Double Loop Interconnection Networks With Minimal Transmission Delay.

Dvora Tzvieli
Louisiana State University and Agricultural & Mechanical College

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Double loop interconnection networks with minimal transmission delay

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DOUBLE LOOP INTERCONNECTION NETWORKS

WITH

MINIMAL TRANSMISSION DELAY

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in

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by

DVORA TZVIELI

B.Sc. Tel-Aviv University, Israel, 1970
M.Sc. Tel-Aviv University, Israel, 1973

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# Table of Contents

Chapter 1: Introduction ................................................................. 1

1.1 Interconnection Networks ................................................................. 1

1.2 Issues in the Design of Interconnection Networks ...................... 4

1.3 Topologies for Interconnection Networks ..................................... 8

1.4 Loop Networks ........................................................................... 15

1.5 Undirected Double Loop Networks .............................................. 20

1.6 Results of This Work .................................................................. 25

Chapter 2: Problem Definition and Preliminary Results ................... 30

2.1 Introduction and Problem Definition .......................................... 30

2.2 Preliminary Observations ............................................................ 32

2.3 Bounds on $D_n^*$, $h_n^*$ ....................................................... 39

2.4 Algorithms for Finding Optimal and Suboptimal Hops ................ 47

2.5 Optimal Hops for the Quartile Points in R[K] ............................ 48

Chapter 3: Sparse Optimal Families ................................................. 52

3.1 Introduction .............................................................................. 52

3.2 Optimal Family $\Phi_1$ ............................................................... 53

3.3 Optimal Family $\Phi_2$ ............................................................... 58

3.4 The Augmented Families $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ ............................ 61
List of Tables

Table 3.1: Membership in Sparse Optimal Families for k=14

................................................................................................................. 68

Table 4.1: Membership in Optimal Families for k=14

................................................................................................................. 79
List of Figures

Figure 1.1: Different Concepts in Memory Sharing Between Processors ..................................................... 3

Figure 1.2: Fully Connected Networks ............................................................. 9

Figure 1.3: The Hypercube ........................................................................... 10

Figure 1.4: A 3-Dimensional CCC Network .................................................. 11

Figure 1.5: The Hypertree and the Pyramid ................................................... 12

Figure 1.6: The Shuffle-Exchange ................................................................... 13

Figure 1.7: The PM2i Functions for n = 8 .................................................... 14

Figure 1.8: A Systolic Array ........................................................................... 15

Figure 1.9: Directed and Undirected Loop Networks .................................... 16

Figure 1.10: The DDLCN Network ................................................................. 18

Figure 1.11: The Daisy Chain Network ......................................................... 18

Figure 1.12: A Chordal Network With n=16, w=5 ....................................... 20

Figure 1.13: The ILLIAC Network With 16 Processors ................................. 21

Figure 1.14: The ILLIAC Interconnection Network Represented as a Loop Network ..................................................... 22

Figure 2.1: G(16,4) .......................................................................................... 30

Figure 2.2: $g_{22,3}$, $g_{22,4}$, and $g_{22,6}$ .......................................................... 34
Figure 2.3: \( G_{41,9}^4, k=4, \) and \( G_{36,8}^4, k=4 \) ................................................. 37

Figure 2.4: \( g_{68,8} \) and \( g_{67,7} \) ................................................................. 43

Figure 2.5: \( g_{70,32}, \) Diam=8, and \( g_{71,33}, \) Diam=8 ............................ 45

Figure 2.6: \( g_{72,25} \) ................................................................................. 50

Figure 3.1: The CPD Transformation (for an odd h) ................................. 54

Figure 3.2: The CPD Transformation (for an even h) ................................. 59

Figure 3.3: CPL Applied to \( g_{25,7} \) ............................................................... 62

Figure 3.4: CPR Applied to \( g_{25,7} \) ............................................................... 63

Figure 3.5: The CPR and CPL Operations Performed on
\( g_{2k^2-k,2k} \) \( k=4 \) ................................................................................. 65

Figure 5.1: \( G_{39,11}^4 \) and the Corresponding Digraph \( H(V,E) \) .............. 101

Figure 5.2: \( H(V,E) \) in Lemma 4.7 (I and II, and III) ............................... 104

Figure 5.3: Row Traversal in \( G_{174,65}^{10} \) and in \( G_{174,110}^{10} \) ................. 107

Figure 5.4: Rows -7,...,7 of \( G_{174,23}^{10} \) and of \( G_{174,110}^{10} \) .................. 108

Figure 6.1: \( g_{1290,622} \) ................................................................................. 130
Abstract

The interconnection network is a critical component in massively parallel architectures and in large communication networks. An important criterion in evaluating such networks is their transmission delay, which is determined to a large extent by the diameter of the underlying graph. The loop network is popular due to its simplicity, symmetry and expandability. By adding chords to the loop, the diameter and reliability are improved. In this work we deal with the problem of minimizing the diameter of double loop networks, which model various communication networks and also the Illiac type Mesh Connected Computer.

A double loop network, (also known as circulant) $G(n,h)$, consists of a loop of $n$ vertices, where each vertex $i$ is also joined by chords to the vertices $i+h \mod n$. $D_n^*$, the minimal diameter of $G(n,h)$, is bounded below by $k$ if $n \in R[k] = \{2k^2 - 2k + 2, ..., 2k^2 + 2k + 1\}$. An integer $n \in R[k]$, a hop $h$ and a network $G(n,h)$ are called optimal (suboptimal) if $\text{Diam } G(n,h) = D_n^* = k(k+1)$.

We determine new infinite families of optimal values of $n$, which considerably improve previously known results. These families are of several different types and cover more than 94% of all values of $n$ up to $\sim 8,000,000$. We conjecture that all values of $n$ are either optimal or suboptimal. Our analysis led to the construction of an algorithm that detects optimal and suboptimal values of $n$. When run on a SUN workstation, it confirmed our conjecture within $\sim 60$ minutes, for all values of $n$ up to $\sim 8,000,000$. Optimal (suboptimal) hops, corresponding to optimal (suboptimal) values of $n$, are provided by a simple construction.
CHAPTER 1

INTRODUCTION

§ 1.1 Interconnection Networks.

The early computer, as designed by von Neumann, consisted of a single processor, a memory and I/O devices. The high cost of hardware, combined with slow speed did not encourage parallelism. As a result, algorithms and programs were all designed for a sequential machine with one processor. Increases in device speed and reductions in physical size have greatly enhanced computer performance and created a new approach to computing. In recent years the field of computer architecture has pushed technology to its limits. Very-Large-Scale-Integrated (VLSI) circuits achieve computing speeds which are close to the physical bound caused by the finite speed of electromagnetic waves.

On the other hand, this very bound limits even the most efficient sequential algorithms. Some problems are so big that no sequential algorithm, even on the fastest possible machine, is capable of solving them in a reasonable amount of time. There is a general agreement that a breakthrough in achieving greater computing power lies potentially in the use of a multiprocessor architecture - a computer system composed of many independent processors. The heuristics behind this is that although each processor's performance is bounded, the number of processors is not bounded. If we succeed in dividing the work between the processors in such a way that many processors work concurrently, and the overhead caused by employing a parallel architecture is kept relatively
small, then we can achieve a considerable performance gain. This gain can be experienced from several aspects:

- Reduced computing time - this is the most obvious gain.
- Enhanced system availability and reduced response time. This could be achieved by devoting some processors to manage the user's interface.
- Higher reliability and fault tolerant computing. A fault in a small percentage of processors in a multiprocessor architecture would degrade performance only slightly, while in a single-processor computer, a fault in the processor could stop execution.
- A multiprocessor system has often the advantage of scalability. The addition of processors could often improve the performance, and in some cases the gain is linear (or near linear) in the number of processors.
- From the manufacturing point of view, the computer system is more cost-effective when it is constructed from a small number of homogeneous building blocks, and has a regular structure.

Today it is feasible to construct a multiprocessor system that interconnects hundreds or even thousands of processors. Task partitioning and the assignment of tasks to individual processors needs to be done very carefully, in order to minimize synchronization problems and communication overhead. A suitable interconnection network thus becomes a critical system component.

Several parallel architectures exist, each of which tries to employ the idea of using many processors to increase the computing power. According to one concept, all the
processors concurrently perform the same instruction, but on different data values. This is the Single Instruction Multiple Data architecture (SIMD). Computers of this type are very efficient, for example, when independent array elements are all given the same treatment. Another concept is that of Multiple Instruction Multiple Data (MIMD) architecture [Fl66]. In this architecture, each processor may perform (concurrently) its own instruction using its own data. The processors of a MIMD machine are interconnected to permit data exchange and synchronization of activities.

Variations exist in the degree of shared memory among parallel processors. Two diagrams of such architectures are shown in Figure 1.1.

**Figure 1.1**

Different Concepts in Memory Sharing Between Processors

(a)  
(b)
In structure (a) a shared architecture is described; both memory and I/O are shared among all processors. Structure (b) is a shared-nothing architecture, in which each processor has its own private memory and I/O channel. In both types of systems, the processors cooperate by exchanging data through the interconnection network and by synchronizing activities. The current trend seems to prefer the shared-nothing approach.

A treatment of parallel architectures can be found in several texts. Examples are Stone [St88] and Hwang and Briggs [Hw84].

§ 1.2 Issues in the Design of Interconnection Networks.

The task of interconnecting n objects, where n may be as large as $10^5$ (and possibly even larger in the future), is not trivial. The interconnection network, consisting of both hardware and software entities, must provide fast and reliable communications at a reasonable cost. This is a critical aspect of parallel computing, and the study of interconnection networks is attracting considerable research effort. In what follows, we shall mention briefly several important issues in the design of interconnection networks. The subject of topologies of networks will be expanded in the next section.

Topology. Networks can be modeled by a graph in which the nodes represent system objects (e.g., processors) and the edges represent communication links. This graph representation is called the network topology, and is perhaps the most fundamental design issue when interconnection networks are considered. Designers seem to favor regular topologies, as they are cheaper to manufacture. Topologies can be static (where each
switching point is connected to a processor) or dynamic (only the switching points on the I/O side are connected to a processor). Among the dynamic topologies we distinguish between single-stage and multi-stage. In a multi-stage network, more than one stage of switches exists, and the network is usually capable of connecting an arbitrary pair of input and output terminals. In the single-stage network, data items may have to recirculate through the network several times before reaching their destination.

Reconfiguration techniques. These are required in dynamic topologies for various needs of allocating processor and communication resources, and to enable matching of algorithms to architectures.

Reliability, or fault tolerance, is the ability of the system to degrade gracefully when communication links fail. In order to achieve fault-tolerance, fault detection and location must be performed before the routing scheme can be applied. For that, a fault model must be developed in order to provide knowledge about the types of faults that need to be tested. Reliability and fault-tolerance are achieved using redundancy of communication paths and port connections, and through fault tolerant switching elements. Usually, a higher node degree implies higher reliability of the system. This, however, often stands in conflict with the cost criterion.

Routing data in the network. Another important design issue is the development of control strategies for the routing of data in the network. The control of data flow can be managed by a centralized controller, or be distributed to individual switching points. Address labeling has to be done before a proper routing path can be specified from a
source to a destination. Two quite different methods exist for implementing data routing. In *circuit switching*, a physical path is actually established between a source and a destination. In *packet switching*, data to be transmitted is divided into "packets" and each packet is routed to its destination according to the availability of routing paths, without establishing a dedicated physical connection. The control strategy largely contributes to the performance of the network.

Network Analysis and simulation are important components of the design process. Measurements must be performed in order to observe network characteristics (especially performance relative to different loads) and to decide upon the adequacy of the network for the intended applications. Important characteristics that need to be examined, and which are independent of communication request distribution, are:

- *overall cost* (which is usually proportional to the number of communication links),
- *reliability*, and
- *combinatorial power*. The questions considered here are the ability of the network to realize any desirable permutation of data between nodes, and the way any particular permutation is performed. It is desirable to have a network that can perform data permutation in a minimal number of steps. This is determined to a large extent by the diameter of the graph underlying the network (see definition in Section 2.1).

Characteristics that are related to communication request distribution are

- *bandwidth* - the expected number of requests accepted per unit time,
- *message delay* and
- *message density per link*. 
Those characteristics are measured to a large extent using simulations on simplified models.

**VLSI network design** is concerned with the implementation of interconnection networks (as well as other components of the computer system) on two dimensional integrated circuits. Quite often, the entire network cannot reside on a single VLSI chip, and a partition of the network into modules has to be achieved. The number of different modular types should be minimized to reduce manufacturing costs. The performance criteria for the VLSI network design include the chip required module area and the transmission delay imposed by the module. When the number of modules used in constructing a network is very large, timing and synchronization schemes have to be developed.

**Programmability.** This important issue deals with the ability of a given architecture to match various algorithms that solve different types of problems. Even when this ability is established, the relative efficiency of such algorithms implemented on different architectures needs to be compared. Every algorithm has a "process graph" which has to be embedded in the topology of the network. Thus, one way to discuss both issues is by considering various characteristics of graph embeddings.

For a discussion of these issues see, for example, the work of Rosenberg [Ro87].

An extensive collection of papers on interconnection networks can be found in a tutorial edited by Wu and Feng [Wu84].

Several popular topologies of networks are briefly discussed in the following section. In Section 1.4 we shall further expand the subject of loop-topologies.
§ 1.3 Topologies for Interconnection Networks.

In this section we very briefly mention several popular interconnection networks. Our selection is by no means exhaustive, and mainly concentrates on static topologies. Throughout this section we denote by \( n \) the number of elements in the network. Elements are usually processors, memories or processing elements which combine a small memory and a processing unit.

The Shared Bus is used to interconnect elements in a one-processor architecture. (See e.g., Thurber et. al. [Th72]). In highly parallel systems, shared busses are not sufficient since it is desirable to enable many different processors to communicate concurrently.

A Fully Connected System would achieve maximal performance. This configuration is impractical when \( n \), the number of processors, is large. Such a system has \( n-1 \) links incident with each element in the network. As a result, the cost of the network, which is proportional to the number of links, grows with \( n^2 \), making it impractical for large systems. Another difficulty with fully connected systems is that current VLSI technology is not able to cope with a very large number of links. A fully connected system is illustrated in Figure 1.2.

The Crossbar Network is a realization of the fully connected network, in which a single communication line corresponds to each node (processor or memory element). Every node is interconnected to all others through crosspoint switches (see Figure 1.2). The number of crosspoint switches is \( n^2 \). This architecture is not feasible with the current
VLSI technology if $n$ is large.

**Figure 1.2**

**Fully Connected Networks**

A fully connected networks with 8 nodes

A crossbar network

**The Hypercube.** This popular interconnection network was suggested by Sullivan and Bashkow [Su77] and has been implemented commercially. Every node $x$ in the hypercube is identified by a binary sequence of length $m$: $x = x_{m-1}x_{m-2}...x_0$, where $x_i = 0$ or 1. Two nodes $x$ and $y$ are connected if and only if their corresponding sequences differ in exactly one digit. The total number of nodes, $n$, is $2^m$; $m$ is called the *dimension* of the cube. Thus each node is connected to $m$ other nodes. The links in the cube network are often represented as $m$ different interconnection functions (see for example Siegel [Si85]):

$$\text{cube}_i(x_{m-1}x_{m-2}...x_{i+1}x_i\bar{x}_{i-1}...x_0) = x_{m-1}x_{m-2}...x_{i+1}\bar{x_i}x_{i-1}...x_0$$

for $0 \leq i < m$. Here $\bar{x_i} = 1 - x_i$. In a SIMD hypercube machine, all active processors communicate concurrently using the same interconnection function. Figure 1.3 shows the labeling of nodes in the three dimensional hypercube and its corresponding interconnection
functions. An illustration of a four dimensional hypercube is shown below.

**Figure 1.3**

The Hypercube

The 3-dimensional cube

The interconnecting functions for the 3-cube

The 4-dimensional hypercube

The \( m \)-dimensional hypercube has a relatively small diameter, \( \log_2 n \), but at the same time the ratio between the number of links \( ([n \log_2 n]/2) \) and the number of nodes is high, which is not cost-effective for a large \( n \). Also, to achieve efficient VLSI design, it is undesirable to have nodes with a high degree \( (m=\log_2 n) \). The hypercube is partitionable into subcubes of lower dimension. This important property enables concurrent processing of distinct tasks.
Cube-Connected-Cycles (CCC). In this modification of the hypercube architecture, suggested by Preparata and Vuillemin [Pr81], every node has degree 3, as compared to degree m in the hypercube. The m-dimensional CCC is obtained by replacing each node x in the m-cube by an m-cycle with nodes y₁,...,yₘ. Every edge eᵢ, 1 ≤ i ≤ m, that would have been incident with x in the m-cube, is incident with a distinct yⱼ, 1 ≤ j ≤ m. Figure 1.4 illustrates the CCC for m=3. In the m-dimensional CCC the number of nodes is \( n = m2^m \), and the number of links is \( \frac{3}{2}n \). The ratio between the number of links and the number of nodes in the CCC is \( 3/2 \), as compared to \( m/2 \) in the hypercube.

Figure 1.4
A 3-Dimensional CCC Network

The Hypertree, suggested by Goodman and Séquin [Go81], is another connection scheme which was inspired by the hypercube. Given a value m, the nodes are organized as a full binary tree with m levels. Thus, \( n = 2^m - 1 \). The root is labeled 1, the left descendent of a node i is labeled 2i, and the right descendent is labeled 2i+1. In addition to the tree links, there are horizontal links between node i and node i+1 (with a wrap around),
provided they are in the same level of the tree. Nodes of the hypertree which are not in level \( m \) have degree 5 (with the exceptions of nodes 1, 2, and 3). The nodes \( 2^{m-1}, \ldots, 2^m-1 \) have degree 3. The total number of links is \( 2n-3 \), and the diameter is \( O(2\log_2 n) \). A hypertree with 5 levels is shown in Figure 1.5.

The Pyramid architecture. The basic pyramid structure was suggested by Dyer and Rosenfeld [Dy81], and is recursively constructed as follows: the root is connected to four descendents, each of which serves as a root to its own sub-pyramid. The nodes in each level form a square lattice (see Figure 1.5). This architecture was motivated by problems in digital image processing. The maximal degree is 9, and the diameter is \( O(\log_4 n) \). A 3 level pyramid is depicted in Figure 1.5.

Figure 1.5

A hypertree with 5 levels

A Pyramid Network
The Shuffle-Exchange Network. In this network (see Johnson [Jo56], Golomb [Go61], and Stone [St81]), the number of nodes $n$ is again $2^m$, and each node is represented by a binary sequence of length $m$, as in the hypercube. The links correspond to two interconnection functions, given by

$$ \text{shuffle} (x_{m-1}x_{m-2}...x_1x_0) = x_{m-2}x_{m-3}...x_1x_0x_{m-1}, \text{ and} $$

$$ \text{exchange} (x_{m-1}x_{m-2}...x_1x_0) = x_{m-1}x_{m-2}...x_1\overline{x}_0. $$

Some models combine several stages of the shuffle-exchange through several layers of switches. An illustration of the shuffle-exchange network for $n=8$ ($m=3$) is given in Figure 1.6, along with an example of a multi-layered version.

**Figure 1.6**

The Shuffle-Exchange

The shuffle-exchange network for $n=8$.

(Solid line is exchange, dashed line is shuffle)

A multi-layered shuffle exchange
The PM2I Network (Plus Minus $2^i$), suggested by Siegel [Si77], consists of $n=2^m$ nodes numbered from zero to $2^m - 1$. There are $2^m - 1$ links to each node $x$ given by the $2m$ interconnection functions:

$$\text{PM}_{2+i}(x) = x + 2^i \mod n,$$

$$\text{PM}_{2-i}(x) = x - 2^i \mod n, 0 \leq i < m.$$ (Note that $\text{PM}_{2+(m-1)}$ and $\text{PM}_{2-(m-1)}$ are equivalent.) Figure 1.7 shows the $\text{PM}_{2+i}$ functions for $n=8$. The $\text{PM}_{2-i}$ functions are obtained by reversing the directions on the arrows.

**Figure 1.7**

The $\text{PM}_{2+i}$ Functions for $n=8$.

The Systolic Array. In this regular scheme suggested by Kung and Leiserson [Ku78], each node is linked to six other nodes, securing a very good fault tolerance.
The Mesh Connected Computer (represented by the ILLIAC network) and various loop networks are discussed in Sections 1.4 and 1.5.

§ 1.4 Loop Networks.

The loop network has been one of the most popular network topologies used in the design and implementation of Local Area Networks (LANs). This network seems especially attractive due to the following properties:

- It is simple and symmetric.
- It is easily expandable. Because of the symmetry, the network can be constructed using uniform, pre-manufactured modules. The same modules can be later used if there is a need to expand the network. For the same reasons, the switching mechanism at each node can be constructed using standard components.
- Relatively simple data routing algorithms can be designed for such networks. Tokens can be passed through the network in a uniform way.
Despite all these obvious virtues, the loop topology has some serious disadvantages:

- It is highly vulnerable to faults in the network. The directed loop has connectivity one so that a fault in one link or processor can disconnect the network. Thus it is very unreliable. (The connectivity is two in the undirected loop).
- The loop topology can suffer from a large transmission delay. The transmission delay is often measured by the diameter of the underlying graph of the network. Let \( x \) and \( y \) represent processors in the network \( G \) and let \( d(x,y) \) denote the number of links in a shortest path connecting \( x \) and \( y \) in \( G \). Then the diameter of the network \( G \) is given by

\[
\text{Diam } G = \max \{ d(x,y) : x, y \text{ are processors in the network } \}.
\]

In a directed loop network with \( n \) processors, the diameter is \( n - 1 \). (The diameter is \( \lceil \frac{n}{2} \rceil \) in an undirected loop network.) The directed and undirected loop networks are shown in Figure 1.9a and 1.9b respectively.

**Figure 1.9a**
A Directed Loop Network

**Figure 1.9b**
An Undirected Loop Network
Both problems mentioned above, high vulnerability and a large diameter, are addressed by adding links to the network. In the extreme case, every processor is directly connected to every other processor in the network (as in the cross-bar connection). This, however, could be too costly and impractical to implement for a large system. Some systems for parallel processing consist of thousands of processing elements and memory modules connected by an interconnection network.

From the points of view of cost and of effective VLSI design, it is desirable to add as few links as possible. Moreover, by adding those links in a uniform way, the nice properties of symmetry, expandability, uniform building blocks for the switching mechanism and uniform token passing are preserved.

Various designs have been suggested in the past, in light of those ideas. Most of them, in the context of LANs, are networks with directed links. In the rest of this section we shall mention design proposals and recent analytical results concerning directed double loop networks. The undirected networks will be discussed in Section 1.5.

Wolf and Liu [Wo78] suggested the Distributed Double Loop Computer Networks (DDLCN). It consists of two oppositely directed loops, hence its name. This network is shown in Figure 1.10.
The Daisy Chain Network was proposed by Grnarov, Kleinrock and Gerla [Gr80]. In this network, shown in Figure 1.11, each processor \( i \) is adjacent through directed links to nodes \( i+1 \) and \( i-2 \mod n \), where \( n \) is the number of processors in the network. The Daisy Network has diameter \( \lceil \frac{n}{3} \rceil + 1 \) and connectivity 2.
Raghavendra, Gerla and Avizienis [Ra85] proposed a loop architecture called *Forward Loop Backward Hop* (FLBH). It has a directed loop on \( n \) nodes and each node \( i \) is connected to node \( i-h \), where \( h = \lceil \sqrt{n} \rceil \). This network was shown to have diameter 
\[
\left\lfloor \frac{n}{h+1} \right\rfloor + h - 1.
\]

Wong and Coppersmith [Wo74] have shown that a lower bound, \( \text{lb}(n) \), on the minimal diameter, \( d(n) \), in the directed loop with indegree = outdegree = 2 (i.e., each node \( i \) is connected to node \( i+1 \) and to one additional node, and each node has exactly two incoming links) is \( \lceil \sqrt{3n} \rceil - 2 \). The networks mentioned above rarely achieve this lower bound. Loop topologies with \( d(n) = \text{lb}(n) \) are given by Erdös and Hsu [Er88], Fiol, Yebra, Alegre and Valero [Fi87] and Hwang and Xu [Hw87]. These authors also give examples of infinitely many values of \( n \) for which the lower bound cannot be achieved.

A special case of the directed loop network with indegree = outdegree = 2 is when each node \( i \) is incident to nodes \( i+1 \) and \( i+h \), for some \( h \). The DDLCN, Daisy and FLBH networks are all examples of this special case. Contrary to our intuition, the problem of determining values of \( n \) for which \( \text{lb}(n) \) is achieved in this case is not simpler to handle than the more general case. Cheng and Hwang [Ch88] give an algorithm to compute the minimal diameter \( d^*(n) \) for such networks. In their survey [Be88a], Bermond, Comellas and Hsu mention a recent result by Coppersmith showing the existence of infinitely many values of \( n \) for which \( d^*(n) - \text{lb}(n) = c \cdot \log^{1/4} n \).

The maximal number of nodes that can be accommodated by a digraph with diameter \( d \) and degree \( \Delta \) can be larger than that allowed by a loop topology. Some well known results are the de-Brujin and the Kautz digraphs, in which the number of nodes is \( \Delta^d \) and
$\Delta^d+\Delta^{d-1}$ respectively (see the surveys [Be86a] and [Be86b]).

§ 1.5 Undirected Double-Loop Networks.

The advantages and disadvantages of the undirected loop are similar to those of the directed one, and so are the remedies. One construction, the Chordal Ring has, in addition to an undirected loop of $n$ processors, chords as follows: (n must be even and w must be odd)

node $i$ is joined to node $i-w \mod n$, if $i$ is even,

node $i$ is joined to node $i+w \mod n$, if $i$ is odd.

An example of a chordal ring with $n=16$, $w=5$ is shown in Figure 1.12.

**Figure 1.12**

A Chordal Network With $n=16$, $w=5$

The chordal ring is a regular graph of degree 3. Arden and Lee [Ar81] show that if $w$ is properly chosen, the diameter of this network is $O(\sqrt{n})$. 
In addition to the context of LANs (Local Area Networks), undirected loop networks arise in the context of *Mesh Connected Computers* (MCC) suited for parallel processing of data, such as the ILLIAC type computers (see Barnes [Ba68]). The ILLIAC interconnection network consists of $N^2$ processors that could be depicted as the elements of an $N$-square matrix; each processor is directly connected through an undirected link to its immediate neighbors in its row and column, and additional wrap-around connections exist. The ILLIAC network with 16 processors is shown in Figure 1.13.

**Figure 1.13**

The ILLIAC Network With 16 Processors

![Diagram of the ILLIAC Network with 16 Processors](image)

The same network can be depicted as a double-loop undirected network, as is shown in Figure 1.14.
The ILLIAC network with \( N^2 \) processors has diameter \( N-1 \). This topology is especially suited to implement parallel algorithms for elliptic partial differential equations, where the values of the derivatives at any grid point are approximated by a difference scheme that averages over values of the function in neighboring points.

In any type of parallel computation on such a network, data has to be routed between the processors. Communication overhead is a major issue in the speed up factor of parallel computations, as opposed to their sequential versions. Each parallel architecture is accompanied by a data routing algorithm that is specific to the network, and utilizes its features to achieve minimal transmission delay, and thus minimize the communication overhead. Examples of such algorithms for the MCC can be found in Nassimi and Sahni [Na80] and in Raghavendra and Kumar [Ra86].

A major factor in the computational efficiency of these algorithms is the diame-
ter of the underlying network: in a worst-case complexity analysis, the diameter
denotes the maximal length of a path that exchanged data would have to travel in the net-
work. As was noticed by several authors, (see the survey by Bermond, Comellas, and
Hsu [Be88a]), the ILLIAC network is not optimal in that respect: one can achieve a
better diameter with the same symmetry in the network, without compromising the con-
nectivity, and without increasing the number of connections per node.

A relaxed structure, or a generalized ILLIAC network, is obtained by letting the
number of nodes be \( n \) (not necessarily a square), and allowing connections of each node
\( i, 0 \leq i \leq n-1 \), to nodes \( i \pm 1, i \pm h \mod n \), where \( h \) is not necessarily \( \sqrt{n} \). This type of network,
which we shall denote by \( G(n,h) \), is also known in the literature as a circulant. Let \( D_n^* \)
denote the minimal diameter of such a network with \( n \) nodes (i.e.,
\[ D_n^* = \min \{ \text{Diam } G(n,h) \mid 2 \leq h \leq n-2 \} \]).

Bermond, Favaron and Maheo [Be88b] have shown that \( G(n,h) \) is the union of two
hamiltonian cycles, and thus, with the historical shadow of the DDLCN, we also adopt
the name Double-Loop Networks for \( G(n,h) \).

A lower bound, \( lb(n) = \lfloor \sqrt{2n-1} \rfloor \), on the minimal diameter \( D_n^* \) has been men-
tioned by several authors: Wong and Coppersmith [Wo74], Beivede, et. al. [Be87], Ber-
mond, Illiades, Peyrat [Be85], Boesch and Wang [Bo85], Du, Hsu, Li and Xu [Du88]. By
introducing the notation \( R[k]=\{2k^2-2k+2,...,2k^2+2k+1\} \), we will see in Chapter 2 that
\( lb(n)=k \) if \( n \in R[k] \). \( R[k] \) contains \( 4k \) values of \( n \) that correspond to each value \( k \) of the lb.
All the above-mentioned authors also noticed that the lower bound is attained when
\( n=2k^2+2k+1 \) (the largest element in \( R[k] \)).
Du, Hsu, Li and Xu [Du88] prove that the lower bound is never achieved for \( n = 2k^2 + 2k \). They also obtain an infinite optimal family of networks \( G(n,h) \), the networks for which \( D_n^* = \text{lb}(n) \). Of the \( 4k \) values of \( n \) that correspond to each value \( k \) for the lower bound, they identify the following 10 values as being optimal:

\[
2k^2 - k - 2, 2k^2 - k - 1, 2k^2 - k, 2k^2 - k + 1, \\
2k^2 - 1, 2k^2, 2k^2 + 1, \\
2k^2 + k, 2k^2 + k + 1, \\
2k^2 + 2k + 1.
\]

It is the goal of this work to identify more optimal values of \( n \), and corresponding values of \( h \), so that \( \text{Diam } G(n,h) = D_n^* \). In attempt to settle this question, all the results mentioned above were independently rediscovered in the initial stages of the work. Many additional results came up during our analysis of the problem. We have been able to show that the vast majority of all values of \( n \) are indeed optimal (more than 94% of all values up to 8,000,000), and that all nonoptimal values of \( n \) up to 8,000,000 are suboptimal, i.e., satisfy \( D_n^* = \text{lb}(n) + 1 \). These results are described in the next section. Chapters 2-6 contain detailed proofs of our results.

If one is willing to sacrifice the edges \( i,i+1 \) of \( G(n,h) \), while maintaining the regularity and the vertex-degree, the lower bound can be met for all values of \( n \). Let \( G(n;h_1,h_2) \) denote the graph with vertex set \( \{0,1,...,n-1\} \) and edge set \( \{i,i+h_1; i,i+h_2 \mid 0 \leq i \leq n-1, \text{ all expressions evaluated modulo } n \} \). It is shown in Beivede, Herrada, Balcazar, and Labarata [Be87], Bermond, Illiades, and Peyrat [Be85], and Boesch and Wang [Bo85] that given \( n \), we can find \( k \) such that \( G(n;k,k+1) \) has the smallest possible diameter.
(which in this case is the lower bound), smallest possible average diameter and largest possible connectivity among all $G(n; h_1, h_2)$. Based on these observations, the authors in [Be87] suggest a related optimized MCC architecture.

It has been shown by Bollobás and de la Vega [Bo82] that a random regular graph has diameter of order $\log_{\Delta-1} n$ with probability close to 1. Bollobás and Chung [Bo88] have shown that a random cubic loop graph obtained by adding a random perfect matching to a cycle has also a diameter of $\log_{\Delta-1} n$.

§ 1.6 Results of This Work.

Our work deals with the double-loop undirected network $G(n, h)$; its vertex set is \{0,...,n-1\}, and each vertex $i$ is connected to the vertices $i\pm 1, i\pm h \text{ mod } n$, for some integer $h$, $1 < h < n-1$. Figure 1.14 shows such a network with $n=16$, $h=4$. Given $n$, we denote by $D_n^*$ the minimal diameter of all such networks $G(n, h)$, and by $R[k]$ the range (i.e., set of integers) \{2k^2-2k+2,...,2k^2+2k+1\}. We also use the notation $n_k = 2k^2+2k+1$.

Our initial aim was to find $D_n^*$, and to find $h_n^*$ such that $\text{Diam } G(n, h_n^*) = D_n^*$.

In Chapter 2 we provide a construction of several grids of lattice points. These grids are then used to:

- Prove that $D_n^* \geq k$, whenever $n \in R[k]$, thus reestablishing the known result about the lower bound on $D_n^*$.

- Show that $\text{Diam } G(n_k, 2k+1) = k$ and $\text{Diam } G(2k^2+k, 2k+1) = k$.

- Prove that $D_n^* \geq k+1$ if $n = n_k - 1$. The proof is different than the one given in [Du88], and was obtained independently. (In Chapter 3 we show that if $n=n_k - 1$
We define $n \in R[k]$, $h$ and $G(n,h)$ as optimal if $\text{Diam } G(n,h) = k$, and as suboptimal if $\text{Diam } G(n,h) = k+1$. In Chapter 2 we also:

- Make the conjecture that for all $n \in R[k]$, $n$ is either optimal or suboptimal.
- Derive upper and lower bounds on optimal and suboptimal hops $h$.
- Design a simple algorithm to detect optimal or suboptimal hops $h$, whenever they exist. The algorithm can in fact be programmed to find values of $h$ such that $\text{Diam } G(n,h) \leq m$, for any $m>0$. If no such values exist, it stops, after performing an exhaustive search.
- Find a large set of optimal hops for a special class of optimal values of $n$ (the "quartile points" of $R[k]$).

In Chapter 3 we define various transformations on the grids we constructed in Chapter 2. Using those transformations, and the results of Chapter 2, we

- Identify several families of optimal values of $n$. These families are infinite, and properly include those obtained by Du, Hsu, Li and Xu [Du88] (see Section 1.5).
- Find that the intersection of each of these families with each $R[k]$ is a set of size $O(\sqrt{k})$, and thus is growing with $k$ (as opposed to the constant size of 10 in the previously known families).
- Show that the relative proportion of these families in each $R[k]$ is $O\left(\frac{1}{\sqrt{k}}\right)$, and therefore tends to 0 as $k \to \infty$. This motivated the name sparse families we chose for them.
In Chapter 4 we use a result about the networks $G(n; h_1, h_2)$, that were mentioned in Section 1.5. The lower bound on the minimal diameter, $D_n$, of $G(n; h_1, h_2)$ is equal to the lower bound on $D_n^*$, the minimal diameter of $G(n, h)$, and is achieved for $n \in R[k]$ in $G(n; k, k+1)$, or in $G(n; k, k-1)$ if $n \leq 2k^2 + 1$. The mapping $i \to q_i$, with $\gcd(q, n) = 1$, is a graph isomorphism between $G(n; h_1, h_2)$ and $G(n; qh_1, qh_2)$. Using this observation, and by proper choice of $q$ and $n$, we

- Derive three large optimal families of values of $n$ and corresponding optimal networks $G(n, h)$. If $n \in R[k]$ is represented by $n = n_k - 1 - p$, these families are characterized by

  1. $\gcd(k, p) = 1$,

  2. $\gcd(k+1, p) = 1$,

  3. $\gcd(k-1, p-4) = 1$, $p > 2k - 2$.

- Show that if $k$ or $k+1$ are prime, these families cover all of $R[k]$, except one or two values of $n$.

- Show that the union of the three families covers 92% of all values of $n$ up to 8,000,000. Because of this and the previous result, we call these families dense families.

- Show that despite the very large size of the dense families, there are infinitely many values of $n$ that belong to the sparse families but do not belong to the dense ones.

In Chapter 5 we generalize the results in Chapters 2-4. We define $\delta$-families that contain values of $n$ satisfying $D_n^* \leq k + \delta$, for some $\delta \geq 0$. Given $n \in R[k]$, and a hop $h$, we
• Identify $\alpha$ and $\beta$ such that $(k+\alpha)h = k+\beta \mod n$.

• Find the general form of solutions $h$ to the above congruence.

• Derive a second congruence that $h$ must satisfy.

• Gain additional knowledge about the structure of the grids that were defined in Chapter 2.

• Use this information to derive sufficient conditions for $D_n^* \leq k+\delta$. These conditions are in the form of implicit constraints involving several mutually dependent parameters, and are derived under two additional assumptions on these parameters. We show that at least in the special case of the dense families, both assumptions hold.

• We simplify the constraints using number theoretic results.

Equipped with those general sufficient conditions for $D_n^* \leq k+\delta$, we proceed and investigate the cases of special interest, namely those with $\delta=0$ or $\delta=1$. In Chapter 6 we

• Derive an infinite number (depending on the parameter $\alpha$) of sufficient conditions for optimality, each one of them defining an optimal family. All the dense families are included here as special cases. These new families cover many of the optimal values of $n$ that were not covered by the sparse or dense families.

• Derive a similar, though much longer list of sufficient conditions for $D_n^* \leq k+1$.

Despite their infinite number, only finitely many of these conditions are applicable in each range $R[k]$. Since every value of $n$ uniquely determines $k$ such that $n \in R[k]$, this set of conditions can be viewed as a battery of programmable tests, (finitely many for each $n$) and only a failure in all of them can leave a doubt about the optimality of $n$. 
The algorithm thus implied for testing optimality is very fast. When run on the SUN workstation, only about an hour was required to cover the range \( n \leq 8,000,000 \). For comparison, the algorithm in Chapter 2 takes more than two weeks of computing time, only to cover the range \( n \leq 20,000 \).

The set of sufficient conditions for \( D_n^* \leq k+1 \) can also be viewed as a set of tests on \( n \). Only an \( n \) that fails all those tests could possibly have \( D_n^* > k+1 \). All the values of \( n \leq 8,000,000 \) passed these tests. Thus, all \( n \) up to 8,000,000 must be either optimal or suboptimal. In the range \( n \leq 8,000,000 \), 93% of all the values of \( n \) satisfy the optimality conditions from Chapter 6. When combined with the sparse families in Chapter 3, the percentage is 94% in the same range.

Our conjecture from Chapter 2 still stands, but the results in Chapter 5 confirm it for very large values of \( n \). Our analysis shows that an optimal double loop network (or a suboptimal network, in a small minority of the cases) can be designed for any application of these networks which seems currently practical. The same analysis also provides values of the hop \( h \) to achieve these optimal designs.
CHAPTER 2

PROBLEM DEFINITION AND PRELIMINARY RESULTS

§ 2.1 Introduction and Problem Definition

In this work we study a special class of interconnection networks, double-loop networks, also referred to by some authors as circulants. (Only a special type of circulants is considered here.) Our network can be described by a graph $G(n,h)$, with vertex-set $V(G) = \{0,1,\ldots,n-1\}$ and edge-set

$E(G) = \{ i,i\pm 1; i,i\pm h \mid i=0,\ldots,n-1; \text{all expressions are evaluated mod } n \}$

and can be represented by a cycle with chords. Figure 2.1 shows an example of such a network, for $n=16$ and $h=4$. We shall refer to $h$ as the hop size, or just the hop.

Figure 2.1

![Diagram of $G(16,4)$]
Let $H$ be an arbitrary graph with vertex set $V$ and edge set $E$. For $u, v \in V$, the distance between $u$ and $v$ is given by

$$d(u, v) = \text{number of edges in a shortest path joining } u \text{ and } v \text{ in } H,$$

the eccentricity of $u$ is given by

$$\text{Ecc}(u) = \max \{ d(u, v) \mid v \in V \}$$

and the diameter of $H$ is given by

$$\text{Diam}(H) = \max \{ \text{Ecc}(u) \mid u \in V \}.$$

Our work deals with the following questions:

1. Given $n$, find $D^* = \min_h \text{Diam } G(n, h)$;
2. Find an $h^*$ for which $\text{Diam } G(n, h^*) = D^*$ ( $h^*$ may not be unique.)

This chapter is organized as follows:

Starting with preliminary facts, observations and notations in Section 2.2, lower bounds (lb) on $D^*_n$ and $h^*_n$ are derived and discussed in Section 2.3. Theorem 2.7 (the lower bound on the minimal diameter) has been mentioned by several authors: Wong and Coppersmith [Wo74], Beivede, et.al. [Be87], Bermond, Illiades and Peyrat [Be85], Boesch and Wang [Bo85], Du, Hsu, Li and Xu [Du88]. It identifies $4k$ values of $n$ that correspond to each value $k$ of the lb. Lemma 2.8, stating the existence of networks $G(n, h)$ for which the lower bound on the diameter is not achieved, is established by methods different from those in [Du88].
Section 2.4 contains an algorithm to find optimal values of \( h \) that minimize the diameter of \( G(n,h) \) whenever \( n \) is optimal, i.e., the lower bound on the diameter is achieved. With slight modifications, that algorithm also finds corresponding "suboptimal" values of \( h \), i.e., whenever \( D_n^* \) exceeds the lower bound by 1.

Section 2.5 provides a method of enriching the set of optimal hops for a special class of optimal values of \( n \). The method can be extended to other values of \( n \), those that admit certain factorizations. This will be done in Chapter 5.

The algorithm in Section 2.4 was programmed, and we found that all the values that were non-optimal in the range which we checked numerically (\( n \leq 26500 \)) turned out to be suboptimal. We were able to cut down a considerable amount of computations using results presented here and in Chapters 3 and 4. This was the initial basis to the conjecture mentioned in Section 2.3. Subsequent work, presented in Chapters 3-6, confirms the conjecture for all values of \( n \) up to 8,000,000.

§ 2.2 Preliminary Observations

Fact 2.1: \( \text{Diam } G(n,h) = \text{Ecc}(0) \).

This follows immediately from the symmetry of \( G(n,h) \).

Let \( F^k \) denote a diamond-shaped frame of lattice points of size \( k \) around the origin in the plane, i.e.

\[
F^k = \{ (i,j) \mid |i| + |j| \leq k, \ i,j \in \mathbb{Z} \}
\]

\( F^k \) is in fact the ball of radius \( k \) around the origin in the \( L_1 \) norm in \( \mathbb{Z}^2 \),
i.e., \| (a, b) \| = |a| + |b|.

Let \( L^m \) denote the \( m \)-th layer of \( F_k \), \( 0 \leq m \leq k \):

\[
L^m = \{ (i, j) \mid |i| + |j| = m, \ i, j \in \mathbb{Z} \}.
\]

Fact 2.2: \( |F_k| = 2k^2 + 2k + 1 \).

Proof: \( |L^0| = 1 \), and by induction on \( m \) one can show that \( |L^m| = 4m, \ m \geq 1 \).

Thus,

\[
|F_k| = |L^0| + |L^1| + \ldots + |L^k| = 1 + 4 + 8 + \ldots + 4k = 2k^2 + 2k + 1. \qed
\]

Given \( G(n, h) \), we construct an infinite grid \( G_{n, h} \) in \( \mathbb{Z}^2 \), labeling each lattice point \((i, j)\) by \( i + jh \mod n \). \( G_{n, h}(i, j) \) will denote the label corresponding to the lattice point \((i, j)\). Every label \( m, 0 \leq m \leq n-1 \), is repeated in \( G_{n, h} \) infinitely many times, resulting in a tesselation of \( \mathbb{Z}^2 \).

Consider the following linear ordering of \( \mathbb{Z}^2 \): Let \( z_1 \in L^m \), \( z_2 \in L^n \). We shall say that \( z_1 < z_2 \) if

(a) \( m < n \), or

(b) \( m = n \) and \( z_1 \) precedes \( z_2 \) in a counterclockwise ordering of \( L^m \)

starting at the \( x \) axis.

We shall identify a basic grid \( g_{n, h} \) as follows: starting with \( G_{n, h} \), we shall keep in \( g_{n, h} \) each label \( m, 0 \leq m \leq n-1 \), only where it occurs first according to the above ordering of \( \mathbb{Z}^2 \). All subsequent points of \( \mathbb{Z}^2 \) in this order that are labeled by \( m \) in \( G_{n, h} \), are unlabeled in \( g_{n, h} \). If point \((i, j)\) is unlabeled in \( g_{n, h} \) we shall say that
$g_{n,h}(i,j)$ is nil (otherwise $g_{n,h}(i,j)=G_{n,h}(i,j)$ ). To simplify our notation, we shall also use $g_{n,h}$ to denote only the labeled part of the grid. In $g_{n,h}$, labels of points in $L^d$ correspond to vertices in the graph $G(n,h)$ that are at distance $d$ from 0. If $m = g_{n,h}(i,j) = i+jh \mod n$, then in $G(n,h)$ vertex $m$ is reachable from 0 via $i$ edges on the cycle (of the form $a,a+1$ or $a,a-1$) and $j$ chord-edges.

We shall further denote by $G_{n,h}^k$ that part of the grid $G_{n,h}$ that lies in $F^k$.

$|G_{n,h}^k|$ and $|g_{n,h}|$ will denote the number of labeled points in each grid, hence by Fact 2.2

$$|g_{n,h}|=n \quad \text{and} \quad |G_{n,h}^k|=|F^k|=2k^2+2k+1.$$  

Let us mention here that the grids just described are not absolutely necessary for what follows, yet they provide a helpful tool for the subsequent presentation.

Figure 2.2 depicts $g_{22,3}$, $g_{22,4}$ and $g_{22,6}$.

Figure 2.2

```
12
  9 10 11
  6 7 8
  3 4 5
20 21 0 1 2
17 18
14 15
13
```

$g_{22,3}$  $g_{22,4}$  $g_{22,6}$

**Observation 2.3:** $\text{Diam } G(n,h) = \min \{ k \mid g_{n,h} \text{ lies in } G_{n,h}^k \}$. 

The grids in Figure 2.2, for example, correspond to graphs $G(n,h)$ with diameters
5, 4 and 3 respectively.

In $G^k_{n,h}$ we shall denote by $C_j$ the row centers on the $y$-axis:

$$C_j = G^k_{n,h}(0,j) = jh \mod n, \ 0 \leq |j| \leq k.$$ 

We shall use the same notation for the grid elements on the $y$-axis in $g_{n,h}$. For example in $g_{22,4}$, $C_{-2}=14$. The row of the grid that contains $C_j$ will be referred to as row $j$, e.g. row 1 of $g_{22,4}$ contains 3,4,5,6 while row 1 of $G^3_{22,4}$ contains 2,3,4,5,6.

The capacity of row $j$ in $G^k_{n,h}$ is the set of labels $\{C_j-(k-|j|),...,C_j+(k-|j|)\}$, i.e. there are $k-|j|$ labels on each side of $C_j$.

**Fact 2.4:**

(a) $\text{Diam } G(2k^2+2k+1, 2k+1) = k.$

(b) $\text{Diam } G(2k^2+k, 2k) = k.$

**Proof:** In both cases $|g_{n,h}| = n > |G_{n,h}^{k-1}| = 2k^2-2k+1$, hence, in view of Observation 2.3, it is sufficient to show that $g_{n,h}$ lies in $G_{n,h}^k$.

(a) Consider $G_{n,h}^k$ where $n=2k^2+2k+1$, $h=2k+1$. (See Figure 2.3.a for the case $k=4$.) We have

$$C_j = \begin{cases} 
(j(2k+1), & j \geq 0 \\
2k^2 + (j+1)(2k+1), & j < 0 
\end{cases}$$

and the following equalities hold:

$$C_{-k+j} - C_j = k+1, \ j=0,...,k$$

$$C_j - C_{-k+j-1} = k, \ j=1,...,k \quad (2.1a)$$
Considering the capacity of the rows involved, equalities (2.1a) imply that in $G_{n,h}^k$

(i) $\{C_j, \ldots, C_{-k+j}\}$ is covered, without any gaps or duplications, by the right half of row $j$ and by the left half of row $(-k+j)$, $j=0, \ldots, k$.

(ii) $\{C_{-k+j-1}, \ldots, C_j\}$ is covered without any gaps or duplications by the right half of row $(-k+j-1)$ and by the left half of row $j$, $j=1, \ldots, k$.

In other words: when the rows of $G_{n,h}^k$ are traversed in the order $0, -k, 1, -k+1, 2, -k+2, \ldots, k-1, -1, k, 0$, starting and ending in the origin, $\{0, \ldots, n-1\}$ is covered without duplication. Using Fact 2.2 we also get

$$|g_{n,h}| = |G_{n,h}^k| = 2k^2 + 2k + 1 = n,$$

and so in this case $G_{n,h}^k$ coincides with $g_{n,h}$.

(b) In this case $n=2k^2+2k$, $h=2k$, and we have

$$C_{-k+j} - C_j = k, \quad j=0, \ldots, k$$
$$C_j - C_{-k+j-1} = k, \quad j=1, \ldots, k.$$  \hspace{1cm} (2.1b)

Equalities (2.1b) imply that when the rows of $G_{n,h}^k$ are traversed in the order mentioned in (a), $\{0, \ldots, n-1\}$ is covered. This time all elements in the upper right diagonal of $G_{n,h}^k$ are duplicated in the lower left diagonal, which accounts for exactly $k+1$ duplications. In Figure 2.3.b $G_{n,h}^k$ for this case is illustrated for $k=4$.

From the above we may conclude that $g_{n,h}$ lies in $G_{n,h}^k$. $\Box$
Notation:

\[ n_k = 2k^2 + 2k + 1, \quad k \geq 0; \]

\[ R[k] = \{ n_{k-1} + 1, \ldots, n_k \}, \quad k \geq 1. \]

Remark: By Fact 2.2, \( |F^k| = n_k \). Note also that the natural numbers are partitioned into \( \{1\} \cup \bigcup_{k \geq 1} R[k] \), where \( R[k] = \{ 2k^2 - 2k + 2, \ldots, 2k^2 + 2k + 1 \} \), and

\[ |R[k]| = 4k. \]

In the range \( R[k] \) we shall distinguish what we will call the "quartile points"

\[ q_1[k] = 2k^2 - k, \quad q_2[k] = 2k^2, \quad q_3[k] = 2k^2 + k \]

and we denote the four "quartiles" by

\[ Q_1[k] = \{ n_{k-1} + 1, \ldots, q_1[k] \}, \quad Q_2[k] = \{ q_1[k] + 1, \ldots, q_2[k] \}, \]

\[ Q_3[k] = \{ q_2[k] + 1, \ldots, q_3[k] \}, \quad Q_4[k] = \{ q_3[k] + 1, \ldots, n_k \}. \]

Note that the "quartiles" are not of equal size. Namely
\begin{align*}
|Q_1[k]| &= k-1, \quad |Q_2[k]| = |Q_3[k]| = k, \quad |Q_4[k]| = k+1.
\end{align*}

**Observation 2.5:** \( \sum_{(i,j) \in L} G_{n,h}(i,j) = 2tn, \quad t \geq 0. \)

Thus, the sum of the elements in the \( t \)-th layer of \( G_{n,h} \) is independent of \( h \).

**Proof:** This follows from the equalities
\begin{align*}
G_{n,h}(t,0) + G_{n,h}(-t,0) &= n, \\
G_{n,h}(t-j,j) + G_{n,h}(-t+j,j) &= (t-j+C_j)+((-t+j)+C_j) = 2C_j, \quad |j| \geq 0, \quad \text{and} \\
C_j + C_{-j} &= n, \quad |j| > 0. \quad \square
\end{align*}

**Observation 2.6:** \( \text{Diam } G(n,h) = \text{Diam } G(n,n-h), \) and therefore
\[
D_n^* = \min \{ \text{Diam } G(n,h) \mid h < \left\lfloor \frac{n}{2} \right\rfloor \}.
\]

In case \( n \) is even and \( h = \frac{n}{2} \), \( G(n,h) \) is a 3-regular graph, hence its diameter would be large compared to other choices of \( h \), for which the graph is 4-regular.

Specifically, in \( G_{n,\frac{n}{2}}, \ C_1 = \frac{n}{2}, \) so in the basic grid \( g_{n,\frac{n}{2}} \) the non-nil labels occur only in rows 0 and 1, forcing it to be very long.

The situation is similar when \( n \) is odd. Letting \( h = \frac{n-1}{2} \) leads to a basic grid \( g_{n,h} \) in which only row 0, the right part of row 1 and the left part of row -1 have non-nil labels.

**Remark:** In fact, \( \text{Diam } G(n, \left\lfloor \frac{n}{2} \right\rfloor) = \left\lfloor \frac{n}{4} \right\rfloor \) for \( n \) even or odd. Results in the next section will imply that this value of the diameter is minimal only if
\( n = 1, 2, 3, 4, 6, 7, 8, 12. \)

\[ \text{\textsection 2.3 Bounds on } D_n^*, \ h_n^* \]

**Theorem 2.7:** (A lower bound for \( D_n^* \))

\[ D_n^* \geq k \text{ for all } n \in \mathbb{R}[k], \text{ and that bound is the best possible.} \]

**Proof:** From Fact 2.2 we know that \( \mathfrak{K}^{k-1} \mid G^k_{n,h} \mid = n_{k-1}. \) Hence, by Observation 2.3, for \( n > n_{k-1}, \ D_n^* \geq k. \) From Fact 2.4 we know that the bound is achieved for \( n = n_k \) and for \( n = q_3[k]. \) \( \square \)

For certain values of \( n \) the inequality in Theorem 2.7 could be strict. In fact we have the following:

**Lemma 2.8:** Let \( n = n_{k-1}, \ k \geq 1. \) Then \( D_n^* \geq k+1. \)

**Proof:** Let \( n = n_{k-1} = 2k^2 + 2k. \) Suppose \( h \) exists such that \( \text{Diam } G(n,h) = k, \) and so, by Observation 2.3, \( G^k_{n,h} \) covers \( g_{n,h}. \) Since \( \mid G^k_{n,h} \mid = n+1, \) exactly one number \( x, x \in \{0, ..., n-1\}, \) is repeated in \( G^k_{n,h}. \) Let us now use Observation 2.5, and the fact that \( x \) is the only repeated element. By equating the sum of the elements in \( G^k_{n,h} \) to \( x + \sum_{i=0}^{n-1} i \) we get

\[ 2n(1+...+k) - x = \frac{n(n-1)}{2}, \text{ or } \]

\[ x = \frac{n}{2} = k^2 + k. \]

Both occurrences of \( x \) must be in the outmost layer \( L^k, \) else their neighbors in more exterior layers of \( G^k_{n,h} \) would have to be repeated as well. Similarly, if
both occurrences of $x$ are at the right end (left end) of two rows in $G_{n,h}^k$, none of which being the top or bottom row, then the two elements to the left (right) of $x$ would also have to be repeated. (The extremal situations are covered by the cases below.) $x$ must therefore be the rightmost element of some row $j_1$, and the leftmost element of another row, $j_2$. Considering the capacity of those rows in $G_{n,h}^k$, we get

$$x = C_{j_1}+(k-|j_1|) = C_{j_2}-(k-|j_2|).$$

Case 1: $0<j_1<k$, $-k<j_2<0$.

Here

$$x = G_{n,h}^k(k-j_1j_1) = G_{n,h}^k(-k-j_2j_2)$$

but then

$$x+h-1 = G_{n,h}^k(k-j_1-1,j_1+1) = G_{n,h}^k(-k-j_2-1,j_2+1)$$

(2.2)

and

$$x-h+1 = G_{n,h}^k(k-j_1+1,j_1-1) = G_{n,h}^k(-k-j_2+1,j_2-1)$$

(2.3)

and so other elements of $0,...,n-1$ must also be repeated - contrary to our assumption.

Case 2: $0<j_1<k$, $j_2=0$ or $j_2=-k$.

Case 3: $-k<j_2<0$, $j_1=0$ or $j_1=k$.

Case 4: $j_1=0$, $j_2=-k$ or $j_1=k$, $j_2=0$.

In each of these cases one of the equalities (2.2) or (2.3) will lead to a contradiction.

Case 5: $j_1=0$, $j_2=0$. 
That would imply $k = n - k$, which is impossible since $n = 2k^2 + 2k$.

Case 6: $j_1 = k$, $j_2 = -k$.

This can be rewritten as $kh - kh \mod n$, hence:

$$h = \epsilon(k + 1), \quad \text{for some integer } \epsilon, 0 < \epsilon < k, \text{ and}$$

$$C_k = C_{-k} = \frac{n}{2} = k(k + 1). \quad \text{(For all } j, C_j + C_{-j} = n.)$$

Having $h = \epsilon(k + 1)$ and $n = 2k(k + 1)$, we must also have

$$C_j \in \{0, k + 1, 2(k + 1), \ldots, (2k - 1)(k + 1)\}, \quad -k \leq j \leq k.$$ 

This set has cardinality $2k$. On the other hand, of the $2k + 1$ $C_j$'s exactly two are equal, and therefore $\{C_j | |j| \leq k\} = \{0, k + 1, 2(k + 1), \ldots, (2k - 1)(k + 1)\}$. In particular, for some $j_0$, $0 < |j_0| < k$, $C_{j_0} = k + 1$. But then $G_{n,h}^k (j_0) = k = G_{n,h}^k (0), \quad \text{contradicting the fact that } x = k(k + 1) \text{ is the only repeated element.}$

Case 7: $-k \leq j_1 \leq 0$, $0 \leq j_2 \leq k$.

The proof mimics the proofs of Cases 1 - 6.

Case 8: $j_1, j_2 > 0$.

Contradiction here follows from the equality

$$G_{n,h}^k (k-j_1, j_1 - 1) = G_{n,h}^k (k-j_2, j_2 - 1).$$

Case 9: $j_1, j_2 < 0$.

The treatment is similar to that of Case 8. □

In fact we can make a stronger assertion. In Theorem 2.6 of Section 2.5 we show that $D_{n-1} = k + 1$. 
Conjecture: For all \( n \in \mathbb{R}[k] \), \( k \leq D_n^* \leq k+1 \).

Using the algorithm in the next section, along with results that appear later (that allowed cutting down a considerable amount of cases), we have verified the conjecture for \( n \leq 26500 \). Later results, presented in Chapters 3-6 confirm the conjecture up to \( n = 8,000,000 \).

In the sequel we shall use the following definition. Let \( n \in \mathbb{R}[k] \).

If \( h_n^* \) exists such that \( D_n^* = \text{Diam} \ G(n,h_n^*) = k \), then \( n, h_n^* \) and \( G(n,h_n^*) \) will be called optimal.

If \( h_n^* \) exists such that \( D_n^* = \text{Diam} \ G(n,h_n^*) = k+1 \), then \( n, h_n^* \) and \( G(n,h_n^*) \) will be called suboptimal.

**Theorem 2.9:** (Lower bounds on optimal h's.)

Let \( n \in \mathbb{R}[k] \), \( h \leq 2k+1 \), be such that \( G(n,h) \) is optimal. Let \( p \) be given by \( n = 2k^2 + 2k - p \).

Then \( h \geq h_0 \), where \( h = \begin{cases} 2k+1 - \lfloor \sqrt{2p+3} \rfloor, & \text{if } h = 2t \\ 2k+1 - \lfloor \sqrt{2p+2} \rfloor, & \text{if } h = 2t+1 \end{cases} \).

**Proof:** Fact 2.4 assures us that such networks indeed exist.

(a) Let \( h = 2t \). In \( g_{n,h} \) the equalities:

\[ C_j - C_{j-1} = 2jt - 2(j-1)t = 2t, \quad j = -k+1, \ldots, k \]

imply that, since \( C_j - t = C_{j-1} + t \),

\[ g_{n,h}(i,j) \text{ is nil when } \begin{cases} j > 0; & i \leq -t \text{ or } i > t \\ j = 0; & |i| > t \\ j < 0; & i \leq -t \text{ or } i \geq t \end{cases} \]
The above situation is shown in Figure 2.4.a, for \( k=6, n=68, h=8, t=4 \).

**Figure 2.4.a**

Let \( S \) denote that part of \( G_{n,h}^k \) which is nil in \( g_{n,h} \). (\( S \) is shaded in Figure 2.4.a.) Let \( |S| \) denote the number of points in \( S \), hence \( |S| = 2(k-t)^2 + 2(k-t) \).

By Observation 2.3, if \( G(n,h) \) is optimal, we must have

\[
n \leq |G_{n,h}^k| - |S|
\]

which, after making the appropriate substitutions, becomes

\[
2k^2 + 2k - p \leq 2k^2 + 2k + 1 - 2\left(k - \frac{h}{2}\right)^2 - 2\left(k - \frac{h}{2}\right).
\]

The bound on \( h \) follows from the solution of this inequality.
(b) By repeating the above arguments for \( h=2t+1 \), we get

\[
\mathcal{g}_{n,h}(i,j) \text{ is nil when } \begin{cases} 
    j>0; & \text{if } i<-t \text{ or } i>t+1 \\
    j=0; & \text{if } |i|>t+1 \\
    j<0; & \text{if } i<-t-1 \text{ or } i>t 
\end{cases},
\]

\[ |S| = 2(k-t)^2, \]

and \( h \) is obtained in a similar fashion to that of \( h=2t \).

The situation is pictured in Figure 2.4.b for \( k=6 \), \( n=67 \), \( h=7 \), \( t=3 \). □

Remark: It will be shown in Theorem 2.11 that for all \( k \), and all \( i \), \( 1 \leq i < \sqrt{2k} \), \( G(n_k-2i^2,h) \) is optimal .

A similar result can be obtained for the suboptimal case:

**Theorem 2.9': (Lower bounds on suboptimal \( h \)'s.)**

Let \( n \in \mathbb{R}[k] \), \( h \leq 2k+3 \) be such that \( G(n,h) \) is suboptimal. Let

\[ n=2k^2+2k-p. \]

Then \( h \geq h' \), where \( h' = \begin{cases} 
    2k+3- \left\lfloor \frac{\sqrt{2p+8k+9}}{2} \right\rfloor, & \text{if } h=2l \\
    2k+3- \left\lfloor \frac{\sqrt{2p+8k+10}}{2} \right\rfloor, & \text{if } h=2l+1 \end{cases} \). □

Again, one can show as in Theorem 2.11, that for every \( k \) there exist \( n \in \mathbb{R}[k] \) such that \( G(n,h) \) is suboptimal.

How large could an optimal hop \( h \) become, given \( n \)? Theorem 2.10 goes beyond Observation 2.6, by providing the upper bound on \( h \).
Theorem 2.10: (Upper bounds on optimal h's)

Let \( n \in \mathbb{R}[k] \) and \( h \) be such that \( G(n,h) \) is optimal. Let \( p \) be given by \( n = 2k^2 + 2k - p \).

Then \( h \leq \bar{h} \), where \( \bar{h} = \begin{cases} k^2 - \left\lfloor \frac{p+1-\sqrt{2p+5}}{2} \right\rfloor, & \text{if } n = 2m \\ k^2 - \left\lfloor \frac{p+1-\sqrt{2p+6}}{2} \right\rfloor, & \text{if } n = 2m+1 \end{cases} \).

Proof: The proof is similar to that of Theorem 2.9, and is based on "area considerations". Rather than repeating the calculations, we shall illustrate the proof using the examples of \( g_{70,32} \) and \( g_{71,33} \), shown in Figure 2.5.a and Figure 2.5.b. They depict the configurations obtained for the cases of \( n \) being even and odd, respectively, when \( n - 2h = \delta \) and \( \delta \) is relatively small, i.e. \( h \) is close to \( \frac{n}{2} \).
In both cases $C_{-2} = \delta$, hence

$$C_{j-2} - C_j = \delta, \quad j \neq 2.$$ 

In the case of an even $n$, all rows of $g_{nj}$ contain at most $\delta$ elements, except for rows 0 and 1, which contain $\delta+1$ elements. (Note that in Figure 2.5.a the location of 4 does not follow the convention to fill layers counterclockwise starting from the x-axis. 4 was relocated, within the same layer, to simplify the structure.)

When $n$ (and therefore $\delta$) is odd, the situation is similar: all rows of $g_{n,h}$ contain at most $\delta$ elements, except for rows -1 and 1 which contain $\delta+1$ elements and row 0 that contains $\delta+2$ elements. In this case, $g_{n,h}$ contains $\left\lceil \frac{\delta}{2} \right\rceil$ full layers.

The upper bound $\bar{h}$ on optimal hops is obtained from the inequality

$$n \leq \lfloor G_{n,h}^k \rfloor - |S|$$

where $S$ and $|S|$ have the same meaning as in the proof of Theorem 2.9. □

In a similar fashion we obtain the next theorem, for which the proof is omitted.

**Theorem 2.10':** (Upper bounds on suboptimal $h$'s)

Let $n \in \mathbb{R}[k]$ and $h$ be such that $G(n,h)$ is suboptimal. Let $p$ be given by $n=2k^2+2k-p$.

Then $h \leq \bar{h}$, where $\bar{h} = \begin{cases} k^2 - \left\lfloor \frac{p+3-\sqrt{2p+8k+13}}{2} \right\rfloor, & \text{if } n=2m \\ k^2 - \left\lfloor \frac{p+3-\sqrt{2p+8k+14}}{2} \right\rfloor, & \text{if } n=2m+1 \end{cases}$.
§ 2.4 Algorithms for Finding Optimal and Suboptimal Hops

Let $n \in \mathbb{R}[k]$. The observations made so far and the bounds derived in Section 2.3 lead to the following simple algorithm for finding optimal hops $h$ for $n$, if indeed they exist. It is based on the idea that if $h$ is optimal for $n$, then $G_{n,h}^k$ covers $g_{n,h}$. Let $\underline{h}$ and $\bar{h}$ denote the lower and upper bound on $h$, as given by Theorem 2.9 and Theorem 2.10, respectively. In Algorithm 2.1 we scan consecutive layers of $G_{n,h}^k$ and keep the labels occurring there in the set $C$.

**Algorithm 2.1** (for finding optimal $h$'s)

\[ N :=\{0,...,n-1\}; \]

For $h := \underline{h}$ to $\bar{h}$ do

\[ C := \emptyset; \]

For $i := -k$ to $k$ do

\[ C := C \cup \{i+jh \mod n\}; \]

If $N - C = \emptyset$ then

mark $h$ as optimal;

end;

**Algorithm 2.2** An algorithm for finding suboptimal $h$'s

In Algorithm 2.1 replace $k$ by $k+1$ and use the bounds of Theorem 2.9' and Theorem 2.10' for $h$ and $\bar{h}$ respectively.
§ 2.5 Optimal Hops for the Quartile Points in $\mathbb{R}[k]$

The next theorem, while not adding any new optimal families, provides a large repertoire of optimal hops for the three quartile points in $\mathbb{R}[k]$:

$$q_1[k] = 2k^2 - k, \quad q_2[k] = 2k^2, \quad q_3[k] = 2k^2 + k.$$  

In the following, $(l,k)$ will denote the greatest common divisor of $l$ and $k$.

**Theorem 2.11:** Let $n = q_i[k]$, $h^+_l = \frac{n}{k} \pm 1 = l(2k-2+i)\pm 1$, $i=1,2,3$, where $1 \leq l < \frac{k}{2}$, $(l,k)=1$. Then $G(n,h^+_l)$ is optimal.

**Proof:** Let $n = q_i[k]$, $0 \leq i \leq 3$, $h = h^+_l = \frac{n}{k} \pm 1$ where $1 \leq l < \frac{k}{2}$ and $(l,k)=1$. Hence $h < \frac{n}{2}$, and we have the congruence

$$kh \equiv k \mod n. \tag{2.4}$$

Since $C_j - C_{-k+j} \equiv jh - (-k+j)h \equiv kh \equiv k \mod n$, we see that in $G_{n,h}^k$

$$C_j - C_{-k+j} = k, \quad j = 0, \ldots, k. \tag{2.5}$$

Also, since $(l,k)=1$, a solution exists to the equation

$$\varepsilon k - j_0 l = 1 \tag{2.6}$$

such that $\varepsilon < l$, $j_0 < k$. Multiply (2.6) by $\frac{n}{k} = 2k-2+i$ to get

$$\varepsilon n - j_0 (h-1) = 2k-2+i.$$  

By substituting $k=C_k$ and $\varepsilon n - j_0 h = C_{-j_0}$ (rightly so, since $0 < \varepsilon n - j_0 h = 2k-2+i - j_0 < n-1$), we get

$$C_{-j_0} - C_k = k-2+i-j_0.$$
which can be extended to
\[ C_{-j_0+j} - C_{k-j} = k-2+i-j_0, \quad j=0,\ldots,k-j_0 \quad \text{and} \]
\[ C_{-j_0+j} - C_{j} = 2k-2+i-j_0, \quad j=1,\ldots,j_0-1 \quad \text{(2.7)} \]

Equalities (2.5) and (2.7) suggest the order in which the rows of \( G_{n,h}^k \) are visited when the set of labels \( \{0,1,\ldots,n-1\} \) is scanned in increasing order:

- 0, k, \(-j_0, k-j_0, k-2j_0,\ldots, \) if \( 2j_0 \geq k+1 \),
- 0, k, \(-j_0, k-j_0, -2j_0,\ldots, \) if \( 2j_0 \leq k \).

Within each row of \( G_{n,h}^k \), the labels form a sequence of consecutive numbers; hence to check whether all of the labels 0,\ldots,n-1 are included, we need to show that no gaps are formed in the row-transitions. Equality (2.5) implies the transition from row \(-k+j\) to row \( j \), for each \( 0 \leq j \leq k \). Using row capacity arguments, as in the proof of Fact 2.4, we see that upon each such transition we have exactly one duplication. Similarly, there are \( 3-i \) duplications on each transition implied by (2.7). There are no gaps on any of those transitions, and there is a total of \( (4-i)k \) duplications, hence \( G_{n,h}^k \) contains all of the labels \( \{0,\ldots,n-1\} \). The case \( h = h_1 \) is similar.

(In Example 2.1 the above is represented for \( k=11 \).)

Suppose now that \( 1 \leq l < \frac{k}{2} \) and \( (k,l) = \alpha > 1 \). Let \( k = \sigma \alpha \), \( l = \sigma \alpha \), \( 2 \leq \sigma \leq \frac{k}{2} \), \( 2 \leq \sigma \leq \frac{1}{2} \). In

When \( h = h_1 \), the equalities (2.5) and (2.7) become

(i) \( C_{-j_0+j} - C_{j} = k, \quad j=0,\ldots,k \)
(ii) \( C_{-j_0+j} - C_{k-j} = k-2+i-j_0, \quad j=0,\ldots,k-j_0 \)
(iii) \( C_{-j_0+j} - C_{j} = 2k-2+i-j_0, \quad j=1,\ldots,j_0-1 \)
that case

\[ C_r = rh = \frac{k}{\alpha} \cdot \left( \frac{n}{k} \cdot l + 1 \right) = \frac{1}{\alpha} \cdot n + r = s \cdot n + r \equiv r \mod n \]

and so \( C_r \leq \frac{k}{2} \). That in turn implies that in \( g_{n,h} \) all rows beyond row \( r-1 \) (and by symmetry, those below row \( n-r+1 \)) are nil. (Such a situation is pictured in Figure 2.6 for \( k=6, n=72, l=2, h=25 \).) Since \( r \leq \frac{k}{2} \), \( |\{ g_{n,h}(i,j) \mid (i,j) \in F^k \}| < n \) and thus \( h \) cannot be optimal. □

**Figure 2.6**

![Figure 2.6](image_url)

**Example 2.1**: Let \( k=11 \). The quartile points of \( R[11] \) are 231, 242, 253. According to Theorem 2.11 all of the following are optimal:
The idea of Theorem 2.11 can be extended to other values of \( n \) as well. This is illustrated in Example 2.2 below.

Example 2.2: Let \( n=2k^2-k-1=(k-1)(2k+1) \) and let \( l \) be such that \( 1<\frac{k-1}{2} \), \( (l,k-1)=1 \). Let \( h=h^+_1+\frac{n}{k-1}l+1 \). By simulating the calculations of Theorem 2.11, we get the following:

\[
(k-1)h \equiv k-1 \mod n,
\]

\[
C_j-C_{k+j+1} = k-1, \quad j=0,...,k,
\]

\[
C_{-j-k}-C_{k-1-j} = k+2-j_0, \quad j=0,...,k-j_0,
\]

\[
C_{-j-k}-C_j = 2k+1-j_0, \quad j=1,...,j_0,
\]

where \( j_0 \) is a solution to the equation

\[
e(k-1)-j_0l=1.
\]

In the special case of \( k=11 \) and \( n=230 \), the possible values for \( l \) are 1 and 3, and the values of optimal hops (\( h^-_1 \) and \( h^+_1 \)) computed here are 22, 24, 68, 70.

The above method will find a set of optimal hops for values of \( n \) that admit certain factorizations. It certainly does not include all possible values of \( n \), and even for those values of \( n \) covered by the method, additional optimal hops might exist.
CHAPTER 3

SPARSE OPTIMAL FAMILIES

§ 3.1 Introduction

Let \( \Theta \) be a family of integers. We shall say that \( \Theta \) is optimal if every \( n \in \Theta \) is optimal, according to the definition preceding Theorem 2.9. We shall also denote by \( \Theta[k] \) the set \( \Theta \cap R[k] \), and define the density of \( \Theta \) by

\[
f_k(\Theta) = \frac{|\Theta[k]|}{|R[k]|} = \frac{|\Theta[k]|}{4k}.
\]

The first optimal family, \( \{n_k \mid k \geq 1\} \), was discovered by Wong and Coppersmith [Wo74], and later rediscovered by several authors (see Section 1.5).

A larger optimal family, call it \( \Lambda \), was obtained by Du, Hsu, Li and Xu [Du88]. \( \Lambda[k] \), the intersection of \( \Lambda \) with the range \( R[k] \), consists of the following 10 values: (here \( q_i[k] \), \( 1 \leq i \leq 3 \), are the quartile points in \( R[k] \), as defined in Section 2.2)

\[
q_1[k]-2, \, q_1[k]-1, \, q_1[k], \, q_1[k]+1, \\
q_2[k]-1, \, q_2[k], \, q_2[k]+1, \\
q_3[k], \, q_1[k]+1, \\
n_k.
\]

In Sections 3.2 and 3.3 we identify new large infinite optimal families, which we call \( \Phi \)-families. We shall show that the union of the \( \Phi \)-families properly includes \( \Lambda \). We shall further prove that each of the \( \Phi \)-families intersects each range \( R[k] \) in a set of cardinality \( O(\sqrt{k}) \). As these cardinalities grow with \( k \), our new \( \Phi \)-
families constitute a considerable improvement over the $\Lambda$-families. The density of the $\Phi$-families is $O(\frac{1}{\sqrt{k}})$, and tends to 0 when $k \to \infty$; this is the reason for naming them "sparse families".

§ 3.2 Optimal Family $\Phi_1$

Let $n,h$ be given, and consider $g_{n,h}$. In the sequel, the upper part of $g_{n,h}$ consists of all rows $j$, $j>0$ and of the right half of row 0, including 0. The lower part of $g_{n,h}$ consists of all rows $j$, $j<0$, and of the left part of row 0, excluding 0.

In the next two lemmas we shall identify special classes of networks along with their diameters. In the theorems that follow, those classes will be used to derive some infinite optimal families of networks.

Lemma 3.1: Let $h=2t+1$, $n=nt+mh$, $t \geq 1$, $m \geq -t+1$.

Then \[ \text{Diam } G(n,h) = t + \left\lceil \frac{m}{2} \right\rceil. \]

Proof: By Fact 2.4 we know that $G(n_t,h)$ has diameter $t$. Consider $g_{n,h}$ (see Figure 3.1). When $n$ is increased to $n_t+h$, the upper part of $g_{n,h}$ is not affected at all, while the lower part is "pushed" one row downwards. This is true because $th<n$, and therefore in the equality $C_j=jh+\varepsilon n$, we have

\[ \varepsilon = \begin{cases} 
0, & \text{if } 0 \leq j \leq t \\
1, & \text{if } -t \leq j \leq -1.
\end{cases} \quad (3.1) \]

In addition to that, the lower part of the external layer can be "folded" onto the upper part. We shall refer to this process as the "Cut and Push Down"
transformation (CPD). An example is shown in Figure 3.1 below.

Figure 3.1
The CPD Transformation (for an odd h)

After one application to \( g_{n,h} \) of CPD, \( g_{n+h,h} \) is completely covered by \( G_{n+h,h}^{t+1} \), and therefore \( G(n_{t+1},h) \) has diameter \( t+1 \) (which is optimal for that \( n \)).

In \( g_{n+h,h} \), the lower half of \( L^{t+1} \) is nil. Another application of CPD would further
increase the current value of \( n \) by \( h \) without affecting the diameter of the corresponding network. This value of \( n \) is still optimal, since
\[
n = n_t + 2h = n_{t+1} - 2 \in R[t+1].
\]

We can now repeat the process. Since \( h \) is constant and \( n \) increases, (3.1) remains true, and thus the diameter is increased by 1 with every two applications of CPD. Similarly, the diameter is decreased by 1 with every two applications of CPU (Cut and Push Up), the inverse of CPD. The CPU transformation causes \( n \) to decrease by \( h \), and it may only be applied while in the resulting net \( n = n_t + mh > h \) (i.e. \( m \geq -t+1 \)).

\[\Box\]

Let \( h = 2t+1 \), where \( t \geq 1 \) is fixed. Consider the set
\[
A_h = \{ n \mid n = n_t + mh, \; m \geq -t + 1 \}
\]

The set \( \{ G(n,h) \mid n \in A_h \} \) constitutes an infinite chain of networks whose diameter is given by Lemma 3.1. Let \( A_h^* \) (\( A_h^{**} \)) denote that part of \( A_h \) that contains optimal (suboptimal) values of \( n \).

**Example 3.1**: The beginning of \( A_9 \) is shown in the following table: (\( k \) in the table is such that \( n \in R[k] \)).
In Theorem 3.2 we shall derive an infinite optimal family by considering, for each $k$, the set $R[k] \cap (\bigcup_{q \geq 1} A_{2q+1}^*)$. This set, by Fact 2.4, is never empty:

$$n_k, q_2[k] \in R[k] \cap A_{2k+1}^*.$$

**Theorem 3.2:** Optimal family $\Phi_1$.

Let $$\Phi_1 = \bigcup_{k \geq 1} \Phi_1[k]$$ where

$$\Phi_1[k] = \{n_k - 2i^2 \mid 0 \leq i < \sqrt{2k}\} \cup \{q_2[k] - (2i^2 + 2i) \mid 0 \leq i \leq -1 + \frac{\sqrt{4k-3}}{2}\}.$$ 

Then (a) $\Phi_1$ is optimal.
(b) For \( n \in \Phi_1[k] \), an optimal hop is given by

\[
h_n^* = \begin{cases} 
2k+1+2i & \text{if } n=n_k-2i^2 \\
2k+(2i+1) & \text{if } n=q_2[k]-(2i^2+2i) 
\end{cases}
\]

(c) \( f_k(\Phi_1) \approx \frac{0.6}{\sqrt{k}} \), for \( k \) sufficiently large.

Proof: Let \( n \in R[k] \), and suppose \( t,m \) exist such that \( n \in A_{2t+1}^* \), i.e. \( n=n_t+mh \), where \( h=2t+1, m \geq -t+1 \). (e.g. when \( n=n_k \) use \( t=k \) and \( m=0 \); when \( n=q_2[k] \) use \( t=k-1 \) and \( m=1 \).) By Lemma 3.1, \( \text{Diam } G(n,h) = t+\lceil \frac{m}{2} \rceil \). \( G(n,h) \) is optimal if

\[
t+\lceil \frac{m}{2} \rceil = k \quad \text{or} \quad t=k-\lceil \frac{m}{2} \rceil .
\]

Since \( n \in R[k] \), there exists a \( 0 \leq p \leq 4k-1 \) such that \( n=n_k-p \). Substituting the above value of \( t \) into the equation

\[
n_k-p = n_t+m(2t+1)
\]

and then solving for \( p \), we get

\[
\begin{cases} 
p=2j^2, & \text{if } m=2j \\
p=2j^2+2(k-j)+1, & \text{if } m=2j-1 
\end{cases}
\]

with the corresponding optimal networks

\[
\begin{cases} 
G(n_k-2j^2,2k+1-2j), & \text{if } m=2j \\
G(q_2[k]-(2j^2-2j),2k+1-2j), & \text{if } m=2j-1
\end{cases}
\]

Letting \( i=\lfloor j \rfloor \), and using the fact that \( 2i^2+2i = 2(i+1)^2-2(i+1) \), we obtain the final form of \( \Phi_1[k] \) and \( h_n^* \).

The sets making up \( \Phi_1[k] \) are disjoint: all values of \( n \) in the first set are odd, while
all those in the second are even. Hence \( f_k(\Phi_1) = \frac{\sqrt{2k+\sqrt{k}}}{4k} = \frac{0.6}{\sqrt{k}} \). \( \square \)

Remarks:

(a) Given \( k \), the last few elements of \( \Phi_1[k] \) (in decreasing order) are:

\[
\{ (n=n_k-p \mid p=0,2,8,18,32,50,72,98,162,\ldots, p<4k) \} \cup \\
\{ (n=2k^2-p \mid p=0,4,12,24,40,60,84,112,\ldots, p<2k-1) \}.
\]

(b) \( \Phi_1 \) is a sparse family in the sense that \( \lim_{k \to \infty} f_k(\Phi_1) = 0 \).

Example 3.2: For \( k=10 \), \( R[k]=\{182,\ldots,221\} \), and we get
\[
\Phi_1[10]= \{ 188, 189, 196, 200, 203, 213, 219, 221 \}
\]
\( f_{10}(\Phi_1) = 0.2 \).

When \( k=20 \), \( R[k]=\{762,\ldots,841\} \), and
\[
\Phi_1[20]= \{ 769, 776, 788, 791, 796, 800, 809, 823, 833, 839, 841 \}
\]
\( f_{20}(\Phi_1) = 0.1375 \).

§ 3.3 Optimal Family \( \Phi_2 \)

In this section, the steps of Section 3.2 are applied to networks with an even hop \( h \).

Lemma 3.3: Let \( h=2t \), \( n=q_3(t)+m \cdot 2t \), \( t \geq 1 \), \( m \geq t+1 \).

Then \( \text{Diam } G(n,h) = t + \lceil \frac{m}{2} \rceil \).

Proof: Proceed as in the proof of Lemma 3.1. Here we start with \( g_{2t^2+2t} \), (see Fact 2.4), and perform CPD (CPU) operations on it. As for the case of odd \( h \), every two
applications of CPD (CPU) increase (decrease) the diameter of the corresponding network by 2, as shown in Figure 3.2 for t=4. □

Remark: Lemma 3.3 defines the chains $A_h$ where $h$ is even, in the same way Lemma 3.1 did for odd $h$.

Figure 3.2

The CPD Transformation (for an even h).
Theorem 3.4: Optimal Family $\Phi_2$.

Let $\Phi_2 = \bigcup_{k \geq 1} \Phi_2[k]$ where

$$\Phi_2[k] = \{ n = q_1[k] - (2i^2 \pm i), n = q_3[k] - (2i^2 \pm i) \mid i \geq 0, n > n_{k-1} \}.$$ 

Then (a) $\Phi_2$ is optimal.

(b) Optimal hops for members of $\Phi_2[k]$ are given by

$$h_n^* = \begin{cases} 
2(k \pm i), & \text{if } n = q_1[k] - (2i^2 + i) \text{ or } n = q_3[k] - (2i^2 + i) \\
2(k \pm i), & \text{if } n = q_1[k] - (2i^2 - i), \text{ or } n = q_3[k] - (2i^2 - i) 
\end{cases}$$

(c) $f_k(\Phi_2) = O\left(\frac{1}{\sqrt{k}}\right)$.

Proof: Analogous to that of Theorem 3.2, using Lemma 3.3. $\square$

Remarks:

(a) Given $k$ we obtain:

$$\Phi_2[k] = \{ n = q_1[k] - p, n = q_3[k] - p \mid p = 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, \ldots, n \in \mathbb{R}[k] \}$$

(b) Again, $\Phi_2$ is a sparse family. Moreover, $\Phi_1 \cup \Phi_2$ is sparse.

(c) $\Phi_1 \cap \Phi_2 \neq \emptyset$, although for a particular $k$ it is possible that $\Phi_1[k] \cap \Phi_2[k] = \emptyset$.

From Example 3.2 and 3 we can see that $\Phi_1[20] \cap \Phi_2[20] = \emptyset$ but $\Phi_1[10] \cap \Phi_2[10] = \{189\}$.

Example 3.3: For $k=10$ and $k=20$ we get

$$\Phi_2[10] = \{182, 184, 187, 189, 190, 195, 200, 204, 207, 209, 210\}, \text{ and } f_{10}(\Phi_2) = 0.275.$$ 

$$\Phi_2[20] = \{765, 770, 774, 775, 777, 779, 780, 784, 792, 799, 805, 810, 814, 817, 819, 820\}, \text{ and}$$
Corollary 3.5: For every \( k \geq 1 \), the quartile points \( q_1[k], q_2[k], q_3[k] \) of \( R[k] \) are optimal.

Proof: By Theorem 3.2 and 5, for every \( k \geq 1 \), \( q_2[k] \in \Phi_1[k] \) and \( q_1[k], q_3[k] \in \Phi_2[k] \), and the networks \( G(q_1[k], 2k) \), \( G(q_3[k], 2k) \) and \( G(q_2[k], 2k+1) \) are optimal. \( \square \)

§ 3.4 The Augmented Families \( \tilde{\Phi}_1 \) and \( \tilde{\Phi}_2 \)

Let \( n = n_k \) and \( h = 2k+1 \) for some \( k \geq 0 \). As shown in the proof of Lemma 3.1, two applications of the CPD operation to \( g_{n,h} \) would produce basic grids with corresponding optimal networks of diameter \( k+1 \). The reason for that, as explained there, is that the lower part of \( L^{k+1} \) is nil in \( g_{n,h} \).

For the very same reason, we can apply a similar operation, CPL (Cut and Push Left) twice to \( g_{n,2k+1} \), resulting in the optimal nets \( G(n+1, 2k+1) \) and \( G(n+2, 2k+1) \). (See Figure 3.3.) In CPL the lower part of the basic grid to which it is applied is pushed to the left, and \( n \) is increased by 1.
In the operation CPR (Cut and Push Right), the lower part of the basic grid is pushed to the right and \( n \) decreases by 1. Applying it to \( \mathfrak{g}_{n,k} \) results in the basic grid corresponding to the network \( G(n_k - 1, 2k + 1) \), which has diameter \( k+1 \) and so is suboptimal. (See Figure 3.4) We can therefore state, in view of Lemma 3.1,

**Theorem 3.6:** For every \( k \geq 1 \), \( n_k - 1 \) is suboptimal. \( \square \)
Lemma 3.7: (a) Let $n \in \Phi_1[k]$ and let $h$ be the smaller of the two optimal hops corresponding to $n$ as given by Theorem 3.2(b), such that

(i) $n \leq q_2[k], \quad$ (ii) $n + h \in \Phi_1[k]$

Then $G(n+1,h)$ and $G(n-1,h)$ are optimal.

(b) For every $k \geq 1$, $G(n_k + 1, 2k + 1)$ and $G(n_k + 2, 2k + 1)$ are optimal.

Proof: (a) When $n$ and $h$ satisfy (i) and (ii), each of the operations CPL, CPR can be applied to $g_{n,h}$ without affecting the diameter. The justification is similar to the discussion following Corollary 3.5, and also relies on the fact that $g_{n,h}$ satisfies (3.1)
on p. ***, and therefore a change in \( n \) does not affect the upper part of the basic grid other than as a result of folding.

(b) For every \( k \geq 1 \), \( g_{n,2k+1} \) admits two CPD operations, resulting in basic grids corresponding to optimal nets with the optimal diameter of \( k+1 \). Therefore, \( g_{n,2k+1} \) would also admit two CPL operations which would result in nets with the same diameter. \( \square \)

**Lemma 3.8:** Let \( n \in \Phi_2[k] \) and let \( h \) be the smaller of the two optimal hops corresponding to \( n \) as given by Theorem 3.4(b). Then

(a) \( G(n+1,h) \) is optimal.

(b) If in addition \( n \) satisfies

(i) \( n < q_2[k] \),  
(ii) \( n+h \in \Phi_2[k] \).

Then \( G(n+2,h) \) and \( G(n-1,h) \) are optimal as well.

**Proof:**

(a) Let \( n \in \Phi_2[k] \) and \( h=2t \) be as in Lemma 3.8. We may view \( g_{n,2t} \) as being derived from \( g_{2t^2+4,t} \) upon applying a number of CPD or CPU operations (see Lemma 3.3). In the proof of FACT 3(b) we showed that the lower left part of the exterior level in \( g_{2t^2+4,t} \) is nil. Any number of applications of CPD (CPU) to it will not affect this situation, (see Figure 3.2): this property of the basic grid is certainly retained under one CPD application that increases the diameter, or one CPU application that would not affect the diameter; folding, as in the proof of Lemma 3.1, accounts for maintaining the same lot nil under further applications of those transformations. As a consequence, one application of CPL to \( g_{n,2t} \) is always possible.
(b) $G(n-1,h)$ is optimal here, since any basic grid $g_{n,h}$ that admits CPD while maintaining the diameter, must also admit CPR. If $n < q_2[k]$, $g_{n+1,h}$ must have a nil lower left part of its exterior level, hence another application of CPL is possible without increasing the diameter. Figure 3.5 illustrates this case.

**Figure 3.5**

The CPR and CPL Operations Performed on $g_{2k^2-k,2k}$, $k=4$

Theorem 3.9 summarizes the "blown up" versions of $\phi_1$ and $\phi_2$, as implied by
Lemma 3.7 and 5.

Theorem 3.9: Let

\[ \Phi_1 = \Phi_1 \cup \{n \pm 1 \mid n \in \Phi_1[k], n \leq q_2[k], n + h^*n \in \Phi_1 \} \cup \{n_{k-1} + 1, n_{k-1} + 2 \mid k \geq 1\} \]

\[ \Phi_2 = \Phi_2 \cup \{n + 1 \mid n \in \Phi_2 \} \cup \{n - 1, n + 2 \mid n \in \Phi_2[k], n < q_2[k], n + h^*n \in \Phi_2 \} \]

where \( h^* \) is the smaller of the two optimal hops given by Theorem 3.2(b) or 5(b), according to each case.

Then (a) Each \( n \in \Phi_1 \cup \Phi_2 \) is optimal.

(b) \( f_k(\Phi_1) \) and \( f_k(\Phi_2) \) are \( O(\frac{1}{\sqrt{k}}) \).\( \square \)

Clearly \( \Phi_1 \) and \( \Phi_2 \) are also sparse families.

Example 3.4:

\[ \Phi_1[10] = \Phi_1[10] \cup \{182,183,187,195,197,199,201\} \]

\[ \Phi_2[10] = \Phi_2[10] \cup \{183,185,186,188,191,192,196,201,205,208,211\} \]

§ 3.5 Summary

- We have presented the optimal infinite families \( \Phi_1, \Phi_2, \Phi_1, \Phi_2 \). Each one of these families intersects each range \( R[k] \) in \( \Phi_i[k], \Phi_i[k], \) such that \( |\Phi_i[k]|, |\Phi_i[k]| = O(\sqrt{k}), i=1,2 \). Thus, although the densities \( f_k(\Phi_i) \) and \( f_k(\Phi_i) \) tend to 0 as \( k \to \infty \), \( \Phi_i[k] \) and \( \Phi_i[k] \) grow with \( k \), for \( i=1,2 \).

- The family presented in [Du88] is covered by the sparse families \( \Phi_i, \Phi_i, i=1,2 \).

When translated to our notation this family, call it \( \Lambda \), is given by
\[
\Lambda = \bigcup_{k \geq 1} \Lambda[k], \quad \text{where}
\]

\[
\Lambda[k] = \{q_i[k]-j : 1 \leq i \leq 3, \ -1 \leq j \leq 3-i \} \cup \{n_k\}.
\]

Using Theorem 2.4, 5 and 7 we can see that

- \(q_1[k] , q_1[k]-1, q_3[k] \in \Phi_2[k]\),

- \(q_1[k]+1, q_1[k]-2 \in \Phi_2[k]\)

- \(q_2[k], n_k \in \Phi_1[k]\)

- \(q_2[k]-1, q_2[k]+1 \in \Phi_1[k]\).

- Table 3.1 illustrates membership in the various sparse families, for \(k=14\).
Table 3.1

Membership in the Sparse Optimal Families for k=14.

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§ 4.1 Introduction

In this chapter we shall expand our knowledge of optimal values of \( n \), which was gained through the sparse optimal families from Chapter 3. We shall identify "dense" optimal families of values of \( n \), along with the corresponding optimal hops \( h \). Although by no means exhaustive, those families cover 92% of all values of \( n \) up to 8,000,000. Also, when \( k \) or \( k+1 \) is a prime, they settle almost completely the question of determining optimality in \( R[k] \). (In this case all but one or two values in \( R[k] \) are optimal.) The results in this chapter were obtained independently by the author and by J-C. Bermond (see [Be88c]).

We begin by presenting several results concerning the graphs \( G(n;h_1,h_2) \), which were mentioned in Section 1.5. These results are needed in the derivation of the dense optimal families. The graphs \( G(n;h_1,h_2) \) have a vertex set \( \{0,1,...,n-1\} \), corresponding to the integers modulo \( n \), and each vertex \( i \) is joined to the four vertices \( i \pm h_1, i \pm h_2 \), where \( h_1, h_2, n \) are positive integers such that \( h_1 < h_2 < \frac{n}{2} \).

Let \( D_n = \min \{ \text{diam } G(n;h_1,h_2) \mid 0 < h_1 < h_2 < \frac{n}{2} \} \).

\( R[k] \) and \( n_k \) have the same meaning throughout this chapter as in Chapters 2 and 3:

\[
R[k] = \{2k^2-2k+1,...,2k^2+2k+1\}, \quad n_k = 2k^2+2k+1.
\]

Theorem 4.1, which was discovered independently by several authors (see [Be85].
[Bo85], [Wo74], [Ye85]), provides a lower bound on $D_n$:

**Theorem 4.1:** For all $n \in \mathbb{R}[k]$, $D_n \geq k$.

The next theorem asserts that the lower bound on $D_n$ can always be achieved: (see [Be85], [Bo85], [Ye85])

**Theorem 4.2:** (i) $\text{diam } G(n; k, k+1) = k$ for all $n \in \mathbb{R}[k]$.

(ii) $\text{diam } G(n; k-1, k) = k$ for all $n \in \mathbb{R}[k], n \leq 2k^2 + 1$.

The networks $G(n,h)$, in which we are interested in this work, correspond to the special case when $h_1 = 1$:

$$G(n,h) = G(n; 1, h), \text{ for } 2 \leq h \leq n-2.$$  

As in previous chapters, we denote

$$D_n^* = \min \{ \text{diam } G(n,h) \mid 2 \leq h \leq n-2 \}.$$  

The lower bound on $D_n^*$, (see Theorem 2.7) equals the one mentioned in Theorem 4.1 for $D_n$, but unlike the case of $G(n; h_1, h_2)$, the lower bound for $D_n^*$ may never be achieved for some values of $n$. (See Lemma 2.8 and Theorem 3.6.)

§ 4.2 Dense Optimal Families

In the following, $\gcd(a, b)$ will denote the greatest common divisor of the integers $a$ and $b$. 
Lemma 4.3: Let $k \geq 1$, $n \in \mathbb{R}[k]$, and let $q$ be relatively prime to $n$. Then

(i) $\text{diam } G(n;qk,q(k+1)) = k$,

(ii) $\text{diam } G(n;q(k-1),qk) = k$, if $n \leq 2k^2+1$;

where $qk$, $q(k+1)$, $q(k-1)$ are all computed modulo $n$.

Proof: (i) Consider the mapping $f: \{0,...,n-1\} \rightarrow \{0,...,n-1\}$ where for each $i$, $0 \leq i \leq n-1$, $f(i) = qi \mod n$. Since $\gcd(q,n)=1$, $f$ is bijective and therefore $f$ is a graph isomorphism mapping $G(n;k,(k+1))$ onto $G(n;qk,q(k+1))$. As the diameter of a graph is preserved under isomorphism, the result follows from (i) in Theorem 4.2. The proof of (ii) is similar. □

Lemma 4.4: Let $n \in \mathbb{R}[k]$, then $n$ is optimal in each of the cases:

(a) $\gcd(n,k)=1$,

(b) $\gcd(n,k+1)=1$,

(c) $\gcd(n,k-1)=1$ and $n \leq 2k^2+1$.

and in each case the associated optimal hop is easily determined.

Proof: Two integers $m$ and $n$ are relatively prime if and only if there exist integers $q$ and $r$ such that $qm - rm = 1$, or equivalently if and only if there exists a $q$ such that $qm \equiv 1 \mod n$. Note that $q$ is necessarily relatively prime to $n$, as $\gcd(q,n) = 1$. Thus, parts (a), (b) and (c) of Lemma 4.4 follow from Lemma 4.3 by choosing $m=k$, $k+1$ and $k-1$ respectively. Furthermore, if $q$ satisfies $qm \equiv 1 \mod n$ then $q+\alpha n$ satisfies the same congruence, and $n-q$ satisfies $(n-q)m \equiv -1 \mod n$, so we can always choose $q$ such that $0 < q < \frac{n}{2}$. □
Example 4.1:

(a) Let \( n = n_k = 2k^2 + 2k + 1 \). We can easily prove that \( n \) is optimal (see Fact 2.4(a)). Indeed, \( \gcd(n, k) = \gcd(n, k+1) = 1 \). The choices \( q = 2k+2 \) and \( q = 2k \) correspond to cases (a) and (b) of Lemma 4.4 respectively, with \( h = 2k+1 \) in both cases.

(b) Similarly, \( n = n_k - 2 = 2k^2 + 2k - 1 \) is optimal. We can either use (a) of Lemma 4.4 with \( q = 2k+2 \) and \( h = 2k+3 \) or (b) with \( q = 2k \) and \( h = 2k-1 \).

(c) \( n = 2k^2 - 3 \) is optimal, using case (c) of the lemma, with \( q = 2k+2 \) and \( h = 2k+3 \).

Let \( n \in \mathbb{R}[k] \). As we have seen in Lemma 4.4, to establish \( n \)'s optimality, the coprimality of \( n \) and \( k \) (or \( k+1 \), or \( k-1 \)) needs to be checked. This would be simplified if the quadratic term \( k^2 \) could be purged from \( n \). In fact, as already hinted in Lemma 2.8 and Theorem 3.6, in each \( \mathbb{R}[k] \) the element \( n_k - 1 = 2k^2 + 2k \) plays a special role, and we will show that it is sufficient to consider \( n \) in terms of its displacement \( p \) from \( n_k - 1 \). In the following theorem we reformulate Lemma 4.4, by identifying three optimal families of values of \( n \). (Recall that a family is said to be an optimal family if all of its members are optimal).

**Theorem 4.5:** Let \( n_k = 2k^2 + 2k + 1 \). The families \( \Psi_i = \bigcup_{k \geq 1} \Psi_i[k], \ 1 \leq i \leq 3 \), are optimal, where

- (a) \( \Psi_1[k] = \{ n = n_k - 1 - p \mid 1 \leq p < 4k - 1, \gcd(k, p) = 1 \} \cup \{ n_k \} \),
- (b) \( \Psi_2[k] = \{ n = n_k - 1 - p \mid 1 \leq p < 4k - 1, \gcd(k + 1, p) = 1 \} \cup \{ n_k \} \),
- (c) \( \Psi_3[k] = \{ n = n_k - 1 - p \mid 2k - 1 \leq p < 4k - 1, \gcd(k - 1, p - 4) = 1 \} \).

In each case, a corresponding optimal hop is given by \( h^* = \min \{ h, n - h \} \) where
(a) $h=2s(k+1)-t+1$, where $(s,t)$ is an integer solution to $sp-tk=1$,

(b) $h=2sk-t+1$, where $(s,t)$ is an integer solution to $t(k+1)-sp=1$,

(c) $h=2s(k+2)-t+1$, where $(s,t)$ is an integer solution to $s(p-4)-t(k-1)=1$,

and where $(s,t)$ is the smallest non-negative solution to the relevant equation.

Proof: (a) By Lemma 4.4(a), $n$ is optimal if $\gcd(n,k)=1$. Letting $n=n_k-1-p=2k(k+1)-p$, this condition is equivalent to $\gcd(p,k)=1$, and hence to the existence of integers $s,t$ such that $sp-tk=1$. Let $q=2s(k+1)-t$, then $qk = 2sk(k+1)-tk = s(n+p)-tk \equiv 1 \mod n$, and according to Lemma 4.4(a), $h=q+1$ is an optimal hop. Choosing $s,t$ to be as small as possible, and then taking $h^* = \min h, n-h$ ensures that $0<h^*<\frac{n}{2}$. $n_k$ is already known to be optimal by Example 4.1. The proofs of parts (b) and (c) are similar to that of part (a); for part (c) note that $n = 2k^2+2k-p = 2(k+2)(k-1)-(p-4)$. □

Example 4.2: Let $k=11$, $n=257$. Here $n_{11}=265$ and $p=7$.

Since $\gcd(p,k)=1$, $n$ is in $\Psi_1$. The equation $7s-11t=1$ has the smallest positive integer solution $s=8$, $t=5$, and so $h=16\cdot12-5+1=188$. $h^* = \min \{188, 257-188\} = 69$. In this case we also have $\gcd(p,k+1)=1$, hence $n \in \Psi_2$. The smallest solution to the equation $12t-7s=1$ is $s=5$, $t=3$ and thus $h^*=108$ is also an optimal hop for $n=257$.

For the subsequent discussion we shall use the quartile points in each range $R[k]$; they were defined in Chapter 2 as

$q_1[k] = 2k^2-k; \quad q_2[k] = 2k^2; \quad q_3[k] = 2k^2+k; \quad q_4[k] = 2k^2+2k=n_k-1.$
Example 4.3: (a) Let $k$ be arbitrary, $n=q_2[k]-1=2k^2-1$ corresponding to $p=2k+1$. Here $(p,k)=(p,k+1)=(p-4,k-1)=1$, hence all parts of Theorem 4.5 are applicable. The solution $(s,t)=(1,2)$ will produce the values $2k+1$ and $2k-1$ for $h^*$, corresponding to parts (a) and (b) of the theorem. The value $h^*=2k+1$ is also obtained from part (c), using $s=k-2$, $t=2k-5$.

(b) For each $k\geq 1$, $q_i[k]$ is not in $\Psi_1[k]$, $i=1,2,3,4$, since those values of $n$ correspond to $p=3k$, $2k$, $k$, $0$ respectively. Yet, $q_1[k]\in\Psi_3[k]$ and $q_3[k]\in\Psi_2[k]$. If $k$ is even then $q_2[k]\in\Psi_2[k]\cap\Psi_3[k]$.

For some values of $k$, the families $\Psi_i[k]$ cover large continuous segments of $R[k]$, and in some cases the determination of optimal values in $R[k]$ is completely settled, as is shown next.

Corollary 4.6: (a) If $k$ is prime then every $n\in R[k]$, $n\neq n_k-1$, is optimal.

(b) If $k+1$ is prime then every $n\in R[k]$, $n\neq n_k-1$ is optimal with the possible exception of $q_2[k]=2k^2-2$.

Proof: (a) If $k$ is prime, $\Psi_1[k] = R[k] - \{q_1[k], q_2[k], q_3[k], q_4[k]\}$. But $q_1[k]$ and $q_3[k]$ are optimal, according to Example 4.3(b), $q_2[k]$ was shown to be optimal in [9], and $q_4[k]=n_k-1$ is not optimal by Lemma 2.8.

(b) In this case $\Psi_2[k] = R[k] - \{q_1[k]-3; q_2[k]-2; q_3[k]-1; q_4[k]\}$, but $q_3[k]-1\in\Psi_1[k]$ (corresponding to $p=k+1$) and $q_1[k]-3\notin\Phi_2[k]$. (See Theorem 3.4 and its ensuing Remark.) □
§ 4.3 Relationships Between Sparse and Dense Optimal Families

- The families $\Psi_1[k]$, $\Psi_2[k]$ and $\Psi_3[k]$ are neither comparable nor exclusive, as show the following examples (compare to Example 4.3):

  \[
  q_{2[k]-1} \in \Psi_1[k] \cap \Psi_2[k] \cap \Psi_3[k], \text{ for each } k \geq 1.
  \]

  \[
  q_3[k] \in \Psi_2[k]-\Psi_1[k]; \quad q_3[k]-1 \in \Psi_1[k]-\Psi_2[k];
  \]

  \[
  q_1[k] \in \Psi_3[k]-\Psi_1[k]; \quad q_1[k]-1 \in \Psi_1[k]-\Psi_3[k];
  \]

  \[
  q_3[k] \in \Psi_2[k]-\Psi_3[k]; \quad q_1[k] \in \Psi_3[k]-\Psi_2[k] \text{ if } k+1 \equiv 0 \text{ mod } 3.
  \]

- Both optimal and suboptimal values of $n$ exist that are not in $\Psi = \Psi_1 \cup \Psi_2 \cup \Psi_3$.

Example 4.4:

(a) By Theorem 3.6 we know that for all $k \geq 1$, $n_k-1$ is not optimal.

(b) $q_2[k]-2$ is not in $\Psi$ whenever $k \geq 4$, $k$ even; this value of $n$ can be optimal (e.g. for $k=4$, $q_2[4]-2=30$ is optimal with $h=8$; for $k=8$ $q_2[8]-2=126$ is optimal with $h=12$), or nonoptimal (e.g. for $k=6$, $q_2[6]-2=70$, see Algorithm 1 in Section 2.4).

(c) Let $k=20$, $p=30$, $n=810=n_{20}-1-30$. Since $p<2k+1$, $(p,k)=10$ and $(p,k+1)=3$, $n$ is not in $\Psi$. However, $810 \in \Phi_2[20]$ (see Example 3.3).

- Despite being very large, (this will be shown in the next section) the $\Psi$ family misses infinitely many values that are covered by the $\Phi$ and $\Phi$ families.
Example 4.5:

(a) Let \( k = i + \alpha i(1+i) \) where \( \alpha, i \geq 2 \). Let \( p = k + i(2i+1) = k + 1 + (i+1)(2i-1) \). By Theorem 3.4(a), \( n = n_k - 1 - p = \Phi_2 \). On the other hand, \( (p,k) \geq i \), \( (p,k+1) \geq i+1 \), and \( p < 2k-1 \). Thus, \( n \) is not a member of \( \Psi \). Note that both \( i \) and \( \alpha \) can assume infinitely many values.

(b) A similar phenomenon occurs if \( k = i + \alpha i(1-i) \) and \( p = k + i(2i-1) = k + 1 + (i-1)(2i+1) \), where \( \alpha \geq 4 \) if \( i = 2 \) or 3 and \( \alpha \geq 3 \) if \( i \geq 4 \).

(c) In particular, let \( k \geq 5 \) be odd such that \( k \equiv 0 \mod 3 \), and let \( p = k + 3 \). Then \( (p,k) = 3 \), \( (p,k+1) \geq 2 \) and \( p < 2k-1 \). Thus \( n = n_k - 1 - p = \Phi_2[k] - \Psi \).

- Table 4.1 illustrates membership in the various dense families, for \( k = 14 \).

§ 4.4 The Relative Size of the \( \Psi \) Families

Let \( \phi \) denotes Euler’s phi-function, i.e.

\[
\phi(k) = |\{ m \in \mathbb{Z} \mid 1 \leq m \leq k \text{ and } (m,k) = 1 \}|,
\]

and recall the definition of the density of a family \( \Theta \):

\[
f_k(\Theta) = \frac{|\Theta[k]|}{|R[k]|} = \frac{|\Theta[k]|}{4k},
\]

where \( \Theta[k] = \Theta \cap R[k] \). For the dense families we obtain
Lemma 4.7:
\[ f_k(\Psi_1) = \frac{\phi(k)}{k}, \quad f_k(\Psi_2) = \frac{\phi(k+1)}{k+1}, \quad f_k(\Psi_2) = \frac{1}{2} \frac{\phi(k-1)}{k-1}. \]

Proof: By Theorem 4.5(a),
\[ |\Psi_1[k] \cap Q_1[k]| = \phi(k)-1, \]
\[ |\Psi_1[k] \cap Q_2[k]| = |\Psi_1[k] \cap Q_3[k]| = \phi(k), \]
\[ |\Psi_1[k] \cap Q_4[k]| = \phi(k)+1. \]
Thus \( |\Psi_1[k]| = 4\phi(k), \) which implies \( f_k(\Psi_1) = \frac{\phi(k)}{k} \). The proofs of the remaining two expressions are similar. □

Define the cumulative density of \( \Psi \) as
\[ F_k(\Psi) = \frac{1}{n_k} \sum_{k \leq k} |R[k]| f_k(\Psi). \]

- A numerical computation of the actual value of \( F_k(\Psi) \) corresponding to the range \( n \leq 8 \cdot 10^6 \) (\( k=2000 \)) shows:
\[ F_{2000}(\Psi_1 \cup \Psi_2) = 0.89, \quad F_{2000}(\Psi) = 0.92. \]
Hence \( \Psi \) covers 92% of all values of \( n \) up to 8,000,000!
Table 4.1

Membership in Optimal Families for k=14.

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§ 5.1 Introduction

In the previous chapters we have introduced the networks $G(n,h)$ and defined $D_n^* = \min_h \text{Diam } G(n,h)$. We have seen that if $n \in \mathbb{R}[\lfloor k \rfloor] = \{2k^2-2k+2, \ldots, 2k^2+2k+1\}$, then a lower bound on $D_n^*$ is $k$. An integer $n \in \mathbb{R}[k]$ was defined as optimal if $D_n^* = k$, and as suboptimal if $D_n^* = k+1$. We have also presented some "sparse" and some "dense" optimal families. It was conjectured in Chapter 2 that every value of $n$ is either optimal or suboptimal. Some trivial upper bounds on $D_n^*$ can be obtained by considering small values of $h$. For example, letting $h=2$ produces a basic grid $g_{n,h}$ (see Section 2.2) in which only the center column, the upper part of column 1 and the lower part of column -1 are non vacant, thus securing the upper bound $D_n^* \leq \left\lfloor \frac{n}{4} \right\rfloor = \frac{k^2}{2}$. In an attempt to gain a more complete characterization of the various networks $G(n,h)$ with respect to their minimal diameters, we shall now use a more general framework. We shall derive sufficient conditions which, if satisfied by $n \in \mathbb{R}[k]$, will guarantee that

$$D_n^* \leq k + \delta, \quad \text{for some constant } \delta \geq 0. \quad (5.1)$$

We shall also obtain families of values of $n$ satisfying (5.1), called $\delta$-families. In the process of deriving the $\delta$-families we shall combine and generalize the different
approaches used in Chapters 3 and 4.

The rest of this section is an outline of the steps taken to derive the $\delta$-families. In Section 5.2 we shall present two "prototype cases": we shall point out the various steps of the derivation in the special case of the dense family $\Psi_1$ from Chapter 4, which is already known to be optimal, and then refer the reader to a second prototype case with different characteristics, which was presented in a different context in Section 2.5.

Let $n \in \mathbb{R}[k]$ be given by $2k^2+2k-p$, and consider $G(n,h)$ where $h$ satisfies $1<h<n-1$. The derivation of the $\delta$-families takes the following two considerations into account:

(a) Algebraic relationships that the relevant hop $h$ must satisfy, and

(b) "Geometric information" concerning the basic grid $g_{n,h}$.

(a) Algebraic Relationships

Step 1: First Congruence

Consider the basic grid $g_{n,h}$ as described in Section 2.2. Let $k+\alpha$ denote that row in $g_{n,h}$ whose leftmost entry is the immediate successor of the rightmost entry of row 0. W.l.o.g. we may assume that $0<k+\alpha<n$: if $k+\alpha<0$ we shall consider $g_{n,h'}$ instead, where $h'=n-h$; $g_{n,h'}$ is a reflection of $g_{n,h}$ with respect to the x-axis. Let $k+\beta$ be the entry in the center of row $k+\alpha$ (i.e. the entry at the grid point $(0,k+\alpha)$). Clearly $0<k+\beta<n$. By the definition of a row center, (see Section 2.2) $h$ (or $h'$) satisfies the congruence
(k+\alpha)h \equiv k+\beta \mod n, \text{ where } n=n_k-1-p \in R[k]. \quad (5.2)

Remark: What can be said about the size of \alpha or \beta relatively to k? First consider the following intuitive observation: roughly speaking, a large \alpha (\alpha that is \text{O}(k) rather than \text{O}(1)) indicates a basic grid \text{g}_{n,h} that is relatively "tall and narrow"; a large \beta (\text{O}(k)) corresponds to a basic grid that is "short and wide". In view of Observation 2.3, none of those is likely to have a "good" diameter, one that is close to the lower bound k. We therefore would like to assume that \alpha and \beta are small relatively to k. (Later on we shall see that assuming \text{\text{-}k<\alpha, \beta<\frac{k}{3}} suffices.) Do such cases indeed exist, and if so, how frequently do they occur?

To answer those questions consider the dense families \Psi_1, \Psi_2 and \Psi_3, presented in Chapter 4. Using Theorem 4.5 we find that the corresponding values of \alpha and \beta are

\Psi_1: \alpha=0, \beta=1;
\Psi_2: \alpha=1, \beta=0;
\Psi_3: \alpha=-1, \beta=0.

In all those cases, the values of \alpha and \beta are small relatively to k. Recall (see Section 4.4) that the dense families comprise 92\% of all values up to 1,000,000. Thus we take this as good evidence that the assumption that \alpha and \beta are small relatively to k is a reasonable condition to impose.

Step 2: An Explicit Expression for h.

Given the congruence (5.2), we shall derive an explicit expression for its solutions, h. This will be done under two additional assumptions, as explained in Section 5.4.
Step 3: Second Congruence.

A second congruence that \( h \) must satisfy, is derived using the expression for \( h \) from Step 2.

(b) "Geometric" considerations.

Step 4: The Structure of \( G_{n,h}^{k+\delta} \).

Based on the two congruences from Steps 1 and 3, we shall identify the relationships between the row centers of \( G_{n,h}^{k+\delta} \), for some \( \delta \geq 0 \), thus gaining knowledge of the structure of \( G_{n,h}^{k+\delta} \).

Step 5: Capacity Considerations.

From Observation 2.3 we know that \( \text{Diam} \ G(n,h) \leq k+\delta \) if and only if \( G_{n,h}^{k+\delta} \) covers \( g_{n,h} \). By examining the structure of \( G_{n,h}^{k+\delta} \), for some \( \delta \geq 0 \), combined with "row capacity" considerations similar to those employed in the proof of Theorem 2.11, we shall derive relationships among \( \alpha, \beta, k, p, \delta \) and other parameters. Those relationships constitute implicit constraints on the various parameters; the satisfaction of those constraints is a sufficient condition for \( n \) to satisfy \( D_n^* \leq k+\delta \).

Step 6: Simplified Constraints for Some "Important" Special Cases.

Using Lemma 5.2 from Section 5.3 we shall be able to limit our attention to some special forms of the constraints derived in Step 5 in Section 5.4.A. Thus we shall be able to eliminate one of the parameters (\( m \)) there, to facilitate subsequent applications of the constraints. This step will not be shown in Prototype Case A, since in
the example presented there already assumes a particular value. Step 6 is not needed for cases of type B, treated in Section 5.4.B.

In Chapter 6 we shall use those relationships and constraints, to elaborate on the special cases of particular interest to us, namely the optimal and the suboptimal families. As we shall see, the results lead to a very detailed picture, for a very large range of values of \( n \). The new optimal and suboptimal families presented in Chapter 6 contribute considerably to close the gap left from our previous work (i.e. the sparse and dense families presented in Chapters 3 and 4).

§ 5.2 Two Prototype Cases

The cases presented here serve as early examples to precede the complete discussion of both types which is carried out in Section 5.4. This is done with the intention of familiarizing the reader with the subsequent process, as these cases have already been discussed before.

5.2.A: Prototype Case A

Consider the optimal family \( \Psi_1 \) presented in Theorem 4.5(a):

\[
\Psi_1 = \bigcup_{k \geq 1} \Psi_1[k], \quad \text{where}
\]

\[
\Psi_1[k] = \{ n : n = nk - p, \gcd(k,p)=1, 1 \leq p \leq 4k-2 \} \cup \{ nk \}.
\]

Let \( n \in \Psi_1[k], \ n \neq nk \).
Step 1: First Congruence

In the proof of Theorem 4.5(a), the optimal hop is given as \( h = q + 1 \) where \( qk \equiv 1 \mod n \). Thus \( h \) satisfies the congruence

\[
kh \equiv k+1 \mod n
\]  

(A.1)

When compared to Step 1 in Section 5.1 we see that \( \alpha = 0, \beta = 1 \), so both \( \alpha \) and \( \beta \) are small.

Step 2: An Explicit Expression for \( h \).

An explicit expression for the optimal hop \( h \) for \( n \) is given in Theorem 4.5(a) quoted above: \( h = 2s(k+1) - t + 1 \) where \( (s,t) \) is a solution to \( sp - tk = 1 \). This can be rewritten, by substituting \( p = p' + \lambda k \) where \( 0 \leq \lambda \leq 3 \) and \( 0 \leq p' < k \) (\( \lambda \) indicates the quartile of \( R[k] \) to which \( n \) belongs). We obtain

\[
h = s(2k+2-\lambda) - t + 1,
\]  

(A.2)

where \( (s,t) \) is a solution to

\[
sp' - tk = 1.
\]  

(A.3)

Note that in this case \( d = 1 \) on the right hand side of (A.3) satisfies \( d = \gcd(k,p) = \gcd(k,p') = \beta - \alpha \). In our subsequent discussion we shall make this an assumption that we impose on the parameters.

Step 3: A Second Congruence

Note that in our notation

\[
n = 2k^2 + 2k - p = 2k(k+1) - (p' + \lambda k) = k(2k+2 - \lambda) - p'.
\]  

(A.4)
Multiplying (A.3) by $2k+2-\lambda$, and using (A.2) and (A.4) we obtain

$$p'(h+t-1)-t(n+p') = 2k+2-\lambda$$

which can be rearranged to yield a second congruence that $h$ satisfies:

$$p'h \equiv 2k+2-\lambda+p' \mod n \quad (A.5)$$

**Step 4: The Structure of $G^k_{n,h}$.**

In the left hand side of (A.1), add and subtract $jh$. Use the definition of the row centers in Section 2.2 to obtain the relationship

$$C_{k-j} - C_{-j} = k+1, \quad j=1,...,k-1. \quad (A.6a)$$

Now use again (A.1), and subtract $kh$ from the left hand side of (A.3), and $k+1$ from the right hand side:

$$-(k-p')h \equiv k+1-\lambda+p' \mod n.$$  

Repeat the process to obtain

$$-(k-p')h - kh \equiv -\lambda+p' \mod n.$$  

By the definition of a row center, the last two congruences imply the relationships

$$C_{-(k-p')} - C_{j} = k+1-\lambda+p', \quad j=0,...,k-p'. \quad (A.6b)$$

$$C_{-(k-p')} - C_{k-j} = -\lambda+p', \quad j=1,...,p'-1. \quad (A.6c)$$

(A.6a)-(A.6c) imply a permutation of the rows $-k+1,...,k+1$ of $G^k_{n,h}$. Each $C_j$ is viewed as a representative of its row; equality (A.6a), for example, would be interpreted as: "go to row $k-j$ after you have reached the right end of row $j". In each row of $G^k_{n,h}$ other than row 0, the entries form a sequence of consecutive numbers. By scanning the rows according to the implied permutation (start with $j=0$ in
(A.6b)), the entries will be in a nondecreasing order. (If rows overlap at the ends, remove one of the overlapping portions.) Note that all the transitions implied by (A.6a) are from bottom rows to top rows. The situation is reversed in (A.6b) and (A.6c). Thus each row transition implied by (A.6a) is followed by one implied by (A.6b) or (A.6c). The restrictions on j were chosen so that the transitions in (A.6b) and (A.6c) are disjoint, and together they represent all transitions with origin in a "top" row and destination in a "bottom" row.

(A.6c) will only be used if \(p' > 1\). Since \(0 \leq \lambda \leq 3\) and \(0 \leq p' < k\), the right hand sides in (A.6) are positive. The only exceptions occur in (A.6c) with \(j=1\), when \([p'=2 \text{ and } \lambda=2 \text{ or } 3]\) or when \([p'=3 \text{ and } \lambda=3]\). In these cases, the two transitions:

\[
C_{k-1} \rightarrow C_{-(k-1)} \text{, implied by (A.6c) with } j=1, \text{ and} \\
C_{-(k-1)} \rightarrow C_1, \text{ implied by (A.6a) with } j=k-1
\]

are combined to form one single transition

\[
C_{k-1} \rightarrow C_1.
\]

Here

\[
C_1 - C_{k-1} = (C_1 - C_{-(k-1)}) + (C_{-(k-1)} - C_{k-1}) = (k+1) + (-\lambda + p') > 0.
\]

Altogether, (A.6a)-(A.6c) represent a permutation of the rows \(-k+1,...,k+1\) of \(G_{n,h}^k\) (rows \(-k+2,...,k-1\) in the exceptional cases mentioned above).

**Step 5: Capacity Considerations.**

We already know from Theorem 4.5(a) that each \(G(n,h)\) in this prototype case is an optimal network. By Observation 2.3, this is equivalent to each \(g_{n,h}\) being completely covered by \(G_{n,h}^k\). This can be verified by comparing the "row capacity" in
\( G_{n,h}^k \) with the "requirements" posed by the right hand sides in (A.6).

5.2.B: Prototype Case B

Consider the optimal family which was discussed in Theorem 2.11 in Section 2.5, consisting of the quartile points \( q_i[k] \), \( 0 \leq i \leq 3 \). Steps 1-5 were actually carried out in the proof of that theorem. It is worthwhile to notice the difference between this case and the case presented in Prototype Case A: There we had \( p' > 0 \) (and hence \( \alpha \neq \beta \)), here \( \alpha = \beta = 0 \) and \( p = (4-i)k \), hence \( p' = 0 \). Equation (A.3) cannot, therefore, be translated to this new environment. What is used instead is equation (2.6).

§ 5.3 Preliminary Number-Theoretic Results

We begin by presenting some preliminary results concerning diophantine equations, which are needed for the subsequent discussion. Theorem 5.1 is a well known result in elementary number theory (see, for example Theorem 2.9 in Burton [Bu80]).

**Theorem 5.1:** Let \( a, b, c \) be integers, and let \( d = \gcd(a, b) \). Then

(a) The equation \( ax - by = c \) has an integer solution \((x, y)\) if and only if \( d \mid c \).

(b) if \( d \mid c \) then all the solutions to \( ax - by = c \) are given by

\[
(x_i, y_i) = (x_0 + i \frac{b}{d}, y_0 + i \frac{a}{d}) \quad i \in \mathbb{Z},
\]

where \((x_0, y_0)\) is one particular solution. □
Remark: Although \((x_0, y_0)\) in Theorem 5.1 could be any particular solution to \(ax-by=c\), it is convenient to choose it so that it satisfies
\[
(0,0) \leq (x_0, y_0) < \left(\frac{b}{d}, \frac{a}{d}\right).
\]
(This can always be done.)

With this choice, the solutions in Theorem 5.1 satisfy
\[
(0,0) \leq (x_i, y_i) < (b,a), \ \text{for} \ 0 \leq i < d.
\]

In the next lemma we prove several properties that the solutions to \(ax-by=c\) (as given by Theorem 5.1 and the ensuing remark) must possess. These properties will be needed in the subsequent discussion.

Lemma 5.2 Let \(a, b, c, d\) be integers such that \(d=gcd(a,b), d|x\). Then the equation \(ax-by=c\) has an integer solution \((x,y)\) satisfying

(a) \(y \equiv 0 \mod d\), if \(gcd\left(\frac{a}{d}, d\right)=1\);

(b) \(y+x \equiv 0 \mod d\), if \(gcd\left(\frac{a+b}{d}, d\right)=1\);

(c) \(y-mx \equiv 0 \mod d\) for some integer \(m, 0 \leq m < d\), if \(gcd\left(\frac{b}{d}, d\right)=1\).

Proof: Let \((x_i, y_i)\) be the solutions to \(ax-by=c\) as given by Theorem 5.1.

(a) An integer \(i\) exists such that \(y_i \equiv 0 \mod d\) if and only if the congruence \(y_0 + i \cdot \frac{a}{d} \equiv 0 \mod d\) has a solution \(i\). \(gcd\left(\frac{a}{d}, d\right)=1\) is a sufficient condition for the solvability of this congruence.

(b) If \(gcd\left(\frac{a+b}{d}, d\right)=1\) then the congruence
\[
\frac{a+b}{d} \equiv -(y_0+x_0) \mod d
\]

is solvable. Let \(i_0\) be the solution, then

\[
y_{i_0}+x_{i_0} \equiv y_0+x_0+i_0 \frac{a+b}{d} \equiv 0 \mod d.
\]

(c) \(y_i-mx_i = y_0-mx_0+i_0 \left( \frac{a}{d} \right) \). If \(\gcd(\frac{b}{d},d)=1\) then the congruence

\[
\frac{b}{d} m \equiv \frac{a}{d} - 1 \mod d
\]

can be solved for \(m\). Let \(m_0\) denote a solution, and let \(i_0 \equiv -(y_0-m_0x_0) \mod d\). Then

\[
y_{i_0}-m_0x_{i_0} \equiv y_0-m_0x_0+i_0 \left( \frac{a}{d} \right) = y_0-m_0x_0+i_0 \equiv 0 \mod d. \quad \square
\]

**Corollary 5.3:** Let \(a, b, c, d\) be integers such that \(d=\gcd(a,b)\), \(dlc\) and \(d\) is a prime power. Then

(a) either \(ax-by=c\) has a solution \((x,y)\) satisfying \(y \equiv 0 \mod d\),

(b) or \(ax-by=c\) has a solution \((x,y)\) satisfying \(y+x \equiv 0 \mod d\).

**Proof:**

If \(\gcd(\frac{a}{d},d)=1\) then (a) follows from Lemma 5.2(a), regardless of \(d\)'s nature.

Otherwise, write \(d=p^a\) where \(p\) is a prime and \(a>0\). Then \(\gcd(\frac{a}{d},d)=p^b\), with \(0<b\) (by hypothesis) and \(b \leq a\). Since \(\gcd(\frac{a}{d},\frac{b}{d})=1\), we must have \(\gcd(\frac{b}{d},d)=1\), and therefore \(\gcd(\frac{a+b}{d},d)=1\). Thus (b) must hold, by Lemma 5.2(b). \( \square \)
Example 5.1 (a) Let \( a=60, b=45 \). Here \( d=\gcd(45,60)=15, \gcd\left(\frac{a}{d},d\right)=\gcd(5,15)=1 \). A particular solution to the equation \( 60x-45y=15 \) is \( (x_0, y_0)=(1,1) \), while \( (x_{11}, y_{11})=(34,45) \) satisfies \( y \equiv 0 \mod 15 \). (The subscripts are chosen in agreement with Theorem 5.1 and the ensuing Remark.)

(b) Now let \( a=45 \) and \( b=60 \). The condition in Lemma 5.2(a) fails, but (b) holds since \( \frac{a+b}{d}=7 \). The solutions to \( 45x-60y=15 \) are given by \( (x_i, y_i)=(3+4i, 2+3i) \). In none of them do we have \( y \equiv 0 \mod 15 \) (else we would have \( 3i \)), but \( (x_{10}, y_{10})=(43, 32) \) satisfies \( y+x \equiv 0 \mod 15 \).

(c) If \( a=150 \) and \( b=90 \), we have \( d=30 \) and the solutions to \( 150x-90y=30 \) are given by \( (x_i, y_i)=(2+3i, 3+5i) \).

The congruence \( y \equiv mx \mod d \) has the solution \( m=2 \), for \( i=29 \), despite the fact that none of the sufficient conditions in Lemma 5.2 and Corollary 5.3 holds.

§ 5.4 Derivation of General \( \delta \)-Families

Let \( k \) be fixed and let \( n \in \mathbb{R}[k], n=\k_1-p, \) where \( p \) is an integer, \( 1 \leq p \leq 4\k-2 \). Notice that we do not consider here the values \( n=n_k \) and \( n=n_k-1 \) corresponding to \( p=-1 \) and \( p=0 \), respectively. By Fact 2.4 and Theorem 3.6, \( n_k \) is optimal and \( n_k-1 \) is suboptimal.

The following discussion involves the parameters \( \alpha, \beta \) and \( h \) that appear in (5.2). Although these parameters were given a meaning in the previous section, no
reference will be made to that meaning until Lemma 5.7. Thus, the reader can relate to what follows as purely arithmetic, temporarily ignoring the semantics.

Given an integer $\alpha$ we observe that

$$n = 2k^2 + 2k - p = 2(k + \alpha)(k - \alpha + 1) - (p - 2\alpha(\alpha - 1)).$$

We shall define $P$, $P'$ and $\lambda$ by

$$P = p - 2\alpha(\alpha - 1), \quad \lambda = \left\lfloor \frac{P}{k + \alpha} \right\rfloor, \quad P' = P - \lambda(k + \alpha), \quad \text{with} \quad 0 \leq P' < k + \alpha. \quad (5.3)$$

Thus

$$n = (k + \alpha)(2k - 2\alpha + 2 - \lambda) - P'. \quad (5.4)$$

Since $0 < p < 4k - 1$, (5.3) implies the following bounds on $\lambda$:

$$- \frac{2\alpha(\alpha - 1)}{k + \alpha} \leq \lambda \leq 3. \quad (5.5)$$

Remarks:

(a) Clearly if $P' > 0$ then $d = \gcd(k + \alpha, P) = \gcd(k + \alpha, P') > 0$. Consider again the example of the dense families (see Remark following (5.2)). In those cases we have

- $\Psi_1$: $\alpha = 0$, $\beta = 1$, $P = p$, $d = \gcd(P, k + \alpha) = \beta - \alpha = 1$, $P' = P - \lambda k$;
- $\Psi_2$: $\alpha = 1$, $\beta = 0$, $P = p$, $d = \gcd(P, k + \alpha) = \alpha - \beta = 1$, $P' = P - \lambda(k + 1)$;
- $\Psi_3$: $\alpha = -1$, $\beta = 0$, $P = p - 4$, $d = \gcd(P, k + \alpha) = \beta - \alpha = 1$, $P' = P - \lambda(k - 1)$.

In all of these examples $P' > 0$, as in Prototype Case A. The complete treatment of this type appears in Section 5.4.A.

(b) Suppose $P' = 0$. Using (5.3) and (5.4) we conclude that $P' = 0$ if and only if $k + \alpha$ divides $n$. By (5.2) we conclude that $P' = 0$ if and only if $\beta = \alpha$. This is the case in
Prototype Case B in Section 5.2. The complete discussion of this type is found in Section 5.4.B.

It is our ultimate goal in this chapter to identify more optimal and suboptimal families. This is achieved by generalizing existing results from previous chapters, of which the dense families is the "largest" in terms of the proportion of values of $n$ to which it applies. The following assumptions, which hold for the dense families, will be made throughout the rest of this chapter:

\[-k < \alpha, \beta < \frac{k}{3}, \]

and

\[d = \text{gcd}(P', k+\alpha) \text{ satisfies } \begin{cases} (i) \ d = \beta - \alpha \quad \text{or} \\ (ii) \ d = \alpha - \beta. \end{cases} \]

It will become evident in Sections 5.5 and 5.6, that many new optimal and suboptimal families can be derived under these assumptions.
5.4.A: Derivation of 5-families with $P'>0$, $(a\neq \beta)$

Step 1: First Congruence

By using (5.7) we can rewrite (5.2) in the form

\[(i) \quad (k+a)h \equiv k+\alpha+d \mod d, \quad \text{or} \]
\[(ii) \quad (k+\alpha)h \equiv k+\alpha-d \mod d \]  \hspace{1cm} (5.2.A)

Step 2: An explicit expression for $h$.

Lemma 5.4: Let $n=n_k-1-p\in R[k]$. Let $\alpha$ be an integer satisfying (5.6). Let $P'>0$ and $\lambda$ be as in (5.3)-(5.5), and let $d=gcd(k+\alpha, P')$ satisfy (5.7). Then each of the congruences (5.2.A)(i) and (ii) has $d$ solutions $h$, $1<h<n-1$, given by

$$h = s(2k-2\alpha+2-\lambda)-t+1,$$ \hspace{1cm} (5.8)

where $(s,t)$ are solutions to

\[(i) \quad sp' - t(k+\alpha) = d \]
\[(ii) \quad sp' - t(k+\alpha) = -d \]  \hspace{1cm} (5.9)

respectively.

Proof: We shall concentrate on the proof of case (i). Case (ii) is similar.

Since $d=gcd(k+\alpha, P')$, (5.9)(i) has a solution $(s,t)$, by Theorem 5.1(a). Substitute this solution into (5.8), then substitute the result into (5.2.A)(i). Using (5.3) and (5.4) we obtain

$$s(n+P')-t(k+\alpha)+k+\alpha = k+\alpha+d+m, \quad \text{for some integer } r.$$

After cancelling common terms and using (5.9)(i), we obtain
\[(s-r)n = 0,\]

which becomes an identity with the obvious choice \(r=s\).

In fact, by Theorem 5.1(b), equation (5.9)(i) has infinitely many solutions given by

\[\begin{align*}
(s_i, t_i) &= \left(s_0 + i \frac{k + \alpha}{d}, t_0 + i \frac{P'}{d}\right),
\end{align*}\]

where \((s_0, t_0)\) is one particular solution to (5.9)(i), and \(i\) is an integer. Let \(h_i = s_i(2k - 2\alpha + 2 - \lambda) - t_i + 1\). Substituting from (5.10), we obtain

\[h_i = h_0 + i \frac{n}{d}.\]

Thus (5.2.A)(i) has exactly one solution \(h_j\) in each interval \([(j-1)\frac{n}{d}, j\frac{n}{d})\), where \(j\) is an integer. Hence there are exactly \(d\) solutions \(h\) in \(\{0, \ldots, n-1\}\). To complete the proof it suffices to show that \(h=0\), \(h=1\) and \(h=n-1\) are not solutions to (5.2.A)(i).

Equation (5.8) with \(h=1\) becomes \(t = s(2k - 2\alpha + 2 - \lambda)\) which, when substituted into (5.9)(i), gives \(-sn = d\). This is impossible since \(d = \gcd(P', k+\alpha)\) must satisfy \(0 < d < k+\alpha\), \(|\alpha|\) is of the order of \(k\), by (5.6), and \(n\) is of order \(k^2\). The impossibility of \(h=0\) and \(h=n-1\) follows in a similar manner. Thus (5.2.A)(i) has exactly \(d\) solutions \(h\) in the set \(\{2, \ldots, n-2\}\). Let \(h_0\) be the smallest of these solutions, corresponding to the solution \((s_0, t_0)\) of (5.9)(i); the \(d\) solutions \(h_i\) are obtained using (5.10) with \(i=0, \ldots, d-1\). (Compare with the Remark following Theorem 5.1.) \(\Box\)

**Example 5.2:** (a) Let \(k=9\), \(p=6\), hence \(n=n_9 - 7 = 174\); let \(\alpha=-1\), so \(P=P'=2\), \(\lambda=0\).

Here \(d = \gcd(8,2) = 2\), \(\beta = d+\alpha = 1\). \(2s-8t=2\) has two solutions satisfying \((0,0) \leq (s,t) < (8,2)\), namely \((s_0,t_0) = (1,0)\) and \((s_1,t_1) = (5,1)\); those solutions
correspond to the hops $h_0=23$ and $h_1=110$. Hop $h_1=110$ can be replaced by its dual, $h'_1=64$.

(b) Repeat the above calculations for $k=9$, $p=22$, $n=158$, $\alpha=-1$. The corresponding parameters are $d=2$, $\beta=1$, $P=18$, $P'=2$, $\lambda=2$. The solutions to (5.9) are as in (a) above, and the corresponding hops are $h_0=21$, $h_1=100$, $h'_1=58$.

Step 3: A Second Congruence

The next lemma establishes an additional congruence that $h$, given by (5.8) and (5.9), must satisfy. This congruence, along with (5.2.A) will be used later on to examine the structure of $G_{n,h}$. Lemma 5.5 also makes reference to a new parameter $m$. A sufficient (but certainly not necessary) condition for the existence of $m$ is given in Lemma 5.2(c).

Lemma 5.5: Let $n$, $h$, $\alpha$, $\beta$, $P'$, $\lambda$, $d$, $s$, $t$ be as in (5.3)-(5.9) above. Suppose an integer $m$ exists satisfying

\begin{enumerate}
\item $t-ms \equiv 0 \mod d$, $1 \leq m \leq d$, if $d=\beta-\alpha$,
\item $t+ms \equiv 0 \mod d$, $0 \leq m \leq d-1$, if $d=\alpha-\beta$.
\end{enumerate}

Then $h$, as given by (5.8), satisfies the congruence

\[ -q(m)h \equiv E(m) \mod n \]  

where the values of $q(m)$ and $E(m)$ are given by
(i) \( q(m) = \frac{m(k+\alpha)-P'}{d} \), \( E(m) = 2k-2\alpha+2-\lambda-m-q(m) \) \( (5.13) \)

(ii) \( q(m) = \frac{m(k+\alpha)+P'}{d} \), \( E(m) = 2k-2\alpha+2-\lambda+m-q(m) \)

and satisfy

(i) \( k-3\alpha-d \leq E(m) \leq 2\frac{k^2-\alpha}{k+\alpha} \)

(ii) \( k-3\alpha \leq E(m) \leq 2\frac{k^2-\alpha}{k+\alpha} + d \) \( (5.14) \)

Proof: We shall only prove case (i). (Case (ii) is similarly proved.) Multiply (5.9)(i) by \( (2k-2\alpha+2-\lambda) \). Use (5.8), (5.4) and (5.7)(i) to get

\[ P'(h+t-1) - t(n+P') = d(2k-2\alpha+2-\lambda) \]

which can be rewritten as

\[ P'h = d(2k-2\alpha+2-\lambda)+P'+tn. \]

Now subtract \( m(k+\beta) = m((k+\alpha)h-sn) \) from both sides, where \( m \) is provided by condition (i) in Lemma 5.5:

\[-[m(k+\alpha)-P']h = d(2k-2\alpha+2-\lambda)-m(k+\beta)+P'+(t-ms)n \]
\[ = d(2k-2\alpha+2-\lambda)-[m(k+\alpha)-P']-md+(t-ms)n. \]

Since \( d = \gcd(k+\alpha,P') \) and \( t-ms \equiv 0 \mod d \), both sides of the last equality are divisible by \( d \). Dividing by \( d \) produces

\[-q(m)h = E(m) + \frac{t-ms}{d}n, \]

where \( q(m) \) and \( E(m) \) are as given by (5.13)(i). Thus (5.12) holds. Since \( 1 \leq m \leq d \) and \(-\alpha \leq -P' < 0\), we must have \( 1 \leq q(m) \leq k+\alpha-1 \). To calculate the bounds on \( E(m) \)
we use (5.7)(i), the bounds on $\lambda$ from (5.5), the bounds on $P'$ from (5.3) and the bounds on $m$ as given in (5.11)(i). This implies the bounds stated in (5.14)(i). □

**Step 4: The Structure of $G_{n,h}^{k+\delta}$.**

Let $n=n_k-1-p\in \mathbb{R}[k]$ be given and let $h$ be given by Lemma (5.3). Based on the congruences (5.2) and (5.14) we shall identify in the next corollary relationships among the row centers $C_j$ in $G_{n,h}$. (Recall that by its definition in Chapter 2, Section 2.2 $G_{n,h}$ is the infinite grid in $\mathbb{Z}^2$, where each point $(i,j)$ is labeled by $i+jh \mod n$.)

Given $\delta \geq 0$, these relationships among the row centers will imply a permutation $\pi$ of the rows of $G_{n,h}^{k+\delta}$, starting and ending at 0, such that when the rows are traversed in the order specified by $\pi$, and after removal of possible overlaps between rows, the labels are ordered in a nondecreasing order. This is stated in Lemma (5.8). The lemma is followed by illustrative examples.

**Corollary 5.6:** Let $n$, $h$, $\alpha$, $\beta$, $m$, $q(m)$, $E(m)$ be as in Lemma 5.5. Then the row-centers $C_j$ in $G_{n,h}$ satisfy the following relationships, for all integers $j$:

(a) $C_{k+\alpha+j} - C_j = k + \beta$ ;

(b) $C_{-q(m)+j} - C_j = E(m)$ ;

(c) $C_{-(k+\alpha+q(m))+j} - C_j = E(m)-(k+\beta)$ , if $E(m)-(k+\beta) > 0$;

(d) $C_{k+\alpha-q(m)+j} - C_j = E(m)+(k+\beta)$ .

**Proof:** By adding and subtracting $jh$ in the left hand side of congruences (5.2) and (5.12) we obtain
\[(k+\alpha+j)h - jh \equiv k+\beta \mod n, \text{ and} \]
\[(-q(m)+j)h - jh \equiv E(m) \mod n.\]

For the proof of (a) and (b) recall that by their definition in Section 2.2,
\[C_i \equiv ih \mod n, \quad 0 \leq C_i \leq n-1.\]
By (5.6), (5.7) and (5.14) we obtain that \(0 < k+\beta < n,\)
\[0 < E(m) < n \text{ and } 0 < E(m)+k+\beta < n.\]
Parts (c) and (d) follow in a similar way, starting from the difference and the sum, respectively, of congruences (5.12) and (5.2). □

Consider now \(G_{n,h}^{k+\delta}\), for some integer \(\delta \geq 0\). In the next lemma we shall define and examine the directed graphs \(H(V,E)\). In \(H(V,E)\), the vertex set \(V\) corresponds to a set of consecutive rows in \(G_{n,h}^{k+\delta}\). Each vertex \(i \in V\) is labeled by the corresponding row center \(C_i\), where \(C_0\) represents both 0 and \(n\). The arcs in \(E\) represent a subset of the relationships between the different centers, as stated in Corollary 5.6. An arc \(ij\) is labeled by (a), (b), (c) or (d) if it represents the corresponding relationship from Corollary 5.6. An example is shown in Figure 5.1 for \(k=4, n=39, \alpha=0\). Following our notation and the steps described so far we obtain in this case \(\beta=1, P'=1, d=1, h=11, m=1, q(m)=3, E(m)=6\). The figure shows \(G_{39,11}^4\) along with the corresponding digraph \(H(V,E)\). Note that \(V=\{-3,\ldots,3\}\).

Since in Corollary 5.6 we only consider relationships with a positive right hand side, a cycle in \(H(V,E)\) can exist only if \(0 \in V\). In the following Lemma we shall identify the sets \(V\) and \(E\) so that \(E\) forms a hamiltonian cycle in \(H\). This will be done separately for the various possible relationships between \(\alpha, \delta\) and \(q(m)\).
Figure 5.1

$G_{39,11}^4$ and the Corresponding Digraph $H(V,E)$

<table>
<thead>
<tr>
<th>5</th>
<th>32 33 34</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 21 22 23 24</td>
<td></td>
</tr>
<tr>
<td>8 9 10 11 12 13 14</td>
<td></td>
</tr>
<tr>
<td>35 36 37 38 0 1 2 3 4</td>
<td></td>
</tr>
<tr>
<td>25 26 27 28 29 30 31</td>
<td></td>
</tr>
<tr>
<td>15 16 17 18 19</td>
<td></td>
</tr>
<tr>
<td>5 6 7</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td></td>
</tr>
</tbody>
</table>

$C_4$: \[ 5 
C_3: \[ 33 \]
C_2: \[ 22 \]
C_1: \[ 11 \]
C_0: \[ 0 \]

end

\[ \downarrow \]

\[ \text{start} \]

Lemmas 5.7: Let $n, h, \alpha, \beta, m, q(m), E(m)$ be as in Lemma 5.5. Let $\delta \geq 0$ and suppose $E(m) - (k+\beta) > 0$.\(^1\) Then in each of the following digraphs $H(V,E)$ the arcs in $E$ form a Hamiltonian cycle:

1. $\delta \geq \alpha - 1$:

\[ V = \{-(k+\alpha-1), \ldots, k+\alpha-1\}; \ i \in E \text{ if and only if} \]

\[ (a) \ j = i + k + \alpha, \quad i = -(k+\alpha-1), \ldots, -1, \]

\(^1\) See the Remark following the proof of this lemma.
(b) \( j = i - q(m) \), \( i = 0, \ldots, q(m) \),

(c) \( j = i - (k + \alpha + q(m)) \), \( i = q(m) + 1, \ldots, k + \alpha - 1 \);

II. \( \delta < \alpha - 1 \), \( q(m) \leq k + \delta \):

\[ V = \{ -(k + \delta), \ldots, k + \delta \}; \text{ \( ij \in E \) if and only if} \]

(a) \( j = i + k + \alpha \), \( i = -(k + \delta), \ldots, -\alpha + \delta \),

(b) \( j = i - q(m) \), \( i = -(\alpha - \delta) + 1, \ldots, \alpha - \delta + q(m) - 1 \),

(c) \( j = i - (k + \alpha + q(m)) \), \( i = \alpha - \delta + q(m), \ldots, k + \delta \);

III. \( \delta < \alpha - 1 \), \( q(m) > k + \delta \):

\[ V = \{ -(k + \delta), \ldots, k + \delta \}; \text{ \( ij \in E \) if and only if} \]

(a) \( j = i + k + \alpha \), \( i = -(k + \delta), \ldots, -(\alpha - \delta) \),

(b) \( j = i - q(m) \), \( i = q(m) - (k + \delta), \ldots, k + \delta \),

(d) \( j = i + (k + \alpha - q(m)) \), \( i = -(\alpha - \delta) + 1, \ldots, q(m) - (k + \delta) - 1 \).

Moreover, this hamiltonian cycle defines a permutation of the rows in \( G_{n,h}^{k+\delta} \) which corresponds to a non-decreasing ordering of the labels in \( G_{n,h}^{k+\delta} \).

**Proof:** First note that the arcs in parts (a), (b), (c) and (d) of each case correspond to the row-center relationships (a), (b), (c) and (d) of Corollary (5.7), respectively (wherever applicable). Therefore, we must have \( C_j - C_i > 0 \) if \( ij \in E \). For each of the cases I, II and III define

\[ I_a = \{ i : \text{arc } ij \text{ is defined in (a)} \} \],

\[ J_a = \{ j : \text{arc } ij \text{ is defined in (a)} \} \].

\( I_b, I_c, I_d \) and \( J_b, J_c, J_d \) are defined in a similar manner. \( I_d = J_d = \emptyset \) in I and II, and
Note that in each of the cases I, II and III, and for each $x=a,b,c,d$, each of the sets $I_x$ and $J_x$ consists of consecutive non-overlapping integers, and the arcs defined in part (x) of the lemma represent a one-to-one correspondence between those sets. Thus $|I_x|=|J_x|$ for $x=a,b,c,d$. Also, it is easy to check that in each of cases I, II and III we have

$$I_a \cup I_b \cup I_c \cup I_d = V; \quad I_x \cap I_y = \emptyset, \quad \text{if } x \neq y.$$  

The following table serves to prove that this is also true of the $J$-sets, namely

$$J_a \cup J_b \cup J_c \cup J_d = V; \quad J_x \cap J_y = \emptyset, \quad \text{if } x \neq y.$$  

<table>
<thead>
<tr>
<th>Case</th>
<th>$x$</th>
<th>$I_x$</th>
<th>$J_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I $\delta \geq \alpha - 1$</td>
<td>a</td>
<td>$-(k+\alpha-1),...,1$</td>
<td>1,...,$k+\alpha-1$</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>0,...,$q(m)$</td>
<td>$-q(m),...,0$</td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>$q(m)+1,...,k+\alpha-1$</td>
<td>$-(k+\alpha-1),...,q(m)-1$</td>
</tr>
<tr>
<td>II $\delta &lt; \alpha - 1$</td>
<td>a</td>
<td>$-(k+\delta),...,-(\alpha-\delta)$</td>
<td>$\alpha-\delta,...,k+\delta$</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>$-(\alpha-\delta)+1,...,(\alpha-\delta)+q(m)-1$</td>
<td>$-(\alpha-\delta)-q(m)+1,...,\alpha-\delta-1$</td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>$\alpha-\delta+q(m),...,k+\delta$</td>
<td>$-(k+\delta),...,-(\alpha-\delta)-q(m)$</td>
</tr>
<tr>
<td>III $\delta &lt; \alpha - 1$</td>
<td>a</td>
<td>$-(k+\delta),...,-(\alpha-\delta)$</td>
<td>$\alpha-\delta,...,k+\delta$</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>$q(m)-(k+\delta),...,q(m)+(k+\delta)$</td>
<td>$-(k+\delta),...,-(q(m)-(k+\delta))$</td>
</tr>
<tr>
<td></td>
<td>d</td>
<td>$-(\alpha-\delta)+1,...,q(m)-(k+\delta)-1$</td>
<td>$-(q(m)-(k+\delta))+1,...,\alpha-\delta-1$</td>
</tr>
</tbody>
</table>

Each vertex in $V$ has indegree and outdegree exactly one. If we start at vertex 0 and follow the arcs, we will traverse all vertices $i$ in $V$, ending up in vertex 0. The corresponding sequence of $C_i$ is strictly increasing (recall that $C_0$ is viewed as 0 or $n$). This defines a hamiltonian cycle in $H(V,E)$. (Since all arcs were used up, this is also the only hamiltonian cycle in $H$.) This hamiltonian cycle represents a permutation of all the rows represented in $V$. In the transition between any two
consecutive rows of $G_{n,h}^{k+\delta}$, according to this permutation, there may be some overlapping of the labels at the ends, or some labels could be missing. By scanning the rows according to this permutation, starting and ending at 0, the labels will be ordered in a nondecreasing order (after removal of overlapping parts).

The three cases are illustrated in Figures 5.2.I, 5.2.II and 5.2.III, respectively. □
Remark: It is essential that in the row permutation of $G_{n,h}^{k+\delta}$ derived in our proof, the sequence of row centers be strictly increasing. If $E(m)-(k+\beta)<0$, we need to avoid the use of (c) in cases I and II of Lemma 5.7. This can be achieved as follows: let $ij\in E$ be an arc in $H$ labeled (c). An examination of the possible values of $j$ in those cases reveals that there must be some arc $jl$ labeled (a). We can now easily modify $H(V,E)$, by defining $H'(V',E')$ with $V'=V-\{j\}$ and
The new arc is labeled (b). (This is verified by adding the right hand sides of (a) and (c) in Corollary 5.6.) Since the resulting relationship between the centers $C_i$ and $C_j$ is one that was already introduced, it will not have any effect on the subsequent discussion. We shall therefore exclude the explicit description of $H(V,E)$ for this case.

Example 5.3: Consider again the values shown in Example 5.2, where $k=9$, $p=6$ (so $n=n_0-7=174$) and $\alpha=-1$. Let $\delta=1$. We obtain the parameters: $k+\alpha=8$, $k+\beta=10$, $P'=2$, $\lambda=0$, $d=2$. The smallest non-negative solution to (5.9)(i) here is $(s_0,t_0)=(1,0)$, with the corresponding hop $h_0=23$. We have $t_0 \equiv 0 \mod d$, so in (5.11)(i) and (5.13)(i) we use $m_0=d=2$. Thus, $q(m_0)=7$, $E(m_0)=13$ and $E(m_0)-(k+\beta)=3$. The relationships given by combining Corollary 5.6 and Lemma 5.7.1 (here $\delta>\alpha-1$) are

(a) $C_{8-j}-C_j = 10$, \quad $j=1,\ldots,7$

(b) $C_{-7+j}-C_j = 13$, \quad $j=0,\ldots,7$

(c) Is vacuous here.

Those relationships induce the permutation of the rows $-7,\ldots,7$ of $G_{174,23}^{10}$ given by:

$$\pi_0 = <0,-7,1,-6,2,-5,3,-4,4,-3,5,-2,6,-1,7,0>.$$

Corresponding to the second solution $(s_1,t_1)=(5,1)$, $h_1=110$, we obtain the quantities $P'=2$, $\lambda=0$, $m_1=1$, $q(m_1)=3$, $E(m_1)=18$, $E(m_1)-(k+\beta)=8$, and the relationships are

(a) $C_{8-j}-C_j = 10$, \quad $j=1,\ldots,7$
(b) $C_{-3+j} - C_j = 18$, \( j=0,...,3 \)

(c) $C_{-3+j} - C_{8-j} = 8$, \( j=1,...,4 \)

inducing the permutation of the rows of $G_{174,110}^{10}$ given by

$$\pi_1 = \langle 0, -3, 5, -6, 2, -1, 7, -4, 4, -7, 1, -2, 6, -5, 3, 0 \rangle.$$

The transitions between the rows of $G_{174,h}^{10}$, as suggested by $\pi_0$ and $\pi_1$ respectively, are shown in Figures 5.3a and 5.3b. Each row is represented by its center. The relationships in (a) are represented by the west-east arrows, those in (b) by the southwest - northeast arrows and those in (c) are represented by the northwest - southeast arrows. For the sake of reference we provide rows -7,...,7 of $G_{174,23}^{10}$ and $G_{174,110}^{10}$ in Figures 5.4a and 5.4b.

**Figure 5.3a**
Row Traversal in $G_{174,65}^{10}$

**Figure 5.3b**
Row Traversal in $G_{174,110}^{10}$
Figure 5.4a - Rows -7,...,7 of $G_{174,23}^{10}$

158 159 160 161 162 163 164  
134 135 136 137 138 139 140 141 142 
110 111 112 113 114 115 116 117 118 119 120 
86 87 88 89 90 91 92 93 94 95 96 97 98 
62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 
38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 
14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 
164 165 166 167 168 169 170 171 172 173 0 1 2 3 4 5 6 7 8 9 10 
142 143 144 145 146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 
120 121 122 123 124 125 126 127 128 129 130 131 132 133 134 135 136 
98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 
76 77 78 79 80 81 82 83 84 85 86 87 88 
54 55 56 57 58 59 60 61 62 63 64 
32 33 34 35 36 37 38 39 40 
10 11 12 13 14 15 16

Figure 5.4b - Rows -7,...,7 of $G_{174,110}^{10}$

71 72 73 74 75 76 77  
134 135 136 137 138 139 140 141 142 
23 24 25 26 27 28 29 30 31 32 33 
86 87 88 89 90 91 92 93 94 95 96 97 98 
149 150 151 152 153 154 155 156 157 158 159 160 161 162 163 
38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 
101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 
164 165 166 167 168 169 170 171 172 173 0 1 2 3 4 5 6 7 8 9 10 
55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 
120 121 122 123 124 125 126 127 128 129 130 131 132 133 134 135 136 
11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 
76 77 78 79 80 81 82 83 84 85 86 87 88 
141 142 143 144 145 146 147 148 149 150 151 
32 33 34 35 36 37 38 39 40 
97 98 99 100 101 102 103
Step 5: Capacity Considerations

Up to this point no consideration was given to the diameter of $G(n,h)$. We would like to identify those $h$'s given by Lemma 5.4 for which $\text{Diam } G(n,h) \leq k+\delta$, for some constant $\delta \geq 0$. As already mentioned before, a necessary and sufficient condition for $\text{Diam } G(n,h) \leq k+\delta$ is that all the labels in $\{0,\ldots,n-1\}$ appear in $G_{n,h}^{k+\delta}$.

Lemma 5.7 provides information about the layout of the labels in $G_{n,h}^{k+\delta}$. Note that since $n \in \mathbb{R}[k]$, $\text{Diam } G(n,h) \geq k$, by Theorem 2.7. The following lemma provides constraints on the various parameters of the network. The constraints are implicit by nature, since they involve parameters that are functionally dependent. The satisfaction of any of those constraints by a set of parameters is equivalent to $\text{Diam } G(n,h) \leq k+\delta$.

Lemma 5.8: Let $k$, $n \in \mathbb{R}[k]$ and $\delta \geq 0$ be given. Let $\alpha$, $\beta$, $\lambda$, $h$, $d$ and $m$ be as in (5.3)-(5.9) and (5.11). Then $\text{Diam } G(n,h) \leq k+\delta$ if and only if

I. $\delta > \alpha - 1$ and

(i) $-2\delta + 1 - \lambda - m \leq 2\alpha \leq 2\delta - 2$, if $d = \beta - \alpha$;

(ii) $-2\delta + 1 - \lambda + m + d \leq 2\alpha \leq 2\delta - 2$, if $d = \alpha - \beta$.

II. $\delta \leq \alpha - 1$ and

(ii) $\max \{-2\delta + 1 - \lambda + m + d, 2\delta + 3\} \leq 2\alpha \leq 2\delta + 1 + d$, if $d = \alpha - \beta$.

(Cases (i) and (ii) here correspond to cases (i) and (ii) in (5.7), (5.9) and (5.11).)

Proof: Consider $G_{n,h}^{k+\delta}$. Let $H(V,E)$ be the digraph defined in Lemma 5.7, and let $\pi$ be the row permutation defined by $H(V,E)$. A complete coverage of $\{0,\ldots,n-1\}$ by
$G_{n,h}^{k+\delta}$ is equivalent to "no labels missing in any of the transitions between consecutive rows in $\pi$." This later condition can be checked by comparing the "space" allowed in $G_{n,h}^{k+\delta}$ for each transition from row $i$ to row $j$ suggested by $\pi$, with the actual "required space", namely $C_j - C_i$. Let us use the relationships between the centers as given by Corollary 5.6 and by Lemma 5.7. In Case I of Lemma 5.7 those conditions are:

1. $\delta \geq \alpha - 1$
2. $(a) \quad C_{k+\alpha+j} - C_j = k + \beta, \quad j = -(k+\alpha-1), \ldots, -1$
3. $(b) \quad C_{-q(m)+j} - C_j = E(m), \quad j = 0, \ldots, q(m)$
4. $(c) \quad C_{-(k+\alpha+q(m))+j} - C_j = E(m) - (k + \beta), \quad j = q(m) + 1, \ldots, k + \alpha - 1.$

In $G_{n,h}^{k+\delta}$, there are $k+\delta-1$ grid points on each side of the row-center $C_i$. Thus the "complete coverage" condition in this case is equivalent to the simultaneous satisfaction of the following inequalities:

1. $\alpha \leq \delta + 1$
2. $(a) \quad [(k+\delta) - (k+\alpha+j)] + [(k+\delta) - j] + 1 \geq k + \beta, \quad j = -(k+\alpha-1), \ldots, -1$
3. $(b) \quad [(k+\delta) - q(m) + j] + [(k+\delta) - j] + 1 \geq E(m), \quad j = 0, \ldots, q(m)$
4. $(c) \quad [(k+\delta) - (k+\alpha+q(m)) + j] + [(k+\delta) - j] + 1 \geq E(m) - (k + \beta), \quad j = q(m) + 1, \ldots, k + \alpha - 1.$

Using the definition of $E(m)$ from Lemma 5.5, we obtain the following inequalities, corresponding to each of the two cases in (5.7):
In each of cases (i) and (ii) we choose the stricter among the upper, and the stricter among the lower bounds on $\alpha$, to get

\begin{align*}
(i) & \quad -2\delta + 1 - \lambda - m + d \leq 2\alpha \leq 2\delta + 1 - d \\
(ii) & \quad -2\delta + 1 - \lambda + m + d \leq 2\alpha \leq 2\delta + 2 \quad \text{(5.15.1)}
\end{align*}

When the process is repeated for cases II and III of Lemma 5.7, that the results for both are identical. Taking again the stricter bound in each case gives in case (i) of (5.7)

\begin{align*}
(i) & \quad 2\delta + 3 \leq 2\alpha \leq 2\delta + 1 - d \\
\end{align*}

which is impossible to solve, as $d > 0$. Thus, for cases II and III we obtain the following necessary and sufficient condition for complete coverage:

\begin{align*}
(ii) & \quad \max \{-2\delta + 1 - \lambda + m + d, 2\delta + 3\} \leq 2\alpha \leq 2\delta + 1 + d. \quad \square \quad (5.15.11)
\end{align*}

If either of the constraints (5.15.1) or (5.15.11) is satisfied by some set of parameters, $h$ is computable by Lemma 5.4 and so, by Lemma 5.8 $\text{Diam } G(n,h) \leq k + \delta$. We can thus form the disjunction of the constraints in (5.15.1) and (5.15.11) to arrive at sufficient conditions for $D^*_n \leq k + \delta$. These conditions are stated in the next theorem.

**Theorem 5.9:** Let $k, n \in \mathbb{R}[k]$ and $\delta \geq 0$ be given. Suppose there exist integers $\alpha, \lambda,$
d and m, as in Lemma 5.8, satisfying

(i) \(-2\delta+1-\lambda-m \leq 2\alpha \leq 2\delta+1-d\) or

(ii) \(-2\delta+1-\lambda+m+d \leq 2\alpha \leq 2\delta+1+d\).

Then \(D_\gamma^* \leq k+\delta\).

**Proof:** In view of the comment preceding the theorem, all we need to show is that conditions (i) and (ii) in the statement of the theorem are obtained by forming the disjunction of the conditions in cases I and II of Lemma 5.8. This is clearly true for (i). Part (ii) is obtained by simply combining the ranges of \(\alpha\) in I(ii) and II(ii) in Lemma 5.8. \(\square\)

**Remark:** Suppose that for some \(n \in \mathbb{R}[k]\) it is known that \(D_\gamma^* \leq k+\delta\). Then a hop \(h\) must exist for which \(\text{Diam } G(n,h) \leq k+\delta\). The sufficient conditions in Theorem 5.9 would also be necessary, if it were possible to prove that \(h\) exists that conforms to the form in Lemma 5.4. Although it is our feeling that this is true, we shall not attempt to prove it here.

**Example 5.4:** (a) Consider again the value \(n=174\) from Examples 5.1(a) and 5.2. In those examples we have chosen \(\alpha=-1\), which corresponds to Case (i) with \(d=2\), \(P'=2\) and \(\lambda=0\). The solutions \(h_0=23\) and \(h_1=110\) correspond to the values \(m_0=2\) and \(m_1=1\) respectively, as was shown in Example 5.3. To see what is the least value of \(\delta\) in this case, substitute all these values into (i) of Theorem 5.9, to get

\[-2\delta-1 \leq -2 \leq 2\delta-1\] for \(h_0 = 23\), and
In both cases the left inequality implies that $\delta \geq 1$, and therefore none of $h_0$, $h_1$ could be optimal.

(b) Repeat these steps for $n=158$ (see Example 5.2(b)). Again $\alpha = -1$, which corresponds to Case (i) with $d=2$, $P'=2$ and $\lambda = 2$. The solutions $h_0 = 21$ and $h_1 = 100$ correspond to the values $m_0 = 2$ and $m_1 = 1$, and from (i) of Theorem 5.9 we get

\[-2\delta - 3 \leq -2 \leq 2\delta - 1 \quad \text{for} \quad h_0 = 21, \quad \text{and} \]

\[-2\delta - 2 \leq -2 \leq 2\delta - 1 \quad \text{for} \quad h_1 = 100. \]

Both inequalities hold for $\delta = 0$, hence, by Theorem 5.9, $\text{Diam} \ G(158,21) \leq 9$ and $\text{Diam} \ G(158,100) \leq 9$. Since $158 \in \mathbb{N}[9]$ we conclude that $n=158$ is optimal.

In their current form, the conditions in Theorem 5.9 are quite complicated to apply, mainly because of the presence of $m$. ($m$ is defined in Lemma 5.5). In Step 6 (Corollary 5.12) we shall derive conditions that are much simpler to apply, using some additional knowledge about the value of $m$ in some special cases. But first we must pause, and go back to treat the case of type B, with $P'=0$.

5.4.B: Derivation of $\delta$-families with $P'=0$ ($\alpha = \beta$)

Let us now examine Case B that was left out earlier, namely that of $\alpha = \beta$ (a prototype case was mentioned in Section 5.2.B). In this case, as was explained after
Theorem 5.1, $P = \lambda(k+\alpha)$ and $P'=0$. The following lemma combines Steps 1-5 for this case. Evidently, Case B is much simpler to treat, but the results it produces correspond to a much smaller set of values of $n$, when compared to those obtained in Case A.

Lemma 5.10 Let $n=(k+\alpha)(2k-2\alpha+2-\lambda)$, where $\alpha$ and $\lambda$ satisfy (5.3), (5.5) and (5.6). Let $s$ be an integer satisfying

$$\gcd(s, k+\alpha)=1, \quad 0 < s < k+\alpha,$$

and let $h$ be given by

$$h = s\frac{n}{k+\alpha}+1 = s(2k-2\alpha+2-\lambda)+1. \quad (5.16)$$

Then $\text{Diam } G(n,h) \leq k+\delta$ if and only if

$$-2\delta+1-\lambda \leq 2\alpha \leq 2\delta+1. \quad (5.17)$$

Proof: $h$ was chosen so that it satisfies the congruence

$$(k+\alpha)h \equiv k+\alpha \mod n \quad (5.2.B)$$

(this corresponds to $\beta=\alpha$ in (5.2), $P'=0$ in (5.3)). A second congruence that $h$ must satisfy is obtained using the fact that $\gcd(s,k+\alpha)=1$. Thus, the equation

$$qs-t(k+\alpha)=1 \quad (5.18)$$

has a solution $(q,t)$ satisfying $(0,0) \leq (q,t) < (k+\alpha,s)$. Let $(q,t)$ be such a solution. Substituting $(q,t)$ into (5.18) and then multiplying it by $\frac{n}{k+\alpha}$, we obtain (using (5.16) and (5.4))
\[ q(h-1)-m=2k-2\alpha+2-\lambda, \]

or

\[ qh \equiv 2k-2\alpha+2-\lambda+q \mod n. \quad (5.19) \]

From congruences (5.2.B) and (5.19), and from their difference, we derive the following relationships between the row-centers of \( G_{n,h} \), (similar to Corollary 5.6):

(a) \( C_{k+\alpha+i}-C_i = k+\alpha \),

(b) \( C_{-(k+\alpha-q)+i}-C_i = k-3\alpha+2-\lambda+q \),

(c) \( C_{-2(k+\alpha)+q+i}-C_i = -4\alpha+2-\lambda+q \).

Let \( \delta \geq 0 \). The digraph \( H(V,E) \) (see explanation preceding Lemma 5.7) is defined here as follows:

\[ V = \{ -(k+\alpha-1),...,-1 \} \; ; \; ij \in E \text{ if and only if:} \]

(a) \( j = k+\alpha+i, \quad i = -(k+\alpha-1),...,-1, \)

(b) \( j = -(k+\alpha-q)+i, \quad i = 0,...,k+\alpha-q, \)

(c) \( j = -[2(k+\alpha)-q]+i, \quad i = k+\alpha-q+1,...,k+\alpha-1. \)

Note that by assumption (5.6), the right hand sides in relationships (a) and (b) are positive. As before (see Remark following Lemma 5.7), if the right hand side in relationship (c) is non-positive, each arc \( ij \) labeled (c) in \( E \) is combined with its "successor" \( jl \) (which must be labeled (a)), to form a new arc \( il \), labeled (b). The old arcs \( ij \) and \( jl \) are removed from \( E \), and vertex \( j \) is removed from \( V \).

As in Corollary 5.6, the arcs in \( E \) form a hamiltonian cycle in \( H \), thus defining a

\[ ^{2} \text{See Remark following this proof.} \]
permutation of the rows of $G_{n,h}^{k+\delta}$ that are represented in $V$ by their centers. If these rows are scanned according to this permutation, then after the removal of possible overlap between successive rows, the labels in those rows will be ordered in a strictly increasing order. If none of the labels $\{0, \ldots, n-1\}$ is missing in this ordering, then clearly $G_{n,h}^{k+\delta}$ covers $g_{n,h}$. This is necessary and sufficient for $\text{Diam } G(n,h) \leq k+\delta$.

Using "row-capacity" considerations, as in Theorem 5.9, we obtain the constraint (5.17). Thus satisfaction of (5.17) is a necessary and sufficient condition for $\text{Diam } G(n,h) \leq k+\delta$. □

Remark: The cases II and III of Lemma 5.7 do not have their parallels for type B cases. The reason is that the upper bound on $\alpha$ in (5.16), which is obtained by applying row-capacity considerations to relationship (a) in the proof of Lemma 5.10, is even stricter than the constraint $\delta \geq \alpha - 1$, which is assumed in case I there.

Example 5.5: Let $k=9$, $p=12$, $n=168$, $\alpha=-1$. Here $P=8$, $P'=0$, $\lambda=1$. Let $s=1$, then from (5.16) we get $h=22$. The solution to (5.18) is $q=1$, $t=0$. The relationships between the centers $C_i$ are

(a) $C_{8-j} - C_{-j} = 8$, \hspace{1cm} $j=1, \ldots, 8$,

(b) $C_{-7+j} - C_j = 13$, \hspace{1cm} $j=0, \ldots, 7$,

(c) is vacuous,

and (5.17) becomes

$$-2\delta \leq -2 \leq 2\delta + 1.$$ 

The smallest value of $\delta$ which satisfies this inequality is $\delta=1$. Thus, we
may conclude that $n=168$ is either optimal or suboptimal. The value $n=168$ is, in fact, optimal. Since $168=q_3(9)-3$, the value $n=168$ belongs to the sparse family $\Phi_2[9]$ from Chapter 3 (see Theorem 3.4).

In the next theorem we state a sufficient condition for $D_n^* \leq k+\delta$, in case $P'=0$. As in Theorem 5.9, no knowledge of a specific $h$ is needed to verify this condition.

**Theorem 5.11** Let $n \in R[k]$, $\alpha$, $\lambda$ and $\delta$ be as in Lemma 5.10, so that

$$-2\delta+1-\lambda \leq 2\alpha \leq 2\delta+1$$

holds. Then $D_n^* \leq k+\delta$.

**Proof**: Let $h$ be given by (5.16). Since (5.17) holds, $\text{Diam } G(n,h) \leq k+\delta$ by Lemma 5.10. $\Box$

We have now identified sufficient conditions for $n \in R[k]$ to belong to a $\delta$-family, for both types of cases, A and B. These conditions are stated in Theorems 5.9 and 5.11. While Theorem 5.9 is general, it is inconvenient to use for the computation of the $\delta$-families, because the knowledge of $m$ (as defined in Lemma 5.5) is required. In Step 6 we derive some special forms of the constraints obtained in Theorem 5.9, thus providing sufficient conditions for $D_n^* \leq k+\delta$ in which the knowledge of $m$ is not required. The cost for switching to those special conditions is that some values of $n$ that belong to a $\delta$-family by Theorem 5.9, may go undetected by the new "tests".

The results stated in Step 6 are combined to cover both types of cases, A and B.
Step 6: Simplified Constraints for Some "Important" Special Cases.

The constraints given in Theorem 5.9 (and in Theorem 5.11) constitute sufficient conditions for a given \( n \in R[k] \) to belong to a general \( \delta \)-family. Those constraints require the existence of, and knowledge about the value of \( m \), which was defined in Lemma 5.5.

If \( n \in R[k] \) and \( \delta \geq 0 \) are given, one could try different values of \( \alpha \), find the corresponding values of \( P, d, \lambda, h \) and \( m \) (as specified in Lemma 5.4 and 5.5), and then check whether the conditions of Theorem 5.9 (or 5.11) are satisfied. However, this process is quite lengthy (it parallels, to some extent, the "trial and error" process in Algorithms 1 and 2 in Chapter 2, where most values of \( h \) are checked, except that here the actual construction of the grids is spared).

Our approach will be different. We shall first find tuples of parameters that satisfy either of the above mentioned constraints, and then match each such tuple to corresponding values of \( n \) in \( R[k] \). There will usually be many values of \( n \) that match each tuple.

In our attempt to find tuples of parameters that satisfy the constraints in Theorems 5.9 and 5.11, we notice that those constraints can also be viewed as implicit bounds on \( \alpha \). The bounds given in Theorem 5.9 depend on \( m \). In those cases when \( m \) exists, some of its possible values will secure better bounds on \( \alpha \), than others. In fact the value \( m = d \) in case (i) and \( m = 0 \) in case (ii) are optimal in the sense that they provide the largest possible range for \( \alpha \).
We shall use Lemma 5.2 in Section 5.3, as it provides a sufficient condition for the existence of \( m \) and sufficient conditions for the existence of an optimal or a near optimal \( m \), in the above sense. Those conditions, after translation to our environment, are incorporated in the next result.

**Corollary 5.12:** Let \( n, \alpha, P, d, \delta \) and \( \lambda \) be as in Theorem 5.9 or Theorem 5.11. Then \( D_n^* \leq k+\delta \) if any one of the following conditions is satisfied:

- (a) \( \gcd\left(\frac{P}{d}, d\right) = 1 \) and
  \[
  \begin{align*}
  -2\delta+1 - \lambda - d &\leq 2\alpha \leq 2\delta+1 - d \\
  \text{or} &
  -2\delta+1 - \lambda + d &\leq 2\alpha \leq 2\delta+1 + d
  \end{align*}
  \]

- (b) \( \gcd\left(\frac{P+k+\alpha}{d}, d\right) = 1 \) and
  \[
  \begin{align*}
  -2\delta+2 - \lambda - d &\leq 2\alpha \leq 2\delta+1 - d \\
  \text{or} &
  -2\delta+2 - \lambda + d &\leq 2\alpha \leq 2\delta+1 + d
  \end{align*}
  \]

- (c) \( \gcd\left(\frac{k+\alpha}{d}, d\right) = 1 \) and
  \[
  \begin{align*}
  -2\delta - \lambda &\leq 2\alpha \leq 2\delta+1 - d \\
  \text{or} &
  -2\delta - \lambda + 2d &\leq 2\alpha \leq 2\delta+1 + d
  \end{align*}
  \]

- (d) \( P = \lambda(k+\alpha) \) and
  \[
  -2\delta+1 - \lambda \leq 2\alpha \leq 2\delta+1
  \]

**Proof:** Consider cases (i) and (ii) of Theorem 5.9, where \( m \) is defined in cases (i) and (ii) of Lemma 5.5, respectively.

Suppose \( \gcd\left(\frac{P}{d}, d\right) = 1 \) (hence also \( \gcd\left(\frac{P'}{d}, d\right) = 1 \)). By Lemma 5.2(a) with \( a=P' \), \( b=k+\alpha \), equation (5.9) must have a solution \((s, t)\) such that \( tm \equiv 0 \pmod{d} \). This corresponds to \( m=d \) in case (i), and \( m=0 \) in case (ii) of (5.11). Condition (a) is obtained by substituting those values of \( m \) in cases (i) and (ii) of Theorem 5.9, respectively.
If \( \gcd\left(\frac{P+k+\alpha}{d}\right) = 1 \), condition (b) follows in a similar way using Lemma 5.2(b). In this case equation (5.9) must have a solution \((s,t)\) such that \( t+s \equiv t-(d-1)s \equiv 0 \mod d \). The result is obtained from Theorem 5.9 by substituting \( m=d-1 \) in (i) and \( m=1 \) in (ii).

If \( \gcd\left(\frac{k+\alpha}{d}\right) = 1 \), the existence of \( m \) in Theorem 5.9 is guaranteed by Lemma 5.2(c). Lemma 5.5 implies that \( 1 \leq m \leq d \) in case (i) and \( 0 \leq m \leq d-1 \) in case (ii). Condition (c) is obtained by taking the "worst" value of \( m \) in each case, namely \( m=1 \) in case (i) and \( m=d-1 \) in case (ii).

Condition (d) is a repetition of Theorem 5.11, included here only for the sake of completeness. □

In Chapter 6 we shall examine special cases of Corollary 5.12, for \( \delta=0 \) and \( \delta=1 \) respectively. Those are the optimal families and the suboptimal families, as defined in Chapter 1.
CHAPTER 6

GENERAL OPTIMAL AND SUBOPTIMAL FAMILIES

§ 6.1 Introduction

In this chapter we shall use the conditions defining the δ-families that were derived in Chapter 5, to elaborate on the special cases of particular interest to us, namely the optimal and the suboptimal families.

In the case of optimal families, we derive in Section 6.2 many additional families of the "dense" type (see Chapter 4). In fact we exhibit infinitely many of them, although, naturally, for each particular k their number is finite. Those families contribute considerably towards a complete characterization of all optimal values of n. Since our proofs are constructive, they can be traced back to construct at least one (and often more than one) optimal hop h corresponding to each value of n that is identified as optimal.

Similarly, we exhibit in Section 6.3 a large list of families that consist of values of n that are either optimal or suboptimal. As it turns out, all the values of n up to 8,000,000 satisfy the sufficient conditions that we present in Section 6.3.

§ 6.2 General Optimal Families

Corollary 5.12 with δ=0 provides sufficient conditions for \( D_n \leq k \). If \( n \in \mathbb{R}(k) \), this means, by Theorem 2.7, that n is optimal. In the proof of the following
theorem, we compute all possible triples of parameters \((\lambda, \alpha, d)\) that satisfy the various constraints in Corollary 5.12, with \(\delta=0\). The set of values of \(n \in R[k]\) that match each such tuple is guaranteed, by Corollary 5.12, to be an optimal family. (To be consistent with notation used in previous chapters we should say that each such set of values of \(n\) is the intersection of some optimal family \(\Theta\) with the range \(R[k]\).) The conditions characterizing each of those families are stated explicitly in the next theorem.

**Theorem 6.1**: Let \(n \in R[k]\) be given by \(n=2k^2+2k-p, 0<p<4k-1\). Let \(-k<\alpha<\frac{k}{3}\) and

\[
P=p-2\alpha(\alpha-1), \quad d=\gcd(k+\alpha, p), \quad e=\gcd\left(\frac{p}{d}, d\right), \quad f=\gcd\left(\frac{p+k+\alpha}{d}, d\right).
\]

Then each of the following conditions is sufficient for the optimality of \(n\):

1. \(p=k, 2k, 3k, 3k+1\);
2. \(\gcd(k, p)=1, p>0\);
3. \(\gcd(k+1, p)=1, p>0\);
4. \(\gcd(k-1, p-4)=1, p>2k+2\);
5. \(d=12\alpha-11, \quad \text{and} \quad \{[c=1, P>0] \text{ or } [c=1, f=1, P>k+\alpha]\}\)
6. \(d=12\alpha-1, \quad \text{and} \quad \{[c=1, P>2(k+\alpha)] \text{ or } [c=1, f=1, P>3(k+\alpha)]\}\)
7. \(d=12\alpha+1, \quad \text{and} \quad \{[c=1, P>3(k+\alpha)]\}\)

**Proof**: The proof consists mainly in identifying all the solutions to the various constraints in Corollary 5.12 with \(\delta=0\). Each such solution is a triple of parameters \((\lambda, \alpha, d)\) that corresponds to a condition listed in the theorem. The set of all values
of $n$ that corresponds to each such condition constitutes a family that is guaranteed to be optimal, by Corollary 5.12.

(0) Consider the constraint in Corollary 5.12(d), with $\delta=0$:

$$1-\lambda \leq 2\alpha \leq 1.$$  \hspace{1cm} (6.1)

(6.1) is only solvable for $1 \leq \lambda \leq 3$. (Recall that by (5.5), $\lambda \leq 3$.) The only solutions $(\lambda, \alpha)$ to (6.1) are given by

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0, -1</td>
</tr>
</tbody>
</table>

These solutions correspond to $p=k$, $2k$, $p=3k$ and $p=3k+1$ respectively.

(1)-(3) If we let $d=1$ in Corollary 5.12, then only part (a) is applicable. In that case the conditions on $\alpha$ are

$$-\lambda \leq 2\alpha \leq 0 \hspace{0.5cm} \text{or} \hspace{0.5cm} 2-\lambda \leq 2\alpha \leq 2.$$  \hspace{1cm} (6.2)

hence $\lambda$ must satisfy $0 \leq \lambda \leq 3$. (See also (5.5).) The only solutions that (6.2) admits are

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0, 1</td>
</tr>
<tr>
<td>1</td>
<td>0, 1</td>
</tr>
<tr>
<td>2</td>
<td>0, 1, -1</td>
</tr>
<tr>
<td>3</td>
<td>0, 1, -1</td>
</tr>
</tbody>
</table>

These results are reformulated in conditions (1), (2), (3) according to $\alpha=0$, 1, -1 respectively. Note that this and the following cases are all of type A, with $P'>0$. 
Thus each pair \((\lambda, \alpha)\) above corresponds to the set of values of \(P\) satisfying
\[ P > \lambda(k+\alpha); \gcd(k+\alpha,P)=1. \]

Now suppose that \(d > 1\). Comparing the ranges of \(\alpha\) in the constraints of parts (a), (b) and (c) of Corollary 5.12, we see that the ranges in parts (a) and (b) are larger than the range allowed in part (c). The constraint in part (c) of Corollary 5.12 with \(\delta = 0\), is
\[ -\lambda \leq 2\alpha \leq 1 - d \quad \text{or} \quad -\lambda + 2d \leq 2\alpha \leq 2. \] (6.3)

By (5.5), \(\lambda \leq 3\), hence (6.3) implies \(d \leq 4\). (since \(d = \gcd(k+\alpha,P)\) by definition, \(d > 0\).)

These values of \(d\) are all prime powers. Thus, as in the proof of Corollary 5.3, at least one of the conditions \(c = \gcd(P/d,d)=1\) or \(f = \gcd(P+k+\alpha/d,d)=1\) must hold. Therefore, in the case of \(\delta = 0\), part (c) of Corollary 5.12 is subsumed by parts (a) and (b).

(4)-(7) Let us now concentrate on conditions (a) and (b) in Corollary 5.12. For \(\delta = 0\) they can be rewritten as:
\[
\begin{align*}
(a) & \quad 1 - \lambda \leq 2\alpha+d \leq 1, \quad \text{if } c = 1, \\
(b) & \quad 2 - \lambda \leq 2\alpha+d \leq 1, \quad \text{if } c > 1 \text{ and } f = 1.
\end{align*}
\] (6.4)

For (6.4)(a) and (b) to be solvable, (and again considering (5.5)) we must have
\[
\begin{align*}
(a) & \quad 0 \leq \lambda \leq 3, \\
(b) & \quad 1 \leq \lambda \leq 3.
\end{align*}
\]

Also note that the solutions to (6.4)(b) with \(\lambda = \lambda_0\) are obtained from the solutions to (6.4)(a) with \(\lambda_0 - 1\). All the solutions to (6.4) are listed in the following table:
These give rise to the conditions listed in (4)-(7). □

Remarks:

(a) Although Theorem 6.1 contains a long list of conditions for optimality, this list may not be complete. The main reason to that is, as was mentioned in the comment preceding Corollary 5.12, the avoidance of the parameter m that appears in Theorem 5.9. Thus for d that is not a prime power, Theorem 5.9 with δ=0, may provide additional sufficient conditions for optimality, according to various values of m.

(b) Note that conditions (1) - (3) in Theorem 6.1 echo the dense families \( T_1, T_2, T_3 \) introduced in Chapter 4, with a slight change in the range of \( \Psi_3 \).

Similarly, the values of \( p \) included in condition (0) of Theorem 6.1 are covered by the sparse families \( \Phi_1 \) and \( \Phi_2 \) from Chapter 3.

(c) Conditions (4) - (7) in Theorem 6.1 define infinitely many new optimal families \( \Psi_i^\alpha, 4 \leq i \leq 7 \), given by

\[
\Psi_i^\alpha = \bigcup_{k \geq 1} \Psi_i^\alpha[k], \quad \text{where}
\]

\[
\Psi_i^\alpha[k] = \{ n=2k^2+2k-p \in \mathbb{R}[k] : P=p-2\alpha(\alpha-1) \text{ satisfies condition (i) in Theorem 6.1} \}.
\]

However, for each given value of \( k \), and for each \( 4 \leq i \leq 7 \), \( \Psi_i^\alpha[k] \) is nonempty only for a finite number of values of \( \alpha \).
(d) The various optimal families often overlap. For example, note that $\Psi_4^0 = \Psi_1$, $\Psi_4^1 = \Psi_2$, $\Psi_6^1 \subseteq \Psi_3$. In fact, the above relationships show that conditions (1)-(3) in Theorem 6.1 are covered by conditions (4)-(7), and could have been excluded from the theorem. We put them there for two reasons:

1. To be able to make the connection to the dense families $\Psi_i$, $1 \leq i \leq 3$.

2. We have seen in Section 4.4 that the dense families cover 92% of all values of $n$ up to $n=8,000,000$ (and this is probably equally true for larger values of $n$). In searching for nonoptimal values in $R[k]$, the application of the tests in (1)-(3) will generally exclude the vast majority of values of $n$ (sometimes, all but one value). This is shown in Example 6.1, for $k=25$.

(e) The sufficient conditions for optimality listed in Theorem 6.1 can be viewed as an algorithm for the detection of optimal values of $n$ in each $R[k]$. Running it shows that 93% of all values of $n$ up to 8,000,000 satisfy at least one of the optimality conditions in the theorem, while 92% belong to the dense families of Chapter 4 (listed here in conditions (1)-(3)). The percentage of optimal values of $n$ that satisfy the conditions in Theorem 6.1 or belong to the sparse families of Chapter 3, is 94%, in the same range.

Example 6.1: Given $k$, we shall search for those values of $n \in R[k]$ that may not be optimal. This will be done by the repeated removal from $R[k]$ of all the values of $n$ that satisfy the various conditions in Theorem 6.1.

Let $k=25$. The values of $n=n_k-1-p$ in $R[25]-\{n_25, n_{25}-1\}$ correspond to all values of $p$ in the set $S = \{1, \ldots, 98\}$. Let $S_0 = S-\{k, 2k, 3k, 3k+1\}$, and
let \( S_i, i=1,\ldots,7, \) denote the set \( S_{i-1} \) from which we have purged all the values of \( p \) which satisfy condition (i) in Theorem 6.1. The resulting set \( S_7 \) contains the only values of \( p \) corresponding to values of \( n \) that may not be optimal, according to Theorem 6.1.

\[
S_0 = S \setminus \{25,50,75,76\};
\]
\[
S_1 = \{5,10,15,20,30,35,40,45,55,60,65,70,80,85,90,95\};
\]
\[
S_2 = \{10,20,30,40,60,65,70,80,90\};
\]
\[
S_3 = \{10,20,30,40,60,70,80,90\};
\]

To apply condition (4) we first need to find an \( \alpha \) such that \( d=12\alpha - 1 \) divides \( k+\alpha \). In our case this only happens when \( \alpha = 2 \) or \(-1\). The values \( p=10 \) and \( p=70 \) both satisfy \( \gcd(27, p-4)=3 \), \( \gcd\left(\frac{p-4}{3}, 3\right)=1 \), \( p-4>0 \). No other value in \( S_3 \) satisfies condition (4). Thus,

\[
S_4 = \{20,30,40,60,80,90\};
\]

Condition (5) is applicable for \( \alpha = 1 \) \( (k+1=26, d=2) \), eliminating 30, 40, 60, 80, 90. Any further applications of this condition will not eliminate 20, so

\[
S_5 = \{20\}.
\]

Since the only remaining value of \( p \) satisfies \( p<k \), we may ignore conditions (6) and (7):

\[
S_6 = S_7 = \{20\}.
\]

Thus the only value of \( n \) in \( R[25] \setminus \{n_{25}, n_{25}-1\} = \{1202,\ldots,1299\} \) that is possibly non-optimal is \( n=n_{25}-1-20=1280 \). In fact \( n=1280 \) is suboptimal,
as will follow from the result in the next section.

Consider $S_3$ of Example 6.1. A comparison with the sparse families $\Phi$ of Chapter 3 shows that all values of $p$ in $S_3$, except $p=10$ and $p=20$, represent values of $n \in \Phi$ as follows:

- $p=30 \leftrightarrow n=q_3[25]-5 \in \Phi_2[25]$;
- $p=40 \leftrightarrow n=q_3[25]-15 \in \Phi_2[25]$;
- $p=60 \leftrightarrow n=q_3[25]-35 \in \Phi_2[25]$;
- $p=70 \leftrightarrow n=q_3[25]-45 \in \Phi_2[25]$;
- $p=80 \leftrightarrow n=q_3[25]-55 \in \Phi_2[25]$;
- $p=90 \leftrightarrow n=q_1[25]-15 \in \Phi_2[25]$;

Thus the only "new" optimal value that was not included in the sparse or dense families of previous chapters is that of $n=1290$ ($p=10$). In the next example we shall use the mechanism presented in Steps 1-6 in Chapter 5 to obtain an optimal hop $h$ so that $\text{Diam } G(1290,h)=25$.

**Example 6.2:** Let $k=25$, $p=10$, $n=1290$. $n$ was identified as optimal using condition (4) in Theorem 6.1. The corresponding set of parameters was $d=3$, $\alpha=2$, $P'=P=6$, $\lambda=0$, $c=1$. The value $c=1$ indicates that this set of parameters was obtained by solving constraint (a) in Corollary 5.12, with $\delta=0$; substitution into the constraint with $\delta=0$ reduces the possibilities to case (ii) there. By (5.7)(ii), $\beta=\alpha-d=-1$. Thus, this set of parameters satisfies constraint (ii) in Theorem 5.9, with the value of $m=0$. To compute the value of an optimal
hop, we now use (5.8) and (5.9)(ii), with the additional requirement that
t=0 \mod d, from (5.11)(ii). Condition (5.9)(ii) in our setting is

\[27t-6s=3.\]

The pertinent solutions are

\[(s_0, t_0) = (4, 1), \quad (s_1, t_1) = (13, 3), \quad \text{and} \quad (s_2, t_2) = (22, 5).\]

Only \((s_1, t_1) = (13, 3)\) satisfies \(t=0 \mod 3\). Thus, the optimal hop is (by
(5.8)) \(h=622\).

Since all remaining conditions in Theorem 6.1 are only applicable for \(\lambda>0\),
they will not contribute any additional optimal hops for this value of \(n\).
Thus, judging by the sufficient conditions for optimality presented in
Theorem 6.1, no other optimal hop (up to taking duals) exists in this case.
(This was verified using Algorithm 1 from Chapter 2.) The basic grid
\(g_{1290,622}\) is illustrated in Figure 6.1.
Figure 6.1

$1290,622$

\[ \begin{array}{c}
70 \\
738 \\
116 \\
784 \\
162 \\
830 \\
208 \\
876 \\
254 \\
922 \\
300 \\
968 \\
346 \\
1014 \\
392 \\
1060 \\
438 \\
1106 \\
484 \\
1152 \\
530 \\
1198 \\
576 \\
1244 \\
622 \\
0 \\
668 \\
46 \\
714 \\
92 \\
760 \\
138 \\
806 \\
184 \\
852 \\
230 \\
899 \\
276 \\
944 \\
323 \\
990 \\
368 \\
1036 \\
414 \\
1082 \\
506 \\
1128 \\
552 \\
1174 \\
920 \\
1220 \\
\end{array} \]

\[ \begin{array}{c}
o = \text{labeled point} \\
x = \text{unlabeled point} \\
\end{array} \]
§ 6.3 General Suboptimal Families

The next special case of Corollary 5.12 that is of particular interest is that of \( \delta = 1 \). In this case, Corollary 5.12 provides sufficient conditions for \( D_n^* \leq k + 1 \), where \( n \in R[k] \). Those values of \( n \) must therefore be either optimal or suboptimal. These sufficient conditions are listed in the next theorem.

Theorem 6.2 Let \( n \in R[k] \) be given by \( n = 2k^2 + 2k - p \), \( 0 < p < 4k - 1 \). Let \( -k < \alpha < \frac{k}{3} \), and

\[
P = p - 2\alpha(\alpha - 1), \quad d = \gcd(k + \alpha, p), \quad c = \gcd\left(\frac{p}{d}, d\right), \quad f = \gcd\left(\frac{p + k + \alpha}{d}, d\right).
\]

Then each of the following conditions is sufficient for \( D_n^* \leq k + 1 \):

1. \( p = 0, k, k + 1, k + 3, 2k, 2k + 2, 3k, 3k + 1, 3k + 3, 3k + 6; \)
2. (i) \( \gcd(k - 2, p - 12) = 6, \quad \gcd\left(\frac{k - 2}{6}, 6\right) = 1, \quad p - 12 > 2(k - 2); \)
   (ii) \( \gcd(k + 4, p - 24) = 6, \quad \gcd\left(\frac{k + 4}{6}, 6\right) = 1, \quad p - 12 > 2(k + 4); \)
3. \( \gcd(k, p) = 1; \)
4. \( \gcd(k + 1, p) = 1; \)
5. \( \gcd(k - 1, p - 4) = 1, \quad p > 4; \)
6. \( \gcd(k + 2, p - 4) = 1, \quad p > 4; \)
7. \( \gcd(k - 2, p - 12) = 1, \quad p > 2k + 8; \)
8. \( d = 12\alpha - 31, \quad \left[ \left( c = 1, \ P > 4(k + \alpha) \right) \text{ or } \left( c > 1, \ f = 1, \ P > 3(k + \alpha) \right) \right]; \)
9. \( d = 12\alpha - 21, \quad \left[ \left( c = 1, \ P > 3(k + \alpha) \right) \text{ or } \left( c > 1, \ f = 1, \ P > 2(k + \alpha) \right) \right]; \)
10. \( d = 12\alpha - 11, \quad \left[ \left( c = 1, \ P > 2(k + \alpha) \right) \text{ or } \left( c > 1, \ f = 1, \ P > -(k + \alpha) \right) \right]; \)
11. \( d = 12\alpha + 1, \quad \left[ \left( c = 1, \ P > -(k + \alpha) \right) \text{ or } \left( c > 1, \ f = 1, \ P > 0 \right) \right]; \)
(12) \( d = 12\alpha + l \), \( \{[c=1, P>0] \text{ or } [c>1, f=1, P>(k+\alpha)]\} \);

(13) \( d = 12\alpha + 21 \), \( \{[c=1, P>(k+\alpha)] \text{ or } [c>1, f=1, P>2(k+\alpha)]\} \);

(14) \( d = 12\alpha + 31 \), \( \{[c=1, P>2(k+\alpha)] \text{ or } [c>1, f=1, P>3(k+\alpha)]\} \);

(15) \( d = 12\alpha + 41 \), \( \{[c=1, P>3(k+\alpha)]\} \).

Proof: As in the proof of Theorem 6.1, we start by a systematical computation of all the solutions to the various constraints in Corollary 5.12, with \( \delta=1 \). Each tuple of parameters in the solution set is represented in the conditions listed in the statement of the theorem. Each one of those conditions corresponds to a set of values of \( n \), which, by Corollary 5.12, must have \( D_n^* \leq k+1 \).

(1) Consider the constraint in Corollary 5.12(d), with \( \delta=1 \):

\[-1 - \lambda \leq 2\alpha \leq 3. \tag{6.5}\]

(6.5) implies \( \lambda \geq -4 \); by (5.5), \( \lambda \leq 3 \). We obtain the solutions to (6.5) by considering separately various values of \( \lambda \in \{-4,...,3\} \). From the resulting set of solutions we then prune all those that do not correspond to values of \( n \) in \( \mathbb{R}[k] \). This is done using the lower bound on \( \lambda \) from (5.5), \( \lambda \geq \frac{-2\alpha(\alpha-1)}{k+\alpha} \). The only admissible solutions to (6.5), are listed below, along with the corresponding values of \( p \):

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \alpha )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0, 1</td>
<td>0</td>
</tr>
<tr>
<td>1, 2</td>
<td>-1, 0, 1</td>
<td>( k, k+1, k+3, 2k, 2k+2 )</td>
</tr>
<tr>
<td>3</td>
<td>-2, -1, 0, 1</td>
<td>( 3k, 3k+1, 3k+3, 3k+6 )</td>
</tr>
</tbody>
</table>

(2) Consider now Part (c) of Corollary 5.12 with \( \delta=1 \). (We must assume that \( d>1 \), else the conditions there would be meaningless.) We obtain the constrains
Constraint (6.6) is only solvable if $\lambda \geq 5$. This, combined with the constraint $\lambda \leq 3$ from (5.5), implies $d \leq 8$. All values of $d$ in the range $2, \ldots, 8$, except for 6, are prime powers, and thus, as in the previous proof, for each $d \neq 6$, condition (c) of Corollary 5.12 with $\delta = 1$ is subsumed by one of conditions (a) or (b) there.

By substituting $d = 6$ in (6.6), we obtain

$$-2 - \lambda \leq 2\alpha \leq -3 \quad \text{or} \quad 10 - \lambda \leq 2\alpha \leq 9,$$

and the only relevant solutions are

$$\lambda = 2, 3; \quad \alpha = -2, 4.$$

When these solutions are substituted back in condition (c) of Corollary 5.12, with $\delta = 1$ and $d = 8$, we obtain the following conjunctions:

$$\gcd\left(\frac{k-2}{6}, 6\right) = 1, \quad \gcd(k-2, p-12) = 6, \quad p-12 > 2(k-2);$$

$$\gcd\left(\frac{k+4}{6}, 6\right) = 1, \quad \gcd(k+4, p-24) = 6, \quad p-24 > 2(k+4).$$

These conditions are listed in part (2).

(3)-(15) Consider now conditions (a) and (b) in Corollary 5.12, with $\delta = 1$. We obtain the constraints

(a) $-1 - \lambda \leq 2\alpha \pm d \leq 3$, if $c = 1$

(b) $-\lambda \leq 2\alpha \pm d \leq 3$, if $c > 1$ and $f = 1$.  \hspace{1cm} (6.7)

(6.7) along with the upper bound on $\lambda$ from (5.5) imply $-4 \leq \lambda \leq 3$. We shall obtain the solutions, as before, by considering different values of $\lambda$. The solutions to (6.7)(b) that correspond to the value $\lambda = \lambda_0$, can be obtained from the solutions to (6.7)(a) corresponding to $\lambda = \lambda_0 - 1$. In the following table we list these solutions; their
full blown forms are listed in parts (8)-(15).

\[
\begin{array}{|c|c|c|}
\hline
\lambda & c=1 & c>1, f=1 \\
\hline
-4 & -3 & 12\alpha -31 \\
-3 & -2 & 12\alpha -31, 12\alpha -21 \\
-2 & -1 & 12\alpha -31, 12\alpha -21, 12\alpha -11 \\
-1 & 0 & 12\alpha -31, 12\alpha -21, 12\alpha -11, 12\alpha -11, 12\alpha -11 \\
0 & 1 & 12\alpha -31, 12\alpha -21, 12\alpha -11, 12\alpha -11, 12\alpha -11, 12\alpha +11 \\
1 & 2 & 12\alpha -31, 12\alpha -21, 12\alpha -11, 12\alpha -11, 12\alpha -11, 12\alpha +11, 12\alpha +21 \\
2 & 3 & 12\alpha -31, 12\alpha -21, 12\alpha -11, 12\alpha -11, 12\alpha +11, 12\alpha +21, 12\alpha +31 \\
3 & - & 12\alpha -31, 12\alpha -21, 12\alpha -11, 12\alpha -11, 12\alpha +11, 12\alpha +21, 12\alpha +31, 12\alpha +41 \\
\hline
\end{array}
\]

The special cases in which \( c=1 \) are listed explicitly in (3)-(7). □

Remarks:

(a) The special cases in which \( d=1 \) are listed explicitly in conditions (3)-(7) of Theorem 6.2 although they are covered in parts (8)-(10). As was explained in Remark (d) following Theorem 6.1, application of the tests in (3)-(7) first facilitates subsequent computations by greatly reducing the number of values of \( p \) that are still in question.

(b) The set of conditions given in Theorem 6.2 is, in fact, a superset of the set of sufficient conditions for optimality given in Theorem 6.1. Every \( n \in \mathbb{R}[k] \) that satisfies any of the conditions in Theorem 6.2 is either optimal or suboptimal. This may be true even of those values of \( n \) that satisfy the conditions in Theorem 6.2, but do not satisfy the conditions in Theorem 6.1, since this last set of conditions is, at this point, not known to also be necessary for optimality.

(c) The criteria in Theorem 6.2 can also be viewed as an algorithm to detect
values of \( n \) that are neither optimal nor suboptimal. This "algorithm" is much more efficient than Algorithm 1 in Section 2.4. While Algorithm 1 requires the scanning of all relevant values of \( h \) for each \( n \), here for each \( n \) we need only consider a relatively small number of conditions. Theorem 6.2 was converted into a PASCAL program. It was run on a SUN workstation, and within \( \sim 60 \) minutes all values of \( n \) up to \( -8,000,000 \) were found to be either optimal or suboptimal. Such determination would have been practically impossible with Algorithm 1 of Chapter 2, since Algorithm 1 would require more than two weeks of computing time only to check the values in the range \( n \leq 20,000 \).

Example 6.3 Let \( k=25, \ p=20, \ n=1280. \) This is the only value, besides \( n_{25}-1 \), that was not shown to be optimal in \( R[25], \) in Example 6.1. This value of \( p \) satisfies condition (6) in Theorem 6.2: \( k+\alpha=27, \ p=16, \ \gcd(27,16)=1, \ p>4. \) This is also equivalent to condition (8) with \( \alpha=2. \) Tracing back the steps of the proof in Theorem 6.2 for this special case, we find that it corresponds to Corollary 5.12(a), case (ii), with \( \delta=1 \) and \( \lambda=0. \) To find \( h \) we first solve (5.9)(ii) and then substitute the solution \( (s,t)=(5,3) \) into (5.8). We obtain \( h=238. \) Thus, by Theorem 6.2, \( D_{25}^* \leq 26. \) In the following table we show all the sufficient conditions of Theorem 6.2 that are satisfied in this case, along with the corresponding values of the hop \( h \), or of the dual hop \( h'=n-h. \)
In conclusion, recall the Conjecture made in Section 2.3. It still stands. However, the results presented here confirm the conjecture for all values of $n$ for which present applications are practical.
CHAPTER 7

SUMMARY

Double loop networks can be used to model various communication networks and also the ILLIAC type Mesh Connected Computer for parallel processing. The simple, symmetric and expandable topology allows a relatively low manufacturing cost. The added chords in the network also improve the reliability and decrease the diameter, which determines to a large extent the transmission delay of messages in the network.

Our work is concerned with minimizing the diameter (and therefore minimizing the transmission delay) of double loop networks. The networks $G(n,h)$ have vertices labeled $0,\ldots,n-1$, and each vertex $i$ is connected to vertices $i\pm 1$ and $i\pm h \mod n$, for some integer $h$, $2\leq h\leq n-2$. For each $n$, the minimal diameter $D^*_{n}$ of $G(n,h)$ is bounded below by $k$ if $n \in R[k]=\{2k^2-2k+2,\ldots,2k^2+2k+1\}$. $n$, $h$ and $G(n,h)$ are called optimal if $\text{Diam } G(n,h) = D^*_{n} = k$, and suboptimal if $\text{Diam } G(n,h) = D^*_{n} = k+1$.

Our results include:

- Upper and lower bounds on optimal and suboptimal hops $h$.
- A simple algorithm that detects optimal or suboptimal hops, if they exist, or declares their absence. This algorithm is only good for relatively small values of $n$. If $n$ is larger than 30,000, the space it requires exceeds the system limits (in the VAX implementation of PASCAL), and its computing time becomes prohibitively long.
• The identification of "sparse" optimal families. These families are infinite and intersect each $R[k]$ in a set of size $O(\sqrt{k})$. Thus, they properly include, and greatly improve, previously known optimal families obtained by Du, Hsu, Li and Xu [Du88].

• The identification of "dense" optimal families. These families are best characterized by representing each $n \in R[k]$ as $n=2k^2+2k-p$. $n \in R[k]$ belongs to one of the dense families if it satisfies at least one of the following conditions:
  
  \begin{align*}
  \text{gcd}(p,k) &= 1, \\
  \text{gcd}(k+1,p) &= 1 \text{ or} \\
  \text{gcd}(k-1,p-4) &= 1 \text{ and } p > 2k-2.
  \end{align*}

  Computations show that 92% of all values of $n<8,000,000$ satisfy at least one of these conditions.

• The identification of a set of implicit constraints involving a set of parameters, the satisfaction of which guarantees $D_n^* < k+\delta$, for some $\delta > 0$. These constraints are obtained through careful examination of the structure of some planar grids that each $G(n,h)$ uniquely determines. In the process of their derivation, some number-theoretic results concerning diophantine equations were needed.

• By substituting $\delta = 0$ or $\delta = 1$ in those constraints we derive conditions that are sufficient for optimality or suboptimality of $n$. These conditions involve testing of the greatest common divisor of two quantities defined by $n$ and by an additional parameter, $\alpha$. The set of values of $n$ that correspond to each one of those conditions, forms a general optimal (or suboptimal) family. There are
several of these families corresponding to each value of $\alpha$ and, overall, these conditions produce an infinite number of optimal and suboptimal families.

- An algorithm based on the general conditions was designed, that detects optimal and suboptimal values of $n$. Using it enables the identification of many optimal values that were left out by the sparse and the dense families. This algorithm is very efficient: within ~60 minutes of computing time on a SUN workstation, we were able to determine that all values of $n$ up to $n=8,000,000$ are either optimal or suboptimal. 93% of the values of $n$ in this range are optimal, according to Theorem 6.1. When the sparse families from Chapter 3 are also incorporated into the algorithm, the percentage of optimal values in the above range increases to 94%.

- Our conjecture that every value of $n$ is either optimal or suboptimal still stands. However, our results confirm it for all values of $n$ that seem currently practical.

Further research is required to prove or disprove our conjecture. (A similar conjecture was made and proved to be wrong in the directed double loop, but it seems that the mathematics involved in this case is quite different.)

Another topic for future research is that of determining the minimal diameter of a similar multistep network, in which each node $i$, $0 \leq i \leq n-1$, is connected to the nodes $i \pm h_1, i \pm h_2, \ldots, i \pm h_t$, for $t > 2$. A lower bound has been derived by Boesch and Wang [Bo85] for this case, that consists of sums over expressions involving binomial coefficients. We have been able to derive a very simple recursive formula for the same lower bound (yielding the same values). To the best of our knowledge,
no other results exist for the multistep undirected case, and the problem is considered very difficult.

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BIBLIOGRAPHY


VITA

Dvora Tzvieli has received her B.Sc. (1970) and M.Sc. (1973) in applied mathematics from the University of Tel-Aviv in Israel. She has taught mathematics in Tel-Aviv University, and in the Technion, Israel Institute of Technology in Haifa, Israel, where she also served as junior-staff coordinator. In January 1985 she started her Ph.D. candidacy in the department of Computer Science at Louisiana State University.

Her areas of interest include parallel architectures and interconnection networks, combinatorics and discrete mathematics. Other areas of interest include topics in AI: search problems and automated reasoning, algorithm design and analysis and computational geometry.

Dvora is married to Dr. Arie Tzvieli and has three children: Gila, Ori and Ziv.
Candidate: Dvora Tzvieli

Major Field: Computer Science

Title of Dissertation: Double Loop Interconnection Networks with Minimal Transmission Delay

Approved:

V. Brooks Reid
Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

Donald N. Koft

Chamaym Fiuma

Leslie P. Jones

James G. Oxley

Date of Examination: July 15, 1988