Ergodic Actions of Semi-Direct Product Groups.

Edgar Navarro Reyes

Louisiana State University and Agricultural & Mechanical College

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Ergodic actions of semi-direct product groups

Reyes, Edgar Navarro, Ph.D.

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SEMI-DIRECT PRODUCT GROUPS

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in

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by

Edgar Navarro Reyes
B.S., University of the Philippines, 1979
M.S., University of the Philippines, 1981
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ABSTRACT

We describe ergodic Borel actions of a semi-direct product group $K \rtimes G$ on a standard Borel space where $G$ is a group acting on a compact group $K$ by automorphisms. In the canonical action of $G$ on the dual $\hat{K}$ of $K$ we assume each $G$-orbit in $\hat{K}$ is finite. This condition is automatically satisfied if $K$ is a compact connected semi-simple Lie group.

We discuss amenable actions and relatively weakly-mixing actions of this semi-direct product group.
INTRODUCTION

In 1942, P. Halmos and J. von Neumann classified measure preserving transformations of a measure space \((X, \mu)\) where \(\mu\) is a probability measure on a separable Borel space \(X\) with no atoms. If \(T\) is such a transformation we can associate a unitary operator \(U\) of \(L^2(X)\) defined by \((Uf)(x) = f(Tx)\). They defined \(T\) to have pure point spectrum if there exists an orthonormal basis \(\{f_n\}\) in \(L^2(X)\) and a sequence \(\{\lambda_n\}\) of complex numbers with absolute value 1 such that \(Uf_n = \lambda_n f_n\). If \(T\) is an ergodic measure preserving transformation on \(X\) then a classic result of Halmos-von Neumann states that a necessary and sufficient condition for \(T\) to have pure point spectrum is that \(X\) is isomorphic to a compact abelian group where the action on this compact group is by translation.

In 1963, G. Mackey extended the Halmos-von Neumann characterization to the case where one has an ergodic action of a locally compact group, rather than the integers \(\mathbb{Z}\) as in the above case. In a similar way, to any measure preserving action of a locally compact group \(G\) on a standard Borel space \(X\) with a probability measure we associate a unitary representation \(\pi\) of \(G\) on \(L^2(X)\) where \((\pi_g f)(x) = f(x \cdot g)\) a.e. \(x\) for each \(g \in G\). Mackey defines the \(G\)-action to have pure point spectrum if \(\pi\) is a direct sum of finite-dimensional irreducible unitary representations. Mackey showed that an ergodic measure preserving action has pure point spectrum iff \(X\) is isomorphic to a homogeneous space \(M_0 \backslash M\) where \(M\) is a compact group and \(M_0\) is a closed subgroup of \(M\) and the action is given by \(M_0 m \cdot g = M_0 m \phi(g)\) where \(\phi\) is a homomorphism from \(G\) onto a dense subgroup of \(M\).
In 1975, R. Zimmer generalized Mackey's description of ergodic actions, of locally compact groups, with pure point spectrum. In so doing, Zimmer studies extensions \( \phi : (X, \mu) \rightarrow (Y, \nu) \) where \( \phi \) is an equivariant map of ergodic G-spaces, and \( \phi_* \mu \sim \nu \). To each extension \( \phi \) we associate a natural cocycle representation. To do this one disintegrates \( \mu \) over the fibers of \( \phi \) and writes \( L^2(X) \) as a direct integral \( \int_Y H_y d\nu(y) \) of Hilbert spaces where \( H_y \) is a Hilbert space over the fiber \( \phi^{-1}(y) \). Without being precise we say that \( \int_Y H_y d\nu(y) \) is a G-invariant Hilbert bundle. Zimmer defines \( X \) to have relatively discrete spectrum over \( Y \) if \( \int_Y H_y d\nu(y) \) can be written as a countable direct sum of G-invariant irreducible finite-dimensional Hilbert bundles.

Zimmer proved that a necessary and sufficient condition for \( X \) to have relatively discrete spectrum over \( Y \) is for \( X \) to be isomorphic to a skew-product \( Y \times_\beta M_0 \backslash M \) where \( \beta : Y \times G \rightarrow M \) is a cocycle with Mackey dense-range into a compact group \( M \), and \( M_0 \) is a closed subgroup of \( M \). This result is known as Zimmer's "Structure Theorem." When \( Y \) is a point this natural cocycle representation on \( L^2(X) = \int_Y H_y d\nu(y) \) is the unitary representation associated to the G-action on \( X \), and this reduces to Mackey's case.

My thesis is a study of ergodic actions of a semi-direct product group \( K \ltimes_s G \) where \( G \) is a locally compact group acting on a compact group \( K \) by automorphisms. Then \( G \) acts canonically on the dual \( \hat{K} \) (i.e. the set of all equivalence classes of all irreducible unitary representations of \( K \)) of \( K \). We will always assume each G-orbit in \( \hat{K} \) is finite. This condition on the G-orbits is automatically satisfied if \( K \) is a compact connected semi-simple Lie group.
If $K \times G$ acts ergodically on a standard Borel space $S$, let 
\[ \hat{S} = S/K \] 
be the space of $K$-orbits. Then $G$ acts canonically on this space of 
$K$-orbits since $K$ is a normal subgroup of $K \times G$. Thus we obtain an extension 
$S \rightarrow \hat{S}$. Then we prove in this thesis that $S$ has relative discrete 
spectrum over $\hat{S}$. In this proof the compactness of $K$ is essential and we use 
some unitary representation theory of compact groups. Then by Zimmer’s 
above result $S$ is isomorphic to a skew-product $\hat{S} \times_\beta M_0 \backslash M$ where $M$ is a 
compact group and $M_0$ is a closed subgroup of $M$, and $\beta : \hat{S} \times K \times G \rightarrow M$ 
is a cocycle with Mackey dense-range.

If we do not assume the finiteness of the $G$-orbits in $\hat{K}$ then $S$ 
may not be isomorphic to such a skew-product. The cocycle $\beta$ becomes simpler 
if we assume there exists a continuous homomorphism $\phi : G \rightarrow K$ such that 
$g \cdot k = \phi(g)k\phi(g)^{-1}$ defines the $G$-action on $K$. And in this case $\beta$ is given by 
$\beta(\hat{s}, kg) = \Lambda(k)\beta(\hat{s}, g)$ where $\Lambda$ is a continuous homomorphism from $K$ into $M$.

We identify some relationships between $K$, $G$, $M_0$, and $M$. If $\beta$ 
is a strict cocycle and $\beta|_{\hat{S} \times G} : \hat{S} \times G \rightarrow M'$ has Mackey dense-range into a closed 
subgroup $M'$ of $M$, then $M$ is the closed subgroup generated by $[\beta(\hat{s}, K) \cup M']$ 
a.e. $\hat{s} \in \hat{S}$ (see Corollary 3.8). Moreover if $\beta$ is a strict cocycle given by 
$\beta(\hat{s}, kg) = \Lambda(k)\beta(\hat{s}, g)$, where $\Lambda$ is a continuous homomorphism from $K$ into $M$ 
then $\Lambda(K)$ is a closed normal subgroup of $M$, $M' \cdot \Lambda(K) = M$, and 
$M_0 \cdot \Lambda(K) = M$(see Theorem 3.1, Corollary 3.9 and Theorem 3.10).

We also study amenable actions and relatively weakly-mixing 
actions of $K \times G$. A group $A$ is amenable if there is a fixed point in every 
compact affine $A$-space. Solvable groups are amenable groups. Zimmer 
introduced the notion of an amenable action in his paper “Amenable Ergodic
Group Actions and an Application to Poisson Boundaries of Random Walks”.

An amenable action is loosely the existence of an invariant section from $X$ to a Borel field of compact spaces. Actions of amenable groups are amenable; on the contrary, $SL(2, \mathbb{R})$ is not an amenable group but its natural action on $\mathbb{R}^2$ is amenable. We prove that an ergodic action of $K \times G$ on $S$ is amenable iff the action of $G$ on $S$ is amenable iff the canonical action of $G$ on the space of $K$-orbits $S/K$ is amenable. The proof of this last statement uses Zimmer’s results on extensions and amenability.

Relatively weakly-mixing is a notion introduced by Zimmer which extends the concept of weakly mixing. By definition an action of a group on a Borel space $X$ with a probability invariant measure is weakly mixing iff the product action on $X \times X$ is ergodic. C. Moore shows in his paper “Ergodicity of Flows on Homogeneous Spaces” that an action is weakly mixing iff the associated unitary representation on $L^2(X)$ contains no finite-dimensional unitary representation except the trivial one. Given an extension $\phi : X \rightarrow Y$, $X$ is said to be relatively weakly-mixing over $Y$ if $\{(x_1, x_2) : \phi(x_1) = \phi(x_2)\}$ with the product action is ergodic. When $Y$ is a point this reduces to the usual weakly mixing. In parallel to Moore’s result on weakly mixing, Zimmer proves that $X$ is relatively weakly mixing over $Y$ iff the natural cocycle representation on the Hilbert bundle $L^2(X)$ over $Y$ contains no finite-dimensional invariant sub-Hilbert bundle except for the trivial one, the constants. In this thesis we show that if $X$ is relatively weakly-mixing over $Y$ where $X$ and $Y$ are ergodic $K \times G$-spaces with probability invariant measures and where $K$ acts trivially on $Y$, then $X$ is $G$-ergodic and as $G$-spaces $X$ is relatively weakly-mixing over $Y$. 

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Group Representation Theory plays an important role in ergodic group actions. It is well-known that an action of $G$ on $X$ with a probability invariant measure is ergodic iff the associated unitary representation of $G$ on $L^2(X)$ contains no 1-dimensional unitary representation except for the trivial one. Recall that we have been assuming throughout that each $G$-orbit in the dual $\hat{K}$ is finite. We prove that a unitary representation $\pi$ of $K \times G$ is unitarily equivalent to a countable sum of finite-dimensional unitary representations if $\pi|G$ is unitarily equivalent to a countable sum of finite-dimensional unitary representations of $G$. 
CHAPTER 1
DEFINITIONS

Throughout this paper a group is a Hausdorff, second countable, locally compact, topological group. All Hilbert spaces will be separable.

Definition 1.1: Let $G$ be a group and let $\mu$ be a finite measure on a standard Borel space $S$. $(S, \mu)$ is called a Borel $G$-space (or we say $G$ acts on $S$) if there exists a Borel map $S \times G \rightarrow S$ such that:

1. $(s \cdot g_1) \cdot g_2 = s \cdot g_1 g_2$
2. $s \cdot e = s$ for all $s \in S$
3. $\mu \cdot g \sim \mu$ for $g \in G$, ($\sim$ denotes equivalent measures) where

$$(\mu \cdot g)(E) = \mu(E \cdot g^{-1}).$$

A measure $\mu$ satisfying (3) of Definition 1.1 is said to be quasi-invariant. $\mu$ is invariant if $\mu \cdot g = \mu$ for each $g \in G$. A Borel subset $E$ of $S$ is called $G$-invariant if $E \cdot g = E$ for $g \in G$. An action of $G$ on $S$ is said to be ergodic if the invariant Borel subsets of $S$ are either $\mu$-null or conull.

A Borel map $p : (S, \mu) \rightarrow (T, \nu)$ between two $G$-spaces is called a $G$-factor map if $p$ is $G$-equivariant (i.e. $p(s \cdot g) = p(s) \cdot g$ for each $s, g$) and $p \cdot \mu \sim \nu$ where $(p \cdot \mu)(E) = \mu(p^{-1}(E))$ where $E$ is a Borel subset of $T$. $S$ is called an extension of $T$ or $T$ is a factor of $S$.

Two $G$-extensions $p : (S, \mu) \rightarrow (T, \nu)$ and $p' : (S', \mu') \rightarrow (T, \nu)$ are called isomorphic extensions of $T$ if there exist conull Borel $G$-invariant subsets $S_0 \subseteq S$, $S_0' \subseteq S'$ and a $G$-equivariant Borel isomorphism $\phi : S_0 \rightarrow S_0'$ such that $p' \circ \phi = p$ and $\phi \cdot \mu \sim \nu$. And two $G$-spaces $S$ and $S'$ are said to be isomorphic if they are isomorphic
Extensions over the trivial action.

**Definition 1.2:** Let \((S, \mu)\) be a \(G\)-space and let \(H\) be another group.

A Borel map \(\alpha : S \times G \rightarrow H\) is called a cocycle if 
\[
\alpha(s, g_1 g_2) = \alpha(s, g_1)\alpha(s, g_1, g_2)
\]
a.e. \(s\) for all \(g_1, g_2 \in G\).

Two cocycles \(\alpha\) and \(\beta\) from \(S \times G\) into \(H\) are said to be equivalent or cohomologous (written \(\alpha \sim \beta\)) if there exists a Borel map 
\[
\phi : S \rightarrow H
\]
such that 
\[
\alpha(s, g) = \phi(s)\beta(s, g)\phi(s \cdot g)^{-1}
\]
a.e. \(s\) for each \(g \in G\).

**Example 1.3:** Let \((S, \mu)\) be a \(G\)-space. Define \(r : S \times G \rightarrow (R^+, \cdot)\) by 
\[
r(s, g) = \frac{d(\mu \cdot g^{-1})}{d\mu}(s)
\]
r is called a Radon-Nikodym cocycle.

Let \(\alpha : S \times G \rightarrow H\) be a cocycle and let \(T\) be an \(H\)-space. 
Then \(S \times T\) with the product measure is a \(G\)-space whose action is given by 
\[
(s, t) \cdot g = (s \cdot g, t \cdot \alpha(s, g))
\]
for almost all \((s, t) \in S \times T\), for each \(g \in G\). We usually denote \(S \times T\) by \(S \times_\alpha T\). This action of \(G\) on \(S \times_\alpha T\) is called the skew-product action defined by \(\alpha\).

It is easy to verify that if \(\alpha \sim \beta\) (as cocycles) then \(S \times_\alpha T\) and \(S \times_\beta T\) are isomorphic \(G\)-spaces. We call \(\alpha\) a cocycle with Mackey dense-range if \(S \times_\alpha H\) is an ergodic \(G\)-space.

We will need the notion of a Hilbert bundle.

**Definition 1.4:** Let \(p : H \rightarrow S\) be a surjective Borel map between standard Borel spaces such that \(p^{-1}(s) = H_s\) is a Hilbert space for all \(s \in S\). For each \(n = 1, 2, \ldots\), let \(f_n : S \rightarrow H\) be a Borel map such that:

1. \(p \circ f_n = \text{id}_s\) for \(n = 1, 2, \ldots\)
2. \(s \rightarrow \langle f_n(s), f_m(s) \rangle_{H_s}\) is Borel for each \(n, m\)

where \(\langle \cdot, \cdot \rangle_{H_s}\) is the inner-product in \(H_s\).
(3) $\overline{LS\{f_n(s) : n = 1, 2, \ldots\}} = H_s$ for each $s \in S$

(4) given a Borel space $X$, a map $F: X \rightarrow H$ is Borel iff \{\(\rho \circ F\)
and \(x \rightarrow (f_n \circ \rho \circ F(x), F(x))\) are Borel maps
for $n=1,2,\ldots$ \}. 

If this is the case, $H$ is called a Hilbert bundle over $S$ and we denote this by $H_s$. 

A Borel section $f$ for a Hilbert bundle is a Borel map $f : S \rightarrow H$ such that $\rho \circ f = \text{id}_s$.

**Definition 1.5:** Let $(S,\mu)$ be a $G$-space and $H$ be a Hilbert bundle over $S$.

Suppose to each $(s,g) \in S \times G$ we assign a continuous linear map $\alpha(s,g) : H_{s\cdot g} \rightarrow H_s$ such that:

(a) $\alpha(s,g)$ is a unitary operator a.e. $s$, for each $g$

(b) if $f$ and $f'$ are bounded Borel sections for $S$ (i.e. there exists an $M$ such that $\|f(s)\|_{H_s} \leq M$ for each $s$)
then $(s,g) \rightarrow (\alpha(s,g)f(s\cdot g), f'(s))_{H_s}$ is a Borel map on $S \times G$

(c) $\alpha(s, g_1 g_2) = \alpha(s, g_1)\alpha(s \cdot g_1, g_2)$ a.e.s, for each $g_1, g_2 \in G$.

Then $\alpha$ is a unitary cocycle representation on the Hilbert
If \( \dim(H_s) < \infty \) for a.e. \( s \), \( \alpha \) is said to be finite dimensional.

Along with the notion of cohomologous cocycles is the concept of cohomologous or equivalent cocycle representations. When \( S \) is a point, a cocycle representation is a unitary representation and equivalent cocycle representations are equivalent unitary representations.

**Definition 1.8:** Let \((S, \mu)\) be a \( G \)-space. Let \( \alpha \) and \( \beta \) be cocycle representations on the Hilbert bundles \( H_s \) and \( H'_s \) respectively. \( \alpha \) and \( \beta \) are cohomologous if for each \( s \in S \) there exists a continuous linear map \( U(s): H_s \rightarrow H'_s \) such that:

1. \( U(s) \) is unitary a.e. \( s \)
2. \( \alpha(s, g) = U(s)^{-1} \beta(s, g) U(s \cdot g) \) a.e. \( s \) for each \( g \in G \)
3. if \( f \) and \( f' \) are bounded Borel sections then
   \[ s \in S \rightarrow (U(s)f(s), f'(s))_{H_s} \]
   is a Borel map.

Let \( p : (X, \mu) \rightarrow (Y, \nu) \) be a \( G \)-factor map between two \( G \)-spaces. To this factor map we associate a natural cocycle representation on a Hilbert bundle over \( Y \). This is an important technique in this paper and we develop it.

Let \( M(X) \) be the set of all finite measures on \( X \). Equip \( M(X) \) with the smallest Borel structure such that for each Borel subset \( E \) of \( X \), \( \mu \in M(X) \rightarrow \mu(E) \) is a Borel map. Then \( M(X) \) is a standard Borel space.
since $X$ is a standard Borel space.

There exists a Borel map $y \mapsto \mu_y$ from $Y$ into $M(X)$ such that $\mu_y$ is concentrated on $p^{-1}(y)$ and $\mu(E) = \int_Y \mu_y(E) d\nu(y)$ for each Borel subset $E$ of $X$. This is written $\mu = \int_Y \mu_y d\nu(y)$, and is called the disintegration of $\mu$ over $Y$.

$$H_x \quad \text{Let be a Hilbert bundle over } X. \text{ For each } y \in Y, \text{ form the}$$

$$X \quad \text{Hilbert space } H_y = \int_X H_x d\mu_y(x). \text{ Then it can be shown} \quad \text{is a Hilbert bundle}$$

$$Y \quad \text{over } Y. \text{ We omit the details.}$$

We now define induced cocycle representations.

**Definition 1.7:** Let $p : (X, \mu) \rightarrow (Y, \nu)$ be a $G$-factor map of $G$-spaces.

Let $\mu = \int_Y \mu_y d\nu(y)$ be the disintegration of $\mu$ over $Y$, and let $\alpha$ be a cocycle representation

on the bundle \( W_x \). Form the Hilbert bundle \( H_y \) where $H_y = \int_X W_x d\mu_y(x)$. We

define a cocycle representation $\beta$ on the bundle \( Y \) by:

$$\beta(y, g)f(x) = \left( \frac{d\mu_y \cdot g}{d(\mu_y \cdot g)}(x \cdot g) \right)^{\frac{1}{2}} \alpha(x, g)f(x \cdot g)$$

$\mu_y$-a.e. $x$, for each $f \in H_y \cdot g$, a.e. $y$, for each $g \in G$.

We call $\beta$ the cocycle representation induced from $\alpha$, and we denote
it by $\beta = \text{Ind}^Y_X \alpha$.

In particular if $\alpha = 1$ is the 1-dimensional identity cocycle

$$X \times \mathbb{C}$$

representation on the Hilbert bundle $\leftarrow X$, then $\text{Ind}^Y_X 1$ is the

natural cocycle representation of the Hilbert bundle $\leftarrow H_y$

over $Y$

where $H_y = \int\mathcal{O} \, \text{Cd}\mu_y(x) = L^2(X,\mu_y)$ for each $y \in Y$.

**Definition 1.8:** Let $p : (X, \mu) \to (Y, \nu)$ be a $G$-factor map of $G$-spaces. If $\text{Ind}^Y_X 1$ is cohomologous to a countable sum of finite-dimensional cocycle representations then $\text{Ind}^Y_X 1$ is said to have $G$-discrete spectrum. $X$ has relative $G$-discrete spectrum over $Y$ if $\text{Ind}^Y_X 1$ has $G$-discrete spectrum.

**Example 1.9:** Let $(X, \mu)$ be a Borel $G$-space and $Y$ a singleton set. Clearly $X \to Y$ is a $G$-factor map. The natural cocycle representation $\pi = \text{Ind}^Y_X 1$ is a unitary representation of $G$ such that

$$\pi : G \to U(L^2(X, \mu))$$

where $(\pi_g f(x)) = \left( \frac{d\mu}{d(\mu \cdot g)}(x \cdot g) \right)^{\frac{1}{2}} f(x \cdot g)$ and $(x, g, f) \in X \times G \times L^2(X, \mu)$.

If $G$ is a compact group then $X$ has $G$-relative discrete spectrum over a singleton set. If $G = SL(2, \mathbb{R})$ and $X = SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R})$ then $X$ has a finite $G$-invariant measure, and $X$ does not have relative $G$-discrete spectrum over a singleton (since $SL(2, \mathbb{R})$ has no finite dimensional unitary representation except for the trivial one).

An important result of Robert Zimmer will play an important role in our analysis of certain ergodic actions.
Structure Theorem (Robert Zimmer).

Let $\phi : X \rightarrow Y$ be a $G$-factor map of ergodic $G$-spaces such that $X$ has relative $G$-discrete spectrum over $Y$. Then there exists a compact group $M$, a closed subgroup $M_0$ of $M$, a cocycle $\beta : Y \times G \rightarrow M$ with Mackey dense-range and an isomorphism $X \cong Y \times_\beta M_0 \backslash M$ of $G$-extensions of $Y$.

Remark 1.10:

In Robert Zimmer's paper "Extensions of Ergodic Group Actions", he shows that in a factor map $Y \times_\beta M_0 \backslash M \rightarrow Y$ where $Y$ is an ergodic $G$-space, $M_0$ a closed subgroup of a compact group $M$, and $\beta : Y \times G \rightarrow M$ a cocycle with Mackey dense-range, then $Y \times_\beta M_0 \backslash M$ has discrete spectrum over $Y$. 
In this chapter we fix a compact group $K$ and another group $G$, and suppose there exists a homomorphism $G \rightarrow \text{Aut}(K)$ from $G$ into the continuous automorphisms of $K$. Let $g \cdot k$ be the image of $k$ under the automorphism defined by $g$ and suppose further that $(g, k) \rightarrow g \cdot k$ is continuous from $G \times K \rightarrow K$. Then the semi-direct product $K \rtimes G$ is the group whose product is:

$$(k_1, g_1)(k_2, g_2) = (k_1(g_1 \cdot k_2), g_1 g_2)$$

and

$$(k_1, g_1)^{-1} = (g_1^{-1} \cdot k_1^{-1}, g_1^{-1})$$

where $(k_i, g_i) \in K \rtimes G$ for $i = 1, 2$.

Note $G$ acts naturally on the dual $\hat{K}$ (i.e. the set of all equivalence classes of all irreducible unitary representations of $K$) of $K$ by:

if $\pi \in \hat{K}$ and $g \in G$ then $(\pi \cdot g)(k) = \pi(g \cdot k)$ for each $k \in K$.

**Proposition 2.1:** Let $\pi$ be a unitary representation of a compact group $K$ acting on $H(\pi)$. Let $H_0$ be a finite dimensional subspace of $H(\pi)$ and let $\gamma \in \hat{K}$. Let $P(\gamma)$ be the $\gamma$-primary projection for $\pi$.

Then $LS\{\pi(k)P(\gamma)H_0 : k \in K\}$ is a finite-dimensional invariant subspace of $H(\pi)$.

**Proof:** Let $H(\gamma)$ be a closed subspace of $H(\pi)$ such that $\pi|_{H(\gamma)} \simeq \gamma$ as unitary representations of $K$. Let $(\cdot, \cdot)$ be the inner-product in $H(\pi)$.

Let $W = LS\{k \rightarrow (v, \pi(k)w) : v, w \in H(\gamma)\}$. Then $W$ is a finite-dimensional $\mathbb{C}$-vector subspace of the continuous complex-valued functions on $K$. 

Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis for \( H(\gamma) \).

Let \( \chi_\gamma(k) = \sum_{i=1}^{n}(\pi(k)e_i, e_i) \).

Let \( V = LS\{k \mapsto \chi_\gamma(k^{-1}k_1) : k_1 \in K\} \). \( V \subseteq W \) and \( V \) is finite-dimensional.

Let \( \{f_1, \ldots, f_m\} \) span \( V \) and fix \( w \in H_0 \).

Since \( P(\gamma) = \text{dim}(\gamma) \int_K \chi_\gamma(k^{-1})\pi(k)dk \) (see Theorem 1.12 of A. Knapp's book "Representation Theory of Semi-simple Groups") then

\[
\pi(k_1)P(\gamma)(w) = \text{dim}(\gamma) \int_K \chi_\gamma(k^{-1})\pi(k_1k)dk(w) = \\
= \text{dim}(\gamma) \int_K \chi_\gamma(k^{-1}k_1)\pi(k)(w)dk \\
= \text{dim}(\gamma) \int_K \sum_{i=1}^{m} a_i f_i(k)\pi(k)(w)dk \quad \text{for some } a_i \in \mathbb{C}.
\]

Thus \( \pi(k_1)P(\gamma)(w) = \sum_{i=1}^{m} \text{dim}(\gamma)a_i\pi(f_i)(w) \), where \( \pi(f_i) = \int_K f_i(k)\pi(k)dk \).

Hence \( \pi(k_1)P(\gamma)(w) \in LS\{\pi(f_1)(w), \ldots, \pi(f_m)(w)\} \).

A consequence of this proposition is:

**Theorem 2.2**: Let \( \pi \) be a unitary representation of \( K \times G \)

acting on \( H \) such that \( H = \bigoplus_{i=1}^{\infty} H_i \), \( \text{dim}H_i < \infty \), and \( H_i \) is \( G \)-invariant for each \( i \). If each \( G \)-orbit in \( \hat{K} \) is finite then \( \pi \) is a countable sum of finite-dimensional unitary representations of \( K \times G \).

**Proof**: Let \( \pi \simeq \sum_{\gamma \in \hat{K}} n(\gamma) \cdot \gamma \) as unitary representations of \( K \) where \( n(\gamma) \in \{0, 1, \ldots, \infty\} \).

Let \( P(\gamma) \) be the \( \gamma \)-primary projection of \( \pi \).

Let \( \chi_\gamma(k) \) be the character of \( \gamma \), and let \( v \in H \).

Then \( \pi_gP(\gamma)v = \text{dim}(\gamma) \int_K \chi_\gamma(k^{-1})\pi(gkg^{-1})\pi(g)(v)dk = \\
= \text{dim}(\gamma \cdot g^{-1}) \int_K \chi_\gamma(g^{-1}k^{-1}g)\pi(k)\pi(g)(v)dk = \\
= \text{dim}(\gamma \cdot g^{-1}) \int_K \chi_{g^{-1}\gamma}(g^{-1}k^{-1})\pi(k)dk\pi(g)(v).\)

Thus \( \pi(g)P(\gamma)(v) = P(\gamma \cdot g^{-1})\pi(g)(v) \) for each \( g \in G \).

Let \( W_{\gamma,i} = \bigoplus_{g \in G} LS\{\pi(k)P(\gamma \cdot g)H_i : k \in K\} \).
Since each $G$-orbit in $\mathcal{K}$ is finite one sees by Proposition 2.1 that $W_{\gamma,i}$ is an orthogonal finite sum of finite-dimensional subspaces. Hence $W_{\gamma,i}$ is a finite-dimensional vector subspace of $H$.

$W_{\gamma,i}$ is $K \times G$-invariant for each $\gamma, i$ since for each $(g, k, v) \in G \times K \times H_i$ we have $\pi(g)\pi(k)P(\gamma)v = \pi(gkg^{-1})\pi(g)P(\gamma)v = \pi(gkg^{-1})P(\gamma \cdot g^{-1})\pi(g)v$, and $\pi(g)v \in H_i$. Let $C \subseteq \mathcal{K}$ meet each $G$-orbit in $\mathcal{K}$ exactly once. $\oplus_{\gamma \in C} W_{\gamma,i} \supseteq H_i$ for each $i$.

Re-label the family $\{W_{\gamma,i}\}_{\gamma \in C, i=1,2,...}$ by $\{W'_j\}_{j=1,2,...}$.

Set $W'_i = W'_1 + W'_2 + ... + W'_j$ for each $j=1,2,...$.

$W'_j$ is not necessarily a direct sum. Each $W'_j$ is still finite-dimensional and $K \times G$-invariant and $W'_j \subseteq W'_{j+1}$ for each $j$.

Also $H = W_1 \oplus (W_2 \cap W_1^\perp) \oplus (W_3 \cap W_2^\perp) \oplus ...$.

**Theorem 2.3:** Let $p : (X, \mu) \longrightarrow (Y, \nu)$ be a $K \times G$-factor map where $\mu$ is $K$-invariant and $K$ acts trivially on $Y$. Let $X$ have relatively $G$-discrete spectrum over $Y$ and suppose each $G$-orbit in $\mathcal{K}$ is finite.

Then $X$ has relative $K \times G$-discrete spectrum over $Y$.

**Proof:** Let $\mu = \int \mu_y d\nu(y)$ be the disintegration of $\mu$ over $Y$. Since $\mu_y$ is $K$-invariant $\nu$-a.e. $y$ we can assume $\mu_y$ is $K$-invariant for each $y$. Let $S$ be the natural $(Y, K \times G)$-cocycle representation.

$L^2(X, \mu_y)$

$S$ acts on the bundle $Y$ such that:

$$(S(y, kg)f)(x) = \left( \frac{d\mu_y g}{d(\mu_y \cdot g)} \right) (x \cdot kg)^{1/2} f(x \cdot kg)$$

$\mu_y$-a.e. $x$, for each $f \in L^2(X, \mu_y g)$, a.e. $y$, for each $(k, g) \in K \times G$.

In particular, $(S(y, k)f)(x) = f(x \cdot k)$ for $\mu_y$-a.e. $x$, for each $f \in L^2(X, \mu_y)$, and for all $(y,k) \in Y \times K$. 
For each \( y \in Y \), \( \pi(y)(k) = S(y,k) \) defines a unitary representation of \( K \) on \( L^2(X,\mu_y) \).

For \( \gamma \in \hat{K} \), set \( P_y(\gamma) \) to be the \( \gamma \)-primary projection of \( \pi(y) \).

Note \( S(y,g)\pi(y \cdot g)(k) = \pi(y)(gkg^{-1})S(y,g) \) a.e. \( y \), for each \( k,g \) since if we fix \( (k,g) \in K \times G \) then for a.e. \( y \) \( S(y,g)\pi(y \cdot g)(k) = S(y,g)S(y \cdot g,k) = S(y,g) = S(y,gkg^{-1})S(y,g) \) a.e. \( y \).

Since \( X \) has relatively \( G \)-discrete spectrum over \( Y \), \( L^2(X,\mu_y) \) is a countable direct sum of Hilbert bundles, such that

\[
\begin{align*}
L^2(X,\mu_y) & \cong \bigoplus_i H_i(y) \\
Y & \cong \bigoplus_i Y \\
Y & \cong \bigoplus_i Y
\end{align*}
\]

and \( \dim H_i(y) < \infty \) a.e. \( y \) for each \( i \), and \( S(y,g)H_i(y \cdot g) = H_i(y) \) a.e. \( y \) for each \( g \in G \).

Now \( S(y,g)\pi(y \cdot g)(k)P_{y,g}(\gamma)H_i(y \cdot g) = \pi(y)(gkg^{-1})P_y(\gamma \cdot g^{-1})H_i(y) \) a.e. \( y \), for each \( (k,g,\gamma) \in K \times G \times \hat{K} \) and each \( i = 1,2,\ldots \).

This above statement follows since if we fix \( (k,g,\gamma) \in K \times G \times \hat{K} \) then:

\[
S(y,g)\pi(y \cdot g)(k)P_{y,g}(\gamma) = \pi(y)(gkg^{-1})S(y,g)P_{y,g}(\gamma) \quad \text{a.e. } y
\]

\[
= \pi(y)(gkg^{-1}) \dim(\gamma) \int_K \chi_\gamma(k_1^{-1})S(y,g)\pi(y \cdot g)(k_1)dk_1
\]

\[
= \pi(y)(gkg^{-1}) \dim(\gamma) \int_K \chi_\gamma(k_1^{-1})\pi(y)(gk_1g^{-1})S(y,g)dk_1 \quad \text{a.e. } y
\]

\[
= \pi(y)(gkg^{-1}) \dim(\gamma) \int_K \chi_\gamma(g^{-1}k_1^{-1}g)\pi(y)(k_1)dk_1S(y,g)
\]

\[
= \pi(y)(gkg^{-1}) \dim(\gamma \cdot g^{-1}) \int_K \chi_{\gamma \cdot g^{-1}}(k_1^{-1})\pi(y)(k_1)dk_1S(y,g)
\]

\[
= \pi(y)(gkg^{-1})P_y(\gamma \cdot g^{-1})S(y,g).
\]
Hence \( S(y,\gamma)\pi(y \cdot \gamma)(k)P_{\gamma\gamma}(\gamma) = \pi(y)(gkg^{-1})P_{\gamma}(\gamma \cdot g^{-1})S(y,\gamma) \) a.e. \( y \).

By Proposition 2.1 and the preceding calculation:

\[
\bigoplus_{g \in G} LS\{\pi(y)(k)P_{\gamma}(\gamma \cdot g)H_i(y) : k \in K\} \quad \text{is a finite-dimensional,}
\]

\( Y \)

is a finite-dimensional, \( G \)-invariant sub-Hilbert bundle

\[
L^2(X,\mu_y)
\]

of \( Y \) for each \( \gamma \in \hat{K} \) and each \( i=1,2,\ldots \). Let \( C \subseteq \hat{K} \) meet each \( G \)-orbit in \( \hat{K} \) exactly once.

Then \( H_i(y) \subseteq \bigoplus_{g \in C} \bigoplus_{k \in G} LS\{\pi(y)(k)P_{\gamma}(\gamma \cdot g)H_i(y) : k \in K\} \) for each \( y \in Y \) and each \( i=1,2,\ldots \).

Label the family of Hilbert bundles

\[
\left\{ \bigoplus_{g \in G} LS\{\pi(y)(k)P_{\gamma}(\gamma \cdot g)H_i(y) : k \in K\} \right\}_{\gamma \in C, i=1,2,\ldots}
\]

by

\[
\left\{ \begin{array}{c}
W_j'(y) \\
Y
\end{array}\right\}_{j=1,2,\ldots}
\]

Set \( W_j(y) = W_1'(y) + W_2'(y) + \ldots + W_j'(y) \) for each \( y \in Y \) and \( j=1,2,\ldots \).

The sum in \( W_j(y) \) is not necessarily a direct sum.

Then we obtain a direct sum of finite-dimensional Hilbert sub-bundles each
Invariant under $K \times, G$ i.e.

\[
L^2(X, \mu_Y) \xrightarrow{\cong} W_1(y) \oplus \left\{ W_2(y) \cap W_1(y) \right\} \oplus \left\{ W_3(y) \cap W_2(y) \right\} \oplus \ldots
\]

Thus $X$ has relatively $K \times, G$-discrete spectrum over $Y$.

For the remainder of this Chapter 2, we will develop and prove a central result of this thesis. Let $K \times, G$ act on $(S, \mu)$. Choose a Borel subset $\hat{S}$ of $S$ such that $\hat{S}$ meets each $K$-orbit in $S$ exactly once (see Theorem 2.9 of E. Effros's paper "Transformation Groups and $C^*$-Algebras").

Define $p : (S, \mu) \rightarrow (\hat{S}, \bar{\mu})$ by $p(s) = \hat{s}$ where $\hat{s}$ is the unique point in $\hat{S}$ such that $\{\hat{s}\} = s \cdot K \cap S$, and $p_*\mu = \bar{\mu}$. $\hat{S}$ inherits the Borel structure of $S$ and we can identify $\hat{S}$ with the space $S/K$ of $K$-orbits in $S$. Let $K$ act trivially on $S$ and let $G$ act on $\hat{S}$ by $\hat{s} \circ g = p(s \cdot g)$ where $\hat{s} \in \hat{S}$ and $g \in G$.

So $K \times, G$ acts naturally on $(\hat{S}, \bar{\mu})$.

Then $p : (S, \mu) \rightarrow (\hat{S}, \bar{\mu})$ is a $K \times, G$-factor map since $p(s \cdot kg) = p(s \cdot k) \circ g = p(s) \circ g = p(s) \circ kg$ for each $(s, k, g) \in S \times K \times G$.

We also use $\circ$ to denote the $K \times, G$-action on $\hat{S}$.

Let $K_\hat{s} = \{k \in K : \hat{s} \cdot k = \hat{s}\}$.

Let $\psi_\hat{s} : K_\hat{s} \setminus K \rightarrow \hat{s} \cdot K$ be a $K$-space Borel isomorphism defined by $K_\hat{s} \cdot k \rightarrow \hat{s} \cdot k$.

($\psi_\hat{s}$ is a Borel map since $K_\hat{s} \setminus K$ has a Borel cross-section).

It is convenient to denote the set $\{(\hat{s}, K_\hat{s} \cdot k) : \hat{s} \in \hat{S}, k \in K\}$ by $f(\hat{s} \times K_\hat{s} \setminus K)$.

$s \mapsto (\hat{s}, K_\hat{s} \cdot k)$ where $\hat{s} \cdot k = s$ is a canonical bijection between $S$ and $f(\hat{s} \times K_\hat{s} \setminus K)$. 
The action of $K \times, G$ on $\int(\hat{s} \times K_{\hat{s}} \setminus K)$ becomes:

$$(\hat{s}, K_{\hat{s}}k) \cdot k_1 = (\hat{s}, K_{\hat{s}}kk_1)$$

$$(\hat{s}, K_{\hat{s}}k) \cdot g = (\hat{s} \circ g, K_{\hat{s}}\circ b(\hat{s}, g)g^{-1}kg)$$

where $b(\hat{s}, g) \in K$ and $\hat{s} \circ g \cdot b(\hat{s}, g) = \hat{s} \cdot g$.

**Proposition 2.4**: If $K \times, G$ acts ergodically on $(S, \mu)$ then $G$ acts ergodically on $(\hat{S}, \hat{\mu})$.

**Proof**: Let $E$ be Borel subset of $\hat{S}$ such that $E \circ g = E$ for each $g \in G$.

For each $(s, k, g) \in E \times K \times, G$, $p(s \cdot k \cdot g) = p(s \cdot g) = \hat{s} \circ g \in E$.

Thus $s \cdot k \cdot g \in p^{-1}(E) = E \cdot K$ for each $(s, k, g) \in E \times K \times, G$.

Hence $E \cdot K$ is a $K \times, G$-invariant Borel subset of $S$.

Thus $\mu(E \cdot K) = 0$ or $\mu(E \cdot K) = 1$.

But $\mu(E \cdot K) = \mu(p^{-1}(E)) = \hat{\mu}(E)$.

So $E$ is a null or conull subset of $\hat{S}$.

Thus $G$ acts ergodically on $(\hat{S}, \hat{\mu})$.

We now state a central result:

**Theorem 2.5**: Let $K \times, G$ act on $S$ where $\mu$ is $K$-invariant and suppose each $G$-orbit in the $K$ is finite. Let $p : (S, \mu) \longrightarrow (\hat{S}, \hat{\mu})$ be the canonical $K \times, G$-factor map where $\hat{S} = S/K$ is the space of $K$-orbits in $S$.

Then $S$ has relatively $K \times, G$-discrete spectrum over $\hat{S}$.

**Proof**: Let $\mu = \int \mu_{\hat{s}} d\hat{\mu}(\hat{s})$ be the disintegration of $\mu$ over $\hat{S}$.

Since $\mu$ is $K$-invariant, then $\mu_{\hat{s}}$ is $K$-invariant $\hat{\mu}$-a.e. $\hat{s} \in \hat{S}$.

We can assume $\mu_{\hat{s}}$ is $K$-invariant for each $\hat{s} \in \hat{S}$.

Let $\xi_{\hat{s}} : (K, m) \longrightarrow (K_{\hat{s}} \setminus K, m_\hat{s})$ be the canonical projection where $m$ is the normalized Haar measure on $K$ and $m_\hat{s} = (\xi_{\hat{s}})_*(m)$.
$m_{\hat{s}}$ is a $K$-invariant measure on $K_{\hat{s}} \backslash K$, i.e. $m_{\hat{s}}(E \cdot k_1) = m_{\hat{s}}(E)$ for each $k_1 \in K$, and $E$ a Borel subset of $K_{\hat{s}} \backslash K$, and for each $\hat{s} \in \hat{S}$.

Consider again the canonical $K$-equivariant Borel isomorphism $\psi_{\hat{s}} : K_{\hat{s}} \backslash K \rightarrow \hat{s} \cdot K$. There exists a non-negative constant $\epsilon(\hat{s}) \geq 0$ for each $\hat{s} \in \hat{S}$ such that $(\psi_{\hat{s}})_* (\epsilon(\hat{s})m_{\hat{s}}) = \mu_{\hat{s}}$.

Define a unitary operator $(\psi_{\hat{s}})_* : L^2(\hat{s} \cdot K, \mu_{\hat{s}}) \rightarrow L^2(K_{\hat{s}} \backslash K, \epsilon(\hat{s})m_{\hat{s}})$ for each $\hat{s} \in \hat{S}$ by $(\psi_{\hat{s}})_*(f) = f \circ \psi_{\hat{s}}$, where $f \in L^2(\hat{s} \cdot K, \mu_{\hat{s}})$.

The natural bundle representation $R$ of $(\hat{S}, K \times_s G)$ is given by:

$$R(\hat{s}, kg) : L^2(S, \mu_{\hat{s} \circ g}) \rightarrow L^2(S, \mu_{\hat{s}})$$

where

$$(R(\hat{s}, kg)f)(s_1) = \left[ \frac{d\mu_{\hat{s} \circ g}}{d(\mu_{\hat{s}} \cdot g)}(s_1 \cdot kg) \right]^\frac{1}{2} f(s_1 \cdot kg)$$

$\mu_{\hat{s}}$-a.e. $s_1$ for each $f \in L^2(S, \mu_{\hat{s} \circ g})$ a.e. $\hat{s}$ for each $(k, g) \in K \times_s G$.

Consider the diagram below:

$$
\begin{array}{ccc}
L^2(S, \mu_{\hat{s} \circ g}) & \rightarrow & L^2(S, \mu_{\hat{s}}) \\
(\psi_{\hat{s} \circ g})_* & \downarrow & (\psi_{\hat{s}})_* \\
L^2(K_{\hat{s} \circ g} \backslash K, \epsilon(\hat{s} \circ g)m_{\hat{s} \circ g}) & \rightarrow & L^2(K_{\hat{s}} \backslash K, \epsilon(\hat{s})m_{\hat{s}})
\end{array}
$$

We write $L^2(K_{\hat{s}} \backslash K)$ for $L(K_{\hat{s}} \backslash K, \epsilon(\hat{s})m_{\hat{s}})$.

By a formal calculation if $f \in L^2(K_{\hat{s} \circ g} \backslash K)$ then

$$(\psi_{\hat{s}})_* R(\hat{s}, kg)(\psi_{\hat{s} \circ g})_*^{-1}f(K_{\hat{s}}k_1) = \left( R(\hat{s}, kg)(\psi_{\hat{s} \circ g})_*^{-1}f \right) \circ \psi_{\hat{s}}(K_{\hat{s}}k_1) =$$

$= R(\hat{s}, kg)(\psi_{\hat{s} \circ g})_*^{-1}f(\hat{s} \cdot k_1)$

$= \left[ \frac{d\mu_{\hat{s} \circ g}}{d(\mu_{\hat{s}} \cdot g)}(\hat{s} \cdot k_1 kg) \right]^\frac{1}{2} f(\psi_{\hat{s} \circ g}^{-1}(\hat{s} \cdot k_1 kg))$

$= \left[ \frac{d\mu_{\hat{s} \circ g}}{d(\mu_{\hat{s}} \cdot g)}(\hat{s} \cdot k_1 kg) \right]^\frac{1}{2} f(K_{\hat{s} \circ g}b(\hat{s}, g)g^{-1}k_1 kg),$

where (as before) $b(\hat{s}, g) \in K$ and $\hat{s} \circ g \cdot b(\hat{s}, g) = \hat{s} \cdot g$. 

We may regard $R$ as a bundle representation where

$$R(s,kg) : L^2(K_{\tilde{s}o}\backslash K) \longrightarrow L^2(K_{\tilde{s}}\backslash K)$$

and

$$(R(s,kg)f)(K_{\tilde{s}}k_1) = \left[ \frac{d\tilde{s}}{d\mu_{\tilde{s}o}}(\tilde{s} \cdot k_1,kg) \right]^{\frac{1}{2}} f(K_{\tilde{s}o}b(\tilde{s},g)g^{-1}k_1,kg)$$

a.e. $K_{\tilde{s}}k_1$ for each $f \in L^2(K_{\tilde{s}o}\backslash K)$, a.e. $\tilde{s}$ for each $(k, g) \in K \times_* G$.

In particular for each $\tilde{s} \in \tilde{S}$ we have a unitary representation of $K$:

$$k \longmapsto R(\tilde{s},k) \in \mathcal{U}(L^2(K_{\tilde{s}}\backslash K))$$

where $(R(\tilde{s},k)f)(K_{\tilde{s}}k_1) = f(K_{\tilde{s}}k_1,k)$

for each $K_{\tilde{s}}k_1 \in K_{\tilde{s}}\backslash K$, for all $f \in L^2(K_{\tilde{s}}\backslash K)$, and for each $k \in K$.

Let $\pi(\tilde{s})(k) = R(\tilde{s},k)$ for each $(\tilde{s},k) \in \tilde{S} \times K$.

For each $\tilde{s} \in \tilde{S}$ we define an isometry (into):

$$(\xi_{\tilde{s}})^* : L^2(K_{\tilde{s}}\backslash K) \longrightarrow L^2(K,m)$$

by $(\xi_{\tilde{s}})^*(f) = \epsilon(\tilde{s})^{\frac{1}{2}} (f \circ \xi_{\tilde{s}})$ where $f \in L^2(K_{\tilde{s}}\backslash K)$.

$(\xi_{\tilde{s}})^*$ is an isometry for:

$$\|(\xi_{\tilde{s}})^* f\|^2_{L^2(K,m)} =$$

$$= \int_K \epsilon(\tilde{s}) |(f \circ \xi_{\tilde{s}})(k)|^2 dm(k)$$

$$= \int_{K_{\tilde{s}}\backslash K} \epsilon(\tilde{s}) |f(K_{\tilde{s}}k)|^2 dm_{\tilde{s}}(K_{\tilde{s}}k)$$

$$= \int_{K_{\tilde{s}}\backslash K} |f(K_{\tilde{s}}k)|^2 d(\epsilon(\tilde{s})m_{\tilde{s}}) (K_{\tilde{s}}k).$$

Hence $\|(\xi_{\tilde{s}})^* f\|^2_{L^2(K,m)} = \|f\|^2_{L^2(K_{\tilde{s}}\backslash K)}$.

Let $\pi$ be the right regular representation of $K$ on $L^2(K,m)$.

$(\xi_{\tilde{s}})^*$ intertwines $\pi(\tilde{s})$ and $\pi$ for each $\tilde{s} \in \tilde{S}$ since if $f \in L^2(K_{\tilde{s}}\backslash K)$ then

$$((\xi_{\tilde{s}})^* \pi(\tilde{s})(k)f)(k_1) = (\pi(\tilde{s})(k)f \circ \xi_{\tilde{s}})(k_1) \cdot \epsilon(\tilde{s})^{\frac{1}{2}}$$

$$= f(K_{\tilde{s}}k_1,k) \cdot \epsilon(\tilde{s})^{\frac{1}{2}}$$
\[ ((\xi_{\hat{s}})^* f)(k_1) = ((\xi_{\hat{s}})^* \pi(\hat{s})(k) f)(k_1) \]

Hence \[ ((\xi_{\hat{s}})^* \pi(\hat{s})(k) f)(k_1) = (\pi(k)(\xi_{\hat{s}})^* f)(k_1) \]
for each \( \hat{s}, k_1 \), and \( k \).

Since \((\xi_{\hat{s}})^* \) intertwines \( \pi(\hat{s}) \) and \( \pi \), then \( L^2(K_{\hat{s}} \setminus K) = \bigoplus_{\gamma \in \hat{K}} L^2_{\gamma}(K_{\hat{s}} \setminus K) \),

where \( L^2_{\gamma}(K_{\hat{s}} \setminus K) \) is the range of the \( \gamma \)-primary projection \( P_{\gamma}(\hat{s}) \) of \( \pi(\hat{s}) \) for each \((\hat{s}, \gamma) \in (\hat{S} \times \hat{K})\).

We remark that \( L^2_{\gamma}(K_{\hat{s}} \setminus K) \) is finite-dimensional for each \((\hat{s}, \gamma) \in \hat{S} \times \hat{K}\).

Note \( R(\hat{s}, kg)\pi(\hat{s} \circ g)(k_1) = \pi(\hat{s})((kg)k_1(kg)^{-1}) R(\hat{s}, kg) \) a.e. \( \hat{s} \) for each \((k, k_1, g) \in K \times K \times G\).

We claim that \( R(\hat{s}, kg)L^2_{\gamma}(K_{\hat{s}o g} \setminus K) \subseteq L^2_{\gamma, (kg)^{-1} - 1}(K_{\hat{s}} \setminus K) \) a.e. \( \hat{s} \) for each \((k, g, \gamma) \in (K \times g \times \hat{K})\), where \( \gamma \cdot (kg)^{-1})(k_1) = \gamma((kg)^{-1}k_1(kg)) \) for all \( k_1 \in K \). Fix \((k, g) \in K \times G \) and \( \gamma \in \hat{K}\).

\[ P_{\gamma, (kg)^{-1}}(\hat{s}) R(\hat{s}, kg) = \dim(\gamma) \int_K \chi_{\gamma}((kg)^{-1}k_1^{-1}(kg)) \pi(\hat{s})(k_1) R(\hat{s}, kg) dk_1 \]
\[ = \dim(\gamma) \int_K \chi_{\gamma}((kg)^{-1}k_1^{-1}(kg)) R(\hat{s}, kg) \pi(\hat{s} \circ g)((kg)^{-1}k_1(kg)) dk_1 \text{ a.e. } \hat{s} \]
\[ = R(\hat{s}, kg) \dim(\gamma) \int_K \chi_{\gamma}(k_1^{-1}) \pi(\hat{s} \circ g)(k_1) dk_1 \]
\[ = R(\hat{s}, kg) P_{\gamma}(\hat{s} \circ g). \]

Hence for each \((k, g, \gamma) \in K \times G \times \hat{K}\),
\[ P_{\gamma, (kg)^{-1}}(\hat{s}) R(\hat{s}, kg) = R(\hat{s}, kg) P_{\gamma}(\hat{s} \circ g) \text{ a.e. } \hat{s}. \]

Thus \[ P_{\gamma, (kg)^{-1}}(\hat{s}) R(\hat{s}, kg) f = R(\hat{s}, kg) P_{\gamma}(\hat{s} \circ g) f \text{, for each } f \in L^2_{\gamma, (kg)^{-1}}(K_{\hat{s}o g} \setminus K) \text{ a.e. } \hat{s}, \]
for each \((k, g, \gamma) \in K \times G \times \hat{K}\).

Hence \( R(\hat{s}, kg)L^2_{\gamma}(K_{\hat{s}o g} \setminus K) \subseteq L^2_{\gamma, (kg)^{-1}}(K_{\hat{s}} \setminus K) \) a.e. \( \hat{s} \), for each \((k, g, \gamma) \in K \times G \times \hat{K}\).

But \( \gamma \cdot (kg)^{-1} = \gamma \cdot g^{-1} \) for each \((k, g) \in K \times G\).

Hence \( R(\hat{s}, kg)L^2_{\gamma}(K_{\hat{s}o g} \setminus K) \subseteq L^2_{\gamma, g^{-1}}(K_{\hat{s}} \setminus K) \) a.e. \( \hat{s} \), for each \((k, g, \gamma) \in K \times G \times \hat{K}\).
each \((k, g, \gamma) \in K \times G \times \hat{K}\). If \(\vartheta\) is a \(G\)-orbit in \(\hat{K}\) then
\[
R(\vartheta, kg) : \bigoplus_{\gamma \in \vartheta} L_{\gamma}^2(K_{\text{so}} \setminus K) \rightarrow \bigoplus_{\gamma \in \vartheta} L_{\gamma}^2(K_{\text{so}} \setminus K)
\]
a.e. \(\hat{s}\), for each \((k, g) \in K \times G\).

Hence, for each \((k, g) \in K \times G\) we have
\[
R(\vartheta, kg) \left( \bigoplus_{\gamma \in \vartheta} L_{\gamma}^2(K_{\text{so}} \setminus K) \right) = \bigoplus_{\gamma \in \vartheta} L_{\gamma}^2(K_{\text{so}} \setminus K)
\]
a.e. \(\hat{s}\).

Let \(\vartheta_1, \vartheta_2, \vartheta_3, \ldots\) be the \(G\)-orbits in \(\hat{K}\).

Then
\[
L_{\gamma}^2(K_{\text{so}} \setminus K) \cong \bigoplus_{\gamma \in \vartheta} L_{\gamma}^2(K_{\text{so}} \setminus K)
\]

since \(L_{\gamma}^2(K_{\text{so}} \setminus K) = \bigoplus_{i=1}^{\infty} L_{\gamma}^2(K_{\text{so}} \setminus K)\) for each \(\hat{s} \in \hat{S}\).

Moreover, \(\bigoplus_{\gamma \in \vartheta} L_{\gamma}^2(K_{\text{so}} \setminus K)\) is finite-dimensional for each \(\vartheta\) and \(\hat{s}\).

Thus the natural bundle representation \(R\) of \((\hat{S}, K \times G)\) has discrete spectrum.

**Corollary 2.6:** Let \(K \times G\) act ergodically on \((S, \mu)\). Let \(\hat{S} = S/K\) be the space of \(K\)-orbits in \(S\) on which \(K \times G\) acts naturally.

If each \(G\)-orbit in \(\hat{K}\) is finite then there exist a compact group \(M\), a closed subgroup \(M_0\) of \(M\), and a cocycle \(\beta : \hat{S} \times K \times G \rightarrow M\) with Mackey dense-range such that \(S \simeq \hat{S} \times \beta M_0 \setminus M\) as \(K \times G\)-extensions of \(\hat{S}\).

**Proof:** We may assume the measure \(\mu\) on \(S\) is \(K\)-invariant otherwise, we replace it by the \(K\)-invariant measure \(\mu'\) defined by \(\mu'(E) = \int_K \mu(E \cdot k) dk\) where \(E\) is a Borel subset of \(S\).

The result now follows from Theorem 2.5, Proposition 2.4, and Robert Zimmer's Structure Theorem.

Let us look at an example to illustrate Corollary 2.6.

**Example 2.7:** Let \(\phi : \mathbb{Z} \rightarrow SU(2)\) be a homomorphism from the integers into
the special unitary group $SU(2)$ defined by

$$\phi(n) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^n \text{ for each } n \in \mathbb{Z}.$$ 

Then $\mathbb{Z}$ acts on $SU(2)$ by inner-conjugation i.e. $n \cdot k = \phi(n)k\phi(-n)$ for each $(n,k) \in \mathbb{Z} \times SU(2)$. We may form the semi-direct product $SU(2) \times \mathbb{Z}$ whose multiplication is given by:

$$(k_1,n_1)(k_2,n_2) = (k_1(n_1 \cdot k_2),n_1 + n_2) \text{ for each } (k_i,n_i) \in SU(2) \times \mathbb{Z}, \ i=1,2.$$ 

Each $\mathbb{Z}$-orbit in $\hat{SU}(2)$ is finite since compact, connected, semi-simple Lie groups (like $SU(2)$) possess only finitely many non-equivalent irreducible representations of a given dimension.

Define an action of $SU(2) \times \mathbb{Z}$ on $T \times SU(2)$ by: if $(t,a) \in T \times SU(2)$ and $(k,n) \in SU(2) \times \mathbb{Z}$, then $(t,a) \cdot k = (t,ak)$ and $(t,a) \cdot n = (t \cdot n,a\phi(n))$ where $\mathbb{Z}$ acts by an irrational rotation on the torus $T$.

We claim $SU(2) \times \mathbb{Z}$ acts ergodically (properly) on $T \times SU(2)$.

Let $E \subseteq T \times SU(2)$ be an $SU(2) \times \mathbb{Z}$-invariant Borel subset.

Let $E^t = \{a \in SU(2)| (t,a) \in E\}$ for each $t \in T$.

For each $t \in T$, $E^t$ is null or conull since $E^t$ is $SU(2)$-invariant.

Let $T_0 = \{ t \in T | E^t \text{ is conull} \}$.

Then note $E^{t^{-1}} = \{ a \in SU(2) : (t \cdot 1,a) \in E \} =$

$= \{ a \in SU(2)| (t,a\phi(3)) \cdot 1 \in E \}$ since $\phi(4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \{ a \in SU(2)| (t,a\phi(3)) \in E \}$

$= \{ a \in SU(2)| (t,a) \in E \} \cdot \phi(3)^{-1}$

$E^{t^{-1}} = E^t \cdot \phi(3)^{-1}$.

Hence $T_0$ is $\mathbb{Z}$-invariant. So $T_0$ is null or conull.

If $m$ and $\nu$ are normalized Haar measures on $SU(2)$ and $T$, respectively then

$$\nu \times m)(E) = \int_{T} m(E^t) d\nu(t) = \int_{T_0} 1 d\nu(t) = \nu(T_0) = 0 \text{ or } 1.$$
Thus each $SU(2) \times \mathbb{Z}$-invariant Borel subset $E$ of $T \times SU(2)$ is either null or conull and we see $SU(2) \times \mathbb{Z}$ acts ergodically (and properly) on $T \times SU(2)$.

This action of $SU(2) \times \mathbb{Z}$ must be given by a skew-product action on $\hat{S} \times _{\beta} M_0 \setminus M$ (as assured by Corollary 2.6) where $\hat{S} = S/K$ and $S = T \times SU(2)$.

In this action of $SU(2) \times \mathbb{Z}$ we may identify $\hat{S}$ with $T$.

Define $\beta : T \times [SU(2) \times \mathbb{Z}] \rightarrow SU(2)$ by $\beta((t, (k, n))) = k\phi(n)$ for $(t, k, n) \in T \times SU(2) \times \mathbb{Z}$, where we are regarding $T$ as $\hat{S}$.

It is easy to verify that $\beta$ is a cocycle.

The skew-product action of $SU(2) \times \mathbb{Z}$ on $T \times _{\beta} SU(2)$ is exactly the action we started with. Indeed, if $(t, a) \in T \times _{\beta} SU(2)$ and $(k, n) \in SU(2) \times \mathbb{Z}$ then

$$(t, a) \cdot (k, n) = (t \circ (k, n), a\beta((k, n))) = (t \circ n, ak\phi(n))$$

$$= (t \cdot n, ak\phi(n)),$$

where $\circ$ denotes the action of $\mathbb{Z}$ on $\hat{S}$.

Corollary 2.6 can fail if we drop the assumption that each $G$-orbit in $\hat{K}$ is finite.

We provide an example.

**Example 2.8:** Let $K \times G = T^2 \times SL(2, \mathbb{Z})$, where $SL(2, \mathbb{Z})$ acts canonically on $T^2$, i.e. if $A \in SL(2, \mathbb{Z})$ and $exp(x, y) = (e^{2\pi i x}, e^{2\pi i y}) \in T^2$ then

$A \cdot exp(x, y) = exp((x, y) \cdot A^t)$, where $(x, y) \in \mathbb{R}$ and $A^t$ is the transpose of $A$.

The action of $SL(2, \mathbb{Z})$ on the dual $\hat{T}^2 = \mathbb{Z}^2$ is given by: if $(n, m) \in \hat{T}^2 = \mathbb{Z}^2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ then $(n, m) \cdot A = (an + cm, bn + dm) \in \hat{T}^2$.

Not all $SL(2, \mathbb{Z})$-orbits in $\hat{T}^2$ are finite since $(n, m) \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = (n, an + m)$ where $a, n, m \in \mathbb{Z}$.

Define an action of $T^2 \times SL(2, \mathbb{Z})$ on $S \equiv T^2$ (with Haar measure) by:

$s \cdot (k, A) = A^{-1} \cdot sk$ where $(s, k, A) \in S \times T^2 \times SL(2, \mathbb{Z})$ and $A^{-1} \cdot sk$ is the action of $SL(2, \mathbb{Z})$ on $T^2$. 


Note $\hat{S} = S/(T^2)$ consists of one point.

The natural cocycle representation $R$ on $(pt., T^2 \times, SL(2, \mathbb{Z}))$ is a (see Example 1.9) unitary representation of $T^2 \times SL(2, \mathbb{Z})$ on $L^2(T^2)$ defined by:

$$(R(k, A)f)(s) = f(s \cdot (k, A))$$

where $f \in L^2(T^2)$ and $(s, k, A) \in T^2 \times T^2 \times SL(2, \mathbb{Z})$.

We regard $\mathbb{Z}$ as a subgroup of $T^2 \times SL(2, \mathbb{Z})$ via the identification $a \in \mathbb{Z} \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in T^2 \times SL(2, \mathbb{Z})$.

Then $R|_{\mathbb{Z}} : \mathbb{Z} \to U(L^2(T^2))$

is given by $(R(a)f)(s_1 + \mathbb{Z}, s_2 + \mathbb{Z}) = f((s_1 + \mathbb{Z}, s_2 + \mathbb{Z}) \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix})$

$$= f\left(\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \cdot \exp(s_1, s_2)\right)$$

$$= f\left(\exp\left((s_1, s_2) \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\right)\right)$$

$$= f(\exp(s_1 - as_2, s_2))$$

Hence $R(a)f(s_1 + \mathbb{Z}, s_2 + \mathbb{Z}) = f(s_1 - as_2 + \mathbb{Z}, s_2 + \mathbb{Z})$ for each $f \in L^2(T^2)$ and $s_1, s_2 \in \mathbb{R}$, $a \in \mathbb{Z}$.

We will show $R|_{\mathbb{Z}}$ is unitarily equivalent to an infinite countable number of copies of the regular representation of $\mathbb{Z}$.

Thus $R|_{\mathbb{Z}}$ is not a countable sum of finite-dimensional unitary representations.

Thus $R$ as a cocycle representation on $(\hat{S}, T^2 \times, SL(2, \mathbb{Z}))$ does not have discrete spectrum.

By Remark 1.10, $S = T^2$ cannot be written as a skew-product $\hat{S} \times_\beta M_0 \backslash M$ as described in Corollary 2.6.

To see $R|_{\mathbb{Z}}$ is equivalent to a countable copy of the regular representation of $\mathbb{Z}$, define for each $s_2 \in \mathbb{R}$ a unitary representation $R^{s_2 + \mathbb{Z}}$ of $\mathbb{Z}$ on $L^2(T)$ by: $R^{s_2 + \mathbb{Z}}(a)f(s_1 + \mathbb{Z}) = f(s_1 - as_2 + \mathbb{Z})$ for $f \in L^2(T)$, and
(a, s_1) \in \mathbb{Z} \times \mathbb{R}, and the measure on T being normalized Haar measure \mu.

Thus \( R_{|\mathbb{Z}} \simeq \int_{\mathbb{R}/\mathbb{Z}} R^{s_2+\mathbb{Z}} d\mu(s_2 + \mathbb{Z}). \)

This equivalence can be verified using the unitary operator described by:
\( L^2(T^2) \simeq L^2(T, L^2(T)) \) where \( f \mapsto [s_2 + \mathbb{Z} \mapsto f(\bullet, s_2 + \mathbb{Z})]. \)

Recall the Fourier transform \( \Lambda : L^2(\mathbb{Z}, \mu_c) \rightarrow L^2(T) \) is defined by
\[ \hat{f}(z) = \sum_{n=-\infty}^{\infty} f(n)z^n \]
where \( f \in L^2(\mathbb{Z}, \mu_c), z \in T \) and \( \mu_c \) is the counting measure.

Then \( R^{s_2+\mathbb{Z}} \simeq \sum_{n=-\infty}^{\infty} e^{-2\pi i s_2 n} \); for if \( f \in L^2(\mathbb{Z}, \mu_c) \), then
\[
\left( \left( \sum_{n=-\infty}^{\infty} e^{-2\pi i s_2 n} \right)_a f \right) ^\wedge (s_1 + \mathbb{Z}) = \sum_{m=-\infty}^{\infty} e^{-2\pi i s_2 m} f(m) e^{2\pi i s_1 m} \\
= \sum_{m=-\infty}^{\infty} e^{2\pi i(s_1 - a s_2) m} f(m) \\
= f^\wedge (s_1 - a s_2 + \mathbb{Z}).
\]
Hence \( \left( \left( \sum_{n=-\infty}^{\infty} e^{-2\pi i s_2 n} \right)_a f \right) ^\wedge (s_1 + \mathbb{Z}) = (R^{s_2+\mathbb{Z}}(f^\wedge))(s_1 + \mathbb{Z}) \) for each \( (a, s_1) \in \mathbb{Z} \times \mathbb{R}. \)

Now \( R_{|\mathbb{Z}} \simeq \int_{\mathbb{R}/\mathbb{Z}} R^{s_2+\mathbb{Z}} d\mu(s_2 + \mathbb{Z}). \)
\[
\simeq \int_{\mathbb{R}/\mathbb{Z}} \sum_{n=-\infty}^{\infty} e^{-2\pi i s_2 n} d\mu(s_2 + \mathbb{Z}) \\
\simeq \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i s_2 n} d\mu(s_2 + \mathbb{Z}) \\
\simeq \sum_{n=-\infty}^{\infty} \int_T z^n d\mu(z) \\
\simeq \sum_{n=-\infty}^{\infty} n \int_T z d\mu(z) \\
R_{|\mathbb{Z}} \simeq \infty \cdot \int_T z d\mu(z).\]
CHAPTER 3
ANALYSIS OF COCYCLES

In this chapter we further the investigation into the structure of semi-direct product actions. To do these we analyze the cocycles for these groups.

Theorem 3.1: Assume $K \times_s G$ acts ergodically on $\hat{S}$ such that $K$ acts trivially on $\hat{S}$. Suppose there exists a continuous homomorphism $\phi : G \rightarrow K$ such that $G$ acts on $K$ by $g \cdot k = \phi(g)k\phi(g^{-1})$ for each $(k, g) \in K \times_s G$. Then a cocycle $\beta : \hat{S} \times K \times_s G \rightarrow M$ with Mackey dense-range into a compact group $M$ is cohomologous to a cocycle $\beta'$ where $\beta'(\hat{s}, k, g) = \Lambda(k)\beta'(\hat{s}, g)$ for each $(\hat{s}, k, g) \in \hat{S} \times K \times G$ where $\Lambda : K \rightarrow M$ is a continuous homomorphism.

Proof: The cocycle $\beta : \hat{S} \times K \times_s G \rightarrow M$ can be assumed to be a strict cocycle i.e. $\beta(\hat{s}, (k_1, g_1)(k_2, g_2)) = \beta(\hat{s}, k_1g_1)\beta(\hat{s} \circ g_1, k_2g_2)$ a.e. $\hat{s}$ for each $(k_1, g_1) \in K \times_s G$ for $i=1,2$.

For each $\hat{s}$, the map $k \mapsto \beta(\hat{s}, k)$ is a continuous homomorphism from $K$ into $M$.

Let $m$ be the normalized Haar measure on $K$.

Let $\Gamma(K, M)$ be the set of all equivalence classes of all Borel maps from $K$ into $M$ where two such maps are equivalent if they are equal $m$-a.e.

Let $d$ be a metric on $M$ whose value is at most 1.

Define a metric $p$ on $\Gamma(K, M)$ by

$$p(f_1, f_2) = \int_K d(f_1(k), f_2(k)) dm(k)$$

where $f_i \in \Gamma(K, M)$ for $i = 1, 2$. 

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Since $M$ is a complete separable metric space, then by a result of C. Moore, $\Gamma(K,M)$ is complete separable metric space (see p.6 of C. Moore's paper "Group Extensions and Cohomology for Locally Compact Groups. III").

Let $F = \{ \phi \in \Gamma(K,M) : \text{there exists a Borel homomorphism } \Phi : K \rightarrow M \text{ such that } \phi = \Phi \text{ a.e.} \}$

Claim 3.2: $F$ is a closed subset of $\Gamma(K,M)$. Hence $F$ is a complete separable metric space and $F$ with its Borel structure is a standard Borel space.

Proof: Let $f_n$ be sequence in $F$ such that $f_n$ converges to $f$ in $\Gamma(K,M)$. Then $f_n$ converges to $f$ in $m$-measure (see Proposition 6 of C. Moore's paper "Group Extensions and Cohomology for Locally Compact Groups. III"). Thus there is a subsequence of $f_n$ which converges point-wise a.e. to $f$.

Hence $f(k_1k_2) = f(k_1)f(k_2)$ a.e. $(k_1, k_2)$.

Then there exists a Borel homomorphism $\Phi : K \rightarrow M$ such that $\Phi = f$ a.e. (see Theorem B.2 of R. Zimmer's book "Ergodic Theory and Semisimple Groups"). Thus $f \in F$ and $F$ is a closed subset of $\Gamma(K,M)$.

Let $K \times M$ act on $F$ by: if $(k, m) \in K \times M$ and $\eta \in F$ then $\eta \cdot (k, m) \in F$ where $(\eta \cdot (k, m))(k_1) = m^{-1}\eta(kk_1k_1^{-1})m$ for $k_1 \in K$.

It is easy to verify $\eta \cdot (k_1, m_1) \cdot (k_2, m_2) = \eta \cdot (k_1k_2, m_1m_2)$ for each $(\eta, k_i, m_i) \in F \times K \times M$, $i = 1, 2$.

Before we proceed we prove a lemma.

Lemma 3.3: The quotient Borel structure of the $K \times M$ - orbits $F/(K \times M)$ in $F$ is a standard Borel space.

Proof:

Since $K \times M$ is a compact group, it is enough to prove that the mapping $F \times (K \times M) \rightarrow F$, defined by the action of $K \times M$ on $F$ is a Borel map (see
Corollary 2.1.21 of R. Zimmer's book "Ergodic Theory and Semisimple Groups" and see Proposition 7.1 of R. Kallman's paper "Certain Quotient Spaces are Countably Separated, III").

Fix \( \Lambda \in F \) and let \((k_i, m_i), (k_0, m_0) \in K \times M\) such that \((k_i, m_i)\) converges to \((k_0, m_0)\). We will show \( \Lambda \cdot (k_i, m_i) \) converges to \( \Lambda \cdot (k_0, m_0) \).

Indeed,

\[
p(\Lambda \cdot (k_i, m_i), \Lambda \cdot (k_0, m_0)) = \int_K d(m_i^{-1}\Lambda(k_i k_{i-1})m_i, m_0^{-1}\Lambda(k_0 k_{0-1})m_0) \, dm(k) \text{ converges to zero as } i \text{ goes to infinity by the Lebesgue Dominated Convergence Theorem.}
\]

Hence \((k, m) \rightarrow \Lambda \cdot (k, m)\) is a continuous map from \( K \times M \) to \( F \).

Let \( \Lambda_n \) be a sequence in \( F \) that converges to a Borel homomorphism \( \Lambda \) in \( F \).

Likewise,

\[
p(\Lambda_n \cdot (k_0, m_0), \Lambda \cdot (k_0, m_0)) = \int_K d(m_0^{-1}\Lambda_n(k_0 k_{0-1})m_0, m_0^{-1}\Lambda(k_0 k_{0-1})m_0) \, dm(k) \text{ converges to zero as } n \text{ goes to infinity(see Proposition 6 of C. Moore's paper "Group Extensions and Cohomology for Locally Compact Groups. III").}
\]

So \( \Lambda \rightarrow \Lambda \cdot (k_0, m_0) \) is a continuous map from \( F \) into itself.

Then \( F \times (K \times M) \rightarrow F \) is continuous in each variable separately.

Then \( F \times (K \times M) \rightarrow F \) is a Borel map (see Lemma 9.2 of G. Mackey's paper "Induced Representations of Locally Compact Groups I").

This proves Lemma 3.3.

The Borel structure on \( \Gamma(K, M) \) is the smallest Borel structure such that

\[
f \in \Gamma(K, M) \rightarrow \int_Y (\psi \circ f)(k) \, dm(k) \text{ is a Borel map}
\]

for all Borel subsets \( Y \) of \( K \) and for all bounded Borel maps of \( \psi \) on \( M \)

(see Proposition 8 of C. Moore's paper "Group Extensions and Cohomology for
By Fubini's Theorem, the mapping

\[ \hat{s} \in \hat{S} \rightarrow \int_Y \psi \circ \beta(\hat{s}, k) \, dm(k). \]

is a Borel map for each Borel subset \( Y \) of \( K \) and for each bounded Borel function \( \psi \) on \( M \).

Thus \( \hat{s} \rightarrow \beta(\hat{s}, \bullet) \) is a Borel map from \( \hat{S} \) into \( \Gamma(K, M) \).

Hence \( \hat{s} \rightarrow \beta(\hat{s}, \bullet) \) is a Borel map from \( \hat{S} \) into \( F \).

Denote the \( K \times M \)-orbit of \( \eta \in F \) by \( \langle \eta \rangle \).

Thus \( \hat{s} \in \hat{S} \rightarrow \langle \beta(\hat{s}, \bullet) \rangle \in F/(K \times M) \) is a \( G \)-invariant Borel map on \( \hat{S} \) since

\[ \beta(\hat{s}, \phi(g)k\phi(g)^{-1}) = \beta(\hat{s}, g)\beta(\hat{s} \circ g, k)\beta(\hat{s}, g)^{-1} \]

for each \((\hat{s}, k, g) \in \hat{S} \times K \times G \). Since \( \hat{S} \) is \( G \)-ergodic and \( F/(K \times M) \) is a standard Borel space, the mapping \( \hat{s} \rightarrow \langle \beta(\hat{s}, \bullet) \rangle \) is essentially constant i.e. there exist a Borel homomorphism \( \Lambda \) from \( K \) into \( M \) and a conull Borel set \( E \) in \( \hat{S} \) such that \( \langle \Lambda \rangle = \langle \beta(\hat{s}, \bullet) \rangle \) for each \( \hat{s} \in E \). Then for each \( \hat{s} \in E \) there exists \((k, m) \in K \times M\)

such that \( \Lambda(k_1) = m^{-1}\beta(\hat{s}, kk_1^{-1})m \) for each \( k_1 \in K \).

Let \( A = \{(\hat{s}, k, m) \in E \times K \times M : \Lambda(k_1) = m^{-1}\beta(\hat{s}, kk_1^{-1})m \text{ for each } k_1 \in K \} \).

A is a Borel subset of \( E \times K \times M \) since by \[**\],

\( (\hat{s}, k, m) \rightarrow m^{-1}\beta(\hat{s}, k \bullet k^{-1})m \) is a Borel map from \( (\hat{S} \times K \times M) \) into \( F \).

The projection of \( A \) into \( E \) is all of \( E \). By a von Neumann selection theorem there exists a conull Borel set \( E' \) in \( E \), and a Borel map \( (K \times M) : E' \rightarrow K \times M \) such that \( (\hat{s}, K(\hat{s}), M(\hat{s})) \in A \) for each \( \hat{s} \in E' \) (see Theorem Z.2 of G. Mackey's book "The Theory of Unitary Group Representations").

Thus \( \Lambda(k_1) = M(\hat{s})^{-1}\beta(\hat{s}, K(\hat{s})k_1, K(\hat{s})^{-1})M(\hat{s}) \) for each \((\hat{s}, k_1) \in E' \times K \).
Define $\beta': \hat{S} \times K \times G \rightarrow M$ by

$$\beta'(\hat{s}, x) = \left( e \cdot \chi(E') \cdot K \times G(\hat{s}, x) \right) \left( \beta(\hat{s}, x) \chi(E') \cdot K \times G(\hat{s}, x) \right)$$

for each $(\hat{s}, x) \in \hat{S} \times K \times G$, and where $e$ is the identity in $M$.

$\beta'$ is a Borel map and $\beta'(\hat{s}, x) = \beta(\hat{s}, x)$ a.e. $\hat{s}$ for each $x \in K \times G$.

Thus $\beta'$ is a cocycle cohomologous to $\beta$.

The Borel map $K \times M$ defined on $E'$ can be extended to be defined on $\hat{S}$ by $(K \times M)(\hat{s}) = (e, e)$ for each $\hat{s} \in (\hat{S} - E')$.

Define $\gamma: \hat{S} \times K \times G \rightarrow M$ to be the Borel map defined by

$$\gamma(\hat{s}, kg) = M(\hat{s})^{-1} \beta'(\hat{s}, kg) M(\hat{s} \circ g)$$

for $(\hat{s}, k, g) \in \hat{S} \times K \times G$.

Then $\gamma \sim \beta'$ where $\sim$ means cohomologous cocycles.

Fix $(k, g) \in K \times G$.

Then $\gamma(\hat{s}, kg) = M(\hat{s})^{-1} \beta'(\hat{s}, k) M(\hat{s}) \beta'(\hat{s}, g) M(\hat{s} \circ g)$ a.e. $\hat{s}$

$$= M(\hat{s})^{-1} \beta(\hat{s}, k) M(\hat{s}) \gamma(\hat{s}, g) \text{ a.e. } \hat{s}$$

$$= M(\hat{s})^{-1} \left[ M(\hat{s}) \Lambda(K(\hat{s})^{-1} k K(\hat{s})) M(\hat{s})^{-1} \right] M(\hat{s}) \gamma(\hat{s}, g) \text{ a.e. } \hat{s}.$$ 

Thus $\gamma(\hat{s}, kg) = \Lambda(K(\hat{s})^{-1} k K(\hat{s})) \gamma(\hat{s}, g)$ a.e. $\hat{s}$ for each $(k, g) \in K \times G$.

Define $\gamma': \hat{S} \times K \times G \rightarrow M$ to be the Borel map defined by

$$\gamma'(\hat{s}, kg) = \Lambda(K(\hat{s})^{-1} k K(\hat{s})) \gamma(\hat{s}, g)$$

for $(\hat{s}, k, g) \in \hat{S} \times K \times G$.

Then $\gamma' \sim \gamma \sim \beta$ as equivalent cocycles.

Define $\beta_1: \hat{S} \times K \times G \rightarrow M$ by:

$$\beta_1(\hat{s}, kg) = (\Lambda \circ K)(\hat{s}) \gamma'(\hat{s}, kg) (\Lambda \circ K)^{-1}(\hat{s} \circ g)$$

for each $(\hat{s}, k, g) \in \hat{S} \times K \times G$.

$\beta_1$ is a cocycle, and $\beta_1 \sim \gamma' \sim \beta$.

Fix $(k, g) \in K \times G$.

Then $\beta_1(\hat{s}, kg) = (\Lambda \circ K)(\hat{s}) \left[ \Lambda(K(\hat{s})^{-1} k K(\hat{s})) \gamma(\hat{s}, g) \right] (\Lambda \circ K)^{-1}(\hat{s} \circ g)$
\[
\beta_1(\tilde{s}, kg) = \Lambda(k)(\Lambda \circ K)(\tilde{s}) \gamma(\tilde{s}, g)(\Lambda \circ K)^{-1}(\tilde{s} \circ g)
\]
\[
= \Lambda(k) ((\Lambda \circ K)(\tilde{s}) \gamma'(\tilde{s}, g)(\Lambda \circ K)^{-1}(\tilde{s} \circ g)) \text{ a.e. } \tilde{s}
\]
\[
= \Lambda(k) \beta_1(\tilde{s}, g).
\]

Thus \( \beta_1(\tilde{s}, kg) = \Lambda(k) \beta_1(\tilde{s}, g) \) a.e. \( \tilde{s} \) for each \((k, g) \in K \times G \).

Define \( \beta_2 : \hat{S} \times K \times G \rightarrow M \) to be the Borel map given

by \( \beta_2(\tilde{s}, kg) = \Lambda(k) \beta_1(\tilde{s}, g) \) for \((\tilde{s}, k, g) \in \hat{S} \times K \times G \).

\( \beta_2 \) is a cocycle such that \( \beta_2 \sim \beta_1 \sim \beta \). In fact \( \beta_2(\tilde{s}, kg) = \Lambda(k) \beta_2(\tilde{s}, g) \)

for each \((\tilde{s}, k, g) \in \hat{S} \times K \times G \).

So the \( \beta \) in Corollary 2.6 can be replaced by \( \beta_2 \). This proves Theorem 3.1. 

In certain cases we will describe the compact group \( M \) in Corollary 2.6. We introduce some notations. Let \( J \) be a subset of a topological group \( P \) then

\(< J > \) denotes the subgroup of \( P \) generated by \( J \), and \( \overline{< J >} \) denotes the closed subgroup generated by \( J \).

**Theorem 3.4:** Let \( M' \) be a closed subgroup of a compact group \( M'' \).

Let \( \beta' : \hat{S} \times G \rightarrow M' \) be a strict cocycle with Mackey dense-range.

Suppose to each \( \tilde{s} \in \hat{S} \) there exist a Borel homomorphism \( \beta'(\tilde{s}, \bullet) : K \rightarrow M'' \)

such that:

1. \( \tilde{s} \in \hat{S} \rightarrow \beta'(\tilde{s}, k) \) is a Borel map for each \( k \in K \),
2. \( \beta'(\tilde{s}, g \cdot k)\beta'(\tilde{s}, g) = \beta'(\tilde{s}, g)\beta'(\tilde{s} \circ g, k) \) for each \((\tilde{s}, k, g) \in (\hat{S} \times K \times G ) \)

and (3) \( \beta'(\tilde{s}, k)\beta'(\tilde{s}, g) \in M'' \) for each \((\tilde{s}, k, g) \in \hat{S} \times K \times G \).

Then there exist a \( G \)-invariant conull Borel set \( \hat{S}' \) in \( \hat{S} \) and a closed subgroup \( M \) of \( M'' \) such that \( M = \overline{< \beta'(\tilde{s}, K) \cup M' >} \) for each \( \tilde{s} \in \hat{S}' \).

Also the map \( \beta : \hat{S}' \times K \times G \rightarrow M \) given by \( \beta(\tilde{s}, kg) = \beta'(\tilde{s}, k)\beta'(\tilde{s}, g) \)

for each \((\tilde{s}, k, g) \in \hat{S}' \times K \times G \) is a strict cocycle with Mackey dense-range.

**Proof:** The proof will be given by a sequence of lemmas.
Lemma 3.5: The map $\beta : \hat{S} \times K \times G \to M''$ given by

$\beta(\hat{s},k) = \beta'(\hat{s},k)\beta'(\hat{s},g)$ is a cocycle.

Proof: The map $\beta$ is well-defined by (3) of Theorem 3.4.

For each $k \in K$, the map $(\hat{s},g) \to \beta'(\hat{s},k)\beta'(\hat{s},g)$ is Borel by (1) of Theorem 3.4. For each $(\hat{s},g) \in \hat{S} \times G$,
the map $k \to \beta'(\hat{s},k)\beta'(\hat{s},g)$ is continuous from $K$ into $M''$.

Thus $\beta$ is a Borel map.

Fix $(\hat{s},k_i,g_i) \in \hat{S} \times K \times G$ for $i = 1,2$.

Then

$\beta(\hat{s},(k_1,g_1)(k_2,g_2)) = \beta(\hat{s},(k_1g_1 \cdot k_2,g_1g_2))$

$= \beta'(\hat{s},k_1(g_1 \cdot k_2))\beta'(\hat{s},g_1g_2)$

$= \beta'(\hat{s},k_1)\beta'(\hat{s},g_1 \cdot k_2)\beta'(\hat{s},g_1)\beta'(\hat{s} \circ g_1,g_2)$

$= \beta'(\hat{s},k_1)\beta'(\hat{s},g_1)\beta'(\hat{s} \circ g_1,k_2)\beta'(\hat{s} \circ g_1,g_2)$ by (2) of Theorem 3.3

$= \beta(\hat{s},(k_1,g_1))\beta(\hat{s} \circ g_1,(k_2,g_2))$.

Hence

$\beta(\hat{s},(k_1,g_1)(k_2,g_2)) = \beta(\hat{s},(k_1g_1))\beta(\hat{s} \circ g_1,k_2g_2)$.

Thus $\beta$ is a strict cocycle.

Let $\Sigma$ be the set of all closed subgroups of $M''$. $\Sigma$ with the Fell topology is a complete separable metric space. The induced Borel structure on $\Sigma$ makes $\Sigma$ into a standard Borel space.

Lemma 3.6: The map $\xi : \hat{S} \to \Sigma$ given by $\xi(\hat{s}) = \overline{\beta'(\hat{s},K) \cup M'}$ is a Borel map.

Proof: Let $d$ be metric on $M$ whose value is at most 1. The Borel structure on $\Sigma$ is the smallest one such that for each $m \in M''$ the map $R \in \Sigma \to d(m,R)$ is Borel where $d(m,R) = \inf\{d(m,p) : p \in R\}$ (See p.67 of Auslander/Moore's American Mathematical Society Memoirs Number 62).

Since $K$ and $M'$ have countable dense sets, there exists a countable number of
Borel maps \( \phi_i : \hat{S} \rightarrow M'' \) such that \( \{ \phi_i(\hat{s}) : i = 1, 2, \ldots \} \) is dense in \( \xi(\hat{s}) \).

That is \( \xi(\hat{s}) = \{ \phi_i(\hat{s}) : i = 1, 2, \ldots \} \), for each \( \hat{s} \in \hat{S} \).

For each \( m \in M'' \),

\[
d(m, \xi(\hat{s})) = \inf \{ d(m, p) : p \in \xi(\hat{s}) \} = \inf \{ d(m, \phi_i(\hat{s})) : i = 1, 2, \ldots \}.
\]

Thus \( \hat{s} \rightarrow d(m, \xi(\hat{s})) \) is Borel for each \( m \in M'' \).

Hence \( \xi : \hat{S} \rightarrow \Sigma \) is a Borel map.

Note \( < \beta(\hat{s} \circ g, K) \cup M' > = < \beta(\hat{s}, K) \cup M' > \) for all \( (\hat{s}, g) \in \hat{S} \times G \) by (2) of Theorem 3.4 and the fact that \( \beta'(\hat{s}, g) \in M' \).

Hence \( \xi(\hat{s} \circ g) = \xi(\hat{s}) \) for all \( (\hat{s}, g) \in S \times G \).

Since \( \hat{S} \) is G-ergodic there exist a closed subgroup \( M \) of \( M'' \) such that

\[
S' = \{ \hat{s} \in \hat{S} : < \beta(\hat{s}, K) \cup M' > = M \}
\]

is Borel conull and G-invariant.

Thus \( \beta(\hat{s}, kg) \in M \) for each \( (\hat{s}, k, g) \in \hat{S}' \times K \times G \).

So \( \beta : \hat{S}' \times K \times G \rightarrow M \) is still a strict cocycle.

We will show \( (\hat{S}' \times \beta M) \) is an ergodic \( K \times G \)-space.

Let \( p : M \rightarrow M/M' \) be given by \( p(m) = mM' \) and \( t : M/M' \rightarrow M \) be a Borel cross-section of \( p \) i.e. \( t(mM') \in mM' \).

Let \( \nu, \nu \) be Haar measures on \( M' \) and \( M \), respectively each of total volume 1.

Let \( p_*(\nu) \) be a measure on \( M/M' \) given by \( p_*(\nu)(E) = \nu(p^{-1}(E)) \) if \( E \) is a Borel subset of \( M/M' \).

Then \( \psi : (M, \nu) \rightarrow (M' \times M/M', \nu_1 \times p_*(\nu)) \) given by

\[
\psi(m) = (tp(m)^{-1} \cdot m, p(m))
\]

is a measure-preserving Borel isomorphism.

Also \( \psi(mm''m') = (m', p(m)) \) where \( m', m'' \in M' \) and \( tp(m) = mm'' \).

Denote \( \overline{\psi} : (\hat{S}' \times M) \rightarrow (\hat{S'} \times M' \times M/M') \) by \( \overline{\psi}(\hat{s}, m) = (\hat{s}, \psi(m)) \).
$\tilde{\psi}$ is again a measure-preserving Borel isomorphism.

And $\tilde{\psi}((\hat{s}, \hat{m} \cdot g)) = \tilde{\psi}((\hat{s} \circ g, m\beta(\hat{s}, g))) = 
= (\hat{s} \circ g, tp(m)^{-1} \cdot m\beta(\hat{s}, g), p(m))$ since $\beta(\hat{s}, g) \in M'$.

Hence $(\hat{s}, tp(m)^{-1} \cdot m, p(m)) \cdot g = (\hat{s} \circ g, tp(m)^{-1} \cdot m\beta(\hat{s}, g), p(m))$ for all $(\hat{s}, m, g) \in \hat{S}' \times M \times G$.

For each $(\hat{s}, m', mm') \in \hat{S}' \times M' \times M/M'$, $(\hat{s}, m', mm') \cdot k = \tilde{\psi}((\hat{s}, mm'\cdot m' \cdot k))$
$= \tilde{\psi}((\hat{s}, mm'\cdot k)\cdot m' \cdot k))$
$= \left(\hat{s}, [tp(mm'\cdot k)\cdot m' \cdot k)^{-1} \cdot mm'\cdot k, p(mm'\cdot k)\cdot m' \cdot k)\right)$.

We have transferred the $K \times G$-action to $\hat{S}' \times M' \times M/M'$.

Let $E$ be a $K \times G$-invariant Borel subset of $(\hat{S}' \times M' \times M/M')$.

Let $E^{mM'} = \{((\hat{s}, m')) \in \hat{S}' \times M' : (\hat{s}, m', mm') \in E\}$ for each $mM' \in M/M'$.

Claim 3.7: $E = \hat{S}' \times M' \times F$ where $F$ is some Borel subset of $M/M'$.

**Proof:** Fix $g \in G$.

If $(\hat{s}, m') \in E^{mM'}$ then $(\hat{s}, m', mm') \cdot g \in E$.

But $(\hat{s}, m', mm') \cdot g = (\hat{s} \circ g, mm'\cdot k, mm')$.

Hence $(\hat{s}, m') \cdot g \in E^{mM'}$. So $E^{mM'}$ is a $G$-invariant Borel subset of $\hat{S}' \times M'$.

Since $\beta : \hat{S} \times G \longrightarrow M'$ is a cocycle with Mackey dense-range, then
so is $\beta : \hat{S}' \times G \longrightarrow M'$.

$E^{mM'}$ is null or conull for each $mM' \in M/M'$.

Let $F = \{mM' \in M/M' : (\bar{\mu} \times \nu_1)(E^{mM'}) = 1\}$.

$F$ is a Borel subset of $M/M'$ and $F^c = \{mM' \in M/M' : (\bar{\mu} \times \nu_1)(E^{mM'}) = 0\}$.

$(\bar{\mu} \times \nu_1 \times p_\ast \nu) \left((\hat{S}' \times M' \times F) - E\right) =
= \int_{M/M'} (\bar{\mu} \times \nu_1) \{(\hat{s}, m') \in \hat{S}' \times M' | mm' \in F \text{ and } (\hat{s}, m') \notin E^{mM'}\} dp_\ast \nu(mm')$
$= \int_{M/M'} 0 dp_\ast \nu(mm') = 0$.

Likewise, $(\bar{\mu} \times \nu_1 \times p_\ast \nu) \left(E - (\hat{S}' \times M' \times F)\right) =$
\[
= \int_{M/M'} (\tilde{\mu} \times \nu) \{(\hat{s}, m') \in \hat{S}' \times M'| mM' \not\in F \text{ and } (\hat{s}, m') \in E^{mM'}\} dp_* \nu(mM')
\]
\[
= \int_{M/M'} 0 dp_* \nu(mM') = 0.
\]
Hence \( E = \hat{S}' \times M' \times F \) a.e. and
\[
\psi^{-1}(M' \times F) \cdot \beta(\hat{s}, g) = \{ m \in M : p(m) \in F \} \cdot \beta(\hat{s}, g)
\]
\[
= \{ m \in M : p(m) \in F \} \cdot \beta(\hat{s}, g)^{-1} \in F
\]
\[
= \{ m \in M : p(m) \in F \} \text{ since } \beta(\hat{s}, g) \in M' \text{ for all } (\hat{s}, g) \in \hat{S} \times G.
\]
Hence \( \psi^{-1}(M' \times F) \cdot \beta(\hat{s}, g) = \psi^{-1}(M' \times F) \) for each \((\hat{s}, g) \in \hat{S} \times G\).

Also \( \beta(\hat{s}, k) \in M \) for each \((\hat{s}, k) \in \hat{S}' \times K\).

Similarly, \( \psi^{-1}(M' \times F) \cdot \beta(\hat{s}, k) = \{ m \in M \mid p(m) \beta(\hat{s}, k)^{-1} \in F \} \) for each \((\hat{s}, k) \in (\hat{S}' \times K)\).

Thus \( \psi(\psi^{-1}(M' \times F) \cdot \beta(\hat{s}, k)) = \{ (tp(m)^{-1} \cdot m, p(m)) \mid p(m) \beta(\hat{s}, k)^{-1} \in F \} \) for all \((\hat{s}, k) \in \hat{S}' \times K\).

For each \((\hat{s}, k) \in \hat{S}' \times K\), let \( R_{\hat{s}, k} = (M' \times F) \triangle \psi(\psi^{-1}(M' \times F) \cdot \beta(\hat{s}, k)) \) where \( A \triangle B = (A - B) \cup (B - A) \). Then
\[
R_{\hat{s}, k} = \{(tp(m_0)^{-1} \cdot m_0, p(m_0)) \in M' \times M/M'| p(m_0) \in F, p(m_0) \beta(\hat{s}, k)^{-1} \not\in F \}
\]
\[
\quad \cup \{ (tp(m_0)^{-1} \cdot m_0, p(m_0)) \in M' \times M/M'| p(m_0) \beta(\hat{s}, k)^{-1} \not\in F, p(m_0) \not\in F \}. \]

Now \((\hat{S}' \times M' \times F) \cdot k = \{(\hat{s}_0, tp(m_0)^{-1} \cdot m_0, p(m_0)) \mid \text{there exists} \)
\((\hat{s}, m', mM') \in (\hat{S}' \times M' \times F) \) such that
\(
\hat{s} = \hat{s}_0 \text{ and } m_0 = mm' \beta(\hat{s}, k) \text{ where } tp(m) = mm' \text{ and } m'' \in M'\}.

Let \( L_k = (\hat{S}' \times M' \times F) \triangle (\hat{S}' \times M' \times F) \cdot k \) for each \( k \in K \).

\( L_k \) is a null set for each \( k \in K \) since \( E = a.e. (\hat{S}' \times M' \times F) \).

Let \( L_{\hat{s}, k} = \{ x \in (M' \times F) \mid (\hat{s}, x) \in L_k \} \) for each \((\hat{s}, k) \in \hat{S}' \times K\).

Claim 3.8: \( R_{\hat{s}, k} \subseteq L_{\hat{s}, k} \) for all \((\hat{s}, k) \in \hat{S}' \times K\). Hence \( R_{\hat{s}, k} \) is a null set a.e. \( \hat{s} \in \hat{S}' \) for each \( k \in K \).

Proof: Fix \((\hat{s}, k) \in \hat{S}' \times K\).
Let \((tp(m_0)^{-1} \cdot m_0, p(m_0)) \in R_{\hat{s}, k}\).

Then either \([p(m_0) \in F \text{ and } p(m_0 \beta(\hat{s}, k)^{-1}) \notin F]\) or
\([p(m_0) \notin F \text{ and } p(m_0 \beta(\hat{s}, k)^{-1}) \in F]\).

Suppose \(p(m_0) \in F \text{ and } p(m_0 \beta(\hat{s}, k)^{-1}) \notin F\).

Then (clearly), \(m_0 \neq mm''m'\beta(\hat{s}, k)\) for each \((\hat{s}, m', mM') \in \hat{S}' \times M' \times F\)
where \(tp(m) = mm''\) and \(m'' \in M'\).

Thus \((tp(m_0)^{-1} \cdot m_0, p(m_0)) \in L^k_{\hat{s}}\).

On the other hand assume \(p(m_0) \notin F \text{ and } p(m_0 \beta(\hat{s}, k)^{-1}) \in F\).

Suppose \(m_0 \neq mm''m'\beta(\hat{s}, k)\) for all \((\hat{s}, m', mM') \in (\hat{S}' \times M' \times F)\) where
\(tp(m) = mm''\) and \(m'' \in M'\).

Then \(m_0 \cdot \beta(\hat{s}, k)^{-1} \neq mm'\) for all \(mM' \in F \text{ and } m' \in M'\).

Thus \(p(m_0 \cdot \beta(\hat{s}, k)^{-1}) \notin F\), a contradiction.

Hence there exists \((\hat{s}, m', mM') \in \hat{S}' \times M' \times F\) such that \(m_0 = mm''m'\beta(\hat{s}, k)\)
where \(tp(m) = mm''\) and \(m'' \in M'\).

Hence \((tp(m_0)^{-1} \cdot m_0, p(m_0)) \in L^k_{\hat{s}}\).

Thus \(R_{\hat{s}, k} \subseteq L^k_{\hat{s}}\) for each \((\hat{s}, k) \in (\hat{S}' \times K)\).

This proves Claim 3.8.

Next note \(M' \times F = \psi^{-1}(M' \times F) \cdot \beta(\hat{s}, k)\) a.e. \(\hat{s} \in \hat{S}'\) for each \(k \in K\)
and thus
\(\psi^{-1}(M' \times F) = \psi^{-1}(M' \times F) \cdot \beta(\hat{s}, k)\) a.e. \(\hat{s} \in \hat{S}'\) for each \(k \in K\).

Hence \(\psi^{-1}(M' \times F) = \psi^{-1}(M' \times F) \cdot \beta(\hat{s}, k)\), a.e. \(k \in K\) a.e. \(\hat{s} \in \hat{S}'\).

Thus \(\psi^{-1}(M' \times F) = \psi^{-1}(M' \times F) \cdot \beta(\hat{s}, k)\) for all \(k \in K\) a.e. \(\hat{s} \in \hat{S}'\),
since \(\{k \in K|\psi^{-1}(M' \times F) \cdot \beta(\hat{s}, k) = \psi^{-1}(M' \times F)\}\) is multiplicatively closed.

Choose \(\hat{s}_0 \in \hat{S}'\) such that:

\((1) \psi^{-1}(M' \times F) = \psi^{-1}(M' \times F) \cdot \beta(\hat{s}_0, k)\) a.e. for all \(k \in K\)
and (2) \( <\beta(s_0, K) \cup M'> = M \).

Since \( \psi^{-1}(M' \times F) = \psi^{-1}(M' \times F) \cdot \beta(s, g) \)
for each \((s, g) \in \hat{S} \times G\), then \( \psi^{-1}(M' \times F) = \circ \cdot \cdot \cdot \psi^{-1}(M' \times F) \cdot m' \) for each \( m' \in M' \) since \( M' = < \beta(s, g) : \hat{s} \in \hat{S}, g \in G > \).

Thus \( \psi^{-1}(M' \times F) \) is essentially invariant under \( < \beta(s_0, K) \cup M' > \). It follows that \( \psi^{-1}(M' \times F) \) is essentially invariant under \( < \beta(s_0, K) \cup M' > = M \).

Hence \( \psi^{-1}(M' \times F) \) is null or conull in \( M \).

But \( M' \times F \) is null or conull in \( M' \times M/M' \).

Thus \( E = \circ \cdot \cdot \cdot \hat{S}' \times M' \times F \) is null or conull in \( (\hat{S}' \times M' \times M/M') \).

Hence each \( K \times G\)-invariant Borel subset \( E \) of \( (\hat{S}' \times M' \times M/M') \) is either null or conull. So \( \hat{S}' \backslash \beta M \) is \( K \times G\)-ergodic.

**Corollary 3.9:** Let \( \beta : \hat{S} \times K \times G \longrightarrow M'' \) be a strict cocycle with Mackey dense-range into a compact group \( M'' \). Let \( M' \) be a closed subgroup of \( M'' \) such that \( \beta_{|\hat{S} \times K} : \hat{S} \times G \longrightarrow M' \) is a cocycle with Mackey dense-range.

Then there exist a conull \( G\)-invariant Borel subset \( \hat{S}' \) of \( \hat{S} \) such that \( M'' = < \beta(s, K) \cup M' > \) for all \( \hat{s} \in \hat{S}' \).

**Proof:** We introduce some notations.

If \( \xi : \hat{S} \times G \longrightarrow M \) is any cocycle, we set \( M_\xi \) to be the closed subgroup of \( M \) generated by \( \xi(\hat{S} \times G) \).

By Theorem 3.4 there exist a conull \( G\)-invariant Borel subset \( \hat{S}' \) of \( \hat{S} \) and a closed subgroup \( M \) of \( M'' \) such that

\[ M = < \beta(s, K) \cup M' > \]

for all \( \hat{s} \in \hat{S}' \).

Also \( \beta_{|\hat{S}' \times K \times G} : \hat{S}' \times K \times G \longrightarrow M \) is a strict cocycle with Mackey dense-range.
Define a cocycle \( \alpha : \hat{S} \times K \times G \to M \) by \( \alpha|_{\hat{S}' \times K \times G} = \beta|_{\hat{S}' \times K \times G} \) and \( \alpha|_{\hat{S}' \times K \times G} = e. \)

Then \( \alpha : \hat{S} \times K \times G \to M \) has Mackey dense-range and \( M_\alpha = M \).

We now regard \( \alpha \) as a cocycle into \( M'' \) i.e. \( \alpha : \hat{S} \times K \times G \to M''. \)

Again \( (M'')_\alpha = M \) and \( \alpha \sim \beta. \)

Since \( \alpha \) and \( \beta \) are equivalent minimal cocycles into \( M'' \), then \( (M'')_\alpha \) is conjugate to \( (M'')_\beta \). Since \( (M'')_\beta = M'' \) and \( (M'')_\alpha = M \),
then \( M'' = M = < \beta(\hat{s}, K) \cup M' > \) for each \( \hat{s} \in \hat{S}' \).

**Corollary 3.10:** Let \( \beta : \hat{S} \times K \times G \to M \) be a strict cocycle with Mackey dense-range into a compact group \( M \) given by \( \beta(\hat{s}, kg) = \Lambda(k)\beta(\hat{s}, g) \) for all \( (\hat{s}, k, g) \in \hat{S} \times K \times G \) where \( \Lambda : K \to M \) is a Borel homomorphism.

Suppose \( M' \) is a closed subgroup of \( M \) such that \( \beta : \hat{S} \times G \to M' \) has Mackey dense-range, then \( M = M'\Lambda(K) \).

**Proof:** By Corollary 3.9, \( M = < \Lambda(K) \cup M' >. \) Also

\( \beta(\hat{s}, g)\Lambda(k)\beta(\hat{s}, g)^{-1} = \Lambda(g \cdot k) \) for all \( (\hat{s}, k, g) \in \hat{S} \times K \times G \) by the cocycle property. Thus \( M' = < \beta(\hat{s}, g) : (\hat{s}, g) \in \hat{S} \times G > \) normalizes \( \Lambda(K). \)

Thus \( < \Lambda(K) \cup M' > = \Lambda(K)M' \) and hence \( < \Lambda(K) \cup M' > = \Lambda(K)M'. \)

**Theorem 3.11:** Let \( S \simeq \hat{S} \times M_0 \setminus M \) be an isomorphism of \( K \times G \)-extensions of \( \hat{S} \) where \( \hat{S} = S/K, \beta : \hat{S} \times K \times G \to M \) is a strict cocycle with Mackey dense-range into a compact group \( M \), and \( M_0 \) is a closed subgroup of \( M \). Suppose \( \beta(\hat{s}, kg) = \Lambda(k)\beta(\hat{s}, g) \) for all \( \hat{s}, k, g \)

where \( \Lambda : K \to M \) is a continuous homomorphism.

Then \( \Lambda(K) \) is a closed normal subgroup of \( M \) such that \( M_0\Lambda(K) = M. \)
Proof: By the cocycle property,

\[ \beta(\hat{s}, g)\Lambda(k)\beta(\hat{s}, g)^{-1} = \Lambda(g \cdot k) \]

for each \( \hat{s}, k, g \).

So \( \Lambda(K) \) is normalized by the group generated by

\( \{\Lambda(k)\beta(\hat{s}, g)|(\hat{s}, k, g) \in \hat{S} \times K \times_* G\} \).

Since \( \beta \) has dense-range \( \Lambda(K) \) is a closed normal subgroup of \( M \).

Let \( p : S \to \hat{S} \) and \( p_1 : \hat{S} \times_* M_0 \setminus M \to \hat{S} \) be canonical maps.

Since \( S \) and \( \hat{S} \times_* M_0 \setminus M \) are isomorphic extensions of \( \hat{S} \) then

there exist \( K \times_* G \)-invariant conull Borel subsets \( S_0 \) and \( (\hat{S} \times_* M_0 \setminus M)_0 \) of \( S \) and \( \hat{S} \times_* M_0 \setminus M \), and a \( K \times_* G \)-equivariant Borel isomorphism

\( \phi : S_0 \to (\hat{S} \times_* M_0 \setminus M)_0 \) such that \( p_1 \circ \phi = p \).

Since \( (\hat{S} \times_* M_0 \setminus M)_0 \) is conull then

\[ (\hat{S} \times_* M_0 \setminus M)_0 \hat{\phi} = \{ x \in M_0 \setminus M \mid (\hat{s}, x) \in (\hat{S} \times_* M_0 \setminus M)_0 \} \]

is conull in \( M_0 \setminus M \) \( \bar{\mu} \) - a.e. \( \hat{s} \in \hat{S} \), where \( p_\star \mu = \bar{\mu} \) (as before).

Choose an \( s \in S_0 \) satisfying (*)

Then \( \phi \) maps \( s \cdot K \) bijectively onto \( \{ \hat{s} \} \times (\hat{S} \times_* M_0 \setminus M)_0 \hat{\phi} \) (since \( p_1 \circ \phi = p \).

But \( \phi(s \cdot K) = \{ \hat{s} \} \times \{ M_0 m \Lambda(K) \} \) for some \( m \in M \).

Hence \( M_0 m \Lambda(K) = (\hat{S} \times_* M_0 \setminus M)_0 \hat{\phi} \).

But \( M_0 \Lambda(K)m = M_0 m \Lambda(K) \). Hence \( M_0 \Lambda(K) \) is conull in \( M_0 \setminus M \).

So \( M_0 \Lambda(K) \) is conull in \( M \). But \( M_0 \Lambda(K) \) is a group itself, thus \( M_0 \Lambda(K) = M \).
CHAPTER 4
AMENABLE AND RELATIVELY WEAKLY-MIXING ACTIONS

In this chapter we discuss amenable actions and weakly mixing actions for semi-direct product groups $K \times_s G$. Let $E$ be a separable Banach space and let $\pi : G \longrightarrow \text{Iso}(E)$ be a continuous homomorphism where $\text{Iso}(E)$ is the group of isometric automorphisms of $E$ with the strong operator topology. $\text{Iso}(E)$ is a separable metric group.

Let $A$ be a non-empty compact convex subset of the unit ball $E_1^*$ of $E^*$ with the weak $^*$-topology such that $\pi^*(g)A \subseteq A$ for each $g \in G$ where $\pi^*$ is the adjoint representation of $\pi$ i.e. $\pi^*(g) = \pi(g^{-1})^*$. Then $A$ is a continuous $G$-space under the action $(g,a) \longrightarrow \pi^*(g)(a)$. This $G$-space is called an affine $G$-space.

A group $G$ is called amenable if there exist a fixed point in every affine $G$-space (Equivalently $G$ is an amenable group iff for each continuous $G$-action on a compact metric space $X$ there is a $G$-invariant probability measure on $X$). The notion of an amenable action was introduced by R. Zimmer in his paper "Amenable Ergodic Group Actions and an Application to Poisson Boundaries of Random Walks."

Let $\alpha : S \times G \longrightarrow \text{Iso}(E)$ be a cocycle where $S$ is a $G$-space.

For each $s \in S$, let $A_s$ a non-empty compact convex subset of the unit ball $E_1^*$ such that $\alpha^*(s,g)A_{s,g} = A_s$ a.e. $s$ for each $g \in G$ where $\alpha^*(s,g) = (\alpha(s,g)^{-1})^*$, and $\{(s,x)|x \in A_s, s \in S\}$ is a Borel subset of $S \times E_1^*$.

Denote such a triple by $(E,\alpha,\{A_s\})$.

Definition 4.1: Let $(S,\mu)$ be a $G$-space. $G$ acts amenably on $S$
if for each triple \((E, \alpha, \{A_x\})\) there exists a Borel map \(\phi : S \rightarrow E_1^*\) such that 
\(\phi(s) \in A_x\) a.e. \(s\), and \(\alpha^*(s, g)\phi(s \cdot g) = \phi(s)\) a.e. \(s\) for each \(g \in G\). Such a \(\phi\) is called an \(\alpha\)-invariant section of \(\{A_x\}\).

We remark that actions of amenable groups are amenable actions.

**Lemma 4.2:** Let \((X, \mu)\) be a \(K \times_s G\)-space such that \(K\) acts trivially on \(X\). If \(G\) acts amenably on \(X\) then \(K \times_s G\) acts amenably on \(S\).

**Proof:** Let \(\alpha : X \times K \times_s G \rightarrow \text{Iso}(E)\) be a cocycle where \(E\) is a separable Banach space. We can assume \(\alpha\) is a strict cocycle (see Corollary 1.8 of R. Zimmer's paper "Amenable Ergodic Group Actions and an Application to Poisson Boundaries of Random Walks").

For each \(x \in X\), let \(A_x\) be a non-empty compact convex subset of the unit ball \(E_1^*\) such that \(\alpha^*(x, kg)A_{x \cdot g} = A_x\) a.e. \(x\) for each \((k, g) \in K \times_s G\), and \(\{(x, v) | x \in X, v \in A_x\}\) is a Borel subset of \(X \times E_1^*\).

Since \(K\) is a compact group (hence amenable) there exists a Borel map 
\(\phi : X \rightarrow E_1^*\) such that \(\phi(x) \in A_x\) a.e. \(x\), and \(\alpha^*(x, k)\phi(x) = \phi(x)\) a.e. \(x\) for each \(k \in K\). Hence \(\alpha^*(x, k)\phi(x) = \phi(x)\) a.e. \(k\) a.e. \(x \in X\).

But the set of all \(k\) such that \(\alpha^*(x, k)\phi(x) = \phi(x)\) is multiplicatively closed.

\(\alpha^*(x, k)\phi(x) = \phi(x)\) for each \(k \in K\) a.e. \(x \in X\).

Let \(X_0\) be a conull Borel subset of \(X\) such that for each \(x \in X_0\), \(\phi(x) \in A_x\) and \(\alpha^*(x, k)\phi(x) = \phi(x)\) for all \(k \in K\).

For each \(x \in X_0\), let \(A'_x = \{v \in A_x | \alpha^*(x, k)v = v \text{ for all } k \in K\}\).

For \(x \in (X - X_0)\), let \(A'_x = \{0\}\).

**Claim 4.3:** \(A'_x\) is a non-empty compact convex subset of \(E_1^*\) for each \(x \in X\).

And \(\{(x, v) | v \in A'_x, x \in X\}\) is a Borel subset of \(X \times E_1^*\).

**Proof:** Clearly \(A'_x\) is a non-empty convex subset of \(E_1^*\) for each \(x \in X\).
Fix \( x \in X_0 \). Let \( v_n \in A'_x \) and suppose \( v_n \) converges to \( v \) in \( E^*_1 \).

Then for all \((e, k) \in E \times K, (\alpha^*(x, k)v)(e) = v(\alpha(x, k^{-1})e)\)

\[
= \lim_n v_n(\alpha(x, k^{-1})(e))
\]

\[
= \lim_n (\alpha^*(x, k)v_n)(e)
\]

\[
= \lim_n v_n(e)
\]

\[
v(e).
\]

Then \( \alpha^*(x, k)v = v \) for all \( k \in K \).

Hence \( v \in A'_x \). Thus \( A'_x \) is compact in \( E^*_1 \) for all \( x \in X_0 \). And trivially

\( A'_x \) is compact in \( E^*_1 \) for all \( x \in X \).

Now \( \{(x, v)|x \in X, v \in A'_x \} = \{(x, v)|x \in X_0, v \in A'_x \} \cup ((X \setminus X_0) \times \{0\}) \).

Let \( \{k_i : i = 1, 2, \ldots \} \) be a countable dense set in \( K \).

Then \( \{(x, v)|x \in X_0, v \in A'_x \} = \{((x, v)|x \in X_0, v \in A_x, \alpha^*(x, k)v = v \ for \ all \ k \in K \}
\]

\[
= \cap_{i=1}^{\infty} \{(x, v)|x \in X_0, v \in A_x, \alpha^*(x, k)v = v \}.
\]

Also \( \{(x, v)|x \in X_0, v \in A_x, \alpha^*(x, k_i)v = v \} = \)

\[
= \{(x, v)|x \in X, v \in A_x \} \cap (X_0 \times E^*_1) \cap \{(x, v)|x \in X, v \in A_x, \alpha^*(x, k)v = v \}.
\]

The mapping \(Iso(E) \times E^*_1 \to E^*_1\) given by \( (T, v) \to T^*(v) \) is continuous.

Therefore \( \{(x, v)|x \in X_0, v \in A_x, \alpha^*(x, k_i)v = v \} \) is a Borel subset of \( X \times E^*_1 \),

for each \( i=1, 2, \ldots \). Hence \( \{(x, v)|x \in X_0, v \in A'_x \} \) is a Borel subset of \( X \times E^*_1 \).

Thus \( \{(x, v)|x \in X, v \in A'_x \} \) is a Borel subset of \( X \times E^*_1 \).

Claim 4.4: \( \alpha^*(x, g)A'_{xzg} = A'_x \) a.e. \( x \) for each \( g \in G \).

**Proof**: Fix \( g \in G \). Choose a conull Borel set \( X'_0 \) in \( X \) such that \( \alpha^*(x, g)A'_{xzg} = A_x \)

for all \( x \in X'_0 \).

Let \( x \in X'_0 \cap X_0 \cap (X_0 \cdot g^{-1}) \) and let \( v \in A'_{xzg} \).

Then for all \( k \in K, \alpha^*(x, k)\alpha^*(x, g)v = \alpha^*(x, kg)v = \)
\[ \alpha^*(x, g) = \alpha^*(x, g^{-1}k)(v) = \alpha^*(x, g)v. \]

Also \( \alpha^*(x, g)v \in A_x \) since \( v \in A_{x,g} \) (since \( x \cdot g \in X_0 \)). Thus \( \alpha^*(x, g)v \in A_x' \).

So \( \alpha^*(x, g)A_{xg}^I \subseteq A_x' \) for all \( x \in X_0' \cap X_0 \cap (X_0 \cdot g^{-1}) \).

Hence \( \alpha^*(x, g)A_{xg}^I \subseteq A_x' \) a.e. \( x \) for each \( g \in G \).

It also follows that \( A_{xg}^I \subseteq \alpha^*(x \cdot g, g^{-1})A_x^I \) for all \( x \in X_0' \cap X_0 \cap (X_0 \cdot g^{-1}) \).

If \( x \in [X_0' \cap X_0 \cap (X_0 \cdot g^{-1})] \cap \{ [X_0' \cap X_0 \cap (X_0 \cdot g^{-1})] \cdot g \} \) then
\[ x \cdot g^{-1} \in X_0' \cap X_0 \cap (X_0 \cdot g^{-1}) \text{ and } A_x^I = A_{xg^{-1}}^I \subseteq \alpha^*(x \cdot g^{-1} \cdot g, g^{-1})A_{xg^{-1}}^I = \alpha^*(x, g^{-1})A_{xg^{-1}}^I. \]

Then \( A_x^I \subseteq \alpha^*(x, g^{-1})A_{xg^{-1}}^I \) for each
\[ x \in [X_0' \cap X_0 \cap (X_0 \cdot g^{-1})] \cap \{ [X_0' \cap X_0 \cap (X_0 \cdot g^{-1})] \cdot g \}. \]

Hence \( A_x^I \subseteq \alpha^*(x, g^{-1})A_{xg^{-1}}^I \) a.e. \( x \) for each \( g \in G \).

\( A_x^I \subseteq \alpha^*(x, g)A_{xg}^I \) a.e. \( x \) for each \( g \in G \).

So \( \alpha^*(x, g)A_{xg}^I = A_x^I \) a.e. \( x \) for each \( g \in G \).

Since \( G \) acts amenably on \( X \) there exist a Borel map \( \psi : X \rightarrow E_1^* \) such that \( \psi(x) \in A_x^I \) a.e. \( x \), and \( \alpha^*(x, g)\psi(x \cdot g) = \psi(x) \) a.e. \( x \) for each \( g \in G \).

In particular \( \psi(x) \in A_x^I \) a.e. \( x \).

For each \( (k, g) \in K \times G \),
\[ \alpha^*(x, kg)\psi(x \cdot g) = \alpha^*(x, k)\alpha^*(x, g)\psi(x \cdot g) = \alpha^*(x, k)\psi(x) = \psi(x) \text{ a.e. } x. \]

So \( \psi \) is an \( \alpha \)-invariant section of \( \{ A_x : x \in X \} \).

Hence \( K \times G \) acts amenably on \( X \). 

**Theorem 4.5:** Let \( K \times G \) acting ergodically on \( S \) and suppose each \( G \)-orbit in \( \hat{K} \) is finite. Let \( \hat{S} = S/K \) be the space of \( K \)-orbits in \( S \). Then \( G \) acts amenably on \( S \) iff \( G \) acts amenably on \( \hat{S} \) iff \( K \times \beta M_0 \) acts amenably on \( S \).

**Proof:** By Corollary 2.6 we have an isomorphism \( S \simeq \hat{S} \times \beta M_0 \setminus M \).
of $K \times_s G$-spaces where $\hat{S} = S/K$, $\beta : \hat{S} \times (K \times_s G) \to M$ is a cocycle with Mackey dense-range into a compact group $M$, and $M_0$ is a closed subgroup of $M$. If $G$ acts amenably on $S$ then by Proposition 2.6 of R. Zimmer's paper "Amenable Ergodic Group Actions and an Application to Poisson Boundaries of Random Walks", $G$ acts amenably on $\hat{S}$. Assume $G$ acts amenably on $\hat{S}$. Then by Lemma 4.2 $K \times_s G$ acts amenably on $\hat{S}$. Since $S$ in an extension of $\hat{S}$ and by Theorem 2.4 of R. Zimmer's paper "Amenable Ergodic Group Actions and an Application to Poisson Boundaries of Random Walks", then $K \times_s G$ acts amenably on $S$. Conversely let $K \times_s G$ act amenably on $S$. Then $K \times_s G$ acts amenably on $\hat{S}$ by the same Proposition 2.6 of R. Zimmer.

Claim 4.6: $G$ acts amenably on $\hat{S}$.

Proof: Let $\beta : \hat{S} \times G \to Iso(E)$ be a cocycle where $E$ is a separable Banach space and let $A_{\hat{s}}$ be a non-empty closed convex subset of $E^*_1$ for each $\hat{s} \in \hat{S}$ such that

$$\{(s,v) : \hat{s} \in \hat{S}, v \in A_{\hat{s}}\}$$

is a Borel subset of $\hat{S} \times E^*_1$ and $\beta^*(\hat{s},g)A_{\hat{s}o g} = A_{\hat{s}}$ a.e. $\hat{s}$ for each $g \in G$.

Define $\alpha : \hat{S} \times K \times_s G \to Iso(E)$ by $\alpha (\hat{s},(k,g)) = \beta(\hat{s},g)$.

Then $\alpha$ is a cocycle.

There exists $\psi : \hat{S} \to E^*_1$, an $\alpha$-invariant section of $\{A_{\hat{s}}\}_{\hat{s}}$ since $K \times_s G$ acts amenably on $\hat{S}$.

Hence $\beta^*(\hat{s},g)\psi(\hat{s} \circ g) = \alpha^*(\hat{s},(k,g))\psi(\hat{s} \circ (k,g)) = \psi(\hat{s})$ a.e. $\hat{s}$ for each $(k,g) \in K \times_s G$.

Since $\psi$ is a $\beta$-invariant section of $\{A_{\hat{s}}\}_{\hat{s}}$, then $G$ acts amenably on $\hat{S}$.

Then $G$ acts amenably on $S$ since $S$ is an extension of $\hat{S}$ (see Theorem 2.4 of
We turn to the concept of relatively weakly mixing. Let \( p : (X, \mu) \longrightarrow (Y, \nu) \) be a \( G \)-factor map of Lebesgue \( G \)-spaces (i.e. \( \mu \) and \( \nu \) are probability \( G \)-invariant measures).

Let \( \mu = \int \mu_y d\nu(y) \) be the disintegration of \( \mu \) over \( Y \).

Let \( X \times Y \times X = \{(x_1, x_2) \in X \times X | p(x_1) = p(x_2)\} \).

Define a measure \( (\mu \times_Y \mu) \) on \( X \times Y \times X \) by \( (\mu \times_Y \mu)(A) = \int_Y (\mu_y \times \mu_y)(A) d\nu(y) \)
where \( A \) is a Borel subset of \( X \times Y \times X \).

\( G \) acts on \( X \times Y \times X \) by \( (x_1, x_2, g) \cdot g = (x_1 \cdot g, x_2 \cdot g) \). Then \( \mu \times_Y \mu \) is a \( G \)-invariant probability measure on \( X \times Y \times X \).

We say \( X \) is \( G \)-relatively weakly mixing over \( Y \) if the action of \( G \) on \( (X \times Y \times X, \mu \times_Y \mu) \) is ergodic.

**Theorem 4.7:** Let \( p : (X, \mu) \longrightarrow (Y, \nu) \) be a \( K \times_s \ ) \)-factor map of ergodic Lebesgue \( K \times_s \ ) \)-spaces such that \( K \) acts trivially on \( Y \). If \( X \) is \( K \times_s \ ) \)-relatively mixing over \( Y \) then \( X \) is \( G \)-ergodic and \( X \) is \( G \)-relatively weakly mixing over \( Y \).

**Proof:** Let \( \mu = \int \mu_y d\nu(y) \) be the disintegration of \( \mu \) over \( Y \). Since \( \mu \) is \( K \times_s \ ) \)-invariant, we can assume each \( \mu_y \) is \( K \)-invariant.

Let \( S \) be the natural cocycle representation on the Hilbert bundle \( L^2(X, \mu_y) \rightarrow Y \).

For each \( y, k, g \),

\[ S(y, kg) : L^2(X, \mu_y) \longrightarrow L^2(X, \mu_y) \]
is a continuous linear operator such that

\[ S(y, kg) f(x) = f(x \cdot kg) \mu_y \text{-a.e. } x \text{ for each } f \in L^2(X, \mu_y), \]
a.e. \( y \) for each \( (k, g) \in K \times G \).

Now \( L^2(Y) = \int K \mathcal{C} d\nu(y) \subseteq \int K L^2(X, \mu_y) d\nu(y) \).

Let \( \mathcal{C}_y = \mathcal{C} \) for all \( y \).

Then \( \begin{array}{c}
\mathcal{C}_y \\
Y
\end{array} \rightarrow \begin{array}{c}
L^2(X, \mu_y) \\
Y
\end{array} \)
is a sub-Hilbert bundle of \( L^2(X, \mu_y) \).

Hence \( S(y, g) : \mathcal{C}_{y,g} \rightarrow \mathcal{C}_y \) is the identity for a.e. \( y \) for each \( g \in G \).

Thus \( S \) restricted to the bundle \( \begin{array}{c}
\mathcal{C}_y \\
Y
\end{array} \) is the identity cocycle representation.

**Claim 4.8:** \( S \) as a natural cocycle representation of \( (Y, G) \) contains no finite-dimensional subcocycle representation of \( G \) other than the identity.

**Proof:** Let \( \begin{array}{c}
V_y \\
Y
\end{array} \) be a non-zero \( G \)-invariant, finite-dimensional sub-Hilbert bundle of \( Y \) i.e. \( S(y, g)V_{y,g} \subseteq V_y \) a.e. \( y \) for each \( g \), and \( 0 \neq \dim(V_y) < \infty \) a.e. \( y \).

We will show \( V_y = \mathcal{C}_y \) a.e. \( y \).

For each \( y \in Y \), let \( \pi(y) \) be the unitary representation of \( K \) given by \( k \rightarrow S(y, k) \).

For each \( \gamma \in \hat{K} \), let \( P_y(\gamma) \) be the \( \gamma \)-primary projection for \( \pi(y) \).
Set \( W_\gamma(y) = \bigoplus_{g \in G} LS\{ \pi(y)(k)P_y(\gamma \cdot g)V_y | k \in K \} \) for each \( \gamma \in \hat{K} \) and \( y \in Y \).

As in the proof of Theorem 2.3, \( W_\gamma(y) \) is a finite-dimensional \( K \times \ast \ G \)-invariant sub-bundle of \( L^2(X, \mu_Y) \) for each \( \gamma \in \hat{K} \).

Let \( C \subseteq \hat{K} \) such that \( C \) meets each \( G \)-orbit in \( \hat{K} \) exactly once.

Suppose \( W_\gamma(y) = 0 \text{ a.e. } y \) for each \( \gamma \in \hat{K} \).

Then \( \bigoplus_{g \in G \setminus \mathcal{C}} P_y(\gamma \cdot g)V_y = 0 \text{ a.e. } y \) for each \( \gamma \in C \).

Hence \( \bigoplus_{\gamma \in C} \bigoplus_{g \in G \setminus \mathcal{C}} P_y(\gamma \cdot g)V_y = 0 \text{ a.e. } y \).

But \( V_y = \bigoplus_{\gamma \in C} \bigoplus_{g \in G \setminus \mathcal{C}} P_y(\gamma \cdot g)V_y \) for all \( y \in Y \).

Hence \( V_y = 0 \text{ a.e. } y \), a contradiction. Thus there exists a \( \gamma \in C \) such that \( W_\gamma(y) \neq 0 \text{ a.e. } y \).

Now \( X \) is \( K \times \ast \ G \)-relatively weakly over \( Y \) iff \( S \) contains no finite-dimensional cocycle representation of \( K \times \ast \ G \) other than the identity (see Corollary 7.10 of Zimmer’s “Ergodic Actions with Generalized Discrete Spectrum”).

Then there exists a unique \( \gamma_1 \in C \) such that \( W_{\gamma_1}(y) \neq 0 \text{ a.e. } y \).

For this unique \( \gamma_1 \in C \), \( W_{\gamma_1}(y) = \mathcal{C}_y \text{ a.e. } y \). And if \( \gamma \in C \) and \( \gamma \neq \gamma_1 \), then \( W_\gamma(y) = 0 \text{ a.e. } y \).

Fix \( y \in Y \) such that \( W_{\gamma_1}(y) = \mathcal{C}_y \) and \( W_\gamma(y) = 0 \text{ for each } \gamma \neq \gamma_1, \gamma \in C \).

Choose \( g_1 \in G \) such that \( W_{\gamma_1}(y) = LS\{ \pi(y)(k)P_y(\gamma_1 \cdot g_1)V_y | k \in K \} \)

If \( v_y \in V_y \) then \( \pi(y)(k)P_y(\gamma_1 \cdot g_1)v_y = \alpha_y \) for some \( \alpha_y \in \mathcal{C}_y \).

So \( P_y(\gamma_1 \cdot g_1)v_y = \pi(y)(k)^{-1}(\alpha_y) = \alpha_y \).

Also \( v_y = \bigoplus_{\gamma \in C} \bigoplus_{g \in G \setminus \mathcal{C}} P_y(\gamma \cdot g)v_y = \bigoplus_{g \in G \setminus \mathcal{C}} P_y(\gamma_1 \cdot g)v_y \).

So \( v_y = P_y(\gamma_1 \cdot g_1)v_y = \alpha_y \).
Hence \( v_y = \alpha_y \in \mathcal{C}_y \).

Hence \( V_y = \mathcal{C}_y \) whenever \( W_{\gamma_1}(y) = \mathcal{C}_y \) and \( W_\gamma(y) = 0 \) for \( \gamma \neq \gamma_1 \) and \( \gamma \in C \).

So \( V_y = \mathcal{C}_y \) a.e \( y \). This proves Claim 4.8.

In particular, \( S \) as a cocycle representation of \((Y,G)\) contains the identity cocycle representation exactly once. Hence \( X \) is \( G \)-ergodic.

So \( p : (X,\mu) \to (Y,\nu) \) is a G-factor map of ergodic Lebesgue G-spaces such that the natural cocycle representation \( S \) of \((Y,G)\) contains no finite-dimensional cocycle representation of \((Y,G)\) other than the identity.

Hence \( X \) is \( G \)-relatively weakly mixing over \( Y \).■


Zimmer, R., Ergodic Actions With Generalized Discrete Spectrum,  

Zimmer, R., Extensions of Ergodic Group Actions,  


VITA

I, Edgar N. Reyes, was born on August 29, 1958 in Manila, Philippines. I received my Bachelor of Science and my Master of Science degrees from the University of the Philippines in 1979 and 1981, respectively.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Edgar N. Reyes

Major Field: Mathematics

Title of Dissertation: Ergodic Actions of Semi-direct Product Groups

Approved:

[Signature]
Major Professor and Chairman

[Signature]
Dean of the Graduate School

EXAMINING COMMITTEE:

[Signature]
Howard Richardson

[Signature]
William A. Gehin

[Signature]
J. R. Dorroh

[Signature]
R. J. Koch

[Signature]
Shappe

Date of Examination:

July 15, 1988