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Option Volatility & Arbitrage Opportunities

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Abstract

This paper develops several methods to estimate a future volatility of a stock in order to correctly price corresponding stock options. The pricing model known as Black-Scholes-Merton is presented with a constant volatility parameter and compares it to stochastic volatility models. It mathematically describes the probability distribution of the underlying stock price changes implied by the models and the consequences. Arbitrage opportunities between stock options of various maturities or strike prices are explained from the volatility smile and volatility term structure.

Chapter 1
Introduction

Derivative instruments have existed since the eighteenth century in the United States of America but until recently, derivative contracts were not standardized, the markets were not regulated, and market participants had trouble buying and selling the contracts from the over-the-counter market on another more active secondary market.

The CBOE, Chicago Board Options Exchange, in 1973 decided to create an exchange for derivative options where the financial instruments started being traded with standardized strike prices and maturity dates. Derivatives traders could now engage or close out an option position at anytime, with lower transaction fees than before, to take their profits or limit their losses. The Options Clearing Corporation (OCC), acting as a third party in a derivative transaction, was created the same year in order to provide stability and financial integrity in the marketplace by ensuring that the obligations of the contracts are fulfilled. The exchange, providing liquidity and more opportunities for derivatives traders became a real success and similar markets quickly appeared around the world.

The multiplication of option exchanges and the increasing volatility in the financial markets encouraged traders to manage their portfolio's risks. Interest rate risks, currency risks, and movements in financial instrument prices became a real concern to financial institutions, hedge funds, proprietary trading firms and other market participants, who still nowadays, spend a major time of their trading activities in managing their different market risk exposures. Risk management has become very sophisticated since then due to the apparition of the derivative products. A financial asset is called a derivative if its value is dependent on another financial asset's price, which is called the underlying asset.

Many derivative products appeared to hedge the rising volatilities in the underlying markets. Financial institutions and market participants came up with new derivative instruments like futures and forward contracts, swaps, warrants, options on stocks, on futures, on currencies, on stock indices, on commodities, and even more sophisticated contracts like credit default swaps (CDS), collateralized debt obligations (CDO), mortgage-backed securities (MBS)... These financial innovations were an answer to the increasing risks in the underlying markets and offer market participants to reach their risk/profit ratio goals at lower costs.

This paper is focusing on one main derivative instrument: the stock options. Pricing stock options has become an important problem in the financial sphere and has attracted a lot of economists and mathematicians.

In 1900, Louis Bachelier suggested a stock option pricing formula in "the theory of speculation", the bases of which is the true assumption that stochastic process is followed by the stock prices. He assumed that Brownian motion is followed by the underlying stock having a drift rate equal to zero. However, his pricing formula was not really representing very well the real world of the stock option derivative markets. His model was giving the possibility of negative stock prices and the derivative options could price more than the underlying asset which is in theory not possible (see section 2.c. Arbitrage opportunities).

Samuelson (1965) tried to improve Bachelier's model by considering a geometric Brownian motion and computing the price of an option as the expected payoff actualized at a continuous rate, the drift of the option.

Then, Myron S. Scholes, Robert C. Merton and Fischer Black used the Itô's lemma (a famous equation discovered by Kiyosi Itô in the 1940's and 1950's to model a continuous stochastic process) to solve the problem of option pricing on an underlying stock which is non-dividend paying. The pricing formula presented by Black-Scholes-Merton in 1973, drastically altered the financial world and is still nowadays used by traders and financial

institutions to correctly price a stock option derivative contract.

Based on more or less realistic assumptions (see section 4.p. The Black-Scholes-Merton model's assumption), the model known as Black-Scholes-Merton has become popular due to the quick and easy computations that the model implies to generate a theoretical stock option price. The famous formula also introduced stochastic computations to the financial sphere to price derivative products. Nowadays, market participants use the model to value a stock option but also to determine the instantaneous implied volatility now, at a given point in time t, pricing an option at time $t + \Delta t$.

Devising a model to price derivative contracts has become a major challenge of the financial world and market participants are still waiting on the next pricing model that will be as easy as the Black-Scholes-Merton one but even more precise to the reality of the markets. In the past few years, many mathematicians and physicians have worked to improve the Black-Scholes-Merton model to make it more realistic to the financial markets. Underlying stock market's jump-diffusion process, stochastic volatility of the underlying asset, a risk-free stochastic interest rate and other assumptions have been studied to enhance the model but always resulting in more complex mathematics and longer computations. With many complex models, is it really necessary to develop even more complex models for the purpose of representing the financial market's trend in an improved way? In accordance with Stix and his book "A calculus of risk" (1998), 20% of losses that a trader suffers in the derivatives markets are due to a wrong estimation of parameters and pricing evaluation of the derivative products. This is the reason why mathematicians focus on finding a better model that will substitute the Black-Scholes-Merton one and will enable traders and financial institutions to lower the percentage of losses due to the mispricing estimations.

Despite the fact that the Black-Scholes-Merton model is the most popular nowadays, some market participants may use a different model or the Black-Scholes-Merton model maybe employed differently from what it was originally intended. This is the reason why it is very difficult to come up with a pricing model that can be used overtime. In fact, a valid pricing model today might not be working under the different market conditions tomorrow.

Derivatives traders and market participants are not all using the same model and have different estimations about future prices. This is why traders, market makers, financial institutions and other market participants have different bids and asks for financial products. The outcome of all the bids is the quoted market price whereas the ask prices entered in the book order of the derivative contracts.

The best model will be the one that best represents the behavior of the derivatives markets and will last over time under all market conditions. Theoretical prices can be calculated quickly and easily with the least assumptions to be made and parameters to estimate. The quickness of the computation is really important in finance, as many traders must take decisions in real time. This is the reason why the Black-Scholes-Merton model, based on some more or less realistic assumptions, is such a success. The precision of the model is also an important factor as the amount of money traded in the derivatives markets is consequent. According to a study by marketwatch.com, there is between $630 trillion and $1.2 quadrillion ($1,200,000,000,000,000,000) invested in derivatives alone between the over-the-counter market and worldwide exchanges.

To get a theoretical option price on a stock that pays no dividends, if the traders use the model of Black-Scholes-Merton, they must implements some inputs in the pricing formula: the stock price underlying the option, the time taken before the option's maturity date, option's strike price to value, an interest rate which is risk free and constant, and underlying stock's constant volatility over the option's life span. The first four parameters

can be observed in the financial markets and therefore are known from traders. The only unknown parameter of the underlying stock that a trader must estimate is the future (or expected) volatility that will occur until expiration of the option.

A constant volatility level is assumed by the Black-Scholes-Merton model over the life of the option which is obviously not a realistic assumption but enables the computations of the model to be quick and easy. However, the model fails to explain many characteristics of the option derivatives markets such as the volatility smile. In fact, there are good reasons to believe that the underlying stock's volatility is stochastic and derivatives markets to be more complex than Black, Scholes and Merton believe it to be.

Using a stochastic volatility model to price option contracts enables to have a more realistic representation of the underlying stock returns as a stochastic process involves the skewness and the kurtosis of a probability distribution. The lognormal distribution is assumed by Black-Scholes-Merton model for the underlying stock returns, which is not necessarily representing the real probability of the stock returns. In fact, the lognormal distribution has a tendency to underestimate a big positive or negative stock return while overestimating a small positive or negative return. However, with a stochastic volatility model, a trader can build a probability distribution for the underlying stock returns that better fits the reality of the underlying stock market as he can use an asymmetric distribution, with lower or higher tails and with a lower or a higher peak. The trader, when using a stochastic volatility model, is also able to explain the volatility smile, which is not the case for a trader using the Black-Scholes-Merton model.

Therefore, as compared to the Black-Scholes-Merton model, stochastic volatility models are more realistic as they describe the volatility smiles observed in the option derivatives markets in a more clear way. However, this type of models require more input estimations from a trader and therefore a higher chance for errors in pricing stock options. Stochastic volatility models as compared to the Black-Scholes-Merton model also require more as-

sumptions and are less time efficient.

This paper's objective is to be able to determine and forecast T-month option's future volatility with strike price K from historical data and implied volatilities to correctly value a stock option and then be able to find arbitrage opportunities between options with different strike prices and maturity dates.

This paper includes six chapters where chapter 1 and chapter 8 are the introduction and conclusion of the thesis respectively.

Chapter 2 is an introduction to the characteristics and the language of options. It describes briefly some inequalities that a call option price must respect at anytime otherwise arbitrage opportunities appear in the option derivatives markets.

Chapter 3 is a description of the different volatilities that a trader must take into consideration when pricing options and a discussion that to predict the future volatility which volatility is the most appropriate to use.

Chapter 4 consists in building the Black-Scholes-Merton differential equation through a procedure studied which is led by stock price. Black-Scholes-Merton pricing formula proof and the put-call parity is given. Moreover, the Black-Scholes-Merton model applying to only European calls and puts on stocks that pay no dividends, a discussion on how to modify the model to price American call furthermore put options on stocks paying dividends or not is included. Finally, the section ends up on a review of Black-Scholes-Merton model assumptions explaining why a stochastic volatility model might better reflect the reality of the option derivatives markets, since it is able to describe the volatility smile of the option contracts.

Chapter 5 talks about the Greek letters and how a trader can hedge his option portfolio to different market risks.

Chapter 6 is a presentation on how traders can estimate the future volatility from historical data and how they can determine the implied volatilities of the market. The volatility surface, volatility term structure, and volatility smile are then discussed in order to find arbitrage opportunities and determine the probability distributions of the underlying stock returns associated with the volatility smiles. Different schemes to name a few the EWMA model, the GARCH(1,1) model and ARCH(m) model to estimate the future volatility of an underlying stock are presented.

Finally, chapter 7 is a presentation of the stochastic volatility models and the Monte Carlo simulation.

Chapter 2
Characteristics of an option contract

2.1 Option contract specifications

An option can be defined as a financial derivative mechanism which gives the right to the investor to purchase or sell an underlying asset's particular quantity at a future date which is pre-determined (called as the option's maturity date) at a particular price (known as strike price or exercise price), but is not an obligation on the investor to do so. The buyer of the option, in other terms, who is long the option, has the right to exercise the option when the maturity date is reached only if this one enables the investor to make a profit by exercising it. If this is not the case and exercising the option results in a loss for the long position, the buyer of the option will let the financial instrument expire and will lose the amount of money spent to purchase the derivative (called the premium of an option).

Now the other party, the option's seller, who is short the option, will be forced to supply the underlying asset to the buyer if he decides to exercise the option at the maturity date. Thus, in case of a European option, the short position is totally dependent of the decision of the buyer and needs to be prepared to any possible scenarios at expiration of the derivative contract.

In American option case, the long position at any time earlier than the maturity date can exercise the option hence giving buyer more chances to show a profit on the contract overtime. This implies that at any time in the option's life the short position should be ready to hand over the underlying asset as soon as it is decided by the long position to exercise the contract. Therefore, an American option as compared to European option

grants more power to the long position and less to the short position where to exercise the option or not is decided by the buyer only at the contract's pre-determined maturity date. Hence, the premium of a European option should be lower than the premium of an American option having similar features.

Nowadays, majority of the options traded over-the-counter are European options however, majority of the options that are traded publicly on exchanges are American options. Throughout this paper, we will implicitly consider all options to be European unless said otherwise.

There exists two types of options: a put option and a call option. A put option does not oblige but gives the right to sell a particular quantity of underlying asset at the maturity date at the exercise price. A call option does not oblige but gives the right to buy a particular quantity of the underlying asset at the strike price at the maturity date. The underlying asset of an option can vary. This can be a simple stock of a firm traded on an exchange, an index, a currency, a commodity or a future.

Options have become a popular financial instrument preventing investors from further losses in order to hedge a stock portfolio or by using these contracts as a real speculating investment by adding some leverage to an investment. No matter what way the investor purchases an option for, options are great tools to manage the risk of a portfolio and any investor must consider these types of contracts to get the best risk/reward ratio possible on an investment strategy.

2.2 Factors affecting the premium of an option

The premium of an option is dependent upon the volatility and the price of the underlying asset, the maturity date, the strike price and the risk-free interest rate which is equivalent to the similar time duration of the option's life. For example, an investor will

use the 90-day Treasury bill rate in order to determine the premium of a 3-month call/put option.

2.2.1 Underlying asset price

An option's premium can be seen as follow:

$$Premium\ option = Intrinsic\ value + Time\ value$$

The intrinsic value is the value of an option if the contract would be exercised immediately. The option's intrinsic value cannot be negative it can only be equal to zero or positive. Such as, a December 90 call for any prices of the underlying asset strictly above $90 will have a positive intrinsic value and an intrinsic value equal to zero for any prices equal to $90 or less. If we look at a June $50 put with the Facebook stock as the underlying asset, the option will have a positive intrinsic value for any Facebook stock prices strictly under $50, and equal to 0 if the stock price of the social media is $50 or more.

A call option and a put option with a positive intrinsic value are considered in-the-money. A put and a call are considered out-of-the-money if they have intrinsic value equals zero. A put and a call option where the price of the underlying asset is equal to the strike price of the derivative contract are considered at-the-money.

More generally, it is possible to come up with a formula for a put and a call intrinsic value.

$$\text{intrinsic value for a put option} : [K - S]_+$$

$$\text{intrinsic value for a call option} : [S - K]_+$$

where S represents the underlying stock price and K represents the option's strike price. The graphs below show the intrinsic value (or payoff) and the profit & loss ("P&L") curves for a long call, long put, short call and a short put option.

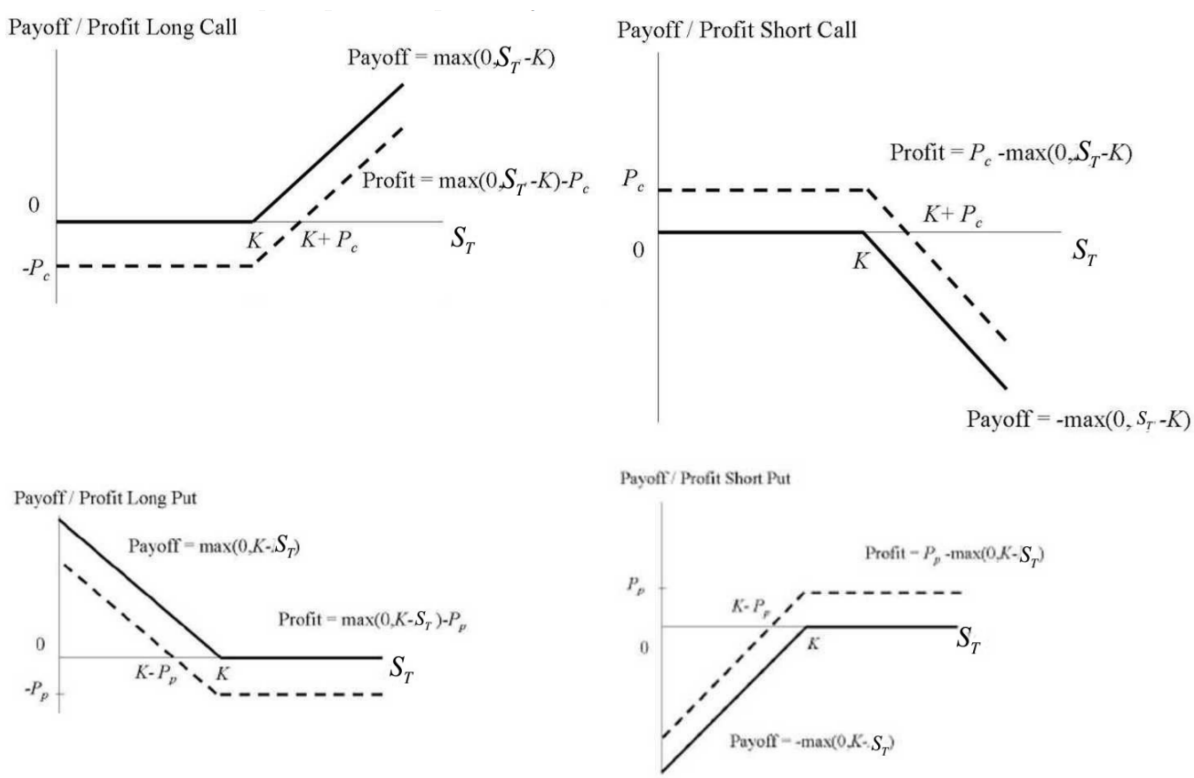

Figure 2.1: Intrinsic value and P&L from an option position at expiration

An option's time value is characterized by the time quantity left earlier than option's expiration and the probability that the contract will be in-the-money at maturity. The investor is willing to pay an additional cost to hold the contract as far there is a chance for him to make a profit by exercising the option. Thus, an option which is intensely out-of-the-money or intensely in-the-money will have a time value close to zero. The maximum time value of an option is achieved as at-the-money option. Indeed, there is a 50% chance to have a positive intrinsic value ($K > S$ for a put, $S > K$ for a call) and a 50% chance to have no intrinsic value ($K < S$ for a put, $S < K$ for a call). The time value, hence, of a financial derivative option is progressively growing as the option contract gets closer and closer at-the-money.

It is interesting to see that a call option which is in-the-money, the decreasing underlying asset price gets closer to the option's strike price, will have more time value but will also lose intrinsic value in the same time. On the other side, an in-the-money put option, the underlying asset price is increasing to get closer to the option's strike price, would have more time value but will lose intrinsic value in the same time. In fact, as the call or put option is moving more towards in-the-money, there is a bigger chance for the option to be exercised at maturity and therefore it is not the willingness of the investors to pay the option's time value where it would be more beneficial for them to straight away sell or purchase on exchange the underlying asset.

The price of an option, as observed from above, is a tradeoff involving time value and intrinsic value. A deeply in-the-money option will have as a result nearly zero time value and a positive intrinsic value. The option has little chance to be worthless before expiration and therefore does not hold any more time value. An deeply out-of-the-money option will have no intrinsic value and a premium which is low and equal to the time value. An at-the-money option premium will only hold an important time value.

2.2.2 Risk-free interest rate

As being long a call option involves paying a premium which is significantly less than buying directly the underlying asset, an investor can invest at the risk-free rate the difference amount, through the option's lifetime. This implies that if the risk-free rate is greater, it is better to buy the derivative option over the underlying asset and therefore the premium will increase.

Mathematically, the difference among the buying of the underlying asset and the actualized call option strike price $S - Ke^{-r(T-t)}$ will get bigger as the risk free rate will increase.

When buying a put option, the buyer of the option will get from the sale of the underlying asset the profits by using the put option at maturity instead of cashing out the sale of the underlying asset right away. A rational investor could invest the amount of the sale of the underlying asset at interest rate which is risk-free through the option's life which is not possible for a buyer of the derivative instrument as he will have to pay a premium and wait until maturity to get money back. Thus, a put option represented by $Ke^{-r(T-t)} - S$ will decrease in value as the risk-free rate will increase.

2.2.3 Lifetime of an option

When investing in option derivatives, the further the date of maturity of the option there is a greater chance that the long position will have to exercise his contract. In fact, the underlying asset price variations rises and the chances that the option will be exercised increases as the expiration date of the option is far in the future. The relationship between the maturity date and the price of a derivative contract is not linear. As time goes, the derivative instrument loses time value. At maturity, the option is only worth its intrinsic value or zero, but no time value is remaining in the option price.

2.2.4 Strike price of the option

If the strike price of a call option is low it has a better chance to be in-the-money at the time of maturity. Conversely, the higher the strike price, the lesser the probability for the call option to be exercised at expiration. However, for a put option, as the exercise price rises, the option has a greater probability to be in-the-money at expiration. The higher the strike price for a put option, the greater the chances that the put option will be exercised at expiration.

2.2.5 Volatility

The standard deviation of percentage change in the underlying asset prices calculated annually is called as the volatility. The long position of a call or a put is speculating on an increase or decrease in the underlying asset price. The higher premium is willingly paid by the trader while the volatility of the underlying asset is important as the option contract has better chances to be in-the-money at expiration. Thus, the more important the volatility, the higher the probability for the option to have some intrinsic value or being out-of-the-money, the higher the premium is.

The risk-free interest rate, the date of maturity, the strike price of an option and the underlying asset price are four factors that are used to determine the option's premium by the investors in the financial markets. Then again, underlying asset volatility is unknown by investors as it implies for a trader to compute the underlying asset's future volatility in the derivative contract's life. Unfortunately, no one is able to forecast and predict the future and especially a future volatility of an underlying asset during a certain period of time of an underlying asset. This means traders need to use a fundamental and technical approach by using charts and graphs to best estimate the underlying asset's future volatility.

It is very difficult to approximate the future volatility and requires traders to be as precise as possible. In fact, a wrong estimation of volatility will produce wrong option prices whose traders will not be able to take advantage of mispriced options.

With a total of five different factors, a trader is able to plug in the data in a theoretical pricing model and estimate the right price of an option. The fact that traders get different prices for the same option contract is due to the volatility of the underlying asset that each trader must estimate. If a trader believes the financial markets are strongly efficient and reflect all public, private and inside information, "the best predictor of the future volatility ought to be the implied volatility. Just how good a predictor of future

volatility is implied volatility?" (Natenberg, 1994, p295). But a trader who agrees with the strong efficiency of the markets is not able to find any investment opportunities as he believes that options are correctly priced. Only traders with different opinions on the underlying asset's volatility as compared to what the marketplace implies about volatility think options are mispriced. Traders are looking for mispriced opportunities between options to sell the overvalued options and buy the undervalued ones. Therefore, traders are trying to come up with better volatility estimates than the implied volatility to implement in the pricing models and decide a trading strategy according to the mispriced instruments.

2.3 Arbitrage opportunities

Arbitrage is an operation which involves no cash outlay by the trader and will result in a sure profit for him. Arbitrage is inconsistent with the hypothesis of a strong efficient market as no sure profit or abnormal return is possible in this type of market when no risk is involved. In strong efficient markets, news are incorporated into prices right away and therefore no profitable arbitrage strategies are possible as the assets are instantaneously correctly priced. However, nowadays, arbitrage is frequently used by derivatives traders while searching for mispriced derivative instruments related to underlying assets or mispriced derivative instruments between them.

To be able to find an arbitrage opportunity with a sure profit and no risk involved, an option must verify several conditions. The scenario where no dividend is paid by the underlying asset, a call option must satisfy all these conditions:

1. An option is a derivative financial instrument with limited risk. The maximum loss that a buyer of such instrument can endure is equal to the option value the investor paid for. Thus, an option premium cannot be negative: $c \geq 0$.

2. At maturity, if the option's strike price K is below the underlying asset price S_T, the trader exercises the call option and is worth an intrinsic value of $S_T - K$. Though, if the strike price of the contract at maturity is above or equal underlying asset price,

the option has no intrinsic value and is worth zero: $c = max(0, S_T - K)$.

3. For a given time period τ, a call premium with strike price K_1 must always be lower than a call premium with strike price K_2 if $K_1 > K_2 : c(S, \tau, K_1) < c(S, \tau, K_2)$.

4. A call option premium cannot be greater in worth as compared to the price of the underlying asset. If it was so then the trader would have bought the underlying asset and not the derivative contract: $c(S, \tau, K) \leq S$. As negative premium of an option is not possible, the option will also be worth nothing if the price of the underlying asset is zero: $c(0, \tau, K) = 0$.

5. $c(S, \tau, K) \geq S - Ke^{-r\tau}$where τ is the elapse time between today and the date of maturity of the option and $Ke^{-r\tau}$ is the actualized call option strike price. This inequality holds because a portfolio with a European call and a loan of amount K at maturity, the call strike price, is no less than the portfolio amount with one underlying asset at maturity.

These conditions also hold for underlying assets paying dividends but one modification must be done on the last condition number 5. Paying a dividend usually implies a decrease in price in the underlying asset by the same amount (at least in a strongly efficient market). If we suppose that investors know when the dividends are being paid, the last condition becomes $c(S, \tau, K) \geq S - Ke^{-r\tau} - Div$. This inequality can be proved as follow: consider 2 portfolios, one with a call c and a loan of amount K at maturity, the other with one underlying asset and a loan equal to the dividend amount. At any time between the present moment and call option expiry date, the call option with the loan of amount K at option expiration date portfolio is worth the same or more than the second portfolio.

Chapter 3
Volatility

Volatility can be described as the percentage changes underlying contract prices. The parameter for the volatility can be seen as a mix of negative and positive rates of return. In mathematical terms, the volatility of an instrument is characterized by the standard deviation and gives no information about the direction of the movement.

When computing the theoretical option value using the Black-Scholes-Merton model, we take volatility as an input which is unknown and therefore traders must estimate this data. Volatility is an important factor, if not the most important, in price determination for the option. The well-known theoretical pricing model, Black-Scholes-Merton, takes volatility as a constant which implies the underlying asset will have the same volatility during the entire life of the option but in reality, it is more appropriate to describe volatility as a stochastic process. The constant volatility assumption over the life of the option is for simplicity and quick computation of theoretical values.

To best estimate the volatility of an option, traders use the historical volatility and the implied volatility given by the marketplace. The volatility factor is the hardest factor to estimate but the more crucial in evaluating option theoretical values, as these derivative contracts change quickly with changes in volatility. Commonly this is known as the volatility input that each trader uses in the model, which gives the trader an idea if the option is overvalued or undervalued.

A trader who anticipates a higher volatility than the implied volatility will buy the options that he believes are undervalued, hoping the marketplace will match the trader's

volatility estimate before the expiration of the options. The trader will therefore profit by selling back the options once the implied volatilities are equal to his estimations. In the case where a trader anticipates a lower volatility than the implied volatility, the trader will find out that options are overvalued and will start selling options. The trader will realize a profit if the implied volatility starts declining to reach out the trader's volatility estimate before the expiration of the options. This is why nowadays, traders in the pit at the Chicago Board Options Exchange (CBOE) do not talk about buying or selling options anymore but buying or selling volatility instead. Thus, traders find overvalued and undervalued options as they come up with different volatility estimations than the ones in the marketplace to then buy and sell the volatility. The more precise the estimation of the volatility, the better the evaluation of the theoretical value of the option.

3.1 The different types of volatilities to consider

3.1.1 Future volatility

The volatility that each trader is interested in knowing since this information helps the traders to decipher and get a theoretical price for the option is the future volatility. The future volatility represents the price changes of the underlying contract in the future till the option expiration date is reached. Future volatility is not known by any trader but they all try to best estimate it.

3.1.2 Historical volatility

Derivatives traders also commonly call historical volatility as realized volatility. When traders try to best guess the future volatility, one possibility is to refer to historical data and see the different price changes of the underlying market in the past. No trader can assume that past volatility will be identically reproduced in the future but this is a great way to start estimating the future volatility. In fact, suppose volatility of an underlying contract for the last five years, has been in the range [10%, 30%] constantly. An estimation

for the future volatility of the underlying contract out of this range would hardly make sense. An approximation in the past years range of 10% and 30% is more likely to be true even though it has the exact same probability for the future volatility to be within the defined interval or not.

To calculate the historical volatility of an underlying contract, various ways can be employed but each way depends on two parameters: the past time span over which to compute the volatility and the elapsed time among consecutive movements in price. For example, a trader can analyze the 10-day underlying asset's return during a period of five years or a 6-month underlying asset's return during a period of ten years.

Longer periods give an average of the past volatility while shorter periods reveal unusual and extreme characteristics about the past volatility. Thus, to fully examine the past volatility, a trader must compare wide variety of scenarios with different time periods and intervals between data. Surprisingly, the interval chosen by the trader does not seem to impact a lot the final result about the historical volatility. In fact,"a contract which is volatile from day to day is likely to be equally volatile from week to week, or month to month" (Natenberg, 1994, p70) but at the end the different data seem to exhibit one general volatility trend and level.

The historical volatility is given by the standard deviation of the return of the underlying asset prior to the option contract issuance. Suppose we omit the dividends, the underlying asset's return can be computed as follow

$$R_T = \frac{S_T - S_t}{S_t} = \frac{dS}{S}$$

where $t < T$. The instantaneous asset's return follows a Brownian geometric movement

such as

$$\frac{dS}{S} = \mu dt + \sigma dv$$

where dv is a Wiener process. This means the variable dv is following a normal law with variance dt and mean equal to zero. This is equivalent to say that R_T is following a lognormal law or $\ln(1 + R_T)$ follows a normal law. A major assumption of the Black-Scholes-Merton model is this, where “price percentage changes are assumed to be normally distributed, the continuous compounding of these price changes will cause the prices at maturity to be lognormally distributed” (Natenberg, 1994, p62-63). Further explanations about the Brownian motion of the underlying asset instantaneous return which is geometric in nature and the lognormally property are given in section 3.b. The process for a stock price, and 3.d. The lognormal properties of stock prices.

This gives the reason why lesser strike price option is cheaper than an option with a higher strike price, where both strike prices are an equal amount away from the underlying instrument price. For example, assume trading is done at $100 for an underlying contract and presume the prices of the underlying contract are normally distributed. As a result, the 90 put and 110 call should have theoretical values which are equal as the two are being out-of-the-money by 10%. Yet, with a lognormal distribution assumption in the Black-Scholes-Merton model, the 90 put will be less expensive as compared to the 110 call. From this observation it can be concluded that a higher upside price movement is allowed by the lognormal distribution as compared to the downside price movement hence there is better chance for 110 call to be in-the-money than the 90 put.

The historical volatility in reality, is not constant as the Black-Scholes-Merton model supposes so, but rather follows a stochastic process. In fact, the historical volatilities of one underlying asset are going to be more or less important at moment t, varying over time, but the Black-Scholes-Merton model will use one volatility constant which best describes the historical movements of the underlying asset.

3.1.3 Forecast volatility

Forecast volatility is an attempt to estimate for an underlying contract the future volatility in the options life span. Common options have expiration dates in 3, 6 or 9 months. Some services, technicians, proprietary trading firms or hedge funds try to forecast volatility for periods similar to common options in the marketplace.

3.1.4 Implied volatility

Establishing upon underlying contract historical volatility and then computing the future volatility has been criticized and every investor is aware that past figures do not predict future figures. This is why traders must use the historical volatility of an underlying contract as a constructive technical indicator. In fact, traders do not blindly and exclusively use this tool to make trade decisions but they do use it to help them make trade decisions combined with other factors, such as implied volatility and news catalyst.

Traders analyze option prices in the marketplace and deduce the volatility which is employed by the theoretical pricing model known as Black-Scholes-Merton by iterating the pricing formula. The volatility determined by the marketplace is called implied volatility. In other words "the implied volatility is the volatility we must feed into our theoretical pricing model to yield a theoretical value identical to the price of the option in the marketplace" (Natenberg, 1994, p72). As the implied volatility is based on the present value of the market, it is a better estimator than the historical volatility. The main idea is that the base for the implied volatility is the options present prices which include future events. It should be noted that implied volatility is subject to change overtime as option prices in the marketplace and market conditions are constantly changing.

Implied volatility has a major drawback that it assumes a diffusion process of the underlying contract. This diffusion process for the underlying asset, in the case of the Black-Scholes-Merton model, is Brownian geometric which does not really well represent the reality. A diffusion process implies price changes are continuous and smooth, with no gaps between consecutive prices. A good example to describe a diffusion process is the temperatures in one location. The temperature might increase from 70F to 73F but for a very small period of time in the middle, the temperature reached 71F and 72F. However, is a diffusion process really representing well price changes in the marketplace? In fact, prices might also follow a jump process. This means prices can jump to new prices overtime without reaching intermediate values.

Most of the theoretical pricing models, whose Black-Scholes-Merton one, assume that prices follow a diffusion process which can be pretty accurate when we look at the trading sessions, prices change with no gaps. However, between two trading sessions, it is common that an underlying contract closes at a particular price and opens at another price the next morning. Even during the trading sessions, an illiquid underlying contract can be subject to a jump process. Thus, the diffusion process assumed by the models make them convenient and easy to use but these models do not truly correspond to the reality as financial markets rather follow a jump-diffusion process.

Few pricing models have been developed assuming a jump-diffusion process in the financial markets but they are not very used by traders due to the fact that they are mathematically more complex and require more inputs, which means more estimations to do and therefore more chances to make mistakes in evaluating the parameters.

Traders do rely on the implied volatility given by the marketplace and particularly pay more attention to the options which are at-the-money as they respond more quickly to volatility changes than any other options (highest vegas are owned by at-the-money options) hence they are better predictor of the future volatility. This method using implied

volatilities to forecast volatilities of underlying contracts is the most commonly used among traders. They use numerical schemes and programs (like the Newton-Rhapson algorithm or by interpolation) to revert the pricing models and deduce implied volatilities and predict future volatilities.

In the literal sense premium is the option price, traders among each other use the word "premium" for referring to implied volatility. This makes more sense as the option price is primarily dependent on the underlying contract volatility characteristics. Thus, traders might say premiums are high meaning the implied volatility is high which is equivalent to a high price for the corresponding option.

For example, suppose a 105 call with \$0.96 as the theoretical value and a price of \$1.34 in the marketplace. The basis for the theoretical value is 16% volatility (the volatility estimate by the trader) whereas the basis for the price is 18.5% volatility (which is the implied volatility). It will be mentioned by the traders that the option being overpriced in terms of volatility rather than dollar value. Thus, traders rather see the 105 call option as being 2.5% overpriced than \$0.38 overpriced.

Once a trader is able to find out by iterating the Black-Scholes-Merton model the implied volatilities, a comparison can be drawn between the implied volatilities and the forecast volatilities to deduce if options are overvalued or undervalued. If implied volatilities are low with respect to forecast volatilities, the trader will prefer to buy options as they will appear to be underpriced. If implied volatilities are high with respect to forecast volatilities, the trader will prefer to sell options as they will look overvalued. Thus, the intelligent trader will take an appropriate trading strategy to play the overpriced and underpriced options by remaining delta neutral during the time of his investment.

3.1.5 Seasonal volatility

Another volatility kind that traders must take into consideration when investing: the seasonal volatility. This particularly affects commodity traders but also other investors dealing with stocks, futures and other instruments. For example, stocks and futures can be very sensitive to volatility factors coming from the season of companies' results. However, the most subject to seasonal volatility are commodities like soybean, gold, wheat, corn where price changes can vary by a lot due to extreme weather conditions. This is why some commodity traders assign higher volatilities during certain time of the year due to events sharply affecting commodity prices and rising price movements.

3.2 Volatility revisited

As we have seen several interpretations for volatility done by the traders, theoretically speaking, the option value primarily depends on the underlying contract volatility which occurs over the option's life span. Some important characteristics about volatility are worth to be notifying.

Volatility, just like prices of the underlying contract, can rise and fall. In fact, if we look at the volatility index (also called the fear index or the VIX) traded on the CBOE, Chicago Board Options Exchange, the VIX quoted 13.96 on April 26,2016, then reached 16.05 on May 4 2016, to then decreased to 13.63 on May 10, 2016. But unlike underlying contract prices, which are apparently not bounded and can move in either direction freely, it is likely to have "an equilibrium to which the volatility always returns" (Natenberg, 1994, p273). Indeed, volatility seems to be bounded and fluctuates into a defined interval. Thus, volatility always seems to reverse and erase previous rise or fall. Therefore, one way to estimate volatility of an underlying asset may be finding a volatility equilibrium which has variations below and above in comparable amounts around this equilibrium. In this case, we say that volatility is mean reverting.

This means that if the volatility is higher than the volatility average, the trader can safely assume that it will go down to its mean value, and if the volatility is below the mean volatility, it is safe to assume by the trader that it will go up to its mean value. Hence volatility fluctuates around its mean value.

Volatility also expresses some trend characteristics. For example, there was an upward trend in the VIX during the month of April 2016 or a downward trend during the period February-March 2016. Moreover, during these major trends, the volatility showed some small fluctuations since volatility decreased and increased in shorter time spans.

As volatility forecast remains a difficult exercise and are often incorrect, traders might approach the problem differently and take a more general approach. Rather than trying to figure out the correct future volatility, a trader may look at the volatility environment in the financial markets and try to find the right strategies corresponding to current volatility climate. For this, a trader must be able to answer several questions like what is the historical trend regarding volatility and historical implied volatility?, what is the stability of the volatility?, what is the underlying contract volatility long-term mean?

Consider for instance that a trader wants to invest in a 3-month to expiration option and tries to coordinate an appropriate volatility strategy. The trader will look at historical volatilities and historical implied volatilities over the past 3-month to determine a coherent strategy. If the historical volatility is declining and is higher as compared to the mean in the long-term, or there is a rise in the historical volatility and is under the long-term historical mean, the trader has some good basis to assume that the volatility eventually reach the long-term historical volatility mean at one point by either rising or falling. At the same time, if the historical implied volatility has a similar trend than the historical volatility, all indicators seem to perfectly indicate a path for the future volatility.

Scenarios are not always this simple in the real life. Historical volatility trends might be in contradiction with historical implied volatilities and this is pretty frequently that a

trader needs to face situations where the historical volatility is higher than the long-term historical mean and falling whereas the historical implied volatility is increasing, or historical volatility is higher (or lower) as compared to the long-term mean furthermore it is increasing (or decreasing). In these situations, the trader is unlikely to have a strong degree of confidence in his volatility estimation as some indicators point out one kind of strategy and the others point to a different type of strategy.

Let's look at a concrete example on how a trader could deal in this type of situation. Suppose a 6-week option and a 19-week option are available in the marketplace. Both the historical volatility and implied volatility for the 6-week option are well above the long-term mean volatility and seem to keep increasing. In the case of the 19-week option, the current implied volatility is higher as compared to the historical volatility which is greater than the mean for the long-term volatility. A trader might get confused by having these kind of contradictory signals as to whether one should sell or purchase the volatility for both options. Nevertheless, a much more acceptable risky strategy called time spread will be devised by the trader. In this case, there is more probability that the volatility for the 19-week option will revert to its mean than the 6-week option therefore the trader would rather short the volatility in the 19-week option than the short-term option. At the same time, the trader could long the 6-week option and would therefore be hedged along a rising volatility continuously over the next 6 weeks of the underlying contract.

By selling the long-term option and purchasing the short-term option, the trader is taking a position with acceptable risk characteristics. The trader's position will not be without risk as if historical volatility falls below its long-term mean or reaches it and implied volatility remains high, the position will show a substantial negative theoretical edge. However, the trader attempted to pick a right strategy that fits his market conditions with an acceptable margin for error and tried to get the best risk/theoretical edge ratio possible.

3.3 Implied vs historical volatility

Implied volatility reflects a consensus volatility among all market participants concerning the underlying contract future volatility in the option remaining life span. As traders expect more fluctuations in the underlying contract, the implied volatility rises; as traders expect fewer fluctuations in the underlying contract, the implied volatility decreases. Market traders assume that the historical volatility is a good data to predict what will happen in the future and therefore a changing historical volatility in the underlying asset will conduct traders to modify the implied volatility. "However, fluctuations in implied volatility are usually less than fluctuations in historical volatility" (Natenberg, 1994, p290).

Moreover, the further the expiration of the option, the more likely the underlying contract volatility will move back to its mean value. This explains why long-term options have forecast volatilities closer to their long-term historical volatility means than short-term options where the forecast volatilities can sometimes be quite different from the long-term mean depending on the market conditions. Thus, as long-term options have a stronger mean reverting characteristic of volatility compared to short-term options, the short-term options implied volatility will rise more as compared to the implied volatility of long-term options. In the same case, if historical volatility falls, the implied volatility of short-term options will decline more as compared to the long-term options implied volatility. Hence, it can be concluded that the main factor affecting long-term option implied volatility is the historical volatility of the underlying contract while significant events and news, affecting the underlying contract to become more volatile, will play a bigger role in the short-term options implied volatility. This is why for short-term options, it is not rare to see an implied volatility which is less correlated to the historical volatility. In some cases, the implied volatility and historical volatility can even give contradictory results to the market traders.

For example, if the marketplace is aware of chances that some important events will take place in the future, there can be an increase in the implied volatility even though historical volatility may have been low comparatively. This is a reason why traders see the implied volatility to drastically drop after an event has happened as all the uncertainty about the event has been removed from the market.

3.4 Implied vs future volatility

As truly believed by several traders, all the information available which affects the underlying asset value is reflected by the price and therefore traders believe that the implied volatility of options should be the best estimation of the future volatilities. But how good is the implied volatility of an option predicting the future volatility?

Once options have expired, traders may look back to the historical implied volatilities and compare them with the real volatilities that actually occurred over the option's life span to assess how accurately the implied volatility of an option estimates the future volatility. By doing this experience several times, traders have come up to a conclusion. When expiration of the option is relatively far in the future, the future volatility is much more stable than it is with few days remaining before expiration of the derivative contract. "This is logical as if we recall the mean reverting characteristics of volatility are much less certain over short periods of time than over long periods" (Natenberg, 1994, p295). In fact, with less days left until expiration a major change in the underlying contract will lead to volatility rising sharply while on the other hand, a quiet underlying asset which sits still in the last few days before expiration in life of an option will cause a sharp drop in volatility. Thus, a trader will act differently according to the amount of time remaining to the option's life.

The trader is aware that volatility is more stable when considering a longer period of time and therefore implied volatility is relatively stable. As expiration approaches, the

volatility of an underlying asset can become very unstable and the implied volatility can be subject to a lot of fluctuations. The main idea here is that market traders try to avoid short-term bad luck.

With a considerable time remaining to expiration, a trader is aware that many events which can affect the underlying asset's volatility differently will compensate between each other giving as a result a relatively stable implied volatility for an option expiring in a long period of time. However, with few days left before expiration, only a limited number of future events affect the implied volatility of the derivative contract giving the restricted number of events an important impact on the implied volatility of the option. One important negative event with few days left to expiration of the option will dangerously impact the trader's strategy as the volatility will move in the opposite direction of the trader's estimation: this is short-term bad luck.

3.5 Conclusion

A trader who has been involved in investing in stocks at the beginning of his career might wonder why the volatility factor is so important in option trading. He probably has been trading and betting on the direction of stock prices and indexes where volatility was not as much a consideration as in option trading. The stock trader is aware that he can pursue some directional strategies in the option market (by being long or short delta) and questions why traders bother with volatility rather than the direction of option premiums. This is due to the fact that traders find simpler to predict volatility rather than forecast market directions.

Moreover, volatility strategies can offer substantial profits and offer a better risk/reward ratio than directional strategies by reducing the trader's risk exposure.

Changing a volatility assumption can lead to drastic effects on options value. In fact, at-the-money options will suffer the highest variation in terms of dollar amount while the

most quickly responding to a change in volatility assumption in terms of percentage change will be the out-of-the-money options.

Given the importance of a good estimation of the volatility, traders spend a considerable amount of time determining the volatility. Gathering all information from the different volatilities previously seen, the intelligent trader will be able to take a smart decision on his final volatility estimation.

The trader will look for option trading strategies with high risk/reward ratio, and which do not result in a catastrophic loss if his volatility estimation is wrong. A trader must always consider the possibility of error and because predicting volatility remains a difficult task, the trader will pick strategies having margin for error higher as compared to rest. It is not possible for a trader to carry on for a long time if his positive theoretical edge disappears when the volatility turns out to be slightly different from his volatility estimations.

Finally, every market has its own volatility characteristics and a trader must not generalize any of these characteristics and apply them blindly to each market. An intelligent trader will mix the technical characteristics of volatility with each feature of volatility of a given market (foreign currencies, interest rates, commodities, futures, stocks) to take a final decision.

Chapter 4
The Black-Scholes-Merton model

4.1 Introduction to generalized Wiener processes

A stochastic process can be defined as changes in value of a variable in an unpredictable way over time. Stock prices in the financial markets are following discrete-variable, discrete-time stochastic processes as they can take discrete values only (since stock prices are given in cent multiples), and price changes can occur only during trading hours on the exchange. However, it can be very practical for many purposes to assume that stock prices follow a continuous-variable (within a defined interval the stock price may assume any value), continuous-time (the stock price can change in value at any time) stochastic process. In fact, this process can be useful to understand the pricing of options and other derivatives instruments.

A Markov process is a stochastic process type in which for future value prediction, only the current value of variable is required. The variable's past values and how the present value has emerged is deduced using the past values is not of interest. Markov process is followed by stock prices. For instance, let the price of Apple stock be $97. A stock trader who tries to predict the future direction for the Apple stock should not consider the past prices of the Apple stock that occurred in past one week, month or year but should rather focus only on one relevant information which is the current stock price $97.

As the stock trader cannot be certain about his future stock price predictions, he must express a probability percentage for each predicted possible outcome. As future stock prices are independent from past prices, the Markov process is the most realistic probability distribution for stock prices as it implies no relationship between past and future data.

A special type of Markov stochastic process is called Wiener process which has a variance rate of 1 per year and mean change of zero. A Wiener process is followed by variable v if:

- the change Δv during a small time period Δt is $\Delta v = \epsilon\sqrt{\Delta t}$
- the values of Δv for any two different intervals of time, Δt, are independent

Hence using the Wiener process definition, the variable Δv has a normal distribution with

$$mean\ of\ \Delta v = 0$$

$$variance\ of\ \Delta v = \Delta t$$

and Markov process is followed by it.

Take into consideration the change variable value v over long period of time $T = v(T) - v(0)$. It can be observed as the aggregate of variable value changes in N small intervals each having a length of Δt where

$$N = \frac{T}{\Delta t}$$

$$v(T) - v(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}$$

where ϵ_i (i=1, 2,..., N) are distributed ϕ(0,1) and are independent between each other (definition of a Wiener process). Obviously, the smaller Δt, the more precise the variable value v changes can be observed. Saying other way round, as Δt→0, the more intervals N we have and a better detailed representation of the changes in variable v value is achieved.

Hence, the time period $T = v(T) - v(0)$ has a normal distribution with

$$mean\ of\ [v(T) - v(0)] = 0$$

$$variance\ of\ [v(T) - v(0)] = N\Delta t = T$$

For example, suppose a variable v that follows a Wiener process. Assume that $t_0 = 25$ years. This means that at the year 1 end, the variable v value is distributed normally having a variance of 1 and a mean of 25. At 10 years end, it is still distributed normally with a variance of 10 and a mean of 25.

A stochastic process can be defined by a variance rate and a drift rate. The variance rate is the variance per unit of time whereas drift rate is the mean change per unit of time. A Wiener process $dv = adt$ in stochastic calculus is equivalent to $\Delta v = a\Delta t$ in the limit as Δt$\rightarrow$0. Explored uptill now, the Wiener process dv has a variance rate of 1 and a drift rate of zero. A zero drift rate implies that at any future time, the variable v expected value is equal to the current value which is zero. The meaning of variance rate equal 1 is that in a time interval T, the variance for the variable v is equivalent to T. A Wiener process, in general form, for a given variable x in terms of dv is presented as

$$dx = adt + bdv \tag{4.1}$$

where a and b are constants.

Let's focus on the equation (4.1) of Wiener process in a general form. The variable x has an expected drift rate given by a per unit of time is shown by the term adt. Thus, the variable x rises by the quantity aT in a period of time T. To add the effect of variations or noise to the trajectory of variable x, the bdv term is added. Thus, the noise present is

equal to the product of b and the Wiener process dv. Wiener process dv has the variance rate per unit time equals 1, hence, bdv has a variance rate b^2 per unit of time. Change in value of the variable x, is given by Δx, over a small interval of time Δt

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t}$$

where ϵ has a standard normal distribution and Δx has a normal distribution with

$$\Delta x \ mean = a\Delta t$$

$$\Delta x \ variance = b^2\Delta t$$

An Itô process is classified as a general form of Wiener process in where the parameters a and b are functions of time t and the underlying variable x. This implies that the Itô process variance rate and expected drift rate are subject to change over time. Mathematically, we can write an Itô process as below

$$dx = a(x,t)dt + b(x,t)dv$$

This Itô process can be classified as a Markov process since the changes at a given time t in variable x are dependent solely on the value of x at that particular time t, and does not use the value of x at previous times. In a small interval of time from t to $t + \Delta t$, the change in variable x is given from x to $x + \Delta x$ where

$$\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t} \tag{4.2}$$

4.2 The stock price process

A stock price does not follow the general form of Wiener process which has constant variance rate and expected drift rate as it fails to take into account stock prices' one

important element. The missing element is known as the expected required rate of return which is not dependent on price of stock.

For example, if expected return per year required by a trader is 12% when the stock prices \$15, the same rate of return will be required when the stock price increases to \$50 given the market conditions remain the same. Therefore, the assumption that expected drift rate is constant is not correct, a better assumption is that the expected return is constant which equals the expected drift rate ratio stock price.

Let S be the stock price at a given time t, the expected drift rate in S can be seen as μS where μ is the expected rate of return parameter which is constant. Hence, in a small time duration Δt, the increase expected in S equals $\mu S \Delta t$. Let's assume there is no uncertainty or variability about the expected return ($b = 0$), the generalized Wiener process equation becomes

$$\Delta S = \mu S \Delta t$$

In the limit, as Δt$\rightarrow$0

$$dS = \mu S dt$$

which is equivalent to

$$\frac{dS}{S} = \mu dt$$

Integrating above equation with limits given by time 0 and time T, the following equation is derived

$$S_T = S_0 e^{\mu T}$$

where the stock price at time 0 and time T are given by S_0 and S_T respectively. Thus, with no uncertainty, the continuously compounded growth rate for stock price is μ per unit of time. There is some uncertainty in reality. It is right to assume that the variation in percentage return over a short time duration Δt is same independent of the stock price.

Hence this implies that the uncertainty level is the same for the investor regarding the percentage return with stock price being either \$15 or \$50. Thus, the change in percentage return standard deviation should be proportionate to the stock price so that the model can be written as

$$dS = \mu S dt + \sigma S dv \tag{4.3}$$

or

$$\frac{dS}{S} = \mu dt + \sigma dv \tag{4.4}$$

where the variable μ represents the expected rate of return of annualized stock and the variable σ represents the stock price volatility.

As a rational investor, the more risk a trader is willing to take, the higher return he is expecting. Thus, the value μ should depend on the risk taken by the trader but also on the interest rates level in the country. As a matter of fact, the expected return increases on any given stock as the interest rates rise in the economy. The variable μ is equal to the r which is the risk-free rate in a world that is risk-neutral. Luckily, the derivative price which is dependent on a stock is not dependent on the parameter μ. Yet, the parameter σ, which represents the volatility of the stock price, plays an important role in price of any derivatives.

The model of the stock price behavior in equation (4.4), known as *geometric Brownian motion*, is a representation of the process of the stock price in the real world. The version of the model in discrete-time is

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t} \tag{4.5}$$

where the term ΔS represents the stock price S change over a short time duration Δt, ϵ follows a standard normal distribution, the parameter σ represents the stock price volatility and the parameter μ represents the expected rate of return of the stock per unit of time.

The left-hand side of the equation represents the stock's rate of return over a short time duration Δt. The expected rate of return is given by the term $\mu\Delta t$. The term $\sigma\epsilon\sqrt{\Delta t}$ is the return's stochastic component. The stochastic component variance, hence the rate of return of whole stock is $\sigma^2\Delta t$. Finally, the discrete-time version of the model shows that $\Delta S/S$ is distributed normally having a mean $\mu\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$

$$\frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma^2\Delta t) \tag{4.6}$$

For example, consider a non-dividend paying stock with continuously compounding expected return of 10% per year and a 20% volatility per year. The stock price process is given by

$$\frac{dS}{S} = 0.10dt + 0.20dv$$

If at a particular time stock price is given by S and stock price increase is given as ΔS in the next short time duration Δt

$$\frac{\Delta S}{S} = 0.10\Delta t + 0.20\epsilon\sqrt{\Delta t}$$

Suppose a time interval of 1 week such that $\Delta t = 0.0192$ years, we have

$$\Delta S = 0.00192S + 0.0277S\epsilon \tag{4.7}$$

4.3 Itô's lemma

The stock option price is a function of time and price of the underlying stock. Generally speaking, derivative instrument price is a function of time and a stochastic variables underlying contract.

Assume the variable x value follows an Itô process given in equation (4.2)

$$dx = a(x,t)dt + b(x,t)dv$$

In this equation, a and b are functions of x and t and dv is a Wiener process. The drift rate is a and the variance of the variable x is b^2. It is shown by Itô's lemma that a function G of variable x and time t follows the process

$$dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt + \frac{\partial G}{\partial x}bdv \tag{4.8}$$

where dv is a Wiener process similar to the Itô process in above equation. Hence an Itô process is followed by G with a drift rate as

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

and a rate of variance given by

$$(\frac{\partial G}{\partial x})^2 b^2$$

From Itô's lemma (4.8) and equation (4.3) of the stock price behavior model, we can deduce the process followed by the function G of S and t

$$dG = (\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial G}{\partial S}\sigma S dv \tag{4.9}$$

4.4 The lognormal property of stock prices

Let's use for the derivation of the process, the Itô's lemma followed by $\ln S$ where process in equation (4.3) is followed by S. Defining $G = \ln S$, the process followed by G of S and t becomes

$$dG = (\mu - \frac{\sigma^2}{2})dt + \sigma dv \tag{4.10}$$

Since in the above equation σ and μ are constant, a generalized Wiener process is followed by $G = \ln S$ with drift rate given as $\mu - \sigma^2/2$ and variance rate σ^2 both of which are constants. Between time 0 and time T, the change in G follows a normal distribution with

variance $\sigma^2 T$ and mean $(\mu - \sigma^2/2)T$. In other words, we have

$$\ln S_T - \ln S_0 \sim \phi[(\mu - \frac{\sigma^2}{2})T\ ,\ \sigma^2 T]$$

or

$$\ln \frac{S_T}{S_0} \sim \phi[(\mu - \frac{\sigma^2}{2})T\ ,\ \sigma^2 T] \tag{4.11}$$

which is equivalent to

$$\ln S_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T\ ,\ \sigma^2 T] \tag{4.12}$$

where stock prices at time 0 and T are given by S_0 and S_T respectively and normal distribution is represented by ϕ.

The equation (4.12) shows that $\ln S_T$ is normally distributed and therefore it implies that a price of stock at time T, given today's price, follows a lognormal distribution with $\sigma\sqrt{T}$ as the standard deviation. Based on the assumption of geometric Brownian motion, the Black-Scholes-Merton model supposes that the percentage changes are continuously compounding in stock price which are lognormally distributed.

Let's take as an example a stock having an initial price of \$30, a volatility rate of 23% annually and an expected return per annum of 18%. Hence the stock price S_T probability distribution in 6 months is given by equation (4.12)

$$\ln S_T \sim \phi[\ln 30 + (0.18 - 0.23^2/2) \times 0.5,\ 0.23^2 \times 0.5]$$

$$\ln S_T \sim \phi[3.478,\ 0.02645]$$

According to the normal distribution table, 1.96 is the approximate value of the 97.5 percentile and hence, the probability that a normal distribution variable has a value lying in 1.96 standard deviations of its mean is 95%

$$3.478 - 1.96\sqrt{0.02645} < \ln S_T < 3.478 + 1.96\sqrt{0.02645}$$

$$23.55 < S_T < 44.56$$

Hence the chances that the stock price will be in the interval [23.55 , 44.56] in a period of 6 months are 95%.

The stock price S_T follows a lognormal distribution and therefore can assume values from zero till infinity. The lognormal distribution unlike the normal distribution is positively skewed and the variance $Var(S_T)$ and the expected value $E(S_T)$ is as follow

$$E(S_T) = S_0 e^{\mu T}$$

$$Var(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

Suppose a stock price of $25, with volatility rate of 35% annually and expected rate of return 15% per annum. The expected price of stock and its variance in 1 year will equal

$$E(S_T) = 25e^{0.15\times 1} = \$29.05$$

$$Var(S_T) = 25^2 e^{2\times 0.15\times 1}(e^{0.35^2\times 1} - 1) = 109.95$$

Thus, the stock price standard deviation in 1 year is $\sqrt{109.95} = 10.49$.

4.5 The continuously compounded rate of return distribution

The stock prices which are lognormally distributed at time T can give useful information regarding the probability distribution of the rate of return, which is continuously compounded, earned on a stock from time 0 to time T. Let the continuously compounded annual rate of return on a stock between 0 and T be defined as x. We have

$$S_T = S_0 e^{xT}$$

such that

$$x = \frac{1}{T} \ln \frac{S_T}{S_0} \tag{4.13}$$

and therefore

$$x \sim \phi(\mu - \frac{\sigma^2}{2}, \ \frac{\sigma^2}{T}) \tag{4.14}$$

Hence the annualized rate of return continuously compounded follows a normal distribution with standard deviation $\sigma/\sqrt{T}$ and mean $\mu - \sigma^2/2$. As there is a rise in T, there is a decrease in standard deviation of x. In fact, the longer the time duration T, the more certain we are about the average rate of return per year for a stock price.

Let's take for example a stock with annual expected rate of return of 15% and annual volatility of 18%. The probability can be calculated for the average rate of return compounded continuously realized over a time period of 4 years is distributed normally with mean equal to

$$0.15 - \frac{0.18^2}{2} = 0.1338$$

or 13.38% per annum and a standard deviation given by

$$\sqrt{\frac{0.18^2}{4}} = 0.09$$

or 9% per annum. The chances that a variable following normal distribution lies within an interval of 1.96 standard deviations of its mean is 95%, we have

$$13.38 - 1.96 \times 9 < x < 13.38 + 1.96 \times 9$$

or

$$-4.26\% < x < +31.02\%$$

4.6 The annualized expected rate of return parameter μ

The expected return for a stock is given by the parameter μ and is required by an investor depends on the risk associated with the stock but also with the economy's interest rates. The greater the interest rates and the risk, the more will be the expected return. It can be seen that stock option value, if written in terms of the underlying contract value, is not dependent on the expected return μ at all. However, it is important to notice that $\mu\Delta t$ is the percentage change expected in the stock price over a small time duration Δt but μ is not the stock's expected return compounded continuously. The annualized continuously compounded return over time duration T is the given variable x in equation (4.13)

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

with expected value

$$E(x) = \mu - \frac{\sigma^2}{2}$$

The difference between the expected continuously compounded return x from μ is important. Let's take huge number of time duration Δt and let S_i be the stock price reached by the end of the ith interval. Now, the stock price returns mean in each interval is a near approximation to μ and this is why $\mu\Delta t$ is a good approximation of the arithmetic mean of the ratio given by $\Delta S_i/S_i$. However, over the whole period of time T, the expected return is stated using a compounding interval Δt which is close to $\mu - \sigma^2/2$ and not close to μ.

Mathematically, we have seen previously that the expected value of a lognormal distribution is

$$E(S_T) = S_0 e^{\mu T}$$

Taking logarithms, we get

$$\ln[E(S_T)] = \ln(S_0) + \mu T$$

Now, the fact that ln is a nonlinear function means that $\ln[E(S_T)] > E[\ln(S_T)]$ so this leads to

$$E[\ln S_T] - \ln S_0 < \mu T$$

as S_0 is known, $\ln S_0 = E[\ln S_0]$ and we obtain

$$E[\ln \frac{S_T}{S_0}] < \mu T$$

which can be written as

$$E(x) = \mu - \frac{\sigma^2}{2} < \mu$$

4.7 The volatility parameter σ

The uncertainty measure about the rates of return of the stock is the volatility parameter σ of a stock. Commonly volatility of a stock can be defined as standard deviation of the stock returns compounded continuously. From the equation (4.6) of the percentage changes in price of the stock over a small time duration, we have

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t,\ \sigma^2 \Delta t)$$

Thus, when Δt, the period of time is small, $\sigma^2 \Delta t$ nearly approximates the variance of the stock price percentage change over the time duration Δt. This implies that $\sigma\sqrt{\Delta t}$ nearly approximates the standard deviation of the stock price percentage change over time period Δt.

Suppose that stock price currently is \$40 and $\sigma = 20\%$ annually. The standard deviation of the stock price percentage change in 1 week is close to

$$20\sqrt{\frac{1}{52}} = 2.77\%$$

Move in the stock price by 1-standard deviation in 1 week is given as $40 \times 0.0277 = 1.1094$.

Future stock price uncertainty which is computed using the standard deviation will increase nearly by the same proportion as the square root of how long the time period in future we consider. For instance, the stock price standard deviation in 16 weeks is nearly equal to four times the standard deviation attained in 1 week.

4.8 The Black-Scholes-Merton differential equation: the idea underlying

The differential equation used in the Black-Scholes-Merton model is an equation which shows the relation between the derivative price and the non-dividend paying stock price.

Consider a portfolio which is riskless having a single position in a derivative contract and other single position in the underlying contract. If arbitrage opportunities are nonexistent, the portfolio returns should equal interest rate r which is risk-free. This portfolio is possible to set up since the same uncertainty affects the derivative price and the underlying contract: the stock price fluctuations. When an appropriate portfolio with a derivative contract and one stock is made, the loss or gain by the stock position is mostly counterbalanced by the loss or gain from the derivative position. By the trading day end, the portfolio value is known without any uncertainty.

Let's assume that a stock price change given by ΔS results in change in the price of a European call option given by Δc such that

$$\Delta c = 0.3\Delta S$$

This implies that the riskless portfolio must contain:

- 0.3 shares in long position
- In one European call option, a short position

Hence, if the price of stock of the option increases by 20 cents, the call option price will increase by 6 cents and therefore, the $0.3 \times 20 = 6$ cents gain on the shares will be compensated by the 6 cents loss on the short option position.

In the Black-Scholes-Merton model, the portfolio is only riskless for instantly small time duration and it should be frequently adjusted by traders.

For example, the relationship between Δc and ΔS is subject to change over time and in our example, the equation $\Delta c = 0.3\Delta S$ today might become $\Delta c = 0.5\Delta S$ in 2 weeks. This means that to retain a portfolio which is riskless, a trader must now own a short position in a European call option and a long position in 0.5 shares. Thus, for portfolio's every short call option, the trader must long an extra 0.2 shares.

Finally, one key element of the pricing formula of Black-Scholes-Merton model is that the return obtained is the interest rate which is risk-free got by riskless portfolio in a very short time duration.

4.9 The Black-Scholes-Merton differential equation: derivation

Take the process of the stock price process to follow equation (4.3) that we have developed earlier

$$dS = \mu S dt + \sigma S dv$$

Now, assuming that c is the call option or some other derivative instruments with underlying price S. The variable c can be defined as function of the time t and S represented in the equation (4.9)

$$dc = \left(\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial c}{\partial S}\sigma S dv \tag{4.15}$$

The discrete version of the two previous equations (4.3) and (4.15) are

$$\Delta S = \mu S \Delta t + \sigma S \Delta v \tag{4.16}$$

and

$$\Delta c = (\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2)\Delta t + \frac{\partial c}{\partial S}\sigma S \Delta v \tag{4.17}$$

where ΔS and Δc are the changes in variable S and c over a small time duration Δt.

Now, we have seen that the Wiener processes underlying c and S in the two previous equations, the $\Delta v (= \epsilon\sqrt{\Delta t})$, are the same and therefore can be eliminated by constructing the following portfolio

$$-1 : \mathit{derivative} \quad \text{and} \quad + \partial c/\partial S : \mathit{shares}$$

The trader holding the given portfolio is long $\partial c/\partial S$ of shares and short one derivative contract. Define Π as the value of the portfolio of the trader. Thus, we have

$$\Pi = -c + \frac{\partial c}{\partial S} S \tag{4.18}$$

The change $\Delta\Pi$ in portfolio value in the time period Δt is given by

$$\Delta\Pi = -\Delta c + \frac{\partial c}{\partial S}\Delta S \tag{4.19}$$

Now into the above equation (4.19) substitute equations (4.16) and (4.17) to get

$$\Delta\Pi = (-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2)\Delta t \tag{4.20}$$

The above equation does not contain a Wiener process Δv, the portfolio is riskless during

the period Δt. Moreover, one of the assumptions is that the riskless portfolio yields the return similar to other risk-free securities which are short-term. If this is not the case, possible arbitrage opportunities appear for the trader. Thus, we have

$$\Delta\Pi = r\Pi\Delta t \tag{4.21}$$

where risk-free interest rate is given by r.

Substituing into the above equation (4.21), the equations (4.18) and (4.20), we have

$$\left(\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(c - \frac{\partial c}{\partial S}S\right)\Delta t$$

so that

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} = rc \tag{4.22}$$

The above equation is the Black-Scholes-Merton differential equation and corresponding to the various derivatives it has many solutions that can be defined with underlying variable S. The solution of the Black-Scholes-Merton differential equation given as a function $c(S, t)$ represents the derivative instrument's theoretical price.

The portfolio used Black-Scholes-Merton differential equation derivation is not risk-free permanently. It is risk-free for an infinitesimally small time duration and needs to be rebalanced as S and t change. The comparative proportions of the underlying stock and the derivative contract need to be frequently adjusted to maintain the riskless portfolio.

4.10 Risk neutral valuation

In pricing of derivatives a very essential instrument is the risk-neutral valuation. In the Black-Scholes-Merton differential equation (4.22), no variables are impacted by investors showing the risk aversion. As a matter of fact, the variables that show in the differential

equation are the stock price volatility, the current stock price, the risk-free rate of interest and the time. None of these variables are dependent on risk preferences of investors.

If the variable μ, the annualized expected return of a stock which does depend on the level of risk taken, were to be appearing in the differential equation, the Black-Scholes-Merton model would be dependent on risk preferences of investors. The variable μ drops out while doing the derivation for the differential equation which brings up an important point. As the level of risk μ does not enter the equation, when evaluating c any combination of risk preferences can be used. Hence leading us to assume that investors are risk neutral.

Taking the scenario with risk neutral behavior of investors, the expected return yielded is the risk-free rate r on each and every investment assets. Furthermore, the future cash flows can be discounted at risk-free interest rate. Hence this assumption that the world is risk neutral reasonably simplifies the derivatives analysis.

Consider a derivative paying off at some time in future. In a world that is risk neutral, we can value the derivative as follow:

- The underlying asset expected rate of return is the risk-free rate r ($i.e.$ $\mu = r$)
- Use the derivative to calculate the expected payoff
- Expected payoff should be discounted at the risk-free interest rate

It should be noted here that evaluation that are risk-neutral is not a real tool for Black-Scholes-Merton differential equation solutions. Yet, the solutions are applicable in every kind of worlds, whether risk neutral or not.

When considering a risk-averse world, two steps out of three from the procedure to value a derivative change. Firstly, the underlying contract's expected return is unequal to the interest rate that is risk-free but equal to a rate r^* where $r^* > r$. Secondly, the discount rate is the expected rate of return r^* and no more the risk-free rate r. Apparently these

two changes exactly offset the effect of each other in this procedure.

4.11 The Black-Scholes-Merton pricing formulas

The most famous Black-Scholes-Merton differential equation solutions are the pricing formulas of Black-Scholes-Merton model for European put and call options. In the book "Pricing Derivatives: The Financial Concepts Underlying the Mathematics of Pricing Derivatives", Ambar Sengupta provides a clear and easy proof of the Black-Scholes-Merton pricing formulas.

Let S_T be the price of stock at time T, and let K be the European call option strike price. Define Q_T, the pricing probability measure when the numeraire is unit cash at time T, and Q_T^*, the pricing probability measure when the numeraire is the asset itself at time T. The maturity of a call option at time T has two possible situations:

- $S_T < K$: means out-of-the-money call and hence option is worthless
- $S_T > K$: the trader receives one underlying asset worth S_T by paying K

The forward priced in time T cash of the European call option is given by

$$E_{Q_T}(S_T)Q_T^*[S_T > K] - KQ_T[S_T > K]$$

or

$$S_0 e^{rT} Q_T^*[S_T > K] - KQ_T[S_T > K]$$

which is equivalent to

$$(FS_T) \qquad \underbrace{(Q_T^*[S_T > K])}_{\text{units of the underlying asset at time } T} \qquad -K\underbrace{(Q_T[S_T > K])}_{\text{in time } T \text{ cash}}$$

where (FS_T) is the underlying stock's forward price at a given time T, which is the call option expiration date.

The European call premium equivalent present value is therefore equal to

$$(FS_T)e^{-rT}Q_T^*[S_T > K] - Ke^{-rT}Q_T[S_T > K]$$

or

$$S_0Q_T^*[S_T > K] - Ke^{-rT}Q_T[S_T > K] \tag{4.23}$$

Assume the percentage price changes in the underlying asset are normally distributed. Thus, the continuous compounding of the percentage price changes cause the price S_T to be lognormally distributed with respect to the forward pricing measure Q_T. Then, we have

$$S_T = e^{Y_T} \tag{4.24}$$

where Y_T is a Gaussian random variable with respect to the measure Q_T. The random variable Y_T has an expected value equal to

$$E_{Q_T}(e^{Y_T}) = e^{E_{Q_T}(Y_T)+\frac{1}{2}\sigma^2_{Y_T}} \tag{4.25}$$

Substituting equation (4.24) into the above equation (4.25), we have

$$E_{Q_T}(S_T) = e^{E_{Q_T}(S_T)+\frac{1}{2}\sigma^2_{\ln S_T}}$$

where $\sigma^2_{\ln S_T} = E_{Q_T}[Y_T - E_{Q_T}(Y_T)]^2$ is the variance of the random variable Y_T. Now, $E_{Q_T}(S_T)$ represents the underlying asset forward price in time T cash. Thus, we have

$$E_{Q_T}(S_T) = e^{E_{Q_T}(Y_T)+\frac{1}{2}\sigma^2_{\ln S_T}}$$

or

$$\ln(FS_T) = E_{Q_T}(Y_T) + \frac{1}{2}\sigma^2_{\ln S_T}$$

which is equivalent to

$$E_{Q_T}(Y_T) = \ln(e^{rT}S_0) - \frac{1}{2}\sigma^2_{\ln S_T}$$

Suppose v_S^Y is the exchange rate between S and Y and consider a financial instrument I_B such that:

$$I_B = \begin{cases} \text{1 unit of numeraire S if event B occurs} \\ \text{0 if event B does not occur} \end{cases}$$

Thus, I_B is worth:

$$I_B = \begin{cases} Q_S(B) \text{ units of } S \\ Q_S(B) v_S^Y E_{Q_Y} \text{ units of } Y \end{cases}$$

From the point of view of Y, I_B is worth $E_{Q_Y}(v_S^Y 1_B)$ units of Y. Then, we have the following equality

$$Q_S(B) E_{Q_Y} v_S^Y = E_{Q_Y}(v_S^Y 1_B)$$

which is equivalent to

$$Q_S(B) = \frac{E_{Q_Y}(v_S^Y 1_B)}{E_{Q_Y} v_S^Y}$$

We know that Y_T is a random variable from Gaussian distribution with mean m and variance σ^2 with respect to the probability measure Q_T. Thus, we have

$$Q_T^*(B) = \frac{E_{Q_T}(e^{Y_T} 1_B)}{E_{Q_T}(e^{Y_T})} \text{ for all events } B$$

which is equivalent to

$$Q_T^*(B) = \frac{\int f e^{Y_T} dQ_T}{E_{Q_T}(e^{Y_T})} = \int f dQ_T^* \tag{4.26}$$

Now, substituing $f = e^{wY_T}$ in the equation (4.26), we obtain

$$\int e^{wY_T} dQ_T^* = \frac{\int e^{wY_T} e^{Y_T} dQ_T}{E_{Q_T}(e^{Y_T})}$$

or

$$\int e^{wY_T} dQ_T^* = \frac{E_{Q_T}(e^{Y_T(w+1)})}{E_{Q_T}(e^{Y_T})} = E_{Q_T^*}(e^{wY_T})$$

where we have

$$E_{Q_T^*}(e^{wY_T}) = e^{(w+1)E(Y_T)+\frac{(w+1)^2\sigma_{Y_T}^2}{2}} E_{Q_T}(e^{-Y_T})$$

or

$$E_{Q_T^*}(e^{wY_T}) = e^{(w+1)E(Y_T)+\frac{(w+1)^2\sigma_{Y_T}^2}{2} - E_{Q_T}(Y_T) - \frac{\sigma_{Y_T}^2}{2}}$$

which is equivalent to

$$E_{Q_T^*}(e^{wY_T}) = e^{w(E_{Q_T}(Y_T)+\sigma_{Y_T}^2)+\frac{1}{2}\sigma_{Y_T}^2 w^2}$$

Therefore, Y_T is a Gaussian random variable with respect to Q_T^* with mean $E_{Q_T}(Y_T)+\sigma_{Y_T}^2$ and variance $\sigma_{Y_T}^2 = \sigma_{\ln S_T}^2$.

Let's recall the equation (4.23) of the European call option present value given by

$$S_0 Q_T^*[S_T > K] - Ke^{-rT} Q_T[S_T > K]$$

Now, as we know that $Y_T = \ln S_T$ and the variable Y_T is Gaussian with mean $E_{Q_T}(Y_T)$ and variance $\sigma_{\ln S_T}^2$, we have

$$Q_T[S_T > K] = Q_T[Y_T > \ln K] = Q_T[z < \frac{E_{Q_T}(Y_T) - \ln K}{\sigma_{\ln S_T}}]$$

or

$$Q_T[S_T > K] = Q_T[z < \frac{\ln(FS_T) - \frac{1}{2}\sigma_{Y_T}^2 - \ln K}{\sigma_{Y_T}}] = Q_T[z < \frac{1}{\sigma_{Y_T}}\ln(\frac{FS_T}{K}) - \frac{1}{2}\sigma_{Y_T}]$$

In this equation z is a standard normal Gaussian variable with mean 0 and variance 1. So $Q_T[S_T > K] = N(d_2)$ where

$$d_2 = \frac{\ln(\frac{FS_T}{K})}{\sigma_{Y_T}} - \frac{1}{2}\sigma_{Y_T}$$

Let's apply the same process to express $Q_T^*[S_T > K]$. The variable Y_T is Gaussian with mean $E_{Q_T}(Y_T) + \sigma_{\ln S_t}^2$ and variance $\sigma_{\ln S_T}^2$ with respect to Q_T^*. We have

$$Q_T^*[S_T > K] = Q_T^*[Y_T > \ln K] = Q_T^*[z < \frac{E_{Q_T}(Y_T) + \sigma_{\ln S_T}^2 - \ln K}{\sigma_{\ln S_T}}]$$

or

$$Q_T^*[S_T > K] = Q_T^*[z < \frac{\ln(FS_T) - \frac{1}{2}\sigma_{Y_T}^2 + \sigma_{Y_T}^2 - \ln K}{\sigma_{Y_T}}] = Q_T^*[z < \frac{\ln(\frac{FS_T}{K}) + \frac{1}{2}\sigma_{Y_T}^2}{\sigma_{Y_T}}]$$

where z is a standard normal Gaussian variable having variance = 1 and mean = 0. So $Q_T^*[S_T > K] = N(d_1)$ where

$$d_1 = \frac{\ln(\frac{FS_T}{K})}{\sigma_{Y_T}} + \frac{1}{2}\sigma_{Y_T}$$

Substituting $Q_T^*[S_T > K]$ and $Q_T[S_T > K]$ into equation (4.23) leads to the pricing formula for the Black-Scholes-Merton model for a European call option

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2) \tag{4.27}$$

where

$$d_1 = \frac{\ln(\frac{e^{rT}S_0}{K})}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \tag{4.28}$$

and

$$d_2 = \frac{\ln(\frac{e^{rT}S_0}{K})}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \tag{4.29}$$

Similarly, we can prove the pricing formula for Black-Scholes-Merton for a European put option which is given by

$$p = Ke^{-rT}N(d_1) - S_0N(d_2) \tag{4.30}$$

where $N(x)$ represents a cumulative probability distribution function for a standard normal distribution having the parameters $\phi(0, 1)$. The variables p and c are the European put and European call price respectively, K is the strike price, S_0 is the stock price at time zero, σ is the stock price volatility, T is the time to maturity of the option and r is the continuously compounded risk-free rate.

Since it is never a good option to exercise early before the maturity date an American call option on a stock that is non-dividend paying (see section 4.o. American options and the Black-Scholes-Merton pricing formulas), the pricing formula for Black-Scholes-Merton model for a European call option also applies for this type of options. However, no exact pricing formulas have been found for American put and call options on stocks which are dividend-paying as these options can be candidate to early exercise.

The Black-Scholes-Merton formula makes use of interest rate r which is theoretically speaking the zero coupon risk-free rate interest rate for a maturity T. r is a function of time and is stochastic under the assumption that at a given time T the stock price S_T follows a lognormal distribution and the volatility σ is estimated appropriately.

Finally, time is normally measured and is calculated as the quantity of trading days remaining in the option's life divided by 252, which are the total trading days in a year.

4.12 Properties of the Black-Scholes-Merton pricing formulas

We now want to have a closer look at the Black-Scholes-Merton formulas. As there is an increase in stock price S_0, it is almost certain that the European call option will be in-the-money and so it should be exercised. In this situation call option is very like the forward contract where the underlying stock will be delivered at maturity with price K of delivery. It is hence expected that the call price equals

$$S_0 - Ke^{-rT}$$

as d_1 and d_2 increase by large amounts and the functions $N(d_1)$ and $N(d_2)$ approach to 1.

Conversely, as there is an increase in stock price S_0, it is almost certain that the European put option p will be out-of-the-money and therefore the derivative price p approaches zero. This is consistent with $N(d_1)$ and $N(d_2)$ function which approach to zero here.

As the volatility parameter σ approaches zero, the underlying stock becomes riskless and the stock price will grow at rate r to be equal to S_0e^{rT} at a given time T. The payoff from the European call option at time T is given by

$$max(S_0e^{rT} - K, 0)$$

The call option's present value is therefore

$$e^{-rT}max(S_0e^{rT} - K, 0) = max(S_0 - Ke^{-rT}, 0)$$

which equals the condition #2 from section 2.c. Arbitrage opportunities.

To show that these results are consistent with the Black-Scholes-Merton pricing formulas, take the case one where $S_0 > Ke^{-rT}$. It is implied by this $\ln(S_0/K) + rT > 0$. As the

volatility parameter σ approaches zero, d_1 and d_2 approach $+\infty$ such that the functions $N(d_1)$ and $N(d_2)$ approach 1. The Black-Scholes-Merton call option formula then becomes

$$c = S_0 - Ke^{-rT}$$

In the case $S_0 < Ke^{-rT}$, we have $\ln(S_0/K) + rT < 0$. As the volatility parameter σ approaches zero, d_1 and d_2 approach $-\infty$ so that the functions $N(d_1)$ and $N(d_2)$ approach zero. Thus, the Black-Scholes-Merton pricing equation gives a zero price for the call option.

The price European call option, as a result, is always $max(S_0 - Ke^{-rT}, 0)$ as the volatility parameter σ approaches zero. A similar proof can be shown for a European put option price such that as σ tends to zero, the price is always $max(Ke^{-rT} - S_0, 0)$.

4.13 The Black-Scholes-Merton formulas and paying dividend stocks

We have assumed so far that no dividends are paid by the underlying contracts during the derivative instruments' lifetime. We now try to adjust the pricing formulas of the Black-Scholes-Merton to consider dividends paid by the underlying contracts. We assume that the dividend amounts and ex dividend dates are known with certainty. For short-life options, these assumptions are reasonable whereas for long-life options, not so much and this is the reason why we would rather assume the dividend yields to be known instead of the cash dividend amounts.

Assume that a price of a stock is addition of the two components: a component that is riskless corresponding to the dividends that are known during the European option lifetime and a second component that is risky. The component that is riskless is the known dividends' present value during the options lifetime discounted at the risk-free rate from the various past ex-dividend dates to the present date. At the time of maturity, the stock has payed the dividends and this would finish the riskless component. If S_0 is equal to the

stock price risky component and risky component follows the volatility parameter σ of the process then the Black-Scholes-Merton formula will be correct .

In terms of being operational, this implies that the Black-Scholes-Merton pricing formulas can be brought into use by lowering the stock price S_0 by all of the dividends present value paid during the option's life. The dividends are discounted at the risk-free rate using the ex-dividend dates to the present time.

4.14 The Put-Call parity

In the book "Pricing Derivatives: The Financial Concepts Underlying the Mathematics of Pricing Derivatives", Ambar Sengupta quickly proves the put-call parity as follow.

As the payoff resulting from a call option is $[S_T - K]_+$ where S_T represents the underlying stock price at time T, the call option maturity time, and K is the option's strike price. The call option forward price is therefore

$$E_{Q_T}([S_T - K]_+)$$

The price of call at time 0 is given as

$$c = e^{-rT} E_{Q_T}([S_T - K]_+)$$

or

$$c = E_{Q_0}(e^{-rT}[S_T - K]_+) \tag{4.31}$$

Similarly, the payoff resulting from a put option is $[K - S_T]_+$ where S_T is the underlying stock price at time T, the put option maturity time, and K is the option's strike price. The put option forward price is given by

$$E_{Q_T}([K - S_T]_+)$$

The price of put at time 0 is therefore

$$p = e^{-rT} E_{Q_T}([K - S_T]_+)$$

or

$$p = E_{Q_0}(e^{-rT}[K - S_T]_+) \tag{4.32}$$

Subtracting the equation (4.32) of a put price at time 0 to the equation (4.31), the call price at time 0, we have

$$c - p = E_{Q_0}(e^{-rT}[S_T - K]_+) - E_{Q_0}(e^{-rT}[K - S_T]_+)$$

or

$$c - p = E_{Q_0}(e^{-rT}([S_T - K]_+ - [K - S_T]_+))$$

which is equivalent to

$$c - p = E_{Q_0}(e^{-rT}[S_T - K])$$

and

$$c - p = E_{Q_0}(e^{-rT} S_T) - E_{Q_0}(e^{-rT} K) = S_0 - e^{-rT} K \tag{4.33}$$

Thus, we obtain the following put-call parity

$$c + e^{-rT} K = p + S_0 \tag{4.34}$$

The above equation giving the put-call parity helps in finding out the put option price using a theoretical price of call option having same date of maturity and strike price, or the call option price using a theoretical price of put option having same date of maturity and strike price.

In case where put-call parity does not hold, an arbitrage opportunity is possible and a trader can make a sure profit with a \$0 investment.

If the right hand side of the equation (4.34) is bigger than the left hand side, the European put option is apparently overvalued while the European call option is undervalued. A trader can then long the call, short the put, short the underlying stock and invest until expiration of the option contracts, a dollar amount such that his entire position does not require any cash outlay at time now.

However, if the the right hand side of (4.34) is smaller than the left hand side, the European put option is underpriced whereas the European call option is overpriced. A trader can then short the call, long the put, long the underlying stock and borrow until expiration of the option contracts, a dollar amount such that his entire position does not require any cash outlay at time now.

In both cases, the trader will close out his position once financial instruments are priced correctly and the put-call parity holds again. He will have realized a profit equal to the synthetic market, which is the absolute value of the figure calculated by subtracting the put price and call price given by equation (4.33).

4.15 American options and the Black-Scholes-Merton pricing formulas

The put-call parity only applies to European options but a pretty similar formula can be obtain for American option prices when no dividends are involved

$$S_0 - K \leq c_{American} - p_{American} \leq S_0 - Ke^{-rT}$$

This inequality puts some lower and upper boundaries for the American options' prices and in case any of American options prices were not respecting it, arbitrage opportunities would appear.

4.15.1 American calls on a non-dividend paying stock

The objective of this section is to show why it is not an optimal choice to exercise an American call option before the date of maturity on a non-dividend paying stock. Let's recall the condition #5 from section 2.c. Arbitrage opportunities, for the European call options on a stock which is non-dividend paying, the lower bound is given as

$$c_{European} \geq S_t - Ke^{-rT}$$

As an American call option has more chances than a European call option to be exercised, the American call option premium must be equal or greater than a European call option. Thus we have

$$c_{American} \geq c_{European} \geq S_t - Ke^{-rT}$$

Consider at $T > 0$, the risk-free interest rate given as $r > 0$. We have

$$c_{American} > S_t - K$$

This means that the American call option premium is at all times higher than the call option intrinsic value before expiration and this is due to the fact that the call option remains some time value which equals the difference among the intrinsic value of option and the premium. Thus, a trader has no reason to exercise early an American call option since it would not be wise to lose the option's time value.

Finally, if the idea to exercise earlier the American call option was optimal, the intrinsic value of option would have been equal to its premium such that $c_{American} = S_t - K$ and therefore the American call option would not retain anymore time value.

To summarize, there exist two reasons for a trader to not exercise early an American call option.

Firstly, the call option provides insurance to the trader against any downside price movements in the underlying contract. In case the trader decides to early exercise the option and the trader owns the underlying stock, the trader is not insured anymore against a decrease in the stock price.

Second of all, if a trader exercises a call option early, he loses the time value of the call option that he should not give up.

Because American call options are never exercised early when no dividends are involved in the underlying contract, they are similar to European call options and we obtain therefore the following inequality

$$max(S_t - Ke^{-rT}, 0) \leq c_{American}, c_{European} \leq S_t$$

4.15.2 American calls on a dividend paying stock

As we have seen previously, an American call option on a stock that is non-dividend paying should never be exercised early. However, in case of a dividend-paying stock, exercising an American call option could be optimal only at a certain time in the option's life with the condition that it is earlier than the going of underlying stock ex-dividend.

Assume that n dividends, denoted D_1, D_2, $...D_n$, are anticipated during the option's life at the ex-dividend dates t_1, t_2,...t_n. One possibility is to exercise the American call option just early to the final date of ex-dividend, that is at time t_n, and hence, the investor gets

$$S(t_n) - K$$

In case of not exercising the call option at time t_n, the price of stock would drop to $S(t_n) - D_n$. The value of the call option then becomes

$$S(t_n) - D_n - Ke^{-r(T-t_n)}$$

where T is the date of maturity of the call option. This leads to if

$$S(t_n) - D_n - Ke^{-r(T-t_n)} \geq S(t_n) - K$$

that is

$$D_n \leq K[1 - e^{-r(T-t_n)}] \tag{4.35}$$

it will not be an optimal choice of exercising the option at time t_n.

However conversely, if

$$D_n > K[1 - e^{-r(T-t_n)}]$$

Then it is best to exercise the option at time t_n for a value of $S(t_n)$ which is sufficiently high. This case occurs most likely when the date of final ex-dividend t_n is very close to the option's maturity date and the amount of cash dividend D_n is significant.

Let's look at the situation of exercising the American call option at time t_{n-1}. In the given scenario, the amount $S(t_{n-1}) - K$ is received by the investor. In case of not exercising the option at time t_{n-1}, there is a fall in the stock price to the level given by $S(t_{n-1}) - D_{n-1}$ and the initial time the investor will consider exercising the option again will be at time t_n, the next ex-dividend date. Hence, this sets option price a lower bound if it is not exercised at time t_{n-1} which is

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n - t_{n-1})}$$

It is lead from here that given that

$$S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n - t_{n-1})} \geq S(t_{n-1}) - K$$

or

$$D_{n-1} \leq K[1 - e^{-r(t_n - t_{n-1})}]$$

Exercising the option immediately prior to time t_{n-1} is not an optimal choice. Likewise, for any $i < n$, if

$$D_i \leq K[1 - e^{-r(t_{i+1} - t_i)}] \tag{4.36}$$

then exercising just before time t_i is not an optimal choice. The inequality (4.36) is approximately equivalent to

$$D_i \leq Kr(t_{i+1} - t_i)$$

Supposing that K is very near to the present stock price, this inequality is satisfied when the risk-free interest rate is more than earning of the dividend on the stock which in reality often happens.

In conclusion, the most likely time an American call option is subject to exercising early is just before the final ex-dividend date given by t_n. Finally, if inequalities (4.35) and (4.36) hold, a trader is certain it is not optimal to exercise early.

4.15.3 American puts on a non-dividend paying stock

The early exercise case is different for American put options on stocks which are non-dividend paying. As a matter of fact, in a specific situation, the early exercise of the put options can be optimal. Any American put options should be early exercised, if they are deeply in-the-money during their lifetimes.

A put option, like a call option, is witnessed as giving insurance to the long position against the underlying contract falling below certain price. However, it is optimal for the long position of a put option to give up the insurance and realize the strike price early, which is as seen previously, not the case for American call options.

Finally, exercising an American put option early seems more striking when the price or volatility of the underlying contract reduce and there is a rise in the risk-free interest rate.

The minimal and maximal value for a European put option on an underlying stock contract which is non-dividend paying are

$$max(Ke^{-rT} - S_t, 0) \leq p_{European} \leq Ke^{-rT}$$

As an American put option is subject to early exercise at anytime when deeply in-the-money, and the American option premium cannot be of greater worth than the strike price, we have

$$max(K - S_t, 0) \leq p_{American} \leq K$$

As the early exercise of American put options is favorable under some circumstances, American put options should always have a higher premium than the corresponding European put options. Moreover, as American put options are sometimes worth less than their intrinsic values, European put options can therefore sometimes trade less than their intrinsic values.

4.16 The assumptions of the Black-Scholes-Merton model

Commonly used by derivatives traders and financial market technicians is the Black-Scholes-Merton model (1973) because of its easy, quick computations and efficiency. However, the pricing model lies on more or less realistic assumptions.

4.16.1 Markets are frictionless

The Black-Scholes-Merton model assumes that markets are frictionless but obviously they are not. First of all, the model assumes that the underlying asset can be freely sold or bought freely without any restrictions. However, there may be preventions against the

sale of stock in the stock market. In the case where short sales are authorized, it might be with certain restrictions on when short selling can be made.

To give an example of a restriction is the uptick rule. Selling stocks is more difficult than buying stocks. While the United States does not completely prohibits it, as done in many markets, there is an uptick rule for traders who want to sell short. Yet it is not guaranteed that an uptick will be present in a single stock.

Second of all, the model assumes that every market participants can lend and borrow money easily, and a single interest rate is applicable to every transactions.

The assumption of lending and borrowing money freely by a trader is a big weakness in the pricing models. Supposing that a trader has funds that are enough for a trade initiation, it is possible that at some point in time later on he may find that he is required to have more funds in the form of margin requirements being increased. Assuming that money was available freely, margin should not have been a problem. Money for the margin could be borrowed by the trader and deposited with the Options Clearing Corporation. As we assume that rate of borrowing and lending rate are equivalent, and since interest is paid on the margin deposit by the Options Clearing Corporation, obtaining margin money should not be a problem; neither any cost will be associated with it.

Traders have a limited borrowing capacity in reality. A trader might be compelled to liquidate a position before the expiration date if he cannot meet a margin requirement. Supposing that a trader has unbounded capacity for borrowing, the reality is that borrowing and lending rates, for most traders are not the same, this can also lead to problems with model generated values strategies. Borrowing margin money at a specific rate by the trader would most likely be deposited at a lower rate with the OCC. The loss due to this difference between the borrowing and lending rates is something of which is ignored by this model. As this difference between borrowing and lending rates increases, the values generated by the model become less reliable.

A major drawback is assuming there are no transaction costs in the frictionless markets hypothesis. This assumption is not very realistic, as every time a trader wants to buy or sell financial instruments, transaction costs have to be endured. These costs can take the form of clearing fees, an exchange membership or brokerage fees. If these transaction costs are high, adjusting too much to maintain a riskless portfolio can affect the theoretical edge of the initial position.

4.16.2 Interest rates are constant over an option's life

It is assumed by the Black-Scholes-Merton model that the risk-free interest rate is constant but this input is obviously a variable which is stochastic and randomly fluctuating over the life of the derivative instruments.

Is the assumption of a fixed and risk-free rate by the model a major concern? While interest rate change will lead to the value of an option position of the trader to change, usually interest rates do not change such that it significantly impacts value of an option, for the the short-term at least. Drastic changes in interest rates are needed to have an impact on any but the most deeply in-the-money options. Interest rates need to be altered by various percentage points to have a significant impact over short time duration.

4.16.3 Volatility is constant over the option's life

It is assumed by the Black-Scholes-Merton model that the underlying contract volatility is a constant parameter over the option's life. Hence, as the volatility input is fed into the pricing model by the trader, he gives his best estimation of the amount of change in price which is expected over the option's life.

"The model assumes that occurrences of each particular magnitude will be evenly distributed over the life of the option" (Natenberg, 1994, p390). However, in reality, large price changes occurring during the beginning part or the latter part of the option's life will affect differently the volatility of the underlying contract. In fact, a trader is going to

encounter moments of high volatilities and other moments of low volatilities rather than an even volatility distribution that the model assumes. Thus, the Black-Scholes-Merton model is unable to differentiate moments of high and low volatilities over the option's life since it supposes that volatility is constant where prices changes are evenly distributed.

Through experience, traders have concluded that order of volatility is important, in particular for options which are at-the-money since they have the greatest gammas. A high volatility period near options expiration will impact at-the-money option more than when the similar high volatility occurs when there is more time in maturity date of option. Traders have concluded that the Black-Scholes-Merton theoretical pricing model tends to minimize at-the-money option values in a market with rising volatility while overpricing at-the-money options in a market with falling volatility.

The volatility of an underlying contract is difficult to predict as it follows a stochastic process and therefore varies overtime. However, we have seen that the volatility parameter is lognormally distributed in the model of Black-Scholes-Merton overcoming the problem of underlying contract prices being negative. In fact, the lognormal distribution allows for an infinite upside move in prices while bounding downside prices by 0. This is a better representation of how prices are actually distributed in the financial markets.

4.16.4 Volatility is independent of the underlying asset price

The market volatility is taken not to be dependent on the underlying contract price by the Black-Scholes-Merton model. Though it seems that the volatility is dependent on the direction in which the movement is taken place in the underlying asset. In some situations, traders expect the market to be either more or less volatile as prices are going up or down.

For example, under certain market conditions, a trader can expect a market to be very volatile when the underlying asset price is decreasing but a low volatility when prices in the underlying market are rising. Because volatility is apparently dependent on the price movements direction in the underlying market, the Black-Scholes-Merton model is flawed

to correctly represent the real volatility of a market instrument.

4.16.5 Underlying stock prices are continuous with no gaps

It is assumed by the Black-Scholes-Merton model that prices for the underlying contract follow a stochastic diffusion process, which is obviously not conform to the reality of the financial markets as prices can jump and undergo unpredictable gaps due to events. These gaps are often happening while financial markets are plummeting or between the closing price and the opening price the morning after where trading the underlying asset was not continuous. This implies that the Black-Scholes-Merton model is less reliable during financial crisis and traders will tend to use less and less the theoretical pricing model during period of huge movements in the financial markets where many price gaps can happen. Traders will rely on their experience and intuition to take trading decisions and consider Black-Scholes-Merton model lesser since the probability of extreme events are underestimated and do not assume gaps in prices. Thus, the diffusion process is not a true representation of the reality of market events implied in the Black-Scholes-Merton model.

Moreover, the diffusion process, following a lognormal distribution and a Brownian geometric movement, do not take into account jumps in underlying asset prices and prevent traders from constantly being delta neutral as jumps in prices often occur after unpredictable events leaving the traders with risky directional

Thus, the model assumes an option trader is continuously delta neutral in his position to eliminate directional risks. The fact that volatility is continuously compounded, the theoretical pricing model assumes that adjustments are made continuously to remain delta neutral at every moment in time. However, as we have seen previously, a gap in the underlying asset price where the option is close to expiration can dangerously affect the delta position of the derivative trader who will become either positive or negative delta. Thus, a

derivative trader must be aware of this flaw in the Black-Scholes-Merton model and must be ready to take adequate actions to rehedge his options portfolio as better as he can to maintain the best riskless hedge by making adjustments.

4.16.6 The normal distribution of the underlying stock prices

It is assumed by the Black-Scholes-Merton model that the underlying contract price changes are normally distributed over small periods of time, leading to a lognormal distribution of underlying prices at expiration. However, many statistical studies have proven that this hypothesis is not necessarily correct. Moreover, they seem to show that extremity tails, representing huge variation in the underlying asset price are underestimated and seem to happen more often than predicted by the Black-Scholes-Merton model.

Even though the normal distribution tails are relatively low compared to the extreme events' real probability in the financial markets, the normal distribution tails are symmetric which might not reflect the reality as well. In fact, financial markets might have more extreme downward movements than extreme upward movement or vice versa. This hypothesis is not taken into consideration in the normal distribution as it predicts few big downward and upward movements in the underlying prices but mainly a huge amount of small variations of the underlying contract price overtime.

Finally, under the hypothesis of a normal distribution, the underlying price can have upward and downward moves of the same magnitude. This means that if a stock is currently pricing $40 and increases by $60 to $100, the stock price can also decrease by -$60 and therefore pricing -$20. Obviously, stocks, commodities or futures and other financial instruments cannot take negative values and this proves that the assumption of a normal distribution in the Black-Scholes-Merton model is clearly flawed.

4.17 Conclusion

Even though it is possible to find theoretical pricing models that better applies to specific financial derivative instruments, like the Black Model (1976) to evaluate options on futures contracts, the Garman-Kohlhagen Model (1983) to evaluate options on foreign currencies or the model of Constant-Elasticity of Variance (CEV) which is based on the connection between price level and volatility of the underlying market, the Black-Scholes-Merton pricing formulas remain the most widely used of all options pricing models for its easy and efficient math computations.

Also, the Black-Scholes-Merton model, which assumes no early exercise of options, poorly suits most of the financial markets where most of the options are American. However, in some markets and especially in the futures option markets,"the early exercise value is so small that there is virtually no difference between values obtained from the Black-Scholes-Merton model and values obtained from an American option pricing model" (Natenberg, 1994, p44) which require more inputs and estimations and therefore less chance to closely approximate real world conditions.

Chapter 5
The Greeks

When selling options in the over the counter market, financial institutions must face the problem of managing the options' risks. If similar options are traded at the same time on an exchange, financial institutions can buy the same option on exchange that was sold by it in the over the counter market in order to neutralize the risks. But when the option is not available on an exchange due to particular characteristics of the derivative to satisfy a client in the over the counter market, like an unusual maturity date or strike price, it becomes more difficult to hedge the risk exposures for the financial institutions. The Greek letters measure different dimensions to the option position risk present and derivative trader's objective is to make the risk he is encountering acceptable by managing the Greeks.

One strategy is that the financial institutions do nothing. This strategy is called as a *naked position* and the financial institution which sold the options hope that they will finish out-of-the-money. At maturity of the options, the financial institution makes a profit equal to the premium of one option times the quantity w of options sold in the over the counter market. However, if the options are in-the-money at expiration, the long position exercises the options, by paying Kw to the financial institution, which in return, must deliver a certain quantity of the underlying stock. The financial institution will have to buy the securities in the underlying market at a higher price than the call option's strike price $S_T > K$, or at a price which is less as compared to the put option's strike price $S_T < K$. This will generate a loss for the financial institution equivalent to $w(S_T - K)$ for short call options and $w(K - S_T)$ for short put options.

Financial institutions can use another strategy apart from the naked position that is to buy directly the underlying stocks when selling the call or put options. If the call or put options are in-the-money at expiration and therefore exercised by the long positions, this strategy, referred as a *covered position* works well. However, if call and put options are out-of-the-money, a covered position can generate a significant loss for the financial institution.

Thus, a good hedge is neither provided by a covered position nor a naked position and this is the reason why most traders and financial institutions use hedging procedures that are more improved involving steps such as gamma, delta, vega, rho and theta. These data, also called the Greeks, allow traders to be aware and anticipate risks that could potentially affect their positions.

5.1 Delta of European stock options

"The delta is a measure of how an option's value changes with respect to a change in the price of the underlying contract" (Natenberg, 1994, p99). The delta of a call option is a figure between 0 and 1. A call option far in-the-money has a delta close to 1 while a call option deeply out-of-the-money has a delta close to 0. A call option which is at-the-money has a delta of 0.5. When shorting a call option, delta values become 0 for being deeply out-of-the-money and -1 for being far in-the-money.

The situation for the put options is pretty similar, except that when a trader is long a put option, he has a negative delta position which is between -1 and 0. This is due to the fact that put options lose value when the underlying market increases, and vice versa. In fact, a trader with a long put position is short the market position as he expects the underlying market to fall. When shorting a put option, the delta values become positive and bounded between 0 and 1. A positive delta position means the trader is betting on

the underlying contract price increase and a negative delta position implies that the trader expects the underlying contract price to decrease.

Derivative call option delta, mathematically speaking, is the "sensitivity of the price of the derivative to changes in the prices of the underlying instrument" (Sengupta, 2005, p104) given by

$$\Delta_{Call} = \frac{\partial c}{\partial S} \tag{5.1}$$

where c represents the derivative call option price and S represents the underlying contract price. Substituting equation (4.27), the price of theoretical European call option from the model of Black-Scholes-Merton, into equation (5.1), we have

$$\Delta_{Call} = \frac{\partial}{\partial S}[S_0 N(d_1) - Ke^{-rT}N(d_2)] = N(d_1) \tag{5.2}$$

where d_1 is defined by equation (4.28) and cumulative distribution function is represented by $N(x)$ for a standard normal distribution implying that the delta of a call option represents the option's probability to be in-the-money. Take as an example, a call option June 75 having delta value as 0.23 has a probability of 23% to be in-the-money, or there is 23% chance that the underlying asset price rises above \$75. The short delta position in a single European call option is equal to $-N(d_1)$.

For a short European call option position, it requires the trader to maintain a long position of $N(d_1)$ shares for each option sold to remain delta hedge. Similarly, with a long position in a European call option, the trader can remain delta neutral by maintaining a short position of $N(d_1)$ shares for each option purchased.

Similarly, we have the put option delta presented as below

$$\Delta_{Put} = \frac{\partial p}{\partial S} \tag{5.3}$$

where p represents the derivative put option price and S represents the underlying contract price. Substituting equation (4.30), the theoretical European put option price from the model of Black-Scholes-Merton, into equation (5.3), we have

$$\Delta_{Put} = \frac{\partial}{\partial S}[Ke^{-rT}N(d_1) - S_0 N(d_2)] = -N(d_2) = N(d_1) - 1 \tag{5.4}$$

where d_1 is defined by equation (4.28) and cumulative distribution function is represented by $N(x)$ for a standard normal distribution.

The put option delta is negative implying that hedging is done for a long position in a put option with an underlying stock long position. Similarly, hedging for a short position in a put option should be done with an underlying stock short position. Finally, substituting equation (5.2) into the equation (5.4), we obtain the following call-put delta relation

$$\Delta_{Put} = \Delta_{Call} - 1$$

The graphs below show the delta variations of a put and a call option with the stock price.

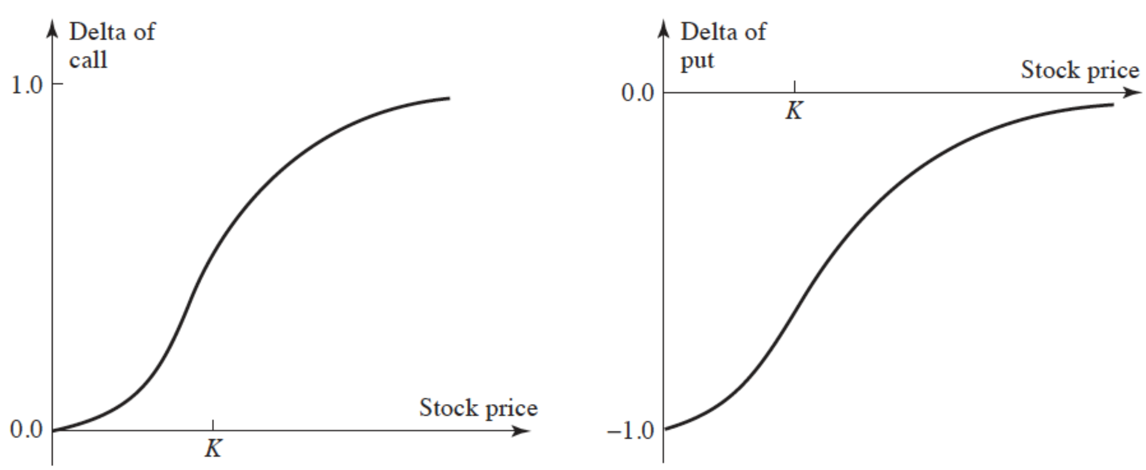

Figure 5.1: Delta variation of a put and a call option with respect to the underlying non-dividend paying stock price

The options portfolio delta or other derivative instruments which depend on a single underlying asset with S as the price is given by

$$\Delta(portfolio) = \frac{\partial \Pi}{\partial S}$$

where $\partial \Pi$ is the portfolio value.

A portfolio with w_i quantity of option i ($1 \leq i \leq n$) has a delta equal to

$$\Delta(portfolio) = \sum_{i=1}^{n} w_i \Delta_i \tag{5.5}$$

where Δ_i is the ith option delta. This formula is useful in calculating the position to be taken by trader in the underlying asset to make his portfolio delta neutral.

For example, consider a long position with strike price K_1 in 100,000 call options, a short position with strike price K_2 (with $K_1 \neq K_2$) in 200,000 call options, and a short position with strike price K_3 in 50,000 put options. The delta of each option are 0.533, 0.468 and -0.508 respectively. The delta of the portfolio is

$$100,000 \times 0.533 - 200,000 \times 0.468 - 50,000 \times (-0.508) = -14,900$$

The above calculation implies that we can make the portfolio delta neutral by buying 14,900 underlying shares.

5.2 Delta hedging

Delta hedging consists in making a portfolio of derivative contracts immune to any price directions of the underlying assets. In order to be delta hedged, a derivatives trader must have a portfolio delta equal or relatively close to zero. From equation (5.5), a portfolio

is delta neutral if

$$\Delta(portfolio) = \sum_{i=1}^{n} w_i \Delta_i = 0$$

Delta hedging enables derivatives traders to enter in a volatility spread position as a strangle, a straddle, a time spread or a butterfly rather than a directional position.

For example, consider the following options and deltas : June 90 put with a delta of -0.16, June 95 put with a delta of -0.30 and June 100 put with a delta of -0.44. Assume the trader does not want to take a directional position but instead a volatility position which means he is willing to bet on the low or high volatility of the underlying market. For this type of position, the trader must have a delta neutral position and can do so by buying 1 June 90 put and 1 June 100 put, and sell 2 June 95 puts. When adding the deltas of this position, we have

$$-0.16 - 2 \times (-0.30) - 0.44 = 0$$

In this scenario, the trader has a long put butterfly betting on low volatility of the underlying market. The highest profit made is \$5 by the trader (given by the spread present between the exercise prices) if the underlying asset is equal to the exercise price of \$95 at expiration and the trader will start losing money as the price of the underlying asset drifts in either direction from \$95.

It should be remembered that an option delta would not be always constant, and hence, the portfolio of the trader may remain delta hedged for a short period of time relatively speaking. It is necessary to adjust the hedge periodically or every time the delta position reaches a positive or negative amount of deltas decided by the trader. This is known as *rebalancing*. The derivatives trader can either long calls, long underlying contracts, short puts to increase the delta of a portfolio or short calls, short underlying contracts, long puts to decrease the delta of a portfolio and remain delta hedged.

When a trader rebalances his position on a regular basis to remain delta hedged, the procedure is called *dynamic hedging*. If a trader initially sets up a delta neutral position and never adjusts his portfolio at later time, the procedure is referred as *static hedging* or *hedge and forget*.

5.3 Theta of European stock options

The theta, Θ, is a loss in value of option over a period of time elapsed. Such as, an option having theta value of 0.10 loses \$0.10 per day in value given that other market conditions remain same. If today's worth of option is \$3.57, then tomorrow it would have a worth of \$3.47. In other words, the portfolio's theta is the rate at which portfolio's value change with respect to the time elapsed with all other parameters and market conditions being unchanged. At times theta is also called as the portfolio's time decay.

It can be seen from the pricing formulas of Black-Scholes-Merton that the European call option theta on a stock that is non-dividend paying is given by

$$\Theta_{Call} = -\frac{X_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) \tag{5.6}$$

where d_1 and d_2 are defined by the equations (4.28) and (4.29) respectively, and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{5.7}$$

is the probability density function for a standard normal distribution.

In case of a European put option on an underlying stock, the theta is

$$\Theta_{Put} = -\frac{X_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(-d_2) \tag{5.8}$$

Since $N(-d_2) = 1 - N(d_2)$, the theta of a put exceeds the theta of the corresponding call by rKe^{-rT}. In the formulas (5.6) and (5.8), the theta is measured in years and therefore, a trader must be careful as time is measured in days in the marketplace. Thus, in order to get theta per calendar day the trader must divide the measured theta by 365, or to get the theta per trading day divide by 252.

All calls and puts options have a negative theta which means that all options lose value and become less valuable as expiration approaches. A negative theta is held by long option position as on the contrary a positive theta is held by a short option position. The theta variation of a call option with respect to the underlying stock price is presented in the figure below.

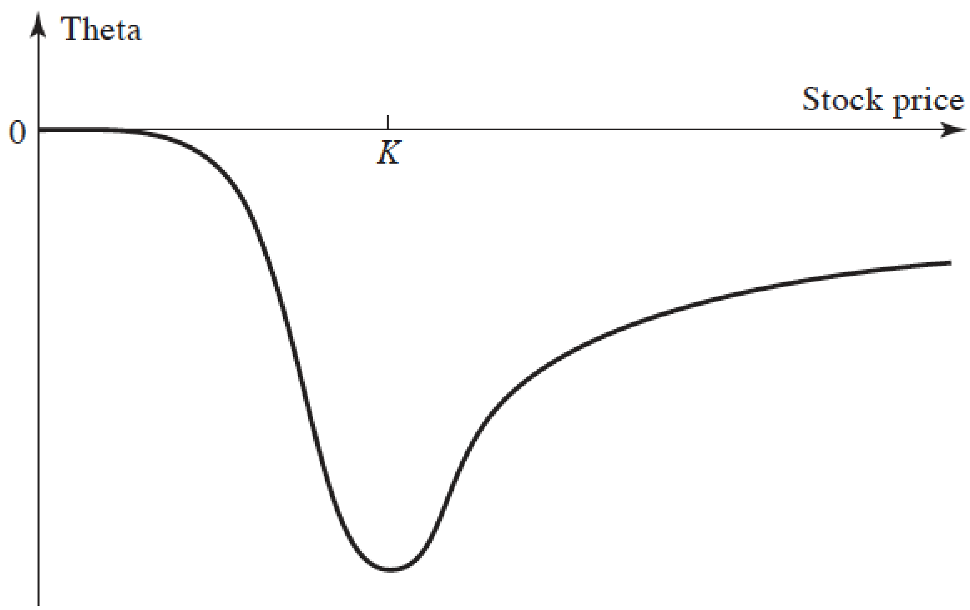

Figure 5.2: Theta variation of European call option along stock price

Theta is close to zero when the underlying stock price is low. Theta is highly negative for at-the-money call options. The larger the underlying stock price becomes, the theta of the call option tends to $-rKe^{-rT}$.

All options have a negative theta but in few cases, it is possible for European options to have a positive theta and therefore the passage of time helps the options gain in value. European futures options, subject to stock type settlement, which have a lower theoretical value as compared to parity may have a theta value which is positive. The value of the European future options will gain value to reach parity (intrinsic value) at expiration of the options. A European put stock option is likely to be affected by a situation of this kind depending on how high the interest rates are and whether the put is trading under parity. Once again, the put stock option premium will rise to parity as expiration approaches.

European options are exception to the rule and can trade under parity and have a positive theta as they do not allow early exercise. If this were the case for an American option, a trader would have an arbitrage opportunity to make a sure profit. Thus, American options cannot have a worth below parity and hence do not have theta values as positive.

Finally, as the future price of the underlying stock is unknown, then time decay is also uncertain. Hence it is wise for a trader to protect a position against price changes of the underlying asset by being delta neutral but there is no sense to be hedged against the passage of time.

5.4 Gamma of European stock options

The gamma, Γ, represents the rate at which an option's delta changes as the underlying asset price moves. Such as, an option having gamma value of 0.10 means that a rise (fall) of each point in the underlying asset price will make the option gain (lose) 0.10 delta.

For a European put or call option on a stock that is non-dividend paying, the equation below gives the gamma

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

where the equation (4.28) of the pricing formula of Black-Scholes-Merton defines d_1 and

the probability density function $N'(x)$ for a standard normal distribution by the equation (5.7). Thus, puts and calls with similar date of maturity and strike price have the exact same positive gamma.

When a trader is long an option, he is long gamma; when a trader is short an option, he is short gamma. Then, a positive gamma position implies that the trader expects the movements in the underlying contract are highly swift independent of the direction; a gamma position which is negative means the trader bets the underlying contract will move slowly regardless of the direction.

The delta of the underlying contract is 1 and remains 1 no matter the price changes and this implies that the gamma of the underlying contract is 0.

As gamma gives the rate at which delta changes, it is a great measure for traders to evaluate how quickly an option can change its directional characteristics, behaving similar to the actual position.

As we have seen previously, the deltas of options are bounded between -1 and 0 or between 0 and 1 depending on if it is a call or a put, and if the trader is long or short the position. The fact that a delta of an option remains between 0 and 1 or -1 and 0 means the gamma of an option would change continuously subtracting or adding deltas with constant amount from the original delta value will make the total delta of the option go out of its range.

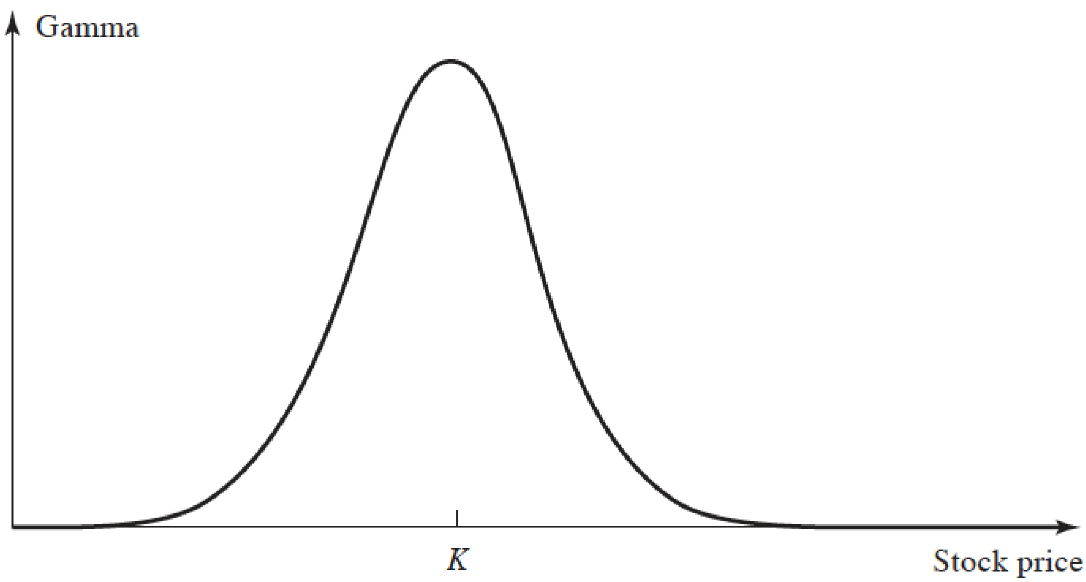

Figure 5.3: Variation of an option's gamma with respect to the underlying stock price

The graph above illustrates the gamma of a long position and shows that the value for gamma is positive always and changes along the stock price. Just like an option time value, gamma of an option reaches the highest point when the derivative contract is at-the-money and then progressively declines as the option becomes out-of-the-money or in-the-money.

When time remaining is less before an option's expiration, an-at-the-money option gamma can increase dramatically. Thus, close to maturity at-the-money options with high gammas are very risky for traders as their position value is very sensitive to changes in the underlying stock price.

The gamma is also subject to a change in volatility assumptions: an increase in volatility in the underlying contract will make options tend to be more at-the-money. A decrease in volatility will move option deltas away from 0.5 and will make options go further in-the-money or further out-of-the-money.

5.5 Making a portfolio Gamma neutral

Suppose a portfolio having gamma equal to Γ and is delta neutral, and a value of gamma being equal to Γ_T for a traded option. If we add to the portfolio a certain amount w_T of traded options, the portfolio's gamma becomes

$$\Gamma(portfolio) = w_T \Gamma_T + \Gamma$$

Therefore to make the portfolio gamma neutral in the traded option a necessary condition is $-\Gamma/\Gamma_T$. As the traded option present in portfolio is expected to change the portfolio's delta, in order to maintain delta neutrality the underlying asset position needs to be changed. Only for a short period of time the portfolio will be gamma neutral and the traded option position needs adjustment, so that it equals $-\Gamma/\Gamma_T$ always in order to maintain a gamma neutral portfolio.

The goal of a delta neutral as well as a gamma neutral portfolio is to correct the hedging error. Delta neutrality is a hedge from comparatively small movements in underlying asset price between rebalancing. A protection is given by gamma neutrality against high movements in this price of stock between rebalancing.

For instance, take a delta hedged portfolio with a gamma value of -3,000. Suppose the gamma of a traded call option is 1.50 and its delta is equal to 0.62. To make portfolio gamma neutral $3,000/1.50 = 2,000$ call options can be purchased. However, the portfolio's delta will change from zero to $2,000 \times 0.62 = 1,240$. Therefore, 1,240 units from the portfolio of the underlying stock should be sold so that it remains delta hedged.

5.6 Relationship between gamma, theta, and delta

A single derivative price that depends on a stock that is non-dividend paying must satisfy the differential equation of Black-Scholes-Merton (4.22). It follows that the Π value

of a portfolio of derivatives can be modeled as the differential equation

$$\frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Since

$$\Theta = \frac{\partial \Pi}{\partial t}, \qquad \Delta = \frac{\partial \Pi}{\partial S}, \qquad \Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

it follows that

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 T = r\Pi$$

For a portfolio that is delta neutral, $\Delta = 0$, hence we get

$$\Theta + \frac{1}{2}\sigma^2 S^2 T = r\Pi$$

It leads to the fact that a high positive gamma value implies a high negative theta; a small negative gamma implies a small positive theta. Thus, gamma and theta are closely related as they represent the risks of similar degree in accordance with movement of market and the time elapsed respectively. All the options are a tradeoff between time decay and market movement. Given that underlying contract volatility aids position of a trader then the time duration will have adverse affects and it is true the other way around.

The gamma is the portfolio's second partial derivative with respect to price of asset

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

The delta position of portfolio will change slowly for a smaller value of gamma, and adjustments need to be less frequent in order to maintain the delta neutrality of the portfolio. In case where gamma is highly negative or highly positive, the portfolio's delta changes instantly with change in underlying asset prices. It then becomes very uncertain for a

trader who needs to adjust its portfolio quite often in order to remain delta hedged.

Let the underlying asset price change be given by ΔS over short time interval, Δt, and the corresponding change in price of portfolio be $\Delta\Pi$. Then, for a delta neutral portfolio, we have

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\Gamma\Delta S^2$$

where Θ is the portfolio's theta.

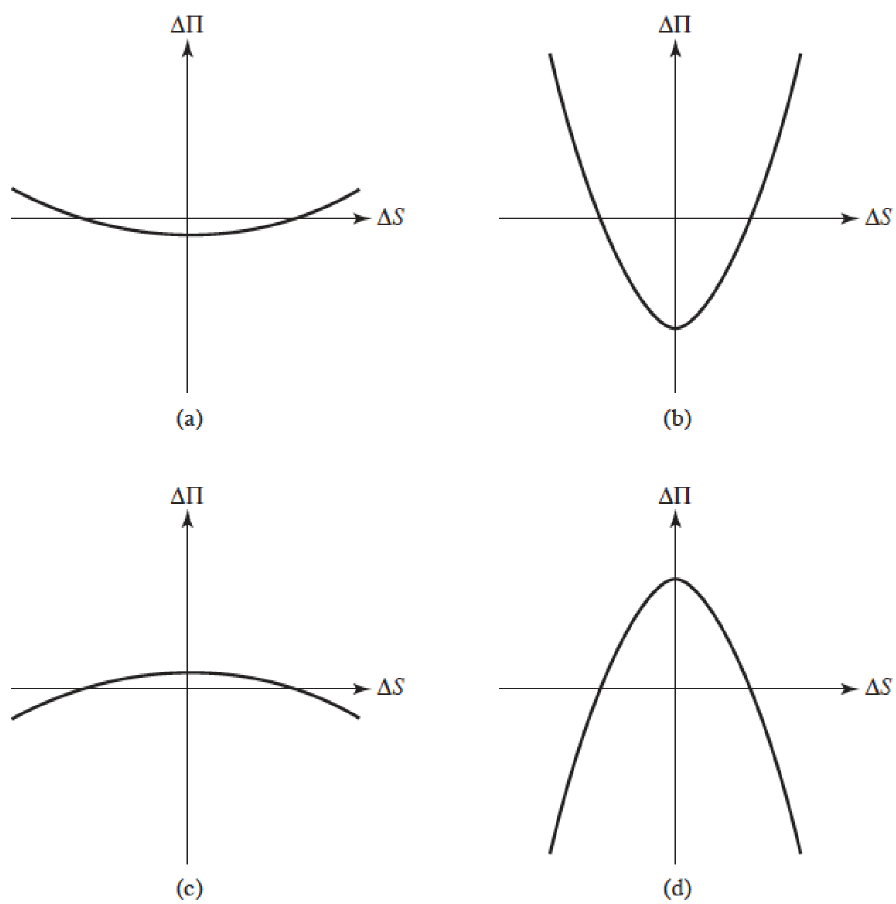

Figure 5.4: Relationship between $\Delta\Pi$ and ΔS in time Δt for a portfolio that is delta neutral with (a) slightly positive gamma, (b) large positive gamma, (c) slightly negative gamma, and (d) large negative gamma

Theta value is likely to be negative when gamma value is positive. The value of portfolio decreases given that price S underlying asset does not change, but if there is a big negative or positive change in S then it increases. Theta is likely to be positive when value

of gamma is negative and hence there is a rise in portfolio value given S remains unchanged but value decreases if S changes by a large negative or positive amount. As the gamma absolute value rises, the portfolio's value sensitivity to S also rises.

5.7 Vega of European stock options

It is assumed by the Black-Scholes-Merton model that the volatility is constant and therefore, calculating vega from the model might seem strange as the volatility in reality follows a stochastic process. On the contrary, the fact is that the vega that is computed using stochastic volatility model is quite same to the vega calculated using Black-Scholes-Merton model. Thus, calculating vega from a constant volatility model is logical.

The vega of an option represents a point change in the option theoretical value for every single percentage point change occurring in the underlying contract volatility.

For example, an option with vega equal to 0.20, with \$4.55 theoretical value at a volatility rate of 21%, will have \$4.75 theoretical value at a volatility rate of 22% or \$4.35 at a volatility of 20%.

For a European put or call option on a stock that is non-dividend paying, vega is

$$\nu = S_0\sqrt{T}N'(d_1)$$

where d_1 is defined by the equation (4.28) of the pricing formula of Black-Scholes-Merton model and the probability density function $N'(x)$ for a standard normal distribution.

A long put or call position vega in an American or European option is positive continuously and changes with the underlying asset price as shown in the following figure.

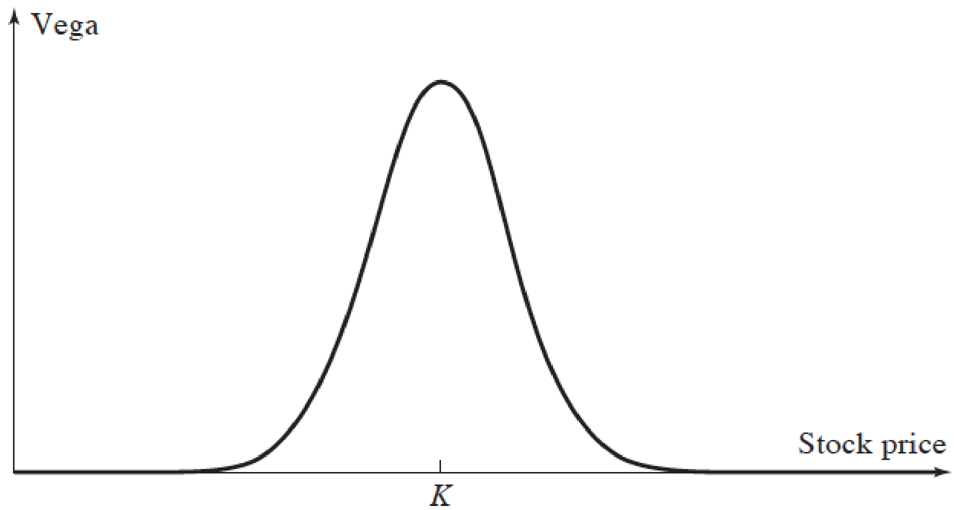

Figure 5.5: Variation of an option's vega with respect to the underlying stock price

We can see from the graph that the highest vegas is for the at-the-money options and hence this changes instantly with changes in volatility in terms of total point change. However, out-of-the-money options are the most affected by changes in volatility in terms of percentage change.

For example, assume a volatility of 15% and an out-of-the-money and at-the-money option with theoretical values of \$0.50 and \$2 respectively. If our estimation of volatility rises to 20%, the premiums will become \$3 and \$1. The at-the-money option shows the greatest increase in points (+\$1) but the out-the-money option shows the greatest percentage increase (+100%).

The vega of an option declines as the derivative instrument gets closer to maturity. Thus, long-term options show more responsiveness to a volatility change as compared with options which are short-term with the same characteristics but a closer expiration date. This brings an important concept about option's evaluation. The further the expiration date of an option, the more time for volatility to take effect.

Traders can take advantage of this situation by doing a time spread strategy when they expect a change in implied volatility. In fact, a trader, who believes implied volatility is

about to increase, can sell a short-term option and buy a long-term option having similar strike prices. In this scenario, according to the time spread strategy, the trader is long. Since the option which is long-term will be more sensitive than the short-term option to the change in implied volatility, the long-term option's theoretical value will increase quicker than the short-term one. This results in an increase in the spread between the two options and therefore generating a profit for the trader when he closes out his position entirely at expiration of the long-term option.

On the other half, a trader will write a time spread (short the long-term option + long the short-term option) if he predicts the implied volatility to decrease in the future.

It is important to notice that in the case of a time spread, a trader has different opinions on the implied volatility and volatility. In fact, a long time spread will become profitable if the underlying market sits still and the implied volatility increases at the same time. A short time spread will become profitable if the underlying market shows some movements but there is a fall in implied volatility.

The vega, ν, of a derivatives portfolio, is defined as rate of change of portfolio value with respect to the underlying asset volatility given by

$$\nu = \frac{\partial \Pi}{\partial \sigma}$$

For a vega that is greatly negative or greatly positive, the value of portfolio responds instantly to small volatility changes. If it approximates to zero, changes in volatility do not have much effect on the value of the portfolio. An underlying asset has a vega equal to zero. On the contrary, a portfolio vega is possible to alter the same way gamma is altered, by addition of a position in traded option. Let ν represent portfolio's vega and ν_T represent the traded option vega, a $-\nu/\nu_T$ position in the traded option would make the portfolio vega neutral immediately. However, a gamma neutral portfolio will not generally be vega

neutral, and the reverse is also true.

Gamma neutrality provides a hedge from big price changes of the underlying asset between rebalancing. Protection against the variable σ is achieved by vega neutrality.

5.8 Rho of European stock options

The rho describes the responsiveness of theoretical value of an option to an interest rate change. Call stock options have a positive rho while put stock options have a negative rho.

For a European call option on a stock that is non-dividend paying, we have

$$\rho_{Call} = KTe^{-rT}N(d_2)$$

where d_2 is defined by the equation (4.29) of the pricing formula of Black-Scholes-Merton model. For a European put option, we have

$$\rho_{Put} = -KTe^{-rT}N(-d_2)$$

Deeply in-the-money options show the greatest rho as their premiums are high and need a consequent cash outlay. Finally, the greater is time in reaching the date of expiration, the higher the value for rho.

Options and underlying contracts that are subject to futures type settlement, like options on futures that do not require any cash outlay until closing out the positions, are not affected by interest rate changes and therefore have a rho equal to 0. All other options subject to stock type settlement will be affected by interest rate changes and therefore will have a rho measure.

The rho of an options portfolio is defined as the rate of change of the portfolio value with respect to the interest rate is

$$\frac{\partial \Pi}{\partial r}$$

It is a measurement of how sensitive the portfolio value is to the interest rate change holding all other parameters and market conditions constant.

5.9 Elasticity of European stock options

The elasticity of an option is the relative percentage sensitivity of the option's theoretical value for specific change in percentage of the underlying contract's price.

For example, let a call have a \$2.50 theoretical value and \$50 be the price of the underlying asset. Let the delta of call be 0.25 and the underlying contract is assumed to rise to \$52. In percentage terms, the call value altered 5 times as quickly as the value of the underlying asset since there was a rise of 4% (2/50) in the underlying contract while the rise in call was 20% (0.5/2.50). In this case, the call option has an elasticity of equal to 5.

The elasticity of an option can be referred as the leverage value of an option and can be computed as follow

$$Elasticity = \frac{Underlying\ price}{Theoretical\ value\ option} \times delta\ option$$

In our example, the computation becomes

$$Elasticity = \frac{50}{2.50} \times 0.25 = 5$$

5.10 The reality of hedging

Ideally, it is possible for a trader to rebalance repeatedly his portfolio for the purpose of maintaining all Greeks equal to zero. In practice, this is not possible. When the portfolio is managed by the trader dependent on a unique underlying asset, delta is made zero or close to zero by the trader at least one time daily by underlying asset trading. Unluckily, a zero vega and a zero gamma are not easy to achieve for a trader since finding options or other derivatives traded in the required amount at competitive prices can be difficult.

A trader must consistently rebalance its portfolio by buying and selling options and underlying contracts for remaining delta neutral. In order to find an option's delta, the trader has plugged in some inputs in the theoretical pricing model to get option's theoretical price but also the various option Greeks. A trader must be aware that the Greeks values he obtained are based on his volatility assumption of the underlying contract and can be changing at any moment. The derivatives trader should have this understanding that his market conditions estimates is the basis for delta neutral position and he cannot guarantee that these estimates are correct. This why many traders use the implied deltas of the option to maintain a delta neutral position.

Finally, one important data that a trader must take into consideration is the theoretical edge. This figure tells him if his position is profitable or not according to the assumptions the trader has made on the market conditions. Obviously, a trader is looking for a positive theoretical edge as he expects to make a profit on his strategy.

Chapter 6
Estimating volatilities

6.1 Using historical data to estimate volatility

For estimating stock price volatility, we take an example having the length of time of fixed intervals τ (every day, week, month) expressed in years. Defining total numbers of observations as n+1 and stock price at the end of the ith interval by S_T where i = 0, 1,...., n. Let

$$u_i = \ln(\frac{S_i}{S_{i-1}}) \quad \text{for } i = 1, 2, ..., n$$

For the standard deviation s of the u_i a reasonable estimate is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}u_i^2 - \frac{1}{n(n-1)}(\sum_{i=1}^{n}u_i)^2}$$

where $\bar{u}$ is the mean of the u_i. The standard deviation of the u_i can be deduced from equation (4.11) $\sigma\sqrt{\tau}$. An approximation of $\sigma\sqrt{\tau}$ is the variable s and therefore, σ can be approximated as $\widehat{\sigma}$ as follow

$$\widehat{\sigma} = \frac{s}{\sqrt{\tau}}$$

The standard error of the approximation $\widehat{\sigma}$ is approximately equal to $\widehat{\sigma}/\sqrt{2n}$.

Choosing the number of observations n can be delicate. The more data n we have, the more accurate σ. The stock's volatility parameter σ is susceptible to change with time and

hence very old data might be irrelevant in predicting a future volatility. Several compromises that seem to work is to make use of the most recent 90 to 180 days for daily closing prices data or alternatively, we can set n as the amount of days to which we apply the volatility. For example, to use volatility estimate on a 9-month option, the last 9 months daily data are used.

Let's look at an example of forecasting the future volatility from historical data. Consider a stock prices sequence during 31 consecutive trading days. Here, n = 30. We obtain the following table:

	A	B	C	D
1	*Day i*	*Closing stock*	*Price relative*	*Daily return*
2		*price* S_i	S_i / S_{i-1}	$u_i = \ln(S_i / S_{i-1})$
3	0	30.00		
4	1	30.65	1.021667	0.021435
5	2	30.68	1.000979	0.000978
6	3	31.72	1.033898	0.033336
7	4	31.24	0.984868	-0.015248
8	5	31.05	0.993918	-0.006101
9	6	30.20	0.972625	-0.027757
10	7	30.20	1.000000	0.000000
11	8	29.84	0.988079	-0.011992
12	9	28.40	0.951743	-0.049461
13	10	28.65	1.008803	0.008764
14	11	28.95	1.010471	0.010417
15	12	29.50	1.018998	0.018820
16	13	29.20	0.989831	-0.010222
17	14	29.20	1.000000	0.000000
18	15	29.00	0.993151	-0.006873
19	16	28.35	0.977586	-0.022669
20	17	28.75	1.014109	0.014011
21	18	28.98	1.008000	0.007968
22	19	29.26	1.009662	0.009615
23	20	29.13	0.995557	-0.004453
24	21	29.48	1.012015	0.011943
25	22	29.35	0.995590	-0.004420
26	23	30.05	1.023850	0.023570
27	24	30.26	1.006988	0.006964
28	25	30.77	1.016854	0.016713
29	26	30.77	1.000000	0.000000
30	27	29.89	0.971401	-0.029016
31	28	29.92	1.001004	0.001003
32	29	30.65	1.024398	0.024106
33	30	30.75	1.003263	0.003257
34	31	31.15	1.013008	0.012924
35			$\Sigma u_i =$	0.037617
36			$\Sigma u_i^2 =$	0.009426

Table 6.1: Computation of future volatility

Thus, the standard deviation estimate of the each day return is

$$\sqrt{\frac{0.009426}{29} - \frac{0.037617^2}{30 \times 29}} = 0.01798$$

or 1.798%. Assuming trading days per year are 252, $\tau = 1/252$, and hence volatility estimate per year is equal to $0.01798\sqrt{252} = 0.2855$ or 28.55%. The standard error of this volatility estimate is

$$\frac{0.2855}{\sqrt{2 \times 30}} = 0.0369$$

or 3.69% per year.

The previous analysis assumes that the stock is not paying any dividends. Nevertheless, the model can be modified for a stock that pays a dividend where during the time interval the return u_i which includes the ex-dividend date is

$$u_i = \ln(\frac{S_i + D}{S_{i-1}})$$

where D is the amount of dividend. In other time intervals the returns remain the same

$$u_i = \ln(\frac{S_i}{S_{i-1}})$$

6.2 Finding the implied volatilities in the Black-Scholes-Merton model

The underlying stock price volatility σ in the pricing formulas of Black-Scholes-Merton is the sole parameter unable to be observed directly and must be therefore estimated by traders from historic stock prices as we have discussed earlier. In practice, however, traders usually work with the implied volatilities. The Black-Scholes-Merton pricing formulas imply these volatilities to give the option prices observed in the marketplace.

To understand how a trader can determine an implied volatility, consider a European call option whose market price is 1.875 when $r = 0.1$, $S_0 = \$21$, $T = 0.25$ and $K = \$20$. Thus, when parameters are substituted in the pricing formula of Black-Scholes-Merton of a call, the option's implied volatility is the value σ that gives $c = 1.875$.

Unfortunately, inverting this formula and expressing σ as a function of S_0, K, r, T is not possible. However, it is possible to use an iterative process to calculate the implied volatility σ. For example, if we enter $\sigma = 0.20$ in the equation (4.27), the value of c is computed equal to 1.76, which is very less. Since we know that c is an increasing function of σ, this implies a higher value of σ. When trying to put $\sigma = 0.30$, we obtain $c = 2.10$ the value being very large. Hence it can be deduced that the implied volatility σ must be in the range [0.20, 0.30]. The implied volatility σ must even lie between 0.20 and 0.25 as $c > 1.875$ when $\sigma = 0.25$. By iteration and shortening the range for σ, it is possible to find the correct implied volatility. For the given example, the implied volatility σ is equal to 0.235, or 23.5% per year.

In order to monitor the opinion of the market about a particular stock's future volatility, implied volatilities are used. Although historical volatilities have the property of backward looking, the implied volatilities look forward and therefore give a better estimation of the future volatilities. Traders in the marketplace commonly state the option's implied volatility instead of its price since the volatility σ has lesser variations as compared to the price of the option.

6.3 The volatility smile

If the expiration date and the risk-free interest rate are fixed and known, the implied volatility should be constant no matter the "moneyness" of the option. This implies that the volatility curve should be horizontal which is not conform to the real life as for each

parity, there is a different level of volatility. Various volatility levels for each parity of the option is not appropriately reflected using the Black-Scholes-Merton model, which being an important factor to be considered in option pricing. Hence, it is important to take under consideration the volatility input as a variable fluctuating over time and consider volatility as a stochastic process.

The Black-Scholes-Merton model is used by the traders but not in the manner that was intended by its founders Black, Scholes and Merton. In fact, instead of assuming that the underlying asset has a constant volatility during the option's life, it is allowed by the traders to use the volatility to price the option so that it is dependent upon its strike price and time remaining until maturity. The option's implied volatility with a particular life duration when written as a function of option's strike price is referred as volatility smile.

Rubinstein (1985, 1994) and Jackwerth and Rubinstein (1996) studied the volatility smile for equity options. As far as the crash of the stock market in October 1987, the traders use the volatility smile to price equity options (both on stock indices and on individual stocks) which can be generalized as follow:

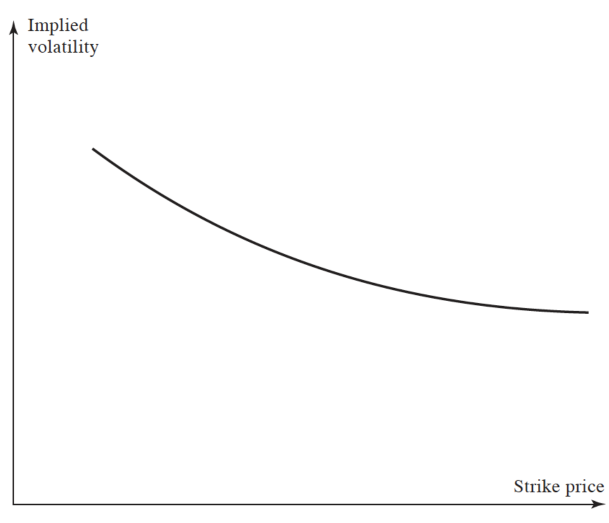

Figure 6.1: Volatility smile for equity options

At times we call this the skewed volatility. Implied volatilities were more independent of strike price before October 1987. As there is a rise in the strike price the implied volatility decreases. To price an option having a low strike price (i.e. a deep in-the-money call or a deep out-of-the-money put) is considerably more as compared to the one used to price a high strike price option (i.e. a deep out-of-the-money call or a deep in-the-money put).

For equity options the volatility smile tallies to the implied probability distribution shown in the graph below by a solid line. A distribution which is lognormal having the same standard deviation and mean as the implied distribution is represented by the dotted line in the graph below.

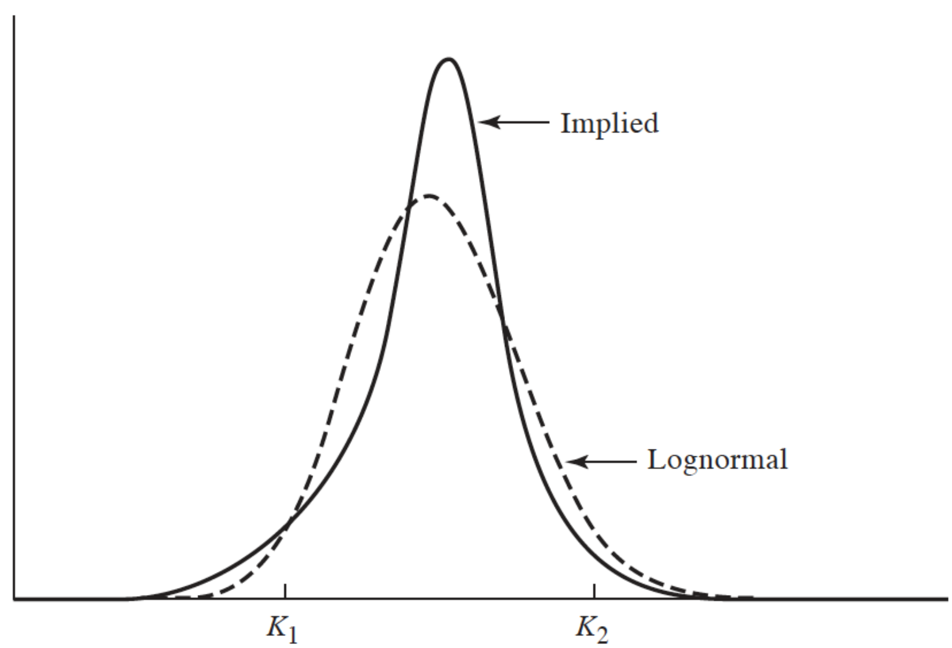

Figure 6.2: Implied distribution and lognormal distribution for equity options

We can see from the graph that the implied distribution as compared to the lognormal distribution has a less heavy right tail and a heavier left tail. When the implied distribution is used a deep out-of-the-money call having a strike price of K_2 has a price that is less as compared to when we use the lognormal distribution. The reason is that the option pays

off only if K_2 is below the stock price, and the probability is lower as compared to the implied probability distribution relative to the lognormal distribution. Hence, the implied distribution should give for the option a lower price. A low price is followed by a lower implied volatility and that is what is observed in the graph of the volatility smile.

Take for consideration a put option which is deep out-of-the-money having a K_1 strike price. If the stock price is less than K_1 only then the option will pay off. The graph of the implied distribution shows that the probability is less for the lognormal distribution as compared to the implied probability distribution. Hence, a relatively high price should be given for the implied distribution, which means that implied volatility will also be relatively high for this put option which can be seen in the volatility smile graph for equity options.

6.3.1 Why the volatility smile is the same for calls and puts

A European put and call option with identical time to maturity and strike price have equal implied volatility. This means that the volatility smile for European calls with a certain maturity is the same as that for European puts with the same maturity. Thus, we do not have to worry about whether the options are calls or puts.

Let's recall the put-call parity from section 4.n. equation (4.34), a relationship between the price of a European put and call option with similar characteristics, when the underlying asset is paying a dividend yield q

$$p + S_0 e^{-qT} = c + Ke^{-rT}$$

where p and c are the European put and call price, K being the strike price, and T representing time to maturity. The underlying stock price is given by the variable S_0, the risk-free interest rate r for maturity T and the asset yield q.

The put-call relationship does not assume any asset price probability distribution in the future however it is based on a no arbitrage idea. In fact, the put-call parity is true

when the asset price distribution is lognormal or not.

Let c_{BSM} and p_{BSM} be European call and put options values for a particular value of the volatility calculated from the Black-Scholes-Merton model. Also let p_{mkt} and c_{mkt} be the market values of these options. Since for the Black-Scholes-Merton model, the put-call parity holds we must have

$$p_{BSM} + S_0 e^{-qT} = c_{BSM} + Ke^{-rT} \tag{6.1}$$

In the arbitrage opportunities being absent, for the market prices the put-call parity also holds, so that

$$p_{mkt} + S_0 e^{-qT} = c_{mkt} + Ke^{-rT} \tag{6.2}$$

Subtracting equation (6.2) from (6.1), we get

$$p_{BSM} - p_{mkt} = c_{BSM} - c_{mkt} \tag{6.3}$$

The above equation shows that when the Black-Scholes-Merton model is employed for pricing a European put option, the dollar pricing error should be equal to the dollar pricing error when pricing a European call option having the same time to maturity and strike price.

Let the implied volatility be 25% for the put option. This implies $p_{mkt} = p_{BSM}$ when a volatility value used in the Black-Scholes-Merton model is 25%. It follows from the last equation (6.3) that $c_{BSM} = c_{mkt}$ when this volatility level is used. The argument concludes that European call option implied volatility is same as the European put option implied volatility given the two options have the same maturity date and strike price. In other words, for a given date of maturity and strike price, the volatility employed in pricing a European call option should be equal to that employed in pricing the European put option

using the model of Black-Scholes-Merton.

Given this does not hold, the option prices reflect an arbitrage opportunity. With same date of maturity and strike price, there can be a profitably execution by a trader doing a conversion if the calls were overpriced with respect to the puts (i.e. implied volatility call > implied volatility put); it will be profitable to execute the reverse conversion by the trader if the calls were underpriced with respect to the puts (i.e. implied volatility put > implied volatility call).

Finally, the fact that for a given strike price and maturity date, we have

$$\textit{implied volatity put} = \textit{implied volatility call}$$

The volatility smile (the relation among strike price for a particular maturity and implied volatility) and the volatility term structure (the relationship between maturity for a particular strike price and implied volatility) must be the same for European puts and European calls.

6.3.2 Alternative ways of characterizing the volatility smile

The link among implied volatility and option's strike price is given by the volatility smile. The relation is dependent on the underlying asset current price. If the underlying stock volatility increases, the volatility smile tends to move up; if the underlying stock volatility falls, the volatility smile tends to move down. Similarly, as there is a rise in stock price, the volatility skew tends to move to the right; when there is a decrease in stock price, the volatility skew tends to move to the left. Thus, in order to obtain a much more stable volatility smile, the volatility smile is computed by taking the relation among the implied volatility and K/S_0 rather than taking the relation among the implied volatility and K.

Suppose a trader wishes to integrate a volatility skew into his theoretical pricing model. What should the trader do? As the price of the underlying contract changes the trader could shift the skew left or right. If several weeks later, the underlying contract was to increase by $4, the constant shape of the skew can be shifted by 4 points to the right. Simultaneously, the trader could lower or raise the skew, using the reference point of at-the-money options to reflect his opinion about the implied volatility being either too low or too high.

It might be a reasonable approach to shift the complete skew given that a trader assumes that the skew will not change even though market conditions are changing. Is it likely? It is probable that the implied volatilities of different exercise prices are dependent on how the marketplace sees the chances of underlying market having large movements. Every movement is relative with respect to the underlying price and to time. An increase or decrease of 8 points in the underlying contract is smaller with the underlying contract at 185 (4.32% move) than with the underlying contract quoting 145 (5.52% move). An increase of decrease of 8 points over a 2-month period is smaller than a 8-point increase or decrease over a 2-week period.

Thus, as market conditions change the volatility skew will probably change. The shape of the skew changes and it becomes more severe with passing time. Its position also changes with the fluctuating underlying contract price and implied volatility. The trader trying to incorporate into a theoretical pricing model the volatility skew is facing a problem. Traders use pricing models to evaluate options but also to manage risk under changing market conditions. To incorporate a skew into the model built by the trader, he should know how the skew is going to vary with market conditions changing.

We have seen earlier that movements in the underlying contract are relative to the underlying contract current price and expiration time left. The relative amount of movement

required to reach an exercise price in the Black-Scholes-Merton model is fully expressed as

$$\frac{1}{\sqrt{T}} \ln \frac{K}{S_0} \tag{6.4}$$

As we employ the Black-Scholes-Merton model to evaluate the various options implied volatilities, and since the Black-Scholes-Merton model expresses strike prices as we have seen above, it is coherent to express the exercise prices skew similarly.

For the purpose of volatility scale generalization, it might be a good idea to express all volatilities for a given skew in terms of theoretical at-the-money option volatility.

Such as take an underlying stock contract price be \$46.68. The option implied volatility with a strike price of \$46.68 (a theoretically at-the-money option) would be about 2.67%. The volatility at each exercise price can be expressed as the difference in that volatility of exercise price and 2.67%. Using this approach, the 51 exercise price, with an implied volatility of 2.95% would be assigned a value of $2.95 - 2.67 = 0.28\%$. The 44 exercise price, with an implied volatility of 2.78% would be assigned a value of $2.78 - 2.67 = 0.11\%$.

When implied volatilities are unchanged this method works well. But assume the stock option market implied volatility were to double to 5.34%. The implied volatilities under these circumstances at every exercise price can double too. The 51 exercise price will have an implied volatility of 5.90% instead of 2.95%. Since in the marketplace implied volatility may alter, a method of linking the overall change in implied volatility to the change in each implied volatility exercise price should be devised. The most simple way to carry this out is expressing at each exercise price the implied volatility as a percentage point of the implied volatility of the at-the-money option.

Suppose an implied volatility of 2.67% for at-the-money option then the 51 exercise price with an implied volatility of 2.95% would get a value of $2.95/2.67 = 110.5\%$. The 44 exercise price, having an implied volatility of 2.78% would get a value of $2.78/2.67 = 104.1\%$

The volatility smile can be computed as the link in the implied volatility and K/F_0 where F_0 is the asset's forward price for a contract having same maturity date as the options under consideration. Traders also define an at-the-money as an option where $K = F_0$ and not as an option where $K = S_0$ as F_0 is the expected stock price on the option's maturity date in a risk neutral world. Thus, equation (6.4) becomes

$$\frac{1}{\sqrt{T}} \ln \frac{K}{F_0} \tag{6.5}$$

and implied volatility at each exercise price can then be expressed by the traders as a percent of the at-the-money implied volatility.

Another possibility is to define a volatility smile as the link between option delta and the implied volatility. In this case, an at-the-money option is defined as a put option with a delta of -0.5 or a call option with a delta of +0.5.

6.4 The volatility term structure and volatility surfaces

Traders allow the implied volatility to depend on time to maturity as well as strike price. Implied volatility is likely to be a function of time to maturity with property of an increasing function given that short-dated volatilities are having lower values in the past. The reason for this is that it is anticipated that in the future volatilities would rise. For the case where short-dated volatilities are high in the past, volatility is likely to be a decreasing function of maturity. This is because that it is expected that in the future the volatility will drop.

Volatility surfaces unite the volatility term structure with volatility smile to estimate the right value of volatilities used in valuing an option with any maturity and strike price. Let's look at an example of the volatility surface for an option on an underlying stock.

	K/S_0				
	0.90	*0.95*	*1.00*	*1.05*	*1.10*
1 month	14.2	13.0	12.0	13.1	14.5
3 month	14.0	13.0	12.0	13.1	14.2
6 month	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

Table 6.2: Volatility surface

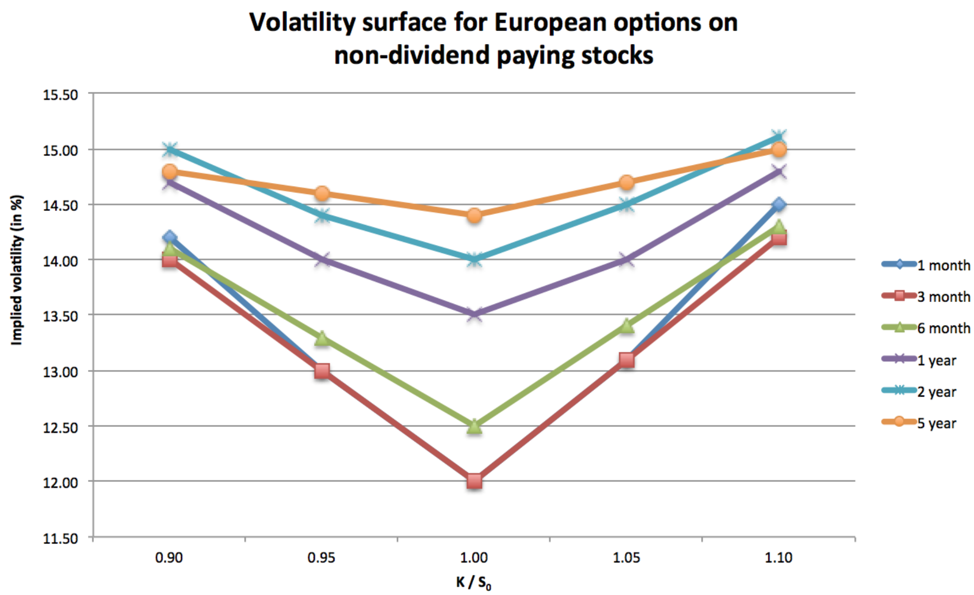

Figure 6.3: Volatility surface for European options on non-dividend paying stocks

One dimension of the table is K/S_0, whereas time to maturity is the other dimension. Implied volatilities computed using the model of Black-Scholes-Merton is shown in graph. It can be concluded from the graph that the volatility smile gets less definite as there is a rise in option maturity. When a new option has to be valued, the trader looks up in the table or graph above.

For example, suppose a trader wishes to evaluate a 9-month option with 1.05 is given as its K/S_0 ratio. Interpolation is done by the trader between 13.4% and 14.0% in order to get a 13.7% volatility value. The Black-Scholes-Merton model uses this volatility to get

the option market price with maturity 9 months and a ratio $K/S_0 = 1.05$.

The volatility smile shape is dependent on the option maturity. The smile tends to be less pronounced as the table illustrates with the maturity of the option rising. When opted by traders to define the volatility smile as a link among equation (6.5) and implied volatility

$$\frac{1}{\sqrt{T}} \ln(\frac{K}{F_0})$$

instead of defining it as a link among K and implied volatility, the volatility smile is much independent of the time to maturity.

A trader making use of the model of Black-Scholes-Merton can use the volatility skew to detect whether options are underpriced or overpriced. To determine the relative value of options between each other and find arbitrage opportunities, the trader can also use the volatility smile. For example, given a maturity date, a trader can short a butterfly if he believes that the options with lower and higher exercise price are overpriced compared to an option with a middle exercise price.

The skew is used to adjust bids and offers by a market maker. When the market maker possesses a gamma position that is highly positive, he can reduce its risk by shifting downward the skew and selling options. The result of this would be offers and bids on all options being reduced. When the market maker is short by a big amount of out-of-the-money puts or calls, the skew can be adjusted by wings being raised. As a result offers and bids would increase on out-of-the-money options.

6.5 The greeks with the volatility smile

The volatility smile complicates the computation of Greek letters. Suppose that the relationship between an option implied volatility and K/S with a specific maturity time remains the same. As the underlying asset price moves, the option's implied volatility also

changes to reflect the moneyness of the option. The previous Greek formulas seen earlier are not correct anymore. Such as call option delta becomes

$$\frac{\partial c_{BSM}}{\partial S} + \frac{\partial c_{BSM}}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}}{\sigma S}$$

where c_{BSM} is the call option price given by Black-Scholes-Merton presented as an asset price S function and the implied volatility σ_{imp}. Let's look at the impact of this new formula for an equity call option. Recall that volatility is a decreasing function of K/S and therefore, as the asset price rises there is a rise in implied volatility so that

$$\frac{\partial \sigma_{imp}}{\partial S} > 0$$

Thus, as compared to the Black-Scholes-Merton model, the delta is greater.

6.6 Determination of implied risk neutral distributions using volatility smiles

The European call option price on an underlying asset with maturity T and strike price K can be presented as follow

$$c = e^{-rT} \int_{S_T=K}^{\infty} (S_T - K) g(S_T) dS_T$$

where the constant interest rate is r, the asset price is presented as S_T at time T, and the risk neutral probability density function is given by g of S_T. We get by differentiating with respect to K one time

$$\frac{\partial c}{\partial K} = -e^{-rT} \int_{S_T=K}^{\infty} g(S_T) dS_T$$

Differentiating again with respect to K gives

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K)$$

This shows that the probability density function of g is given by

$$g(K) = e^{rT} \frac{\partial^2 c}{\partial K^2} \tag{6.6}$$

This result, from Breeden and Litzenberger (1978), allows risk neutral probability distributions to be estimated from volatility smiles. Assume that c_1, c_2 and c_3 are the European call option prices of T-year with strike prices of $K - \delta$, K, and $K + \delta$ respectively. Assuming δ is small, an estimate of $g(K)$ obtained by approximating the partial derivative in the equation (6.6) is

$$e^{rT} \frac{c_1 + c_3 - 2c_2}{\delta^2}$$

6.7 Different weighting schemes for the volatility estimate

We are now going to look at some models for which volatilities are not constant such as the autoregressive conditional heteroscedasticity (ARCH), exponentially weighted moving average (EWMA), and generalized autoregressive conditional heteroscedasticity (GARCH). Volatility is comparatively low during some periods, while in others, it may be relatively high. Volatility variations through time are recorded by these models.

6.7.1 Estimating volatility

Define σ_n as the market variable volatility on day n by estimation at day n-1 end. On day n, the volatility square σ_n^2 is the variance rate. Suppose at the end of day i the market variable value is S_i. We have

$$u_i = \ln \frac{S_i}{S_{i-1}}$$

where the variable u_i is the return compounded continuously during day i. An unbiased variance rate estimate on each day is σ_n^2, using m observations which are most recent on

the u_i is

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2 \tag{6.7}$$

where $\bar{u}$ is the mean of the u_i given by

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}$$

For the variance rate σ_n^2 simplifying the formula (6.7) by following the steps below let:

- Define the u_i as the market variable percentage change during the day i so that

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}}$$

- $\bar{u}$ is taken as zero
- replace $m-1$ by m

Thus, the new variance rate formula becomes

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-1}^2 \tag{6.8}$$

6.7.2 The ARCH(m) model

The equation (6.8) gives equal weight to u_{n-1}^2, u_{n-2}^2,..., u_{n-m}^2. Estimating the volatility present level, σ_n, is the purpose here. Hence, the most recent data plays a more significant role. Following is the model which satisfies this condition

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

The α_i represents the variable that gives weight for each individual observation i days ago.

The α_i's are having positive values. The sum of weight should equal 1

$$\sum_{i=1}^{m} \alpha_i = 1$$

It is possible to extend the model assuming that a long-run average variance rate exists and some weight should be given to this. The model becomes

$$\sigma_n^2 = \gamma V_L \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where V_L represents the variance rate in the long run and the weight assigned to V_L is γ. Since the sum of the weight should equal to 1, we have

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

This model was suggested by Engle (1982) and is called the ARCH(m) model. The variance estimate is based on an average variance in the long-run and m number of observations. The older an observation, the less weight it is given.

6.7.3 The EWMA model

A particular case of the ARCH model is the exponentially weighted moving average (EWMA) model in which the weights α_i decrease exponentially as moving in past time. In particular, we have $\alpha_{i+1} = \lambda \alpha_i$, where λ represents a constant lying between values 0 and 1. Thus, to update volatility estimates the formula turns out to be

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2 \tag{6.9}$$

where σ_n is the estimate for a variable volatility for day n and is computed using σ_{n-1} and u_{n-1} represents daily percentage change that is the most recent in the variable.

Let's prove that the equation (6.9) corresponds to weights that decrease exponentially. It is possible to rewrite the formula (6.9) as follow by substituting σ_{n-1}^2 to get

$$\sigma_n^2 = \lambda[\lambda\sigma_{n-2}^2 + (1-\lambda)u_{n-2}^2] + (1-\lambda)u_{n-1}^2$$

or

$$\sigma_n^2 = (1-\lambda)(u_{n-1}^2 + \lambda_{n-2}^2) + \lambda^2\sigma_{n-2}^2$$

In a similar way substituting for σ_{n-2}^2 gives

$$\sigma_n^2 = (1-\lambda)(u_{n-1}^2 + \lambda u_{n-2}^2 + \lambda^2 u_{n-3}^2) + \lambda^3\sigma_{n-3}^2$$

Continuing in this way gives

$$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-i}^2 + \lambda^m\sigma_{n-m}^2$$

For bigger values of m, the term $\lambda^m\sigma_{n-m}^2$ is small enough to be ignored and hence the EWMA model equation and the following equation are the same

$$\sigma_n^2 = \sum_{i=1}^{m}\alpha_i u_{n-i}^2$$

with $\alpha_i = (1-\lambda)\lambda^{i-1}$. Moving back in time the weights for the u_i decrease at a rate given by λ. Hence every weight is λ times the last weight.

To maintain attraction about using EWMA model is that a small amount of data needs to be stored. For a given time, the current variance rate estimate and the latest observations on the market variable are the only values to be stored. Then, the estimate of the variance rate is updated as a new observation on the market variable is obtained while the variance rate previous values and the older market variable values can be ignored.

The model of EWMA is designed to track volatility changes. Assume a large movement in the market variable during day $n-1$ so that u_{n-1}^2 is big. This leads the current volatility estimate to go up. The λ value shows how the daily volatility estimate is affected by the very latest daily percentage change. A low λ value leads to greater weight being given to the variable u_{n-1}^2 when calculating σ_n. In this case, volatility estimates are highly volatile. A value of λ that is high (i.e. a value close to 1) produces daily volatility estimates which are slow responding to the daily percentage changes being most recent.

The bank JP Morgan made a database called RiskMetrics which uses the $\lambda = 0.94$ for EWMA model to update estimates for daily volatility. The bank found out that the value 0.94 for λ is the one that gives the variance rate closest estimate to the variance rate that is realized.

6.7.4 The GARCH(1,1) model

Invented in 1986 by Bollerslev, the GARCH(1,1) model enables to calculate σ_n^2 using an average variance rate V_L from the long-run and from σ_{n-1} and u_{n-1}. The equation for GARCH(1,1) is

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \tag{6.10}$$

where γ, α and β are the assigned weights for V_L, u_{n-1}^2 and σ_{n-1}^2 respectively. The weights must sum to 1, thus we have $\gamma + \alpha + \beta = 1$. A particular case of GARCH(1,1) model is the EWMA model taking the values $\gamma = 0$, $\alpha =$ 1-λ and $\beta = \lambda$.

It is indicated by the GARCH(1,1) model that σ_n^2 is based on the latest variance rate estimate and latest observation of u^2. The GARCH(p,q) model that is more general computes σ_n^2 from the latest p observations on u^2 and latest variance rate estimates q. Today, the most used GARCH model is the GARCH(1,1).

Let $w = \gamma V_L$. The GARCH(1,1) model (equation (6.10)) becomes

$$\sigma_n^2 = w + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

This model form is employed for parameters estimation. Once we estimate w, α and β, the term 1-α-β is computed. Next computation for V_L the long-term variance is done as w/γ. For GARCH(1,1) model to be stable, $\alpha + \beta < 1$ else weight becomes negative that is being applied to the long-term variance.

Consider the model of GARCH(1,1) from daily data as

$$\sigma_n^2 = 0.000002 + 0.13 u_{n-1}^2 + 0.86 \sigma_{n-1}^2$$

where $\alpha = 0.13$, $w = 0.000002$ and $\beta = 0.86$. Because $\gamma = 1 - \alpha - \beta$, we have $\gamma = 0.01$. Because $w = \gamma V_L$, we have $V_L = 0.0002$. Thus, the model implies a long-run daily average variance of 0.0002. The corresponding volatility is equal to $\sqrt{0.0002} = 0.014$ or 1.4% per day. Now assume that the volatility estimate is 1.6% per day on day n-1, so that we have $\sigma_{n-1}^2 = 0.016^2 = 0.000256$ and that on day n-1 the market variable decreased by 1%, such that $u_{n-1}^2 = 0.01^2 = 0.0001$. Then we have

$$\sigma_n^2 = 0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023516$$

The new estimate of the volatility is therefore $\sqrt{0.00023516} = 0.0153$ or 1.53% per day.

6.7.5 The weights γ, α, and β

Substituting for σ_{n-1}^2 in the equation of the GARCH(1,1) model gives

$$\sigma_n^2 = w + \alpha u_{n-1}^2 + \beta(w + \alpha u_{n-2}^2 + \beta \sigma_{n-2}^2)$$

or

$$\sigma_n^2 = w + \beta w + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \beta^2 \sigma_{n-2}^2$$

Substituting for σ_{n-2}^2 gives

$$\sigma_n^2 = w + \beta w + \beta^2 w + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \alpha\beta^2 u_{n-3}^2 + \beta^3 \sigma_{n-3}^2$$

Thus, if we keep on extending the formula, it can be observed that the weight applied to u_{n-1}^2 is $\alpha\beta^{i-1}$. There is an exponential decrease in weights at rate β interpreted as the "decay rate". In the EWMA model, it is similar to λ.

For example, if $\beta = 0.9$, then u_{n-2}^2 is only 90% as important as u_{n-1}^2; u_{n-3}^2 is 81% as important as u_{n-1}^2 and so on. The GARCH(1,1) model is very similar to the EWMA model in a sense that they both assign exponentially declining weights to past u^2. However, in the GARCH(1,1) model, the average volatility in the long-run is given some weight.

6.7.6 Choosing between the models

Variance rates, in practice, are likely to be mean reverting. It is recognized in the GARCH(1,1) model that with time the variance is likely to get pulled back to an average level of V_L in the long-run. The GARCH(1,1) model is the same as the model with the variance V following the stochastic process

$$dV = a(V_L - V)dt + \xi V dv$$

where we measure time in days, $a = 1 - \alpha - \beta$ and $\xi = a\sqrt{2}$. This model is mean reverting. There is a drift in the variance pulling it back at rate a to V_L. When $V < V_L$, the variance has a positive drift; when $V > V_L$, the variance has a negative drift. While mean reversion is part of the GARCH(1,1) model, the EWMA model does not show this behavior. Hence, the GARCH(1,1) model is better adapted than the EWMA model.

6.7.7 The maximum likelihood method to estimate parameters

We are now going to see how we can estimate the parameters w, α and β in GARCH(1,1). When $w = 0$, the GARCH(1,1) model tends to be like the EWMA model. The GARCH(1,1) model becomes unstable when w is negative hence the EWMA model becomes more appealing. Estimating parameters using the historical data is known as the maximum likelihood method. Parameter values in such models are selected to maximize the data occurring chances.

Firstly, estimating for a variable X, the constant variance is the main objective from m observations on X by the maximum likelihood method. Suppose the distribution underlying is normal having zero mean and take u_1, u_2,..., u_m as the observations. Denoting variable X variance by v. The probability density function for X given below is defined as the likelihood of u_i being observed when $X = u_i$

$$\frac{1}{\sqrt{2\pi v}} e^{\frac{-u_i^2}{2v}}$$

The likelihood of occurring of m observations in the same order as they are observed in is

$$\prod_{i=1}^{m} [\frac{1}{\sqrt{2\pi v}} e^{\frac{-u_i^2}{2v}}] \tag{6.11}$$

The best estimate of v, by the maximum likelihood method, is hence the value that maximizes this expression. The expression (6.11) being maximized is similar to the logarithm of the expression being maximized. Thus, we want to maximize

$$\sum_{i=1}^{m} [-\ln(v) - \frac{u_i^2}{v}] \tag{6.12}$$

or

$$-m\ln(v) - \sum_{i=1}^{m} \frac{u_i^2}{v} \tag{6.13}$$

Let the differentiation of expression (6.13) with respect to v equal to zero, for v the maximum likelihood estimator is

$$\frac{1}{m}\sum_{i=1}^{m} u_i^2$$

Now, let's see how one trader can use the maximum likelihood method for GARCH(1,1) model parameter estimation. Let $v_i = \sigma_i^2$ be the estimated variance for day i. Suppose the conditional probability distribution of u_i on the variance is a normal distribution. Thus, for the GARCH(1,1) model, the best parameters are the ones that maximize

$$\prod_{i=1}^{m}[\frac{1}{\sqrt{2\pi v_i}} e^{\frac{-u_i^2}{2v_i}}]$$

Now take logarithms, this is the same as maximizing

$$\sum_{i=1}^{m}[-\ln(v_i) - \frac{u_i^2}{v_i}] \tag{6.14}$$

The expression (6.14) is the same as (6.12) except that to search iteratively v is replaced by v_i and for finding the parameters w, α, β in the model for expression maximization. We can implement the GARCH(1,1) method using the solver tool of Excel for searching the parameters values that maximize the likelihood function.

6.7.8 Forecasting future volatility using GARCH(1,1) model

The estimation for variance rate for day n done at the end of day n-1, when GARCH(1,1) is used is given by equation (1.48) or

$$\sigma_n^2 = (1-\alpha-\beta)V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$$

which is the same as

$$\sigma_n^2 - V_L = \alpha(u_{n-1}^2 - V_L) + \beta(\sigma_{n-1}^2 - V_L)$$

In the future on day $n + t$,

$$\sigma_{n+t}^2 - V_L = \alpha(u_{n+t-1}^2 - V_L) + \beta(\sigma_{n+t-1}^2 - V_L)$$

The expected value of u_{n+t-1}^2 is σ_{n+t-1}^2. Hence

$$E[\sigma_{n+t}^2 - V_L] = (\alpha + \beta)E[\sigma_{n+t-1}^2 - V_L]$$

where expected value is denoted by E. Using this equation repeatedly yields

$$E[\sigma_{n+t}^2 - V_L] = (\alpha + \beta)^t(\sigma_n^2 - V_L)$$

or

$$E[\sigma_{n+t}^2] = V_L + (\alpha + \beta)^t(\sigma_n^2 - V_L) \tag{6.15}$$

The equation (6.15) on day $n + t$ forecasts the volatility using the information given by the end of day $n - 1$. If $\alpha + \beta < 1$, the term $(\alpha + \beta)^t(\sigma_n^2 - V_L)$ for increasing t becomes progressively smaller.

The figure below shows that when current variance rate is different from V_L, the trajectory of the expected path followed by the variance rate. As we have seen that the variance rate gives the mean reversion with a reversion rate of $1 - \alpha - \beta$ and a reversion level of V_L.

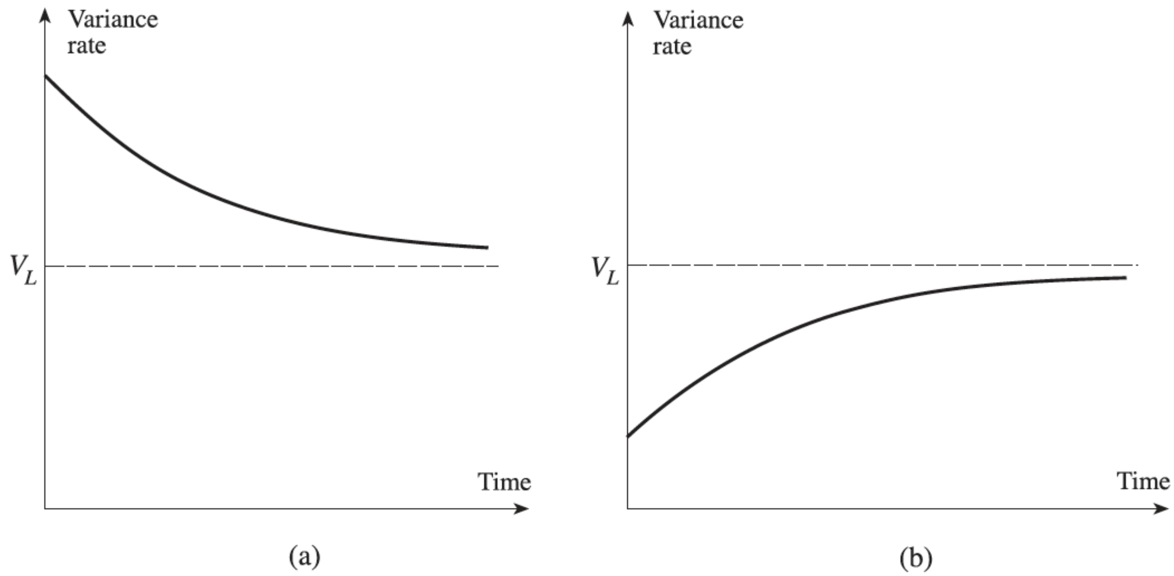

Figure 6.4: Variance rate expected path when (a) current variance rate is above long-term variance rate and (b) current variance rate is below long-term variance rate

The future forecast for the variance rate approaches V_L as we look very far in the future. Thus, this analysis shows the stability of the GARCH(1,1) model when $\alpha + \beta < 1$. Finally, if $\alpha + \beta > 1$, the long-term average variance weight is negative and the process is mean fleeing and not mean reverting.

6.7.9 The volatility term structure

Let be day n. Then we define

$$V(t) = E(\sigma_{n+t}^2)$$

and

$$a = \ln \frac{1}{\alpha + \beta}$$

so the equation (6.15)

$$E[\sigma_{n+t}^2] = V_L + (\alpha + \beta)^t (\sigma_n^2 - V_L)$$

becomes

$$V(t) = V_L + e^{-at}[V(0) - V_L]$$

Here, $V(t)$ is the instantaneous variance rate estimate in t number of days. The variance rate per day average between today and time T is as follow

$$\frac{1}{T}\int_0^T V(t)dt = V_L + \frac{1 - e^{-aT}}{aT}[V(0) - V_L]$$

The closer is the value to V_L as T grows larger.

Defining $\sigma(T)$ as the annual volatility rate that should be used for valuing a T-day option with the model of GARCH(1,1). Taking 252 trading days per year, $\sigma(T)^2$ is the average variance rate per day multiplied by 252 so that

$$\sigma(T)^2 = 252(V_L + \frac{1 - e^{-aT}}{aT}[V(0) - V_L]) \tag{6.16}$$

On the model of GARCH(1,1) basing results, the equation (6.16) enables a trader to find out a relationship among the option implied volatilities on a same asset and their maturities. This is called a volatility term structure. Thus, given a strike price and a volatility term structure, a trader is able to identify arbitrage opportunities between overpriced and underpriced options of the same strike price and different maturities. For example, a trader will be able to identify long and short time spreads from the volatility term structure.

In order to see how the actual volatility term structure responds to volatility changes we use the volatility term structure. The volatility term structure is downward-sloping if the current volatility exceeds the long-term volatility. The GARCH(1,1) model estimates a volatility term structure that is upward-sloping if the current volatility is less than the long-term volatility.

Let's look at the impact of volatility changes on the volatility term structure. The average variance rate per year equation (6.16) can also be presented as

$$\sigma(T)^2 = 252[V_L + \frac{1 - e^{-aT}}{aT}(\frac{\sigma(0)^2}{252} - V_L)]$$

Changing $\sigma(0)$ by $\Delta\sigma(0)$, $\sigma(T)$ changes by approximately

$$\frac{1 - e^{-aT}}{aT}\frac{\sigma(0)}{\sigma(T)}\Delta\sigma(0)$$

This process enables a trader to determine how much risk exposure he has due to instantaneous volatility changes in the underlying market.

Option life (days)	10	30	50	100	500
Increase in volatility (%)	0.97	0.92	0.87	0.77	0.33

Table 6.3: Impact of a 100 basis point change in the instantaneous volatility predicted from GARCH(1,1)

From this table based on S&P 500 options of varying maturities, a trader is therefore able to say that an increase of 1% in the instantaneous volatility increases the volatility of a 10-day option by 0.97%, 0.92% for a 30-day option and so on. If the implied volatilities of the different maturity options do not increase by the same percentage amount predicted by the GARCH(1,1) model, some arbitrage opportunities are then available to market participants.

Chapter 7
Stochastic volatility models: an alternative to the Black-Scholes-Merton model

7.1 Stochastic volatility models

An alternative to the Black-Scholes-Merton model that is assuming constant volatility over the option's life is a model where the geometric Brownian motion volatility parameter is known as a function of time. Followed by the asset price, the risk-neutral process is then

$$dS = (r - q)Sdt + \sigma(t)Sdv \tag{7.1}$$

If the variance rate is taken to be equal to the average variance rate during the option's life, then the Black-Scholes-Merton formulas are correct.

The variance rate is the volatility square. Suppose that in a total duration of 1-year during the first 6 months the stock volatility will be 20% and 30% during the next 6 months. The average variance rate is $0.5 \times 0.20^2 + 0.5 \times 0.30^2 = 0.065$. Then, using the Black-Scholes-Merton formulas is right with a variance rate of 0.065, corresponding to a volatility of $\sqrt{0.065} = 25.5\%$.

The equation (7.1) assumes that a trader can perfectly predict the asset's instantaneous volatility. In practice, volatility varies stochastically and therefore a more complex model is needed with two stochastic variables: the volatility of stock price and stock price itself.

One model that has been used by financial participants is

$$\frac{dS}{S} = (r - q)dt + \sqrt{V}dv_S$$

$$dV = a(V_L - V)dt + \xi V^{\alpha} dv_V$$

where dv_S and dv_V are Wiener processes and a, V_L, ξ and α are constants. The variance rate has a drift that pulls it back to a level V_L at rate a.

In "The Pricing of Options on Assets with Stochastic Volatilities", Hull and White demonstrate that when volatility is uncorrelated with the asset price but stochastic, the European call option premium equals Black-Scholes-Merton price integrated over the average variance rate probability distribution during the life of the option

$$\int_0^{\infty} c(\bar{V})g(\bar{V})d\bar{V}$$

where $\bar{V}$ is the variance rate average value, the Black-Scholes-Merton price is c expressed as a function of $\bar{V}$ and g in a risk-neutral world is the probability density function of $\bar{V}$. It can be shown using this results that at-the-money or close to the money options are overvalued using Black-Scholes-Merton, and options that are deep in or deep out-of-the-money are undervalued.

We discussed earlier alternative models, EWMA and GARCH(1,1) models, to characterize a stochastic volatility model. In "The GARCH Option Pricing Model", the author, J.-C Duan, shows that for a consistent option pricing model it is possible to use GARCH(1,1) model. In fact, stochastic volatility models can be used to model exotic options and plain vanilla prices.

For options with less than a year left to maturity, using a stochastic volatility model impact is quite small in absolute terms but larger for deep out-of-the-money options in percentage terms. For options with longer maturities (over a year), using a stochastic volatility process makes more sense as the impact becomes progressively larger as the option's life rises.

Finally, using a stochastic volatility process impact is generally quite large on the performance of delta hedging. Traders are aware of this and therefore monitor their exposure to volatility changes by calculating vega.

7.2 Monte Carlo Simulation

We usually use for derivatives where the payoff is dependent on the history of the underlying variable or where there are several underlying variables the Monte Carlo simulation. When used to value an option, Monte Carlo simulation makes use of the risk neutral valuation result. In order to obtain the expected payoff in a risk neutral world sample paths are used and this payoff is then discounted at the interest rate that is risk-free.

Take for example a derivative that depends on a single market variable S providing a payoff at given time T. Taking constant interest rates, the derivative can be value as follow:

- Assuming a risk neutral world, we sample a random path for S
- Compute the derivative payoff
- To get many sample values of the payoff from the derivative in a risk neutral world repeat the steps one and two above
- Compute sample payoffs mean to get an expected payoff estimate in a risk neutral world

- This expected payoff is discounted at the risk-free rate to get a derivative price estimate

Assuming underlying stock price variable is following the process in a risk neutral world given by equation (4.3)

$$dS = \mu S dt + \sigma S dv$$

where a Wiener process is represented by dv, the expected return in a risk neutral world is μ, and the volatility parameter is σ. For simulating the path S, we can divide the life of the derivative into N short time intervals Δt and hence the above equation can be approximated as

$$S(t + \Delta t) - S(t) = \mu S(t)\Delta t + \sigma S(t)\epsilon\sqrt{\Delta T}$$

where the value of S at time t is given by $S(t)$, a random sample from a normal distribution with standard deviation of 1 and mean zero is represented by ϵ.

We can then compute at time Δt the value of S from the starting value of S, at time $2\Delta t$ compute the value S from previous computation at time Δt and so on. Practically, it is a better idea to simulate $\ln S$ rather than simulating S. From equation (4.10) and Itô's lemma, the $\ln S$ follows a process given by

$$d\ln S = (\mu - \frac{\sigma^2}{2})dt + \sigma dv$$

hence

$$\ln S(t + \Delta t) - \ln S(t) = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma\epsilon\sqrt{\Delta t}$$

or equally can be written as

$$S(t + \Delta t) = S(t)e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\epsilon\sqrt{\Delta t}} \tag{7.2}$$

We use the equation (7.2) for constructing a path for S.

Working with $\ln S$ is more accurate than working with S. Also, if μ and σ are constant, then

$$\ln S(T) - \ln(0) = (\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}$$

is true for all T. It follows that

$$S(T) = S(0)e^{(\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}} \tag{7.3}$$

The equation (7.3) can be used to price derivatives with nonstandard payoff at time T.

7.2.1 Probability distribution of the Monte Carlo simulation

A simulation done to obtain random outcomes for a stochastic process is called as Monte Carlo simulation. Let's recall the previous example where the stock expected return annual rate is 0.10% and the volatility annual rate is 0.20%. The stock price change over 1 week is given by equation (4.7)

$$\Delta S = 0.00192S + 0.0277S\epsilon$$

Simulations for a path for the stock price over 10 weeks can be done by sampling repeatedly for ϵ from $\phi(0, 1)$ and putting this into the equation (4.7).

	A	B	C
1 2	Stock price at start of period	Random sample for ε	Change in stock price during period
3	100.0000	0.9004	2.6862
4	102.6862	0.7430	2.3104
5	104.9967	-0.5994	-1.5418
6	103.4549	0.1847	0.7280
7	104.1829	-1.6288	-4.5005
8	99.6823	0.2804	0.9655
9	100.6479	-0.3709	-0.8407
10	99.8072	-0.2621	-0.5329
11	99.2743	0.8492	2.5258
12	101.8001	0.0377	0.3016
13	102.1017	0.9340	2.8375
14	An = A(n-1) + C(n-1)	NORMSINV(RAND())	0.00192A + 0.0277AB

Table 7.1: Stock price simulation for $\sigma = 0.20$ and $\mu = 0.10$ during 1-week periods, given that initial stock price is S = \$100

This table shows one path for a stock price over a 10-week period. It is significant that the given table presents a single possible pattern for the movements in stock price and hence various price movements result by various random samples. We can use any small time interval Δt for the simulation. The final stock price of \$102.1017 can be regarded as a distribution of stock prices from random sample by 10 weeks end. When simulations in stock price movements are done repeatedly, at the end of 10 weeks, a different probability distribution is obtained. The trader can then compute the payoff from the option derivative contract for each probability distribution obtained.

7.2.2 Strengths and weaknesses of the Monte Carlo simulation

The Monte Carlo simulation prime benefit is that we can use it for calculating the payoff when it is dependent on only on the final value of S as well as when it is dependent on the path being followed by the underlying variable S. Such as we can use the Monte Carlo simulation when the payoffs are dependent on the average value of S between 0 and time T. Payoffs can occur during all the derivative life the end.

Efficiency of Monte Carlo simulation is greater numerically as compared to other procedures when stochastic variables exceed or equal to three. A benefit of using Monte Carlo simulation is that it provides for the estimates a standard error as well. The result accuracy is dependent on the number of trials for the Monte Carlo simulation. The mean μ and the standard deviation σ given by the simulation for the discounted payoffs can be computed too. The variable μ corresponds to the simulation's estimate of the derivative value. The estimate standard error is given by

$$\frac{\sigma}{\sqrt{M}} \tag{7.4}$$

where the number of trials is represented by M. For the price f of the derivative a 95% confidence interval is

$$\mu - \frac{1.96\sigma}{\sqrt{M}} < f < \mu + \frac{1.96\sigma}{\sqrt{M}}$$

The Monte Carlo simulation major disadvantage is that it is very time consuming computationally and is not adapted for situations where early exercise is a possibility.

7.2.3 Monte Carlo simulation and the Greeks

The Greek letters can be calculated in the case of a Monte Carlo simulation. Suppose that a trader looks at the partial derivative c with respect to S where c represents the call option premium and S the underlying stock price. The trader applies the Monte Carlo simulation to estimate the derivative value given by $\hat{c}$. An increase in S, by ΔS, is then made and a new value for the derivative, $\hat{c^*}$, is similarly computed to $\hat{c}$. Hence for the hedge parameter an estimate is

$$\frac{\hat{c^*} - \hat{c}}{\Delta S}$$

To minimize (7.4), the estimate standard error should correspond to N representing the number of time intervals used, the random samples that are used, and the number of trials M to calculate both $\hat{c}$ and $\hat{c^*}$.

Chapter 8
Conclusion

Many traders and financial institutions have experienced huge losses due to the use of derivatives. In fact, in 1998, a major hedge fund collapsed, the Long-Term Capital Management (LTCM) where Myron S. Scholes and Robert C. Merton were key people of the firm, and more recently, the bankruptcy of the fourth biggest bank in the United States in 2008, Lehman Brothers, were the result of underestimating the risks of these derivative instruments. Some non-financial corporations due to some of the losses have declared that they plan to reduce their use of derivatives and might even eliminate it at all. This will be very adverse because derivatives provide firms and financial institutions a very efficient way to manage their risks and achieve their risk/profit goals at lower transaction costs.

This brings up an important point. The use of derivatives can be either for speculating or for hedging; meaning, we can use them either to take risks or to reduce risks. Primary reason for losses due to derivatives is that they were not used correctly. Financial institutions and hedge funds that were supposed to hedge their risks instead speculated with the high leverage of these instruments.

From these losses a major lesson that should be learnt is the significance of controlling and managing risks taken while using derivatives. The Black-Scholes-Merton model, allowing traders to price theoretically European put and call options on stocks that are non-dividend paying, offers market participants an easy and efficient way to manage their option portfolio's risks through the Greek letters. In fact, most of option derivatives traders nowadays follow a delta neutral strategy to be protected against the direction of the underlying asset prices and instead bet on the volatility of the underlying asset prices. Derivatives traders can also choose to be gamma neutral in order to make an option portfolio's delta

rate of change equal to zero, or vega neutral to hedge their portfolio against time sensitivity and implied volatility respectively.

However, when using a different model like stochastic volatility models to price stock options, the computation of the Greeks is more time consuming as it implies to iterate the Monte Carlo simulation several times by making the one variable (delta for a directional hedge, gamma for a volatility hedge, vega for an implied volatility hedge and rho for an interest rate hedge) to calculate vary.

The Black-Scholes-Merton model is based on more or less realistic assumptions. One main critic that can be made is about the assumption that this model take the underlying asset returns following a normal distribution over small periods of time, resulting in underlying prices at options expiration following a lognormal distribution. This hypothesis made by the model implies that the underlying stock volatility remains unchanged over the derivative instrument life and fails to explain the volatility smile. In fact, for a given maturity date, options have different volatilities which are having different strike prices resulting in a graph with a shape of a smile. Thus, the use of stochastic volatility models can be justified when pricing options as they better represent financial market conditions with the mean reverting phenomenon of volatility and the capacity to explain the volatility smile of stock options.

However, even if stochastic volatility models are better adapted to derivatives markets, these models imply long computations with more parameters to estimate to price and hedge an option portfolio. Thus, derivatives traders rather use the time efficient Black-Scholes-Merton model where less estimation errors can be made.

The assumption of the normal distribution of underlying asset price changes made by the Black-Scholes-Merton model tends to overprice close-to-the-money or at-the-money options while deep out-of-the-money options are undervalued. Once again, stochastic volatil-

ity models, to the detriment of longer computations, give a better representation of the underlying asset returns as the probability distribution can be skewed positively or negatively with an unusual tall and pointed or low and flat peak to give more probabilities to certain market events than the normal distribution does.

Since employing a theoretical pricing model, like a stochastic volatility model or the Black-Scholes-Merton model, is based on assumptions more or less accurate to the real world and requires a trader to make some estimations on inputs, the new derivatives trader might feel that taking the right decision based on the theoretical pricing model is just a chance matter. In the short-run luck might be helpful however in the long-run, better performing traders are the ones who will put some efforts in comprehending the model along with its every problem. This is the reason why using a pricing model remains among traders the best way to manage risk and evaluate options.

Finally, the reader of this paper might be thinking that trading and evaluating options is merely an arithmetic calculations series. The options evaluation mechanism forms any trader's education integral part, but mathematical models provide the trader with tools to make decisions. A successful trader for options is one who knows when science ends and where intuition, market fell or experience begins. If a trader is basing his decisions intensely on the model calculations and putting intuition aside he would end up in a misfortune. Only a trader who understands the limitations of the models used will end up being successful.

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