matrix conforming to a template, the column indexed by an element $z \in Z$ is constructed by adding a column indexed by an element of Y_1 to the column indexed by z in a matrix respecting the template. If M conforms to Φ', let $Z_R \cup Z_S = Z \subseteq E(M)$, where Z_R consists of the elements of Z indexing columns constructed by adding a column indexed by an element of R, and where Z_S consists of the elements of Z indexing columns constructed by adding a column indexed by an element of S. Note that the set of elements of M represented by columns with nonzero entries in rows indexed by X_2 consists of $Z_S \cup T$. Scale the columns indexed by T and the rows indexed by X_2 by -1. To see that $M \in \mathcal{M}(\Phi)$, note that the columns indexed by elements of T can now be constructed as columns indexed by elements of Z in a matrix conforming to Φ, and the columns indexed by elements of Z_S can now be constructed as graphic columns (defined in Section 3.1) indexed by elements of $E - (C \cup Z \cup Y_0 \cup Y_1)$ in a matrix conforming to Φ. Moreover, columns indexed by elements of $E - (C \cup Z \cup Y_0 \cup Y_1 \cup)$, or Z_R, or C, or U in Φ' can be constructed as columns indexed by elements of $E - (C \cup Z \cup Y_0 \cup Y_1)$, or Z, or C, or Y_0, respectively in Φ. ■

Recall that in a Y-template $\mathrm{YT}(P_0, P_1)$, the rows of P_0 and P_1 are indexed by a set X.

Definition 4.2.9. A Y-template $\mathrm{YT}(P_0, P_1)$ is *complete* if P_0 contains $D_{|X|}$ as a submatrix.

Recall from Definition 3.3.1 the definition of minor equivalence.

Lemma 4.2.10. *Every Y-template is minor equivalent to a complete template.*

Proof. This follows immediately from the following construction. ∎

Lemma 4.2.11. *Let $\Phi = \mathrm{YT}(P_0, P_1)$, and let P_0^- be the result of removing from P_0 all graphic columns of weight 2. The template Φ is minor equivalent to $\Phi' = \mathrm{YT}([P_0^-|D_{|X|}], P_1)$.*

Proof. It suffices to show that every matroid conforming to Φ is a minor of a matroid conforming to Φ', and vice-versa. By appending to P_0 the columns of $D_{|X|}$ not already contained in P_0, one easily sees that every matroid conforming to Φ is a minor of a matroid conforming to Φ'.

We now must show that every matroid conforming to Φ' is a minor of a matroid conforming to Φ. Note that every matroid conforming to Φ' is a restriction of a matroid M' with a representation matrix of the following form:

<table>
<tr><td colspan="2">0</td><td>columns from $[I|P_1]$</td><td>P_0^-</td><td>$D_{|X|}$</td></tr>
<tr><td>I_n</td><td>D_n</td><td>unit columns</td><td colspan="2">0</td></tr>
</table>

Note that the following matrix conforms to Φ:

<table>
<tr><td></td><td></td><td>Z_1</td><td>Z_2</td><td>$\cdots$</td><td>$Z_{|X|}$</td><td></td><td></td></tr>
<tr><td colspan="2">0</td><td>$I_{|X|}$</td><td>$I_{|X|}$</td><td>$\cdots$</td><td>$I_{|X|}$</td><td>columns from $[I|P_1]$</td><td>P_0</td></tr>
<tr><td>I_n</td><td>D_n</td><td>0</td><td>0</td><td>$\cdots$</td><td>0</td><td>unit columns</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>$1 \cdots 1$</td><td>$1 \cdots 1$</td><td>$\ddots$</td><td>$1 \cdots 1$</td><td>0</td><td>0</td></tr>
</table>

Let M be the matroid represented by this matrix. Contract the i-th element of each Z_i, pivoting on the bottom nonzero entry in each case. The resulting matroid can be obtained by adding elements in parallel to existing elements of M'. Thus, M' is a minor of the resulting matroid, and every matroid conforming to Φ' is a minor of a matroid conforming to Φ. ∎

Lemma 4.2.12. *Every Y-template $\Phi = \mathrm{YT}(P_0, P_1)$ is semi-strongly equivalent to a Y-template $\Phi' = \mathrm{YT}(P_0', P_1')$ where the sum of the rows of P_1' is $[1, \ldots, 1]$ and the sum of the rows of P_0' is the zero vector. If Φ is complete, so is Φ'.*

Proof. This follows immediately from the following construction. ∎

Lemma 4.2.13. *Let $x_1, x_2, \ldots$ be the rows of P_0, let $y_1, y_2, \ldots$ be the rows of P_1, and let $y_* = [1, \ldots, 1] - \Sigma y_i$. Let $P_1' = \left[\begin{array}{c} P_1 \\ \hline y_* \end{array}\right]$ and let*

$$P_0' = \left[\begin{array}{c|c|c} I & P_1 & P_0 \\ \hline -1 \cdots -1 & -\Sigma y_i & -\Sigma x_1 \end{array}\right]$$

Then $\Phi = \mathrm{YT}(P_0, P_1)$ is semi-strongly equivalent to $\Phi' = \mathrm{YT}(P_0', P_1')$.

Proof. Every matroid virtually conforming to Φ is a restriction of a rank-r universal matroid for Φ with a representation matrix as follows, with rows and columns indexed by B and E, respectively. Here $x \in B - X$. (If $r = |X|$, then append a zero row with index x.) We denote a zero vector by $\mathbf{0}$ and a vector of all 1s as $\mathbf{1}$.

X	$\mathbf{0}$	0	0	0	I	P_1	I	P_1	$\cdots$	I	P_1	I	P_1	P_0
x	1	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\ldots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
	$\mathbf{0}$	I	$D_{r-\lvert X\rvert-1}$	$-I$	0	0	$\mathbf{1}$	$\mathbf{1}$	$\ddots$	$\mathbf{1}$	$\mathbf{1}$	0	0	0

Choose some element $x \in B - X$. (If $r = |X|$, then append a zero row with index x.) From the row indexed by x, subtract all rows indexed by elements of X and, to the row indexed by x, add all rows indexed by elements in $B - (X \cup x)$. The result is the following (after scaling the fourth "block" of columns from $-I$ to I).

X	$\mathbf{0}$	0	0	0	I	P_1	I	P_1	$\cdots$	I	P_1	I	P_1	P_0
x	1	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	y_*	$\mathbf{0}$	y_*	$\ldots$	$\mathbf{0}$	y_*	$\mathbf{-1}$	$-\Sigma y_i$	$-\Sigma x_i$
	$\mathbf{0}$	I	$D_{r-\lvert X\rvert-1}$	I	0	0	$\mathbf{1}$	$\mathbf{1}$	$\ddots$	$\mathbf{1}$	$\mathbf{1}$	0	0	0

Rearranging the columns, we obtain the following, which is a representation matrix for the rank-r universal matroid for Φ'.

$X \cup x$	0	0	0	I	P_1'	I	P_1'	$\cdots$	I	P_1'	I	P_1'	P_0'
	I	$D_{r-\lvert X\rvert-1}$	$-I$	0	0	$\mathbf{1}$	$\mathbf{1}$	$\ddots$	$\mathbf{1}$	$\mathbf{1}$	0	0	0

Note that if Φ is a complete template, then the presence of $\left[\begin{array}{c} I \\ \hline -1 \cdots -1 \end{array}\right]$ in P_0' implies that Φ' is complete also. ∎

Definition 4.2.14. The Y-template $\mathrm{YT}([P_0|D_{|X|}],[\emptyset])$ is the complete, lifted template *determined* by P_0 and is denoted by Φ_{P_0}.

Note that a matroid virtually conforming to Φ_{P_0} is a restriction of a matroid with a representation matrix of the following form:

$$\begin{array}{|c|c|c|c|c|}\hline \multicolumn{2}{|c|}{0} & \text{unit columns} & P_0 & D_{|X|} \\ \hline I_n & D_n & \text{unit or zero columns} & \multicolumn{2}{c|}{0} \\ \hline \end{array}$$

By scaling appropriately, we may assume that the bottom submatrix of unit columns actually consists of the negatives of unit columns. Thus a rank-r matroid conforming to Φ_{P_0} is a restriction of a matroid with a representation matrix of the following form.

$$\begin{array}{|c|c|c|}\hline I_r & D_r & P_0 \\ \cline{3-3} & & 0 \\ \hline \end{array}$$

The matroid represented by this matrix is the rank-r universal matroid of Φ_{P_0}.

Lemma 4.2.15. *The following are true.*

(i) Every complete, lifted Y-template is semi-strongly equivalent to a complete, lifted Y-template determined by a matrix the sum of whose rows is the zero vector.

(ii) Conversely, let Φ be the complete, lifted Y-template determined by a matrix P_0 the sum of whose rows is the zero vector. Choose any one row of P_0. Then Φ is semi-strongly equivalent to the complete, lifted Y-template determined by the matrix obtained from P_0 by removing that row.

Proof. This follows directly from Lemma 4.2.13. ■

Lemma 4.2.16. *Let Q_1 and Q_2 be matrices over a field $\mathbb{F}$, each with m rows. Let $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_n\in\mathbb{F}^m$ be column vectors. If $P_1=\left[\begin{array}{c} Q_1 \\ \hline 0\end{array}\right]$ and P_0 is of the form*

$$\begin{array}{|c|c|c|c|c|c|}\hline \boldsymbol{v}_1\cdots\boldsymbol{v}_1 & \boldsymbol{v}_2\cdots\boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\cdots\boldsymbol{v}_n & Q_2 & H \\ \cline{1-5} \multicolumn{4}{|c|}{\textit{unit or zero columns}} & 0 & \\ \hline \end{array},$$

where H is a column submatrix of $D_{|X|}$, then $\mathcal{M}_w(\mathrm{YT}(P_0,P_1))\subseteq\mathcal{M}_w(\mathrm{YT}(P_0',P_1'))$, where $P_0'=[Q_2|D_{|X|}]$ and $P_1'=[Q_1|-\boldsymbol{v}_1|-\boldsymbol{v}_2|\cdots|-\boldsymbol{v}_n]$.

Proof. Let $\Phi=\mathrm{YT}(P_0,P_1)$, and let $\Phi'=\mathrm{YT}(P_0',P_1')$. Every matroid virtually conforming to Φ is represented by a column submatrix of a matrix of the following form, where $V=[\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n]$ and where $X=X_1\cup X_2$.

<table>
<tr><th></th><th>A</th><th>B</th><th>C</th><th>D</th><th>F</th><th>G</th><th>J</th></tr>
<tr><td>X_1</td><td>0</td><td>unit or zero columns</td><td>0</td><td>columns from Q_1</td><td>columns from V</td><td>Q_2</td><td rowspan="2">$D_{|X|}$</td></tr>
<tr><td>X_2</td><td>0</td><td>0</td><td>unit or zero columns</td><td>0</td><td>unit or zero columns</td><td>0</td></tr>
<tr><td></td><td>$D_{r-|X|}$</td><td colspan="3">unit or zero columns</td><td colspan="3">0</td></tr>
</table>

Every matroid virtually conforming to Φ' is represented by a column submatrix of a matrix of the following form.

<table>
<tr><th>A'</th><th>B'</th><th>C'</th><th>D'</th><th>F'</th></tr>
<tr><td>0</td><td>unit or zero columns</td><td>columns from P_1'</td><td>Q_2</td><td>$D_{|X_1|}$</td></tr>
<tr><td>$D_{r-|X_1|}$</td><td colspan="2">unit or zero columns</td><td colspan="2">0</td></tr>
</table>

Let J_1 be the subset of J indexing columns both of whose nonzero entries occur in rows indexed by X_1; let J_2 be the subset of J indexing columns both of whose nonzero entries occur in rows indexed by X_2; and let J_3 be the subset of J indexing columns that have one nonzero entry in each of the sets of rows indexed by X_1 and X_2. To prove the result, it suffices to show that every matroid virtually conforming to Φ is a restriction of a matroid virtually conforming to Φ'. To do this, scale each of the rows indexed by X_2 and each of the columns indexed by $C \cup F \cup J_2$ by -1. Then we can choose $A' = A \cup C \cup J_2$, and $B' = B \cup J_3$, and $C' = D \cup F$, and $D' = G$, and $F' = J_1$. ■

4.3 Algebraic Equivalence of Y-Templates

Let $\mathbb{F}$ and $\mathbb{F}'$ be fields, and let $\Phi = \mathrm{YT}(P_0, P_1)$ be a Y-template over $\mathbb{F}$. This section is motivated by the following question: When is the class of abstract matroids associated with members of $\mathcal{M}(\Phi)$ contained in the class of (abstract) $\mathbb{F}'$-representable matroids?

If a matrix P_0 has a zero column or a pair of columns that are scalar multiples of each other, then no matroid conforming to $\mathrm{YT}(P_0, P_1)$ is simple. Since we use templates to study highly connected (and therefore simple) matroids, we will assume in this section that no matrix P_0 (or P_0') has a zero column or a pair of columns that are scalar multiples of each other. In fact, the results in this section still hold in the general case, but a bit more care must be taken in order to prove the them.

Definition 4.3.1. Let P_0, P_0', P_1, and P_1' be matrices with the same number of rows such that P_0 and P_0' have the same zero-nonzero pattern, such that P_1 and P_1' have the same zero-nonzero pattern, such that P_0 and P_1 have entries from a field $\mathbb{F}$, and such that P_0' and P_1' have entries from a field $\mathbb{F}'$. Let $\Phi = \mathrm{YT}(P_0, P_1)$ and $\Phi' = \mathrm{YT}(P_0', P_1')$. Then Φ and Φ' are *pattern-compatible* templates.

Definition 4.3.2. Let $\Phi = \mathrm{YT}(P_0, P_1)$ and $\Phi' = \mathrm{YT}(P_0', P_1')$ be pattern-compatible templates, and let M_r and M_r' be the rank-r universal matroids of Φ and Φ', respectively. Let r be the smallest integer such that $\widetilde{M_r} \neq \widetilde{M_r'}$, if such an r exists.

Let $S \subseteq E(M_r) = E(M'_r)$ be a set of minimum size such that S is a circuit in M_r but independent in M'_r (or vice-versa). Then S is called a *distinguishing set* for Φ and Φ'. We call M_r and M'_r the *critical matroids* for the pair (Φ, Φ').

Lemma 4.3.3. *Let $\Phi = \mathrm{YT}(P_0, P_1)$ and $\Phi' = \mathrm{YT}(P'_0, P'_1)$ be pattern-compatible templates, with critical matroids M_r and M'_r. A distinguishing set S for Φ and Φ' contains none of the elements of $E(M_r) - (C \cup Y_0 \cup Y_1 \cup Z)$.*

Remark 4.3.4. The elements of $E(M_r) - (C \cup Y_0 \cup Y_1 \cup Z)$ are the elements indexing the graphic columns of $[I_{r-|X|}|D_{r-|X|}]$ usually written close to the bottom left of a matrix virtually conforming to Φ.

Proof of Lemma 4.3.3. Suppose otherwise, so $e \in S \cap (E(M_r) - (C \cup Y_0 \cup Y_1 \cup Z))$. Without loss of generality, let S be a circuit in M_r and independent in M'_r. Note that $S - e$ is a circuit in M_r/e and independent in M'_r/e. Thus, $\widetilde{M_r/e} \neq \widetilde{M'_r/e}$. On the other hand, from the structure of M_r and M'_r, we can see that M_r/e and M'_r/e have the exact same parallel classes. Note that $M_{r-1} = M'_{r-1}$ if and only if $M_r/e = M'_r/e$. By the assumption that M_r and M'_r are critical matroids, $\widetilde{M_r/e} = \widetilde{M'_r/e}$, a contradiction. ■

Lemma 4.3.5. *Let $\Phi = \mathrm{YT}(P_0, P_1)$ and $\Phi' = \mathrm{YT}(P'_0, P'_1)$ be pattern-compatible templates, with critical matroids M_r and M'_r, with representation matrices A_r and A'_r, respectively. Let S be a distinguishing set for Φ and Φ', and let A_S and A'_S be the column submatrices of A_r and A'_r, respectively, whose columns are indexed by S. Each nonzero row of A_S and A'_S has at least two nonzero entries.*

Proof. Suppose otherwise. Then the restrictions of M_r and M'_r to S each contain a coloop, which is impossible since S is a circuit in either M_r or M'_r. ■

Lemma 4.3.6. *Let $\Phi = \mathrm{YT}(P_0, P_1)$ and $\Phi' = \mathrm{YT}(P'_0, P'_1)$ be pattern-compatible templates, with critical matroids M_r and M'_r. Let $\bar{x}^T$ be a column of $[I|P_1]$. A distinguishing set S contains at most one of the elements represented by the columns of the following column submatrix of the representation matrix of M_r:*

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{\bar{x} \quad \bar{x} \quad \cdots \quad \bar{x}} \\ \hline 0 & \\ \vdots & I_{r-|X|} \\ 0 & \end{array}\right]$$

Proof. Without loss of generality, let S be a circuit in M_r and independent in M'_r. Suppose for a contradiction that S contains two of the elements indexing the columns of the submatrix given in the claim. Call these elements e and f. Note that e and f form a triangle with some element $g \in E(M_r) - (C \cup Y_0 \cup Y_1 \cup Z)$. By circuit elimination, M_r contains a circuit $C_1 \subseteq (S - e) \cup g$. On the other hand, the triangle $\{e, f, g\}$ must be the only circuit of M'_r contained in $S \cup g$. Thus, C_1 is

independent in M'_r. By minimality of S, we must have $C_1 = (S-e) \cup g$. Therefore, C_1 is a distinguishing set. The fact that $g \in C_1$ contradicts Lemma 4.3.3. ■

Recall the definition of algebraic equivalence (Definition 3.3.1).

Lemma 4.3.7. *Let $\Phi = \mathrm{YT}(P_0, P_1)$ and $\Phi' = \mathrm{YT}(P'_0, P'_1)$ be Y-templates over the fields $\mathbb{F}$ and $\mathbb{F}'$, respectively. Let P_1 and P'_1 be $c \times d$ matrices. Then Φ and Φ' are algebraically equivalent if and only if the (abstract) vector matroids of the following matrices are equal (not just isomorphic), where the bottom submatrices each have $\lfloor \frac{c+d}{2} \rfloor$ rows.*

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline I & I & P_1 & \cdots & I & P_1 & P_1 & P_0 \\ \hline & 1\cdots 1 & 1\cdots 1 & & & & & \\ 0 & & & \ddots & & & 0 & 0 \\ & & & & 1\cdots 1 & 1\cdots 1 & & \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline I & I & P'_1 & \cdots & I & P'_1 & P'_1 & P'_0 \\ \hline & 1\cdots 1 & 1\cdots 1 & & & & & \\ 0 & & & \ddots & & & 0 & 0 \\ & & & & 1\cdots 1 & 1\cdots 1 & & \\ \hline \end{array}$$

Proof. Suppose Φ and Φ' are algebraically equivalent. Let M_r and M'_r be the universal matroids of rank r for Φ and Φ', respectively, for $r = |X| + \lfloor \frac{c+d}{2} \rfloor$. Then the abstract vector matroids of the matrices given in the result are restrictions of $\widetilde{M_r} = \widetilde{M'_r}$. By deleting all other elements from M_r and M'_r, we see that the given matrices have equal abstract vector matroids.

Conversely, suppose the given matrices have equal abstract vector matroids. It suffices to show that, for every positive integer $r \geq |X|$, we have $\widetilde{M_r} = \widetilde{M'_r}$. We will show this by induction on r. For $r = |X|$, note that by deleting the same subset of the ground sets of the matroids represented by the given matrices, we obtain $\widetilde{M}([I|P_1|P_0]) = \widetilde{M}([I|P'_1|P'_0])$. Moreover, the fact that $\widetilde{M}([I|P_1|P_0]) = \widetilde{M}([I|P'_1|P'_0])$ implies that Φ and Φ' are pattern-compatible.

For the inductive step, let $r > |X|$. If $\widetilde{M_r} \neq \widetilde{M'_r}$, then there are critical matroids $M_{r'}$ and $M'_{r'}$ for some $r' \leq r$. By the induction hypothesis, $r' = r$. Let S be the distinguishing set for (Φ, Φ'). Without loss of generality, we assume that S is a circuit in M_r and is independent in M'_r. Let A_S be the column submatrix of the representation matrix of M_r with columns indexed by S.

By Lemma 4.3.6, $|S \cap Z| \leq c+d$. By Lemma 4.3.3, the only columns indexed by elements of S that have nonzero entries in the rows indexed by elements of $B - X$ are those in Z. By Lemma 4.3.5, each row of A_S has at least two nonzero entries. This implies that the number of rows of A_S indexed by elements of $B - X$ that have nonzero entries is at most $\lfloor \frac{c+d}{2} \rfloor$. The result follows. ■

Lemma 4.3.8. *Let Φ and Φ' be complete, lifted Y-templates over the fields $\mathbb{F}$ and $\mathbb{F}'$, respectively, determined by the matrices P_0 and P'_0, respectively. Then Φ and Φ'*

are algebraically equivalent if and only if $\widetilde{M}([I|P_0|D_{|X|}])$ *and* $\widetilde{M}([I|P_0'|D_{|X|}])$ *are equal (not just isomorphic).*

Proof. Suppose Φ and Φ' are algebraically equivalent. The matroids represented by the matrices given in the lemma are the extremal matroids of rank $|X|$ for Φ and Φ' and are therefore equal.

Conversely, suppose the given matrices have equal abstract vector matroids. Suppose for a contradiction that Φ and Φ' are not algebraically equivalent. Then, for some rank r, the pair (Φ, Φ') has critical matroids M_r and M_r' and distinguishing set $S \subseteq E(M_r) = E(M_r')$ such that, after swapping Φ and Φ' if necessary, S is a circuit in M_r and independent in M_r'. Suppose S contains some element e indexing a column with a nonzero entry in a position other than the first $|X|$ entries. Then $S - e$ is a circuit in M_r/e and independent in M_r'/e. Thus, $\widetilde{M_r/e} \neq \widetilde{M_r'/e}$. The parallel classes of M_r/e and M_r'/e are exactly the same, and their simplifications are M_{r-1} and M_{r-1}', implying that $\widetilde{M_{r-1}} \neq \widetilde{M_{r-1}'}$. On the other hand, since M_r and M_r' are critical matroids, $\widetilde{M_{r-1}} = \widetilde{M_{r-1}'}$, a contradiction.

Therefore, a distinguishing set must be contained in the elements indexing $[I|P_0|D_{|X|}]$ and $[I|P_0'|D_{|X|}]$, but the abstract vector matroids of these matrices are equal. Thus, by contradiction, Φ and Φ' are algebraically equivalent. ■

Definition 4.3.9. Let Φ_0 be the frame template over a field $\mathbb{F}$ with all groups trivial and all sets empty. We call this template the *trivial template*. Note that $\Phi_0 = \mathrm{YT}([\emptyset], [\emptyset])$.

Lemma 4.3.10. *Let* $\Phi = \mathrm{YT}([P_0|D_{|X|}], P_1)$ *be a complete* Y*-template over a field* $\mathbb{F}$ *such that, for each* $M \in \mathcal{M}(\Phi)$*, the abstract matroid* $\widetilde{M}$ *is representable over a field* $\mathbb{F}'$*. Then there is a complete* Y*-template* $\Phi' = \mathrm{YT}([P_0'|D_{|X|}], P_1')$ *over* $\mathbb{F}'$ *that is algebraically equivalent to* Φ*. Moreover,* Φ' *can be chosen so that the following statements hold.*

1. *The templates* Φ *and* Φ' *are pattern-compatible.*
2. *If an entry of* P_1 *is a* 1*, then the corresponding entry in* P_1' *is also a* 1*.*

Proof. Recall that the rows of P_0 are indexed by the elements of X. If $X = \emptyset$, then Φ is the trivial template over $\mathbb{F}$. In this case, we take Φ' to be the trivial template over $\mathbb{F}'$. Since a matroid conforms to the trivial template if and only if it is graphic, it is clear that Φ and Φ' are algebraically equivalent.

Thus, we may assume that $X \neq \emptyset$. Let $|X| = c > 0$. If P_1 is a $c \times d$ matrix, let r be the greater of $c+2$ and $c + \lfloor \frac{c+d}{2} \rfloor$. We consider the largest simple matroid M of rank r virtually conforming to Φ. This matroid is represented by a matrix A of the following form.

I_r	0	I_c	P_1	$\cdots$	I_c	P_1	P_1	P_0	D_c
	D_{r-c}	$1\cdots1$	$1\cdots1$	$\ddots$			0	0	0
					$1\cdots1$	$1\cdots1$			

If $\widetilde{M}$ is representable over $\mathbb{F}'$, then $\widetilde{M}$ can be represented by a matrix $A' = [I_r|*]$ over $\mathbb{F}'$ with the same zero-nonzero pattern as A. For an integer n, we will use D'_n to denote some matrix with the same zero-nonzero pattern as D_n such that the first nonzero entry of each column is a 1. A well-known result (see Oxley [23, Theorem 6.4.7]) implies that we may scale the rows and columns of A' so that it is of the following form, where the stars represent arbitrary matrices not necessarily equal to each other.

<table>
<tr><td rowspan="3">I_r</td><td>0</td><td>0</td><td>I_c</td><td>$*$</td><td>$\cdots$</td><td>$*$</td><td>$*$</td><td>$*$</td><td>$*$</td><td>D'_c</td></tr>
<tr><td>$1 \quad \cdots \quad 1$</td><td>$0 \quad \cdots \quad 0$</td><td rowspan="2">$1 \cdots 1$</td><td rowspan="2">$1 \cdots 1$</td><td rowspan="2">$\ddots$</td><td rowspan="2">$1 \cdots 1$</td><td rowspan="2">$1 \cdots 1$</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">0</td></tr>
<tr><td>$-I_{r-c-1}$</td><td>D'_{r-c-1}</td></tr>
</table>

Now, let $[x_1, x_2, \ldots, x_c]^T$ be a column of $A_1[X, Y_1]$ in Φ. Thus, $[x_1, x_2, \ldots, x_c]^T$ is either a unit column or a column of P_1. Note that, since $r \geq c+2$, the following is a linearly dependent column submatrix of A, where the first column corresponds to one of the columns of A' used for the negative identity matrix.

$$\begin{bmatrix} 0 & x_1 & x_1 \\ \vdots & \vdots & \vdots \\ 0 & x_c & x_c \\ 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

Let the following matrix be the corresponding submatrix of A'. If the element represented by the first column is contracted, then the elements represented by the other two columns form a parallel pair.

$$\begin{bmatrix} 0 & x_{1,1} & x_{1,2} \\ \vdots & \vdots & \vdots \\ 0 & x_{c,1} & x_{c,2} \\ 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_{i,1} = x_{i,2}$ for all i with $1 \le i \le c$. Similarly, the following is a linearly dependent column submatrix of A.

$$\begin{bmatrix} 0 & x_1 & x_1 \\ \vdots & \vdots & \vdots \\ 0 & x_c & x_c \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

Contracting the element representing the first column shows that, in the corresponding submatrix of A' given below, $[x_{1,1}, \dots, x_{c,1}]^T$ must be a scalar multiple of $[x_{1,2}, \dots, x_{c,2}]^T$.

$$\begin{bmatrix} 0 & x_{1,1} & x_{1,2} \\ \vdots & \vdots & \vdots \\ 0 & x_{c,1} & x_{c,2} \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, there are matrices P_0' and P_1', with the same zero-nonzero patterns as P_0 and P_1, respectively, such that A' is the following matrix.

<table>
<tr><td rowspan="3">I_r</td><td>0</td><td>0</td><td>I_c</td><td>P_1'</td><td>$\cdots$</td><td>I_c</td><td>P_1'</td><td>P_1'</td><td>P_0'</td><td>D_c'</td></tr>
<tr><td>$1 \ \cdots \ 1$</td><td>$0 \ \cdots \ 0$</td><td>$1 \cdots 1$</td><td>$1 \cdots 1$</td><td rowspan="2">$\ddots$</td><td></td><td></td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">0</td></tr>
<tr><td>$-I_{r-c-1}$</td><td>D_{r-c-1}'</td><td></td><td></td><td>$1 \cdots 1$</td><td>$1 \cdots 1$</td></tr>
</table>

Since $c \ge 1$ and since $r > c + 1$, for each column of D_{r-c-1}' and D_c', there is some $x \in \mathbb{F}'$ and a column submatrix of A' that, restricted to its nonzero rows, is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ x & 0 & 1 \end{bmatrix}$, up to permuting rows and columns. This matrix is linearly dependent if and only if $x = -1$. Since this is the case for the corresponding submatrix of A, it is also the case for A'. Thus, $D_c' = D_c$ and $D_{r-c-1}' = D_{r-c-1}$.

Let $\Phi' = \mathrm{YT}([P_0'|D_c], P_1')$. Since $\widetilde{M}(A) = \widetilde{M}(A')$ and since $r \ge c + \lfloor \frac{c+d}{2} \rfloor$, Lemma 4.3.7 implies that Φ and Φ' are algebraically equivalent. By construction, Φ and Φ' are pattern-compatible, so statement (1) of the result holds. We now prove statement (2). Suppose a column of P_1 has a 1 as one of its entries. Without loss of generality, we may assume that this column is of the form $[1, x_2, \dots, x_t, 0, \dots, 0]^T$, with each $x_i \neq 0$. Let $[x_1', x_2', \dots, x_t', 0, \dots, 0]^T$ be the corresponding column of

P_1'. Note that the following column submatrices of A and A', respectively, each represent the same circuit of $\widetilde{M}$.

$$
\begin{array}{|c|cc|}
\hline
0\cdots 0 & 1 & 1 \\
\hline
 & 0 & x_2 \\
I_{t-1} & \vdots & \vdots \\
 & 0 & x_t \\
\hline
 & 0 & 0 \\
 & \vdots & \vdots \\
0 & 0 & 0 \\
 & 1 & 1 \\
 & 0 & 0 \\
 & \vdots & \vdots \\
 & 0 & 0 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{|c|cc|}
\hline
0\cdots 0 & 1 & x_1' \\
\hline
 & 0 & x_2' \\
I_{t-1} & \vdots & \vdots \\
 & 0 & x_t' \\
\hline
 & 0 & 0 \\
 & \vdots & \vdots \\
0 & 0 & 0 \\
 & 1 & 1 \\
 & 0 & 0 \\
 & \vdots & \vdots \\
 & 0 & 0 \\
\hline
\end{array}
$$

By contracting the elements that index the columns of the identity submatrix, we see that the remaining two columns must represent a parallel pair. Thus, we must have $x_1' = 1$, and (2) holds. ■

4.4 Minimal Nontrivial Templates

We define a preorder $\preceq$ on the set of frame templates over a field $\mathbb{F}$ as follows.

Definition 4.4.1. We say $\Phi \preceq \Phi'$ if $\mathcal{M}_w(\Phi) \subseteq \mathcal{M}_w(\Phi')$. This is indeed a preorder since reflexivity and transitivity follow from the subset relation. We may obtain a partial order by considering equivalence classes of templates, with equivalence as defined at the end of Section 4.1. However, the templates themselves, rather than equivalence classes, are the objects we work with in this dissertation.

Recall from Definition 4.3.9 that Φ_0 is the frame template with all groups trivial and all sets empty. In general, we say that a template Φ is *trivial* if $\Phi \preceq \Phi_0$. It is easy to see that for any template Φ, we have $\Phi_0 \preceq \Phi$. Therefore, if $\Phi \preceq \Phi_0$, then actually $\Phi \sim \Phi_0$. In this section, we find a collection of nontrivial templates that are minimal with respect to the preorder $\preceq$ given in Definition 4.4.1.

Definition 4.4.2. We define the following templates for a finite field $\mathbb{F} = \mathrm{GF}(p^m)$.

- If $|\mathbb{F}| \geq 3$ and $k \in (\mathbb{F} - \{0, 1\})$, let $\Phi_{Y_1}(k)$ be the Y-template $\mathrm{YT}([\emptyset], [k])$.
- If $\mathbb{F}$ has characteristic 2, let Φ_{Y_1} be the Y-template $\mathrm{YT}([\emptyset], [1, 1]^T)$.
- Let Φ_C be the template over $\mathbb{F}$ with all groups trivial and all sets empty except that $|C| = 1$ and $\Delta \cong \mathbb{Z}/p\mathbb{Z}$.
- Let Φ_X be the template over $\mathbb{F}$ with all groups trivial and all sets empty except that $|X| = 1$ and $\Lambda \cong \mathbb{Z}/p\mathbb{Z}$.

- Let Φ_{Y_0} be the template over $\mathbb{F}$ with all groups trivial and all sets empty except that $|Y_0| = 1$ and $\Delta \cong \mathbb{Z}/p\mathbb{Z}$.
- For each $k \in (\mathbb{F} - \{0\})$, let Φ_{CXk} be the template with $Y_0 = Y_1 = \emptyset$, with $|C| = |X| = 1$, with $\Delta \cong \Lambda \cong \mathbb{Z}/p\mathbb{Z}$, with Γ trivial, and with $A_1 = [k]$. We abbreviate Φ_{CX1} to Φ_{CX}.
- If n is a prime dividing $p^m - 1$, let Φ_n be the template with all sets empty and all groups trivial except that Γ is the cyclic subgroup of $\mathbb{F}^\times$ of order n.

Note that each of the templates given in Definition 4.4.2 is refined (often in a trivial or vacuous way). If $\mathbb{F}$ has odd characteristic, it is not too difficult to see that $U_{2,4}$ virtually conforms to each Φ_{CXk}, each Φ_n, each $\Phi_{Y_1}(k)$ and each of Φ_C, Φ_X, and Φ_{Y_0}. Similarly, if $\mathbb{F}$ has characteristic 2, it is not difficult to see that the Fano matroid F_7 virtually conforms to each Φ_{CXk}, as well as to each of Φ_C, Φ_X, Φ_{Y_0}, and Φ_{Y_1} and that $U_{2,4}$ virtually conforms to each Φ_n and to each $\Phi_{Y_1}(k)$. Therefore, these templates are nontrivial.

Lemma 4.4.3. *Let $\mathbb{F}$ be a field, and let $k_1, k_2 \in (\mathbb{F} - \{0,1\})$. The Y-templates $\Phi_{Y_1}(k_1)$ and $\Phi_{Y_1}(k_2)$ over $\mathbb{F}$ are algebraically equivalent.*

Proof. Lemma 4.3.7 implies that it suffices to show that the vector matroids of $\begin{bmatrix} 1 & \alpha_1 & 1 & \alpha_1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & \alpha_2 & 1 & \alpha_2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ are equal. This is clearly true. (If we index both ground sets with $\{1,2,3,4\}$ from left to right, then the circuits of both matroids are $\{1,2\}$, $\{1,3,4\}$, and $\{2,3,4\}$.) ■

Our goal in defining reductions and weak conforming is essentially to perform operations on matrices while leaving the Γ-frame submatrix intact. The following lemma does not contribute to that goal, so we will only make occasional use of it.

Lemma 4.4.4.

(1) For fields $\mathbb{F}$ with odd characteristic, if $k \in (\mathbb{F}_p - \{0,1\})$, then $\Phi_{Y_1}(k) \preceq \Phi_X$.

(2) For fields $\mathbb{F}$ with odd characteristic, we have $\Phi_{Y_1}(-1) \preceq \Phi_C$.

(3) For fields $\mathbb{F}$ of characteristic 2, we have $\Phi_{Y_1} \preceq \Phi_X$.

(4) For fields $\mathbb{F}$ of characteristic 2, we have $\Phi_{Y_1} \preceq \Phi_C$.

(5) For every field $\mathbb{F}$, we have $\Phi_{Y_0} \preceq \Phi_C$.

(6) For every field $\mathbb{F}$ and $k \in (\mathbb{F} - \{0\})$, we have $\Phi_C \preceq \Phi_{CXk}$.

(7) For every field $\mathbb{F}$ and $k \in (\mathbb{F} - \{0\})$, we have $\Phi_X \preceq \Phi_{CXk}$.

Proof. For (1), let A be a matrix conforming to $\Phi_{Y_1}(k)$. Then $A[B - X, E]$ is a $\{1\}$-frame matrix, and every entry of the row indexed by X is either $0, 1$, or k, each of which is contained in $\mathbb{F}_p$. Therefore, A conforms to Φ_X.

For (2), every matroid M conforming to $\Phi_{Y_1}(-1)$ is a restriction of the matroid obtained from the vector matroid of the following matrix by contracting c:

				c
0	1	$-1\cdots-1$	$0\cdots0$	1
	0	$0\cdots0$	$-1\cdots-1$	1
$\{1\}$-frame matrix	0	I	I	0

Removing c from this matrix, we obtain a $\{1\}$-frame matrix. Therefore, M conforms to Φ_C.

For (3), note that a simple matroid M of rank r virtually conforming to Φ_{Y_1} is a restriction of the vector matroid of a binary matrix A of the following form:

0	1	0	1	$1\cdots1$	$0\cdots0$	$1\cdots1$
	0	1	1	$0\cdots0$	$1\cdots1$	$1\cdots1$
$\{1\}$-frame matrix		0		I	I	I

If we label the sets of rows and columns of A as B and E respectively, and the first row as x, then we see that $A[B-x,E]$ is a $\{1\}$-frame matrix. If we let $X=\{x\}$, then we see that M conforms to Φ_X.

For (4), consider the matrix shown above for (3). Note that it is obtained by contracting c in the following binary matrix:

							c
0	0	0	1	$0\cdots0$	$0\cdots0$	$1\cdots1$	1
	1	0	0	$1\cdots1$	$0\cdots0$	$0\cdots0$	1
	0	1	0	$0\cdots0$	$1\cdots1$	$0\cdots0$	1
$\{1\}$-frame matrix		0		I	I	I	0

Removing c from this matrix, we obtain a $\{1\}$-frame matrix. Therefore, M conforms to Φ_C.

For (5), a matroid M conforming to Φ_{Y_0} is the vector matroid of a matrix of the following form, where $\bar{v}$ is a column vector all of whose entries are contained in $\mathbb{F}_p$:

$\{1\}$-frame matrix	$\bar{v}$

Let A be the matrix below. Label its sets of rows and columns as B and E respectively, and let c be the last column, with $C=\{c\}$.

0	1	1
$\{1\}$-frame matrix	0	$-\bar{v}$

Note that M is isomorphic to $M(A)/C$. Since $A[B,E-C]$ is a $\{1\}$-frame matrix, we see that M conforms to Φ_C.

For (6), let A be a matrix conforming to Φ_C and let $M=M(A)/C$ be the corresponding matroid conforming to Φ_C. If the column of A indexed by C is a zero column, then construct the matrix $\bar{A}$ by appending a row that is a vector of

weight one, indexed by X, whose only nonzero entry is k in the column indexed by C. One readily sees that $\bar{A}$ conforms to Φ_{CXk} and that the corresponding matroid $M(\bar{A})/C$ is equal to M. Otherwise, if the column of A indexed by C has a nonzero entry, then scale this column so that one of the entries is k. All other nonzero entries in that row are either 1 or -1, which are both contained in $\mathbb{F}_p$; therefore, the resulting matrix conforms to Φ_{CXk} by considering that row to be indexed by X.

For (7), every matroid M conforming to Φ_X is the vector matroid of a matrix of the following form, where $\bar{v}$ is a row vector all of whose entries are contained in $\mathbb{F}_p$:

$$\begin{array}{|c|}\hline \bar{v} \\ \hline \\ \Gamma\text{-frame matrix} \\ \\ \hline \end{array}$$

Consider the following matrix A, whose last column is indexed by $\{c\} = C$:

$$\begin{array}{|c|c|}\hline \bar{v} & k \\ \hline 0 & 1 \\ \hline & \\ \Gamma\text{-frame matrix} & 0 \\ & \\ \hline \end{array}$$

The matroid M is isomorphic to $M(A)/c$, which conforms to Φ_{CXk}. ■

Lemma 4.4.5. *Let Φ be a template with $y \in Y_1$. Let Φ' be the template obtained from Φ by removing y from Y_1 and placing it in Y_0. Then $\Phi' \preceq \Phi$.*

Proof. Every matrix respecting Φ' virtually respects Φ since a matrix virtually respecting Φ can have zero columns in Z. Thus, every matroid conforming to Φ' virtually conforms to Φ. ■

We call the operation described in Lemma 4.4.5 a *y-shift*.

Definition 4.4.6. Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template over a finite field $\mathbb{F}$. We say that Φ is in *standard form* if there are disjoint sets C_0, C_1, X_0, and X_1 such that $C = C_0 \cup C_1$, such that $X = X_0 \cup X_1$, such that $A_1[X_0, C_0]$ is an identity matrix, and such that $A_1[X_1, C]$ is a zero matrix.

Figure 4.2, with the stars representing arbitrary matrices, shows a matrix that virtually respects a template in standard form. Note that if Φ is in standard form, $|C_0| = |X_0|$. Also note that any of C_0, C_1, X_0, or X_1 may be empty.

Lemma 4.4.7. *Every frame template $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is equivalent to a frame template in standard form.*

Proof. Choose a basis C_0 for $M(A_1[X, C])$, and let $C_1 = C - C_0$. Repeatedly perform operation (5) to obtain a template Φ' where $A_1[X, C_0]$ consists of an identity matrix on top of a zero matrix. Each use of operation (5) results in an

<table>
<tr><td></td><td></td><td>Z</td><td>Y_0</td><td>Y_1</td><td>C_0</td><td>C_1</td></tr>
<tr><td>X_0</td><td>columns from $\Lambda|X_0$</td><td>0</td><td colspan="2" rowspan="2">$*$</td><td>I</td><td>$*$</td></tr>
<tr><td>X_1</td><td>columns from $\Lambda|X_1$</td><td>0</td><td colspan="2">0</td></tr>
<tr><td></td><td>Γ-frame matrix</td><td>unit or zero columns</td><td colspan="4">rows
from Δ</td></tr>
</table>

FIGURE 4.2. Standard Form

equivalent template; therefore, $\Phi \sim \Phi'$. Let $X_0 \subseteq X$ index the rows of the identity matrix, and let $X_1 \subseteq X$ index the rows of the zero matrix. Since C_0 is a basis for $M(A_1[X, C])$, the matrix $A_1[X, C_1]$ must be a zero matrix as well. Thus, Φ' is in standard form. ■

Throughout the rest of this section, we will implicitly use Lemma 4.4.7 to assume that all templates are in standard form. Also, the operations (1)-(12) to which we will refer throughout the rest of this dissertation are the operations (1)-(8) from Proposition 4.1.2 and (9)-(12) from Definition 4.1.3.

Lemma 4.4.8. *Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template over $\mathbb{F} \cong \mathrm{GF}(p^m)$ for some prime p. If Γ is nontrivial, then $\Phi_n \preceq \Phi$ for some prime n dividing $p^m - 1$.*

Proof. Perform operations (2) and (3) on Φ to obtain the following template:

$$(\Gamma, C, X, Y_0, Y_1, A_1, \{0\}, \{0\}).$$

On this template, repeatedly perform operation (7), then (8), then (4), and then (11) to obtain the following template:

$$(\Gamma, \emptyset, X_1, \emptyset, \emptyset, [\emptyset], \{0\}, \{0\}).$$

Now, on this template, repeatedly perform operation (6) to obtain the following template:

$$\Phi' = (\Gamma, \emptyset, \emptyset, \emptyset, \emptyset, [\emptyset], \{0\}, \{0\}).$$

Since Γ is a subgroup of the multiplicative group of a field, it is a cyclic group. If $|\Gamma|$ is a prime n, then $\Phi' = \Phi_n$ and we are finished. Otherwise, let n be a prime dividing $|\Gamma|$ and let α be a generator for Γ. Then the subgroup Γ' generated by $\alpha^{|\Gamma|/n}$ has order n. Perform operation (1) on Φ' to obtain the template

$$(\Gamma', \emptyset, \emptyset, \emptyset, \emptyset, [\emptyset], \{0\}, \{0\}),$$

which is Φ_n. ■

Lemma 4.4.9. *If $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is a frame template over $\mathbb{F} \cong \mathrm{GF}(p^m)$ for some prime p with $\Lambda|X_1$ nontrivial, then $\Phi_X \preceq \Phi$.*

Proof. Perform operations (1), (2), and (3) on Φ to obtain the following template, where λ is an element of Λ with $\lambda_x \neq 0$ for some $x \in X_1$:

$$(\{1\}, C, X, Y_0, Y_1, A_1, \{0\}, \langle\lambda\rangle).$$

On this template, repeatedly perform operation (7), then (8), then (4), and then (10) until the following template is obtained:

$$(\{1\}, \emptyset, X_1, \emptyset, \emptyset, [\emptyset], \{0\}, \langle\lambda\rangle).$$

On this template, repeatedly perform operation (5) to obtain a template that is identical to the previous one except that the support of λ contains only one element of X_1. On this template, repeatedly perform operation (6) to obtain the following template, where $x \in X_1$:

$$(\{1\}, \emptyset, \{x\}, \emptyset, \emptyset, [\emptyset], \{0\}, \mathbb{Z}/p\mathbb{Z}).$$

This template is Φ_X. ■

Lemma 4.4.10. *If $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is a frame template over $\mathbb{F} \cong \mathrm{GF}(p^m)$ for some prime p, then either $\Phi_C \preceq \Phi$ or Φ is equivalent to a template with $C_1 = \emptyset$.*

Proof. Suppose there is an element $\delta \in \Delta|C$ that is not in the row space of $A_1[X, C]$. Repeatedly perform operations (4) and (10) on Φ until the following template is obtained:

$$(\{1\}, C, X, \emptyset, \emptyset, A_1[X, C], \Delta|C, \Lambda).$$

On this template, perform operations (2) and (3) to obtain the following template:

$$(\{1\}, C, X, \emptyset, \emptyset, A_1[X, C], \langle\delta\rangle, \{0\}).$$

Every matrix virtually respecting this template is row equivalent to a matrix virtually respecting a template that is identical to the previous template except that there is the additional condition that $\delta|C_0$ is a zero vector. Note that $\delta|C_1$ is nonzero since, in the previous template, δ was not in the row space of $A_1[X, C]$. Now, on the current template, repeatedly perform operation (7) and then operation (6) to obtain the following template:

$$\Phi' = (\{1\}, C_1, \emptyset, \emptyset, \emptyset, [\emptyset], \langle\delta|C_1\rangle, \{0\}).$$

Now, every matroid M conforming to Φ' is obtained by contracting C_1 from $M(A)$, where A is a matrix conforming to Φ'. By contracting any single element $c \in C_1$, where $\delta_c \neq 0$, we turn the rest of the elements of C_1 into loops. So $C_1 - c$ is deleted to obtain M. Thus, M conforms to the template

$$(\{1\}, \{c\}, \emptyset, \emptyset, \emptyset, [\emptyset], \mathbb{Z}/p\mathbb{Z}, \{0\}),$$

which is Φ_C. Similarly, the converse is true that any matroid conforming to Φ_C conforms to Φ'. Thus, $\Phi_C \sim \Phi' \preceq \Phi$.

Now suppose that every element of $\Delta|C$ is in the row space of $A_1[X,C]$. Thus, contraction of C_0 turns the elements of C_1 into loops, and contraction of C_1 is the same as deletion of C_1. By deleting C_1 from every matrix virtually conforming to Φ, we see that Φ is equivalent to a template with $C_1 = \emptyset$. ■

Lemma 4.4.11. *If $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is a frame template over $\mathbb{F} \cong \mathrm{GF}(p^m)$, then one of the follwing is true:*

- $\Phi_C \preceq \Phi$
- *Φ is equivalent to a template with $\Lambda|X_1$ nontrivial and $\Phi_X \preceq \Phi$*
- *Φ is equivalent to a template with $\Lambda|X_0$ nontrivial and $\Phi_{CXk} \preceq \Phi$ for some $k \in \mathbb{F} - \{0\}$*
- *Φ is equivalent to a template with Λ trivial and $C = \emptyset$.*

Proof. By Lemmas 4.4.9 and 4.4.10, we may assume that $\Lambda|X_1$ is trivial and that $C_1 = \emptyset$. Perform operation (1) to obtain the following template:

$$(\{1\}, C, X, Y_0, Y_1, A_1, \Delta, \Lambda).$$

First, suppose there exist elements $\delta \in \Delta|C_0$ and $\lambda \in \Lambda|X_0$ such that $\sum \delta_i\lambda_i = k^{-1} \neq 0$. Thus, $\Lambda|X_0$ is nontrivial. Repeatedly perform operations (4) and (10) on Φ until the following template is obtained:

$$(\{1\}, C_0, X, \emptyset, \emptyset, A_1[X, C_0], \Delta|C_0, \Lambda).$$

On this template, repeatedly perform operation (6) to obtain the following template:

$$\Phi' = (\{1\}, C_0, X_0, \emptyset, \emptyset, A_1[X_0, C_0], \Delta|C_0, \Lambda|X_0).$$

Perform operations (2) and (3) on Φ' to obtain the following template:

$$(\{1\}, C_0, X_0, \emptyset, \emptyset, A_1[X_0, C_0], \langle\delta\rangle, \langle\lambda\rangle).$$

A matroid conforming to this template is obtained by contracting C_0. Let $a, b \in \mathbb{F}_p$. If $a\delta$ is in the row labeled by r and $b\lambda$ is in the column labeled by c, then when C_0 is contracted, $-abk$ is added to the entry of the Γ-frame matrix in row r and column c. We see then that this template is equivalent to Φ_{CXk}, where b is used to replace $b\lambda$ and a is used to replace $a\delta$.

Thus, we may assume that for every element $\delta \in \Delta|C_0$ and $\lambda \in \Lambda|X_0$, we have $\sum \delta_i\lambda_i = 0$. This implies that contraction of C has no effect on the Γ-frame matrix. So Φ is equivalent to a template with $\Lambda|X_0$ trivial. Therefore, since $\Lambda|X_1$ is trivial, we see that Λ is trivial. Note that operation (7) is a reduction that produces an equivalent template, since C must be contracted to produce a matroid that conforms to a template. By repeatedly performing operation (7), we obtain a template equivalent to Φ with $C = \emptyset$. ■

Lemma 4.4.12. *If $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is a frame template over $\mathbb{F} \cong \mathrm{GF}(p^m)$ with Λ trivial and with $C = \emptyset$, then either $\Phi_{Y_0} \preceq \Phi$ or Φ is equivalent to a template with Δ trivial.*

Proof. First, perform operation (1) to obtain the following template:

$$(\{1\}, C, X, Y_0, Y_1, A_1, \Delta, \{0\})$$

Now, we first consider the case where there is an element $\delta \in \Delta$ that is not in the row space of $A_1 = A_1[X_1, (Y_0 \cup Y_1)]$. Recall that a y-shift is the operation described in Lemma 4.4.5. Repeatedly perform y-shifts to obtain the following template, where $Y_0' = Y_0 \cup Y_1$:

$$(\{1\}, \emptyset, X, Y_0', \emptyset, A_1, \Delta, \{0\}).$$

On this template, perform operation (3) to obtain the following template:

$$(\{1\}, \emptyset, X, Y_0', \emptyset, A_1, \langle \delta \rangle, \{0\}).$$

Choose a basis B' for $M(A_1)$. By performing elementary row operations on every matrix virtually respecting Φ, we may assume that $A_1[X, B']$ consists of an identity matrix with zero rows below it and that $\delta|B'$ is the zero vector. By assumption, there is some element $y \in (Y_0' - B')$ such that δ_y is nonzero. Thus, we can repeatedly perform operation (11) to obtain the following template:

$$(\{1\}, \emptyset, X, B' \cup \{y\}, \emptyset, A_1[X, B' \cup \{y\}], \langle \delta | (B' \cup \{y\} \rangle, \{0\}).$$

Now, we can repeatedly perform operation (6) and then operation (12) to obtain the following template, which is Φ_{Y_0}:

$$(\{1\}, \emptyset, \emptyset, \{y\}, \emptyset, [\emptyset], \mathbb{F}_p, \{0\}).$$

Now suppose that every element $\delta \in \Delta$ is in the row space of $A_1 = A_1[X, (Y_0 \cup Y_1)]$. Since Λ is trivial, by performing elementary row operations on every matrix virtually respecting Φ, we obtain a template equivalent to Φ with Δ trivial. ■

Lemma 4.4.13. *If Φ is a frame template over $\mathbb{F} \cong \mathrm{GF}(p^m)$ with Δ trivial, then Φ is equivalent to a template Φ' where $A_1[X, Y_1]$ is a matrix with every column nonzero and where no column is a copy of another. Moreover, if $\mathbb{F} = \mathrm{GF}(2)$, then $M(A_1[X, Y_1])$ is simple.*

Proof. Let A be a matrix that virtually conforms to Φ. Since Δ is trivial, the columns of A indexed by elements of Z are formed by placing a column of $A_1[X, Y_1]$ on top of a unit column or a zero column. These columns can be made using any copy of the same column of $A_1[X, Y_1]$, so only one copy is needed. If any column of $A_1[X, Y_1]$ is a zero column, then any column indexed by an element of Z that is made with this zero column can also be made as a column indexed by an element

of $E-(Z\cup Y_0\cup Y_1\cup C)$ and choosing for the element of Λ the zero vector. Thus, no zero columns of $A_1[X,Y_1]$ are needed.

In the binary case, $M(A_1[X,Y_1])$ has no parallel elements because any such elements index copies of the same column. Also, $M(A_1[X,Y_1])$ has no loops because every column of $A_1[X,Y_1]$ is nonzero. Therefore, $M(A_1[X,Y_1])$ is simple. ∎

Lemma 4.4.14. *Let $\Phi=(\Gamma,C,X,Y_0,Y_1,A_1,\Delta,\Lambda)$ be a frame template over a finite field $\mathbb{F}$ with Λ and Δ trivial. If $M(A_1[X_1,(Y_0\cup Y_1)])$ has a circuit Y' with $|Y'\cap Y_1|\geq 3$, then either $\Phi_{Y_1}\preceq\Phi$ or $\Phi_{Y_1}(k)\preceq\Phi$ for some $k\in(\mathbb{F}-\{0,1\})$.*

Proof. A matroid conforming to Φ is obtained by contracting C. Since Λ and Δ are trivial, we may assume that $C=X_0=\emptyset$ and therefore that $X=X_1$. Perform operation (1), and then repeatedly perform operations (4) and (10) on Φ to obtain the following template:

$$(\{1\},\emptyset,X,Y_0\cap Y',Y_1\cap Y',A_1[X,Y'],\{0\},\{0\}).$$

Choose a 3-element subset Y'' of $Y'\cap Y_1$. Repeatedly perform y-shifts to obtain the following template:

$$(\{1\},\emptyset,X,Y'-Y'',Y'',A_1[X,Y'],\{0\},\{0\}).$$

On this template, repeatedly perform operation (12) to obtain the following template:

$$(\{1\},\emptyset,X',\emptyset,Y'',A_1[X',Y''],\{0\},\{0\}),$$

where X' is the subset of X that remains after $Y'-Y''$ is contracted. On this template, repeatedly perform operations (5) and (6) to obtain the following template, where X'' is a 2-element subset of X' and where α and β are nonzero:

$$(\{1\},\emptyset,X'',\emptyset,Y'',\begin{bmatrix}1&0&\alpha\\0&1&\beta\end{bmatrix},\{0\},\{0\})$$

First, we consider the case where $\alpha=\beta=1$. If $\mathbb{F}$ has characteristic 2, then this template is Φ_{Y_1}, and we are done. If $\mathbb{F}$ has odd characteristic, then perform a y-shift on the last column of $\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}$. Perform operation (12) to contract that column. The result is $\Phi_{Y_1}(-1)$.

Thus, we may assume that α and β are not both 1. Without loss of generality, $\alpha\neq 1$. Perform a y-shift on the second column of $\begin{bmatrix}1&0&\alpha\\0&1&\beta\end{bmatrix}$. Perform operation (12) to contract that column. The result is $\Phi_{Y_1}(\alpha)$. ∎

Lemma 4.4.15. *Let Φ be a refined frame template over a finite field $\mathbb{F}=\mathrm{GF}(p^m)$. Then one of the following is true:*

(i) $\Phi'\preceq\Phi$ for some $\Phi'\in\{\Phi_X,\Phi_C,\Phi_{Y_0}\}\cup\{\Phi_{CXk}:k\in\mathbb{F}-\{0\}\}\cup\{\Phi_n:n$ is a prime dividing $p^m-1\}$

(ii) Φ *is a* Y*-template.*

Proof. Suppose (i) does not hold. By Lemma 4.4.8, we may assume that Γ is trivial. By Lemma 4.4.11, we may assume that Λ is trivial and $C = \emptyset$. By Lemma 4.4.12, we may assume that Δ is trivial. By Definition 4.2.1, these facts, combined with the assumption that Φ is refined, imply that Φ is a Y-template. ■

Theorem 4.4.16. *Let* Φ *be a refined frame template over a finite field* $\mathbb{F} = \mathrm{GF}(p^m)$*. Then at least one of the following is true:*

(i) $\Phi_0 \sim \Phi$

(ii) $\Phi' \preceq \Phi$ *for some* $\Phi' \in \{\Phi_X, \Phi_C, \Phi_{Y_0}, \Phi_{Y_1}\} \cup \{\Phi_{CXk} : k \in \mathbb{F} - \{0\}\} \cup \{\Phi_n : n$ *is a prime dividing* $p^m - 1\} \cup \{\Phi_{Y_1}(k) : k \in (\mathbb{F} - \{0, 1\})\}$.

Proof. Suppose that (ii) does not hold. By Lemma 4.4.15, Φ is a Y-template. Thus, $\Phi = \mathrm{YT}(P_0, P_1)$ for some matrices P_0 and P_1 over $\mathbb{F}$. Therefore, the matrix A_1 is of the form

$$\begin{array}{c|c|c} & Y_1 & Y_0 \\ \hline I & P_1 & P_0 \end{array}.$$

By Lemma 4.4.14, every circuit of $M(A_1) = M(A_1[X_1, (Y_0 \cup Y_1)])$ has an intersection with Y_1 of size at most 2. Thus, each column of P_1 has at most one nonzero entry, and each column of P_0 has at most two nonzero entries. By Lemma 4.4.13, every column of P_1 must have exactly one nonzero entry $\alpha \neq 1$. If such a column exists, repeatedly perform operations (4), (10), and (6) to obtain the Y-template $\mathrm{YT}([\emptyset], [\alpha])$, which is $\Phi_{Y_1}(\alpha)$. Thus, we may assume that P_1 is an empty matrix. By column scaling, we may assume that each nonzero column of P_0 contains a 1 as an entry. Suppose that a column of P_0 contains a second nonzero entry $\alpha \neq -1$ (or $\alpha \neq 1$ in characteristic 2). By appropriate use of y-shifts and operation (12), we may reduce the matrix $[I|P_0]$ to the matrix $\left[\begin{array}{cc|c} 1 & 0 & \alpha \\ 0 & 1 & 1 \end{array}\right]$, with the first two columns labeled by elements of Y_1 and the last column labeled by an element of Y_0. By performing operation (12), we obtain the Y-template $\mathrm{YT}([\emptyset], [-\alpha])$, which is $\Phi_{Y_1}(-\alpha)$, contradicting the assumption that (ii) does not hold. Thus, every nonzero column of P_0 contains a 1, and if a column contains a second nonzero entry, that entry must be -1. Therefore, P_0 is a $\{1\}$-frame matrix.

Therefore, a simple matroid of rank r virtually conforming to Φ is a restriction of a matrix of the form

<table>
<tr><td colspan="2">0</td><td>unit columns</td><td>$D_{|X|}$</td></tr>
<tr><td>$I_{r-|X|}$</td><td>$D_{r-|X|}$</td><td>unit or zero columns</td><td>0</td></tr>
</table>

By scaling appropriately, we may assume that the bottom submatrix of unit columns actually consists of the negatives of unit columns. Thus a rank-r matroid conforming to Φ is a restriction of a matroid with a representation matrix of the form $[I_r|D_r]$. Thus, every matroid conforming to Φ is graphic, implying that (i) holds. ∎

4.5 Extremal Functions

A significant portion of the next two chapters (as well as the work of Nelson and Walsh [22]) involves the use of the structure theory of Geelen, Gerards, and Whittle to obtain results about the extremal functions (also called growth rate functions) of classes of represented matroids. We will prove that these results are true subject to Hypothesis 3.2.3. Recall from Section 1.3 that the *extremal function* for a minor-closed class $\mathcal{M}$, denoted by $h_{\mathcal{M}}(r)$, is the function whose value at an integer $r \geq 0$ is given by the maximum number of elements in a simple represented matroid in $\mathcal{M}$ of rank at most r. Also recall from Theorem 1.3.1 and the discussion following it that a minor-closed class $\mathcal{M}$ is *quadratically dense* if, for some $c \in \mathbb{R}$, we have $\binom{r+1}{2} \leq h_{\mathcal{M}}(r) \leq cr^2$ for all r. Moreover, a quadratically dense class contains the class of graphic matroids.

Geelen and Nelson proved the next result. (In fact, their result is a bit more detailed, but the following result follows from theirs.) Recall that $\varepsilon(M) = |\mathrm{si}(M)|$; that is, $\varepsilon(M)$ is the number of rank-1 flats of M.

Theorem 4.5.1 ([11, Theorem 6.1])**.** *Let $\mathcal{M}$ be a quadratically dense minor-closed class of matroids and let $p(x)$ be a real quadratic polynomial with positive leading coefficient. If $h_{\mathcal{M}}(n) > p(n)$ for infinitely many $n \in \mathbb{Z}^+$, then for all integers $r, s \geq 1$ there exists a vertically s-connected matroid $M \in \mathcal{M}$ satisfying $\varepsilon(M) > p(r(M))$ and $r(M) \geq r$.*

The next result is from Nelson and Walsh [22].

Lemma 4.5.2 ([22, Lemma 2.2])**.** *Let $\mathbb{F}$ be a finite field, let $f(x)$ be a real quadratic polynomial with positive leading coefficient, and let $k \in \mathbb{N}_0$. If $\mathcal{M}$ is a restriction-closed class of $\mathbb{F}$-represented matroids and if, for all sufficiently large n, the extremal function of $\mathcal{M}$ at n is given by $f(n)$, then for all sufficiently large r, every rank-r matroid $M \in \mathcal{M}$ with $\varepsilon(M) = f(r)$ is vertically k-connected.*

The next lemma is an easy observation.

Lemma 4.5.3. *Every frame template is strongly equivalent to a Y-reduced frame template such that no column of $A_1[X, Y_1]$ is contained in Λ.*

Proof. By Lemma 3.3.5, every template is strongly equivalent to some Y-reduced template $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$. Every element of Z indexes a column constructed by placing a column of $A_1[X, Y_1]$ on top of an identity column. If such a column is made from a column of $A_1[X, Y_1]$ that is a copy of an element of Λ, then the column can also be obtained in $E - (Z \cup C \cup Y_0 \cup Y_1)$ by choosing an identity column for the portion of the column coming from the Γ-frame matrix.

Thus, that element of Y_1 is unnecessary, and a template strongly equivalent to Φ can be obtained from Φ by removing that element of Y_1. ■

We will now show that, for all sufficiently large ranks, the extremal function for the set of matroids conforming to a frame template is given by a quadratic polynomial. We will call a largest simple matroid of a given rank that conforms to a template an *extremal matroid* of the template.

Lemma 4.5.4. *Let $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a Y-reduced frame template, with reduction partition $X = X_0 \cup X_1$, such that no column of $A_1[X, Y_1]$ is contained in Λ. Let $|\widehat{Y_0}|$ denote the number of columns of $A_1[X, Y_0]$ that are not contained in Λ. Let $|\widehat{\Lambda}|$ denote the maximum number of nonzero elements of Λ that pairwise are not scalar multiples of each other. And let t denote the difference between $|X_1|$ and the rank of the matrix $A_1[X_1, C \cup Y_0 \cup Y_1]$. If $r \geq 2|C| + |X| - t + 2$, then the size of a rank-r extremal matroid of Φ is $ar^2 + br + c$, where*

$$a = \frac{1}{2}|\Gamma||\Lambda|,$$

$$b = \frac{1}{2}|\Gamma||\Lambda|(2|C| + 2t - 2|X| - 1) + |\Lambda| + |Y_1|,$$

and

$$c = \frac{1}{2}(|C| + t - |X|)[|\Gamma||\Lambda|(|C| + t - |X| - 1) + 2|\Lambda| + 2|Y_1|] + |\widehat{\Lambda}| + |\widehat{Y_0}|.$$

Proof. An extremal matroid M of Φ is obtained by contracting C and deleting Y_1 from the vector matroid of some matrix A that conforms to Φ. Let $r_C = r_{M(A)}(C)$. Then $r(M(A)\backslash Y_1) = r + r_C$, and the number of rows of A is $r + r_C + t$. We wish to calculate the largest possible size of a simple matroid of the form $M(A)\backslash Y_1$, where A conforms to Φ and where $r(M(A)\backslash Y_1/C) = r$. Since A has $r + r_C + t$ rows, the number of rows of the Γ-frame submatrix of A is $r + r_C + t - |X|$, which we abbreviate as n. Let $n \geq 1$. Thus, A has at least $|X| + 1$ rows and rank at least $|X| - t + 1$, and $r \geq |X| - r_C - t + 1$.

In $E - (Z \cup C \cup Y_0 \cup Y_1)$, there are $|\Gamma||\Lambda|\binom{n}{2}$ distinct possible columns where the Γ-frame matrix has two nonzero entries per column. There are $|\Lambda|n$ distinct possible columns where the Γ-frame matrix has one nonzero entry per column. And there are $|\widehat{\Lambda}|$ distinct possible nonzero columns where the Γ-frame matrix is a zero column because including all of the elements of Λ would result in a matroid that is not simple.

The size of Z is at most $|Y_1|n$ since there are that many possible distinct possible columns.

The entire sets C and Y_0 are always contained in $M(A)$, but if any columns of $A_1[X, Y_0]$ are contained in Λ, then the corresponding element of $E-(Z \cup C \cup Y_0 \cup Y_1)$ must be deleted in order for the matroid to be simple. Therefore, adding together the elements of $E - (Z \cup C \cup Y_0 \cup Y_1)$, the elements of Z, and the elements of

$C \cup Y_0$, we see that

$$\varepsilon(M(A)\backslash Y_1) = |\Gamma||\Lambda|\binom{n}{2} + |\Lambda|n + |\widehat{\Lambda}| + |Y_1|n + |C| + |\widehat{Y_0}|.$$

If $C = \emptyset$, then $M(A)\backslash Y_1 = M(A)\backslash Y_1/C$. Keeping in mind that $n = r + r_C + t - |X|$, some arithmetic shows that this proves the result in the case where $C = \emptyset$. Thus, we now assume $C \neq \emptyset$.

One can see that $\varepsilon(M(A)\backslash Y_1)$ increases as r_C increases, since $n = r + r_C + t - |X|$. Thus, to achieve maximum density, we should take C to be independent, if possible. This can easily be achieved since $\Delta|C = \Gamma(\mathbb{F}_p^C)$ in a Y-reduced template. Thus, we take r_C to be equal to $|C|$ and $n = r + |C| + t - |X|$. In fact, let $A[B - X, C]$ be equal to

$$\begin{array}{|c|}
\hline
I_{|C|} \\
\hline
I_{|C|} \\
\hline
I_{|C|} \\
\hline
1 \cdots 1 \\
1 \cdots 1 \\
\hline
0 \\
\hline
\end{array}$$

This implies that $n \geq 3|C| + 2$ and, therefore, $r = n - |C| + |X| - t \geq 2|C| + |X| - t + 2$.

Claim 4.5.4.1. *For every pair $\{e, f\} \subseteq E - (C \cup Y_1)$, the set $C \cup \{e, f\}$ is independent in $M(A)$.*

Proof. Note that every column of $A[B - X, E - (C \cup Y_1)]$ has at most two nonzero entries. So the columns of $A[B - X, E - (C \cup Y_1)]$ labeled by e and f have nonzero entries in at most four rows. We proceed by induction on $|C|$. Suppose $|C| = 1$. Since the single column of $A[B - X, C]$ has five nonzero entries, there is a unit row in $A[B - X, C \cup \{e, f\}]$ whose nonzero entry is in C. This implies that $r(C \cup \{e, f\}) = r(\{e, f\}) + 1$. Since e and f are not parallel elements, $C \cup \{e, f\}$ must be independent.

Now suppose $|C| > 1$. Then there are at least $3|C| \geq 6$ unit rows in $A[B - X, C]$. Since e and f have nonzero entries in at most four rows, this implies that there is a unit row in $A[B - X, C \cup \{e, f\}]$ with its nonzero entry in a column labeled by some element $c \in C$. Thus $r(C \cup \{e, f\}) = r((C - c) \cup \{e, f\}) + 1$. By the induction hypothesis, $(C - c) \cup \{e, f\}$ is independent. Therefore, $C \cup \{e, f\}$ is independent also. $\square$

Claim 4.5.4.1 implies that, when C is contracted, the resulting matroid is still simple. Thus, $\varepsilon(M) = \varepsilon(M(A)\backslash Y_1) - |C| =$

$$|\Gamma||\Lambda|\binom{n}{2} + |\Lambda|n + |\widehat{\Lambda}| + |Y_1|n + |\widehat{Y_0}|$$

Some arithmetic, recalling that $n = r + |C| + t - |X|$, shows that this implies the result. ■

Although Lemma 4.5.4 is about Y-reduced templates, the fact that every frame template is strongly equivalent to a Y-reduced template implies that, for *every* frame template Φ, the size of a rank-r extremal matroid of Φ is given by a quadratic polynomial in r.

Lemma 4.5.5. *Suppose Hypothesis 3.2.3 holds, and let $\mathbb{F}$ be a finite field. Let $\mathcal{M}$ be a quadratically dense minor-closed class of $\mathbb{F}$-represented matroids, and let $\{\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t\}$ be the set of templates given by Hypothesis 3.2.3. For all sufficiently large r, the extremal matroids of $\mathcal{M}$ are the extremal matroids of the templates in some subset of $\{\Phi_1, \ldots, \Phi_s\}$. Moreover, $\{\Phi_1, \ldots, \Phi_s\}$ can be chosen so that it consists entirely of refined templates.*

Proof. Let p be the characteristic of $\mathbb{F}$. Since $\mathcal{M}$ is a quadratically dense minor-closed class and since $\varepsilon(\mathrm{PG}(r-1, \mathbb{F}_p)) = \frac{p^r-1}{p-1}$, for all sufficiently large r, no member of $\mathcal{M}$ contains $\mathrm{PG}(r-1, \mathbb{F}_p)$ as a minor. By Hypothesis 3.2.3 , there are a pair of integers k, n such that every simple vertically k-connected member of $\mathcal{M}$ with an $M(K_n)$-minor is a member of at least one of the classes $\mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s)$. Moreover, by Theorem 3.3.8, the integers k and n and templates $\Phi_1, \ldots, \Phi_s$ can be chosen so that the templates are refined.

By Lemmas 4.5.3 and 4.5.4, for every frame template Φ and for all sufficiently large r, the size of a rank-r extremal matroid of Φ is given by a quadratic polynomial in r. Thus, for all sufficiently large r, the size of the largest simple rank-r matroid that conforms to some template in $\{\Phi_1, \ldots, \Phi_s\}$ is given by a quadratic polynomial $h'_{\mathcal{M}}(r)$.

By definition, $h_{\mathcal{M}}(r) \geq h'_{\mathcal{M}}(r)$. We wish to show that equality holds for all sufficiently large r. Suppose otherwise. Then, for infinitely many r, we have $h_{\mathcal{M}}(r) > h'_{\mathcal{M}}(r)$. Theorem 4.5.1, with $h'_{\mathcal{M}}(r)$ playing the role of $p(n)$ and with k playing the role of s, implies that, for infinitely many r, there is a vertically k-connected rank-r matroid $M_r \in \mathcal{M}$ with $\varepsilon(M_r) > h'_{\mathcal{M}}(r)$. Thus, these M_r do not conform to any template in $\{\Phi_1, \ldots, \Phi_s\}$. By Hypothesis 3.2.3, these M_r contain no $M(K_n)$ minor. However, by Theorem 1.3.2, there is an integer c such that $\varepsilon(M_r) \leq cm$. This contradicts the fact that $\varepsilon(M_r) > h'_{\mathcal{M}}(r)$ for all r. By contradiction, we determine that $h_{\mathcal{M}}(r) = h'_{\mathcal{M}}(r)$, for all sufficiently large r.

Therefore, we know that, for all sufficiently large r, the extremal function $h_{\mathcal{M}}(r)$ is given by a quadratic polynomial. Now, Lemma 4.5.2 implies that, for all sufficiently large r, the rank-r extremal matroids of $\mathcal{M}$ are vertically k-connected. Thus, by Hypothesis 3.2.3, it suffices to show that, for all sufficiently large r, the largest simple matroids of rank r contain $M(K_n)$ as a minor. Suppose otherwise. Then, for infinitely many r, the largest simple matroids in $\mathcal{M}$ of rank r have no $M(K_n)$-minor. By Theorem 1.3.2, for infinitely many r, the largest simple matroids in $\mathcal{M}$ of rank r have size at most cr, for some integer c. This contradicts the quadratic density of $\mathcal{M}$. ■

The next lemma deals with a technicality involving virtual conforming. Note that there are templates such that the largest simple matroid of rank r conforming to the template is also the largest simple matroid of rank r virtually conforming to the template. The most obvious such templates are those with $Y_1 = \emptyset$. For another set of examples, let Φ be a template with $Y_1 = \{y_1, y_2 \ldots, y_n\}$ and let $Y' = \{y'_1, y'_2, \ldots, y'_n\} \subseteq Y_0$ such that, for each $i \leq n$, we have $A_1[X, y_i] = A_1[X, y'_i]$ and for each $\delta \in \Delta$, we have $\delta_{y_i} = \delta_{y'_i}$.

Lemma 4.5.6. *Suppose Hypothesis 3.2.3 holds, and let $\mathbb{F}$ be a finite field. Let $\mathcal{M}$ be a quadratically dense minor-closed class of $\mathbb{F}$-represented matroids, and let $\{\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t\}$ be the set of templates given by Hypothesis 3.2.3. For all sufficiently large r, the extremal matroids of $\mathcal{M}$ are the largest simple matroids that virtually conform to the templates in some subset of $\{\Phi_1, \ldots, \Phi_s\}$.*

Proof. By Lemma 4.5.5, we know that for all sufficiently large r, the extremal matroids of $\mathcal{M}$ are the largest simple matroids that conform to to the templates in some subset $T \subseteq \{\Phi_1, \ldots, \Phi_s\}$. By Lemma 4.1.4, a matroid that virtually conforms to a template in this set is a minor of some matroid that conforms to it. Since every matroid conforming to the template is in the minor-closed class $\mathcal{M}$, every matroid virtually conforming to the template is also in $\mathcal{M}$. The size of the largest simple matroid that virtually conforms to a template is at least the size of the largest simple matroid that conforms to the template. Thus, T must consist of templates where the largest simple matroids conforming to the template are the same as the largest simple matroids virtually conforming to the template. ■

Consider the example given immediately before the previous lemma. The largest simple matroid (virtually) conforming to Φ is the same as the largest simple matroid virtually conforming to the template obtained from Φ by deleting Y' from Y_0. In practice, when using templates to determine the extremal function of a minor-closed class, we will usually consider this other template, rather than Φ itself.

Chapter 5: Applications to Binary Matroids

In this chapter, we give some applications of the previous two chapters to the class of binary matroids. Every binary matroid is uniquely representable over GF(2), as well as over all other fields over which it is representable. Therefore, for simplicity of terminology, we make no distinction between abstract matroids and represented matroids in the binary case. Sections 5.1–5.6 deal with the classes of even-cycle and even-cut matroids. We show that only a few excluded minors suffice to characterize the highly connected members of these classes (subject to Hypothesis 3.2.2). Subject to Hypothesis 3.2.3, Section 5.7 answers the following question: What minor-closed classes of binary matroids have the same extremal function as the class of graphic matroids? Finally, Section 5.8 gives applications to the class of 1-flowing matroids, which are binary matroids with a certain property that generalizes the max-flow min-cut property of graphs.

5.1 Even-Cycle and Even-Cut Matroids

The complete lists of excluded minors for the classes of even-cycle matroids and even-cut matroids are currently unknown. Irene Pivotto and Gordon Royle [30] have found nearly 400 different excluded minors for the class of even-cycle matroids. We will show that, subject to Hypothesis 3.2.2, a highly connected binary matroid M of sufficient size is an even-cycle matroid if and only if it contains no minor isomorphic to one of three matroids. Similarly, subject to that same hypothesis, a highly connected binary matroid M of sufficient size is an even-cut matroid if and only if it contains no minor isomorphic to one of two matroids.

An even-cycle matroid is a binary matroid of the form $M = M\binom{w}{D}$, where $D \in \mathrm{GF}(2)^{V\times E}$ is the vertex-edge incidence matrix of a graph $G = (V, E)$ and $w \in \mathrm{GF}(2)^E$ is the characteristic vector of a set $W \subseteq E$. The pair (G, W) is an *even-cycle representation* of M. The edges in W are called *odd* edges, and the other edges are *even* edges. *Resigning* at a vertex u of G occurs when all the edges incident with u are changed from even to odd and vice-versa. This corresponds to adding the row of the matrix corresponding to u to the characteristic vector of W. Therefore, resigning at a vertex does not change an even-cycle matroid. A pair of vertices u, v of G is a *blocking pair* of (G, W) if (G, W) can be resigned so that every odd edge is incident with u or v. We will say that an even-cycle matroid has a blocking pair if it has an even-cycle representation with a blocking pair.

In her PhD thesis [29], Pivotto gives several descriptions of even-cut matroids, each of which can serve as a definition. The most practical definition for our purposes follows. An *even-cut matroid* is a matroid M that can be represented by a binary matrix with a row whose removal results in a matrix representing a cographic matroid. One can also think of an even-cut matroid as arising from a *graft*,

---

The content of Sections 5.7 and 5.8, as well as a portion of Section 5.4, were published in Kevin Grace and Stefan H.M. van Zwam, Templates for binary matroids, *SIAM Journal on Discrete Mathematics* **31** (2017), 254–282.

which is a pair (G,T), where G is a graph and T is a subset of $V(G)$ of even cardinality whose members are called *terminals*. The collection of inclusion-wise minimal edge cuts $\delta(U)$, where $U \subseteq V(G)$ and $|U \cap T|$ is even is the collection of circuits for an even-cut matroid. The graft (G,T) is an *even-cut representation* of that matroid.

In order to state our results on even-cycle and even-cut matroids, we need a few more definitions. If $\{F_1, F_2, \ldots, F_n\}$ is a collection of matroids, denote by $\mathcal{EX}(F_1, F_2, \ldots, F_n)$ the class of *binary* matroids with no minor contained in the set $\{F_1, F_2, \ldots, F_n\}$. We denote by $\mathrm{PG}(3,2)\backslash e$, or $\mathrm{PG}(3,2)_{-2}$, or $\mathrm{PG}(3,2)\backslash L$, respectively, the matroid obtained by deleting from $\mathrm{PG}(3,2)$ one element, or two elements, or the three points of a line. Note that $\mathrm{PG}(3,2)\backslash L$ is the vector matroid of the following matrix and, therefore, is the even-cycle matroid represented by the graph in Figure 5.1, with odd edges printed in bold.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

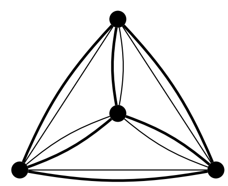

FIGURE 5.1. Even-Cycle Representation of $\mathrm{PG}(3,2)\backslash L$

We define L_{19} to be the dual of the cycle matroid of the graph obtained from K_7 by deleting two adjacent edges, and we define L_{11} to be the vector matroid of the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Finally, let H_{12} be the matroid with the even-cycle representation given in Figure 5.2. Again, odd edges are printed in bold.

We will prove the following theorems in Section 5.4.

Theorem 5.1.1. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that a k-connected binary matroid with at least $2k$ elements is an even-cycle matroid if and only if it is contained in $\mathcal{EX}(\mathrm{PG}(3,2)\backslash e, L_{19}, L_{11})$.*

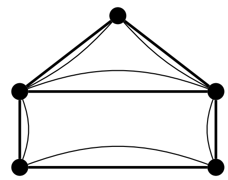

FIGURE 5.2. Even-Cycle Representation of H_{12}

Theorem 5.1.2. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that a k-connected binary matroid with at least $2k$ elements is an even-cycle matroid with a blocking pair if and only if it is contained in $\mathcal{EX}(\mathrm{PG}(3,2)\backslash L, M^*(K_6))$.*

We will prove the following theorem in Section 5.6.

Theorem 5.1.3. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that a k-connected binary matroid with at least $2k$ elements is an even-cut matroid if and only if it is contained in $\mathcal{EX}(M(K_6), H_{12}^*)$.*

Pivotto [29, Section 2.4.2] showed that the class of even-cycle matroids with a blocking pair consists of the duals of the members of the class of matroids with an even-cut representation with at most four terminals. Moreover, it is well-known that, for every positive integer k, a matroid M is k-connected if and only if M^* is k-connected. Therefore, Theorem 5.1.2 immediately implies the following result.

Corollary 5.1.4. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that a k-connected binary matroid with at least $2k$ elements has an even-cut representation with at most four terminals if and only if it is contained in $\mathcal{EX}((\mathrm{PG}(3,2)\backslash L)^*, M(K_6))$.*

In Section 5.2, we prove that $\mathrm{PG}(3,2)\backslash e$, L_{19}, L_{11}, $\mathrm{PG}(3,2)\backslash L$, $M(K_6)$, and H_{12}^* are indeed excluded minors for the respective classes given in the theorems above. The next several sections of the chapter are devoted to the converse statements. Much of the work required to prove these statements involves analysis of specific templates, showing that either one of these excluded minors can be constructed using the template or that the template is highly structured—to the point that only even-cycle matroids (or even-cycle matroids with a blocking pair, or even-cut matroids) can be constructed using the template. The finite case checks involved in this process are by and large carried out using the SageMath software system [34]. The technical lemmas proved by the computations will be given in Sections 5.3 and 5.5. In Section 5.4, we prove Theorems 5.1.1 and 5.1.2, and we also give the extremal function of $\mathcal{EX}(\mathrm{PG}(3,2))$, subject to Hypothesis 3.2.3. In Section 5.6, we prove Theorem 5.1.3.

The results listed in this section and the techniques used to prove them give no indication of how large the value for k must be. The sets of matroids in our theorems are not unique, and their members do not necessarily need to be excluded minors for the classes we study. For example, L_{19} and $M^*(K_6)$ can be replaced with $M^*(K_n)$ for $n > 6$, and $M(K_6)$ can be replaced with $M(K_n)$ for $n > 6$. We chose

the small matroids that we did because they are actually excluded minors for the various classes. Presumably, this comes at the cost of a larger value for k.

5.2 Excluded Minors

In this section, we will establish that the matroids given in Section 5.1 are indeed excluded minors for the various classes of matroids. Section A.2 in the Appendix gives the SageMath code for the functions `is_even_cycle` and `is_even_cut`, which test whether a binary matroid is even-cycle or even-cut, respectively. The code is based on the fact that an even-cycle matroid M can be represented by a binary matrix with a row whose removal results in a matrix representing a graphic matroid. Thus, there is some binary extension N of M on ground set $E(M) \cup \{e\}$ such that N/e is graphic. Therefore, to check if a binary matroid M is even-cycle, it suffices to check if N/e is graphic for some binary extension N of M. If this is false for all such N, then M is not even-cycle. The even-cut case is analogous. Section A.2 also gives the SageMath code for the functions `is_even_cycle_excluded_minor` and `is_even_cut_excluded_minor`, which test whether a binary matroid is an excluded minor for the class of even-cycle matroids and even-cut matroids, respectively.

Theorem 5.2.1. *Each of the matroids* $\mathrm{PG}(3,2)\backslash e$*,* L_{19}*, and* L_{11} *is an excluded minor for the class of even-cycle matroids.*

Proof. The largest even-cycle matroid of rank r has a representation obtained by putting an odd edge in parallel with every even edge of K_r, and by adding an odd loop. Therefore, the size of the largest even-cycle matroid of rank r is $2\binom{r}{2}+1 = r^2 - r + 1$. Therefore, the matroid $\mathrm{PG}(3,2)\backslash e$, which has rank 4 and size 14 is too large to be even-cycle. Deletion of any element from $\mathrm{PG}(3,2)\backslash e$ results in the unique matroid (up to isomorphism) obtained from $\mathrm{PG}(3,2)$ by deleting two elements. This is exactly the largest simple even-cycle matroid of rank 4, as described above. Thus, deletion of any element from $\mathrm{PG}(3,2)\backslash e$ results in an even-cycle matroid. To see that contraction of any element from $\mathrm{PG}(3,2)\backslash e$ results in an even-cycle matroid, note that every binary matroid of rank 3 is even-cycle since removal of any row results in a matrix that obviously has at most two nonzero entries per column.

The fact that L_{19} and L_{11} are excluded minors for the class of even-cycle matroids was verified using SageMath. Section A.2 gives the code used to define L_{19} and L_{11}, as well as the code used to check that they are excluded minors. The code returned `True` in both cases. ■

Theorem 5.2.2. *The matroids* $M(K_6)$ *and* H_{12}^* *are excluded minors for the class of even-cut matroids.*

Proof. This was verified using SageMath. The code for the computations can be found in Section A.2. ■

In order to prove that $\mathrm{PG}(3,2)\backslash L$ and $M^*(K_6)$ are excluded minors for the class of even-cycle matroids with a blocking pair, we need a few more lemmas. Recall the

definition of the Y-template Φ_{Y_1} from Definition 4.4.2. Also, recall the definition of the rank-r universal matroid for a Y-template, found in Definition.4.2.3.

Definition 5.2.3. Let X_r be the rank-r universal matroid for Φ_{Y_1}.

Note that X_r is the largest simple matroid of rank r that virtually conforms to Φ_{Y_1}, since $Y_0 = \emptyset$ in Φ_{Y_1}.

Lemma 5.2.4. *The class $\mathcal{M}_v(\Phi_{Y_1})$ is the class of even-cycle matroids with a blocking pair. This class is minor-closed.*

Proof. Every simple matroid M virtually conforming to Φ_{Y_1} is a restriction of X_r for some r.

Label the rows of A_r as $1, \ldots, r$. Add to the matrix row $r+1$, which is the sum of rows $2, \ldots, r$. This does not change the matroid X_r. We see that X_r is an even-cycle matroid (G, W), where row 1 is the characteristic vector of W and rows $2, \ldots, r+1$ form the incidence matrix of G. Moreover, every edge in W is incident with the vertex corresponding to either row 2 or row $r+1$. Thus, every matroid virtually conforming to Φ_{Y_1} has an even-cycle representation with a blocking pair. Conversely, every matroid that has an even-cycle representation with a blocking pair $\{u, v\}$ virtually conforms to Φ_{Y_1}, by making u correspond to the second row and making v correspond to row $r+1$, which can be removed without changing the matroid.

By resigning whenever we wish to contract an element represented by an odd edge, it is not difficult to see that the class of matroids having an even-cycle representation with a blocking pair is minor-closed. ∎

Lemma 5.2.5. *A matroid is an even-cycle matroid with a blocking pair if and only if its cosimplification also is.*

Proof. The class of even-cycle matroids with a blocking pair is minor-closed; therefore the cosimplification of an even-cycle matroid with a blocking pair will be such a matroid as well.

For the converse, let M be even-cycle with a blocking pair, and consider an even-cycle representation of M with a blocking pair. It suffices to consider coextensions N of M, with $E(N) = E(M) \cup e$ and such that either $\{e, f\}$ is a series pair of N or e is a coloop of N. First, we consider the case where $\{e, f\}$ is a series pair. If f is represented by an even edge in the even-cycle representation of M, then e and f in N are represented by edges obtained by subdividing f in M. This has no effect on the blocking pair. If f is represented by an odd edge other than a loop, we resign at a vertex in the blocking pair that is incident with f. This maintains the blocking pair, but now f is represented by an even edge as above. Now consider the case where f is represented by an odd loop. Since M is even-cycle with a blocking pair, Lemma 5.2.4 and Definition 5.2.3 imply that M is a restriction of a matroid represented by a matrix of the following form:

	f					
0	1	0	1	$1\cdots1$	$0\cdots0$	$1\cdots1$
	0	1	1	$0\cdots0$	$1\cdots1$	$1\cdots1$
$\{1\}$-frame matrix	0			I	I	I

Since N contains $\{e, f\}$ as a series pair, N is a restriction of a matroid N' represented by a matrix of the following form:

e		f					
0	0	1	0	1	$1\cdots1$	$0\cdots0$	$1\cdots1$
0		0	1	1	$0\cdots0$	$1\cdots1$	$1\cdots1$
0	$\{1\}$-frame matrix	0			I	I	I
1	$0\cdots\cdots\cdots\cdots0$	1	0	0	$0\cdots0$	$0\cdots0$	$0\cdots0$

By Lemma 5.2.4, N' is even-cycle with a blocking pair. Therefore, so is N.

Lastly, we consider the case where e is a coloop of N. Then N can be represented by a graph obtained from the graph representing M by adding a new vertex and joining it to any other vertex with an even edge. The blocking pair is maintained. ■

Theorem 5.2.6. *The matroids* $\mathrm{PG}(3,2)\backslash L$ *and* $M^*(K_6)$ *are excluded minors for the class of even-cycle matroids with a blocking pair.*

Proof. Note that X_4 is the matroid obtained from $\mathrm{PG}(3,2)$ by deleting an independent set of size 3. Therefore, $\mathrm{PG}(3,2)\backslash L$ is not a restriction of X_4. By Lemma 5.2.4, $\mathrm{PG}(3,2)\backslash L$ is not an even-cycle matroid with a blocking pair. However, since $X_3 = \mathrm{PG}(2,2) = F_7$, all binary matroids of rank at most 3 are even-cycle matroids with blocking pairs. Therefore, $\mathrm{PG}(3,2)\backslash L/e$ is even-cycle with a blocking pair for each element e of $\mathrm{PG}(3,2)\backslash L$. Moreover, by deleting any element from $\mathrm{PG}(3,2)\backslash L$, we obtain a restriction of X_4. Therefore, $\mathrm{PG}(3,2)\backslash L$ is an excluded minor for the class of even-cycle matroids with a blocking pair.

By Theorem 5.2.2, $M(K_6)$ is not an even-cut matroid. Recall from Section 5.1 that the dual of an even-cycle matroid with a blocking pair is an even cut matroid. Therefore, $M^*(K_6)$ is not an even-cycle matroid with a blocking pair. It remains to show that $M^*(K_6\backslash e)$ and $M^*(K_6/e)$ are even-cycle with a blocking pair. By Lemma 5.2.5, $M^*(K_6/e)$ has an even-cycle representation with a blocking pair if and only if $M^*(K_5)$ does. Even-cycle representations of $M^*(K_5)$ and $M^*(K_6\backslash e)$, with odd edges printed in bold, are given in Figure 5.3. Each of these representations have blocking pairs. ■

5.3 Some Technical Lemmas Proved with SageMath: Even-Cycle Matroids

In this section, we list several technical lemmas that we need to prove Theorems 5.1.1 and 5.1.2. The reader may prefer to move on to Section 5.4, referring to Section 5.3 as necessary. Recall from Definition 4.2.14 that the complete, lifted Y-template $\mathrm{YT}([D_{|X|}|P_0], [\emptyset])$ determined by a matrix P_0 is denoted by Φ_{P_0}.

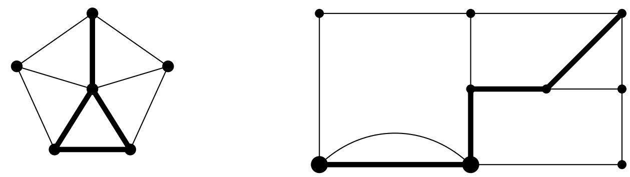

FIGURE 5.3. Even-Cycle Representations of $M^*(K_5)$ and $M^*(K_6\backslash e)$

Lemma 5.3.1. *If P_0 is a binary matrix that contains a column with five or more nonzero entries, then $\mathcal{M}(\Phi_{P_0}) \nsubseteq \mathcal{EX}(\mathrm{PG}(3,2)\backslash e)$.*

Proof. The function `complete_Y_template_matrix`, the SageMath code for which is found in A.3, builds the matrix $[I_n|D_n|P_0]$, where n is the number of rows of an input matrix `P_0`. We use this function to build $[I_n|D_n|P_0]$, where $P_0 = [1,1,1,1,1]^T$. We then test if M contains $\mathrm{PG}(3,2)\backslash e$ as a minor by looking for a subset S of the ground set of M such that $r(M/S) = 4$ and $|\mathrm{si}(M/S)| \geq 14$. If this subset exists, then M must contain $\mathrm{PG}(3,2)\backslash e$ as a minor. The code for this computation is below; it returned $\{$`15`$\}$. In the Python programming language, on which SageMath is based, a set of size n has elements labeled $0, 1, \ldots, n-1$. Therefore, $\{$`15`$\}$ means that the sixteenth element should be contracted to obtain $\mathrm{PG}(3,2)\backslash e$.

```
P0 = Matrix(GF(2), [[1],
                    [1],
                    [1],
                    [1],
                    [1]])
A = complete_Y_template_matrix(P0)
M = Matroid(field=GF(2), matrix=A)

#   This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
```

∎

Lemma 5.3.2. *If P_0 contains the submatrix*

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

then $\mathcal{M}(\Phi_{P_0}) \nsubseteq \mathcal{EX}(\mathrm{PG}(3,2)\backslash e)$.

Proof. The proof is similar to that of Lemma 5.3.1. The code returned {`21, 22`}. ■

Lemma 5.3.3. *If* P_0 *contains the submatrix*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

then $\mathcal{M}(\Phi_{P_0}) \nsubseteq \mathcal{EX}(\mathrm{PG}(3,2)_{-2})$ *and* $\mathcal{M}(\Phi_{P_0}) \nsubseteq \mathcal{EX}(L_{11})$.

Proof. The following SageMath code was used to check for L_{11} as a minor of $M = M([[I_6|D_6|P_0]])$. The code also checks for $\mathrm{PG}(3,2)_{-2}$ as a minor of M. We test if M contains $\mathrm{PG}(3,2)_{-2}$ as a minor by looking for a subset S of the ground set of M such that $r(M/S) = 4$ and $|\mathrm{si}(M/S)| \geq 13$, rather than 14.

```
P0 = Matrix(GF(2), [[1,1,0],
                    [1,1,0],
                    [1,0,1],
                    [1,0,1],
                    [0,1,1],
                    [0,1,1]])
A=complete_Y_template_matrix(P0)
M=Matroid(field=GF(2), matrix=A)
M.has_minor(L11)

#   This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break
```

The code returned `True` and {`0, 21`}. ■

Lemma 5.3.4. *If* P_0 *contains the submatrix*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

then $\mathcal{M}(\Phi_{P_0}) \nsubseteq \mathcal{EX}(\mathrm{PG}(3,2)\backslash e)$.

Proof. The proof is similar to that of Lemma 5.3.1. The code returned `{0,22}`. ■

Lemma 5.3.5. *If P_0 contains the submatrix*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then $\mathcal{M}(\Phi_{P_0}) \not\subseteq \mathcal{EX}(\mathrm{PG}(3,2)\backslash e)$.

Proof. The proof is similar to that of Lemma 5.3.1. The code returned `{0,30,23}`. ■

Lemma 5.3.6. *If P_0 contains the submatrix*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

then $\mathcal{M}(\Phi_{P_0}) \not\subseteq \mathcal{EX}(\mathrm{PG}(3,2)\backslash e)$.

Proof. The proof is similar to that of Lemma 5.3.1. The code returned `{0,23}`. ■

Lemma 5.3.7. *If P_0 contains the submatrix*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

then $\mathcal{M}(\Phi_{P_0}) \not\subseteq \mathcal{EX}(\mathrm{PG}(3,2)_{-2})$.

Proof. The proof is similar to that of Lemma 5.3.1, except that we look for a subset S of the ground set of $M = M([I_6|D_6|P_0])$ such that $r(M/S) = 4$ and $|\mathrm{si}(M/S)| \geq 13$, rather than 14. The code returned `{0,21}`. ■

Lemma 5.3.8. *If P_0 contains the submatrix*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

then $\mathcal{M}(\Phi_{P_0}) \nsubseteq \mathcal{EX}(\mathrm{PG}(3,2)_{-2})$.

Proof. The proof is similar to that of Lemma 5.3.1, except that we look for a subset S of the ground set of $M = M([I_5|D_5|P_0])$ such that $r(M/S) = 4$ and $|\mathrm{si}(M/S)| \geq 13$, rather than 14. The code returned $\{\mathtt{0}\}$. ■

5.4 Even-Cycle Matroids

Before we can prove the results listed in Section 5.1, we need some information about $\mathcal{EX}(\mathrm{PG}(3,2))$.

Lemma 5.4.1.

(i) The set of matroids conforming to $\mathcal{M}(\Phi_X)$ is contained in $\mathcal{EX}(\mathrm{PG}(3,2))$.

(ii) Let Φ be a refined frame template such that $\mathcal{M}_w(\Phi) \subseteq \mathcal{EX}(\mathrm{PG}(3,2))$. Then either $\Phi \sim \Phi_X$ or Φ is a Y-template.

Proof. The class of matroids conforming to Φ_X is exactly the class of even-cycle matroids. This class is minor-closed. The largest simple, even-cycle matroid of rank r has an even-cycle representation obtained from the graph K_r by adding to each even edge an odd edge in parallel as well as adding one odd loop to the graph. Therefore, the class of even-cycle matroids has an extremal function of $2\binom{r}{2} + 1 = r^2 - r + 1$. Thus, the largest simple, even-cycle matroid of rank 4 has size 13. Since $\mathrm{PG}(3,2)$ has size 15, we have $\mathcal{M}(\Phi_X) \subseteq \mathcal{EX}(PG(3,2))$. This proves (i).

To prove (ii), let

$$\Phi = (\{1\}, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$$

be a refined binary frame template such that $\mathcal{M}_w(\Phi) \subseteq \mathcal{EX}(\mathrm{PG}(3,2))$. Consider the graft matroid $M(K_6, V(K_6))$. A straightforward computation shows that, by contracting the nongraphic element, we obtain $\mathrm{PG}(3,2)$. Therefore, $\Phi_{Y_0} \npreceq \Phi$. By Lemma 4.4.4, we also have $\Phi_C \npreceq \Phi$ and $\Phi_{CX} \npreceq \Phi$.

Recall the definition of standard form from Definition 4.4.6. Lemma 4.4.7 implies that we may assume Φ is in standard form. Since $\Phi_C \npreceq \Phi$, by Lemma 4.4.10 we may assume that $C_1 = \emptyset$. Also, by Lemma 4.4.11, since $\Phi_{CX} \npreceq \Phi$ and $\Phi_C \npreceq \Phi$, either $\Lambda|X_1$ is nontrivial and $\Phi_X \preceq \Phi$ or Λ is trivial and $C = \emptyset$.

First, suppose that Λ is trivial and $C = \emptyset$. Since $\Phi_{Y_0} \npreceq \Phi$, Lemma 4.4.12 implies that Φ is equivalent to a template with Δ trivial. In the binary case, Γ is

always trivial, so Φ is a refined template with all groups trivial. Therefore, Φ is a Y-template. This is one of the possible conclusions of (ii).

Thus, we may assume that $\Lambda|X_1$ is nontrivial and $\Phi_X \preceq \Phi$. Suppose $\big|\Lambda|X_1\big| > 2$. On the template

$$\Phi = (\{1\}, C_0, Y_0, Y_1, A_1, \Delta, \Lambda),$$

perform operation (3) and then repeatedly perform operations (4) and (10) to obtain the template

$$(\{1\}, C_0, X, \emptyset, \emptyset, A_1[X, C_0], \{0\}, \Lambda).$$

Then repeatedly perform operations (5) and (7) to obtain

$$(\{1\}, \emptyset, X_1, \emptyset, \emptyset, [\emptyset], \{0\}, \Lambda|X_1).$$

Since $\Lambda|X_1$ has size greater than 2, it contains a subgroup Λ' isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Perform operation (2) to obtain the template

$$(\{1\}, \emptyset, X_1, \emptyset, \emptyset, [\emptyset], \{0\}, \Lambda');$$

then repeatedly perform operations (5) and (6) to obtain

$$(\{1\}, \emptyset, X', \emptyset, \emptyset, [\emptyset], \{0\}, \Lambda''),$$

where $|X'| = 2$ and Λ'' is the additive group generated by $[1,0]^T$ and $[0,1]^T$. One readily sees that $\mathrm{PG}(3,2)$ conforms to this template. Therefore, $|\Lambda| = 2$. We may perform row operations so that Λ is generated by $[1, 0 \dots, 0]^T$. Let Σ be the element of X such that $\Lambda|\{\Sigma\}$ is nonzero.

Now, suppose there is an element $\bar{x} \in \Delta$ that is not in the row space of A_1. Perform operations (2) and (3) on Φ to obtain

$$(\{1\}, C_0, X, Y_0, Y_1, A_1, \{0, \bar{x}\}, \{0\}).$$

Now, by a similar argument to the one used in the proof of Lemma 4.4.12, we have $\Phi_{Y_0} \preceq \Phi$. Since we already know this is not the case, we deduce that every element of Δ is in the row space of A_1.

Let $\bar{x} \in \Delta|C_0$ and $\bar{y} \in \Lambda$ be such that there are an odd number of natural numbers i such that $\bar{x}_i = \bar{y}_i = 1$. Then we call the ordered pair $(\bar{x}, \bar{y})$ a *pair of odd type*. Otherwise, $(\bar{x}, \bar{y})$ is a *pair of even type*. Suppose $(\bar{x}, \bar{y})$ is a pair of odd type with $\bar{y}|X_1$ a zero vector. By performing operations (2) and (3) and repeatedly performing operations (4) and (10), we obtain

$$(\{1\}, C_0, X, \emptyset, \emptyset, A_1[X, C], \{0, \bar{x}\}, \{0, \bar{y}\}),$$

which is equivalent to Φ_{CX}. We already know this is not the case. Therefore, for every pair $(\bar{x}, \bar{y})$ of odd type, $\bar{y}|X_1 = [1, 0, \dots, 0]^T$.

Suppose $\bar{x} \in \Delta|C$ and $\bar{y}_1, \bar{y}_2 \in \Lambda$ are such that $\bar{y}_1|X_1 = \bar{y}_2|X_1 = [1, 0, \dots, 0]^T$, such that $(\bar{x}, \bar{y}_1)$ is a pair of odd type, and such that $(\bar{x}, \bar{y}_2)$ is a pair of even type.

Then $(\bar{y}_1+\bar{y}_2)|X_1$ is the zero vector, and $(\bar{x}, \bar{y}_1+\bar{y}_2)$ is a pair of odd type. Therefore, either all pairs $(\bar{x}, \bar{y}) \in \Delta|C \times \Lambda$ are of even type, in which case Φ is equivalent to a template with $\Lambda|X_0$ trivial and $C = \emptyset$, or if $(\bar{x}, \bar{y})$ is a pair of odd type, then $(\bar{x}, \bar{z})$ is of odd type for every $\bar{z} \in \Lambda$ with $\bar{z}|X_1$ nonzero. In this case, consider any matrix virtually conforming to Φ. After contracting C, we can restore the Γ-frame matrix by adding Σ to each row where the Γ-frame matrix has been altered. Therefore, Φ is equivalent to a template with $\Lambda|X_0$ trivial and $C = \emptyset$.

So we now have that

$$\Phi = (\{1\}, \emptyset, X, Y_0, Y_1, A_1, \Delta, \Lambda),$$

with Λ generated by $[1, 0 \ldots, 0]^T$ and with every element of Δ in the row space of A_1. We will now show that, in fact, Φ is equivalent to a template with Δ trivial. On Φ, perform y-shifts to obtain the following template, where $Y_0' = Y_0 \cup Y_1$:

$$\Phi' = (\{1\}, \emptyset, X, Y_0', \emptyset, A_1, \Delta, \Lambda).$$

By repeatedly performing operation (5) and then operation (6) on this template, we may assume that A_1 has the following form, with the star representing an arbitrary binary matrix and $\bar{v}$ representing an arbitrary row vector:

$$\left[\begin{array}{c|c} 0 \cdots 0 & \bar{v} \\ \hline I_{|X|-1} & * \end{array}\right].$$

Also, since $\Lambda|(X - \{\Sigma\})$ is trivial, we may perform row operations on every matrix conforming to Φ' to obtain a template

$$\Phi'' = (\{1\}, \emptyset, X, Y_0', \emptyset, A_1, \Delta'', \Lambda),$$

so that every element of Δ'' has 0 for its first $|X| - 1$ entries. Since every element of Δ was in the row space of A_1, the only possible nonzero element of Δ'' is the row vector with 0 for its first $|X| - 1$ entries and whose last $|Y_0'| - |X| + 1$ entries form the row vector $\bar{v}$. Note that operations (5) and (6) and the row operations we performed on every matrix conforming to Φ' each changes a template to an equivalent template. Thus, we may assume that $\bar{v}$ is nonzero and that $\Delta'' = \{\mathbf{0}, \bar{v}\}$ because otherwise, Φ is equivalent to a template with Δ trivial. So, for some $y \in Y_0'$, we have $\bar{v}_y = 1$. On the template Φ'', repeatedly perform operation (11) and then operation (10) to obtain the following template:

$$\Phi''' = (\{1\}, \emptyset, \{\Sigma\}, \{y\}, \emptyset, [1], \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}),$$

which is Φ_{CX}. Since we already know that $\Phi_{CX} \not\preceq \Phi$, we have shown that Φ must be equivalent to a template with Δ trivial. So we may assume that Φ is the following template, with Λ generated by $[1, 0, \ldots, 0]^T$.

$$\Phi = (\{1\}, \emptyset, X, Y_0, Y_1, A_1, \{0\}, \Lambda)$$

Now, let us consider the structure of the matrix A_1. By repeated use of operation (5), we may assume that A_1 is of the following form, with the top row indexed by Σ, with $*$ representing an arbitrary row vector, and with each L_i representing an arbitrary binary matrix:

	Y_1		Y_0
$0\cdots0$	$0\cdots0$	$1\cdots1$	$*$
I	L_0	L_1	L_2

Suppose either L_0 or L_1 has a column with two or more nonzero entries. Let y be the element of Y_1 that indexes that column, and let Y' be the union of $\{y\}$ with the subset of Y_1 that indexes the columns of the identity submatrix of $A_1[X, Y_1]$. Repeatedly perform operations (4) and (10) on Φ to obtain

$$(\{1\}, \emptyset, X, \emptyset, Y', A_1, \{0\}, \Lambda).$$

On this template, repeatedly perform y-shifts, operation (11), and operation (6) to obtain

$$(\{1\}, \emptyset, X', \emptyset, Y'', \begin{bmatrix} 0 & 0 & x \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \{0\}, \Lambda),$$

where $x = i$ if y indexes a column of L_i and where X' and Y'' index the set of rows and columns, respectively, of the matrix $\begin{bmatrix} 0 & 0 & x \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

The following matrix conforms to this template. By contracting the columns printed in bold, we obtain PG$(3, 2)$.

$$\left[\begin{array}{cccccccccccccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0
\end{array}\right].$$

This shows that L_0 and L_1 consist entirely of unit and zero columns. Thus, by Lemma 4.4.13, L_0 is an empty matrix and L_1 consists entirely of distinct unit columns. Therefore, A_1 is of the following form:

	Y_1		Y_0
$0\cdots0$	$0\cdots0$	$1\cdots1$	$*$
I	0	I	Q_1
0	I	0	Q_2

with each Q_i representing an arbitrary binary matrix.

Let Q be the submatrix of A_1 consisting of Q_1 and Q_2. Suppose that Q has a column c, indexed by the element $y \in Y_0$ with three or more nonzero entries.

Repeatedly perform operation (10) on Φ to obtain the template

$$\Phi' = (\{1\}, \emptyset, X, \{y\}, Y_1, A_1[X, Y_1 \cup \{y\}], \{0\}, \Lambda).$$

Let $c = \left[\begin{array}{c} c_1 \\ \hline c_2 \end{array}\right]$, with c_1 a column of Q_1 and c_2 a column of Q_2. Consider the following cases:

Case 1. The vector c_1 has three nonzero entries.

Case 2. The vector c_1 has two nonzero entries, and c_2 has one nonzero entry.

Case 3. The vector c_1 has one nonzero entry, and c_2 has two nonzero entries.

Case 4. The vector c_2 has three nonzero entries.

In Case i, repeatedly perform y-shifts and operation (11) to obtain the template

$$\Phi_i'' = (\{1\}, \emptyset, X', \{y\}, Y_1', A_{1,i}, \{0\}, \Lambda),$$

where $A_{1,i}$ is the matrix defined below with rows indexed by X' and columns indexed by $Y_1' \cup \{y\}$. In each case, the last column is indexed by y, and it turns out that the value of x does not matter.

$$A_{1,1} = \left[\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 1 & 1 & 1 & x \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \quad A_{1,2} = \left[\begin{array}{ccc|cc|c} 0 & 0 & 0 & 1 & 1 & x \\ \hline 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A_{1,3} = \left[\begin{array}{ccc|c|c} 0 & 0 & 0 & 1 & x \\ \hline 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \quad A_{1,4} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & x \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

In Case i, the matrix below virtually conforms to Φ_i''. By contracting the columns printed in bold, we obtain PG$(3, 2)$.

Case 1:

$$\left[\begin{array}{ccc|cccccccccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & \boldsymbol{x} \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \mathbf{1} \\ \hline 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{array}\right]$$

Case 2:

$$\left[\begin{array}{ccccccc|ccccccccc|c} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \mathbf{0} & 1 & 1 & 0 & 0 & 1 & 1 & \boldsymbol{x} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{0} & 1 & 0 & 1 & 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{0} & 0 & 1 & 0 & 1 & 0 & 1 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 0 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 & 1 & 1 & 1 & 1 & \mathbf{0} \end{array}\right]$$

Case 3:

$$\left[\begin{array}{ccccc|cccccccccccc|c}
0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{x} \\
0 & 0 & 0 & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \mathbf{1} \\
0 & 0 & 0 & \mathbf{0} & \mathbf{0} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \mathbf{1} \\
0 & 0 & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \mathbf{1} \\
\hline
1 & 0 & 1 & \mathbf{1} & \mathbf{0} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
0 & 1 & 1 & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
0 & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \mathbf{0}
\end{array}\right]$$

Case 4:

$$\left[\begin{array}{ccccc|cccccccccccc|c}
1 & 0 & \mathbf{1} & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{x} \\
0 & 0 & \mathbf{0} & 0 & \mathbf{0} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \mathbf{1} \\
0 & 0 & \mathbf{0} & 0 & \mathbf{0} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \mathbf{1} \\
0 & 0 & \mathbf{0} & 0 & \mathbf{0} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \mathbf{1} \\
\hline
0 & 0 & \mathbf{1} & 0 & \mathbf{0} & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
0 & 0 & \mathbf{0} & 0 & \mathbf{1} & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} \\
0 & 1 & \mathbf{0} & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \mathbf{0}
\end{array}\right]$$

Therefore, every column of Q has at most two nonzero entries. This implies $\Phi \preceq \Phi_X$ because every matrix conforming to Φ has a row that results in a $\{1\}$-frame matrix if removed. Since we already have $\Phi_X \preceq \Phi$, we have $\Phi \sim \Phi_X$. ■

Since we have some information about $\mathcal{EX}(\mathrm{PG}(3,2))$, let us compute its extremal function, subject to Hypothesis 3.2.3.

Theorem 5.4.2. *Suppose Hypothesis 3.2.3 holds. For all sufficiently large r, the extremal function of $\mathcal{EX}(\mathrm{PG}(3,2))$ is r^2-r+1, and the extremal matroids of rank r are the extremal matroids of rank r for the class of even-cycle matroids.*

Proof. Since $\mathcal{EX}(\mathrm{PG}(3,2))$ contains all graphic matroids but not all binary matroids, it is a quadratically dense class, by Theorem 1.3.1. Therefore, by Lemma 4.5.5, the extremal matroids of $\mathcal{EX}(\mathrm{PG}(3,2))$ are the extremal matroids of the templates in some subset of the templates $\{\Phi_1,\dots,\Phi_s\}$, where $\{\Phi_1,\dots,\Phi_s,\Psi_1,\dots,\Psi_t\}$ is the set of templates given by Hypothesis 3.2.3 for $\mathcal{EX}(\mathrm{PG}(3,2))$.

By Lemma 4.5.4, the size of a rank-r extremal matroid of a template $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is $\frac{1}{2}|\Gamma||\Lambda|r^2 + br + c$ for some constants b and c. In the binary case $|\Gamma| = 1$ always. By Lemma 5.4.1, if Φ is a template such that $\mathcal{M}(\Phi) \subseteq \mathcal{EX}(\mathrm{PG}(3,2))$, either $\Phi \sim \Phi_X$, in which case $|\Lambda| = 2$, or Φ is a Y-template, in which case $|\Lambda| = 1$. Therefore, for sufficiently large r, the extremal matroid of rank r for $\mathcal{EX}(\mathrm{PG}(3,2))$ is the rank-r extremal matroid for Φ_X. This matroid is the extremal matroid of rank r for the class of even-cycle matroids and has size r^2-r+1, as explained in the proof of Lemma 5.4.1. ■

We now prove Theorem 5.1.1.

Proof of Theorem 5.1.1. Let $\mathcal{M} = \mathcal{EX}(\mathrm{PG}(3,2)\backslash e, L_{19}, L_{11})$. The class of even-cycle matroids is contained in $\mathcal{M}$ since $\mathrm{PG}(3,2)\backslash e$, L_{19}, and L_{11} are excluded minors for the class of even-cycle matroids. We need to prove that there exists a positive integer k such that the reverse inclusion holds for k-connected matroids on at least $2k$ elements.

Let $\mathcal{T} = \{\Phi_1, \dots \Phi_s, \Psi_1, \dots, \Psi_t\}$ be the set of refined templates and k the positive integer given by Corollary 4.1.5 for the class $\mathcal{M}$. Consider a template $\Psi \in \{\Psi_1, \dots, \Psi_t\}$. Recall that every matroid coconforming to Ψ must be contained in the minor-closed class $\mathcal{M}$. Every cographic matroid is a minor of a matroid that coconforms to Ψ. Therefore, Ψ does not exist since $\mathcal{M}$ does not contain L_{19}, which is cographic. Thus, $t = 0$ and $\mathcal{T} = \{\Phi_1, \dots \Phi_s\}$. Because $\mathrm{PG}(3,2)\backslash e$, L_{19}, and L_{11} are simple matroids, it suffices to consider the simple matroids conforming to these templates.

Every matroid containing $\mathrm{PG}(3,2)$ as a minor of course also contains $\mathrm{PG}(3,2)\backslash e$ as a minor. Therefore, Lemma 5.4.1 implies that, for any template $\Phi \in \{\Phi_1, \dots \Phi_s\}$, either $\Phi \sim \Phi_X$ or Φ is a Y-template. We will show that in fact $\Phi \preceq \Phi_X$. In this case, we will be able to assume that $\mathcal{T} = \{\Phi_X\}$, since $\mathcal{M}(\Phi_X)$ is the class of even-cycle matroids and is therefore minor-closed.

Since $\mathcal{M}$ is minor-closed, it suffices to consider a set of templates $\{\Phi'_1, \dots \Phi'_s\}$ such that Φ'_i is minor equivalent to Φ_i. Therefore, by Remark 4.2.7 and Lemma 4.2.10, we may assume that each Y-template $\Phi \in \mathcal{T}$ is the complete, lifted Y-template Φ_{P_0} determined by some matrix P_0. By Lemma 4.2.15, we may assume that every column of P_0 has entries whose sum is 0. Therefore, every column has an even number of nonzero entries. Lemma 5.3.1 implies that no column of P_0 has five nonzero entries. Therefore, every column of P_0 has exactly four nonzero entries (since graphic columns with two nonzero entries are already assumed in a complete Y-template).

Suppose two columns of P_0 have supports whose intersection is empty or has size 1. Then, P_0 contains the submatrix forbidden by Lemma 5.3.2. Therefore, every pair of columns v_1, v_2 in P_0 have supports whose intersection has size at least 2. We wish to show that the intersection of the supports of all of the columns of P_0 must have size 2. Suppose otherwise. Then P_0 contains one of the following submatrices.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

These submatrices are forbidden by Lemmas 5.3.3–5.3.6, respectively.

Therefore, there are two rows of P_0 that consist entirely of 1s. By Lemma 4.2.15, Φ is equivalent to the complete, lifted Y-template determined by a matrix of the

following form, which is obtained by removing one of those two rows.

$$\left[\begin{array}{c} 1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 1 \\ \hline \text{two nonzeros per column} \end{array}\right]$$

Every matroid conforming to this template is even-cycle because removing the top row of any matrix conforming to the template results in a matrix representing a graphic matroid. This completes the proof. ■

Now, we prove Theorem 5.1.2.

Proof of Theorem 5.1.2. Let $\mathcal{M} = \mathcal{EX}(\mathrm{PG}(3,2)\backslash L, M^*(K_6))$. The class of even-cycle matroids with a blocking pair is contained in $\mathcal{M}$ since $\mathrm{PG}(3,2)\backslash L$ and $M^*(K_6)$ are excluded minors for the class of even-cycle matroids with a blocking pair. We need to prove that there exists a positive integer k such that the reverse inclusion holds for k-connected matroids on at least $2k$ elements.

Let $\mathcal{T} = \{\Phi_1, \dots \Phi_s, \Psi_1, \dots, \Psi_t\}$ be the set of refined templates and k the positive integer given by Corollary 4.1.5 for the class $\mathcal{M}$. Similarly to the last proof, since $M^*(K_6)$ is cographic, $\mathcal{T} = \{\Phi_1, \dots \Phi_s\}$.

Since $\mathcal{M} \subseteq \mathcal{EX}(\mathrm{PG}(3,2))$, Lemma 5.4.1 implies that, for every template $\Phi \in \mathcal{T}$, either $\Phi \sim \Phi_X$ or Φ is a Y-template. Since $\mathcal{M}(\Phi_X)$ is the class of even-cycle matroids, which contains $\mathrm{PG}(3,2)\backslash L$, every template in $\mathcal{T}$ is a Y-template. Since $\mathcal{M}$ is minor-closed, it suffices to consider a set of templates $\{\Phi'_1, \dots \Phi'_s\}$ such that Φ'_i is minor equivalent to Φ_i. Therefore, by Remark 4.2.7 and Lemma 4.2.10, we may assume that each Y-template $\Phi \in \mathcal{T}$ is the complete, lifted Y-template Φ_{P_0} determined by some matrix P_0.

The matroid $\mathrm{PG}(3,2)\backslash L$ is a restriction of $\mathrm{PG}(3,2)_{-2}$. Therefore, any matroid containing $\mathrm{PG}(3,2)_{-2}$ contains $\mathrm{PG}(3,2)\backslash L$ also. The proof of Theorem 5.1.1, along with the fact that Lemma 5.3.3 excludes a template from $\mathcal{EX}(\mathrm{PG}(3,2)_{-2})$ in addition to $\mathcal{EX}(L_{11})$, implies that every pair of columns of P_0 must have supports whose intersection has size at least 2. Moreover, Lemma 5.3.7 implies that every pair of columns of P_0 must have supports whose intersection has size at least 3. We wish to show that the intersection of the supports of all of the columns of P_0 must have size 3. Suppose otherwise. Then P_0 contains the following submatrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

This submatrix is forbidden by Lemma 5.3.8. Therefore, there are three rows of P_0 that consist entirely of 1s. By Lemma 4.2.15, Φ is equivalent to the complete, lifted Y-template determined by a matrix of the following form, which is obtained by removing one of those three rows.

$$\left[\begin{array}{c} 1 \cdots\cdots\cdots\cdots\cdots 1 \\ 1 \cdots\cdots\cdots\cdots\cdots 1 \\ \hline \text{unit columns} \end{array}\right]$$

By Lemma 4.2.16, every matroid weakly conforming to this template also weakly conforms to Φ_{Y_1} and is therefore an even-cycle matroid with a blocking pair, by Lemma 5.2.4. This completes the proof. ∎

The next two theorems are proved in essentially the same way as Theorems 5.1.1 and 5.1.2, respectively. We omit the proofs.

Theorem 5.4.3. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that a vertically k-connected binary matroid with an $M(K_n)$-minor is contained in $\mathcal{EX}(\mathrm{PG}(3,2)\backslash e, L_{19}, L_{11})$ if and only if it is an even-cycle matroid.*

Theorem 5.4.4. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that a vertically k-connected binary matroid with an $M(K_n)$-minor is contained in $\mathcal{EX}(\mathrm{PG}(3,2)\backslash L, M^*(K_6))$ if and only if it is an even-cycle matroid with a blocking pair.*

5.5 Some Technical Lemmas Proved with SageMath: Even-Cut Matroids

In this section, we will list several technical lemmas that we will need to prove Theorem 5.1.3. As was the case with Section 5.3, the computations use the SageMath software system. The reader may prefer to move on to Section 5.6, referring to Section 5.5 as necessary. In Lemmas 5.5.1–5.5.10, Ψ is a template with $C = \emptyset$, with Λ trivial, and with $\Delta = \{0, \bar{x}\}$ for some row vector $\bar{x}$. Moreover, there are partitions $Y_1 = Y_1' \cup Y_{1,0} \cup Y_{1,1}$ and $Y_0 = Y_{0,0} \cup Y_{0,1}$ such that $\bar{x}_y = 1$ if and only if $y \in Y_{1,1} \cup Y_{0,1}$ and such that A_1 is of the following form:

$$\begin{array}{ccccc} Y_1' & Y_{1,0} & Y_{1,1} & Y_{0,0} & Y_{0,1} \\ \hline \multicolumn{1}{|c|}{I} & \multicolumn{1}{c|}{A_{Y_1}} & \multicolumn{1}{c|}{B_{Y_1}} & \multicolumn{1}{c|}{A_{Y_0}} & \multicolumn{1}{c|}{B_{Y_0}} \\ \hline \end{array}.$$

Section A.4 in the Appendix gives code for a function `matrix_from_template`, to be used in SageMath, which builds the largest possible matrix A conforming to such a template whose vector matroid is a simple matroid of rank $r + |X|$. The variable `B_Jrows` specifies the number of row indices $b \in B$ for which $A[b, Y_0 \cup Y_1] = \bar{x}$. As an example, we will give the code used to prove Lemma 5.5.1. The proofs of Lemmas 5.5.2, 5.5.3, and 5.5.5–5.5.10 are similar (verified directly from SageMath) and are omitted. Recall that H_{12} was defined in Section 5.1.

Lemma 5.5.1. *If B_{Y_1} contains the submatrix $[1, 0]$, then $\mathcal{M}_w(\Psi) \not\subseteq \mathcal{EX}(H_{12})$.*

Proof. Below is the code necessary to prove this result.

```
Y1I = identity_matrix(GF(2),1)
Y1A = Matrix(GF(2), [])
Y1B = Matrix(GF(2), [[1,0]])
Y0A = Matrix(GF(2), [])
Y0B = Matrix(GF(2), [])
A = matrix_from_template(4, Y1I, Y1A, Y1B, Y0A, Y0B, 3)
M = Matroid(A)
```

```
M.has_minor(H12)
```

This code returned the following.

```
Template with C empty, Lambda trivial,Y1 consisting
of matrices I, Y1A, Y1B, with  3  all-one
rows below B and then all-zero rows, Y0 consisting
of matrices A, B with  3  all-one rows
below B and then all-zero rows.

True
```

■

Lemma 5.5.2. *If $[A_{Y_1}|B_{Y_1}]$ contains either of the following submatrices, with the column to the left of the vertical line contained in A_{Y_1}, and the column to the right of the vertical line contained in B_{Y_1}, then $\mathcal{M}_w(\Psi) \not\subseteq \mathcal{EX}(H_{12})$.*

$$\left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \end{array}\right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 0 \end{array}\right]$$

Lemma 5.5.3. *If A_{Y_1} contains either of the following submatrices, then $\mathcal{M}_w(\Psi) \not\subseteq \mathcal{EX}(H_{12})$.*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Lemma 5.5.4. *If A_1 contains either of the following matrices, with the column on the left indexed by an element of Y_1 and the column on the right is indexed by an element of Y_0, then $\mathcal{M}_w(\Psi) \not\subseteq \mathcal{EX}(H_{12})$.*

$$\left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array}\right]$$

Proof. If the column on the left is contained in A_{Y_1} and the column on the right is contained in A_{Y_0}, then these matrices are forbidden because contraction of the element indexing this column of A_{Y_0} produces a new A_{Y_1}, containing a column originally in the identity matrix, that contains one of the submatrices listed in Lemma 5.5.3. Since we may choose the zero vector for every element of Δ, we also have $\mathcal{M}_w(\Psi) \not\subseteq \mathcal{EX}(H_{12})$ if these submatrices are contained in $[A_{Y_1}|B_{Y_0}]$, in $[B_{Y_1}|B_{Y_0}]$, or in $[B_{Y_1}|A_{Y_0}]$. ■

Lemma 5.5.5. *If $[A_{Y_1}|B_{Y_0}]$ contains either of the following submatrices, with the column to the left of the vertical line contained in A_{Y_1}, and the column to the right of the vertical line contained in B_{Y_0}, then $\mathcal{M}_w(\Psi) \nsubseteq \mathcal{EX}(H_{12})$.*

$$\left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \end{array}\right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 0 \end{array}\right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \end{array}\right]$$

Lemma 5.5.6. *If $[B_{Y_1}|A_{Y_0}]$ contains either of the following submatrices, with the column to the left of the vertical line contained in B_{Y_1}, and the column to the right of the vertical line contained in A_{Y_0}, then $\mathcal{M}_w(\Psi) \nsubseteq \mathcal{EX}(H_{12})$.*

$$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array}\right]$$

Lemma 5.5.7. *If $[B_{Y_1}|B_{Y_0}]$ contains any of the following submatrices, with the column to the left of the vertical line contained in B_{Y_1}, and the column to the right of the vertical line contained in B_{Y_0}, then $\mathcal{M}_w(\Psi) \nsubseteq \mathcal{EX}(H_{12})$.*

$$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \end{array}\right]$$

Lemma 5.5.8. *If A_{Y_0} contains the following submatrix, then $\mathcal{M}_w(\Psi) \nsubseteq \mathcal{EX}(H_{12})$.*

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Lemma 5.5.9. *If B_{Y_0} contains the following submatrix, then $\mathcal{M}_w(\Psi) \nsubseteq \mathcal{EX}(H_{12})$.*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Lemma 5.5.10. *If A_{Y_0} contains the following submatrix, then $\mathcal{M}_w(\Psi) \nsubseteq \mathcal{EX}(H_{12})$.*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

5.6 Even-Cut Matroids

In this section, we prove Theorem 5.1.3. Recall that we use the following definition: An *even-cut matroid* is a matroid M that can be represented by a binary

matrix with a row whose removal results in a matrix representing a cographic matroid. Thus, there is some binary extension N of M on ground set $E(M) \cup \{e\}$ such that N/e is cographic. Thus, to check if a binary matroid M is even-cut, it suffices to check if N/e is cographic for some binary extension N of M. It will be useful to consider the dual situation. Therefore, it suffices to check if there is a binary coextension N^* of M^* such that $N^* \backslash e$ is graphic. If this is the case, then $M^* \in \mathcal{M}(\Phi_C)$. We see then that $\mathcal{M}^*(\Phi_C)$ is exactly the class of even-cut matroids. Recall from Lemma 4.4.4 that $\Phi_{Y_1} \preceq \Phi_C$. This property reflects the fact, first observed by Pivotto [29], that even-cycle matroids with a blocking pair are duals of even-cut matroids.

Recall that H_{12} is the matroid with the even-cycle representation given in Figure 5.2. Thus, H_{12} is the vector matroid of the binary matrix below. In that matrix, the top row is the sign row.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Lemma 5.2.2 shows that the class of even-cut matroids is contained in the class $\mathcal{EX}(M(K_6), H_{12}^*)$. Theorem 5.1.3 is the claim that for sufficiently highly connected matroids, the reverse inclusion holds. We will prove Theorem 5.1.3 after giving a definition and proving some lemmas.

Definition 5.6.1. Let $|C| = 2$ and let Δ be the subgroup of $\mathrm{GF}(2)^C$ generated by $[1, 0]$ and $[0, 1]$. The template Φ_C^2 is given by

$$\Phi_C^2 = (\{1\}, C, \emptyset, \emptyset, \emptyset, [\emptyset], \Delta, \{0\}).$$

Recall the definition of standard form from Definition 4.4.6.

Lemma 5.6.2. *For a template Φ in standard form, either $\Phi_C^2 \preceq \Phi$ or Φ is equivalent to a template with $|C_1| \leq 1$, where C_1 is as in Figure 4.2.*

Proof. There are three cases to consider.

Case 1: Every element of $\Delta|C$ is in the row space of $A_1[X_0, C]$. Then contraction of C_0 turns the elements of C_1 into loops, and contraction of C_1 is the same as deletion of C_1. By deleting C_1 from every matrix virtually conforming to Φ, we see that Φ is equivalent to a template with $C_1 = \emptyset$.

Case 2: There is exactly one element $\bar{x} \in \Delta|C$ that is not in the row space of $A_1[X_0, C]$. Then contraction of C_0 turns the elements of C_1 into parallel elements. Thus, contraction of some element $c \in C_1$ turns the elements of $C_1 - \{c\}$ into loops, and contraction of $C_1 - \{c\}$ is the same as deletion of $C_1 - \{c\}$. By deleting $C_1 - \{c\}$ from every matrix virtually conforming to Φ, we see that Φ is equivalent to a template with $|C_1| = 1$.

Case 3: There are distinct elements $\bar{x}$ and $\bar{y}$ in $\Delta|C$ that are not in the row space of $A_1[X_0, C]$. Index the elements of C_0 by $\{1, 2, \ldots, n\}$ and the elements of X_0 by

$\{d_1, d_2, \ldots, d_n\}$. Let S_x and S_y be the supports of $\bar{x}|C_0$ and $\bar{y}|C_0$, respectively. Then the support of $(\bar{x}+\bar{y})|C_0$ is the symmetric difference $S_x \triangle S_y$. First, suppose that for every pair of elements $\bar{x}$ and $\bar{y}$ in $\Delta|C$ that are not in the row space of $A_1[X_0, C]$, we have that $\bar{x}+\bar{y}$ is in the row space of $A_1[X, C]$. Since the rows of $A_1[X_0, C]$ are linearly independent, it must be that the zero vector is equal to

$$\sum_{i \in S_x \triangle S_y} A_1[\{d_i\}, C] + \bar{x} + \bar{y} = \sum_{i \in S_x} A_1[\{d_i\}, C] + \bar{x} + \sum_{i \in S_y} A_1[\{d_i\}, C] + \bar{y}$$

and therefore, since we are working in characteristic 2,

$$\sum_{i \in S_x} A_1[\{d_i\}, C] + \bar{x} = \sum_{i \in S_y} A_1[\{d_i\}, C] + \bar{y}.$$

Thus, contraction of C_0 projects $\bar{x}$ and $\bar{y}$ onto the same element of $\mathrm{GF}(2)^{C_1}$. Moreover, this is true for every pair of elements of $\Delta|C$ that are not in the row space of $A_1[X_0, C]$. Therefore, the same argument used for Case 2 shows that Φ is equivalent to a template with $|C_1| = 1$.

Therefore, we may assume that there are elements $\bar{x}$ and $\bar{y}$ in $\Delta|C$ that are not in the row space of $A_1[X_0, C]$ and such that $\bar{x}+\bar{y}$ is also not in the row space of $A_1[X_0, C]$. Repeatedly perform operations (4) and (10) on Φ until the following template is obtained:

$$(\{1\}, C, X, \emptyset, \emptyset, A_1[X, C], \Delta|C, \Lambda).$$

On this template, perform operations (2) and (3) to obtain the following template:

$$(\{1\}, C, X, \emptyset, \emptyset, A_1[X, C], \langle \bar{x}, \bar{y} \rangle, \{0\}).$$

By performing elementary row operations, we see that every matrix virtually respecting this template is row equivalent to a matrix virtually respecting the following template, where $\bar{x}'|C_0$ and $\bar{y}'|C_0$ are zero vectors:

$$\Phi' = (\{1\}, C, X, \emptyset, \emptyset, A_1[X, C], \langle \bar{x}', \bar{y}' \rangle, \{0\}).$$

Note that $\bar{x}'|C_1$, $\bar{y}'|C_1$, and $(\bar{x}'+\bar{y}')|C_1$ are nonzero since $\bar{x}$, $\bar{y}$, and $\bar{x}+\bar{y}$ were not in the row space of $A_1[X_0, C]$ in the original template Φ. Also, we must have $\bar{x}' \neq \bar{y}'$ because otherwise, $\bar{x}' + \bar{y}' = \mathbf{0}$, contradicting the assumption that $\bar{x}+\bar{y}$ was not in the row space of $A_1[X_0, C]$ in Φ. Now, on Φ', repeatedly perform operation (7) and then operation (6) to obtain the following template:

$$\Phi'' = (\{1\}, C_1, \emptyset, \emptyset, \emptyset, [\emptyset], \langle \bar{x}'|C_1, \bar{y}'|C_1 \rangle, \{0\}).$$

Now, every matroid M conforming to Φ'' is obtained by contracting C_1 from $M(A)$, where A is a matrix conforming to Φ''. Thus, if there are any elements of C_1 that are parallel elements in $M(A)$, contracting one of these elements turns the rest of the parallel class into loops. So these elements are deleted to obtain M.

Thus, Φ'' is equivalent to a template where these elements have been deleted from C. There are two cases to consider. First, if it is the case that either the supports of $\bar{x}'$ and $\bar{y}'$ are disjoint or that one support is contained in the other, then in the resulting template, $|C| = 2$ and $\Delta = \langle [1, 0], [0, 1] \rangle$. So this resulting template is Φ_C^2. In the other case, $\bar{x}'$ and $\bar{y}'$ have intersecting supports but neither is contained in the other. In this case, Φ'' is equivalent to the following template with $|C_1| = 3$:

$$\Phi''' = (\{1\}, C_1, \emptyset, \emptyset, \emptyset, [\emptyset], \langle [1, 1, 0], [1, 0, 1] \rangle, \{0\}).$$

However, by contracting any element of C, the other two become parallel. Thus, by contracting a second element, the third becomes a loop. Therefore, the third element is deleted to obtain a matroid conforming to Φ'''. Thus, $\Phi_C^2 \sim \Phi''' \preceq \Phi$. ■

Lemma 5.6.3. *If Φ is a template in standard form, with $|C_1| = 1$ and with $\Lambda|X_1$ trivial, then $\Phi_{CX} \preceq \Phi$ or Φ is equivalent to a template with $C = \emptyset$.*

Proof. We consider two cases, depending on whether $\Delta|C$ contains an element that is not in the row space of $A_1[X_0, C]$.

Case 1: Every element of $\Delta|C$ is in the row space of $A_1[X_0, C]$. Let A be a matrix that conforms to Φ. When C_0 is contracted from $M(A)$, each element of C_1 becomes a loop and can therefore be deleted rather than contracted. Thus, Φ is equivalent to a template Φ' with $C_1 = \emptyset$. Suppose there exist elements $\bar{x} \in \Delta|C_0$ and $\bar{y} \in \Lambda|X_0$ such that there are an odd number of natural numbers i with $\bar{x}_i = \bar{y}_i = 1$. Repeatedly perform operations (4), (10), and (6) on Φ' to obtain the following template:

$$(\{1\}, C_0, X_0, \emptyset, \emptyset, A_1[X_0, C_0], \Delta|C_0, \Lambda).$$

Then perform operations (2) and (3) to obtain the following template:

$$(\{1\}, C_0, X_0, \emptyset, \emptyset, A_1[X_0, C_0], \{\mathbf{0}, \bar{x}\}, \{\mathbf{0}, \bar{y}\}).$$

Any matroid conforming to this template is obtained by contracting C_0 from $M(A)$, where A is a matrix conforming to Φ. Recall that $A[B - X_0, E - C_0]$ is a frame matrix. If $\bar{x}$ is in the row labeled by r and $\bar{y}$ is in the column labeled by c, then when C_0 is contracted, 1 is added to the entry of the frame matrix in row r and column c. Otherwise, the entry remains unchanged when C is contracted. We see then that this template is equivalent to Φ_{CX}, where 1s are used to replace $\bar{x}$ and $\bar{y}$.

Thus, we may assume that for every element $\bar{x} \in \Delta|C_0$ and $\bar{y} \in \Lambda|X_0$, there are an even number of natural numbers i such that $\bar{x}_i = \bar{y}_i = 1$. This implies that contraction of C has no effect on the frame matrix. So Φ', and therefore Φ are equivalent to a template with $\Lambda|X_0$ trivial. In this case, we see that repeated use of operation (7) produces a template equivalent to Φ with $C = \emptyset$.

Case 2: There is an element $\delta \in \Delta|C$ that is not in the row space of $A_1[X_0, C]$. Since $|C_1| = 1$, every element of $\Delta|C$ not in the row space of $A_1[X_0, C]$ becomes a

1 after C_0 is contracted, and every element that is in the row space becomes a 0. Therefore, we may assume that the column vector $A_1[X, C_1]$ is a zero vector and that an element of $\Delta|C$ has a 0 as its final entry if it is in the row space and a 1 otherwise.

If $\bar{x} \in \Delta|C$ and $\bar{y} \in \Lambda|X_0$ are such that there are an odd number of natural numbers i such that $\bar{x}_i = \bar{y}_i = 1$, then we call the ordered pair $(\bar{x}, \bar{y})$ a *pair of odd type*. Otherwise, $(\bar{x}, \bar{y})$ is a *pair of even type*. Consider a matrix A virtually conforming to Φ and contract C from $M(A)$. The effect on the elements of Δ is a change of basis followed by a projection into a lower dimension. Therefore, a group structure is maintained. Let us call the resulting group Δ'. There are two subcases to check.

Subcase a: Suppose there exists a pair $(\bar{x}, \bar{y})$ of odd type. If $\bar{x}$ is in the row space of $A_1[X_0, C]$, or if $\bar{x}$ is not in the row space of $A_1[X_0, C]$ but $(\delta, \bar{y})$ is a pair of even type, then we will show that $\Phi_{CX} \preceq \Phi$. On Φ, repeatedly perform operations (4), (10), and (6) as needed to obtain the following template:

$$(\{1\}, C, X_0, \emptyset, \emptyset, A_1[X_0, C], \Delta|C, \Lambda).$$

Then perform operations (2) and (3) to obtain the following template:

$$\Phi' = (\{1\}, C, X_0, \emptyset, \emptyset, A_1[X_0, C], \langle \bar{x}, \delta \rangle, \{\mathbf{0}, \bar{y}\}).$$

Consider the following matrix conforming to Φ_{CX}:

$$\begin{array}{|c|c|c|}
\hline
0 \cdots 0 & 1 \cdots 1 & 1 \\
\hline
 & & 1 \\
 & & \vdots \\
\text{frame} & \text{frame} & 1 \\
\text{matrix} & \text{matrix} & 0 \\
 & & \vdots \\
 & & 0 \\
\hline
\end{array}$$

The matrix below conforms to Φ' and results in the same matroid when C is contracted:

$$\begin{array}{|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{C_0} & \multicolumn{1}{c}{C_1} \\
\hline
 & & & 0 \\
0 & \bar{y} \cdots \bar{y} & I & \vdots \\
 & & & 0 \\
\hline
0 \cdots 0 & 0 \cdots 0 & \multicolumn{2}{c|}{\delta} \\
\hline
 & & \multicolumn{2}{c|}{\bar{x}} \\
 & & \multicolumn{2}{c|}{\vdots} \\
\text{frame} & \text{frame} & \multicolumn{2}{c|}{\bar{x}} \\
\cline{3-4}
\text{matrix} & \text{matrix} & \multicolumn{2}{c|}{} \\
 & & \multicolumn{2}{c|}{0} \\
 & & \multicolumn{2}{c|}{} \\
\hline
\end{array}$$

Therefore, we may assume that an element $\delta' \in \Delta|C$ is in the row space of $A_1[X_0, C]$ if and only if $(\delta', \bar{y})$ is a pair of even type. Moreover, this is true for every nonzero element of $\Lambda|X_0$. Thus, if $\bar{y_1}$ and $\bar{y_2}$ are nonzero elements of $\Lambda|X_0$, then both $(\delta, \bar{y_1})$ and $(\delta, \bar{y_2})$ are pairs of odd type, since δ is not in the row space of $A_1[X_0, C]$. This implies that $(\delta, \bar{y_1} + \bar{y_2})$ is a pair of even type. But we have just shown that this implies that $\bar{y_1} + \bar{y_2}$ is the zero vector. Thus $\bar{y_1} = \bar{y_2}$ and $\Lambda|X_0 = \{\mathbf{0}, \bar{y}\}$ in the original template Φ. By a similar argument, $\bar{x} = \delta$ and $\Delta|C = \{\mathbf{0}, \delta\}$ in the original template Φ. Therefore, Φ is equivalent to a template with $|C_0| = |X_0| = 1$, with $A_1[X_0, C_0] = [1]$, with $A_1[X_0, C_1] = [0]$, with $\Lambda|X_0 = \{[0], [1]\}$, and with $\Delta|C = \{[0, 0], [1, 1]\}$. We may now assume that the original template Φ was of this form and will study this template Φ in this form but before any operations have been performed on it.

We will show that Φ is equivalent to the following template Φ', with $C = \emptyset$, obtained by adjoining an element y to Y_1 and letting $A_1[X_1, y]$ be the zero vector. We will define Δ'' below.

$$\Phi' = (\{1\}, \emptyset, X_1, Y_0, Y_1 \cup y, A_1[X_1, Y_0 \cup Y_1 \cup y], \Delta'', \{0\})$$

Recall that Δ' is the group obtained from Δ after C is contracted. Let Δ'' be the subgroup of $\mathrm{GF}(2)^{Y_0 \cup Y_1 \cup y}$ consisting of all the row vectors obtained by adjoining to any element of Δ' either a zero or a 1. So $|\Delta''| = 2|\Delta'|$. Let A be a matrix that virtually conforms to Φ. Recall that columns indexed by elements of Z are formed by adding a column indexed by Y_1 to a column indexed by Z in a matrix that respects Φ. If $A[B - X, C]$ is the zero matrix, then $M(A)/C$ conforms to Φ' because we may simply choose never to use y to build a column indexed by Z.

Otherwise, choose an element r of $B - X$ such that $A[r, C] = [1, 1]$. Let S be the subset of $B - (X \cup r)$ such that $s \in S$ if and only if $A[s, C] = [1, 1]$. Let $T = B - (X \cup S \cup r)$. The effect on the frame matrix of contracting C from $M(A)$ is to remove r and to add a 1 to each entry $A_{s,c}$ of the frame matrix where $s \in S$ and where c is an element of $E - (C \cup Y_0 \cup Y_1 \cup Z)$ with $A_{r,c} = 1$. Let $\hat{A}$ be the matrix that results from A by contracting C. Recall that every column of the frame matrix $A[B - X, E - (C \cup Y_0 \cup Y_1 \cup Z)]$ contains at most two nonzero entries. Thus, for a column c with $A_{r,c} = 1$, the column $A[B - (X \cup r), c]$ must be either a unit column or a zero column. Therefore, there are several possibilities for $\hat{A}[B - (X \cup r), c]$. Either $\hat{A}[S, c] = [1, \ldots, 1]^T$ and $\hat{A}[T, c] = [0, \ldots, 0]^T$, or $\hat{A}[S, c] = [1, \ldots, 1]^T$ and $\hat{A}[T, c]$ is an identity column, or $\hat{A}[S, c]$ is the complement of an identity column and $\hat{A}[T, c] = [0, \ldots, 0]^T$. This exact same situation can be obtained with Φ' using the new column y. Thus, Φ is equivalent to Φ', a template with $C = \emptyset$.

Subcase b: Therefore, we may assume that every pair of elements $(\bar{x}, \bar{y}) \in \Delta \times (\Lambda|X_0)$ is a pair of even type. Thus, contraction of C_0 has no effect on the frame matrix. This implies that Φ is equivalent to a template with Λ trivial. By repeated use of operation (7), we obtain a template equivalent to Φ with $C_0 = \emptyset$, with $|C_1| = 1$, and with $\Delta|C = \{[0], [1]\}$. Using an argument similar to the one used at the end of Subcase a, we see that Φ is equivalent to a template with $C = \emptyset$ by adjoining an element to Y_1. ∎

Lemma 5.6.4. *Let $\Phi = (\{1\}, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a template. If*

$$\Phi' = (\{1\}, C', X, \emptyset, \emptyset, A_1, \Delta, \Lambda),$$

where $C' = Y_0 \cup Y_1 \cup C$, then every matroid conforming to Φ' is a minor of a matroid conforming to Φ.

Proof. Let $\Phi'' = (\{1\}, C, X, Y_0', \emptyset, A_1, \Delta, \Lambda)$, where $Y_0' = Y_0 \cup Y_1$. By Lemma 4.4.5, $\Phi'' \preceq \Phi$. Any matroid conforming to Φ' is obtained from a matroid conforming to Φ'' by contracting Y_0'. ■

We now prove Theorem 5.1.3.

Proof of Theorem 5.1.3. Let $\mathcal{M} = \mathcal{EX}(M(K_6), H_{12}^*)$, and let $\mathcal{T} = \{\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t\}$ be the set of refined templates given by Corollary 4.1.5 for $\mathcal{M}$. Consider a template $\Phi \in \{\Phi_1, \ldots, \Phi_s\}$. Recall that every matroid conforming to Φ must be contained in the minor-closed class $\mathcal{M}$. By Lemma 4.4.7, we may assume that each of these templates is in standard form. Every graphic matroid is a minor of a matroid that conforms to Φ. Since $\mathcal{M}$ does not contain $M(K_6)$, it must be the case that Φ does not exist. Thus, $s = 0$ and $\mathcal{T} = \{\Psi_1, \ldots, \Psi_t\}$. Therefore, we will study the highly connected matroids in $\mathcal{M}$ by considering their dual matroids which virtually conform to some template $\Psi \in \{\Psi_1, \ldots, \Psi_t\}$. Because $M(K_6)$, and H_{12}^* are cosimple matroids, it suffices to consider cosimple matroids in $\mathcal{M}$. Thus, it suffices to consider simple matroids that are duals of matroids in $\mathcal{M}$. Therefore, we only consider simple matroids conforming to Ψ.

Let $\Psi = (\{1\}, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a template in $\mathcal{T}$. We know that $\mathcal{M}^*(\Phi_C)$ is the class of even-cut matroids. Therefore, we may assume that $\Phi_C \in \{\Psi_1, \ldots, \Psi_t\}$, and if any template $\Psi \preceq \Phi_C$, we may discard Ψ form the set $\{\Psi_1, \ldots, \Psi_t\}$. Since H_{12} is an even-cycle matroid, H_{12} conforms to Φ_X. Thus, we have $\Phi_X \not\preceq \Psi$. By Lemma 4.4.9, $\Lambda|X_1$ is trivial. Moreover, by Lemma 4.4.4, we have $\Phi_{CX} \not\preceq \Psi$.

The following matrix conforms to Φ_C^2, with C indexing the last two columns:

$$\left[\begin{array}{cccccccccccc|cc}
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right].$$

By contracting C, we obtain the following matrix A with $M(A)$ conforming to Φ_C^2:

$$A = \begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}.$$

By adding the first and third rows to the fifth row, we see that this matrix represents H_{12}. Therefore, $\Phi_C^2 \not\preceq \Psi$ and by Lemma 5.6.2, we may assume that $|C_1| \leq 1$.

Since $\Phi_{CX} \not\preceq \Psi$, Lemma 5.6.3 implies that Ψ is equivalent to a template with $C = \emptyset$. Hence we will assume from now on that $C = \emptyset$.

Since $C_0 = \emptyset$, we have $X_0 = \emptyset$. Also, we have seen that $\Lambda|X_1$ is trivial. Therefore, Λ is trivial. By performing elementary row operations on every matrix respecting Ψ, we may assume that A_1 is of the following form, with $Y_1 = Y_1' \cup Y_1''$:

$$\begin{array}{ccc} Y_1' & Y_1'' & Y_0 \\ \hline \multicolumn{1}{|c|}{I} & \multicolumn{1}{c|}{P_1} & \multicolumn{1}{c|}{P_0} \\ \hline \end{array}\ .$$

Also, by elementary row operations, we may assume that $\Delta|Y_1'$ is trivial.

We will now show that $|\Delta| \leq 2$. Suppose otherwise. Then Δ contains a subgroup Δ' isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Repeatedly perform y-shifts and operation (12) and then perform operation (3) to obtain the following template:

$$(\{1\}, \emptyset, \emptyset, Y_1'' \cup V_1, \emptyset, [\emptyset], \Delta', \{0\}).$$

By Lemma 5.6.4, if $C' = Y_1'' \cup V_1$, then every matroid conforming to the following template is a minor of a matroid conforming to Ψ:

$$(\{1\}, C', \emptyset, \emptyset, \emptyset, [\emptyset], \Delta', \{0\})$$

The latter template is equivalent to Φ_C^2 since $\Delta' \cong \langle [1, 0], [0, 1] \rangle$. By contradiction, we deduce that $|\Delta| \leq 2$. Therefore, there is at most one nonzero element $\bar{x} \in \Delta$. Let $Y_{1,i}$ consist of the elements $y \in Y_1''$ such that $\bar{x}_y = i$. Similarly, let $Y_{0,i}$ consist of the elements $y \in Y_0$ such that $\bar{x}_y = i$. Thus, A_1 is of the following form, where $Y_1 = Y_1' \cup Y_{1,0} \cup Y_{1,1}$ and where $Y_0 = Y_{0,0} \cup Y_{0,1}$:

$$\begin{array}{ccccc} Y_1' & Y_{1,0} & Y_{1,1} & Y_{0,0} & Y_{0,1} \\ \hline \multicolumn{1}{|c|}{I} & \multicolumn{1}{c|}{A_{Y_1}} & \multicolumn{1}{c|}{B_{Y_1}} & \multicolumn{1}{c|}{A_{Y_0}} & \multicolumn{1}{c|}{B_{Y_0}} \\ \hline \end{array}\ .$$

By Lemma 5.5.1, each row of B_{Y_1} consists either entirely of 0s or entirely of 1s. Any duplicate columns in either $[I|A_{Y_1}]$ or B_{Y_1} produce the same columns in a matrix virtually conforming to Ψ. Therefore, we may assume that $|Y_{1,1}| \leq 1$, that every column of A_{Y_1} contains at least two nonzero entries, and that no column of A_{Y_1} is a copy of another. Since we are only considering templates to which simple matroids conform, we may assume that no column of A_{Y_0} is a copy of another and also that no column of B_{Y_0} is a copy of another. By Lemma 5.5.2, either $Y_{1,0}$ or $Y_{1,1}$ is empty. If $|Y_{1,0}| \geq 2$, then A_{Y_1} contains one of the submatrices below, all of which are forbidden by Lemma 5.5.3. Therefore, $|Y_{1,0}| \leq 1$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

By Lemma 5.5.5, if $|Y_{1,0}| = 1$, then $Y_{0,1} = \emptyset$. By Lemma 5.5.6, if $|Y_{1,1}| = 1$, then each column of A_{Y_0} contains at most two nonzero entries.

Recall that a binary matroid M conforms to Φ_C if there is some binary coextension N of M on ground set $E(M) \cup \{e\}$ such that $N \backslash e$ is graphic. Thus, checking if a binary matroid conforms to Φ_C amounts to checking if some row can be added to the matrix to make the resulting matroid graphic. There are four cases to check:

Case I: $|Y_{1,0}| = 1$

Case II: $|Y_{1,1}| = 1$

Case III: $Y_{1,0} = Y_{1,1} = Y_{0,1} = \emptyset$

Case IV: $Y_{1,0} = Y_{1,1} = \emptyset$ and $Y_{0,1} \neq \emptyset$

In the diagrams of matrices below, we will use the abbreviations *n.p.c.* and *z.p.c.* to stand for "nonzero entries per column" and "zeros per column," respectively.

Case I: Since $|Y_{1,0}| = 1$, the arguments above imply that $Y_{1,1} = Y_{0,1} = \emptyset$. We study the matrix A_{Y_0}. Let $X = R \cup S$, where R is the set of rows where A_{Y_1} has its nonzero entries. We will show that $\Psi \preceq \Phi_C$ by appending to every matrix virtually conforming to Ψ a row indexed by d so that the resulting matroid is graphic. The row indexed by d also has a 1 in the column indexed by $Y_{1,0}$, and we will add this row to every row in R. After the row indexed by d has been added to every row in R, the entire resulting matrix (with rows indexed by $B \cup d$ and columns indexed by E) will have at most two nonzero entries per column and will therefore represent a graphic matroid. The form of A_{Y_0} itself (without the row indexed by d) is determined by Lemma 5.5.4.

First, we study A_{Y_0} when $|R| = 2$.

d	0	0	$1 \cdots 1$
R	0	1 n.p.c.	2 n.p.c.
S	≤ 2 n.p.c.	≤ 1 n.p.c	≤ 1 n.p.c.

Next, we study A_{Y_0} when $|R| = 3$.

d	0	0	0	$1 \cdots 1$
R	0	1 n.p.c.	2 n.p.c.	3 n.p.c.
S	≤ 2 n.p.c.	≤ 1 n.p.c	0	≤ 1 n.p.c

Next, we study A_{Y_0} when $|R| \geq 4$. Here, J denotes a matrix where every entry is a 1.

d	0	0	0	$1 \cdots 1$	$1 \cdots 1$
R	0	1 n.p.c.	2 n.p.c.	1 z.p.c.	J
S	≤ 2 n.p.c.	≤ 1 n.p.c	0	0	≤ 1 n.p.c

Case II: Since $|Y_{1,1}| = 1$, the arguments above imply that $|Y_{1,0}| = \emptyset$ and that each column of A_{Y_0} has at most two nonzero entries. We study the matrix B_{Y_0}. By Lemma 5.5.7, the submatrices below, with the column to the left of the vertical line contained in B_{Y_1}, and the column to the right of the vertical line contained in B_{Y_0}, are forbidden.

$$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \end{array}\right]$$

This fact, along with Lemma 5.5.4, determines the form of B_{Y_0}.

Let $X = R \cup S$, where R is the set of rows where B_{Y_1} has its nonzero entries. We will show that $\Psi \preceq \Phi_C$ by appending to every matrix virtually conforming to Ψ a row indexed by d so that the resulting matroid is graphic. The row indexed by d also has a 1 in the column indexed by $Y_{1,0}$, and we are adding this row to every row in R, as well as to every row indexed by an element of $B - X$ where the nonzero element $\bar{x}$ of Δ is used.

First, we study the case when $R = \emptyset$.

	$Y_{0,1}$	$Y_{0,0}$
d	$1 \cdots 1$	$0 \cdots 0$
S	≤ 1 n.p.c.	A_{Y_0} (≤ 2 n.p.c.)

Now we study the case when $R \neq \emptyset$. Here J denotes a matrix where every entry is a 1.

<table>
<tr><th></th><th colspan="2">$Y_{0,1}$</th><th>$Y_{0,0}$</th></tr>
<tr><td>d</td><td>$1 \cdots 1$</td><td>$1 \cdots 1$</td><td>$0 \cdots 0$</td></tr>
<tr><td>R</td><td>1 z.p.c.</td><td>J</td><td rowspan="2">A_{Y_0}
(≤ 2 n.p.c.)</td></tr>
<tr><td>S</td><td>0</td><td>≤ 1 n.p.c.</td></tr>
</table>

Case III: Since $Y_{1,0} = Y_{1,1} = Y_{0,1} = \emptyset$, we have $Y_1 = Y_1'$ and $Y_0 = Y_{0,0}$. The submatrices below are forbidden from A_{Y_0} because by deleting the rest of Y_0 and contracting the elements of Y_0 indexing the two given columns, we produce one of the submatrices forbidden by Lemma 5.5.3.

$$Q_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} Q_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} Q_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} Q_4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

If every column of A_{Y_0} has at most two nonzero entries, then $\Psi \sim \Phi_0 \preceq \Phi_C$ and can be discarded. Thus, we may assume that there is a column of A_{Y_0} with at least three nonzero entries. Let H be the submatrix of A_{Y_0} consisting of all the columns with at most two nonzero entries.

Let y be an element of Y_0 such that $A_{Y_0}[X, \{y\}]$ has a maximum number of nonzero entries among all elements of Y_0. Let $X = R \cup S$, where $(A_{Y_0})_{r,y} = 1$ for each $r \in R$ and $(A_{Y_0})_{s,y} = 0$ for each $s \in S$. We will prove the following.

Claim 5.6.4.1. *Let v be a column of A_{Y_0} such that $v|R$ has at least two zeros. Then either v has at most two nonzero entries, or $v|R$ has exactly two zeros and $v|S$ is a zero vector.*

Proof. If there is a column v of A_{Y_0} such that $v|R$ has exactly two zeros, then if $|R| = 3$, the fact that Q_2 is forbidden implies that v has at most two nonzero entries. If $|R| > 3$, then the fact that Q_1 is forbidden implies that $v|S$ is a zero vector. Now, if there is a column v of A_{Y_0} such that $v|R$ has at least three zeros,

then since Q_4 is forbidden, $v|R$ has at most two nonzero entries. Since Q_1, Q_2, and Q_3 are forbidden (corresponding to when $v|R$ has two, one, or zero nonzero entries), v has at most two nonzero entries. □

Suppose there are two elements other than y that index columns v_1 and v_2 of A_{Y_0} with $|R|$ nonzero entries. Since Q_2 is forbidden, $v_1|R$ and $v_2|R$ each have at most one zero. Since we are only considering simple matroids conforming to Ψ, we have that $v_1|R$ and $v_2|R$ each have exactly one zero. If $v_1|R$ and $v_2|R$ have their zeros in different rows, then again since Q_2 is forbidden, the nonzero entries in $v_1|S$ and $v_2|S$ must be in the same row. Thus, we may divide this case into three subcases:

1. There is at least one column other than the one indexed by y with $|R|$ nonzero entries, and all such columns have a zero in the same row r of R.

2. There are at least two columns other than the one indexed by y with $|R|$ nonzero entries, and all such columns have a nonzero entry in the same row s of S.

3. No column other than the one indexed by y has $|R|$ nonzero entries.

In subcase (1), we need to determine the structure of the columns v such that $v|R$ has at least two zeros. In fact, by Claim 5.6.4.1, either v is a column of H or $v|R$ has exactly two zeros and $v|S$ is a zero vector. If v is not a column of H but is such that $v|R$ has exactly two zeros, then we must have $|R| \geq 5$. Since Q_1 is forbidden, v has a zero in the row indexed by r. Therefore, $A_{Y_0}[X, Y_0 - y]$ has the following form, where J denotes a matrix where every entry is a 1:

<table>
<tr><td>$R-r$</td><td>J</td><td>1 z.p.c.</td><td rowspan="3">H</td></tr>
<tr><td>r</td><td>$0 \cdots 0$</td><td>$0 \cdots 0$</td></tr>
<tr><td>S</td><td>≤ 1 n.p.c.</td><td>0</td></tr>
</table>

Below, we append the row d to the matrix, where d also has a 1 in the entry in the column of y. By adding d to every row in $R - r$, we see that the resulting matroid is graphic. Thus, in this subcase, $\Psi \preceq \Phi_C$.

<table>
<tr><td>d</td><td>$1 \cdots 1$</td><td>$1 \cdots 1$</td><td>$0 \cdots 0$</td></tr>
<tr><td>$R-r$</td><td>J</td><td>1 z.p.c.</td><td rowspan="3">H</td></tr>
<tr><td>r</td><td>$0 \cdots 0$</td><td>$0 \cdots 0$</td></tr>
<tr><td>S</td><td>≤ 1 n.p.c.</td><td>0</td></tr>
</table>

We now consider subcase (2). Suppose there is some column v of A_{Y_0} such that $v|R$ has two zeros. If v has more than two nonzero entries, then Claim 5.6.4.1 implies that $v|S$ is a zero vector. By Lemma 5.5.8, along with the facts that Q_1 is forbidden and that we are only considering templates to which simple matroids conform, no such column v can exist. Therefore, $A_{Y_0}[X, Y_0 - y]$ has the following form:

$$\begin{array}{r|c|c|}
\cline{2-3}
R & \text{1 z.p.c.} & \\
\cline{2-2}
s & 1\cdots1 & H \\
\cline{2-2}
S-s & 0 & \\
\cline{2-3}
\end{array}$$

Below, we append the row d to the matrix, where d also has a 1 in the entry in the column of y. By adding d to every row in $R \cup \{s\}$, we see that the resulting matroid is graphic. Thus, in this subcase, $\Psi \preceq \Phi_C$.

$$\begin{array}{r|c|c|}
\cline{2-3}
d & 1\cdots1 & 0\cdots0 \\
\cline{2-3}
R & \text{1 z.p.c.} & \\
\cline{2-2}
s & 1\cdots1 & H \\
\cline{2-2}
S-s & 0 & \\
\cline{2-3}
\end{array}$$

We now consider subcase (3). First, suppose that there is a column w of A_{Y_0} such that $w|R$ has at least two zeros and at least three nonzero entries (so $|R| \geq 5$). By Claim 5.6.4.1, $w|R$ has exactly two zeros and $w|S$ is a zero vector. Since Q_2 is forbidden, every pair of such columns must have zero entries in a common row. By Lemma 5.5.10, all such columns must have a zero in the same row r of R. Since Q_1 is forbidden, a column v such that $v|R$ has exactly one zero must also have its zero in row r. Since we are only considering simple matroids, there is at most one such column v. Therefore, $A_{Y_0}[X, Y_0 - y]$ has the following form, with or without v:

$$\begin{array}{r|c|c|c|}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{v} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{2-4}
 & 1 & & \\
R-r & \vdots & \text{1 z.p.c} & \\
 & 1 & & H \\
\cline{2-3}
r & 0 & 0\cdots0 & \\
\cline{2-3}
 & 0 & & \\
S & \vdots & 0 & \\
 & 0 & & \\
\cline{2-4}
\end{array}$$

We append the row d to the matrix, where d also has a 1 in the entry in the column of y. By adding d to every row in $R - r$, we see that the resulting matroid is graphic.

$$\begin{array}{r|c|c|c|}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{v} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{2-4}
d & 1 & 1\cdots1 & 0\cdots0 \\
\cline{2-4}
 & 1 & & \\
R-r & \vdots & \text{1 z.p.c} & \\
 & 1 & & H \\
\cline{2-3}
r & 0 & 0\cdots0 & \\
\cline{2-3}
 & 0 & & \\
S & \vdots & 0 & \\
 & 0 & & \\
\cline{2-4}
\end{array}$$

Therefore, we may assume that no column w of A_{Y_0} exists such that $w|R$ has two zeros and at least three nonzero entries. Thus, Claim 5.6.4.1 implies that $A_{Y_0}[X, Y_0 - y]$ is of the following form:

R	1 z.p.c.	H
S	0	

We append the row d to the matrix, where d also has a 1 in the entry in the column of y. By adding d to every row in R, we see that the resulting matroid is graphic. Thus, in this subcase, $\Psi \preceq \Phi_C$.

d	$1 \cdots 1$	$0 \cdots 0$
R	1 z.p.c.	H
S	0	

Case IV: If any column of A_{Y_0} has three nonzero entries, then by contracting that element of $Y_{0,0}$ we make a column of the identity matrix into a column of A_{Y_1} with two nonzero entries. Since $Y_{0,1}$ is nonempty, this is forbidden by Lemma 5.5.5. Therefore, each column of A_{Y_0} contains at most two nonzero entries.

The matrices Q_5 and Q_6 below are forbidden from B_{Y_0} because by contracting one of the corresponding elements of Y_0, we obtain, using a column of the identity matrix, a submatrix forbidden by Lemma 5.5.7. The matrix Q_7 below is forbidden by Lemma 5.5.9.

$$Q_5 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} Q_6 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} Q_7 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Let y be an element of Y_0 such that $B_{Y_0}[X, \{y\}]$ has a maximum number of nonzero entries among all elements of $Y_{0,1}$. Let $X = R \cup S$, where $(B_{Y_0})_{r,y} = 1$ for each $r \in R$ and $(B_{Y_0})_{s,y} = 0$ for each $s \in S$. If $|R|$ is 0 or 1, then we append to each matrix A virtually conforming to Ψ a row which is the characteristic vector of $Y_{0,1}$. If we add this row to each row of A where $\bar{x}$ has been used as the element of Δ, we see that the resulting matroid is graphic. Therefore, we may assume that $|R| \geq 2$.

Since Q_5 is forbidden, each column of $B_{Y_0}[R, Y_{0,1}]$ contains at most two zeros. Since Q_6 is forbidden, every column w such that $w|R$ has two zeros must be such that $w|S$ is a zero vector.

Suppose there are two columns v_1, v_2 of B_{Y_0}, in addition to the column indexed by y, with $|R|$ nonzero entries. Since Q_6 is forbidden, $v_1|R$ and $v_2|R$ each have at most one zero. Since we are only considering simple matroids conforming to Ψ, $v_1|R$ and $v_2|R$ each have exactly one zero. If $v_1|R$ and $v_2|R$ have their zeros in different rows, then again since Q_6 is forbidden, the nonzero entries in $v_1|S$ and $v_2|S$ must be in the same row. Thus, we have the same three subcases as we did in Case III. In each subcase, we will determine the structure of $B_{Y_0}[X, Y_{0,1} - y]$. Since we are only considering simple matroids conforming to Ψ, we may assume that no column of B_{Y_0} is a copy of another.

Let us consider subcase (1). If there is a column v of $B_{Y_0}[R, Y_{0,1}]$ with two zeros, then since Q_6 is forbidden one of the zeros of v must be in row r. Therefore, $B_{Y_0}[X, Y_{0,1} - y]$ is of the following form, where J denotes a matrix where every entry is a 1:

$R - r$	1 z.p.c.	J
r	$0 \cdots 0$	$0 \cdots 0$
S	0	≤ 1 n.p.c.

Append to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$. By adding this characteristic vector to each row of A where $\bar{x}$ has been used as the element of Δ as well as to each row of $R - r$, we see that the resulting matroid is graphic. Therefore, $\Psi \preceq \Phi_C$.

Now, we consider subcase (2). Then there are columns v_1 and v_2 of B_{Y_0}, other than the column indexed by y, with $|R|$ nonzero entries. Suppose w is a column of B_{Y_0} such that $w|R$ has two zeros. Since Q_6 is forbidden, w must have a zero in each of the rows where v_1 and v_2 have their zeros. But then B_{Y_0} contains Q_7. Therefore, no column of $B_{Y_0}[R, Y_{0,1}]$ has two zeros. Therefore, recalling that each column of $B_{Y_0}[R, Y_{0,1}]$ has at most two zeros, we have that $B_{Y_0}[X, Y_{0,1} - y]$ is of the following form:

R	1 z.p.c.
s	$1 \cdots 1$
$S - s$	0

Append to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$. By adding this characteristic vector to each row of A where $\bar{x}$ has been used as the element of Δ as well as to each row of $R \cup \{s\}$, we see that the resulting matroid is graphic. Therefore, $\Psi \preceq \Phi_C$.

Now, we consider subcase (3). If $|R| = 2$, then since Q_7 is forbidden B_{Y_0} (including the column indexed by y) is a submatrix, obtained by deleting columns or any rows but the first two, of one of the following matrices :

$$T_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \quad T_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

Append to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$. If B_{Y_0} is a submatrix of T_i, then add this characteristic vector to the first i rows of A as well as to every row of A where $\bar{x}$ has been used as the element of Δ. We see that the resulting matroid is graphic. Therefore, $\Psi \preceq \Phi_C$. Thus, we may assume that $|R| > 2$.

Recall that each column of $B_{Y_0}[R, Y_{0,1}]$ has at most two zeros. Suppose there is a column w of B_{Y_0} such that $w|R$ has exactly two zeros. Since Q_6 and Q_7 are forbidden, all columns of $B_{Y_0}[X, Y_{0,1} - y]$ must have a zero in the same row r of

R. Since we are only considering simple matroids, there is at most one column v where $v|R$ has exactly one zero. Therefore, $B_{Y_0}[X, Y_{0,1} - y]$ has the following form, with or without v:

$$\begin{array}{r|c|c|} & v & \\ \hline R-r & \begin{matrix}1\\ \vdots \\ 1\end{matrix} & 1 \text{ z.p.c} \\ \hline r & 0 & 0\cdots 0 \\ \hline S & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & 0 \\ \hline \end{array}$$

Append to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$ and add this characteristic vector to every row of $R - r$ as well as to every row of A where $\bar{x}$ has been used as the element of Δ. We see that the resulting matroid is graphic.

Therefore, we may assume that every column of $B_{Y_0}[R, Y_{0,1} - y]$ has exactly one zero. Thus, $B_{Y_0}[X, Y_{0,1} - y]$ is of the following form:

$$\begin{array}{r|c|} \hline R & 1 \text{ z.p.c.} \\ \hline S & 0 \\ \hline \end{array}$$

Append to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$. By adding this characteristic vector to each row of A where $\bar{x}$ has been used as the element of Δ as well as to each row of R, we see that the resulting matroid is graphic. Therefore, $\Psi \preceq \Phi_C$. This completes the proof. ■

The next theorem is proved in essentially the same way as Theorem 5.1.3. We omit the proof.

Theorem 5.6.5. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that a cyclically k-connected binary matroid with an $M^*(K_n)$-minor is contained in $\mathcal{EX}(M(K_6), H_{12}^*)$ if and only if it is an even-cut matroid.*

Recall from Section 5.1 that the class of even-cycle matroids with a blocking pair consists of the duals of the members of the class of matroids with an even-cut representation with at most four terminals. Therefore, the following corollary follows immediately from dualizing Theorem 5.4.4.

Corollary 5.6.6. *Suppose Hypothesis 3.2.3 holds. Then there exists $k, n \in \mathbb{Z}_+$ such that a cyclically k-connected binary matroid with an $M^*(K_n)$-minor is contained in $\mathcal{EX}((\mathrm{PG}(3,2)\backslash L)^*, M(K_6))$ if and only if it has an even-cut representation with at most four terminals.*

5.7 Classes with the Same Extremal Function as the Graphic Matroids

Since the extremal function for the class of graphic matroids is $\binom{r+1}{2}$, the Growth Rate Theorem (Theorem 1.3.1) implies that, if F is a nongraphic binary matroid,

$$h_{\mathcal{EX}(F)}(r) \geq \binom{r+1}{2}.$$

Kung, Mayhew, Pivotto, and Royle [17] pose the following question: For which nongraphic binary matroids F of rank 4 does equality hold above for all but finitely many r? Geelen and Nelson answer this question in [11]. Let N_{12} be the matroid formed by deleting a three-element independent set from $\mathrm{PG}(3,2)$. The nongraphic binary matroids F of rank 4 for which $h_{\mathcal{EX}(F)}(r) \approx \binom{r+1}{2}$ are exactly the nongraphic restrictions of N_{12}. We present here an alternate proof. Both proofs allow us to answer the question when F is a matroid of any rank, not just rank 4.

If f and g are real-valued functions of a real variable, then we write $f(x) \approx g(x)$ to denote that $f(x) = g(x)$ for all x sufficiently large, and we say that f and g are *eventually equal*. We will prove the following theorem after proving several lemmas.

Theorem 5.7.1. *Suppose Hypothesis 3.2.3 holds. Let $\mathcal{M}$ be a minor-closed class of binary matroids. Then $h_{\mathcal{M}}(r) \approx \binom{r+1}{2}$ if and only if $\mathcal{M}$ contains all graphic matroids but does not contain $\mathcal{M}_v(\Phi_{Y_1})$.*

For $r \geq 2$, let A_r be the following binary matrix, where we choose for the Γ-frame matrix the matrix representation of $M(K_{r-1})$, so that the identity matrices are $(r-2) \times (r-2)$ matrices.

$0\cdots0$	$0\cdots0$	1	0	1	$1\cdots1$	$0\cdots0$	$1\cdots1$
$0\cdots0$	$0\cdots0$	0	1	1	$0\cdots0$	$1\cdots1$	$1\cdots1$
I_{r-2}	D_{r-2}		0		I_{r-2}	I_{r-2}	I_{r-2}

Note that $M(A_r)$ is the largest simple matroid of rank r that virtually conforms to Φ_{Y_1}.

Recall from Definition 5.2.3 that X_r is the rank-r universal matroid for Φ_{Y_1}.

Lemma 5.7.2. *Every simple, rank-r matroid M that is a minor of a matroid virtually conforming to Φ_{Y_1} is a restriction of X_r.*

Proof. By Lemma 5.2.4, M is a restriction of some $X_{r'}$. So M has an even-cycle representation (G, W) with a blocking pair $\{u, v\}$. Let w be the characteristic vector of W. There are $r' - r$ rows in the matrix $A_{r'}[(V \cup w) - v, E(M)]$ whose deletion does not alter the matroid M. After these rows are deleted, the resulting matrix is a submatrix of A_r. ■

Lemma 5.7.3. *Every matroid virtually conforming to Φ_{Y_1} is a minor of a matroid conforming to Φ_{Y_0}.*

Proof. By Lemma 4.4.4, we have $\Phi_{Y_1} \preceq \Phi_C$. Every matroid conforming to Φ_C is obtained by contracting an element from a matroid conforming to Φ_{Y_0}. ■

Proof of Theorem 5.7.1. First, suppose $h_{\mathcal{M}}(r) \approx \binom{r+1}{2}$. By the Growth Rate Theorem, $\mathcal{M}$ contains all graphic matroids. For $r \geq 1$, we have $|X_r| = \binom{r-1}{2} + 3r - 3$, which for $r > 2$ is greater than $\binom{r+1}{2}$. Thus, $\mathcal{M}$ does not contain $\mathcal{M}_v(\Phi_{Y_1})$.

Now, let $\mathcal{M}$ be a minor-closed class of binary matroids that contains all graphic matroids but does not contain $\mathcal{M}_v(\Phi_{Y_1})$. Hypothesis 3.2.3 gives for $\mathcal{M}$ a set $\{\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t\}$ of binary frame templates. By Theorem 3.3.8, these templates can be chosen so that they are refined. By Lemma 4.5.5, $h_{\mathcal{M}}(r)$ is equal to the size of the largest simple matroid of rank r that conforms to some template in $\{\Phi_1, \ldots, \Phi_s\}$. Since $\mathcal{M}$ contains the class of graphic matroids, $h_{\mathcal{M}}(r) \geq \binom{r+1}{2}$. Suppose for a contradiction that $h_{\mathcal{M}}(r) > \binom{r+1}{2}$. Then there is a nontrivial template $\Phi \in \{\Phi_1, \ldots, \Phi_s\}$. Combining Theorem 4.4.16 with Lemma 4.4.4, we see that either $\Phi_{Y_0} \preceq \Phi$ or $\Phi_{Y_1} \preceq \Phi$. Since $\mathcal{M}$ does not contain $\mathcal{M}_v(\Phi_{Y_1})$, we must have $\Phi_{Y_0} \preceq \Phi$. However, by Lemma 5.7.3, this implies $\mathcal{M}_v(\Phi_{Y_1}) \subseteq \mathcal{M}$. Therefore, by contradiction, we conclude that $h_{\mathcal{M}}(r) \approx \binom{r+1}{2}$, completing the proof. ∎

Corollary 5.7.4. *Suppose Hypothesis 3.2.3 holds. Let F be a simple, binary matroid of rank r. Then $h_{\mathcal{EX}(F)}(r) \approx \binom{r+1}{2}$ if and only if F is a nongraphic restriction of X_r.*

Proof. By Theorem 5.7.1, $h_{\mathcal{EX}(F)}(r) \approx \binom{r+1}{2}$ if and only if $\mathcal{EX}(F)$ contains all graphic matroids but does not contain $\mathcal{M}_v(\Phi_{Y_1})$. The condition that $\mathcal{EX}(F)$ contains all graphic matroids is equivalent to the condition that F is nongraphic. By Lemma 5.7.2, the condition that $\mathcal{EX}(F)$ does not contain $\mathcal{M}_v(\Phi_{Y_1})$ is equivalent to the condition that F is a restriction of X_r. ∎

Note that $X_4 = N_{12}$; so this answers the question posed in [17].

5.8 1-flowing Matroids

The 1-flowing property is a generalization of the max-flow min-cut property of graphs; in this section, we characterize the highly connected 1-flowing matroids, subject to Hypotheses 3.2.2 and 3.2.3. See Seymour [33] or Mayhew [18] for more of the background and motivation concerning 1-flowing matroids. We follow the notation and exposition of [18].

Definition 5.8.1. Let e be an element of a matroid M. Let c_x be a non-negative integral capacity assigned to each element $x \in E(M) - e$. A flow is a function f that assigns to each circuit C containing e a non-negative real number f_C with the constraint that for each $x \in E - e$, the sum of f_C over all circuits containing both e and x is at most c_x. We say that M is *e-flowing* if, for every assignment of capacities, there is a flow whose sum over all circuits containing e is equal to the minimum of

$$\sum_{x \in C^* - e} c_x$$

among cocircuits C^* containing e. If M is e-flowing for each $e \in E(M)$, then M is *1-flowing.*

Since the 1-flowing property is a generalization of a property of graphs, graphic matroids are 1-flowing. In fact, cographic matroids are also 1-flowing, as shown by Seymour [33].

Proposition 5.8.2. *All graphic and cographic matroids are* 1*-flowing.*

The matroid T_{11} is the even-cycle matroid obtained from K_5 by adding a loop and making every edge odd, including the loop. In [33], Seymour showed the following.

Proposition 5.8.3. *The class of* 1*-flowing matroids is minor-closed. Moreover,* $\mathrm{AG}(3,2)$*,* $U_{2,4}$*,* T_{11}*, and* T_{11}^* *are excluded minors for the class of* 1*-flowing matroids.*

Seymour [33] conjectured that these are the only excluded minors.

Conjecture 5.8.4 (Seymour's 1-flowing Conjecture)**.** *The set of excluded minors for the class of* 1*-flowing matroids consists of* $\mathrm{AG}(3,2)$*,* $U_{2,4}$*,* T_{11}*, and* T_{11}^**.*

Since $U_{2,4}$ is an excluded minor for the class of 1-flowing matroids, all such matroids are binary.

Theorem 5.8.5. *Suppose Hypothesis 3.2.2 holds. Then there exists* $k \in \mathbb{Z}_+$ *such that every* k*-connected,* 1*-flowing matroid with at least* $2k$ *elements is either graphic or cographic.*

Proof. Recall the minimal nontrivial templates with respect to the preorder $\preceq$, as described in Definition 4.4.2 and Theorem 4.4.16. Also, recall that we abbreviate Φ_{CX1} to Φ_{CX}.

The matroid $\mathrm{AG}(3,2)$ conforms to Φ_{Y_1} since it is a restriction of N_{12}. Indeed, consider the matrix representing N_{12} that virtually conforms to Φ_{Y_1}. Add the rows labeled by X in this matrix to one of the other rows. Then we can see the matrix representation $[I_4|J_4 - I_4]$ of $\mathrm{AG}(3,2)$ as a restriction of N_{12}. Also, it is not difficult to see that $\mathrm{AG}(3,2)$ can be obtained from a matroid conforming to Φ_{Y_0} by contracting Y_0. Thus, $\mathcal{EX}(\mathrm{AG}(3,2))$ contains neither $\mathcal{M}(\Phi_{Y_0})$ nor $\mathcal{M}(\Phi_{Y_1})$.

Let Φ be a refined template such that $\mathcal{M}(\Phi) \subseteq \mathcal{EX}(\mathrm{AG}(3,2))$. Thus, $\mathcal{M}(\Phi)$ contains neither $\mathcal{M}(\Phi_{Y_0})$ nor $\mathcal{M}(\Phi_{Y_1})$. By Lemma 4.4.4, $\mathcal{M}(\Phi)$ does not contain $\mathcal{M}(\Phi_C)$, $\mathcal{M}(\Phi_X)$, or $\mathcal{M}(\Phi_{CX})$ either. In the binary case, $\Phi_{Y_1}(k)$, Φ_n, and Φ_{CXk} for $k \neq 1$ do not exist. Therefore, by Theorem 4.4.16, Φ is trivial. Since $\mathrm{AG}(3,2)$ is self-dual, every template Ψ such that $\mathcal{M}^*(\Psi) \subseteq \mathcal{EX}(\mathrm{AG}(3,2))$ is also trivial. By Corollary 4.1.5, there is a positive integer k such that, if M is a simple, k-connected, 1-flowing matroid of size at least $2k$, then either M or M^* weakly conforms to the trivial template. A matroid conforms to the trivial template if and only if it is graphic. The result follows from the fact that a matroid is graphic if and only if its simplification also is graphic. ∎

The next theorem has a similar proof to that of Theorem 5.8.5. We omit the details except to note that Corollary 4.1.6 is used instead of Corollary 4.1.5.

Theorem 5.8.6. *Suppose Hypothesis 3.2.3 holds. Then there exist* $k, n \in \mathbb{Z}_+$ *such that every vertically* k*-connected,* 1*-flowing matroid with an* $M(K_n)$*-minor*

is graphic and every cyclically k-connected, 1-flowing matroid with an $M^*(K_n)$-minor is cographic.

It follows from Theorems 5.8.5 and 5.8.6 that Seymour's 1-flowing Conjecture holds for highly connected matroids, subject to Hypotheses 3.2.2 and 3.2.3, respectively. We make this precise in the following corollary. Recall that $\mathcal{EX}(F_1, F_2, \ldots, F_n)$ denotes the class of binary matroids with no minor contained in $\{F_1, F_2, \ldots, F_n\}$. Since every matroid in $\mathcal{EX}(F_1, F_2, \ldots, F_n)$ is binary, there is no need to include $U_{2,4}$ as one of the F_i.

Corollary 5.8.7. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that a k-connected matroid M with at least $2k$ elements is 1-flowing if and only if $M \in \mathcal{EX}(\mathrm{AG}(3,2), T_{11}, T_{11}^*)$.*

Proof. Let k be the value given by Theorem 5.8.5. By Proposition 5.8.3, the class of 1-flowing matroids is contained in $\mathcal{EX}(\mathrm{AG}(3,2), T_{11}, T_{11}^*)$. For the converse, note that $\mathcal{EX}(\mathrm{AG}(3,2), T_{11}, T_{11}^*) \subseteq \mathcal{EX}(\mathrm{AG}(3,2))$. The proof of Theorem 5.8.5 shows that the k-connected members of $\mathcal{EX}(\mathrm{AG}(3,2))$ with at least $2k$ elements are either graphic or cographic and therefore 1-flowing, by Proposition 5.8.2. ■

The proofs of the next two corollaries are similar to that of Corollary 5.8.7 and are omitted.

Corollary 5.8.8. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that a vertically k-connected matroid M with an $M(K_n)$-minor is 1-flowing if and only if $M \in \mathcal{EX}(\mathrm{AG}(3,2), T_{11}, T_{11}^*)$.*

Corollary 5.8.9. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that a cyclically k-connected matroid M with an $M^*(K_n)$-minor is 1-flowing if and only if $M \in \mathcal{EX}(\mathrm{AG}(3,2), T_{11}, T_{11}^*)$.*

Chapter 6: Applications to Golden-Mean Matroids and Other Classes of Quaternary Matroids

A matroid is *quaternary* if it is representable over the field GF(4). In this chapter, we apply the results of Chapters 3 and 4 to certain subclasses of the quaternary matroids—particularly the golden-mean matroids, which consists of matroids representable over both GF(4) and GF(5). We denote the class of golden-mean matroids by $\mathcal{GM}$.

Let $\mathcal{P}$ be the set of prime numbers, and let $\mathcal{AC}_4$ denote the class of quaternary matroids whose characteristic set (defined in Section 6.1) is $\mathcal{P} \cup \{0\}$. Let $\mathcal{AF}_4$ be the class of matroids representable over all fields of size at least 4, and let $\mathcal{SL}_4$ denote the class of quaternary matroids M for which there exists a prime power q' such that M is representable over all fields of size at least q'. We show that, subject to Hypothesis 3.2.3, the extremal function for the class of golden-mean matroids, as well as for $\mathcal{AC}_4$, for $\mathcal{AF}_4$, and for $\mathcal{SL}_4$ is $\binom{r+3}{2} - 5$, verifying a conjecture of Archer [1] for matroids of sufficiently large rank.

We also completely characterize the highly connected matroids in these classes, subject to Hypothesis 3.2.2. We prove the following.

Theorem 6.0.1. *Suppose Hypothesis 3.2.2 holds. There exists $k \in \mathbb{Z}_+$ such that every k-connected member of $\mathcal{AC}_4$ with at least $2k$ elements is contained in exactly one of $\mathcal{AF}_4$, $\mathcal{GM} - \mathcal{AF}_4$, and $\mathcal{SL} - \mathcal{AF}_4$ and such that every k-connected member of $\mathcal{SL}_4$ with at least $2k$ elements is representable over all fields of size at least 7.*

As in some previous chapters, the use of the word *matroid* in this chapter will refer to represented matroids, unless abstract matroids are explicitly mentioned. However, when we say that a matroid has some property, we mean that the represented matroid we are referring to has an associated abstract matroid with that property. In particular, when we say that a matroid is representable over a field $\mathbb{F}'$, we mean that the $\mathbb{F}$-represented matroid (usually $\mathbb{F} = \mathrm{GF}(4)$ in this chapter) has an associated abstract matroid that is representable over $\mathbb{F}'$. In this chapter, if A is a matrix with rows and columns labeled by sets B and E, respectively, and if $\{a_1, a_2, \ldots, a_n\} \subseteq E$, then $[a_1, a_2, \ldots, a_n]$ denotes the column submatrix $A[B, \{a_1, a_2, \ldots, a_n\}]$ of A.

6.1 Characteristic Sets

The *characteristic set* of a matroid M is the set $\mathcal{K}(M) = \{k : M \text{ is representable over some field of characteristic } k\}$. Let $\mathcal{P}$ denote the set of prime numbers. We will denote by $\mathcal{AC}_q$ the class of GF(q)-represented matroids with characteristic set $\mathcal{P} \cup \{0\}$. (The notation $\mathcal{AC}$ stands for "all characteristics.") Combining results of Rado [31, Theorem 6] and Vámos [40] gives the following.

Theorem 6.1.1. *If M is a matroid, then either $0 \in \mathcal{K}(M)$ and $\mathcal{P} - \mathcal{K}(M)$ is finite, or $0 \notin \mathcal{K}(M)$ and $\mathcal{K}(M)$ is finite.*

For an abstract matroid M, we construct a system of polynomial equations that has a solution over a field $\mathbb{F}$ if and only if M is $\mathbb{F}$-representable. We start with four observations. First, for any $r \times |E(M)|$ representation matrix A of the rank-r matroid M, an $r \times r$ submatrix D has $\det(D) \neq 0$ if and only if the set of column labels is a basis of M. Second, given a basis B of M, we can perform row operations such that the submatrix of A corresponding to B is an identity matrix. Third, the fundamental circuits of M with respect to B determine exactly which entries of A are zero (see Oxley [23, Proposition 6.4.1]). Fourth, we can choose some entries of the remaining matrix arbitrarily. These entries correspond to a maximal forest of the *fundamental graph of M with respect to B* (see [23, Theorem 6.4.7]). We will set all these entries to 1.

Introduce variables $x_1, \ldots, x_s$, one for each matrix entry not determined by the above observations, and variables $y_1, \ldots, y_t$, one for each basis of M. Fill a matrix A' over $\mathbb{Z}[x_1, \ldots, x_s, y_1, \ldots, y_t]$ with zeroes, ones, and the x_i as described. Let S be a system of polynomials, one for each r-subset $X \subseteq E(M)$, given by

$$\begin{cases} \det(D) & \text{if } X \text{ is not a basis of } M \\ \det(D)y_i - 1 & \text{if } X \text{ is the } i^{\text{th}} \text{ basis of } M, \end{cases}$$

where D is the $r \times r$ submatrix of A' corresponding to X. From the construction we have the following:

Theorem 6.1.2. *We can assign elements of $\mathbb{F}$ to the variables $x_1, \ldots, x_t, y_1, \ldots, y_s$ such that all polynomials in S evaluate to 0 if and only if M has a representation over $\mathbb{F}$.*

We can deduce information about the representability of M by studying the *ideal* generated by S. Baines and Vámos developed an algorithm [2, Theorem 3.5] to determine the set of characteristics of the fields over which such a system has a solution. This algorithm involves the Gröbner basis of the ideal generated by S. The author implemented this algorithm in SageMath. The code can be found in Section A.5 and is essentially identical to code written by Dillon Mayhew [19].

It is well-known that the Fano matroid F_7 is $\mathbb{F}$-representable if and only $\mathbb{F}$ has characteristic 2. (See, for example, Oxley [23, Proposition 6.4.8].) It is also well-known that, for each $e \in E(F_7)$, the matroids F_7/e and $F_7\backslash e$ are graphic and therefore regular. Therefore, since representability over a field is preserved by duality, F_7 and F_7^* are both excluded minors for $\mathcal{AC}_4$.

We introduce here six additional excluded minors for $\mathcal{AC}_4$, none of which are binary. Three of them have characteristic set $\{2\}$, and three have characteristic set $(\mathcal{P} - \{3\}) \cup \{0\}$. In order to show that these matroids are indeed excluded minors, we use the function `CharacteristicSet` (found in Section A.5). Let α be a solution to the equation $x^2 + x + 1 = 0$ over GF(4). Then the other solution is $\alpha^2 = \alpha + 1 = \alpha^{-1}$.

Definition 6.1.3. We define V_1, V_2, V_3, P_1, P_2, and P_3 to be the vector matroids of the following quaternary matrices.

$$V_1 = M\left(\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & \alpha^2 & \alpha^2 & \alpha & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & \alpha & 1 & \alpha \end{bmatrix} \end{array}\right)$$

$$V_2 = M\left(\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & \alpha^2 & \alpha & \alpha & 1 \\ 0 & 0 & 1 & 0 & \alpha & \alpha^2 & \alpha & 1 \\ 0 & 0 & 0 & 1 & \alpha^2 & \alpha^2 & \alpha & \alpha \end{bmatrix} \end{array}\right)$$

$$V_3 = M\left(\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & \alpha & 1 & \alpha^2 & \alpha^2 & 1 \\ 0 & 0 & 1 & 0 & \alpha & 1 & \alpha & 1 & \alpha^2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}\right)$$

$$P_1 = M\left(\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & \alpha & \alpha^2 & 1 & \alpha & \alpha^2 \end{bmatrix} \end{array}\right)$$

$$P_2 = M\left(\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & \alpha & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & \alpha^2 & 1 & \alpha^2 & 0 \end{bmatrix} \end{array}\right)$$

$$P_3 = M\left(\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \alpha & \alpha^2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & \alpha & \alpha^2 & 1 & \alpha \end{bmatrix} \end{array}\right)$$

Geometric representations of V_1 and V_2, essentially the work of James Oxley [24], are given in Figures 6.1 and 6.2. Note that P_1 is the result of deleting from the rank-3 Dowling geometry $Q_3(\mathrm{GF}(4)^\times)$ two joints and an additional point in the closure of the two joints, and that P_3 is the result of deleting from $Q_3(\mathrm{GF}(4)^\times)$ two joints and a point (other than the third joint) not in the closure of the two joints.

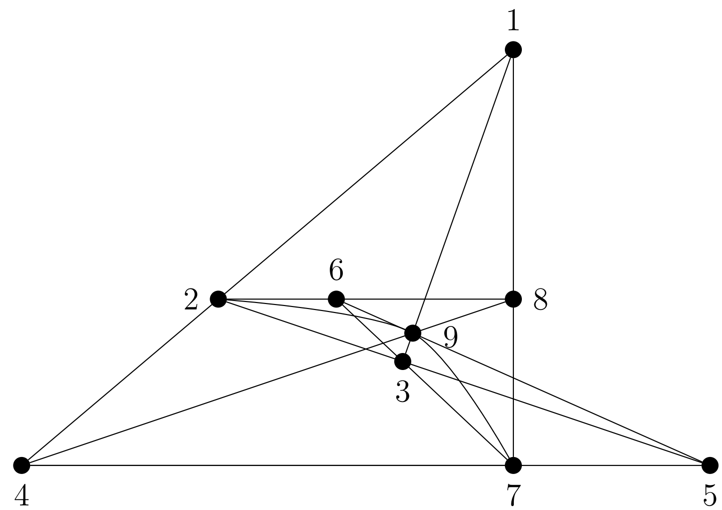

FIGURE 6.1. A Geometric Representation of V_1

Theorem 6.1.4. *The matroids V_1, V_2, and V_3 each have $\{2\}$ as their characteristic set. The matroids P_1, P_2, and P_3 each have $(\mathcal{P} - \{3\}) \cup \{0\}$ as their characteristic set. All six of these matroids are excluded minors for the class $\mathcal{AC}_4$.*

Proof. As shown in Section A.5, the author ran the function `CharacteristicSet` on each of these six matroids and obtained the characteristic sets claimed. For each of the six matroids, the author also ran the following code, where `V1` is replaced by `V2`, `V3`, `P1`, `P2`, and `P3`.

```
for e in V1.groundset():
    if CharacteristicSet(V1\e)!= [0]:
        print e
print ‘done’
```

In each case, we found that the characteristic set of $M\backslash e$ is $\mathcal{P} \cup \{0\}$ for every matroid $M \in \{V_1, V_2, V_3, P_1, P_2, P_3\}$ and every element $e \in E(M)$.

Since V_1, P_1, and P_3 each have rank 3, we see that the characteristic set of M/e is $\mathcal{P} \cup \{0\}$ for every matroid $M \in \{V_1, P_1, P_3\}$ and every element $e \in E(M)$, since every rank-2 matroid is representable over all sufficiently large fields. Using the SageMath code `V2.is_isomorphic(V2.dual())` and `P2.is_isomorphic(P2.dual())`, we see that V_2 and P_2 are self-dual. Since every single-element deletion of these matroids has characteristic set $\mathcal{P} \cup \{0\}$, every single-element contraction of these matroids has the same characteristic set.

It remains to check the single-element contractions of V_3. The following code establishes that every single-element contraction of V_3 has characteristic set $\mathcal{P} \cup \{0\}$.

```
for e in V3.groundset():
    if CharacteristicSet(V3/e)!= [0]:
        print e
print ‘done’
```

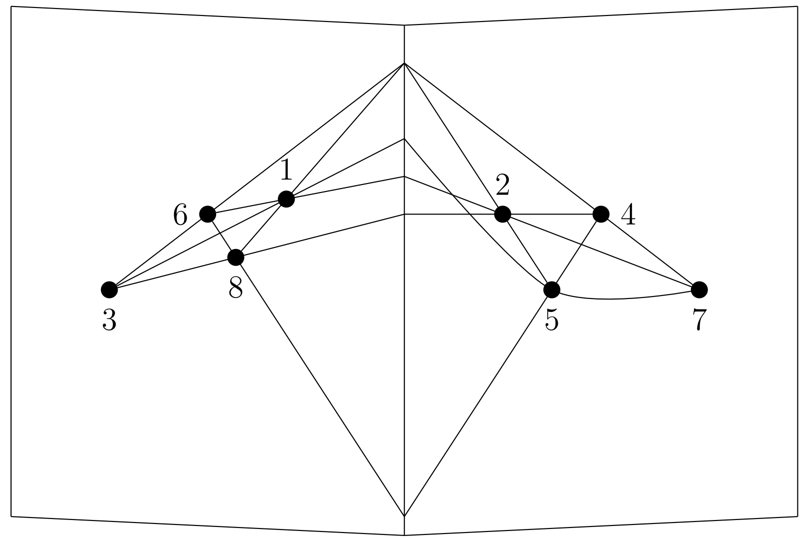

FIGURE 6.2. A Geometric Representation of V_2

This completes the proof that V_1, V_2, V_3, P_1, P_2, and P_3 are excluded minors for the class $\mathcal{AC}_4$. ■

6.2 Golden-Mean Matroids

Let $\tau = \frac{1+\sqrt{5}}{2}$ be the *golden mean*, the positive solution to the equation $x^2 - x - 1 = 0$ over $\mathbb{R}$. A *golden-mean matroid* is a matroid that can be represented by a matrix over $\mathbb{R}$ where every nonzero subdeterminant is $\pm\tau^i$ for some $i \in \mathbb{Z}$. We will denote the class of golden-mean matroids by $\mathcal{GM}$. The following characterization of golden-mean matroids was originally an unpublished result of Dirk Vertigan. Pendavingh and Van Zwam [26, Theorem 4.9] published a proof later.

Theorem 6.2.1. *The following are equivalent for a matroid M:*

- *M is golden-mean.*
- *M is representable over* GF(4) *and* GF(5)*.*
- *M is representable over* GF(5)*, over* GF(p^2) *for all primes p, and over* GF(p) *for all primes p such that $p \equiv \pm 1 \pmod 5$.*

This characterization of the golden-mean matroids is analogous to characterizations of several other classes of matroids, including regular matroids, near-regular matroids, dyadic matroids, and $\sqrt[6]{1}$-matroids. This characterization also immediately leads to the following result that will be fundamental to the rest of this chapter.

Corollary 6.2.2. *Every golden-mean matroid is contained in $\mathcal{AC}_4$.*

In GF(5), the unique solution to the equation $x^2 - x - 1 = 0$ is 3. In fields of characteristic 2, the equation $x^2 - x - 1 = 0$ is the same as $x^2 + x + 1 = 0$, which we have already seen has α and α^2 as solutions in GF(4).

Recall from Section 1.3 that the extremal function for a class $\mathcal{M}$ of matroids is denoted by $h_{\mathcal{M}}(r)$ and is the function whose value at an integer $r \geq 0$ is given by the maximum number of elements in a simple matroid in $\mathcal{M}$ of rank at most r. Archer [1] and Welsh [42] have studied the extremal function of $\mathcal{GM}$. Archer conjectured that the extremal function for $\mathcal{GM}$ is

$$h_{\mathcal{GM}}(r) = \begin{cases} \binom{r+3}{2} - 5 & if \quad r \neq 3 \\ 11 & if \quad r = 3 \end{cases}$$

Archer showed that indeed $h_{\mathcal{GM}}(3) = 11$ and that the unique maximum-sized golden-mean matroid of rank 3 is the Betsy Ross matroid B_{11} (see figure 6.3). He

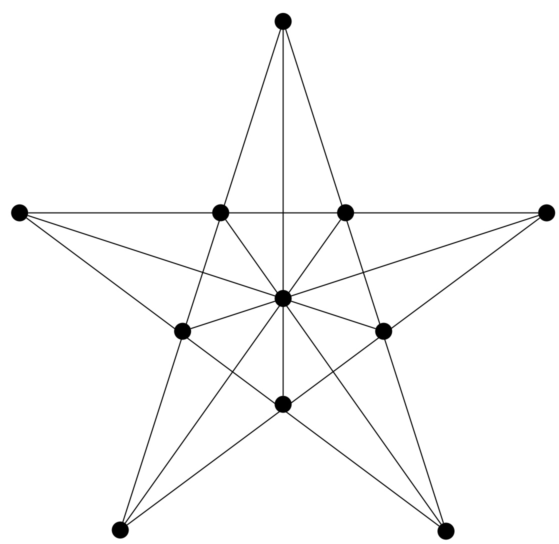

FIGURE 6.3. A Geometric Representation of the Betsy Ross Matroid

further conjectured that there are three families of matroids that are the maximum-sized golden-mean matroids for rank $r \neq 3$. Welsh denoted these three conjectured maximum-sized golden-mean matroids of rank $r \neq 3$ by T_r^2, G_r, and HP_r. Welsh also gave the matrix representations over GF(4) for T_r^2, G_r, and HP_r given in Definition 6.2.3.

Definition 6.2.3. The matroids T_r^2, G_r, and HP_r are the vector matroids of the following matrices over GF(4).

$$T_r^2 = M\left(\begin{array}{|c|c|c|c|c|} \hline & 0 \cdots 0 & 1 \cdots 1 & \alpha \cdots \alpha & \alpha^2 \cdots \alpha^2 \\ \cline{2-5} I_r & D_{r-1} & I_{r-1} & I_{r-1} & I_{r-1} \\ \hline \end{array}\right)$$

$$G_r = M\left(\begin{array}{|c|c|c|c|c|c|ccc|} \hline & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 & \alpha \cdots \alpha & 0 \cdots 0 & 1 & 1 & 1 \\ I_r & 0 \cdots 0 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 & \alpha \cdots \alpha & 1 & \alpha & \alpha^2 \\ \cline{2-9} & D_{r-2} & I_{r-2} & I_{r-2} & I_{r-2} & I_{r-2} & & 0 & \\ \hline \end{array}\right)$$

$$HP_r = M\left(\left[\begin{array}{c|c|c|c|c|c|ccc} & 0\cdots0 & \alpha\cdots\alpha & 0\cdots0 & \alpha\cdots\alpha & \alpha^2\cdots\alpha^2 & 1 & 1 & 1 \\ I_r & 0\cdots0 & 0\cdots0 & 1\cdots1 & \alpha\cdots\alpha & 1\cdots1 & 1 & \alpha & \alpha^2 \\ \cline{2-9} & D_{r-2} & I_{r-2} & I_{r-2} & I_{r-2} & I_{r-2} & & 0 & \end{array}\right]\right)$$

We will discuss partial fields in Section 6.5; however, to the reader familiar with partial fields, we note that Welsh [42] intended to define T_r^2, G_r, and HP_r as matroids over the golden-mean partial field, rather than over GF(4). But this was done incorrectly because the matrices given in [42, Figure 2.1] for G_r and HP_r are not actually golden-mean matrices. It suffices for our purposes to define T_r^2, G_r, and HP_r as matrices over GF(4), as we have done in Definition 6.2.3 and as Welsh did in [42, Figure 2.5].

In the representation of HP_r given above, scale the top row by α^2, the first column by α, and the last three columns by α. Then rearrange the last three columns to obtain the following.

$$HP_r = M\left(\left[\begin{array}{c|c|c|c|c|c|ccc} & 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1 & \alpha\cdots\alpha & 1 & 1 & 1 \\ I_r & 0\cdots0 & 0\cdots0 & 1\cdots1 & \alpha\cdots\alpha & 1\cdots1 & 1 & \alpha & \alpha^2 \\ \cline{2-9} & D_{r-2} & I_{r-2} & I_{r-2} & I_{r-2} & I_{r-2} & & 0 & \end{array}\right]\right)$$

Thus, for $r \geq 2$, the GF(4)-represented matroids corresponding to T_r^2, G_r, and HP_r are, respectively, the largest simple matroids of rank r virtually conforming to the Y-template $\mathrm{YT}(P_0, P_1)$ over GF(4), where

$$P_0 = [\emptyset] \qquad P_1 = [\alpha, \alpha^2],$$

$$P_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \end{bmatrix} \qquad P_1 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix},$$

and

$$P_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad P_1 = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}.$$

(This last P_0 only needs one column because, with virtual conforming and scaling, $[1, \alpha, 0, \dots, 0]^T$ and $[1, \alpha^2, 0, \dots, 0]^T$ can be thought of as elements of Z, rather than Y_0.) We will call these templates $\Phi(T_r^2)$, $\Phi(G_r)$, and $\Phi(HP_r)$, respectively.

Proposition 6.2.4. *The matroids T_r^2, G_r, and HP_r are golden-mean matroids.*

Proof. By Definition 6.2.3, T_r^2, G_r, and HP_r are representable over GF(4). By Theorem 6.2.1, it suffices to show that they are representable over GF(5). Since these matroids are respectively the largest simple matroids of rank r that virtually conform to complete Y-templates, it suffices to find complete Y-templates over GF(5) that are abstractly equivalent to these templates.

We claim that $\Phi(T_r^2)$ is abstractly equivalent to the template $\mathrm{YT}([\emptyset], [3, 4])$ over GF(5). By Lemma 4.3.7, it suffices to show that the $\widetilde{M}(A) = \widetilde{M}(A')$, where $A = \begin{bmatrix} 1 & 1 & \alpha & \alpha^2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and $A' = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Both of these matroids are $U_{2,4}$, with the

collection of circuits consisting of all sets of size 3. Therefore, the abstract matroids are equal.

We claim that $\Phi(G_r)$ is abstractly equivalent to the template

$$\mathrm{YT}\left(\begin{bmatrix}1 & 1 & 1\\ 4 & 2 & 3\end{bmatrix}, \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}\right)$$

over GF(5) and that $\Phi(HP_r)$ is abstractly equivalent to the template

$$\mathrm{YT}\left(\begin{bmatrix}1\\ 4\end{bmatrix}, \begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}\right)$$

over GF(5). Note that $4 = -1$ in GF(5), so these are complete templates. By Lemma 4.3.7 (and simplification), it suffices to show that the vector matroids of

$$\begin{bmatrix}1 & 0 & 1 & 0 & \alpha & 0 & 1 & 0 & \alpha & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & 0 & \alpha & 0 & 1 & 0 & \alpha & 1 & \alpha & \alpha^2\\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 3 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 3 & 4 & 2 & 3\\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$$

are equal and that the vector matroids of

$$\begin{bmatrix}1 & 0 & 1 & \alpha & 1 & 0 & 1 & \alpha & 1 & 0 & 1 & \alpha & 1\\ 0 & 1 & \alpha & 1 & 0 & 1 & \alpha & 1 & 0 & 1 & \alpha & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 0 & 1 & 3 & 1 & 0 & 1 & 3 & 1 & 0 & 1 & 3 & 1\\ 0 & 1 & 3 & 1 & 0 & 1 & 3 & 1 & 0 & 1 & 3 & 1 & 4\\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0\end{bmatrix}$$

are equal.

This is easily verified with SageMath (see Section A.6). ∎

Recall that F_7, F_7^*, V_1, V_2, V_3, P_1, P_2, and P_3 are excluded minors for $\mathcal{AC}_4$. This fact will be used in the next several proofs.

Lemma 6.2.5. *If $\Phi = (\Gamma, C, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ is a refined quaternary template such that $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$, then Φ is a Y-template.*

Proof. We will prove this result using Lemma 4.4.15. Consider the following representation of F_7: $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$. Choose one of these rows and let $X = \{x\}$ be the set indexing it. The remaining two rows form a $\{1\}$-frame matrix. Thus, F_7 conforms to Φ_X. Taking the same representation, if we index the last column by Y_0, the rest of the columns form a $\{1\}$-frame matrix. Thus, F_7 conforms to Φ_{Y_0}. Therefore, $\Phi_X \not\preceq \Phi$, and $\Phi_{Y_0} \not\preceq \Phi$. Moreover, by Lemma 4.4.4, $\Phi_C \not\preceq \Phi$, and $\Phi_{CXk} \not\preceq \Phi$ for each $k \in \mathrm{GF}(4) - \{0\}$.

The multiplicative group of GF(4) has size 3. Therefore, the only possible prime divisor of $|\Gamma|$ is 3. The class $\mathcal{M}(\Phi_3)$ is exactly the class of quaternary matroids with representation matrices such that every column has at most two nonzero entries. The matroids P_1 and P_3 are such matroids; therefore $\Phi_3 \not\preceq \Phi$. Therefore, by Lemma 4.4.15, Φ is a Y-template. ■

6.3 Extremal Functions

If an extremal matroid for a minor-closed class virtually conforms to a template, then we call that template an *extremal template* for the class. Let Φ be a template such that $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$. By Lemma 6.2.5, Φ is a Y-template $\mathrm{YT}(P_0, P_1)$. By Lemma 4.4.13, we may assume that every column of P_1 is nonzero and that no column of $[I_{|X|}|P_1]$ is a copy of another column. If P_1 is a $c \times d$ matrix and if $\varepsilon(M([I_{|X|}|P_1|P_0])) = |\widehat{Y}|$, then the largest simple matroid M of rank $r \geq |X| = c$ virtually conforming to Φ has $r - c + \binom{r-c}{2}$ elements in $E(M) - (Y_0 \cup Y_1 \cup Z)$ and $(c+d)(r-c) + |\widehat{Y}|$ elements in $Z \cup Y_0$. Thus,

$$\varepsilon(M) = r - c + \binom{r-c}{2} + (c+d)(r-c) + |\widehat{Y}|.$$

Some arithmetic shows

$$\varepsilon(M) = \frac{1}{2}r^2 + \frac{2d+1}{2}r - \frac{c^2 + c + 2dc - 2|\widehat{Y}|}{2}.$$

Thus, for all sufficiently large ranks, to find the extremal templates for $\mathcal{AC}_4$, we must find P_1 with as many (distinct) columns as possible. We now analyze the structure of P_1 when $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$.

Recall that applying an automorphism of a field $\mathbb{F}$ to every entry in a matrix A over $\mathbb{F}$ results in a matrix whose vector matroid is equal to that of A. The only nontrivial automorphism of GF(4) maps α to α^2 and α^2 to α (while leaving 0 and 1 fixed). In the rest of this chapter, when we say "up to a field automorphism," this is the automorphism we refer to.

Lemma 6.3.1. *Suppose $\Phi = \mathrm{YT}(P_0, P_1)$ is a template such that $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$. Then none of the matrices listed in Table 6.1 are submatrices of P_1. Moreover, none of the matrices obtained from those in Table 6.1 by replacing α with α^2 and vice-versa are submatrices of P_1.*

TABLE 6.1. Forbidden Matrices

Matrix	Rank	Excluded Minor	Matrix	Rank	Excluded Minor
$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	3	F_7	$B = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}$	4	V_2
$C = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}$	4	F_7^*	$D = \begin{bmatrix} \alpha \\ \alpha \\ \alpha^2 \end{bmatrix}$	4	V_2
$E = \begin{bmatrix} \alpha & 0 \\ \alpha & 0 \\ 0 & \alpha \end{bmatrix}$	5	F_7^*	$F = \begin{bmatrix} \alpha & 0 \\ \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}$	5	F_7^*
$G = \begin{bmatrix} 1 & 0 \\ \alpha & \alpha \end{bmatrix}$	3	F_7	$H = \begin{bmatrix} 1 & 0 \\ \alpha & \alpha^2 \end{bmatrix}$	4	F_7^*
$I = \begin{bmatrix} 1 & \alpha^2 \\ \alpha & 1 \end{bmatrix}$	4	P_2	$J = \begin{bmatrix} \alpha & \alpha \\ \alpha & 0 \end{bmatrix}$	4	P_2
$K = \begin{bmatrix} 1 & 1 \\ \alpha & \alpha^2 \end{bmatrix}$	3	F_7	$L = \begin{bmatrix} \alpha^2 & 0 \\ \alpha & 0 \\ 0 & \alpha \end{bmatrix}$	5	P_2
$M = \begin{bmatrix} \alpha & \alpha^2 & 0 \\ \alpha & 0 & \alpha^2 \end{bmatrix}$	3	F_7	$N = \begin{bmatrix} \alpha^2 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$	4	P_2

Proof. If one of the matrices listed in Table 6.1 is a submatrix of P_1, then we can perform deletion (operation (4) of Proposition 4.1.2) and y-shifts (Lemma 4.4.5) followed by contraction (operation (11) of Definition 4.1.3) on elements of Y_1 to obtain a template where the matrix from Table 6.1 is itself P_1 (rather than just a submatrix).

Now we show that, if P_1 is one of the matrices in Table 6.1, then $\mathcal{M}(\Phi) \nsubseteq \mathcal{AC}_4$. We verify this with SageMath, using the functions `Y_template_matrix` (the same function found in Section A.6 and used in Section 6.3) and `MinorCheck`, which is found in Section A.7. The rank listed in the table is the rank of the matroid virtually conforming to the template which contained the excluded minor given in the table. For example, the following code was used for matrix E.

```
AY0 = Matrix(GF4, [])
AY1 = Matrix(GF4, [[a,0],
                   [a,0],
                   [0,a]])
A = Y_template_matrix(5, AY0, AY1)
A
M=Matroid(field=GF4, matrix=A)
MinorCheck(M)
```

It returned `'Fano dual'`. The last statement of the lemma follows from the fact that field isomorphisms preserve (abstract) representable matroids. ■

The next lemma involves larger matrices where the technique used in the previous lemma is significantly more time-consuming.

Lemma 6.3.2. *Suppose* $\Phi = \mathrm{YT}(P_0, P_1)$ *is a template such that* $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$. *Then* P_1 *contains neither* $\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$ *nor* $\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$ *as a submatrix.*

Proof. Rather than testing for each of the excluded minors for which the function `MinorCheck` tests, we instead check only for the minor P_2. In fact, we contract an element first and simplify.

```
AY0 = Matrix(GF4, [])
AY1 = Matrix(GF4, [[a,0,0],
                   [0,a,0],
                   [0,0,a]])
A = Y_template_matrix(6, AY0, AY1)
A
M=Matroid(field=GF4, matrix=A)
((M/29).simplify()).has_minor(P2)
```

Now, this SageMath computation shows that the first matrix listed in the result is forbidden from the matrix P_1. If we call this matrix A, then scaling the last row of $[I_3|A]$ by α^2 and rearranging the columns results in the template corresponding to the second matrix listed in the result. ■

Lemma 6.3.3. *Suppose* $\Phi = \mathrm{YT}(P_0, P_1)$ *is a template such that* $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$. *Then* Φ *is equivalent to a template* $\mathrm{YT}(P_0', P_1')$, *where* P_1' *is, after its zero rows are removed, a submatrix of one of the following matrices, up to field automorphism.*

$$\begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \\ \alpha & 0 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha \\ \alpha & 1 \end{bmatrix}, \begin{bmatrix} \alpha^2 & \alpha^2 \\ \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \begin{bmatrix} \alpha^2 & \alpha \\ \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}, \begin{bmatrix} \alpha^2 & \alpha \\ \alpha & \alpha^2 \end{bmatrix}$$

Proof. In this proof, the letters A, B, ..., N will refer to the matrices in Table 6.1.

Combining Lemma 4.2.12 with the fact that A, B, C, and D cannot be submatrices of P_1, we may assume that every column of P_1 is, up to field automorphism and permuting of rows, either $[1, \alpha, \alpha, 0, \ldots, 0]^T$ or $[\alpha, \alpha^2, 0, \ldots, 0]^T$.

We now analyze the possible forms P_1 can take if it contains exactly two columns. Tables 6.2–6.5 list matrices that are candidates for being submatrices of P_1. If such a matrix cannot occur as a submatrix of P_1, the second column of the table indicates the presence of a submatrix forbidden by Lemma 6.3.1. Table 6.2 considers the case where one column is of the type $[1, \alpha, \alpha, 0, \ldots, 0]^T$ and the other column is of the type $[\alpha, \alpha^2, 0, \ldots, 0]^T$. Up to a field automorphism, this is the same as the case where one column is of the type $[1, \alpha^2, \alpha^2, 0, \ldots, 0]^T$ and the other column is of the

TABLE 6.2. Candidate Matrices—Case 1

Candidate Matrix	Forbidden Matrix	Candidate Matrix	Forbidden Matrix
$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & 0 \\ 0 & \alpha \\ 0 & \alpha^2 \end{bmatrix}$	E	$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & \alpha \\ 0 & \alpha^2 \end{bmatrix}$	G
$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & \alpha^2 \\ 0 & \alpha \end{bmatrix}$	H	$\begin{bmatrix} 1 & \alpha \\ \alpha & 0 \\ \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}$	F
$\begin{bmatrix} 1 & \alpha^2 \\ \alpha & 0 \\ \alpha & 0 \\ 0 & \alpha \end{bmatrix}$	E	$\begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \\ \alpha & 0 \end{bmatrix}$	None
$\begin{bmatrix} 1 & \alpha^2 \\ \alpha & \alpha \\ \alpha & 0 \end{bmatrix}$	J	$\begin{bmatrix} 1 & 0 \\ \alpha & \alpha \\ \alpha & \alpha^2 \end{bmatrix}$	H

TABLE 6.3. Candidate Matrices—Case 2

Candidate Matrix	Forbidden Matrix	Candidate Matrix	Forbidden Matrix
$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & 0 \\ 0 & 1 \\ 0 & \alpha \\ 0 & \alpha \end{bmatrix}$	E	$\begin{bmatrix} 1 & 1 \\ \alpha & 0 \\ \alpha & 0 \\ 0 & \alpha \\ 0 & \alpha \end{bmatrix}$	E
$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & \alpha \\ 0 & \alpha \\ 0 & 1 \end{bmatrix}$	J	$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & 1 \\ 0 & \alpha \\ 0 & \alpha \end{bmatrix}$	E
$\begin{bmatrix} 1 & 1 \\ \alpha & \alpha \\ \alpha & 0 \\ 0 & \alpha \end{bmatrix}$	J	$\begin{bmatrix} 1 & \alpha \\ \alpha & \alpha \\ \alpha & 0 \\ 0 & 1 \end{bmatrix}$	J
$\begin{bmatrix} 1 & 0 \\ \alpha & \alpha \\ \alpha & \alpha \\ 0 & 1 \end{bmatrix}$	G	$\begin{bmatrix} 1 & \alpha \\ \alpha & \alpha \\ \alpha & 1 \end{bmatrix}$	None

type $[\alpha, \alpha^2, 0, \ldots, 0]^T$. Table 6.3 considers the case where both columns are of the type $[1, \alpha, \alpha, 0, \ldots, 0]^T$. Up to a field automorphism, this is the same as the case

TABLE 6.4. Candidate Matrices—Case 3

Candidate Matrix	Forbidden Matrix
$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & 0 \\ 0 & 1 \\ 0 & \alpha^2 \\ 0 & \alpha^2 \end{bmatrix}$	F
$\begin{bmatrix} 1 & 1 \\ \alpha & 0 \\ \alpha & 0 \\ 0 & \alpha^2 \\ 0 & \alpha^2 \end{bmatrix}$	F
$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & \alpha^2 \\ 0 & \alpha^2 \\ 0 & 1 \end{bmatrix}$	H
$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \alpha & 1 \\ 0 & \alpha^2 \\ 0 & \alpha^2 \end{bmatrix}$	F (up to field automorphism)
$\begin{bmatrix} 1 & 1 \\ \alpha & \alpha^2 \\ \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}$	K
$\begin{bmatrix} 1 & \alpha^2 \\ \alpha & \alpha^2 \\ \alpha & 0 \\ 0 & 1 \end{bmatrix}$	H (up to field automorphism)
$\begin{bmatrix} 1 & 0 \\ \alpha & \alpha^2 \\ \alpha & \alpha^2 \\ 0 & 1 \end{bmatrix}$	H
$\begin{bmatrix} 1 & \alpha^2 \\ \alpha & \alpha^2 \\ \alpha & 1 \end{bmatrix}$	I

where both columns are of the type $[1, \alpha^2, \alpha^2, 0, \ldots, 0]^T$. Table 6.4 considers the case where one column is of the type $[1, \alpha, \alpha, 0, \ldots, 0]^T$ and the other column is of the type $[1, \alpha^2, \alpha^2, 0, \ldots, 0]^T$. Table 6.5 considers the case where both columns are of the type $[\alpha, \alpha^2, 0, \ldots, 0]^T$.

TABLE 6.5. Candidate Matrices—Case 4

Candidate Matrix	Forbidden Matrix	Candidate Matrix	Forbidden Matrix
$\begin{bmatrix} \alpha^2 & 0 \\ \alpha & 0 \\ 0 & \alpha \\ 0 & \alpha^2 \end{bmatrix}$	L	$\begin{bmatrix} \alpha^2 & \alpha \\ \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}$	None
$\begin{bmatrix} \alpha^2 & \alpha^2 \\ \alpha & 0 \\ 0 & \alpha \end{bmatrix}$	None	$\begin{bmatrix} \alpha^2 & \alpha \\ \alpha & \alpha^2 \end{bmatrix}$	None

We see from Tables 6.2-6.5 that the only matrices of two columns that are allowed to be contained in P_1 are the ones claimed in the result. Up to field automorphism and permuting of rows and columns, the only matrices with three columns such that every pair of columns consists of one of the permissible matrices are the following.

$$\begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & \alpha^2 & 0 \\ \alpha & 0 & \alpha^2 \end{bmatrix}, \begin{bmatrix} \alpha & \alpha^2 & \alpha \\ \alpha^2 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}, \begin{bmatrix} \alpha^2 & \alpha^2 & \alpha^2 \\ \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \begin{bmatrix} \alpha^2 & \alpha^2 & \alpha \\ \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

The first two of these matrices contain, respectively, the forbidden matrices M and N. The last two matrices contain, respectively, the matrices forbidden by Lemma 6.3.2. This completes the proof of the lemma. ∎

Recall from Section 5.7 that if f and g are real-valued functions of a real variable, then we write $f(x) \approx g(x)$ to denote that $f(x) = g(x)$ for all x sufficiently large, and we say that f and g are eventually equal.

Theorem 6.3.4. *Suppose Hypothesis 3.2.3 holds. For all sufficiently large r, the extremal matroids of $\mathcal{AC}_4$ and $\mathcal{GM}$ are T_r^2, G_r, and HP_r. Thus, we have*

$$h_{\mathcal{AC}_4}(r) \approx h_{\mathcal{GM}}(r) \approx \binom{r+3}{2} - 5.$$

Proof. Since graphic matroids are regular, $\mathcal{AC}_4$ and $\mathcal{GM}$ each contain the class of graphic matroids. Also note that there is no finite field $\mathbb{F} = \mathrm{GF}(p^k)$ such that $\mathcal{AC}_4$ is contained in the class of $\mathbb{F}$-representable matroids. Therefore, by Theorem 1.3.1, both $\mathcal{AC}_4$ and $\mathcal{GM}$ are quadratically dense.

By Lemma 4.5.6, the extremal matroids of $\mathcal{AC}_4$ are the largest simple matroids that virtually conform to some template in the set $\{\Phi_1, \ldots, \Phi_s\}$ whose existence is implied by Hypothesis 3.2.3. By Lemma 6.2.5, these are Y-templates.

Let $\Phi = \mathrm{YT}(P_0, P_1)$ be an extremal template for $\mathcal{AC}_4$. We know from the discussion at the beginning of this section that the extremal templates for $\mathcal{AC}_4$ are those templates where P_1 has the most columns. By Lemma 6.3.3, P_1 has two columns. Recall from Definition 4.2.4 that semi-strongly equivalent templates have the same

universal matroids. Also recall from Section 4.5 that the largest simple matroid of a given rank that conforms to a template is an *extremal matroid* of the template. Since the extremal matroids of a template are obtained by simplifying the universal matroids, Lemma 4.2.12 implies that we may assume that each column of P_0 has entries that sum to 0 and that each column of P_1 has entries that sum to 1.

Claim 6.3.4.1. *If $\Phi = \mathrm{YT}(P_0, P_1)$ is an extremal template for $\mathcal{AC}_4$, then P_0 and P_1 can be chosen so that P_1 has no zero rows. That is, P_1 can be chosen so that it is exactly one of the matrices given in Lemma 6.3.3.*

Proof. Let $\mathbf{v}$ and $\mathbf{w}$ be the columns of the matrix that results when the zero rows of P_1 are removed. Suppose P_0 contains a column with a nonzero entry in one of the rows corresponding to a zero row of P_1. By scaling, we may assume that that entry is 1. Then P_0 and P_1 (with columns indexed by Y_0 and Y_1) contain the following submatrix.

$$\begin{array}{cc|c} \multicolumn{2}{c}{Y_1} & \multicolumn{1}{c}{Y_0} \\ \hline \multicolumn{1}{|c|}{\mathbf{v}} & \mathbf{w} & \multicolumn{1}{c|}{\mathbf{u}} \\ \hline \multicolumn{1}{|c|}{0} & 0 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

Since the matrix $A_1[X, Y_1]$ contains an identity matrix in addition to P_1, by contracting this element of Y_0, we obtain the following submatrix in $A_1[X, Y_1]$.

$$\begin{array}{|c|c|c|} \multicolumn{3}{c}{Y_1} \\ \hline \mathbf{v} & \mathbf{w} & \mathbf{u} \\ \hline \end{array}$$

Since P_1 can have at most two columns, either $\mathbf{u}$ is a unit column or $\mathbf{u}$ is equal to $\mathbf{v}$ or $\mathbf{w}$.

Thus, P_0 must be of the following form, where T is an arbitrary matrix the sum of whose rows is the zero vector.

$$\begin{array}{|c|c|c|c|} \hline \mathbf{v} \dots \mathbf{v} & \mathbf{w} \dots \mathbf{w} & \text{unit columns} & T \\ \hline \multicolumn{3}{|c|}{\text{unit columns}} & 0 \\ \hline \end{array}$$

In fact, since Φ is an extremal template, P_0 must be the following for some positive integer n.

$$\begin{array}{|c|c|c|c|} \hline \mathbf{v} \dots \mathbf{v} & \mathbf{w} \dots \mathbf{w} & & T \\ \cline{1-2} \cline{4-4} I & I & \smash{\raisebox{1.5ex}{D_n}} & 0 \\ \hline \end{array}$$

The rank-r universal matroid of this template is isomorphic to the rank-r universal matroid of $\mathrm{YT}([D_m|T], [\mathbf{v}|\mathbf{w}])$, where m is the number of rows in $[\mathbf{v}|\mathbf{w}]$. Thus, these two templates are semi-strongly equivalent, and we may choose $\Phi = \mathrm{YT}([D_m|*], [\mathbf{v}|\mathbf{w}])$. $\square$

Thus, by Claim 6.3.4.1, it suffices to show that when P_1 is one of the matrices listed in Lemma 6.3.3, then the largest possible matroids virtually conforming to any of the corresponding templates are T_r^2, G_r, and HP_r. By Lemma 4.2.12, we may assume that the sum of the rows of P_0 is the zero vector. We see then that if

P_1 is one of the matrices with three rows listed in Lemma 6.3.3, then the largest possible matroids are obtained when P_0 is

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ \alpha & \alpha^2 & 1 & 0 & 1 \\ \alpha^2 & \alpha & 0 & 1 & 1 \end{bmatrix}.$$

Similarly, if $P_1 = \begin{bmatrix} \alpha^2 & \alpha \\ \alpha & \alpha^2 \end{bmatrix}$, then the largest possible matroids are obtained when P_0 is $A_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Before we can analyze what happens when P_1 is one of the matrices listed in Lemma 6.3.3, we need two more claims.

Claim 6.3.4.2. *The extremal matroids for* $\Phi = \mathrm{YT}\left(\begin{bmatrix} 1 & 1 \\ 1 & \alpha^2 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \end{bmatrix}\right)$ *are isomorphic to those of* $\Phi(G_r)$.

Proof. The rank-r extremal matroid of $\Phi(G_r)$ (obtained by simplifying the universal matroid) has the following representation matrix.

$$\begin{array}{|c|c|c|c|c|c|ccccc|} \hline 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & \alpha\cdots\alpha & 0\cdots0 & 1 & 0 & 1 & 1 & 1 \\ 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & \alpha\cdots\alpha & 0 & 1 & 1 & \alpha & \alpha^2 \\ \hline I_{r-|X|} & D_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & \multicolumn{5}{c|}{0} \\ \hline \end{array}$$

To the first row of this matrix, add all other rows. Then scale the first row by α. The result is the following.

$$\begin{array}{|c|c|c|c|c|c|ccccc|} \hline \alpha\cdots\alpha & 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 1\cdots1 & \alpha & \alpha & 0 & 1 & \alpha^2 \\ 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & \alpha\cdots\alpha & 0 & 1 & 1 & \alpha & \alpha^2 \\ \hline I_{r-|X|} & D_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & \multicolumn{5}{c|}{0} \\ \hline \end{array}$$

Scale the last five columns so that their first nonzero entries are 1 and reorder the columns of the entire matrix to obtain the following, which is a representation matrix of the rank-r extremal matroid of Φ.

$$\begin{array}{|c|c|c|c|c|c|ccccc|} \hline 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1 & \alpha\cdots\alpha & 1 & 0 & 1 & 1 & 1 \\ 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & \alpha\cdots\alpha & 0\cdots0 & 0 & 1 & 1 & \alpha & \alpha^2 \\ \hline I_{r-|X|} & D_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & \multicolumn{5}{c|}{0} \\ \hline \end{array}$$

□

Claim 6.3.4.3. *The extremal matroids for* $\Phi = \mathrm{YT}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}\right)$ *are isomorphic to those of* $\Phi(G_r)$.

Proof. Consider the representation matrix for the rank-r extremal matroid of $\Phi(G_r)$ given at the beginning of the proof of Claim 6.3.4.2. Scaling the second row by α^2, we obtain the following.

$$\begin{array}{|c|c|c|c|c|c|ccccc|} \hline 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & \alpha\cdots\alpha & 0\cdots0 & 1 & 0 & 1 & 1 & 1 \\ 0\cdots0 & 0\cdots0 & 0\cdots0 & \alpha^2\cdots\alpha^2 & 0\cdots0 & 1\cdots1 & 0 & \alpha^2 & \alpha^2 & 1 & \alpha \\ \hline I_{r-|X|} & D_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & \multicolumn{5}{c|}{0} \\ \hline \end{array}$$

Scale the fourth from last column to once again make it $[0,1,0,\ldots,0]^T$ and reorder the columns to obtain the following, which is the rank-r extremal matroid for Φ.

$$\begin{array}{|c|c|c|c|c|c|ccccc|}
\hline
0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & \alpha\cdots\alpha & 0\cdots0 & 1 & 0 & 1 & 1 & 1\\
0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & \alpha^2\cdots\alpha^2 & 0 & 1 & 1 & \alpha & \alpha^2\\
\hline
I_{r-|X|} & D_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & I_{r-|X|} & \multicolumn{5}{c|}{0}\\
\hline
\end{array}$$

□

We are now ready to analyze what happens when P_1 is one of the matrices listed in Lemma 6.3.3.

Claim 6.3.4.4. *If* $P_1 = \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \\ \alpha & 0 \end{bmatrix}$, *then the rank-$r$ extremal matroid for* $\mathrm{YT}(A_3, P_1)$ *is G_r for all $r \geq 3$.*

Proof. By Lemma 4.2.13, $\Phi = \mathrm{YT}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \end{bmatrix}\right)$ is semi-strongly equivalent to

$$\Phi' = \mathrm{YT}\left(\begin{bmatrix} 1 & 0 & 1 & \alpha & 1 & 1 & 1 \\ 1 & 1 & \alpha^2 & \alpha & 0 & \alpha^2 & \alpha \\ 0 & 1 & \alpha & 0 & 1 & \alpha & \alpha^2 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \\ \alpha & 0 \end{bmatrix}\right),$$

with the middle row playing the role of the bottom row in Lemma 4.2.13. Therefore, their universal matroids are isomorphic. By simplifying these universal matroids, we obtain the extremal matroids for Φ and Φ'. By scaling and reordering columns, we see that the extremal matroids for Φ' are isomorphic to those of $\mathrm{YT}(A_3, P_1)$.

By simplifying the universal matroids for Φ, we also see that the extremal matroids for Φ are isomorphic to those of $\mathrm{YT}\left(\begin{bmatrix} 1 & 1 \\ 1 & \alpha^2 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \end{bmatrix}\right)$, which are G_r by Claim 6.3.4.2. □

Claim 6.3.4.5. *If* $P_1 = \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha \\ \alpha & 1 \end{bmatrix}$, *then the rank-$r$ extremal matroid for* $\mathrm{YT}(A_3, P_1)$ *is HP_r for all $r \geq 3$.*

Proof. By Lemma 4.2.13, $\Phi = \mathrm{YT}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}\right)$ is semi-strongly equivalent to

$$\Phi' = \mathrm{YT}\left(\begin{bmatrix} 1 & 0 & 1 & \alpha & 1 & 1 & 1 \\ 1 & 1 & \alpha^2 & \alpha^2 & 0 & \alpha^2 & \alpha \\ 0 & 1 & \alpha & 1 & 1 & \alpha & \alpha^2 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha \\ \alpha & 1 \end{bmatrix}\right),$$

with the middle row playing the role of the bottom row in Lemma 4.2.13. Therefore, their universal matroids are isomorphic. By simplifying these universal matroids, we obtain the extremal matroids for Φ and Φ'. By scaling and reordering columns, we see that the extremal matroids for Φ' are isomorphic to those of $\mathrm{YT}(A_3, P_1)$.

By simplifying the universal matroids for Φ, we also see that the extremal matroids for Φ are isomorphic to those of $\mathrm{YT}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1 & \alpha\\ \alpha & 1\end{bmatrix}\right)$, which is $\Phi(HP_r)$. $\square$

Claim 6.3.4.6. *If* $P_1 = \begin{bmatrix}\alpha^2 & \alpha^2\\ \alpha & 0\\ 0 & \alpha\end{bmatrix}$, *then the rank-$r$ extremal matroid for* $\mathrm{YT}(A_3, P_1)$ *is G_r for all $r \geq 3$.*

Proof. By Lemma 4.2.13, $\Phi(G_r) = \mathrm{YT}\left(\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\end{bmatrix}, \begin{bmatrix}\alpha & 0\\ 0 & \alpha\end{bmatrix}\right)$ is semi-strongly equivalent to

$$\Phi' = \mathrm{YT}\left(\begin{bmatrix}1 & 1 & \alpha & \alpha & 0 & \alpha^2 & \alpha\\ 1 & 0 & \alpha & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & \alpha & 1 & \alpha & \alpha^2\end{bmatrix}, \begin{bmatrix}\alpha^2 & \alpha^2\\ \alpha & 0\\ 0 & \alpha\end{bmatrix}\right),$$

with the top row playing the role of the bottom row in Lemma 4.2.13. Therefore, their universal matroids are isomorphic. By simplifying these universal matroids, we obtain the extremal matroids for $\Phi(G_r)$ and Φ'. By scaling and reordering columns and simplifying the universal matroids for Φ', we see that the extremal matroids for Φ' are isomorphic to those of $\mathrm{YT}(A_3, P_1)$. Since the extremal matroids for $\Phi(G_r)$ are G_r, this proves the claim. $\square$

Claim 6.3.4.7. *If* $P_1 = \begin{bmatrix}\alpha^2 & \alpha\\ \alpha & 0\\ 0 & \alpha^2\end{bmatrix}$, *then the rank-$r$ extremal matroid for* $\mathrm{YT}(A_3, P_1)$ *is G_r for all $r \geq 3$.*

Proof. By Lemma 4.2.13, $\Phi = \mathrm{YT}\left(\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\end{bmatrix}, \begin{bmatrix}\alpha & 0\\ 0 & \alpha^2\end{bmatrix}\right)$ is semi-strongly equivalent to

$$\Phi' = \mathrm{YT}\left(\begin{bmatrix}1 & 1 & \alpha & \alpha^2 & 0 & \alpha^2 & \alpha\\ 1 & 0 & \alpha & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & \alpha^2 & 1 & \alpha & \alpha^2\end{bmatrix}, \begin{bmatrix}\alpha^2 & \alpha\\ \alpha & 0\\ 0 & \alpha^2\end{bmatrix}\right),$$

with the top row playing the role of the bottom row in Lemma 4.2.13. Therefore, their universal matroids are isomorphic. By simplifying these universal matroids, we obtain the extremal matroids for Φ and Φ'. By scaling and reordering columns, we see that the extremal matroids for Φ' are isomorphic to those of $\mathrm{YT}(A_3, P_1)$. By Claim 6.3.4.3, the extremal matroids of Φ are isomorphic to those of $\Phi(G_r)$, which are G_r. $\square$

Claim 6.3.4.8. *If* $P_1 = \begin{bmatrix}\alpha^2 & \alpha\\ \alpha & \alpha^2\end{bmatrix}$, *then the rank-$r$ extremal matroid for* $\mathrm{YT}(A_2, P_1)$ *is T_r^2 for all $r \geq 2$.*

Proof. By Lemma 4.2.13, $\Phi(T_r^2) = \mathrm{YT}\left([\emptyset], \begin{bmatrix}\alpha & \alpha^2\end{bmatrix}\right)$ is semi-strongly equivalent to

$$\Phi' = \mathrm{YT}\left(\begin{bmatrix}1 & \alpha & \alpha^2\\ 1 & \alpha & \alpha^2\end{bmatrix}, \begin{bmatrix}\alpha^2 & \alpha\\ \alpha & \alpha^2\end{bmatrix}\right),$$

with the top row playing the role of the bottom row in Lemma 4.2.13. Therefore, their universal matroids are isomorphic. By simplifying these universal matroids, we obtain the extremal matroids for $\Phi(T_r^2)$ and Φ'. By scaling and reordering columns, we see that the extremal matroids for Φ' are isomorphic to those of $\mathrm{YT}(A_2, P_1)$. Since the extremal matroids for $\Phi(T_r^2)$ are T_r^2, this proves the claim. □

Combining Claim 6.3.4.1 with Claims 6.3.4.4–6.3.4.8, we see that the extremal matroids for $\mathcal{AC}_4$ are T_r^2, G_r, and HP_r. Since $\mathcal{GM} \subseteq \mathcal{AC}_4$, we have $h_{\mathcal{GM}}(r) \leq h_{\mathcal{AC}_4}(r)$. By Proposition 6.2.4, T_r^2, G_r, and HP_r are all golden-mean matroids. Thus, we have $h_{\mathcal{GM}}(r) \approx h_{\mathcal{AC}_4}(r)$. It is easily verified (see also [42]) that $\varepsilon(T_r^2) = \varepsilon(G_r) = \varepsilon(HP_r) = \binom{r+3}{2} - 5$. This completes the proof of the theorem. ■

6.4 Maximal Templates

In this section and Section 6.7, we will determine a collection $\mathcal{T}$ of Y-templates over GF(4) such that, for each template $\Phi \in \mathcal{T}$, we have $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$ and such that, for every refined template Φ' with $\mathcal{M}(\Phi') \subseteq \mathcal{AC}_4$, there is a template $\Phi \in \mathcal{T}$ such that $\Phi' \preceq \Phi$. Our motivation is to use Hypotheses 3.2.2 and 3.2.3 and Corollaries 4.1.5 and 4.1.6 to study $\mathcal{AC}_4$ and $\mathcal{GM}$; however we will not refer to these hypotheses and corollaries again until later in Section 6.7. This will allow the next several results to be free from some of the inherent technicalities in those hypotheses and corollaries, and it will also illustrate that the results in this section are independent of the hypotheses.

In the next definition, let e_i be a unit column whose nonzero entry is in the i^{th} row.

Definition 6.4.1. Let $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ be column vectors with the same number m of entries. Let A be a matrix whose columns can be scaled so that the matrix is of the following form, where $*$ represents an arbitrary matrix and where each $\mathbf{v}_i$ appears at least once.

<table>
<tr><td>$\mathbf{v}_1 \cdots \mathbf{v}_1$</td><td>$\mathbf{v}_2 \cdots \mathbf{v}_2$</td><td>$\cdots$</td><td>$\mathbf{v}_n \cdots \mathbf{v}_n$</td><td>$*$</td></tr>
<tr><td colspan="4">unit columns</td><td>0</td></tr>
</table>

We say that all of the columns of the matrix of the form $\mathbf{v}_i + e_k$, where $i \in \{1, \dots, n\}$ and $k > m$, are *semi-parallel* to each other. If $\mathbf{v}$ is an additional column not contained in A, and if $\mathbf{v}$ can be scaled so that it is of the form $\mathbf{v}_j + e_\ell$, where $j \in \{1, \dots, n\}$ and $\ell > m$, then we say that $\mathbf{v}$ is also semi-parallel to the existing columns of A of the form $\mathbf{v}_i + e_k$, where $i \in \{1, \dots, n\}$ and $k > m$. Moreover, we call $\mathbf{v}$ a *semi-parallel extension* of A.

Definition 6.4.2. Let $Q' = \left[\frac{Q}{I}\right]$, where no column of Q is a zero column or the negative of a unit column (which, of course, is a unit column itself in characteristic 2), where no row of Q is a zero row, and where no column of Q' is a semi-parallel extension of the matrix resulting from Q' by removing that column. We call any matrix that can be obtained from Q' by permuting rows, permuting columns, and scaling columns a *contractible* matrix.

Lemma 6.4.3. *Let $\Phi = \mathrm{YT}(P_0, P_1)$ be such that $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$. Suppose P_0 contains a contractible submatrix Q', as given in Definition 6.4.2. Then Q must be a submatrix of one of the matrices listed in Lemma 6.3.3 (up to permutations and field isomorphism). In particular, Q and Q' have at most two columns.*

Proof. From Φ, elements of Y_0 can be deleted (operation (10) of Definition 4.1.3) and elements of Y_1 can be y-shifted (see Lemma 4.4.5) and contracted (operation (12) of Definition 4.1.3) to obtain $\mathrm{YT}(Q', [\emptyset])$. Then the remaining elements of Y_0 can be contracted (operation (12) of Definition 4.1.3) to obtain $\mathrm{YT}([\emptyset], Q)$. By Lemma 6.3.3, the result holds. ■

Note that in the previous lemma, the requirement in 6.4.2 about semi-parallel extensions is necessary because, otherwise, contracting the semi-parallel columns results in equal columns of P_1. Whenever columns of P_1 are equal, all but one of them can be discarded to produce a new template strongly equivalent to the original template.

Definition 6.4.4. We define the following matrices.

$$I = \begin{bmatrix} 1 & 1 & \alpha & \alpha & \alpha & \alpha^2 \\ \alpha & \alpha & \alpha & \alpha & \alpha^2 & \alpha \\ \alpha & \alpha & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad II = \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & \alpha^2 & 0 \\ \alpha & 0 & \alpha^2 \\ 1 & 0 & \alpha \\ 0 & 1 & 1 \end{bmatrix} \quad III = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha & \alpha \\ \alpha & \alpha^2 & 1 & \alpha^2 & \alpha^2 \\ \alpha & 0 & \alpha^2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$IV = \begin{bmatrix} 1 & 1 & \alpha & \alpha & \alpha & \alpha^2 \\ \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha^2 & \alpha \\ \alpha & \alpha & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 1 & \alpha^2 & \alpha & \alpha^2 \\ \alpha & \alpha & 0 & \alpha^2 \\ \alpha & 1 & \alpha^2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad VI = \begin{bmatrix} 1 & \alpha^2 & 1 & \alpha & \alpha^2 \\ \alpha & \alpha & \alpha^2 & 1 & 0 \\ \alpha & 1 & \alpha^2 & \alpha & \alpha \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$VII = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha & 0 \\ \alpha & \alpha^2 & \alpha & 0 & \alpha^2 \\ \alpha & 1 & 0 & \alpha^2 & \alpha \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad VIII = \begin{bmatrix} 1 & 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad IX = \begin{bmatrix} 1 & \alpha & 1 & 0 \\ \alpha & \alpha^2 & \alpha^2 & \alpha^2 \\ \alpha & 1 & \alpha^2 & \alpha \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha^2 & \alpha \\ \alpha^2 & 0 & \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 1 & \alpha & \alpha^2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad XI = \begin{bmatrix} \alpha & \alpha & \alpha^2 & \alpha^2 \\ \alpha^2 & 0 & 0 & \alpha \\ 0 & \alpha^2 & \alpha & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$XII = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha^2 & \alpha & \alpha \\ \alpha^2 & 0 & \alpha^2 & \alpha & \alpha^2 & 0 \\ 0 & \alpha^2 & 1 & 1 & 0 & \alpha^2 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} XIII = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha & \alpha^2 \\ \alpha^2 & 0 & \alpha^2 & \alpha^2 & 0 \\ 0 & \alpha^2 & 1 & 0 & \alpha \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$XIV = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha & \alpha^2 & \alpha \\ \alpha^2 & 0 & \alpha^2 & \alpha^2 & \alpha & 0 \\ 0 & \alpha^2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \alpha^2 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} XV = \begin{bmatrix} \alpha & \alpha & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha & \alpha \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$XVI = \begin{bmatrix} \alpha & \alpha^2 & 0 & \alpha \\ \alpha^2 & \alpha & \alpha^2 & 0 \\ 1 & 0 & \alpha & \alpha^2 \\ 0 & 1 & 1 & 1 \end{bmatrix} XVII = \begin{bmatrix} 1 & \alpha^2 & \alpha^2 \\ \alpha & \alpha & 0 \\ \alpha & 1 & \alpha \\ 1 & 0 & 1 \end{bmatrix}$$

Lemma 6.4.5. *Let Φ be a refined frame template over* GF(4) *such that $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$. Then there is a complete, lifted Y-template Φ' that is determined by a column submatrix of one of matrices I–XVI listed in Definition 6.4.4 (up to permutations and field isomorphism) such that every matroid conforming to Φ is a minor of a matroid conforming to Φ'.*

Note that, in the statement of Lemma 6.4.5, we do not claim that $\mathcal{M}(\Phi') \subseteq \mathcal{AC}_4$. This is indeed true but will not be proved until Section 6.7. Also, note that seventeen matrices are listed in Definition 6.4.4, but Lemma 6.4.5 only deals with the first sixteen of them. We will not see matrix $XVII$ again until Section 6.7. Recall the definition of a template minor from Definition 4.1.3. If a template Φ' is obtained from a Y-template $\Phi'' = \mathrm{YT}(P_0, P_1)$ by removing columns from P_0, then Φ' is a template minor of Φ''. If $\Phi \preceq \Phi'$, then $\Phi \preceq \Phi''$. Throughout the proof of Lemma 6.4.5, whenever we permute the rows of P_0 we automatically scale the columns so that the last nonzero entry in each column is 1.

Proof of Lemma 6.4.5. Recall from Lemma 6.2.5 that, if Φ is a template such that $\mathcal{M}(\Phi) \subseteq \mathcal{AC}_4$, then Φ is a Y-template. Combining Remark 4.2.7 and Lemma 4.2.10, we see that every Y-template is minor equivalent to a complete, lifted Y-template Φ_{P_0} determined by some matrix P_0. Moreover, by Lemma 4.2.15, we may assume that the sum of the rows of P_0 is the zero vector. By Lemma 6.4.3, $[1,1,1]^T$, and $[a, a, a^2, 1]^T$, and $[1, \alpha, \alpha^2, 1]^T$ are all forbidden from P_0. Thus, up to column scaling, permuting rows, and field isomorphism, each column of P_0 must be of the form $[\alpha, \alpha^2, 1, 0, \ldots, 0]^T$ or $[1, \alpha, \alpha, 1, 0, \ldots, 0]^T$. (Graphic columns are already assumed in a complete template.)

Recall that the weight of a vector is its number of nonzero entries. There are five cases to check.

1. The matrix P_0 contains a contractible submatrix with two columns each of which have weight 4.

2. Case 1 does not hold, but P_0 contains a contractible submatrix with one column of weight 4 and another column of weight 3.

3. Neither Case 1 nor Case 2 holds, but P_0 contains a column of weight 4.

4. Every column of P_0 has weight 3; there are weight-3 columns of P_0 with supports whose intersection has size exactly 1.

5. Every column of P_0 has weight 3, and there are no pairs of columns with supports whose intersection has size 1.

We analyze these five cases in Claims 6.4.5.1–6.4.5.5 as follows.

Claim 6.4.5.1. *The result holds in Case 1.*

Proof. In Case 1, P_0 contains a contractible submatrix with two columns each of which have weight 4. By Lemmas 6.4.3 and 6.3.3, this contractible submatrix can be chosen to be $\begin{bmatrix} 1 & \alpha \\ \alpha & \alpha \\ \alpha & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Moreover, if a column of P_0 with four nonzero entries has a nonzero entry outside of the first five rows, then this column must be semi-parallel to one of the two known weight-4 columns. If a weight-3 column has a nonzero entry outside of the first five rows, scale the column so that this entry is a 1. Then the fact that P_0 contains $[1, \alpha, \alpha, 1, 0, \dots, 0]^T$ implies that the first entry in the weight-3 column must be α and that the third entry must be α^2 or 0. But the fact that P_0 contains $[\alpha, \alpha, 1, 0, 1, 0, \dots, 0]^T$ implies that the third entry must be α, a contradiction. Thus, P_0 must be a column submatrix of the following matrix for some positive integer m. This column submatrix contains columns 1 and 2 because we are in Case 1.

$$
\begin{array}{|cccc|c|c|c|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\hline
1 & \alpha & 1 & \alpha & 1\cdots 1 & \alpha\cdots\alpha & \\
\alpha & \alpha & \alpha & \alpha & \alpha\cdots\alpha & \alpha\cdots\alpha & \text{columns} \\
\alpha & 1 & \alpha & 1 & \alpha\cdots\alpha & 1\cdots 1 & \text{of} \\
1 & 0 & 0 & 1 & 0\cdots 0 & 0\cdots 0 & \text{weight 3} \\
0 & 1 & 1 & 0 & 0\cdots 0 & 0\cdots 0 & \\
\hline
\multicolumn{4}{|c|}{0} & I_m & I_m & 0 \\
\hline
\end{array}
$$

Now suppose that one of the weight-3 columns has a nonzero entry in the fourth or fifth row. By symmetry, we may assume that it is the fifth row. Scale this weight-3 column so that the entry in the fifth row is a 1. Because of column 1, Lemma 6.4.3 implies that either the first entry or the fourth entry of the weight-3 column must be α. If the fourth entry is α, then α^2 must be either the second or third entry of the column. In this case, the weight-3 column forms a contractible submatrix with column 2 that results in a forbidden matrix when contracted. (These forbidden matrices are, respectively, matrix J and the matrix that results from matrix H

after a field isomorphism, where matrices J and H are from Table 6.1. Recall that a matrix is contractible if its rows and columns can be permuted and its columns can be scaled to be of the form given in Definition 6.4.2. Permuting and scaling is necessary here and in several other places in the proof.) Thus, the first entry of the weight-3 column is α. Either the second or third entry must be α^2. Thus, P_0 contains one of the following submatrices.

$$
\begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & \alpha & \alpha^2 \\ \alpha & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & \alpha & 0 \\ \alpha & 1 & \alpha^2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}
$$

If P_0' is either of these matrices, we find an excluded minor in the vector matroid of $[I_5|D_5|P_0']$. We do this using SageMath, particularly the function called `complete_Y_template_matrix`, which is found in Section A.3 and was already used in Section 5.3. Since each column of P_0 has entries whose sum is zero, Lemma 4.2.15 implies that we may consider the template determined by the matrix obtained by removing one row from P_0. We accomplish this by contracting the element representing the first column in the identity matrix I_5 and simplifying. In the Python programming language, on which SageMath is based, this first column is labeled as `0`. For example, for the first matrix listed above, we used the following code.

```
GF4 = GF(4, 'a')
a=GF4.gens()[0]

P0 = Matrix(GF4,  [[1,a,a],
                    [a,a,a^2],
                    [a,1,0],
                    [1,0,0],
                    [0,1,1]])
A=complete_Y_template_matrix(P0)
M=Matroid(field=GF4, matrix=A)
MinorCheck((M/0).simplify())
```

This code returned `'V2'`. Running the same code but for the second matrix above also returns `'V2'`.

Thus, P_0 is a column submatrix of the following matrix for some positive integer m. Columns 1 and 2 are included in P_0, while columns 3 and 4 may not be.

$$
\begin{array}{cccc|c|c|cc}
1 & 2 & 3 & 4 & & & & \\
\hline
1 & \alpha & 1 & \alpha & 1\cdots 1 & \alpha\cdots\alpha & \alpha & \alpha^2 \\
\alpha & \alpha & \alpha & \alpha & \alpha\cdots\alpha & \alpha\cdots\alpha & \alpha^2 & \alpha \\
\alpha & 1 & \alpha & 1 & \alpha\cdots\alpha & 1\cdots 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0\cdots 0 & 0\cdots 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0\cdots 0 & 0\cdots 0 & 0 & 0 \\
\hline
\multicolumn{4}{c|}{0} & I_m & I_m & \multicolumn{2}{c}{0}
\end{array}
$$

Regardless of the value of m and regardless of whether columns 3 and 4 are included in P_0, Lemma 4.2.16 implies that $\Phi_{P_0} \preceq \Phi'$ where

$$\Phi' = \mathrm{YT}\left(\begin{bmatrix} \alpha & \alpha^2 & 1 & 1 & 0 \\ \alpha^2 & \alpha & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha \\ \alpha & 1 \end{bmatrix}\right).$$

Then Remark 4.2.7 implies that Φ' is equivalent to a template minor of the complete, lifted Y-template determined by matrix I listed in Definition 6.4.4. □

Claim 6.4.5.2. *The result holds in Case 2.*

Proof. In Case 2, Case 1 does not hold, but P_0 contains a contractible submatrix with one column of weight 4 and another column of weight 3. By Lemmas 6.4.3 and 6.3.3, this contractible submatrix can be chosen to be

$$\begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \\ \alpha & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}.$$

Recall from the discussion immediately preceding Section 6.1 that this matrix can be denoted by $[1, 2]$. Suppose P_0 contains an additional column v, that is not a semi-parallel extension of the matrix $[1, 2]$, with a nonzero entry in a row other than the first five rows. Since Case 1 does not hold, v cannot have weight 4 and therefore has weight 3. By Lemmas 6.4.3 and 6.3.3, the intersection of the supports of columns 1 and v must have size 2. Therefore, v has at most one nonzero entry in a row other than the first four rows. By column scaling, let $v = [a, b, c, d, 0, 1, 0 \ldots 0]^T$, where two members of $\{a, b, c, d\}$ are 0. If $d = 0$, then since v is not a semi-parallel extension of $[1, 2]$, we obtain a contractible submatrix with three columns, which is forbidden. Therefore, $d \neq 0$. We must also have $c \neq 0$ because otherwise, we obtain a contractible submatrix with three columns. But then columns 2 and v have disjoint supports, which contradicts the combination of Lemmas 6.4.3 and 6.3.3.

Therefore, we deduce that every column of P_0 either is semi-parallel to 1 or 2 or has all of its nonzero entries in the first five rows. First we will analyze the structure of the columns that have all of their nonzero entries in the first five rows. Let Q be the matrix obtained by restricting P_0 to the first five rows and to the columns whose nonzero entries are all in the first five rows.

Now, let v be a column of Q of weight 4 that is not semi-parallel to column 1. Since we are not in Case 1, Lemmas 6.4.3 and 6.3.3 imply that the supports of columns v and 1 must be equal. Scale v so that its last nonzero entry is a 1. One

other entry is a 1, and the other two nonzero entries are either both α or both α^2. Therefore, v is one of the columns of the following matrix.

$$\begin{array}{c} \begin{array}{ccccc} 3 & & & & \end{array} \\ \begin{bmatrix} \alpha^2 & 1 & \alpha & \alpha & \alpha^2 \\ 1 & \alpha^2 & 1 & \alpha & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

By Lemmas 6.4.3 and 6.3.3, each of these columns, other than the one labeled by 3, forms a forbidden contractible submatrix with column 2.

Now, let v be a column of Q of weight 3 whose support is contained in the support of column 1. Scale v so that its last nonzero entry is a 1. One of its other nonzero entries is α, and the remaining nonzero entry is α^2. Thus, v is one of the columns of the following matrix.

$$\begin{array}{c} \begin{array}{cccccccc} 4 & 5 & 6 & 7 & a & b & c & d \end{array} \\ \begin{bmatrix} \alpha & \alpha^2 & \alpha & \alpha^2 & \alpha & \alpha^2 & 0 & 0 \\ \alpha^2 & \alpha & \alpha^2 & \alpha & 0 & 0 & \alpha & \alpha^2 \\ 1 & 1 & 0 & 0 & \alpha^2 & \alpha & \alpha^2 & \alpha \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

For each column $v \in \{a, b, c, d\}$, we use SageMath to test for excluded minors in the rank-5 universal matroid conforming to the template determined by the matrix $[1, 2, v]$. We use the functions `complete_Y_template_matrix` and `MinorCheck`, as we did in the proof of Claim 6.4.5.1. Running this code with columns a, b, c, and d, returns the excluded minors F_7, P_2, F_7, and P_2, respectively.

Now, let v be a column, other than column 2, of weight 3 whose support is not contained in the support of column 1. Scale v so that its last entry is 1. By Lemmas 6.4.3 and 6.3.3, the supports of columns 1 and v must have an intersection of size 2. Moreover, either the first or fourth entry of v must be α and either the second or third entry of v must be α^2. Thus, v is one of the columns of the following matrix.

$$\begin{array}{c} \begin{array}{ccc} 8 & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & \alpha \\ 0 & \alpha^2 & 0 \\ \alpha^2 & 0 & \alpha^2 \\ \alpha & \alpha & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Again, we use SageMath to test for excluded minors contained in the rank-5 universal matroid conforming to the template determined by the matrix $[1, 2, v]$, when v is either e or f. In both cases, we found that F_7^* is an excluded minor.

We see then that, if P_0 has at most five rows, then it must be a column submatrix of the following matrix, where column 9 is included because columns semi-parallel to column 1 have not yet been ruled out.

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha & \alpha^2 & \alpha & \alpha^2 & 0 & 1 \\ \alpha & \alpha^2 & 1 & \alpha^2 & \alpha & \alpha^2 & \alpha & 0 & \alpha \\ \alpha & 0 & \alpha^2 & 1 & 1 & 0 & 0 & \alpha^2 & \alpha \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & \alpha & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Now, call this matrix R. Then $[I_5|D_5|R]$ has $5 + \binom{5}{2} + 9 = 24$ columns, which are labeled as $0, \ldots, 23$ in SageMath. Therefore, columns $3, \ldots, 9$ of R are labeled as $17, \ldots, 23$ in SageMath. If we enter R into SageMath, then the following code gives pairs of columns $\{v, w\}$ of R such that $P_0 = [1, 2, v, w]$ is forbidden.

```
 R=Matrix(GF4,[
[1,a,  a^2,a,  a^2,a,  a^2,0,  1],
[a,a^2,1,  a^2,a,  a^2,a,  0,  a],
[a,0,  a^2,1,  1,  0,  0,  a^2,a],
[1,0,  1,  0,  0,  1,  1,  a,  0],
[0,1,  0,  0,  0,  0,  0,  1,  1]])
A=complete_Y_template_matrix(R)
M=Matroid(field=GF4, matrix=A)

U=Set(range(17,24))
F=[]
for S in Subsets(U,2):
    if not MinorCheck(
         (M/0\(U.difference(S))).simplify())=='None found':
         F.append(S)
print F
```

The forbidden pairs obtained are $\{3,5\}$, $\{3,7\}$, $\{3,8\}$, $\{3,9\}$, $\{4,8\}$, $\{5,7\}$, $\{5,8\}$, $\{6,8\}$, $\{7,8\}$, $\{7,9\}$, and $\{8,9\}$. Therefore, if P_0 contains column 8, it must be a submatrix of $[1,2,8]$, which is matrix II from Definition 6.4.4, and if P_0 contains column 3, it must be a submatrix of $[1,2,3,4,6]$, which is matrix III.

Now, consider the matrix $[1,2,7]$. Swap the first and second row, swap the third and fourth row, and scale. The result is the following.

$$\begin{bmatrix} 1 & \alpha^2 & \alpha \\ \alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

After a field isomorphism, this is $[1, 2, 5]$. Thus, if column 7 is contained in Q, we can obtain a template, equivalent up to field isomorphism, determined by a matrix including column 5 instead. Since $\{5, 7\}$ is a forbidden pair, we need not consider any matrices containing column 7. Therefore, if Q contains neither column 3 nor column 8, then Q can be chosen to be a submatrix of $[1, 2, 4, 5, 6, 9]$, which is matrix IV (up to permuting of columns).

Therefore, P_0 is a column submatrix of a matrix of the following form, where Q is either II, III, or IV.

$$\begin{array}{c|c|c}
 & V & W \\ \hline
 & 1\cdots 1 & \alpha\cdots\alpha \\
 & \alpha\cdots\alpha & \alpha^2\cdots\alpha^2 \\
Q & \alpha\cdots\alpha & 0\cdots 0 \\
 & 0\cdots 0 & 0\cdots 0 \\
 & 0\cdots 0 & 0\cdots 0 \\ \hline
0 & I & I
\end{array}$$

If Q is matrix II, then both V and W must be empty because otherwise a forbidden (by Lemmas 6.4.3 and 6.3.3) contractible matrix with two columns is formed with column 8. If Q contains column 3 and is therefore matrix III, then the presence of column 3 implies that $V = \emptyset$. Thus, if P_0' and P_1' are, respectively, the following matrices, then Lemma 4.2.16 implies that $\Phi_{P_0} \preceq \Phi'$, where $\Phi' = \mathrm{YT}([P_0'|D_4], P_1')$.

$$\begin{bmatrix} 1 & \alpha^2 & \alpha & \alpha \\ \alpha & 1 & \alpha^2 & \alpha^2 \\ \alpha & \alpha^2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha \\ \alpha^2 \\ 0 \\ 0 \end{bmatrix}$$

By Remark 4.2.7, Φ' is equivalent to a template minor of the complete, lifted Y-template determined by matrix III.

Finally, if Q is matrix IV, let P_0' and P_1' be, respectively, the following matrices. By Lemma 4.2.16, $\Phi_{P_0} \preceq \Phi'$, where $\Phi' = \mathrm{YT}([P_0'|D_3], P_1')$

$$\begin{bmatrix} \alpha & \alpha^2 \\ \alpha^2 & \alpha \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \\ \alpha & 0 \end{bmatrix}$$

By Remark 4.2.7, Φ is equivalent to a template minor of the complete, lifted Y-template determined by matrix IV. This completes Case 2. $\square$

Claim 6.4.5.3. *The result holds in Case 3.*

Proof. In Case 3, neither Case 1 nor Case 2 holds, but P_0 contains a column of weight 4. By Lemmas 6.4.3 and 6.3.3, this column can be chosen to be $[1,\ \alpha,\ \alpha,\ 1,\ 0,\ \ldots,\ 0]^T$. Label this column 10. Since neither Case 1 nor Case 2 holds, all other columns must either be semi-parallel to column 10 or must have a support contained in the support of column 10. Let us consider columns v whose supports

are contained in the support of column 10. If v has weight 4, then scale so that the last nonzero entry is a 1. One of the other nonzero entries must be a 1, and the other two nonzero entries are either both α or both α^2. If v has weight 3, then scale so that the last nonzero entry is a 1. Then one of the other nonzero entries is α, and the other nonzero entry is α^2. Therefore, when restricted to the rows in the support of column 10, v must be one of columns $11 - 23$ in the matrix below. We choose these labels because they match the labels that will be used in SageMath below.

$$\begin{array}{c}
\begin{array}{cccccccccccccc} 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \end{array} \\
\begin{bmatrix}
1 & 1 & \alpha & \alpha^2 & \alpha & \alpha^2 & \alpha & \alpha^2 & \alpha & \alpha^2 & \alpha & \alpha^2 & 0 & 0 \\
\alpha & \alpha^2 & 1 & 1 & \alpha & \alpha^2 & \alpha^2 & \alpha & \alpha^2 & \alpha & 0 & 0 & \alpha & \alpha^2 \\
\alpha & \alpha^2 & \alpha & \alpha^2 & 1 & 1 & 1 & 1 & 0 & 0 & \alpha^2 & \alpha & \alpha^2 & \alpha \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}$$

If P_0 contains a column semi-parallel to column 10, then since neither Case 1 nor Case 2 holds, P_0 must be a column submatrix of a matrix of the following form.

$$\begin{array}{|c|cc|}
\hline
1 \cdots 1 & \alpha & \alpha^2 \\
\alpha \cdots \alpha & \alpha^2 & \alpha \\
\alpha \cdots \alpha & 1 & 1 \\
\hline
I & \multicolumn{2}{c|}{0} \\
\hline
\end{array}$$

If $P_0' = \begin{bmatrix} \alpha & \alpha^2 \\ \alpha^2 & \alpha \\ 1 & 1 \end{bmatrix}$, and $P_1' = \begin{bmatrix} 1 \\ \alpha \\ \alpha \end{bmatrix}$, then by Lemma 4.2.16 $\Phi_{P_0} \preceq \Phi'$, where $\Phi' = \mathrm{YT}([P_0'|D_3], P_1')$. By Remark 4.2.7, Φ' is equivalent to a template minor of the complete, lifted Y-template determined by the following matrix.

$$\begin{bmatrix}
1 & \alpha & \alpha^2 \\
\alpha & \alpha^2 & \alpha \\
\alpha & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}$$

This is a submatrix of matrix I.

Therefore, we may assume that no column of P_0 is semi-parallel to column 10. Thus, we may assume that P_0 has exactly four rows. We used SageMath to look for subsets $S = \{v_1, v_2, \ldots, v_n\}$ of $\{11, \ldots, 23\}$ such that the complete, lifted Y-template determined by the matrix $[10, v_1, v_2, \ldots, v_n]$ is forbidden from P_0 by Lemmas 6.4.3 and 6.3.3. Then we found maximal subsets of $\{11, \ldots, 23\}$ that contain no such set S. The code for these computations is found in Section A.8. These computations returned a collection of 40 sets that we list as follows: $\{17, 20, 21\}$, $\{18, 11, 13, 22\}$, $\{17, 14, 22, 15\}$, $\{18, 11, 21\}$, $\{18, 19, 22\}$, $\{19, 20, 14, 15\}$, $\{17, 11, 12, 21\}$, $\{13, 22, 23\}$, $\{11, 19, 20\}$, $\{18, 12, 21, 13\}$, $\{16, 17, 12\}$, $\{17, 18, 11, 15\}$, $\{11, 14, 15\}$, $\{16, 17, 18\}$, $\{14, 11, 21, 22\}$, $\{19, 14, 22\}$, $\{16, 18, 19\}$, $\{12, 21, 23\}$, $\{15, 11, 20, 23\}$, $\{17, 20, 15\}$, $\{20, 21, 14\}$, $\{11, 12, 13\}$, $\{16, 11, 23\}$, $\{16, 18, 13\}$, $\{16, 19, 20,$

$23\}, \{20, 21, 13\}, \{17, 11, 22\}, \{16, 11, 20, 13\}, \{21, 22, 23\}, \{19, 22, 23\}, \{18, 19, 12\}$, $\{16, 17, 20\}$, $\{16, 12, 13, 23\}$, $\{16, 11, 19, 14\}$, $\{11, 19, 12, 23\}$, $\{17, 18, 21, 22\}$, $\{15, 22, 23\}$, $\{16, 17, 14\}$, $\{20, 21, 23\}$, $\{18, 19, 15\}$. However, we show now that several of the corresponding matrices determine equivalent templates and therefore, some of these sets can be discarded.

Note that in the matrix $[10, 11, \ldots, 23]$ above, swapping the second and third rows and scaling the columns so that the last nonzero entry of each column is 1, we obtain the matrix $[10, 11, 14, 15, 12, 13, 17, 16, 20, 21, 18, 19, 23, 22]$. Therefore, for example, since $\{17, 20, 21\}$ is one of the sets, we may discard the set $\{16, 18, 19\}$. Table 6.6 reduces the number of sets from 40 to 20.

TABLE 6.6. Discarding Twenty Redundant Sets

Set to Keep	Set to Discard	Set to Keep	Set to Discard
$\{17, 20, 21\}$	$\{16, 18, 19\}$	$\{18, 11, 13, 22\}$	$\{15, 11, 20, 23\}$
$\{17, 14, 22, 15\}$	$\{16, 12, 13, 23\}$	$\{18, 11, 21\}$	$\{11, 19, 20\}$
$\{18, 19, 22\}$	$\{20, 21, 23\}$	$\{19, 20, 14, 15\}$	$\{18, 21, 12, 13\}$
$\{17, 11, 12, 21\}$	$\{16, 11, 19, 14\}$	$\{13, 22, 23\}$	$\{15, 22, 23\}$
$\{16, 17, 12\}$	$\{16, 17, 14\}$	$\{17, 18, 11, 15\}$	$\{16, 11, 20, 13\}$
$\{11, 14, 15\}$	$\{11, 12, 13\}$	$\{16, 17, 18\}$	$\{16, 17, 20\}$
$\{14, 11, 21, 22\}$	$\{11, 19, 12, 23\}$	$\{19, 14, 22\}$	$\{12, 21, 23\}$
$\{17, 20, 15\}$	$\{16, 18, 13\}$	$\{20, 21, 14\}$	$\{18, 19, 12\}$
$\{16, 11, 23\}$	$\{17, 11, 22\}$	$\{16, 19, 20, 23\}$	$\{17, 18, 21, 22\}$
$\{20, 21, 13\}$	$\{18, 19, 15\}$	$\{21, 22, 23\}$	$\{19, 22, 23\}$

Similarly, if we reverse the order of the rows of $[10, \ldots, 23]$, then we obtain the matrix [10, 11, 13, 12, 15, 14, 23, 22, 21, 20, 19, 18, 17, 16]. Table 6.7 reduces the number of sets from 20 to 14.

TABLE 6.7. Discarding Six Redundant Sets

Set to Keep	Set to Discard	Set to Keep	Set to Discard
$\{17, 20, 21\}$	$\{18, 19, 22\}$	$\{17, 14, 22, 15\}$	none
$\{17, 11, 12, 21\}$	$\{18, 11, 13, 22\}$	$\{16, 17, 12\}$	$\{23, 22, 13\}$
$\{11, 14, 15\}$	none	$\{14, 11, 21, 22\}$	$\{17, 18, 11, 15\}$
$\{17, 20, 15\}$	$\{19, 14, 22\}$	$\{16, 11, 23\}$	none
$\{20, 21, 13\}$	none	$\{18, 11, 21\}$	none
$\{19, 20, 14, 15\}$	none	$\{16, 17, 18\}$	$\{21, 22, 23\}$
$\{20, 21, 14\}$	none	$\{16, 19, 20, 23\}$	none

There are 14 sets left, and now we show that each of them is contained in one of the matrices listed in the statement of the result. Recall that each of the sets listed above gives a set of columns contained in P_0 in addition to column 10. Recall that

whenever the rows of P_0 are permuted, we will automatically scale the columns of the matrix so that the last nonzero entry of each column is 1.

For $[10, 17, 21, 21]$, put the rows in order $3, 1, 4, 2$. The result is $[11, 21, 17, 16]$. Then a field isomorphism results in $[10, 18, 16, 17]$, which is a submatrix of matrix IV.

For $[10, 17, 14, 22, 15]$, put the rows in order $2, 4, 3, 1$. The resulting matrix is $[12, 21, 11, 17, 10]$, which is matrix VI, up to permuting columns.

Matrix VI is $[10, 17, 11, 12, 21]$.

For $[10, 16, 17, 12]$, swap rows 2 and 3. The result is $[10, 17, 16, 14]$, which is a submatrix of matrix I.

Matrix $VIII$ is $[10, 11, 14, 15]$.

For $[10, 14, 11, 21, 22]$, put the rows in order $4, 2, 3, 1$. The resulting matrix is $[10, 13, 11, 20, 16]$. A field isomorphism results in $[11, 12, 10, 21, 17]$, which is matrix VI, up to permuting columns.

Matrix V is $[10, 17, 20, 15]$.

Matrix IX is $[10, 16, 11, 23]$.

For $[10, 20, 21, 13]$, put the rows in order $4, 1, 3, 2$. The result is $[15, 16, 17, 11]$. A field isomorphism results in $[14, 17, 16, 10]$, which is a submatrix of matrix I.

For $[10, 18, 11, 21]$, put the rows in order $2, 1, 4, 3$. The result is $[11, 17, 10, 22]$. A field isomorphism results in $[10, 16, 11, 23]$, which is matrix IX.

For $[10, 19, 20, 14, 15]$, put the rows in order $4, 2, 1, 3$. The resulting matrix is $[13, 16, 20, 10, 11]$. A field isomorphism results in $[14, 17, 21, 11, 10]$, which is matrix VI, up to permuting columns.

The matrix $[10, 16, 17, 18]$ is a submatrix of matrix IV.

For $[10, 20, 21, 14]$, put the rows in order $3, 4, 1, 2$. The result is $[11, 17, 16, 15]$. A field isomorphism results in $[10, 16, 17, 14]$, which is a submatrix of matrix I.

Matrix VII is $[10, 16, 19, 20, 23]$.

This completes Case 3. $\square$

Claim 6.4.5.4. *The result holds in Case 4.*

Proof. In Case 4, every column of P_0 has weight 3, and there are weight-3 columns of P_0 with supports whose intersection has size exactly 1. By column scaling and permuting the rows of P_0, we may assume that the two weight-3 columns, restricted to their nonzero rows, are

$$\begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} \alpha & \alpha \\ \alpha^2 & 0 \\ 0 & \alpha^2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}.$$

Suppose P_0 has another column v with a nonzero entry outside of the first five rows, such that v is not a semi-parallel extension of the matrix $[1, 2]$. Without loss of generality, we may assume that this nonzero entry is in the sixth row, and that

for some values a, b, c, d, e, the column is scaled to be $v = [a, b, c, d, e, 1, 0 \ldots, 0]^T$. By Lemmas 6.4.3 and 6.3.3, P_0 has no contractible submatrix with three columns. Therefore, $d \neq 0$ or $e \neq 0$. Without loss of generality, say $e \neq 0$. Therefore, only one member of $\{a, b, c, d\}$ is nonzero. One can then easily check that all possibilities for v result either in $[1, 2, v]$ being a contractible submatrix with three columns or in v having support disjoint from that of either column 1 or column 2. Both of these outcomes are forbidden by Lemmas 6.4.3 and 6.3.3.

Now, suppose that there is a column that is a semi-parallel extension of $[1, 2]$ with a nonzero entry outside of the first five rows. Therefore P_0 is a submatrix of a matrix of the following form for some positive integer m and with either $A \neq \emptyset$ or $B \neq \emptyset$. The submatrix contains columns 1 and 2, but it may not contain one or both of columns 3 and 4.

$$
\begin{array}{cccc|c|c|c}
1 & 2 & 3 & 4 & A & B & C \\
\hline
\alpha & \alpha & \alpha & \alpha & \alpha \cdots \alpha & \alpha \cdots \alpha & \\
\alpha^2 & 0 & \alpha^2 & 0 & \alpha^2 \cdots \alpha^2 & 0 \cdots 0 & \text{weight-3} \\
0 & \alpha^2 & 0 & \alpha^2 & 0 \cdots 0 & \alpha^2 \cdots \alpha^2 & \text{columns} \\
1 & 0 & 0 & 1 & 0 \cdots 0 & 0 \cdots 0 & \\
0 & 1 & 1 & 0 & 0 \cdots 0 & 0 \cdots 0 & \\
\hline
\multicolumn{4}{c|}{0} & I_m & I_m & 0
\end{array}
$$

Without loss of generality, let $B \neq \emptyset$, so $u = [\alpha, 0, \alpha^2, 0, 0, 1, 0, \ldots, 0]^T$ is a column of B. Then let $v = [a, b, c, d, e, 0, \ldots, 0]^T$ be a column of C. Since v has weight 3, exactly two of $\{a, b, c, d, e\}$ are 0. If $a = b = 0$, or $a = d = 0$, or $b = c = 0$, or $b = d = 0$ (with $v \neq 2$), or $c = d = 0$, then $[1, u, v]$ is a contractible matrix with three columns. If $a = c = 0$, then columns u and v have disjoint supports. By Lemmas 6.4.3 and 6.3.3, all of those outcomes are forbidden. If $a = e = 0$, then scale v so that $d = 1$. If $(b, c) = (\alpha, \alpha^2)$, then the function `MinorCheck` shows that $M([I_6|D_6|P_0])$ contains F_7^*. If $(b, c) = (\alpha^2, \alpha)$, then `MinorCheck` shows that $M([I_6|D_6|P_0])$ contains P_2. Thus, we have that $a \neq 0$ and $e = 0$.

Suppose that either P_0 contains column 3 or that $A \neq \emptyset$, as well as $B \neq \emptyset$. Then P_0 contains either column 3 or some vector $t = [\alpha, \alpha^2, 0, 0, \ldots, 0, 1, 0 \ldots, 0]^T$. If v is either $[\alpha^2, \alpha, 0, 1, 0, \ldots, 0]^T$ or $[\alpha^2, 0, \alpha, 1, 0, \ldots, 0]^T$, then $[2, 3, v]$ or $[2, t, v]$ is a contractible matrix with three columns. Therefore, P_0 is a submatrix of either Q or S below.

$$
Q = \begin{array}{ccc|c|cccc}
1 & 2 & 4 & B & & & & \\
\hline
\alpha & \alpha & \alpha & \alpha \cdots \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha^2 \\
\alpha^2 & 0 & 0 & 0 \cdots 0 & \alpha^2 & \alpha & \alpha & 0 \\
0 & \alpha^2 & \alpha^2 & \alpha^2 \cdots \alpha^2 & 1 & 1 & 0 & \alpha \\
1 & 0 & 1 & 0 \cdots 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \cdots 0 & 0 & 0 & 0 & 0 \\
\hline
\multicolumn{3}{c|}{0} & I_m & \multicolumn{4}{c}{0}
\end{array}
$$

$$
S = \begin{array}{|cccc|c|c|cc|}
\multicolumn{1}{c}{1} & 2 & 3 & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{B} & x & \multicolumn{1}{c}{y} \\
\hline
\alpha & \alpha & \alpha & \alpha & \alpha\cdots\alpha & \alpha\cdots\alpha & \alpha & \alpha^2 \\
\alpha^2 & 0 & \alpha^2 & 0 & \alpha^2\cdots\alpha^2 & 0\cdots 0 & \alpha^2 & \alpha \\
0 & \alpha^2 & 0 & \alpha^2 & 0\cdots 0 & \alpha^2\cdots\alpha^2 & 1 & 1 \\
1 & 0 & 0 & 1 & 0\cdots 0 & 0\cdots 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0\cdots 0 & 0\cdots 0 & 0 & 0 \\
\hline
\multicolumn{4}{|c|}{0} & I_m & I_m & \multicolumn{2}{c|}{0} \\
\hline
\end{array}
$$

We show that, in both cases, we may assume that P_0 has at most five rows. Suppose P_0 is a submatrix of Q. Then by Lemma 4.2.16, $\Phi_{P_0} \preceq \mathrm{YT}([P_0'|D_4], P_1')$, where P_0' is obtained from Q by deleting the columns indexed by $\{2\} \cup B$ and by restricting to the first four rows and where $P_1' = [\alpha, 0, \alpha^2, 0]^T$. By Remark 4.2.7, $\mathrm{YT}([P_0'|D_4], P_1')$ is equivalent to a template minor of the complete, lifted Y-template determined by a matrix with five rows, and each column of this matrix has exactly three nonzero entries.

Now suppose P_0 is a submatrix of S. Then by Lemma 4.2.16, we have $\Phi_{P_0} \preceq \mathrm{YT}([P_0'|D_3], P_1')$, where P_0' is obtained from S by restricting to the first three rows and columns x and y, and where $P_1' = \begin{bmatrix} \alpha & \alpha \\ \alpha^2 & 0 \\ 0 & \alpha^2 \end{bmatrix}$. By Remark 4.2.7, $\mathrm{YT}([P_0'|D_3], P_1')$ is equivalent to a template minor of the complete, lifted Y-template determined by a matrix with five rows, and each column of this matrix has exactly three nonzero entries.

Therefore, we may assume that P_0 has at most five rows. Thus, if we scale the columns of P_0 so that the last nonzero entry of each column is 1, then P_0 contains columns 15 and 16 in the matrix below, and every column of P_0 is one of the columns in the two matrices below.

$$
\begin{array}{c}
\begin{array}{cccccccccc}
15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24
\end{array} \\
\begin{bmatrix}
\alpha & \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha & \alpha^2 & \alpha & \alpha^2 & \alpha^2 \\
\alpha^2 & 0 & \alpha^2 & \alpha & \alpha & \alpha^2 & \alpha & 0 & 0 & 0 \\
0 & \alpha^2 & 1 & 1 & 0 & 0 & 0 & \alpha^2 & \alpha & \alpha \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccccccc}
25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34
\end{array} \\
\begin{bmatrix}
\alpha & \alpha^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & \alpha^2 & \alpha & \alpha^2 & \alpha & \alpha^2 & 0 & 0 \\
0 & 0 & \alpha^2 & \alpha & \alpha^2 & \alpha & 0 & 0 & \alpha & \alpha^2 \\
\alpha^2 & \alpha & 1 & 1 & 0 & 0 & \alpha^2 & \alpha & \alpha^2 & \alpha \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

We used SageMath to look for subsets $S = \{v_1, v_2, \ldots, v_n\}$ of $\{17, \ldots, 34\}$ such that the complete, lifted Y-template determined by the matrix $[15, 16, v_1, v_2, \ldots, v_n]$ is forbidden from P_0. Then we found maximal subsets of $\{17, \ldots, 34\}$ that contain

no such set S. The code for these computations is found in Section A.8. These computations returned a collection of 13 sets that we list as follows: $\{18,26,22,23\}$, $\{17,18,20,22\}$, $\{24,17,22\}$, $\{17,20,21,25\}$, $\{25,19,22\}$, $\{24,17,20\}$, $\{18,19,22\}$, $\{24,26,22\}$, $\{25,26,20,22\}$, $\{25,19,20\}$, $\{18,19,20\}$, $\{24,26,20\}$, $\{24,19\}$.

Note that in the matrix $[15,16,17,\ldots,34]$ above, swapping the second and third rows, swapping the fourth and fifth rows, and scaling the columns so that the last nonzero entry of each column is 1, we obtain the matrix [16, 15, 18, 17, 24, 22, 23, 20, 21, 19, 26, 25, 30, 29, 28, 27, 34, 33, 32, 31].

Therefore, for example, since $\{17,20,21,25\}$ is one of the sets, we may discard the set $\{18,26,22,23\}$. Table 6.8 reduces the number of sets from 13 to 8.

TABLE 6.8. Discarding Five Redundant Sets

Set to Keep	Set to Discard	Set to Keep	Set to Discard
$\{17,20,21,25\}$	$\{18,26,22,23\}$	$\{17,18,20,22\}$	None
$\{24,17,22\}$	$\{18,19,20\}$	$\{25,19,22\}$	$\{24,26,20\}$
$\{24,17,20\}$	$\{18,19,22\}$	$\{24,26,22\}$	$\{25,19,20\}$
$\{25,26,20,22\}$	None	$\{24,19\}$	None

There are 8 sets left, and now we show that each of them is contained in one of the matrices listed in the statement of the result. Up to permuting of columns, the matrices X–XIV are, respectively, the matrices $[15,16,24,17,22]$, $[15,16,24,19]$, $[15,16,17,18,20,22]$, $[15,16,24,17,20]$, and $[15,16,17,20,21,25]$.

For $[15,16,25.19,22]$, put the rows in order $1,5,4,3,2$. The resulting matrix is $[24,19,18,16,23]$. Then a field isomorphism results in $[16,15,17,24,22]$, which is matrix XI, up to permuting columns.

For $[15,16,24,26,22]$, put the rows in order $1,4,5,3,2$. The resulting matrix is $[19,24,16,18,21]$. Then a field isomorphism results in $[15,16,24,17,20]$, which is matrix X, up to permuting columns.

For $[15,16,25,26,20,22]$, put the rows in order $1,5,4,2,3$. The resulting matrix is $[23,21,18,17,19,24]$. Then a field isomorphism results in $[22,20,17,18,15,16]$, which is matrix XII, up to permuting columns. This completes Case 4. $\square$

Claim 6.4.5.5. *The result holds in Case 5.*

Proof. In Case 5, every column of P_0 has weight 3, and there are no pairs of columns with supports whose intersection has size 1. If the intersection of the supports of all of the columns of P_0 consists of the same two rows, then P_0 is a submatrix of a matrix of the following form, for some positive integer m.

$$\left[\begin{array}{c|c} \alpha\cdots\alpha & \alpha^2\cdots\alpha^2 \\ \alpha^2\cdots\alpha^2 & \alpha\cdots\alpha \\ \hline I_m & I_m \end{array}\right]$$

Then, by Lemma 4.2.16, $\Phi_{P_0} \preceq \Phi' = \mathrm{YT}(D_2, P_1')$, where $P_1' = \begin{bmatrix} \alpha & \alpha^2 \\ \alpha^2 & \alpha \end{bmatrix}$. By Remark 4.2.7, Φ' is equivalent to a template minor of the complete, lifted Y-template determined by matrix XV.

Therefore, we may assume that P_0 has three columns such that the intersections of the supports of each pair of columns has size 2, but such that the three intersections are distinct. By permuting rows, we may assume without loss of generality that two of the columns are $\begin{bmatrix} \alpha & a \\ \alpha^2 & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, where $\{a, b\} = \{\alpha, \alpha^2\}$. The third column must have a nonzero entry in exactly one of the first two rows and both of the third and fourth rows. If there is a column v with a nonzero entry outside of the first four rows, then one of the first three columns will have a support whose intersection with the support of v has size 1, contradicting Case 5. Therefore, we may assume that P_0 has exactly four rows. Therefore, the three given columns are of the following form, where $\{a, b, c, d\} \subseteq \{\alpha, \alpha^2\}$.

$$\begin{bmatrix} \alpha & a & c \\ \alpha^2 & b & 0 \\ 1 & 0 & d \\ 0 & 1 & 1 \end{bmatrix}$$

We now show that (a, b) can be taken to be (α^2, α). Suppose otherwise; then $(a, b, c, d) = (\alpha, \alpha^2, \alpha^2, \alpha)$ or $(a, b, c, d) = (\alpha, \alpha^2, \alpha, \alpha^2)$. If $(a, b, c, d) = (\alpha, \alpha^2, \alpha^2, \alpha)$, then putting the rows in order $1, 4, 3, 2$ and scaling, we obtain

$$\begin{bmatrix} \alpha^2 & \alpha^2 & \alpha \\ 0 & \alpha & \alpha^2 \\ \alpha & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

whose columns can be permuted to be in the desired form. Now suppose that $(a, b, c, d) = (\alpha, \alpha^2, \alpha, \alpha^2)$. Putting the rows in order $1, 3, 4, 2$ and scaling, we obtain

$$\begin{bmatrix} \alpha^2 & \alpha^2 & \alpha \\ \alpha & 0 & \alpha^2 \\ 0 & \alpha & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

whose columns can be permuted to be in the desired form.

Therefore, P_0 is a submatrix of the following matrix that includes columns 10 and 11.

$$\begin{array}{c} \begin{array}{cccccccc} 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \end{array} \\ \begin{bmatrix} \alpha & \alpha^2 & \alpha^2 & \alpha & \alpha & \alpha^2 & 0 & 0 \\ \alpha^2 & \alpha & \alpha & \alpha^2 & 0 & 0 & \alpha & \alpha^2 \\ 1 & 0 & 1 & 0 & \alpha^2 & \alpha & \alpha^2 & \alpha \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

We used SageMath to look for subsets $S = \{v_1, v_2, \ldots, v_n\}$ of $\{12, \ldots, 17\}$ such that the complete, lifted Y-template determined by the matrix $[10, 11, v_1, v_2, \ldots, v_n]$ is forbidden from P_0. Then we found maximal subsets of $\{12, \ldots, 17\}$ that contain no such set S. The code for these computations is found in Section A.8. These computations returned a collection of 12 sets that we list as follows: $\{13, 15\}$, $\{17, 13\}$, $\{17, 12\}$, $\{12, 15\}$, $\{17, 14\}$, $\{12, 13\}$, $\{16, 12\}$, $\{16, 13\}$, $\{12, 14\}$, $\{13, 14\}$, $\{14, 15\}$, $\{16, 17\}$.

From the discussion above, we may assume that P_0 contains $[c, 0, d, 1]^T$, where $\{c.d\} = \{\alpha, \alpha^2\}$. Therefore, we may assume that P_0 contains column 14 or 15. Thus, we need only consider the following sets: $\{13, 15\}$, $\{12, 15\}$, $\{17, 14\}$, $\{12, 14\}$, $\{13, 14\}$, $\{14, 15\}$.

For $[10, 11, 13, 15]$, put the rows in order $1, 2, 4, 3$. The result is $[13, 12, 10, 14]$. This is a submatrix of matrix XII. (The submatrix is contained in the first four rows of matrix XII.)

For $[10, 11, 12, 15]$, a field isomorphism results in $[12, 13, 10, 14]$, which is a submatrix of matrix XII, up to permuting columns. (The submatrix is contained in the first four rows of matrix XII.)

Matrix XVI is $[10, 11, 17, 14]$.

The matrix $[10, 11, 13, 14]$ is a submatrix of matrix XIV, up to permuting columns. (The submatrix is contained in the first three rows and the last row of matrix XIV.)

For $[10, 11, 12, 14]$, put the rows in order $1, 2, 4, 3$. The result is $[13, 12, 11, 15]$. Then a field isomorphism results in $[11, 10, 13, 14]$, which we just saw is a submatrix of matrix XIV, up to permuting columns.

For $[10, 11, 14, 15]$, put the rows in order $1, 2, 4, 3$. The result is $[13, 12, 10, 14]$, which we have seen is a submatrix of matrix XIV, up to permuting columns. This completes Case 5. $\square$

Lemma 6.4.5 follows from Claims 6.4.5.1–6.4.5.5. $\blacksquare$

6.5 Partial Fields: Definition and Examples

Although Lemma 6.4.5 says much about the matrices I–$XVII$ and how they relate to the class $\mathcal{AC}_4$, we have yet to show that, if P_0 is one of matrices I–$XVII$, then $\mathcal{M}(\Phi_{P_0}) \subseteq \mathcal{AC}_4$. In order to do this, we will use the theory of partial fields. Partial fields were introduced by Semple and Whittle [36] to study classes $\mathcal{M}$ of matroids such that a matroid $M \in \mathcal{M}$ if and only if M is representable by a matrix over a field such that every nonzero subdeterminant of that matrix is an element of some multiplicative subgroup of the field. The class of golden-mean matroids is such a class. Other examples include the regular matroids, near-regular matroids, dyadic matroids, and $\sqrt[6]{1}$-matroids.

For the next several definitions, we follow Pendavingh and Van Zwam [27].

Definition 6.5.1. A *partial field* is a pair $\mathbb{P} = (R, G)$, where R is a commutative ring with identity and G is a subgroup of the multiplicative group $R^\times$ of R such that $-1 \in G$. When $\mathbb{P}$ is referred to as a set, then it is the set $G \cup \{0\}$.

A partial field $\mathbb{P}$ behaves very much like a field, except that, for $p, q \in \mathbb{P}$, the sum $p+q$ need not be an element of $\mathbb{P}$. Note that, if $\mathbb{F}$ is a field, then $(\mathbb{F}, \mathbb{F}^{\times})$ is a partial field.

Definition 6.5.2. A matrix A with entries in $\mathbb{P}$ is a $\mathbb{P}$-*matrix* if $\det(A') \in \mathbb{P}$ for every square submatrix A' of A. If M is a matroid of rank r on ground set E and there exists an $r \times E$ $\mathbb{P}$-matrix A such that $M = M(A)$, then we say that M is *representable* over $\mathbb{P}$ or, more briefly, $\mathbb{P}$-*representable*.

Definition 6.5.3. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be partial fields. A function $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$ is a *partial-field homomorphism* if

- $\varphi(1) = 1$;
- $\varphi(pq) = \varphi(p)\varphi(q)$ for all $p, q \in \mathbb{P}_1$; and
- $\varphi(p+q) = \varphi(p) + \varphi(q)$ for all $p, q \in \mathbb{P}_1$ such that $p + q \in \mathbb{P}_1$.

We will call a partial field homomorphism $\mathbb{P}_1 \to \mathbb{P}_2$ *trivial* if $\mathbb{P}_2$ is the trivial partial field $(\{0\}, \{0\})$. For a function $f : \mathbb{P}_1 \to \mathbb{P}_2$ and a matrix A over $\mathbb{P}_1$, we denote by $f(A)$ the matrix obtained by applying f to each entry of A. The proof of the next theorem is found in [36, Corollary 5.3] as well as [27, Corollary 2.9].

Theorem 6.5.4. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be partial fields and let $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$ be a nontrivial homomorphism. If A is a $\mathbb{P}_1$-matrix, then $\varphi(A)$ is a $\mathbb{P}_2$-matrix and $M(\varphi(A)) = M(A)$.*

If G is a group and $g_1, g_2, \dots, g_n \in G$, then we denote by $\langle g_1, g_2, \dots, g_n \rangle$ the subgroup of G generated by $\{g_1, g_2, \dots, g_n\}$. We now give several examples of partial fields that will be used later in this dissertation.

Example 6.5.5. The 2-*regular* partial field is

$$\mathbb{U}_2 = (\mathbb{Q}(\alpha_1, \alpha_2), \langle -1, \alpha_1, \alpha_2, \alpha_1 - 1, \alpha_2 - 1, \alpha_1 - \alpha_2 \rangle),$$

where α_1 and α_2 are indeterminates. This partial field has also been called the 2-*uniform* partial field.

Example 6.5.6. The 2-*cyclotomic* partial field is $\mathbb{K}_2 = (\mathbb{Q}(\alpha), \langle -1, \alpha, \alpha-1, \alpha+1 \rangle)$, where α is an indeterminate.

Example 6.5.7. Let τ be the positive root of $x^2 - x - 1$ over $\mathbb{R}$. The *golden-mean* partial field is $\mathbb{G} = (\mathbb{Z}[\tau], \langle -1, \tau \rangle)$. Note that $\{\tau+1, \tau-1\} \subseteq \mathbb{G}$ because $\tau + 1 = \tau^2$ and $\tau - 1 = \tau^{-1}$.

The previous examples of partial fields have been studied before (for example in [36], [26], and [27]). However, the next definition introduces a partial field that has not previously appeared in the literature, as far as the author can tell.

Definition 6.5.8. The *Pappus* partial field is

$$\mathbb{P}_{\text{Pap}} = (\mathbb{Q}(\alpha), \langle -1, \alpha, \alpha+1, \alpha-1, \alpha+2, 2\alpha+1 \rangle),$$

where α is an indeterminate.

Lemma 6.5.9. *There is a homomorphism from $\mathbb{P}_{Pap}$ to every field $\mathbb{F}$ such that $\mathbb{F} = \mathrm{GF}(4)$ or $|\mathbb{F}| \geq 7$.*

Proof. If $\mathbb{P}_1 = (R_1, G_1)$ and $\mathbb{P}_2 = (R_2, G_2)$ are partial fields and $\varphi : R_1 \to R_2$ is a ring homomorphism such that $\varphi(G_1) \subseteq G_2$, then the restriction of φ to G_1 is a partial field homomorphism.

Let $\mathbb{P}_1 = (R_1, G_1)$, where $R_1 = \mathbb{Z}\left[\alpha, \frac{1}{\alpha}, \frac{1}{\alpha+1}, \frac{1}{\alpha-1}, \frac{1}{\alpha+2}, \frac{1}{2\alpha+1}\right]$ and where $G_1 = \langle -1, \alpha, \alpha+1, \alpha-1, \alpha+2, 2\alpha+1\rangle$. Let $\varphi_1 : \mathbb{P}_{\mathrm{Pap}} \to \mathbb{P}_1$ be the partial field homomorphism given by the identity map on the set $\mathbb{P}_{\mathrm{Pap}}$.

The composition of partial field homomorphisms is a again a partial field homomrphism. Therefore, to construct a partial field homomrphism from $\mathbb{P}_{\mathrm{Pap}}$ to a field $\mathbb{F}$, it suffices to construct a ring homomorphism $\varphi_2 : R_1 \to \mathbb{F}$ defined by $\varphi_2(\alpha) = x$, for some $x \in \mathbb{F}$ such that x, $x+1$, $x-1$, $x+2$, and $2x+1$ are all nonzero. Then $\varphi_2 \circ \varphi_1 : \mathbb{P}_{\mathrm{Pap}} \to \mathbb{F}$ is a partial field homomorphism.

If $\mathbb{F}$ is a field other than a prime field (so $|\mathbb{F}| = 4$ or $|\mathbb{F}| > 7$) and $\mathbb{F}_p$ is its prime subfield, let $x \in \mathbb{F} - \mathbb{F}_p$. If $\mathbb{F}$ is a prime field of size 7 or larger, let $x = 2$. ■

The next two lemmas have proofs similar to that of Lemma 6.5.9. Alternatively, see the proofs of [35, Proposition 3.1] and [26, Lemma 4.14], respectively.

Lemma 6.5.10. *There is a homomorphism from the 2-regular partial field $\mathbb{U}_2$ to every field $\mathbb{F}$ such that $|\mathbb{F}| \geq 4$.*

Lemma 6.5.11. *There is a homomorphism from the 2-cyclotomic partial field $\mathbb{K}_2$ to every field $\mathbb{F}$ such that $|\mathbb{F}| \geq 4$.*

The next theorem is [26, Theorem 4.9].

Theorem 6.5.12. *There is a homomorphism from the golden-mean partial field $\mathbb{G}$ to $\mathrm{GF}(5)$, to $\mathrm{GF}(p^2)$ for every prime p, and to $\mathrm{GF}(p)$ for every prime p such that $p \equiv \pm 1 (\mathrm{mod}\ 5)$.*

Recall that we denote the set of prime numbers by $\mathcal{P}$. Moreover, we denote by $\mathcal{AC}_4$ the class of quaternary matroids with characteristic set $\mathcal{P} \cup \{0\}$, and we denote by $\mathcal{GM}$ the class of golden-mean matroids.

Corollary 6.5.13. *The following are true.*

(i) If a matroid M is representable over $\mathbb{P} \in \{\mathbb{U}_2, \mathbb{K}_2\}$, then it is representable over all fields of size at least 4.

(ii) If a matroid M is representable over $\mathbb{P} \in \{\mathbb{P}_{Pap}, \mathbb{U}_2, \mathbb{K}_2\}$, then it is representable over $\mathrm{GF}(4)$ and all fields of size at least 7.

(iii) If a matroid M is representable over $\mathbb{P} \in \{\mathbb{U}_2, \mathbb{K}_2, \mathbb{G}\}$, then $M \in \mathcal{GM}$.

(iv) If a matroid M is representable over $\mathbb{P} \in \{\mathbb{P}_{Pap}, \mathbb{U}_2, \mathbb{K}_2, \mathbb{G}\}$, then $M \in \mathcal{AC}_4$.

Proof. By Theorem 6.5.4, to prove that M is representable over some field $\mathbb{F}$, it suffices to prove that M is representable over a partial field $\mathbb{P}$ such that there is a

homomorphism from $\mathbb{P}$ to $\mathbb{F}$. By Lemmas 6.5.10 and 6.5.11, if $\mathbb{P} \in \{\mathbb{U}_2, \mathbb{K}_2\}$, there is a homomorphism from $\mathbb{P}$ to every field of size at least 4. Thus, (i) holds. Lemma 6.5.9, combined with (i), implies (ii). To prove (iii), let M be representable over $\mathbb{P} \in \{\mathbb{U}_2, \mathbb{K}_2, \mathbb{G}\}$. By Theorem 6.2.1, it suffices to show that M is representable over GF(4) and GF(5). This follows from (i) and Theorem 6.5.12. Finally, (ii) and (iii) imply (iv). ■

6.6 Partial Fields and Templates

The next lemma is important for understanding the relationship between partial fields and Y-templates. Recall from Definition 4.2.14 that the complete, lifted Y-template determined by a matrix P_0 is denoted by Φ_{P_0}.

Lemma 6.6.1. *Let P_0 be a matrix with m rows over some field $\mathbb{F}$. Suppose $M = \widetilde{M}([I_m|D_m|P_0])$ is representable over a partial field $\mathbb{P}$. Then every matroid in $\mathcal{M}(\Phi_{P_0})$ is $\mathbb{P}$-representable.*

Proof. We note, without proof, that Lemma 4.3.8 can be generalized to partial fields[1].

Since $\widetilde{M}([I_r|D_r|P_0])$ is $\mathbb{P}$-representable, there is a matrix $A = [I_m| * |P_0']$ over $\mathbb{P}$ that represents M. Here P_0' has the same zero-nonzero pattern as P_0, and the matrix can be scaled so that $*$ is of the form $\left[\begin{array}{c|c} 1\cdots 1 & 0\cdots 0 \\ \hline -I_{m-1} & D'_{m-1}\end{array}\right]$, where D'_{m-1} has nonzero entries in the same locations as D_{m-1} and is scaled so that the first nonzero entry in each column is a 1. It is not hard to see that, in order to have the circuits of M, the matrix D'_{m-1} must indeed be D_{m-1}. Thus $A = [I_m|D_m|P_0']$ for some matrix P_0' over $\mathbb{F}'$. By Lemma 4.3.8, every matroid in $\mathcal{M}(\Phi_{P_0})$ can be represented by a column submatrix of a matrix over $\mathbb{P}$ of the following form.

$$\left[\begin{array}{c|c|c} I_r & D_r & \begin{array}{c} P_0' \\ \hline 0 \end{array}\end{array}\right]$$

■

Therefore, if P_0 is one of the matrices I–$XVII$ given in Definition 6.4.4, we will show that $M = \widetilde{M}([I_r|D_r|P_0])$ is representable over $\mathbb{U}_2$, $\mathbb{K}_2$, $\mathbb{G}$, or $\mathbb{P}_{\text{Pap}}$. To do this, we use a series of functions implemented in SageMath. The code for these functions can be found in the Appendix in Section A.9. If A is a matrix whose entries are contained in a ring R, then we say that R is the *base ring* of A.

Lemma 6.6.2. *Let P_0 be a matrix that contains a submatrix of the form $[1, \alpha, \alpha]^T$ (up to field isomorphism and permuting rows). If $\mathcal{M}(\Phi_{P_0})$ is algebraically equivalent to a template $\mathcal{M}(\Phi'_{P_0})$ over a field $\mathbb{F}$, then the corresponding submatrix of P_0' must be $[-1, -x, x, 1]^T$ for some $x \in \mathbb{F} - \{0, 1\}$.*

[1] This is based on the fact that partial field representability is closed under generalized parallel connections (see [20, Theorem 3.1]).

Proof. Consider the following submatrix of $[I_4|D_4|P_0]$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

If the elements represented by the second and third columns are contracted, we see that the elements represented by the last two columns become a parallel pair. In the corresponding submatrix of P_0', the nonzero entries of the fifth column are 1 and -1. Therefore, the entries of P_0' corresponding to the 1s of the last column above, must be a 1 and a -1 also. ∎

The function `complete_template_representation` takes as input a matrix `P0` over GF(4). It returns a pair of matrices `A4` and `Avar`. The matrix `A4` is the matrix $[I_r|D_r|P_0]$ over GF(4), where each column of P_0 has been scaled so that the last nonzero entry is 1. The matrix `Avar` has entries from a polynomial ring $\mathbb{Z}[z_0, z_1, \ldots, z_n]$ for some n, and it is of the form $[I_r|D_r|P_0']$ for some matrix P_0' with the same zero-nonzero pattern as P_0. The symbols $z_0, z_1, \ldots, z_n$ are indeterminates. In order for `Avar` to be a representation of M, there will be certain relationships between the indeterminates. The columns of P_0' are scaled so that the last nonzero entry is a 1. If there is a second 1 in a column of P_0, then the corresponding entry of P_0' is -1, by Lemma 6.6.2.

The function `zero_determinant_ideal` takes as input a matroid `M` and a matrix `A`, over a ring R, of the form $A = [I_r|A']$, where $r = r(M)$ and where the rows and columns of I_r are indexed by a set B. For each column, indexed by c, and row, indexed by b, of A' the entry $A'_{b,c}$ is nonzero if and only if the basis element with nonzero entry in row b is in the B-fundamental circuit of c. For each size-r subset of the ground set of M that is not a basis, we compute the determinant of the corresponding square submatrix of `Avar`. (We call these size-r sets *nonbases.*) In order for `Avar` to represent `M`, these determinants should be 0. To do this, we need the quotient ring of R modulo the ideal generated by all of these determinants. The function returns a Gröbner basis for this ideal.

The function `check_partial_field` takes as input a matroid `M`, a matrix `A`, and an ordering `E` of the elements of `M`, as in the function `zero_determinant_ideal`. It also takes as an argument a list of generators of a multiplicative group G where $\mathbb{P} = (R, G)$ is some partial field. These generators should be elements of the fraction field of the base ring of A. The function determines if `A` is a $\mathbb{P}$-matrix that represents `M`.

6.7 The Highly Connected Matroids in $\mathcal{AC}_4$

We begin this section with seventeen lemmas—one for each of the matrices I–$XVII$ listed in Definition 6.4.4. For the first two lemmas, we will give the details of the SageMath code used to obtain the results. For the rest of the lemmas, we will only give the sketch of the proof, making statements that implicitly refer to

SageMath. The proofs of all of the lemmas follow the basic pattern explained in Section 6.6.

Lemma 6.7.1. *Let P_0 be matrix I listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{G}$-representable. Moreover, M is only representable over a field if it contains a solution to the equation $x^2 - x - 1 = 0$.*

Proof. The following code was used to determine the relations that must be satisfied between the nonzero entries of a matrix representing M. Besides 1 and -1, these entries are called $z_0, z_1, \ldots, z_{11}$, as explained in Section 6.6. (The entries in the leftmost columns are assigned the variables first. Within a column, the uppermost entry is assigned a variable first.)

```
P0 = Matrix(GF4, [[1,1,a,a,a,a^2],
                  [a,a,a,a,a^2,a],
                  [a,a,1,1,1,1],
                  [1,0,1,0,0,0],
                  [0,1,0,1,0,0]])
(A4,Avar)=complete_template_representation(P0)
M = Matroid(A4)
E = M.groundset_list()
I = zero_determinant_ideal(M, Avar)
I
Avar.base_ring().inject_variables()
Avar.change_ring(Avar.base_ring().fraction_field())
```

The function `zero_determinant_ideal` returned the ideal with the Gröbner basis $\{z_{11}^2 + z_{11} - 1,\ z_0 + z_{11},\ z_1 - z_{11},\ z_2 + z_{11},\ z_3 - z_{11},\ z_4 - z_{11},\ z_5 + z_{11},\ z_6 - z_{11},\ z_7 + z_{11},\ z_8 + z_{11},\ z_9 - z_{11} + 1,\ z_{10} + z_{11} + 1\}$. Take the quotient ring of the fraction field of the base ring of `Avar` modulo this ideal. The fact that $z_{11}^2 + z_{11} - 1$ and $z_2 + z_{11}$ are in the ideal implies that, in the quotient ring, $z_2^2 - z_2 - 1 = 0$. Thus, z_2 is a solution to the equation $x^2 - x - 1 = 0$ in the quotient ring. Thus, any field over which M has a representation must contain a solution to this equation. Moreover, $z_2 + 1 = z_2^2$ and $z_2 - 1 = z_2^{-1}$. In fact, we also have $z_2^2 - 2z_2 = z_2^{-1}$. Therefore, although the only generators for the golden-mean partial field are -1 and z_2, the ideal allows us to include $z_2 + 1$, $z_2 - 1$, and $z_2^2 - 2z_2$ as generators. The following code returns `True` and therefore confirms that `Avar` is a $\mathbb{G}$-matrix after we pass from its base ring to the quotient ring.

```
Avar2 = Avar(z0=z2,z1=-z2,z3=-z2,z4=-z2,z5=z2,z6=-z2,z7=z2,
z8=z2,z9=-z2-1,z10=z2-1,z11=-z2)
check_partial_field(M, Avar2, M.groundset_list(),
[-1,z2,z2+1,z2-1,z2^2 - 2z2],[])
```

∎

Lemma 6.7.2. *Let P_0 be matrix II listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The following code was used to determine the relations that must be satisfied between the nonzero entries of a matrix representing M. Besides 1 and -1, these entries are called $z_0, z_1, \ldots, z_5$, as explained in Section 6.6. (The entries in the leftmost columns are assigned the variables first. Within a column, the uppermost entry is assigned a variable first.)

```
P0 = Matrix(GF4, [[1,a  ,0],
                  [a,a^2,0],
                  [a,0  ,a^2],
                  [1,0  ,a],
                  [0,1  ,1]])
(A4,Avar)=complete_template_representation(P0)
M = Matroid(A4)
E = M.groundset_list()
I = zero_determinant_ideal(M, Avar)
I
Avar.base_ring().inject_variables()
Avar.change_ring(Avar.base_ring().fraction_field())
```

The function `zero_determinant_ideal` returned the ideal with the Gröbner basis $\{z_1z_5 + z_5 + 1,\ z_0 + z_1,\ z_2 - z_5,\ z_3 + z_5 + 1,\ z_4 + z_5 + 1\}$. Take the quotient ring of the fraction field of the base ring of `Avar` modulo this ideal. If z_0, $z_0 + 1$, and $z_0 - 1$ are generators of a multiplicative group, then $1/z_0$, $1/(z_0+1)$, and $1/(z_0-1)$ are in the group and can also be added to the list of generators. The following code returns `True` and therefore confirms that `Avar` is a $\mathbb{K}_2$-matrix after we pass from its base ring to the quotient ring.

```
Avar2 = Avar(z1=-z0,z2=1/(z0-1),z3=-z0/(z0-1),z4=-z_0/(z_0-1),
z_5=1/(z_0-1))
check_partial_field(M, Avar2, M.groundset_list(),
[-1,z_0,z_0-1,z_0+1,1/z_0,1/(z_0+1),1/(z_0-1)],[])
```

■

Lemma 6.7.3. *Let P_0 be matrix III listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_1z_5 - 1,\ z_1z_9 + z_1 + z_9,\ z_5z_9 + z_9 + 1,\ z_0 + z_1,\ z_2 + z_9 + 1,\ z_3 - z_9,\ z_4 + z_5,\ z_6 + z_9 + 1,\ z_7 - z_9,\ z_8 + z_9 + 1\}$. We solve for the variables in terms of z_0 and obtain $z_1 = -z_0$, $z_2 = 1/(z_0 - 1)$, $z_3 = -z_0/(z_0 - 1)$, $z_4 = 1/z_0$, $z_5 = -1/z_0$, $z_6 = 1/(z_0 - 1)$, $z_7 = -z_0/(z_0 - 1)$, $z_8 = 1/(z_0 - 1)$, and $z_9 = -z_0/(z_0-1)$. We check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_0,\ z_0 + 1,\ z_0 - 1,\ 1/z_0,\ 1/(z_0 + 1),\ 1/(z_0 - 1)\}$. ■

Lemma 6.7.4. *Let P_0 be matrix IV listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_3z_9+z_3+z_9,\ z_3z_{11}+z_3-1,\ z_9z_{11}+2z_9+1,\ z_0+z_3,\ z_1-z_3,\ z_2+z_3,\ z_4+z_9+1,\ z_5-z_9,\ z_6+z_9+1,\ z_7-z_9,\ z_8+z_9+1,\ z_{10}+z_{11}+1\}$. We solve for the variables in terms of z_{11} and obtain $z_0=-1/(z_{11}+1)$, $z_1=1/(z_{11}+1)$, $z_2=-1/(z_{11}+1)$, $z_3=1/(z_{11}+1)$, $z_4=-(z_{11}+1)/(z_{11}+2)$, $z_5=-1/(z_{11}+2)$, $z_6=-(z_{11}+1)/(z_{11}+2)$, $z_7=-1/(z_{11}+2)$, $z_8=-(z_{11}+1)/(z_{11}+2)$, $z_9=-1/(z_{11}+2)$, and $z_{10}=-z_{11}-1$. We check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_{11},\ z_{11}+1,\ z_{11}+2,\ 1/z_{11},\ 1/(z_{11}+1),\ 1/(z_{11}+2)\}$. (Here, $z_{11}+1$ is plays the role of α in Example 6.5.6.) ■

Lemma 6.7.5. *Let P_0 be matrix V listed in Definition 6.4.4. Then the abstract matroid $M=\widetilde{M}([I_4|D_4|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_1z_5+z_1-z_5,\ z_1z_7+1,\ z_5z_7+z_5+1,\ z_0+z_1,\ z_2-z_7,\ z_3+z_7+1,\ z_4+z_5+1,\ z_6+z_7\}$. We solve for the variables in terms of z_3 and obtain $z_0=-1/(z_3+1)$, $z_1=1/(z_3+1)$, $z_2=-(z_3+1)$, $z_4=-(z_3+1)/z_3$, $z_5=1/z_3$, $z_6=z_3+1$, and $z_7=-(z_3+1)$. We check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_3,\ z_3+1,\ z_3+2,\ 1/z_3,\ 1/(z_3+1),\ 1/(z_3+2)\}$. (Here, z_3+1 is plays the role of α in Example 6.5.6.) ■

Lemma 6.7.6. *Let P_0 be matrix VI listed in Definition 6.4.4. Then the abstract matroid $M=\widetilde{M}([I_4|D_4|P_0])$ is $\mathbb{G}$-representable. Moreover, M is only representable over a field if it contains a solution to the equation $x^2-x-1=0$.*

Proof. The ideal is generated by $\{z_9^2-z_9-1,\ z_0+z_9,\ z_1-z_9,\ z_2+z_9-1,\ z_3-z_9+2,\ z_4+z_9-1,\ z_5-z_9+1,\ z_6+z_9,\ z_7-z_9,\ z_8+z_9+1\}$. We solve for the variables in terms of z_9 and obtain $z_0=-z_9$, $z_1=z_9$, $z_2=-z_9+1$, $z_3=z_9-2$, $z_4=-z_9+1$, $z_5=z_9-1$, $z_6=-z_9$, $z_7=z_9$, and $z_8=-z_9-1$. Since $z_9^2-z_9-1$ is in the ideal, we have $z_9+1=z_9^2$, $z_9-1=1/z_9$, $z_9-2=-z_9^{-2}$, $-z_9^2+2=-z_9^{-1}$, and $-z_9^2+2z_9+1=z_9$. Therefore, we check that the matrix is a $\mathbb{G}$-matrix by checking the partial field generated by $\{-1,\ z_9,\ z_9+1,\ z_9-1,\ z_9-2,\ -z_9^2+2,\ -z_9^2+2z_9+1\}$.

The fact that $z_9^2-z_9-1$ is in the ideal implies that M is only representable over fields that contain a solution to the equation $x^2-x-1=0$. ■

Lemma 6.7.7. *Let P_0 be matrix VII listed in Definition 6.4.4. Then the abstract matroid $M=\widetilde{M}([I_4|D_4|P_0])$ is $\mathbb{G}$-representable. Moreover, M is only representable over a field if it contains a solution to the equation $x^2-x-1=0$.*

Proof. The ideal is generated by $\{z_9^2-z_9-1,\ z_0+z_9,\ z_1-z_9,\ z_2-z_9+2,\ z_3+z_9-1,\ z_4-z_9+1,\ z_5+z_9,\ z_6-z_9,\ z_7+z_9+1,\ z_8+z_9+1\}$. We solve for the variables in terms of z_9 and obtain $z_0=-z_9$, $z_1=z_9$, $z_2=z_9-2$, $z_3=-z_9+1$, $z_4=z_9-1$, $z_5=-z_9$, $z_6=z_9$, $z_7=-z_9-1$, $z_8=-z_9-1$. Since $z_9^2-z_9-1$ is in the ideal, we have $z_9+1=z_9^2$, $z_9-1=1/z_9$, $z_9-2=-z_9^{-2}$, $-z_9^2+2=-z_9^{-1}$, $2z_9^2-2z_9-1=1$, $2z_9+1=z_9^3$, and $-z_9^2+2z_9+1=z_9$. Therefore, we check that the matrix is a $\mathbb{G}$-matrix by checking the partial field generated by $\{-1,\ z_9,\ z_9+1,\ z_9-1,\ z_9-2,\ -z_9^2+2,\ 2z_9^2-2z_9-1,\ 2z_9+1,\ -z_9^2+2z_9+1\}$.

The fact that $z_9^2-z_9-1$ is in the ideal implies that M is only representable over fields that contain a solution to the equation $x^2-x-1=0$. ■

Lemma 6.7.8. *Let P_0 be matrix $VIII$ listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_4|D_4|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_5z_7 - 1,\ z_0 - z_5,\ z_1 + z_5,\ z_2 - z_7,\ z_3 + z_7,\ z_4 + z_5,\ z_6 + z_7\}$. We solve for the variables in terms of z_7 and obtain $z_0 = 1/z_7$, $z_1 = -1/z_7$, $z_2 = z_7$, $z_3 = -z_7$, $z_4 = -1/z_7$, $z_5 = 1/z_7$, $z_6 = -z_7$. Note that $-z_7^3 + z_7^2 + z_7 - 1 = -(z_7 - 1)^2(z_7 + 1)$. Therefore, we check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_7,\ z_7 + 1,\ z_7 - 1,\ 1/z_7,\ 1/(z_7 + 1),\ 1/(z_7 - 1),\ -z_7^3 + z_7^2 + z_7 - 1\}$. ■

Lemma 6.7.9. *Let P_0 be matrix IX listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_4|D_4|P_0])$ is $\mathbb{P}_{Pap}$-representable.*

Proof. The ideal is generated by $\{z_3z_7 + z_3 + z_7,\ z_0 + z_7,\ z_1 - z_7,\ z_2 + z_3 + 1,\ z_4 + z_7 + 1,\ z_5 - z_7 - 1,\ z_6 + z_7 + 1\}$. We solve for the variables in terms of z_7 and obtain $z_0 = -z_7$, $z_1 = z_7$, $z_2 = -1/(z_7 + 1)$, $z_3 = -z_7/(z_7 + 1)$, $z_4 = -z_7 - 1$, $z_5 = z_7 + 1$, and $z_6 = -z_7 - 1$. We check that the matrix is a $\mathbb{P}_{\mathrm{Pap}}$ matrix by checking the partial field generated by $\{-1,\ z_7,\ z_7 + 1,\ z_7 - 1,\ z_7 + 2,\ 2z_7 + 1,\ 1/z_7,\ 1/(z_7 + 1),\ 1/(z_7 - 1),\ 1/(z_7 + 2)\}$. ■

Lemma 6.7.10. *Let P_0 be matrix X listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_7z_9 - 1,\ z_0 + z_9 + 1,\ z_1 - z_9,\ z_2 + z_9 + 1,\ z_3 - z_9,\ z_4 + z_9 + 1,\ z_5 - z_9,\ z_6 + z_7 + 1,\ z_8 + z_9 + 1\}$. We solve for the variables in terms of z_9 and obtain $z_0 = -z_9 - 1$, $z_1 = z_9$, $z_2 = -z_9 - 1$, $z_3 = z_9$, $z_4 = -z_9 - 1$, $z_5 = z_9$, $z_6 = -(z_9 + 1)/z_9$, $z_7 = 1/z_9$, $z_8 = -z_9 - 1$. We check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_9,\ z_9 + 1,\ z_9 - 1,\ 1/z_9,\ 1/(z_9 + 1),\ 1/(z_9 - 1)\}$. ■

Lemma 6.7.11. *Let P_0 be matrix XI listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_3z_7 - 1,\ z_0 + z_3 + 1,\ z_1 - z_3,\ z_2 + z_3 + 1,\ z_4 + z_7 + 1,\ z_5 - z_7,\ z_6 + z_7 + 1\}$. We solve for the variables in terms of z_7 and obtain $z_0 = -(z_7+1)/z_7$, $z_1 = 1/z_7$, $z_2 = -(z_7+1)/z_7$, $z_3 = 1/z_7$, $z_4 = -z_7-1$, $z_5 = z_7$, and $z_6 = -z_7-1$. There is an optional argument for the function `check_partial_field` called `extra_determinants` that allows us to enter a list of polynomials that are known to be products of the generators of the partial field but that we do not wish to include as a generator in order to reduce the running time of the function. We know that $z_7^4 - 2z_7^2 + 1 = (z_7 + 1)^2(z_7 - 1)^2$. Therefore, we check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_7,\ z_7 + 1,\ z_7 - 1,\ 1/z_7,\ 1/(z_7 + 1),\ 1/(z_7 - 1)\}$, with the list of extra determinants consisting of $(z_7^4 - 2z_7^2 + 1)/z_7^2$ and $(-z_7^4 + 2z_7^2 - 1)/z_7^2$. ■

Lemma 6.7.12. *Let P_0 be matrix XII listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_7z_{11}-1,\ z_0+z_{11}+1,\ z_1-z_{11},\ z_2+z_{11}+1,\ z_3-z_{11},\ z_4+z_{11}+1,\ z_5-z_{11},\ z_6+z_7+1,\ z_8+z_{11}+1,\ z_9-z_{11},\ z_{10}+z_{11}+1\}$. We solve for the variables in terms of z_{11} and obtain $z_0=-z_{11}-1$, $z_1=z_{11}$, $z_2=-z_{11}-1$, $z_3=z_{11}, z_4=-z_{11}-1$, $z_5=z_{11}$, $z_6=-(z_{11}+1)/z_{11}$, $z_7=1/z_{11}$, $z_8=-z_{11}-1$, $z_9=z_{11}$, and $z_{10}=-z_{11}-1$. We check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_{11},\ z_{11}+1,\ z_{11}-1,\ 1/z_{11},\ 1/(z_{11}+1),\ 1/(z_{11}-1)\}$. ■

Lemma 6.7.13. *Let P_0 be matrix $XIII$ listed in Definition 6.4.4. Then the abstract matroid $M=\widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_7z_9-1,\ z_0+z_7+1,\ z_1-z_7,\ z_2+z_7+1,\ z_3-z_7,\ z_4+z_7+1,\ z_5-z_7,\ z_6+z_7+1,\ z_8+z_9+1\}$. We solve for the variables in terms of z_7 and obtain $z_0=-z_7-1$, $z_1=z_7$, $z_2=-z_7-1$, $z_3=z_7$, $z_4=-z_7-1$, $z_5=z_7$, $z_6=-z_7-1$, $z_8=-(z_7+1)/z_7$, and $z_9=1/z_7$. We check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1, z_7, z_7+1, z_7-1, 1/z_7, 1/(z_7+1), 1/(z_7-1)\}$. ■

Lemma 6.7.14. *Let P_0 be matrix XIV listed in Definition 6.4.4. Then the abstract matroid $M=\widetilde{M}([I_5|D_5|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_{11}^2+z_9,\ z_0+z_{11}+1,\ z_1-z_{11},\ z_2+z_{11}+1,\ z_3-z_{11},\ z_4+z_{11}+1,\ z_5-z_{11},\ z_6+z_{11}+1,\ z_7-z_{11},\ z_8+z_9+1,\ z_{10}+z_{11}+1\}$. We solve for the variables in terms of z_{11} and obtain $z_0=-z_{11}-1$, $z_1=z_{11}$, $z_2=-z_{11}-1$, $z_3=z_{11}$, $z_4=-z_{11}-1$, $z_5=z_{11}$, $z_6=-z_{11}-1$, $z_7=z_{11}$, $z_8=(z_{11}+1)(z_{11}-1)$, $z_9=-z_{11}^2$, and $z_{10}=-z_{11}-1$. We check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_{11},\ z_{11}+1,\ z_{11}-1\}$. ■

Lemma 6.7.15. *Let P_0 be matrix XV listed in Definition 6.4.4. Then the abstract matroid $M=\widetilde{M}([I_4|D_4|P_0])$ is $\mathbb{U}_2$-representable.*

Proof. The ideal is generated by $\{z_0+z_3+1,\ z_1-z_3,\ z_2+z_3+1,\ z_4+z_7+1,\ z_5-z_7,\ z_6+z_7+1\}$. We solve for the variables in terms of z_3 and z_7 and obtain $z_0=-z_3-1$, $z_1=z_3$, $z_2=-z_3-1$, $z_4=-z_7-1$, $z_5=z_7$, and $z_6=-z_7-1$. We check that the matrix is a $\mathbb{U}_2$-matrix by checking the partial field generated by $\{-1,\ z_3,\ z_7,\ z_3+1,\ z_7+1,\ z_7-z_3\}$. (Here, z_3+1 and z_7+1 play the roles of α_1 and α_2 in Example 6.5.5.) ■

Lemma 6.7.16. *Let P_0 be matrix XVI listed in Definition 6.4.4. Then the abstract matroid $M=\widetilde{M}([I_4|D_4|P_0])$ is $\mathbb{U}_2$-representable.*

Proof. The ideal is generated by $\{z_1z_5+z_3+z_5+1,\ z_3z_5-z_3z_7-z_5z_7-z_7,\ z_1z_7+z_3,\ z_0+z_1+1,\ z_2+z_3+1,\ z_4+z_5+1,\ z_6+z_7+1\}$. We solve for the variables in terms of z_1 and z_3 and obtain $z_0=-z_1-1$, $z_2=-z_3-1$, $z_4=(z_3-z_1)/(z_1+1)$, $z_5=-(z_3+1)/(z_1+1)$, $z_6=(z_3-z_1)/z_1$, and $z_7=-z_3/z_1$. We check that the matrix is a $\mathbb{U}_2$-matrix by checking the partial field generated by $\{-1,\ z_1,\ z_3,\ z_1+1,\ z_3+1,\ z_3-z_1,\ 1/z_1,\ 1/(z_1+1)\}$. (Here, z_1+1 and z_3+1 play the roles of α_1 and α_2 in Example 6.5.5.) ■

Lemma 6.7.17. *Let P_0 be matrix $XVII$ listed in Definition 6.4.4. Then the abstract matroid $M = \widetilde{M}([I_4|D_4|P_0])$ is $\mathbb{K}_2$-representable.*

Proof. The ideal is generated by $\{z_3z_5+z_5-1,\ z_0+z_5,\ z_1-z_5,\ z_2+z_3+1,\ z_4+z_5+1\}$. We solve for the variables in terms of z_5 and obtain $z_0 = -z_5$, $z_1 = z_5$, $z_2 = -1/z_5$, $z_3 = -(z_5-1)/z_5$, and $z_4 = -z_5 - 1$. We check that the matrix is a $\mathbb{K}_2$-matrix by checking the partial field generated by $\{-1,\ z_5,\ z_5+1,\ z_5-1,\ 1/z_5,\ 1/(z_5+1),\ 1/(z_5-1)\}$. ■

The proofs of the next two theorems are essentially identical to each other. We give the proof of Theorem 6.7.18 but omit the proof of Theorem 6.7.19.

Theorem 6.7.18. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that, for every k-connected member M of $\mathcal{AC}_4$ with at least $2k$ elements, either M or M^* is a minor of the vector matroid of a matrix of the form below, where P_0 is one of matrices I–XVI listed in Definition 6.4.4, up to a field isomorphism.*

$$\begin{array}{|c|c|c|}\hline \multirow{2}{*}{I_r} & \multirow{2}{*}{D_r} & P_0 \\ \cline{3-3} & & 0 \\ \hline \end{array}$$

Theorem 6.7.19. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that every simple vertically k-connected member of $\mathcal{AC}_4$ with an $M(K_n)$-minor is a minor of the vector matroid of a matrix of the form above, and every cosimple cyclically k-connected member of $\mathcal{AC}_4$ with an $M^*(K_n)$-minor is a minor of the dual of the vector matroid of a matrix of the form above, where P_0 is one of matrices I–XVI listed in Definition 6.4.4, up to a field isomorphism.*

Proof of Theorem 6.7.18. Recall the definition of weak conforming from Definition 4.1.3. Also recall that $\mathcal{M}_w(\Phi)$ is the set of matroids weakly conforming to a template Φ. By Corollary 4.1.5 (with $m = 3$, since the characteristic set of $\mathrm{PG}(2,2) = F_7$ is $\{2\}$), there is a set of refined templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ such that

1. $\mathcal{AC}_4$ contains each of the classes $\mathcal{M}_w(\Phi_1), \ldots, \mathcal{M}_w(\Phi_s)$,
2. $\mathcal{AC}_4$ contains the duals of the represented matroids in each of the classes $\mathcal{M}_w(\Psi_1),\ldots,\mathcal{M}_w(\Psi_t)$, and
3. if M is a simple k-connected member of $\mathcal{AC}_4$ with at least $2k$ elements, then either M is a member of at least one of the classes $\mathcal{M}_w(\Phi_1), \ldots, \mathcal{M}_w(\Phi_s)$, or M^* is a member of at least one of the classes $\mathcal{M}_w(\Psi_1), \ldots, \mathcal{M}_w(\Psi_t)$.

Lemma 6.4.5 (and the fact that $\mathcal{AC}_4$ is closed under duality) implies that these templates can be chosen so that each of them is the complete, lifted Y-template determined by a submatrix of one of matrices I–XVI. Now, consider the complete, lifted Y-templates determined by a matrix P_0, where P_0 is one of matrices I–XVI themselves (rather than a submatrix). Let m be the number of rows of P_0. By Lemmas 6.7.1–6.7.16, $M = \widetilde{M}([I_m|D_m|P_0])$ is representable over $\mathbb{G}$, $\mathbb{K}_2$, $\mathbb{U}_2$, or

$\mathbb{P}_{\text{Pap}}$. Therefore, by Corollary 6.5.13, $M \in \mathcal{AC}_4$. Lemma 6.6.1 then implies that $\mathcal{M}(\Phi_{P_0}) \subseteq \mathcal{AC}_4$.

Therefore, we may take $\{\Phi_1, \ldots, \Phi_s\}$ and $\{\Psi_1, \ldots, \Psi_t\}$ both to consist of the complete, lifted Y-templates determined by matrices I–XVI. Again, let P_0 be one of these matrices. The rank-r universal matroids conforming to Φ_{P_0} (of which every matroid conforming to Φ_{P_0} is a restriction) are represented by matrices of the form given in the statement of the theorem. ■

6.8 The Highly Connected Golden-Mean Matroids

In this section, we characterize the highly connected golden-mean matroids, subject to Hypotheses 3.2.2 and 3.2.3. To do this, we need some information about the *Pappus matroid*, which has the geometric representation given in Figure 6.4.

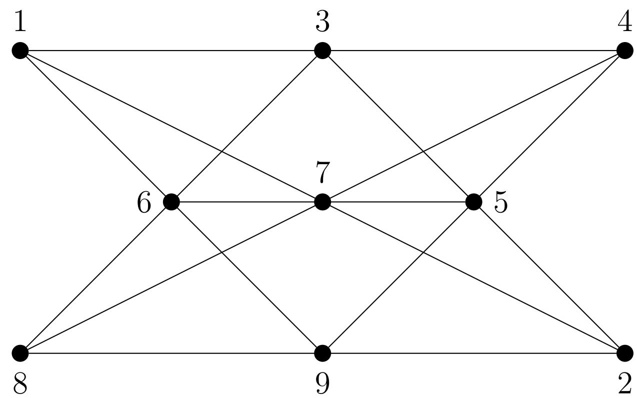

FIGURE 6.4. A Geometric Representation of the Pappus Matroid

Lemma 6.8.1.

(i) The Pappus matroid is a minor of a matroid conforming to Φ_{P_0}, where P_0 is matrix IX.

(ii) The Pappus matroid is representable over a field $\mathbb{F}$ if and only if $\mathbb{F} = \mathrm{GF}(4)$ or $|\mathbb{F}| \geq 7$.

(iii) If P_0 is any proper column submatrix of matrix IX, then it is also a submatrix of matrix VI or VII (up to field isomorphism and permuting of rows and columns).

Proof. To prove (i), let P_0 be matrix IX. Note that the vector matroid of $[I_4|D_4|P_0]$ virtually conforms to Φ_{P_0} and is therefore a minor of a matroid conforming to Φ_{P_0}. Contract the element represented by the first column and simplify. The result is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & \alpha & \alpha^2 & \alpha^2 & \alpha^2 \\ 0 & 1 & 0 & 1 & 0 & 1 & \alpha & 1 & \alpha^2 & \alpha \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Delete the element represented by the fourth column, and the result is

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \alpha & \alpha^2 & \alpha^2 & \alpha^2 \\ 0 & 1 & 0 & 0 & 1 & \alpha & 1 & \alpha^2 & \alpha \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \end{array}$$

which represents the Pappus matroid with the column indices corresponding to the labels in Figure 6.4.

The result in (ii) is stated without proof in Oxley [23, Appendix]. Alternatively, (i), combined with Lemma 6.7.9 implies that the Pappus matroid is representable over GF(4) and all fields of size at least 7. Since it contains a $U_{3,5}$-minor, it is not representable over GF(2) or GF(3). The fact that it is not representable over GF(5) was verified using SageMath.

To prove (iii), label the columns of matrix IX from left to right as a, b, c, and d and the rows as 1, 2, 3, 4. The matrix $[a, b, c]$, after a field isomorphism, is contained in matrix VI. The matrix $[a, b, d]$ is contained in matrix VII. If we put the rows of $[a, c, d]$ in order $4, 3, 2, 1$ and perform a field isomorphism, the resulting matrix is contained in matrix VI. If we put the rows of $[b, c, d]$ in order $1, 3, 2, 4$ and perform a field isomorphism, the resulting matrix is contained in matrix VII. ■

The proofs of the next two theorems are essentially identical to each other. We give the proof of Theorem 6.8.2 but omit the proof of Theorem 6.8.3.

Theorem 6.8.2. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that, for every k-connected golden-mean matroid M with at least $2k$ elements, either M or M^* is a minor of the vector matroid of a matrix of the form below, where P_0 is one of matrices I–$VIII$ or $X - XVI$ listed in Definition 6.4.4, up to a field isomorphism.*

$$\begin{array}{|c|c|c|} \hline \multirow{2}{*}{I_r} & \multirow{2}{*}{D_r} & P_0 \\ \cline{3-3} & & 0 \\ \hline \end{array}$$

Theorem 6.8.3. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that simple every vertically k-connected golden-mean matroid with an $M(K_n)$-minor is a minor of the vector matroid of a matrix of the form above, and every cosimple cyclically k-connected golden-mean matroid with an $M^*(K_n)$-minor is a minor of the dual of the vector matroid of a matrix of the form above, where P_0 is one of matrices I–$VIII$ or X–XVI listed in Definition 6.4.4, up to a field isomorphism.*

Proof of Theorem 6.8.2. We wish to find the templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ for $\mathcal{GM}$ whose existence are implied by Corollary 4.1.5. Since $\mathcal{GM} \subseteq \mathcal{AC}_4$, it follows from combining Lemma 6.4.5 and Theorem 6.7.18 that we may take each of $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ to be the complete, lifted Y-template determined by some column submatrix P_0 of one of matrices I–XVI.

Consider the complete, lifted Y-templates determined by a matrix P_0, where P_0 is one of matrices I–$VIII$ or $X - XVI$. Let m be the number of rows of P_0. By Lemmas 6.7.1–6.7.8 and Lemmas 6.7.10–6.7.16, $M = \widetilde{M}([I_m|D_m|P_0])$ is representable over $\mathbb{G}$, $\mathbb{K}_2$, or $\mathbb{U}_2$. Therefore, by Corollary 6.5.13, $M \in \mathcal{GM}$. Lemma 6.6.1 then implies that $\mathcal{M}(\Phi_{P_0}) \subseteq \mathcal{GM}$.

By Lemma 6.8.1, P_0 cannot be matrix IX (because every golden-mean matroid is GF(5)-representable). Lemma 6.8.1 also states that if P_0 is a proper column submatrix of matrix IX, then it must also be a submatrix of matrix VI or matrix VII; therefore, in that case, P_0 has already been analyzed above.

Therefore, we may take $\{\Phi_1, \ldots, \Phi_s\}$ and $\{\Psi_1, \ldots, \Psi_t\}$ both to consist of the complete, lifted Y-templates determined by matrices I–$VIII$ and $X - XVI$. Similarly to the proof of Theorem 6.7.18, we see that every simple k-connected member of $\mathcal{GM}$ with at least $2k$ elements is a minor of a matroid represented by a matrix of the form given in the statement of the theorem. ■

6.9 The Quaternary Matroids Representable over All Sufficiently Large Fields

If q is a prime power, let $\mathcal{AF}_q$ be the class of matroids representable over all fields of size at least q, and let $\mathcal{SL}_q$ denote the class of GF(q)-representable matroids M for which there exists a prime power q' such that M is representable over all fields of size at least q'. The abbreviations $\mathcal{AF}$ and $\mathcal{SL}$ stand for "all fields" and "sufficiently large," respectively. Clearly, $\mathcal{AF}_q \subseteq \mathcal{SL}_q \subseteq \mathcal{AC}_q$. In this section, first we characterize the highly connected members of $\mathcal{AF}_4$ and $\mathcal{SL}_4$, subject to Hypotheses 3.2.2 and 3.2.3. Then we determine the extremal functions and extremal matroids for these classes. To do this, we will need a few additional lemmas to differentiate between $\mathcal{AC}_4$ and $\mathcal{SL}_4$.

Lemma 6.9.1. *There are infinitely many fields that do not contain a solution to the equation $x^2 - x - 1 = 0$.*

Proof. Let p be a prime other than 2 or 5. Solving $x^2 - x - 1 = 0$ in GF(p), we obtain $x = (1+\alpha)2^{-1}$, where $\alpha^2 = 5$ in GF(p). Thus, $x^2 - x - 1 = 0$ has a solution in GF(p) if and only if there is a solution to $x^2 \equiv 5 \pmod{p}$. A well-known result in number theory, known as quadratic reciprocity and first proved by Gauss [6], implies that $x^2 \equiv 5 \pmod{p}$ has a solution if and only if $x^2 \equiv p \pmod{5}$ has a solution. This is the case precisely when $p \equiv \pm 1 \pmod{5}$. Therefore, to prove the result, it suffices to show that there are infinitely many primes p such that $p \equiv 2 \pmod{5}$ or $p \equiv 3 \pmod{5}$. This follows from another well-known number theoretic result known as Dirichlet's theorem on arithmetic progressions [5], which implies that if a and b are coprime integers, then there are infinitely many primes p such that $p \equiv a \pmod{b}$. ■

Lemma 6.9.2. *Let P_0 be a column submatrix of either matrix I, matrix VI, or matrix VII, and let m be the number of rows of P_0. Either P_0 is a submatrix of one of matrices II–V or $VIII$–$XVII$ (up to field isomorphism and permuting of*

rows and columns), or $\widetilde{M}([I_m|D_m|P_0])$ is only representable over fields that contain a solution to the equation $x^2 - x - 1 = 0$.

Proof. To show that $\widetilde{M}([I_m|D_m|P_0])$ is only representable over fields that contain a solution to the equation $x^2 - x - 1 = 0$, the argument is similar to the proofs of Lemmas 6.7.1, 6.7.6, and 6.7.7 and uses the same functions, which are `complete_template_representation` and `zero_determinant_ideal`, found in Section A.9. Therefore, for each column submatrix P_0 of matrices I, VI, and VII, we either go through this process, or we show that P_0 is a column submatrix of one of matrices II–V or $VIII$–$XVII$. As in Section 6.4, whenever we permute the rows of P_0, we automatically scale each column so that the last nonzero entry is 1.

First, we consider matrix I. Label the columns of matrix I from left to right as a, b, c, d, e, f and the rows from top to bottom as $1, 2, 3, 4, 5$. If P_0 is a column submatrix of matrix I that contains neither c nor d, then P_0 is a submatrix of matrix IV. Also, if we swap the last two rows of $[a, b, c, e, f]$, we obtain $[b, a, d, e, f]$. Therefore, we may assume that P_0 contains d. Consider $P_0 = [a, d]$. We use the following code.

```
P0 = Matrix(GF4, [[1,a],
                  [a,a],
                  [a,1],
                  [1,0],
                  [0,1]])
(A4,Avar)=complete_template_representation(P0)
M = Matroid(A4)
E = M.groundset_list()
I = zero_determinant_ideal(M, Avar)
I
```

The ideal includes $z_3^2 - z_3 - 1$, meaning that in any representation of the matroid `M = Matroid(A4)` over a field $\mathbb{F}$, the entry corresponding to z_3 must be a solution to the equation $x^2 - x - 1 = 0$ in $\mathbb{F}$. Now, by swapping the last two rows of $[b, c]$, we obtain $[a, d]$. Therefore, we may assume that P_0 does not contain $[b, c]$. Thus, P_0 is a column submatrix of either $[b, d, e, f]$ or $[c, d, e, f]$. For $[c, d, e, f]$, put the rows in order $3, 2, 1, 4, 5$; the result is a submatrix of matrix IV. Thus, to complete the analysis of matrix I, it suffices to consider submatrices of $[b, d, e, f]$. For $[b, d, f]$, the ideal contains $z_5^2 + z_5 - 1$. Therefore, $(z_5^{-1})^2 - z_5^{-1} - 1 = 0$. For $[b, d, e]$, put the rows in order $3, 2, 1, 4, 5$. The result is $[d, b, f]$, which we just analyzed. The matrix $[b, e, f]$ is a submatrix of matrix IV. For $[d, e, f]$, put the rows in order $3, 2, 1, 4, 5$. The result is $[b, f, e]$, which we just analyzed.

Now we consider matrix VI. Label the columns of matrix VI from left to right as a, b, c, d, e and the rows as $1, 2, 3, 4$. If we reorder the rows as $3, 4, 1, 2$, then the columns a, b, c, d, e become c, e, a, d, b. Therefore, we may discard sets that contain column c but not column a. For the matrix $[a, b, c, d]$, the ideal includes $z_7^2 - z_7 - 1$. For $[a, b, c, e]$, the ideal contains $z_7^2 - z_7 - 1$. For $[a, b, d, e]$, the ideal

contains $z_7^2 - z_7 - 1$. For $[a, c, d, e]$, the ideal contains $z_7^2 - z_7 - 1$. Thus, every column submatrix of matrix VI with four columns corresponds to a template to which only conform matroids that are only representable over fields that contain a solution to the equation $x^2 - x - 1 = 0$. We now consider column submatrices of matrix VI with three columns. The matrix $[a, b, c]$ is a submatrix of matrix IX (after a field isomorphism). After swapping rows 2 and 3, $[a, b, d]$ is a submatrix of matrix I, which has already been analyzed. Matrix $XVII$ is $[a, b, e]$. After swapping rows 2 and 3, $[a, c, d]$ is a submatrix of matrix $VIII$. After putting the rows of $[a, c, e]$ in order $2, 1, 4, 3$ and performing a field isomorphism, the resulting matrix is a submatrix of matrix IX. For $[a, d, e]$, put the row in order $4, 3, 2, 1$. The resulting matrix is a submatrix of matrix III. We saw above that we need not consider submatrices that contain c but not a. Thus, the only remaining submatrix with three columns to check is $[b, d, e]$. Put the rows of $[b, d, e]$ in order $2, 1, 3, 4$. The result is a submatrix of matrix IX.

Now we consider matrix VII. Label the columns of matrix VII from left to right as a, b, c, d, e and the rows as 1.2.3.4. For $[a, b, c, d]$, the ideal contains $z_5^2 + z_7$ and $z_5 * z_7 + z_5 + z_7$. Combining these, we see that $z_5^2 + z_5 - 1 = 0$. Therefore, $(z_5^{-1})^2 - z_5^{-1} - 1 = 0$. For $[a, b, c, e]$, the ideal contains $z_7^2 - z_7 - 1$. For $[a, b, d, e]$, the ideal contains $z_7^2 - z_7 - 1$. For $[a, c, d, e]$, the ideal contains $z_5 * z_7 - z_5 + z_7$ and $z_7^2 + z_5$. Combining these, we see that $z_7^2 - z_7 - 1$. The matrix $[b, c, d, e]$ is matrix XVI, up to reordering the columns. Therefore, we now need only check submatrices that contain a and contain a total of three columns. For $[a, b, c]$, swap rows 2 and 3. The result is matrix $XVII$, up to reordering columns. For $[a, b, d]$, swap rows 2 and 3. The result is a submatrix of matrix IV. The matrix $[a, b, e]$ is a submatrix of matrix IX. By putting the rows of $[a, c, d]$ in order $2, 1, 4, 3$, we obtain a submatrix of matrix IX. By putting the rows of $[a, c, e]$ in order $2, 1, 4, 3$ and performing a field isomorphism, we obtain the matrix $XVII$. By putting the rows of $[a, d, e]$ in order $4, 2, 3, 1$, we obtain matrix $XVII$. This completes the proof. ■

We now characterize the highly connected members of $\mathcal{SL}_4$, subject to Hypotheses 3.2.2 and 3.2.3. The proofs of the next two theorems are essentially identical to each other. We give the proof of Theorem 6.9.3 but omit the proof of Theorem 6.9.4.

Theorem 6.9.3. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that, for every k-connected member M of $\mathcal{SL}_4$ with at least $2k$ elements, either M or M^* is a minor of the vector matroid of a matrix of the form below, where P_0 is one of matrices II–V or VIII–XVII listed in Definition 6.4.4, up to a field isomorphism.*

$$\begin{array}{|c|c|c|}\hline I_r & D_r & P_0 \\ \cline{3-3} & & 0 \\ \hline\end{array}$$

Theorem 6.9.4. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that every simple vertically k-connected member of $\mathcal{SL}_4$ with an $M(K_n)$-minor is a minor of the vector matroid of a matrix of the form above, and every cosimple*

cyclically k-connected member of $\mathcal{SL}_4$ with an $M^*(K_n)$-minor is a minor of the dual of the vector matroid of a matrix of the form above, where P_0 is one of matrices II–V or $VIII$–$XVII$ listed in Definition 6.4.4, up to a field isomorphism.

Proof of Theorem 6.9.3. We wish to find the templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ for $\mathcal{SL}_4$ whose existence is implied by Corollary 4.1.5. Since $\mathcal{SL}_4 \subseteq \mathcal{AC}_4$, it follows from combining Lemma 6.4.5 and Theorem 6.7.18 that we may take each of $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ to be the complete, lifted Y-template determined by some column submatrix P_0 of one of matrices I–XVI.

Consider the complete, lifted Y-templates determined by a matrix P_0, where P_0 is one of matrices II–V or $VIII$–$XVII$. Let m be the number of rows of P_0. By Lemmas 6.7.2–6.7.5, 6.7.8–6.7.17, $M = \widetilde{M}([I_m|D_m|P_0])$ is representable over $\mathbb{K}_2$, $\mathbb{U}_2$, or $\mathbb{P}_{\mathrm{Pap}}$. Therefore, by Corollary 6.5.13, $M \in \mathcal{SL}_4$. Lemma 6.6.1 then implies that $\mathcal{M}(\Phi_{P_0}) \subseteq \mathcal{SL}_4$.

However, if P_0 is matrix I, VI, or VII, Lemma 6.7.1, 6.7.6, or 6.7.7, respectively, combined with Lemma 6.9.1, implies that there are infinitely many fields $\mathbb{F}$ such that $\mathcal{M}(\Phi_{P_0})$ is not contained in the class of $\mathbb{F}$-representable matroids. Therefore, $\mathcal{M}(\Phi_{P_0})$ is not contained in $\mathcal{SL}_4$. Moreover, if P_0 is any column submatrix of matrix I, VI, or VII, then Lemma 6.9.2 and 6.9.1 imply that P_0 must be a submatrix of one of matrices II–V or IX–$XVII$.

Therefore, we may take $\{\Phi_1, \dots, \Phi_s\}$ and $\{\Psi_1, \dots, \Psi_t\}$ both to consist of the complete, lifted Y-templates determined by matrices II–V and $VIII$–$XVII$. Similarly to the proof of Theorem 6.7.18, we see that every simple k-connected member of $\mathcal{SL}_4$ with at least $2k$ elements is a minor of a matroid represented by a matrix of the form given in the statement of the theorem. ■

Now we characterize the highly connected members of $\mathcal{AF}_4$, subject to Hypotheses 3.2.2 and 3.2.3. The proofs of the next two theorems are essentially identical to each other. We give the proof of Theorem 6.9.5 but omit the proof of Theorem 6.9.6.

Theorem 6.9.5. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that, for every k-connected member M of $\mathcal{AF}_4$ with at least $2k$ elements, either M or M^* is a minor of the vector matroid of a matrix of the form below, where P_0 is one of the proper column submatrices of matrix IX that contains three columns, or P_0 is one of matrices II–V, $VIII$, or X–$XVII$ listed in Definition 6.4.4, up to a field isomorphism.*

$$\begin{array}{|c|c|c|}\hline \multirow{2}{*}{I_r} & \multirow{2}{*}{D_r} & P_0 \\ \cline{3-3} & & 0 \\ \hline \end{array}$$

Theorem 6.9.6. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that every simple vertically k-connected member of $\mathcal{AF}_4$ with an $M(K_n)$-minor is a minor of the vector matroid of a matrix of the form above, and every cosimple cyclically k-connected member of $\mathcal{AF}_4$ with an $M^*(K_n)$-minor is a minor of the dual of the vector matroid of a matrix of the form above, where P_0 is one of the*

proper column submatrices of matrix IX that contains three columns, or P_0 is one of matrices II–V, $VIII$, or X–$XVII$ listed in Definition 6.4.4, up to a field isomorphism.

Proof of Theorem 6.9.5. We wish to find the templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ for $\mathcal{AF}_4$ whose existence is implied by Corollary 4.1.5. Since $\mathcal{AF}_4 \subseteq \mathcal{SL}_4$, it follows from Theorem 6.9.3 that we may take each of $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ to be the complete, lifted Y-template determined by some column submatrix P_0 of one of matrices II–V or $VIII$–$XVII$.

Consider the complete, lifted Y-templates determined by a matrix P_0, where P_0 is one of matrices II–V, $VIII$, or X–$XVII$. Let m be the number of rows of P_0. By Lemmas 6.7.2–6.7.5, 6.7.8, and 6.7.10–6.7.17, $M = \widetilde{M}([I_m|D_m|P_0])$ is representable over $\mathbb{K}_2$ or $\mathbb{U}_2$. Therefore, by Corollary 6.5.13, $M \in \mathcal{AF}_4$. Lemma 6.6.1 then implies that $\mathcal{M}(\Phi_{P_0}) \subseteq \mathcal{AF}_4$.

However, if P_0 is matrix IX, Lemma 6.8.1 implies that $\mathcal{M}(\Phi_{P_0})$ is not contained in the class of GF(5)-representable matroids. Therefore, $\mathcal{M}(\Phi_{P_0})$ is not contained in $\mathcal{AF}_4$. However, if P_0 is any column submatrix of matrix IX, then P_0 must be a submatrix of either matrix VI or matrix VII, by Lemma 6.8.1. Thus, Theorem 6.8.2 and the fact that golden-mean matroids are representable over GF(5) imply that $\mathcal{M}(\Phi_{P_0})$ is contained in the class of GF(5)-representable matroids. The fact that P_0 is a submatrix of matrix IX implies that the members of $\mathcal{M}(\Phi_{P_0})$ are representable over all fields of size at least 7. Thus, they are representable over all fields of size at least 4, and $\mathcal{M}(\Phi_{P_0}) \subseteq \mathcal{AF}_4$.

Therefore, we may take $\{\Phi_1, \ldots, \Phi_s\}$ and $\{\Psi_1, \ldots, \Psi_t\}$ both to consist of the complete, lifted Y-templates determined by the column submatrices of matrix IX with three columns and those determined by matrices II–V and $VIII$, and X–$XVII$. Similarly to the proof of Theorem 6.7.18, we see that every simple k-connected member of $\mathcal{AF}_4$ with at least $2k$ elements is a minor of a matroid represented by a matrix of the form given in the statement of the theorem. ■

Before leaving this section, we determine the extremal functions and extremal matroids of $\mathcal{SL}_4$ and $\mathcal{AF}_4$. Recall the definitions of the matroids T_r^2, G_r, and HP_r from Definition 6.2.3. Also recall, from the discussion following Definition 6.2.3, that the T_r^2, G_r, and HP_r are the largest simple matroids of rank r virtually conforming to the templates $\Phi(T_r^2)$, $\Phi(G_r)$, and $\Phi(HP_r)$, respectively.

Theorem 6.9.7. *Suppose Hypothesis 3.2.3 holds. For all sufficiently large r, the extremal matroids of $\mathcal{SL}_4$ and $\mathcal{AF}_4$ are T_r^2 and G_r. Thus, we have*

$$h_{\mathcal{SL}_4}(r) \approx h_{\mathcal{AF}_4}(r) \approx \binom{r+3}{2} - 5.$$

Proof. Recall from Theorem 6.3.4 that the extremal matroids for $\mathcal{AC}_4$ are T_r^2, G_r, and HP_r, each of which have size $\binom{r+3}{2} - 5$. Since $\mathcal{AF}_4 \subseteq \mathcal{SL}_4 \subseteq \mathcal{AC}_4$, it suffices to show that for all r, we have $T_r^2 \in \mathcal{AF}_4$ and $G_r \in \mathcal{AF}_4$ and that for some r, we have $HP_r \notin \mathcal{SL}_4$.

Combining Remark 4.2.7, Lemma 4.2.11, and Lemma 4.2.15, we see that $\Phi(T_r^2)$, $\Phi(G_r)$, and $\Phi(HP_r)$ are, respectively, minor equivalent to the complete, lifted Y-templates determined by the following matrices.

$$\begin{bmatrix} \alpha^2 & \alpha \\ \alpha & \alpha^2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha^2 & \alpha^2 & \alpha^2 & \alpha \\ \alpha & 0 & 1 & 1 \\ 0 & \alpha & \alpha & \alpha^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \alpha & \alpha \\ 1 & \alpha \\ \alpha & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the first of these matrices, swap the first two rows and the result is a submatrix of matrix XVI. Combining Lemma 6.7.16 with Lemma 6.6.1, we see that $T_r^2 \in \mathcal{AF}_4$ for every r. For the second of the above matrices, scale each column so that its last nonzero entry is 1. Then perform a field isomorphism. The result is a submatrix of matrix XII. Combining Lemma 6.7.12 with Lemma 6.6.1, we see that $G_r \in \mathcal{AF}_4$ for every r. For the third matrix, swap the first two rows, and call the resulting matrix P_0. We saw in the proof of Lemma 6.9.2 that $M = \widetilde{M}([I_5|D_5|P_0])$ is only representable over fields that contain a solution to $x^2 - x - 1 = 0$. The minor equivalence of Φ_{P_0} and $\Phi(HP_r)$ implies that $HP_r \notin \mathcal{SL}_4$ for some r. ■

6.10 Summary

We end this chapter by proving Theorem 6.0.1 and making a few additional observations.

Proof of Theorem 6.0.1. Let k_1, k_2, k_3, and k_4 be the values for k given by Theorems 6.7.18, 6.8.2, 6.9.3, and 6.9.5, and let $k = \max\{k_1, k_2, k_3, k_4\}$. If $\mathcal{M}$ is a minor-closed class of matroids, let $\mathcal{M}(k)$ denote the set of k-connected members of $\mathcal{M}$ with at least $2k$ elements.

Combining Theorems 6.7.18, 6.8.2, and 6.9.3, we see that $\mathcal{AC}_4(k) = \mathcal{GM}(k) \cup \mathcal{SL}_4(k)$. Combining Theorem 6.9.3 with the lemmas in Section 6.7, we see that the members of $\mathcal{SL}_4(k)$ are representable over all fields of size at least 7. Therefore, a member of $\mathcal{SL}_4(k)$ is a member of $\mathcal{AF}_4(k)$ if and only if it is representable over GF(5), implying that it is a member of $\mathcal{GM}(k)$. Thus, $\mathcal{AC}_4(k)$ is the disjoint union $\mathcal{AF}_4(k) \cup (\mathcal{GM}(k) - \mathcal{AF}_4(k)) \cup (\mathcal{SL}_4(k) - \mathcal{AF}_4(k))$. This completes the proof. ■

By Theorem 6.0.1, if M is a large, highly connected member of $\mathcal{AC}_4$, then the set of fields of size at least 4 over which M is representable is one of exactly three sets. This is in contrast to the general case where there are infinitely many such sets. This is shown (for example) as follows. Oxley, Vertigan, and Whittle [25] showed that, for all r, the rank-r free spike is representable over all finite fields of non-prime order. Therefore, the rank-r free spike is a member of $\mathcal{AC}_4$. However, Geelen, Oxley, Vertigan, and Whittle [12] showed that the rank-r free spike is not representable over GF(p) for all primes $p \leq r + 1$.

Recall the definition of semi-strong equivalence found in Definition 4.2.4. It follows easily that semi-strongly equivalent Y-templates are minor equivalent. Therefore, in Theorems 6.7.18, 6.7.19, 6.8.2, 6.8.3, 6.9.3, 6.9.4, 6.9.5, and 6.9.6, we may

replace each of the templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ with a template that is semi-strongly equivalent to it. Lemma 4.2.15 states that a complete, lifted Y-template determined by a matrix P_0, the sum of whose rows is the zero vector, is semi-strongly equivalent to the template determined by the matrix obtained by P_0 by removing a row. Therefore, if $I', II', \ldots, XVII'$ are the matrices obtained from $I, II, \ldots, XVII$ by removing the first row, then the following results are immediate; the conciseness of having matrices with fewer rows is perhaps preferable in many situations.

Corollary 6.10.1. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that, for every k-connected member M of $\mathcal{AC}_4$ with at least $2k$ elements, either M or M^* is a minor of the vector matroid of a matrix of the form below, where P_0 is one of matrices I'–XVI', up to a field isomorphism.*

$$\begin{array}{|c|c|c|}\hline \multirow{2}{*}{I_r} & \multirow{2}{*}{D_r} & P_0 \\ \cline{3-3} & & 0 \\ \hline \end{array}$$

Corollary 6.10.2. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that every simple vertically k-connected member of $\mathcal{AC}_4$ with an $M(K_n)$-minor is a minor of the vector matroid of a matrix of the form given in Corollary 6.10.1, and every cosimple cyclically k-connected member of $\mathcal{AC}_4$ with an $M^*(K_n)$-minor is a minor of the dual of the vector matroid of a matrix of the form given in Corollary 6.10.1, where P_0 is one of matrices I'–XVI', up to a field isomorphism.*

Corollary 6.10.3. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that, for every k-connected golden-mean matroid M with at least $2k$ elements, either M or M^* is a minor of the vector matroid of a matrix of the form given in Corollary 6.10.1, where P_0 is one of matrices I'–$VIII'$ or X'–XVI', up to a field isomorphism.*

Corollary 6.10.4. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that every simple vertically k-connected golden-mean matroid with an $M(K_n)$-minor is a minor of the vector matroid of a matrix of the form given in Corollary 6.10.1, and every cosimple cyclically k-connected golden-mean matroid with an $M^*(K_n)$-minor is a minor of the dual of the vector matroid of a matrix of the form given in Corollary 6.10.1, where P_0 is one of matrices I'–$VIII'$ or X'–XVI', up to a field isomorphism.*

Corollary 6.10.5. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that, for every k-connected member M of $\mathcal{SL}_4$ with at least $2k$ elements, either M or M^* is a minor of the vector matroid of a matrix of the form given in Corollary 6.10.1, where P_0 is one of matrices II'–V' or $VIII'$–$XVII'$, up to a field isomorphism.*

Corollary 6.10.6. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that every simple vertically k-connected member of $\mathcal{SL}_4$ with an $M(K_n)$-minor is a minor of the vector matroid of a matrix of the form given in Corollary 6.10.1, and every cosimple cyclically k-connected member of $\mathcal{SL}_4$ with an $M^*(K_n)$-*

minor is a minor of the dual of the vector matroid of a matrix of the form given in Corollary 6.10.1, where P_0 is one of matrices II'–V' or $VIII'$–$XVII'$, up to a field isomorphism.

Corollary 6.10.7. *Suppose Hypothesis 3.2.2 holds. Then there exists $k \in \mathbb{Z}_+$ such that, for every k-connected member M of $\mathcal{AF}_4$ with at least $2k$ elements, either M or M^* is a minor of the vector matroid of a matrix of the form given in Corollary 6.10.1, where P_0 is one of the proper column submatrices of matrix IX' that contains three columns, or P_0 is one of matrices II'–V', $VIII'$, or X'–$XVII'$, up to a field isomorphism.*

Corollary 6.10.8. *Suppose Hypothesis 3.2.3 holds. Then there exist $k, n \in \mathbb{Z}_+$ such that every simple vertically k-connected member of $\mathcal{AF}_4$ with an $M(K_n)$-minor is a minor of the vector matroid of a matrix of the form given in Corollary 6.10.1, and every cosimple cyclically k-connected member of $\mathcal{AF}_4$ with an $M^*(K_n)$-minor is a minor of the dual of the vector matroid of a matrix of the form given in Corollary 6.10.1, where P_0 is one of the proper column submatrices of matrix IX that contains three columns, or P_0 is one of matrices II'–V', $VIII'$, or X'–$XVII'$, up to a field isomorphism.*

The results in this chapter lead to excluded-minor results of a similar nature to those of Theorems 5.1.1, 5.1.2, and 5.1.3. Two of these results follow easily from the fact that there was a fairly small number of excluded minors for $\mathcal{AC}_4$ used to establish the results in this chapter (recall Definition 6.1.3). We also need to recall the fact that V_2 and P_2 are self-dual.

Theorem 6.10.9. *Suppose Hypothesis 3.2.2 holds. There exists $k \in \mathbb{Z}_+$ such that a k-connected quaternary matroid with at least $2k$ elements is contained in $\mathcal{AC}_4$ if and only if it contains none of the following matroids as minors: F_7, F_7^*, V_1, V_1^*, V_2, V_3, V_3^*, P_1, P_1^*, P_2, P_3, P_3^*.*

Theorem 6.10.10. *Suppose Hypothesis 3.2.2 holds. There exists $k \in \mathbb{Z}_+$ such that a k-connected quaternary matroid with at least $2k$ elements is golden-mean if and only if it contains none of the following matroids as minors: F_7, F_7^*, V_1, V_1^*, V_2, V_3, V_3^*, P_1, P_1^*, P_2, P_3, P_3^*, the Pappus matroid, the dual of the Pappus matroid.*

Analogous results for $\mathcal{AF}_4$ and $\mathcal{SL}_4$ should be fairly straightforward but require additional case analysis that has not yet been done.

As a final remark, we note that the words "up to a field isomorphism" in the corollaries in this section (and the theorems in the previous sections on which they are based) can be dropped if we are content to deal with abstract matroids rather than represented matroids.

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Appendix A: SageMath Code

A.1 Code for Section 2.2

```
from itertools import combinations
from sage.matroids.advanced import *
from itertools import product

# The complete ternary Dowling geometry:
def DowlingGeometry(r):
    A = Matrix(GF(3), r, r*r)
    i = 0
    for j in range(r):
        A[j,i] = 1
        i += 1
    for j in range(r-1):
        for k in range(j+1,r):
            A[j,i] = -1
            A[k,i] = 1
            i += 1
            A[j,i] = -1
            A[k,i] = -1
            i += 1
    return Matroid(A)

def vname(path, ray, index):
    return path + str(ray) + '_' + str(index)
def ename(path, ray, index):
    return path + str(ray) + '_' + str(index)

def perturbed_gadget_matroid(perturb_matrix, gadgets_per_ray,
                             num_rays=1):
    """
    Create the matrix representing "rays" of gadgets joined by graph
    edges. In the case where perturb_matrix=[0 0 0 0], num_rays = 4,
    and gadgets_per_ray=[5,2,2,2], the result is M(J''') from
    Figure 5.

    Edges are encoded <path><ray>_<position>. So element 'p2_1' is the
    second edge on path 'p' from the third ray (counts start at 0).
    The other paths are q and r.

    gadgets_per_ray is a vector if num_rays > 1
```

```
The gadget elements are labeled <label><ray>_<position>. The
labels are d,e,f,g. The ‘‘root’’ gadget is ‘d0_0, e0_0, f0_0,
g0_0’. In Figure 5, the root gadget is labeled F4.
"""
gadget_block = Matrix(GF(3), [[0,1,0,1],
                              [0,1,1,0],
                              [0,0,1,1],
                              [1,1,1,1]])  # matrix from Lemma
                                           # 2.2.1 with the first
                                           # six columns removed
if num_rays == 1:
    gadgets_per_ray = [gadgets_per_ray]
num_rows = perturb_matrix.nrows() +
           sum(4 * g for g in gadgets_per_ray) -
           4 * (num_rays - 1) # Count root gadget only once.
num_cols = sum(7 * (g - 1) for g in gadgets_per_ray) + 4
            # Each gadget other than the root gadget accounts for
            # seven elements -- the four elements of the gadget
            # and the three edges joining the gadget to the next
            # gadget. The root gadget gives four additional
            # elements.
A = Matrix(GF(3), num_rows, num_cols)
groundset = []
vtx = {’p0_0’: 0, ’q0_0’: 1, ’r0_0’: 2}  # Maps path vertices to
                                         # matrix rows
imax = 3  # first unused row
for ray in range(num_rays):
    vtx[vname(’p’,ray,0)] = vtx[’p0_0’]
    vtx[vname(’q’,ray,0)] = vtx[’q0_0’]
    vtx[vname(’r’,ray,0)] = vtx[’r0_0’]
    for i in range(1, gadgets_per_ray[ray]):
        vtx[vname(’p’,ray,i)] = imax
        vtx[vname(’q’,ray,i)] = imax + 1
        vtx[vname(’r’,ray,i)] = imax + 2
        imax += 3
j = 0  # first unfilled column
# Create the paths
for ray in range(num_rays):
    for i in range(gadgets_per_ray[ray] - 1):
        # first path
        A[vtx[vname(’p’,ray,i)],j] = -1
        A[vtx[vname(’p’,ray,i+1)],j] = 1
        j += 1
        groundset.append(ename(’p’,ray,i))
```

```
            # second path
            A[vtx[vname('q',ray,i)],j] = -1
            A[vtx[vname('q',ray,i+1)],j] = 1
            j += 1
            groundset.append(ename('q',ray,i))
            # third path
            A[vtx[vname('r',ray,i)],j] = -1
            A[vtx[vname('r',ray,i+1)],j] = 1
            j += 1
            groundset.append(ename('r',ray,i))
    first_gadget = 0  # "root" gadget only gets added once
    for ray in range(num_rays):
        for k in range(first_gadget, gadgets_per_ray[ray]):
            first_gadget = 1 # "root" gadget only gets added once
            # glue on the gadgets
            elts = 'defg'
            for l in range(4):
                A[vtx[vname('p',ray,k)], j + l] = gadget_block[0, l]
                A[vtx[vname('q',ray,k)], j + l] = gadget_block[1, l]
                A[vtx[vname('r',ray,k)], j + l] = gadget_block[2, l]
                A[imax, j + l] = gadget_block[3, l]
            groundset.extend([ename(e,ray,k) for e in elts])
            imax += 1
            # put in the perturbation
            A.set_block(num_rows - perturb_matrix.nrows(), j,
                        perturb_matrix)
            j += 4
    return A, groundset

def check_perturbation(P):
    # Now we add a perturbation:
    A, gs = perturbed_gadget_matroid(P, [5,2,2,2], 4)
    M = Matroid(gs, A)
    contract_set = []
    # create negative loops on vertices of root gadget:
    contract_set.extend(['e1_1', 'f1_1', 'g1_1', 'e2_1', 'f2_1',
                         'g2_1', 'e3_1', 'f3_1', 'g3_1', 'p1_0',
                         'q2_0', 'r3_0'])
    # contract one element from each gadget on the main ray, except
    # the last:
    contract_set.extend(['d0_1', 'e0_2', 'f0_3', 'g0_4'])
    # contract the edges of the main ray:
    contract_set.extend([ename('p',0,i) for i in range(4)])
    contract_set.extend([ename('q',0,i) for i in range(4)])
```

```
    contract_set.extend([ename('r',0,i) for i in range(4)])
    N = (M / contract_set).simplify()
    return DowlingGeometry(N.rank()).has_minor(N) # We check if N is
                                                  # signed-graphic.

# We need to construct a list of all possible perturbations.
# We begin the list with the rank-0 perturbation.
perturbation_list=[Matrix(GF(3), [[0,0,0,0]])]

# Now we add all possible rank-1 perturbations.
bases_already_checked=[]
X=set([0,1,2,3])
for B in Subsets(X,1):
    nonB=X.difference(B)
    for T in product(range(3), repeat=3): # There are 3 entries
                                          # outside the identity
                                          # matrix.
        A=Matrix(GF(3), 1, 4)
        i=0
        for b in B:  # We make an identity matrix with columns
                     # labeled by the basis B.
            A[i,b]=1
            i=i+1
        A[0,list(nonB)[0]]=T[0]
        A[0,list(nonB)[1]]=T[1]
        A[0,list(nonB)[2]]=T[2]
        if all((A.matrix_from_columns(S)).det()==0 for S in
               bases_already_checked):
            # We only need to add a matrix to the list of
            # perturbations if it contains no basis for which we
            # already found all possible perturbations. This is
            # because row operations allow us to transform the
            # submatrix indexed by that basis into an identity matrix.
            perturbation_list.append(A)
    bases_already_checked.append(B)

# Now we add all possible rank-2 perturbations.
bases_already_checked=[]
X=set([0,1,2,3])
for B in Subsets(X,2):
    nonB=X.difference(B)
    for T in product(range(3), repeat=4): # There are 4 entries
                                          # outside the identity
                                          # matrix.
```

```
        A=Matrix(GF(3), 2, 4)
        i=0
        for b in B: # We make an identity matrix with columns
                    # labeled by the basis B.
            A[i,b]=1
            i=i+1
        A[0,list(nonB)[0]]=T[0]
        A[1,list(nonB)[0]]=T[1]
        A[0,list(nonB)[1]]=T[2]
        A[1,list(nonB)[1]]=T[3]
        if all((A.matrix_from_columns(S)).det()==0 for S in
               bases_already_checked):
            # We only need to add a matrix to the list of
            # perturbations if it contains no basis for which we
            # already found all possible perturbations. This is
            # because row operations allow us to transform the
            # submatrix indexed by that basis into an identity matrix.
            perturbation_list.append(A)
    bases_already_checked.append(B)

# Now we add all possible rank-3 perturbations.
bases_already_checked=[]
X=set([0,1,2,3])
for B in Subsets(X,3):
    nonB=X.difference(B)
    for T in product(range(3), repeat=3): # There are 3 entries
                                          # outside the identity
                                          # matrix.
        A=Matrix(GF(3), 3, 4)
        i=0
        for b in B: # We make an identity matrix with columns
                    # labeled by the basis B.
            A[i,b]=1
            i=i+1
        A[0,list(nonB)[0]]=T[0]
        A[1,list(nonB)[0]]=T[1]
        A[2,list(nonB)[0]]=T[2]
        if all((A.matrix_from_columns(S)).det()==0 for S in
               bases_already_checked):
            # We only need to add a matrix to the list of
            # perturbations if it contains no basis for which we
            # already found all possible perturbations. This is
            # because row operations allow us to transform the
            # submatrix indexed by that basis into an identity matrix.
```

```
            perturbation_list.append(A)
    bases_already_checked.append(B)

# Now we add the only possible rank-4 perturbation.
perturbation_list.append(matrix.identity(GF(3),4))

# We check all possible perturbations to see if any are
# signed-graphic matroids.
if any(check_perturbation(perturbation_list[t]) != False
       for t in range(len(perturbation_list))):
    print "signed-graphic matroid"
else:
    print "No perturbation is signed-graphic."
```

The CoCalc online interface returned the following:

```
No perturbation is signed-graphic.
```

A.2 Code for Section 5.2

The following functions, respectively, test whether a binary matroid is even-cycle, even-cut, an excluded minor for even-cycle matroids, or an excluded minor for even-cut matroids.

```
from sage.matroids.advanced import *
def is_even_cycle(M, certificate=False):
    """
    Check if matroid M is an even-cycle matroid. If certificate=True,
    also return the matroid N such that M = N \ e, and N / e is
    graphic.
    """
    if not isinstance(M, BinaryMatroid):
        raise TypeError("This function only works on binary matroids")
    e = newlabel(M.groundset())
    for N in M.linear_extensions(e):
        if (N / e).is_graphic():
            if certificate:
                return (True, N, e)
            else:
                return True
    return False

def is_even_cut(M, certificate=False):
    """
    Check if matroid M is an even-cut matroid. If certificate=True,
    also return the matroid N such that M = N \ e, and N / e is
```

```
    cographic.
    """
    if not isinstance(M, BinaryMatroid):
        raise TypeError("This function only works on binary matroids")
    e = newlabel(M.groundset())
    for N in M.linear_extensions(e):
        if (N / e).dual().is_graphic():
            if certificate:
                return (True, N, e)
            else:
                return True
    return False

def is_even_cycle_excluded_minor(M):
    """
    Check whether M is an excluded minor for the class of even-cycle
    matroids.
    """
    if is_even_cycle(M):
        return False
    for e in M.groundset():
        if not is_even_cycle(M \ e) or not is_even_cycle(M / e):
            # not minimal
            return False
    return True

def is_even_cut_excluded_minor(M):
    """
    Check whether M is an excluded minor for the class of even-cycle
    matroids.
    """
    if is_even_cut(M):
        return False
    for e in M.groundset():
        if not is_even_cut(M \ e) or not is_even_cut(M / e):
            # not minimal
            return False
    return True
```

The following code defines the matroids L_{19}, L_{11}, $M^*(K_6)$, and H_{12}, It also shows that L_{19} and L_{11} are excluded minors for the class of even-cycle matroids and that $M^*(K_6)$, and H_{12}^* are excluded minors for the class of even-cut matroids. Each of these tests returned `True`.

```
A = Matrix(GF(2), [[1,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0],
```

```
                    [0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0],
                    [0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0],
                    [0,0,0,1,0,0,0,0,0,0,0,0,0,1,0,0,0,1,0],
                    [0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,1],
                    [0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,1,0,0,0],
                    [0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0],
                    [0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,1,0],
                    [0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1],
                    [0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0],
                    [0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,1,0],
                    [0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1,0,0,1],
                    [0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1,1,0]])
L19=Matroid(field=GF(2), matrix=A)
is_even_cycle_excluded_minor(L19)

A = Matrix(GF(2), [[1,0,0,0,0,0,1,0,1,0,1],
                    [0,1,0,0,0,0,1,0,0,1,1],
                    [0,0,1,0,0,0,0,1,1,1,0],
                    [0,0,0,1,0,0,0,1,1,0,1],
                    [0,0,0,0,1,0,0,1,0,1,1],
                    [0,0,0,0,0,1,0,0,1,1,1]])
L11=Matroid(field=GF(2), matrix=A)
is_even_cycle_excluded_minor(L11)

A = Matrix(GF(2), [[1,0,0,0,0,1,1,1,1,0,0,0,0,0,0],
                    [0,1,0,0,0,1,0,0,0,1,1,1,0,0,0],
                    [0,0,1,0,0,0,1,0,0,1,0,0,1,1,0],
                    [0,0,0,1,0,0,0,1,0,0,1,0,1,0,1],
                    [0,0,0,0,1,0,0,0,1,0,0,1,0,1,1]])
MK6=Matroid(field=GF(2), matrix=A)
is_even_cut_excluded_minor(MK6)

A = Matrix(GF(2), [[1,0,1,0,1,0,1,0,1,0,1,0],
                    [0,0,0,0,1,1,1,1,0,0,0,0],
                    [0,0,0,0,1,1,0,0,1,1,0,0],
                    [1,1,0,0,0,0,0,0,0,0,1,1],
                    [0,0,1,1,0,0,1,1,0,0,1,1]])
H12=Matroid(field=GF(2), matrix=A)
is_even_cut_excluded_minor(H12.dual())
```

A.3 Code for Section 5.3

```
def complete_Y_template_matrix(P0):
    k=P0.nrows()
```

```
    num_elts=k+k*(k-1)/2+P0.ncols()
    F=P0.base_ring()
    A = Matrix(F, k, num_elts)
    i = 0
    # identity in front
    for j in range(k):
        A[j,j] = 1
    i = k
    # all pairs
    for S in Subsets(range(k),2):
        A[S[0],i]=1
        A[S[1],i]=-1
        i = i + 1
    # Columns from Y0
    for l in range(P0.ncols()):
        for j in range(k):
            A[j, i] = P0[j, l]
        i=i+1
    return A
```

```
A = Matrix(GF(2), [[1,0,0,0,0,0,1,0,1,0,1],
                   [0,1,0,0,0,0,1,0,0,1,1],
                   [0,0,1,0,0,0,0,1,1,1,0],
                   [0,0,0,1,0,0,0,1,1,0,1],
                   [0,0,0,0,1,0,0,1,0,1,1],
                   [0,0,0,0,0,1,0,0,1,1,1]])
L11=Matroid(field=GF(2), matrix=A)
```

A.4 Code for Section 5.5

```
def matrix_from_template(r, Y1I, Y1A, Y1B, Y0A, Y0B, B_Jrows):
    # Each of Y1A, Y1B, Y0A, and Y0B either must have the same
    # number of rows as Y1I or must be an empty matrix."
    print "Template with C empty, Lambda trivial,Y1 consisting"
    print "of matrices I, Y1A, Y1B, with " , B_Jrows, " all-one"
    print "rows below B and then all-zero rows, Y0 consisting"
    print "of matrices A, B with ", B_Jrows, " all-one rows"
    print "below B and then all-zero rows."
    c = Y1I.nrows()
    A = Matrix(GF(2), r + c, binomial(r+1,2)+(r+1)*Y1I.ncols()+
    (r+1)*Y1A.ncols()+(r+1)*Y1B.ncols()+Y0A.ncols()+Y0B.ncols())
    i = 0
    for j in range(c,r+c):
```

```
    A[j,i] = 1
    i = i + 1
for j in range(c, r-1 + c):
    for k in range(j+1,r+c):
        A[j,i] = 1
        A[k,i] = 1
        i = i + 1
for j in range(Y1I.ncols()):
    for k in range(c):
        A[k,i] = Y1I[k,j]
    i = i + 1
    for l in range(r):
        for k in range(c):
            A[k,i] = Y1I[k,j]
        A[c+l, i] = 1
        i = i + 1
for j in range(Y1A.ncols()):
    for k in range(c):
        A[k,i] = Y1A[k,j]
    i = i + 1
    for l in range(r):
        for k in range(c):
            A[k,i] = Y1A[k,j]
        A[c+l, i] = 1
        i = i + 1
for j in range(Y1B.ncols()):
    for k in range(c):
        A[k,i] = Y1B[k,j]
    for k in range(B_Jrows):
        A[c+k,i] = A[c+k,i] + 1
    i = i + 1
    for l in range(r):
        for k in range(c):
            A[k,i] = Y1B[k,j]
        A[c+l, i] = 1
        for k in range(B_Jrows):
            A[c+k,i] = A[c+k,i] + 1
        i = i + 1
for j in range(Y0A.ncols()):
    for k in range(c):
        A[k,i] = Y0A[k,j]
    i = i + 1
for j in range(Y0B.ncols()):
    for k in range(c):
```

```
            A[k,i] = Y0B[k,j]
        for k in range(c, c+B_Jrows):
            A[k,i] = 1
        i = i + 1
    return A
```

Below, we give an example of the use of this function.

```
def print_matrix(A):
    s = ""
    for i in range(A.nrows()):
        s += "["
        for j in range(A.ncols()):
            s += str(A[i,j])
        s += "]\n"
    print s

Y1I = identity_matrix(GF(2),3)
Y1A = Matrix(GF(2), [[1,1],
                     [1,1],
                     [0,1]])
Y1B = Matrix(GF(2), [[1,1],
                     [0,0],
                     [0,1]])
Y0A = Matrix(GF(2), [[1,1,1,1],
                     [0,1,0,1],
                     [1,0,0,1]])
Y0B = Matrix(GF(2), [[1,1,1,0],
                     [0,1,0,1],
                     [1,0,0,1]])
A=matrix_from_template(4, Y1I, Y1A, Y1B, Y0A, Y0B, 2)
print_matrix(A)
```

The code above returned the following.

```
Template with C empty, Lambda trivial,Y1 consisting
of matrices I, Y1A, Y1B, with  2  all-one
rows below B and then all-zero rows, Y0 consisting
of matrices A, B with  2  all-one rows
below B and then all-zero rows.
[0000000001111100000000000111111111111111111111111110]
[0000000000000011111000001111111111000000000000001010101]
[0000000000000000000111110000011111000001111110011001]
[1000111000010000100001000010000100010111101110000 1111]
[0100100110001000010000100001000010011011110110000 1111]
[0010010101000100001000010000100001000010000100000000 0]
```

[00010010110000100001000010000100001000010000100000000]

A.5 Code for Section 6.1

By Theorem 6.1.1, if $\mathcal{K}(M)$ is the characteristic set of a matroid M, then either $0 \in \mathcal{K}(M)$ and $\mathcal{P} - \mathcal{K}(M)$ is finite, or $0 \notin \mathcal{K}(M)$ and $\mathcal{K}(M)$ is finite. The function `CharacteristicSet` below takes a matroid M as input. If $0 \notin \mathcal{K}(M)$ and $\mathcal{K}(M) = \{p_1, p_2 \ldots, p_n\}$, then the output is $[p_1, p_2, \ldots, p_n]$. If $0 \in \mathcal{K}(M)$ and $\mathcal{P} - \mathcal{K}(M) = \{p_1, p_2 \ldots, p_n\}$, then the output is $[0, -p_1, -p_2, \ldots, -p_n]$. Thus, if $M \in \mathcal{AC}_4$, then `CharacteristicSet(M)` is `[0]`.

```
def CharacteristicSet(M, InputBasis = []):
    if len(InputBasis)==0:
        InputBasis=list(M.basis())
    Basis=set(InputBasis)
    CoBasis=M.groundset().difference(Basis)
    CB=list(CoBasis)
    B=list(Basis)
    r=M.rank()
    cr=M.corank()
    FundCircs=[]
    numedges=0
    for i in range(cr):
        Circ=M.fundamental_circuit(Basis,CB[i])
        FundCircs.append(Circ)
        numedges=numedges+len(Circ)-1
        # This is determining the number of edges in the fundamental
        # graph of M. Each vertex v representing an element
        # e in the cobasis is incident with each vertex representing
        # an element in the basis and also in thefundamental
        # circuit of e. Thus, the number of edges incident with v
        # is the size of the fundamental circuit minus 1.
    numvars=numedges-(r+cr-len(M.components()))
    # M.components() returns an iterable containing the components
    # of the matroid. A component is an inclusionwise maximal
    # connected subset of the matroid. A subset is connected if the
    # matroid resulting from deleting the complement of that subset
    # is connected. Oxley (Theorem 6.4.7) shows that
    # r+cr-len(M.components()) is the number of nonzero entries in
    # the cobasis that can be scaled to be 1. The other nonzero
    # entries require variables.
    Vars=[]
    for i in range(numvars):
        Vars.append('x'+str(i))
    # Vars is [x0,x1,x2,...]
```

```
R=PolynomialRing(ZZ,Vars,order='degrevlex')
D=matrix.zero(R,r,cr)
# D will be the matrix such that [I|D] is a partial representation
# of M.
for i in range(r):
    for j in range(cr):
        if B[i] in FundCircs[j]:
            D[i,j]=1
G=BipartiteGraph(D)
# G is the fundamental graph of M associated with Basis.
T=[]
for C in G.connected_components_subgraphs():
    T.extend(C.random_spanning_tree())
    # The random_spanning_tree function only works for connected
    # graphs, so it needs to be run for each component.
# Next, we assign variables (from Vars) to nonzero entries in
# the matrix (other than the ones corresponding to edges in the
# spanning trees.)
varcount=0
for i in range(r):
    for j in range(cr):
        if D[i,j]==1:
            if not ((j,i+cr) in T or (i+cr,j) in T):
                D[i,j]=Vars[varcount]
                varcount=varcount+1
Zero=[]
NonZero=list(R.gens())
# Next, find all square submatrices.
for numrows in range(2,min(r,cr)+1):
    for RowSet in Subsets(range(r),numrows):
        for ColSet in Subsets(range(cr),numrows):
            Candidate=Basis
            # For each submatrix, we will append the basis columns
            # with nonzero entries in the other rows.
            for x in RowSet:
                Candidate=Candidate.difference({B[x]})
                # We don't need the other basis columns.
            for y in ColSet:
                Candidate=Candidate.union({CB[y]})
                # We need to append the cobasis columns of
                # the submatrix.
            Det=D[list(RowSet),list(ColSet)].determinant()
            if M.rank(Candidate)==r: # Candidate is a basis of M.
                if Det!=1 and Det!=-1:
```

```
                # If a subdeterminant is 1 or -1, that's what
                # it will be regardless of what field the matrix
                # is viewed over. There's no need to consider
                # them.
                    if not Det in NonZero:
                        if not -Det in NonZero:
                            NonZero.append(Det)
            else: # Candidate is not a basis of M.
                if not Det in Zero:
                    if not -Det in Zero:
                        Zero.append(Det)
for i in range(len(NonZero)):
    Vars.append('y'+str(i))
S=PolynomialRing(ZZ,Vars,order='degrevlex')
Dummies=[]
for i in range(len(Vars)):
    if i>=numvars:
        Dummies.append(S.gen(i)) # Dummies consists of [y0,y1,...].
NewPolys=Zero
for i in range(len(NonZero)):
    NewPolys.append(NonZero[i]*Dummies[i]-1)
    # y0, y1, etc. are inverses of the nonzero polynomials.
Grob1=S.ideal(NewPolys).groebner_basis('macaulay2:gb')
if 1 in Grob1:
    Characteristic=[]  # One direction of Oxley 6.8.10.
else: # The rest is based on Theorem 3.5 of Baines and Vs.
    k=0
    for g in Grob1:
        if g.degree()==0:
            k=g.constant_coefficient()
    if k>0:
        Characteristic=k.prime_divisors()
    else:
        LCs=[]
        for g in Grob1:
            LCs.append(g.lc())
        gamma=lcm(LCs)
        if gamma==1:
            Characteristic=[0]
        else:
            NewGenerators=[]
            for g in Grob1:
                NewGenerators.append(g)
            NewGenerators.append(gamma)
```

```
                Grob2=S.ideal(NewGenerators).groebner_basis('macaulay2:gb')
                k=0
                for g in Grob2:
                    if g.degree()==0:
                        k=g.constant_coefficient()
                if k==1:
                    Characteristic=[0]
                    for p in gamma.prime_divisors():
                        Characteristic.append(-p)
                elif k==gamma:
                    Characteristic=[0]
                else:
                    Characteristic=[0]
                    for p in gamma.prime_divisors():
                        if not p.divides(k):
                            Characteristic.append(-p)
    return Characteristic
```

To determine the characteristic sets of V_1, V_2, and V_3, we ran the following code.

```
GF4 = GF(4, 'a')
a = GF4.gens()[0]

A = Matrix(GF4, [[1,0,1,1,1,1],
                 [1,1,a+1,a+1,a,0],
                 [0,1,1,a,1,a]])
V1=Matroid(field=GF4, reduced_matrix=A)

A = Matrix(GF4, [[1,1,1,0],
                 [a+1,a,a,1],
                 [a,a+1,a,1],
                 [a+1,a+1,a,a]])
V2=Matroid(field=GF4, reduced_matrix=A)

A = Matrix(GF4, [[1,0,1,1,1],
                 [a,1,a+1,a+1,1],
                 [a,1,a,1,a+1],
                 [0,1,1,0,0]])
V3=Matroid(field=GF4, reduced_matrix=A)

CharacteristicSet(V1)
CharacteristicSet(V2)
CharacteristicSet(V3)
```

This code returned `[2]` for each of these three matroids.

To determine the characteristic sets of P_1, P_2, and P_3, we ran the following code.

```
GF4 = GF(4, 'a')
a = GF4.gens()[0]

A = Matrix(GF4, [[0,1,1,1,1,1,0,0,0],
                 [0,1,a,0,0,0,1,1,1],
                 [1,0,0,1,a,a+1,1,a,a+1]])
P1=Matroid(field=GF4, matrix=A)

A = Matrix(GF4, [[1,0,1,1],
                 [a,1,1,1],
                 [1,1,0,1],
                 [a+1,1,a+1,0]])
P2=Matroid(field=GF4, reduced_matrix=A)

A = Matrix(GF4, [[0,1,1,1,1,1,1,0,0],
                 [0,1,a,a+1,0,0,0,1,1],
                 [1,0,0,0,1,a,a+1,1,a]])
P3=Matroid(field=GF4, matrix=A)

CharacteristicSet(P1)
CharacteristicSet(P2)
CharacteristicSet(P3)
```

This code returned [0,-3] for each of these three matroids.

A.6 Code for Section 6.2

The next function builds the largest matrix of rank r virtually conforming to the Y-template $\mathrm{YT}(AY0, AY1)$.

```
def Y_template_matrix(r, AY0, AY1):
    """
    Generate the largest matrix of rank r conforming to the
    relevant template.
    """
    if AY0.base_ring()==AY1.base_ring():
        F=AY0.base_ring()
    else:
        raise ValueError("AY0 and AY1 must be matrices over "
                         "the same field.")
    if AY1.nrows()!=0:
        k=AY1.nrows()
        if AY1.nrows()!=AY0.nrows():
            if AY0.nrows()==0:
                pass
            else:
```

```
            raise ValueError("AY0 and AY1 must have the "
                             "same number of rows or one "
                             "of them must be empty.")
    if AY1.nrows()==0:
        k=AY0.nrows()
    n=r-k
    num_elts=n*(n+1)/2+(n+1)*k+(n+1)*AY1.ncols()+AY0.ncols()
    A = Matrix(F, r, num_elts)
    i = 0
    # identity in front
    for j in range(n):
        A[j+k,j] = 1
    i = n
    # all pairs
    for S in Subsets(range(k,r),2):
        A[S[0],i] = 1
        A[S[1],i] = -1
        i = i + 1
    # Identity columns from Y1 all-zero below it
    for j in range(k):
        A[j,i]=1
        i=i+1
    # Identity columns in Y1 with identity columns below
    for j in range (k):
        for m in range(k,r):
            A[j,i]=1
            A[m,i]=1
            i=i+1
    # Rest of Y1
    for l in range(AY1.ncols()):
        for j in range(k):
            A[j,i] = AY1[j,l]
        i = i + 1
        for m in range(n):
            A[k+m, i] = 1
            for j in range(k):
                A[j,i] = AY1[j,l]
            i = i + 1

    # Columns from Y0
    for l in range(AY0.ncols()):
        for j in range(k):
            A[j, i] = AY0[j, l]
        i=i+1
```

```
    return A
```

The following code was used to show that the relevant templates discussed in Section 6.2 are indeed abstractly equivalent and therefore that G_r is a golden-mean matroid, for each r.

```
AY0 = Matrix(GF4, [[1,1,1],
                   [1,a,a+1]])
AY1 = Matrix(GF4, [[a,0],
                   [0,a]])
A=Y_template_matrix(4, AY0, AY1)
A
M=Matroid(field=GF4, matrix=A)

AY0 = Matrix(GF(5), [[1,1,1],
                     [4,2,3]])
AY1 = Matrix(GF(5), [[3,0],
                     [0,3]])
A=Y_template_matrix(4, AY0, AY1)
A
N=Matroid(field=GF(5), matrix=A)

M.equals(N)
```

The previous code returned `True`. Similarly, the following code was used to show that HP_r is a golden-mean matroid, for each r.

```
AY0 = Matrix(GF4, [[1],
                   [1]])
AY1 = Matrix(GF4, [[1,a],
                   [a,1]])
A=Y_template_matrix(4, AY0, AY1)
A
M=Matroid(field=GF4, matrix=A)

AY0 = Matrix(GF(5), [[1],
                     [4]])
AY1 = Matrix(GF(5), [[1,3],
                     [3,1]])
A=Y_template_matrix(4, AY0, AY1)
A
N=Matroid(field=GF(5), matrix=A)

M.equals(N)
```

This code also returned `True`.

A.7 Code for Section 6.3

The next function checks for excluded minors for $\mathcal{AC}_4$. To reduce the amount of contractions necessary, it checks for excluded minors of rank 4 before those of rank 3.

```
F1 = matroids.named_matroids.Fano()
F2 = (F1).dual()

def MinorCheck(M):
    if M.has_minor(F2):
        return 'Fano dual'
    else:
        if M.has_minor(V2):
            return 'V2'
        else:
            if M.has_minor(P2):
                return 'P2'
            else:
                if M.has_minor(F1):
                    return 'Fano'
                else:
                    if M.has_minor(V1):
                        return 'V1'
                    else:
                        if M.has_minor(P1):
                            return 'P1'
                        else:
                            if M.has_minor(V3):
                                return 'V3'
                            else:
                                if M.has_minor(P3):
                                    return 'P3'
                                else:return 'None found'
```

A.8 Code for Section 6.4

The following code was used in the proof of Claim 6.4.5.3. First, we build the matrix containing all possible columns of weight 4.

```
R=Matrix(GF4,[
[1,1,  a,a^2,a,a^2,a,  a^2,a,  a^2,a,  a^2,0,  0],
[a,a^2,1,1,  a,a^2,a^2,a,  a^2,a,  0,  0,  a,  a^2],
[a,a^2,a,a^2,1,1,  1,  1,  0,  0,  a^2,a,  a^2,a],
[1,1,  1,1,  1,1,  0,  0,  1,  1,  1,  1,  1,  1]])
A=complete_Y_template_matrix(R)
```

```
M=Matroid(field=GF4, matrix=A)
```

Then we find the forbidden sets of columns.

```
U=Set(range(11,24))
F=[]
# We find forbidden sets of size 1.
for S in Subsets(U,1):
    if not MinorCheck(
    (M/0\(U.difference(S))).simplify())=='None found':
        F.append(S)
# We find forbidden sets of size 2.
# We only consider sets not containing smaller forbidden sets.
for S in Subsets(U,2):
    if not any(T.issubset(S) for T in F):
        if not MinorCheck(
        (M/0\(U.difference(S))).simplify())=='None found':
            F.append(S)
# We find forbidden sets of size 3.
# We only consider sets not containing smaller forbidden sets.
for S in Subsets(U,3):
    if not any(T.issubset(S) for T in F):
        if not MinorCheck(
        (M/0\(U.difference(S))).simplify())=='None found':
            F.append(S)
# We find forbidden sets of size 4.
# We only consider sets not containing smaller forbidden sets.
for S in Subsets(U,4):
    if not any(T.issubset(S) for T in F):
        if not MinorCheck(
        (M/0\(U.difference(S))).simplify())=='None found':
            F.append(S)
```

Then we find the maximal subsets containing no forbidden subsets.

```
Max=set([])
working_list=[U]
while len(working_list) > 0:
    T=working_list.pop()
    if all(S.issubset(T) is false for S in F):
        # If T contains no forbidden subsets, then it may be one of
        # the maximal subsets we seek.
        Max.add(frozenset(T))
    else:
        for S in F:
            if S.issubset(T):
```

```
                # If T contains a forbidden set S, then we replace T
                # with the subsets obtained by deleting from T each
                # element of S. Therefore, none of these sets
                # contain S. These resulting sets are appended to
                # working_list.
                for t in S:
                    working_list.append(T.difference({t}))
                break # Now, we repeat the process with
                      # another set T.
nonmax=set([])
for Q in Max:
    if any(Q.issubset(R) for R in Max.difference({Q})):
        nonmax.add(Q)
Max=Max.difference(nonmax)
Max
```

The following code was used in the proof of Claim 6.4.5.4. First, we build the matrix containing all possible columns of weight 3.

```
R=Matrix(GF4,[
[a  ,a  ,a  ,a^2,a^2,a  ,a^2,a  ,a^2,a^2,
 a  ,a^2,0  ,0  ,0  ,0  ,0  ,0  ,0  ,0],
[a^2,0  ,a^2,a  ,a  ,a^2,a  ,0  ,0  ,0  ,
 0  ,0  ,a  ,a^2,a  ,a^2,a  ,a^2,0  ,0],
[0  ,a^2,1  ,1  ,0  ,0  ,0  ,a^2,a  ,a  ,
 0  ,0  ,a^2,a  ,a^2,a  ,0  ,0  ,a  ,a^2],
[1  ,0  ,0  ,0  ,1  ,0  ,0  ,1  ,1  ,0  ,
 a^2,a  ,1  ,1  ,0  ,0  ,a^2,a  ,a^2,a],
[0  ,1  ,0  ,0  ,0  ,1  ,1  ,0  ,0  ,1  ,
 1  ,1  ,0  ,0  ,1  ,1  ,1  ,1  ,1  ,1]])
A=complete_Y_template_matrix(R)
M=Matroid(field=GF4, matrix=A)
```

Then we find the forbidden sets of columns.

```
U=Set(range(17,35))
F=[]
# We find forbidden sets of size 1.
for S in Subsets(U,1):
    if not MinorCheck(
    (M/0\(U.difference(S))).simplify())=='None found':
        F.append(S)
# We find forbidden sets of size 2.
# We only consider sets not containing smaller forbidden sets.
for S in Subsets(U,2):
    if not any(T.issubset(S) for T in F):
```

```
        if not MinorCheck(
        (M/O\(U.difference(S))).simplify())=='None found':
            F.append(S)
# We find forbidden sets of size 3.
# We only consider sets not containing smaller forbidden sets.
for S in Subsets(U,3):
    if not any(T.issubset(S) for T in F):
        if not MinorCheck(
        (M/O\(U.difference(S))).simplify())=='None found':
            F.append(S)
```

Then we find the maximal subsets containing no forbidden subsets.

```
Max=set([])
working_list=[U]
while len(working_list) > 0:
    T=working_list.pop()
    if all(S.issubset(T) is false for S in F):
        # If T contains no forbidden subsets, then it may be one of
        # the maximal subsets we seek.
        Max.add(frozenset(T))
    else:
        for S in F:
            if S.issubset(T):
                # If T contains a forbidden set S, then we replace T
                # with the subsets obtained by deleting from T each
                # element of S. Therefore, none of these sets
                # contain S. These resulting sets are appended to
                # working_list.
                for t in S:
                    working_list.append(T.difference({t}))
                break # Now, we repeat the process with
                      # another set T.
# It is possible that some sets in Max are subsets of other sets in
# Max. The following code removes those subsets from Max.
nonmax=set([])
for Q in Max:
    if any(Q.issubset(R) for R in Max.difference({Q})):
        nonmax.add(Q)
Max=Max.difference(nonmax)
Max
```

The following code was used in the proof of Claim 6.4.5.5. First, we build the matrix containing all possible columns.

```
R=Matrix(GF4,[
```

```
[a  ,a^2,a^2,a  ,a  ,a^2,0  ,0  ],
[a^2,a  ,a  ,a^2,0  ,0  ,a  ,a^2],
[1  ,0  ,1  ,0  ,a^2,a  ,a^2,a  ],
[0  ,1  ,0  ,1  ,1  ,1  ,1  ,1  ]])
A=complete_Y_template_matrix(R)
M=Matroid(field=GF4, matrix=A)
```

Then we find the forbidden sets of columns.

```
U=Set(range(12,18))
F=[]
# We find forbidden sets of size 1.
for S in Subsets(U,1):
    if not MinorCheck(
    (M/0\(U.difference(S))).simplify())=='None found':
        F.append(S)
# We find forbidden sets of size 2.
# We only consider sets not containing smaller forbidden sets.
for S in Subsets(U,2):
    if not any(T.issubset(S) for T in F):
        if not MinorCheck(
        (M/0\(U.difference(S))).simplify())=='None found':
            F.append(S)
# We find forbidden sets of size 3.
# We only consider sets not containing smaller forbidden sets.
for S in Subsets(U,3):
    if not any(T.issubset(S) for T in F):
        if not MinorCheck(
        (M/0\(U.difference(S))).simplify())=='None found':
            F.append(S)
```

Then we find the maximal subsets containing no forbidden subsets.

```
Max=set([])
working_list=[U]
while len(working_list) > 0:
    T=working_list.pop()
    if all(S.issubset(T) is false for S in F):
        # If T contains no forbidden subsets, then it may be one of
        # the maximal subsets we seek.
        Max.add(frozenset(T))
    else:
        for S in F:
            if S.issubset(T):
                # If T contains a forbidden set S, then we replace T
                # with the subsets obtained by deleting from T each
```

```
                # element of S. Therefore, none of these sets
                # contain S. These resulting sets are appended to
                # working_list.
                for t in S:
                    working_list.append(T.difference({t}))
                break # Now, we repeat the process with
                      # another set T.
# It is possible that some sets in Max are subsets of other sets in
# Max. The following code removes those subsets from Max.
nonmax=set([])
for Q in Max:
    if any(Q.issubset(R) for R in Max.difference({Q})):
        nonmax.add(Q)
Max=Max.difference(nonmax)
Max
```

A.9 Code for Section 6.6

The following functions are explained in Section 6.6 and used repeatedly in Section 6.7

```
from itertools import product
from functools import reduce
import operator

GF4 = GF(4, 'a')
a=GF4.gens()[0]

def complete_template_representation(P0, vars_prefix='z'):
    """
    Return a pair of matrices, one over GF(4) and one over a
    polynomial ring over ``ZZ'' such that the first is a matrix
    conforming to the appropriate template, and there are maps
    from the second to every actual representation.

    We assume that the input matrix is over GF(4). Adapting
    this code for a general finite field would essentially
    consist of removing the code that maps 1s to -1s.
    """
    (r, c0) = P0.dimensions()
    # Create a rescaled copy of P0 with the last nonzero in
    # each column equal to 1.
    P0scaled = copy(P0)
    lastnonzero = [-1]*c0
    for ccount in range(c0):
```

```
    rcount = P0.nrows() - 1
    while P0[rcount, ccount] == 0:
        rcount -= 1
    lastnonzero[ccount] = rcount
    P0scaled.rescale_col(ccount, P0[rcount,ccount]^(-1))
# Determine the entries and their locations in the matrix
# over the polynomial ring.
unknowns = {}
num_unknowns = 0
# We have 0 -> 0, last 1 in column -> 1, and (in
# characteristic 2) every other 1 -> -1.
for j in range(c0):
    for i in range(r):
        if (i < lastnonzero[j]) and (P0scaled[i,j]
        != 0) and (P0scaled[i,j] != 1):
            unknowns[(i,j)] = num_unknowns
            num_unknowns += 1
vars = [vars_prefix + str(i) for i in range(num_unknowns)]
outring = PolynomialRing(ZZ, vars, order='degrevlex')
# Size of output:
c=(r+1)*r/2+c0
A4 = Matrix(GF4, r, c)
Avar = Matrix(outring, r, c)
# For entries in P0, the map between the known entries (other
# than the last nonzero entry in each column) is as follows.
FourToZZ = {GF4(0): 0, GF4(1): -1}
# Create the output matrices.
j = 0  # column index
# Start with the identity.
for i in range(r):
    A4[i,i]=1
    Avar[i,i]=1
#Next, non-identity graphic columns
j = r
for S in Subsets(range(r),2):
    A4[S[0],j] = 1
    A4[S[1],j]=-1
    Avar[S[0],j]=1
    Avar[S[1],j]=-1
    j += 1
# Finally, we copy over P0
for k in range(c0):
    for i in range(r):
        A4[i,j] = P0scaled[i,k]
```

```
            if (i,k) in unknowns:
                Avar[i,j] = vars[unknowns[(i,k)]]
            elif i == lastnonzero[k]:
                Avar[i,j] = 1
            else:
                Avar[i,j] = FourToZZ[P0scaled[i,k]]
        j += 1
    return A4, Avar

def zero_determinant_ideal(M, A, E=None, include_inverses=False,
single_var_inverses=False, basis_vars_prefix='Bi',
gb_method="macaulay2:gb"):
    """
    Compute the ideal generated by determinants of A that must be
    zero in order for M=M(A).

    This ideal differs from the Baines and Vs ideal by not
    including information about bases being invertible.
    The optional argument ``include_inverses'' will include this
    information as well.

    INPUT:

    - ``M'' -- a matroid
    - ``A'' -- a matrix over a polynomial ring such that, if every
    nonzero entry of the matrix is replaced by 1, the result is a
    partial representation of M. This is the case if
    (A4,Avar) == complete_template_representation(P0), M == M(A4),
    and A == Avar.
    - ``E'' (default: ``None'') -- an ordering of the set of
    elements of ``M'', defaults to ``M.groundset_list()'' if that
    method exists.
    - ``include_inverses'' (default: ``False'') -- whether to
    include relations indicating that determinants of bases must
    be units.
    - ``single_var_inverses'' (default: ``False'') -- if true, only
    one variable gets added which symbolizes the inverse of the
    product of all determinants of bases.
    - ``basis_vars_prefix'' (default: `` 'Bi' '') -- The name of
    these inverse variables equals this followed by a number.
    - ``gb_method'' (default: ``"macaulay2:gb"'') -- the Grbner
    basis algorithm to be used.
    """
    # Map subsets to column indices.
```

```
col_index_map = {}
if E is None:
    E = M.groundset_list()
for i in range(len(E)):
    col_index_map[E[i]] = i
R = A.base_ring()
rels = []
# Compute determinants of A that correspond to nonbases of M.
for B in M.nonbases():
    determ = A[:,[col_index_map[e] for e in B]].determinant()
    # Make sure nonbases have zero determinant.
    if determ != 0:  # Only nontrivial relations need to be added.
        rels.append(determ)
print "nonbasis computation complete"
if include_inverses:
    b_rels = []
    num_bases = 0
    print "basis computation commences"
    for B in M.bases():
        determ = A[:, [col_index_map[e] for e in B]].determinant()
        if determ != 1 and determ != -1:  # already units, so we
                                          # cut down on variables
            b_rels.append(determ)
            num_bases += 1
    print "basis computation complete"
    if single_var_inverses:
        R2 = ZZ[tuple([basis_vars_prefix]) + R.gens()]
        print R2
        # Product of all nonzero determinants is invertible.
        rels.append(R2(reduce(operator.mul, b_rels, 1))
        *R2(basis_vars_prefix) - R2(1))
    else:
        R2 = ZZ[tuple(basis_vars_prefix + str(i) for i in
        range(num_bases)) + R.gens()]
        rels = [R2(x) for x in rels]
        for i in range(num_bases):
            rels.append(R2(b_rels[i]) * R2(
            basis_vars_prefix + str(i)) - R2(1))
    print "ideal construction complete"
else:  # no inverses
    R2 = R
Grob1 = R2.ideal(rels).groebner_basis(gb_method)
print "Grbner basis computed"
return Grob1
```

```
def check_partial_field(M, A, E=None, generators=[-1],
extra_determinants=[], max_power=None):
    """
    Check if, for all bases X of M, det(A[X]) is a product of the
    generators of total degree at most max_power (default: rank of
    matrix), or in the set extra_determinants.

    The set extra_determinants typically contains elements of the
    form ‘g+p’, where ‘g’ is a product of the generators and ‘p’ a
    member of the zero determinant ideal.

    We assume that the first generator of the partial field is -1.

    Note that, if this function is used in isolation of the
    previous two functions, there is no guarantee that the
    submatrices of A corresponding to the nonbases of M have
    zero determinants.
    """
    if generators[0] != -1:
        raise ValueError("The first generator should be -1.")
    col_index_map = {}
    if E is None:
        E = M.groundset_list()
    for i in range(len(E)):
        col_index_map[E[i]] = i
    if max_power is None:
        max_power = A.nrows()
    # Generate list of candidate determinants.
    candidate_determinants = []
    # The exponent on -1 can be taken to be either 0 or 1. The
    # exponents on the other generators each can be taken to be
    # less than ‘max_power’. (If some generator of the partial field
    # appears as a factor of a determinant of A more than ‘max_power’
    # times, then we can include that determinant in
    # extra_determinants after it causes the function to return
    # ‘False’.)
    exponents = [[0,1]] + [range(max_power)]*(len(generators)-1)
    for exponents_vector in product(*exponents):
        # We only check candidate determinants where the sum of the
        # exponents on the generators (other than -1) is at most
        # ‘max_power’. (Again, more determinants can be added to the
        # list of extra determinants if necessary.)
        if sum(exponents_vector) <= max_power + exponents_vector[0]:
```

```
        x = 1
        for i in range(len(generators)):
            x *= generators[i]^exponents_vector[i]
        candidate_determinants.append(x)
candidate_determinants.extend(extra_determinants)
# Check if determinants corresponding to bases appear in the
# list of candidate determinants.
for B in M.bases():
    determ = A[:,[col_index_map[e] for e in B]].determinant()
    if determ not in candidate_determinants:
        return (False, B, determ)
return True
```

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Vita

Kevin Grace was born in 1984 in Yonkers, New York, and moved with his parents to Pensacola, Florida, at fourteen years of age. He graduated from high school at Pensacola Christian Academy and completed a Bachelor of Science degree from Pensacola Christian College, with a double major in Mathematics Education and History Education. He served on the mathematics faculty of Pensacola Christian College for seven years. During that time, he earned a Master of Science degree in Mathematics from the University of South Alabama in Mobile. He came to Louisiana State University in 2013 and is currently a candidate for the degree of Doctor of Philosophy in Mathematics. In the fall of 2018, he will begin a three-year Research Fellowship at the Heilbronn Institute for Mathematical Research and will be based at the University of Bristol in the United Kingdom.