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ON STRUCTURES OF LARGE ROOTED GRAPHS

A Dissertation

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in

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by
Shilin Wang
B.S. in Applied Math, South China Normal University, 2009
M.S. in Operational Research and Cybernetics, South China Normal University, 2012
M.S. in Math, Louisiana State University, 2014
M.S. in Applied Statistics, Louisiana State University, 2017
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Abstract

A rooted graph is a pair (G, R), where G is a graph and $R \subseteq V(G)$. There are two research topics in this thesis. One is about unavoidable substructures in sufficiently large rooted graphs. The other is about characterizations of rooted graphs excluding specific large graphs.

The first topic of this thesis is motivated by Ramsey Theorem, which states that K_n and $\overline{K}_n$ are unavoidable induced subgraphs in every sufficiently large graph. It is also motivated by a classical result of Oporowski, Oxley, and Thomas, which determines unavoidable large 3-connected minors. We first determine unavoidable induced subgraphs, and unavoidable subgraphs in connected graphs with sufficiently many roots. We also extend this result to generalized rooted connected graphs. Secondly, we extend these results to rooted graphs of higher connectivity. In particular, we determine unavoidable subgraphs of sufficiently large rooted 2-connected graphs. Again, this result is extended to generalized rooted 2-connected graphs.

The second topic of this dissertation is motivated by two results of Robertson and Seymour, let's only talk about path and star. In the first result they established that graphs without a long path subgraph are precisely those that can be constructed using a specific operation within a bounded number of iterations, starting from the trivial graph. In the second result they showed that graphs without a large star minor are those that are subdivisions of graphs with bounded number vertices. We consider similar problems for path, star and comb. We have some theorems on characterizations of rooted connected graphs excluding a heavy path, a large (nicely) confined comb, a large (nicely) confined star, which are similar to those of Robertson and Seymour. Moreover, our results strengthen their related results.

1 Introduction

This dissertation is about structures of large rooted graphs. Before stating our main results, we need some necessary definitions. In this chapter, we begin with a brief summary of basic definitions on graph minor. Then we introduce basic lemmas which are used in the proof of later theorems in this thesis. Finally we list some related results on unavoidable substructures in large graphs, and on characterizations of excluding certain large graphs. An outline of this dissertation will be given in section 1.6.

1.1 Preliminaries

In this section, we provide some basic definitions and lemmas in graph theory, which are used in the thesis. Undefined terminology can be found in [2].

Let $\mathbb{N}$ denote the set of positive integers. We will use symbols $\emptyset, \cup, \cap, \in, \subseteq, \setminus, +$ for set operations in the usual sense. We also remark that $A \cup \{v\}$ is abbreviated to $A + v$.

A *graph* G consists of an ordered pair (V, E) of disjoint finite sets, where V is not empty, and an incidence relation such that each member of E is *incident* with one or two members of V. Members of E and V are called *edges* and *vertices*, respectively. The number of vertices of G is its *order*, and is denoted by $|G|$, which is exactly $|V|$. If e is an edge of G or v is a vertex of G, instead of writing $e \in E$ or $v \in V$ we may simply write $e \in G$ or $v \in G$. If an edge e is incident with a vertex x, then x is called an *end* of e. If two distinct vertices x, y are incident with a common edge, or if two distinct edges x, y are incident with a common vertex, then we say that x, y are *adjacent*. An edge with only one end is a *loop*. If two non-loop edges have the same ends, then they are *parallel* and each is a *parallel edge* in G.

A graph G is *simple* if it has neither loops nor parallel edges. A graph with only one vertex is called *trivial*. All graphs considered in this thesis are simple, except where otherwise noted. For a simple graph $G = (V, E)$, each edge $e \in E$ consists of two distinct ends x and y, and is denoted by $e = xy$. For a vertex $v \in V$, a *neighbor* of v is a vertex adjacent with v. The set of neighbors of v is denoted by $N_G(v)$. We also remark that $N_G(v)$ is abbreviated to $N(v)$ when the dependence on G is clear. The number of edges incident with v is called the *degree* of v, and is denoted by $d_G(v)$, which is exactly $|N_G(v)|$. The maximal degree of all vertices in G is denoted by $\triangle(G)$. When $X \subseteq V$, we denote by $N_G(X)$ the set of vertices outside X that are adjacent to at least one vertex in X. That is to say, $N_G(X) = \bigcup_{v \in X} N_G(v) \backslash X$. The *complement* $\overline{G}$ of G is the graph on V such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

Let $G = (V, E)$ be a graph. *Deleting an edge* e from G means deleting e from E, and is denoted by $G \backslash e$. *Deleting a vertex* v from G means deleting v from V and deleting all edges incident with v from E, and is denoted by $G \backslash v$. For any two distinct $e, f \in E$, it is clear that $G \backslash e \backslash f = G \backslash f \backslash e$. Thus we can define the deletion of a set of edges. For any $X \subseteq E$, *deleting* X from G means deleting every edge of X from G, and is denoted by $G \backslash X$. Similarly, for any two distinct vertices $x, y \in V$, we also have $G \backslash x \backslash y = G \backslash y \backslash x$. Naturally, for any $X \subseteq V$, *deleting* X from G means deleting every vertex of X from G, and is denoted by $G \backslash X$. For non-adjacent vertices $x, y \in V$, let $G + xy = (V, E \cup \{xy\})$.

A *subgraph* of a graph $G = (V, E)$ is a graph obtained from G by deleting a vertex set and an edge set. If $G' = (V', E')$ is a subgraph of G and G' contains all the edges $xy \in E$ with $x, y \in V'$, then $G' = G \backslash (V \backslash V')$ and G' is called *an induced subgraph* of G. We write $G' = G[V']$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We call G_1 and G_2 *isomorphic*, and write $G_1 \simeq G_2$, if there exists a bijection $\varphi : V_1 \to V_2$ with $xy \in E_1 \Leftrightarrow \varphi(x)\varphi(y) \in E_2$ for all $x, y \in V_1$. Such a map φ is called an *isomorphism*. We do not normally distinguish between isomorphic graphs. Thus we usually write $G_1 = G_2$ rather than $G_1 \simeq G_2$. We set $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ and $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. We set $G_1 \backslash G_2 = G_1[V_1 \backslash V_2]$. If $V_1 \cap V_2 = \emptyset$, then we call G_1 and G_2 *disjoint*.

If all the vertices of a graph G are pairwise adjacent, then G is called a *complete* graph. A complete graph with n vertices is denoted by K_n. A complete subgraph of a graph is called a *clique*. Let G_1, G_2 be disjoint graphs and let $k \geq 0$ be an integer. For $i = 1, 2$, let S_i be a clique of G_i with $V(S_i) = \{v_1^i, v_2^i, \cdots, v_k^i\}$. Let a graph G be obtained from the union of G_1 and G_2 by identifying v_i^1 with v_i^2 $(1 \leq i \leq k)$, identifying $v_i^1 v_j^1$ with $v_i^2 v_j^2$ $(1 \leq i < j \leq k)$, and deleting a (possibly empty) set of identified edges. Then we call G a *k-sum* of G_1 and G_2, and we write $G = G_1 \oplus_k G_2$ by which we mean that G is one of many k-sums of G_1 and G_2, since $G_1 \oplus_k G_2$ does not produce a unique outcome.

Let G be a graph. We call G a *bipartite* graph if $V(G)$ admits a partition into two sets X, Y such that each edge has its ends in different set. That is, vertices in the same partition set must not be adjacent. We call G a *complete bipartite* graph if every vertex of X is adjacent to every vertex of Y. If $|X| = m$ and $|Y| = n$, then this complete bipartite graph is denoted by $K_{m,n}$.

A *path* is a graph $P = (V, E)$ such that $V = \{v_0, v_1, \cdots, v_n\}$ and $E = \{v_0v_1, \cdots, v_{n-1}v_n\}$. We call vertices v_0, v_n *ends* of P and each vertex in $V \backslash v_0 \backslash v_n$ an *interior vertex* of P. To specify the path we often write $P = v_0v_1 \cdots v_n$. The number of edges of P is called the *length* of P. A path of length n is denoted by P_n. Two or more paths are *independent* if any common vertex of any two of these paths must

be their ends. Let G be a graph. For any two vertices x and y of G, an *xy-path* of G is a subgraph of G that is a path with ends x, y. In general, for any two subsets X and Y of $V(G)$, an *XY-path* of G is an xy-path such that $V(P) \cap X = \{x\}$ and $V(P) \cap Y = \{y\}$.

A *cycle* is a graph $C = (V, E)$ such that $V = \{v_0, v_1, \cdots, v_n\}$ and $E = \{v_0v_1, v_1v_2, \cdots, v_{n-1}v_n, v_nv_0\}$. The number of vertices of C is called the *length* of C. A *triangle* is a cycle of length 3. A cycle of length n is denoted by C_n.

A graph G is *connected* if G contains an xy-path for every pair of vertices x, y of G. A graph G is *k-connected* if $|G| > k$ and $G \backslash X$ is connected for any $X \subseteq V(G)$ with $|X| < k$. Note that every graph G can be expressed as the union of disjoint connected subgraphs, which we call *components* of G. With this terminology, we may say that G is the disjoint union of its components.

A *forest* is a graph without cycles. A *tree* is a connected forest. Vertices of degree one in a tree are its *leaves*. A *spanning tree* T of a graph G is a subgraph of G such that T is a tree and T contains all the vertices of G.

A *star* is a complete bipartite graph $K_{1,n}$, for $n \geq 3$. The *size* of this star is n. The unique non-leaf vertex of a star is called its *center vertex*.

The following lemma is a well known result in graph theory, which is the frequency used application of Theorem 1.5.1 of [2]. It shows the existence of spanning trees in every connected graph.

Lemma 1.1.1. *Every connected graph has at least one spanning tree.*

A *separation* of a graph G is an unordered edge-disjoint pair (G_1, G_2) of subgraphs of G such that $G_1 \cup G_2 = G$. If $|G_1 \cap G_2| = k$, then k is called the order of the separation and the separation is called a *k-separation*. If a k-separation (G_1, G_2) satisfies either $|G_1| = k$ or $|G_2| = k$, then (G_1, G_2) is called a *trivial k-separation*.

Let (G_1, G_2) be a separation of a graph G and let $V_i \subseteq V(G_i)$ for $i = 1, 2$. Then we say that (G_1, G_2) separates V_1 from V_2.

The following theorem is one of the pillars of graph theory, which is Theorem 3.3.1 of [2]. It characterizes the obstruction for having many disjoint paths between two vertex sets in a graph.

Theorem 1.1.2 (Menger, 1927)**.** *Let $G = (V, E)$ be a graph and $X, Y \subseteq V$. Then G does not have k disjoint XY-paths if and only if G has a k'-separation separating X from Y with $k' < k$.*

A connected graph G with at least one edge is *non-separable* if G does not have a nontrivial 1-separation. Equivalently, either G is 2-connected or G has only one edge. Let G be a connected graph. A vertex v of G is called a *cut vertex* if it is the common vertex of the two parts of a nontrivial 1-separation of G. A *block* of G is a maximal non-separable subgraph of G. Let A be the set of cut vertices of G, and $\mathcal{B}$ be the set of all blocks of G. Notice that $\mathcal{B} \neq \emptyset$ as long as $E(G) \neq \emptyset$. The *block graph* of G is the graph on $A \cup \mathcal{B}$ formed by the edges aB with $a \in B$.

The following lemma, which can be found in Proposition 3.1.1 of [2], characterizes the structure of the block graph of any nontrivial connected graph.

Lemma 1.1.3. *The block graph of a nontrivial connected graph is a tree.*

Let G be a connected graph and H be a subgraph of G. By an *H-bridge* we mean a connected subgraph B of $G \backslash E(H)$ that satisfies either one of the following two conditions:

(i) $|E(B)| = 1$ and $V(B) \subseteq V(H)$;

(ii) there exists a connected component C of $G \backslash H$ such that $E(B)$ consists of all edges incident with at least one vertex of C.

Vertices belong to both B and H are called *feet* of B.

Let $X \subseteq V(G)$. By an *X-bridge* we mean that $H = (X, \emptyset)$, an X-bridge is one such H-bridge.

1.2 Minor and Topological Minor

Many results in this thesis will be about minors and topological minors. In this section, we define these terms and provide some lemmas on the relationship between them. We remark that, in this section, we allow loops and parallel edges in a graph.

Let $e = xy$ be an edge of a graph $G = (V, E)$. *Contracting* e means identifying x with y in $G \backslash e$, and is denoted by G/e. For any two distinct edges $e, f \in E$, it is clear that $G/e/f = G/f/e$. Hence we can define the contraction of any $X \subseteq E$ as contracting every edge in X (in any order), and denote the result by G/X.

A graph H is a *minor* of a graph G if H is obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions (where the order of the operations could be arbitrary). If H is a minor of G, then we write $H \leq_m G$. A graph G is called *H-free*, where H is a graph, if no minor of G is isomorphic to H. The following lemma is on the transitivity of the minor relation.

Lemma 1.2.1. *Let G_1, G_2, G_3 be graphs. Then $G_1 \leq_m G_2$ and $G_2 \leq_m G_3$ imply $G_1 \leq_m G_3$.*

Proof. We know that G_2 is obtained from G_3 by a sequence of vertex deletions, edge deletions, and edge contractions, and G_1 is obtained from G_2 by a sequence of vertex deletions, edge deletions, and edge contractions. Hence G_1 can be obtained from G_3 by a sequence of vertex deletions, edge deletions, and edge contractions. That is to say, $G_1 \leq_m G_3$. $\square$

Instead of defining a minor by local operations, we can also define it by a global structure. The key idea is, vertices in a minor can be traced back to connected subgraphs of the original graph.

Lemma 1.2.2. *A graph H is a minor of a graph G if and only if G has a set $\{G_v : v \in V(H)\}$ of disjoint connected subgraphs and a set $F = \{f_e : e \in E(H)\}$ of edges such that F is disjoint from every $E(G_v)$, and for every edge $e = uv \in E(H)$, the two ends of f_e are in G_u and G_v, respectively.*

Proof. First we claim that for any connected graph $L = (V, E)$, $L/E = K_1$. It is obvious that L/e is connected for any edge $e \in E$, otherwise, there are at least two components of L/e, which means that there are at least two components of L, which contradicts the assumption that L is connected. It follows that L/E is connected. Since there is no edge in L/E, then L/E is a single vertex graph K_1, we are done with this claim. Next we start proving the lemma using the above claim.

The backward implication easily follows from the above claim and the definition of a minor. Next we consider the forward implication.

By the definition of a minor, we know that H is obtained from a subgraph of G by contracting edges. Then there exists $Z \subseteq V(G)$ and disjoint $X, Y \subseteq E(G \backslash Z)$ such that $H = (G\backslash Z)\backslash X/Y$. Take $G' = G\backslash Z\backslash X$. Then $H = G'/Y$, which means that H is a minor of G'. Now we prove that there exist $F \subseteq E(G')$ and a set of subgraphs $\{G_v : v \in V(H)\}$ of G' that satisfy the description in the lemma. Since G' is a subgraph of G, it follows that F is a subset of $E(G)$ and each G_v is a subgraph of G, which would prove the lemma.

Since $H = G'/Y$, then $(Y, E(H))$ is a partition of $E(G')$. Let $G'' = G'\backslash E(H) = (V(G'), Y)$. Note that there is no edge in G''/Y. By the above claim, there exists a one-to-one correspondence between vertices of G''/Y and components of G''.

On the other hand, $G''/Y = G'\backslash E(H)/Y = (G'/Y)\backslash E(H) = H\backslash E(H)$, which implies that there exists a one-to-one correspondence between vertices of H and components of G''. Let components of G'' be G_v, for $v \in V(H)$, and let $F = E(H)$. If $e = uv$, then ends of e belong to G_u and G_v before contracting Y, as required. $\square$

The pair $(\{G_v\}, \{f_e\})$ from the last theorem is called a *model* of the minor H of G.

Let $e = xy$ be an edge of a graph G. *Subdividing* e means deleting e from G and then adding a new vertex z and two new edges zx, zy, this operation is also called *edge subdivision*. A *subdivision* of a graph G is a graph obtained from G by subdividing edges repeatedly.

If a graph G has a subgraph that is a subdivision of a graph H, then H is called a *topological minor* of G and we write $H \leq_t G$. If two edges $e, f \in E(G)$ have a common end $v \in V(G)$ and they are the only edges incident with v, then e, f are in *series* and each of them is a *series edge* in G. A *series contraction* is the operation of contracting a series edge. Clearly, a series contraction is the reverse operation of subdividing an edge. This observation implies the next lemma immediately.

Lemma 1.2.3. *Let H and G be graphs. Then $H \leq_t G$ if and only if H is obtained from a subgraph of G by series contractions.*

The following lemma, which is a corollary of Lemma 1.2.2, is on the relationship between topological minor and minor.

Lemma 1.2.4. *Let G be a graph.*

(1) Every topological minor of G is also a minor of G;

(2) if H is a minor of G and the maximum degree of H is at most 3, then H is a topological minor of G.

Proof. (1) Since every series contraction is a contraction, this result follows from Lemma 1.2.3.

(2) By Lemma 1.2.2, G has a set $\{G_v : v \in V(H)\}$ of disjoint connected subgraphs and a set $F = \{f_e : e \in E(H)\}$ of edges satisfying Lemma 1.2.2. Let U be the set of vertices that are incident with at least one edge in F. If for each G_v, there exists a vertex w in G_v such that G_v has independent paths from w to each vertex in $V(G_v) \cap U$, then H can be obtained from G by deleting vertices and edges not in those independent paths and then contracting series edges in those independent paths, which implies $H \leq_t G$. Next we prove the existence of w in each G_v. Since the maximal degree of H is at most 3, $|V(G_v) \cap U| \leq 3$. Suppose $|V(G_v) \cap U| \leq 1$, we can choose w to be any vertex in G_v. Suppose $V(G_v) \cap U = \{a, b\}$, then there exists a path between a, b in G_v, we may choose any vertex in such path as w. Suppose $V(G_v) \cap U = \{a, b, c\}$. In this case, G_v has a path P between a, b and another path Q between c and P. Then the common vertex of P and Q is w. □

1.3 Unavoidable large graphs

In this section, we introduce some results on large substructures that are necessarily presented in every large enough graph. We will say that these substructures are unavoidable in a sufficiently large graph. Historically, the first result on unavoidable graphs was established by Ramsey in 1930 [5]. The following is the most common formulation of this result, which can be found in Theorem 9.1.1 of [2].

Theorem 1.3.1 (Ramsey 1930). *There exists a function $f_{1.3.1}(n)$ such that every graph of order at least $f_{1.3.1}(n)$ contains either K_n or its complement $\overline{K}_n$ as an induced subgraph.*

Other than choosing a better function $f_{1.3.1}(n)$, Theorem 1.3.1 is already the best possible. However, if the input graph is known to be connected, then it is possible

to improve the theorem by replacing the disconnected $\overline{K}_n$ in the output with some connected large graphs. This result is the content of the following theorem and its proof is provided later as Lemma 1.5.2.

Theorem 1.3.2. *There exists a function $f_{1.3.2}(n)$ such that every connected graph of order at least $f_{1.3.2}(n)$ has an induced subgraph isomorphic to P_n, $K_{1,n}$ or K_n.*

Naturally, one may try to extend Theorem 1.3.2 to graphs of higher connectivity. In the next four theorems, we state the corresponding results for k-connected graphs when $k = 2, 3, 4, 5$. The problem is open for connectivity exceeding five.

In the following theorem, we will not determine the unavoidable induced subgraphs, like what we did in the previous two theorems, since these graphs do not admit a simple and explicit description. Instead, we will determine the unavoidable minors or topological minors.

The next is the result for 2-connected graphs, which is Proposition 9.4.2 in [2].

Theorem 1.3.3. *There exists a function $f_{1.3.3}(n)$ such that every 2-connected graph of order at least $f_{1.3.3}(n)$ contains C_n or $K_{2,n}$ as a topological minor.*

For any integers $h \geq 1$ and $n \geq 3$, let $W(h, n)$ denote the graph obtained from the disjoint union of C_n and $\overline{K}_h$ by adding all possible edges between them. In particular, $W(1, n)$ is known as a *wheel* and $W(2, n)$ is known as a *double-wheel.* Oporowski, Oxley and Thomas determined the unavoidable large 3-connected minors [4].

Theorem 1.3.4. *There exists a function $f_{1.3.4}(n)$ such that every 3-connected graph of order at least $f_{1.3.4}(n)$ contains $W(1, n)$ or $K_{3,n}$ as a minor.*

In the same paper [4], Oporowski, Oxley and Thomas also determined all the unavoidable large 4-connected minors. For any integer $n \geq 5$, let C_n^2 denote the graph obtained from C_n by joining vertices whose distance is two in C_n.

Theorem 1.3.5. *There exists a function $f_{1.3.5}(n)$ such that every 4-connected graph of order at least $f_{1.3.5}(n)$ contains $W(2,n)$, C_n^2, or $K_{4,n}$ as a minor.*

Shantanam [9] identified thirty (families of) graphs $H_1(n), \cdots, H_{30}(n)$ and he proved that these are all the unavoidable large 5-connected minors. Since we do not use this result, we will not explicitly define these thirty graphs.

Theorem 1.3.6. *There exists a function $f_{1.3.6}(n)$ such that every 5-connected graph of order at least $f_{1.3.6}(n)$ contains $H_1(n), \cdots,$ or $H_{30}(n)$ as a minor.*

In this dissertation, we consider similar problems for rooted graphs, which we formally introduce in Chapter 2 and Chapter 3. We will study rooted connected and 2-connected graphs and those of generalized rooted graphs. In particular, we will extend Theorem 1.3.2.

1.4 Excluding large graphs

In this section we explain what we mean by excluding a large graph. We will also present a few well known results of this type. In the following discussion we talk about the subgraph relation, but the same idea applies to other graphs containment relations like induced subgraph and minor.

Let $\mathcal{H}$ be a class of graphs. We consider the problem of characterizing classes $\mathcal{G}$ of graphs such that $\mathcal{G} \not\supseteq \mathcal{H}$. In general, since members of $\mathcal{H}$ could be totally unrelated, there are no effective ways to characterize $\mathcal{G}$. However, suppose $H_1, H_2, \cdots$ is a sequence of graphs such that H_i is a subgraph of H_{i+1}, for all integers $i \geq 1$. Suppose $\mathcal{H}$ is the class of graphs H such that H is a subgraph of some H_i. Then the problem of characterizing classes $\mathcal{G} \not\supseteq \mathcal{H}$ is more tractable.

Since $\mathcal{H}$ is *closed* under taking subgraphs, that is, subgraphs of members of $\mathcal{H}$ remain members of $\mathcal{H}$, it is reasonable to assume that $\mathcal{G}$ is also closed under taking

subgraphs (otherwise the problem would still be hard). Then it is clear that the following statements are equivalent

(i) $\mathcal{G} \not\supseteq \mathcal{H}$;

(ii) $\mathcal{G} \not\ni H_i$ for at least one i;

(iii) $\mathcal{G} \subseteq \mathcal{H}_i$ for at least one i, where $\mathcal{H}_i$ is the set of all graphs that do not contain H_i as a subgraph;

(iv) $\mathcal{G} \subseteq \mathcal{H}_i$ for a sufficiently large i .

Therefore, characterizing classes $\mathcal{G} \not\supseteq \mathcal{H}$ is the same, in a sense, as characterizing graphs that do not contain an H_i subgraph for a sufficiently large i. This problem is often called loosely the problem of characterizing graphs without a large H_i subgraph. In this sense, we can talk about excluding large graphs.

The following example illustrates how we use this language. Let $\mathcal{P} = \{P_1, P_2, \cdots\}$ be the set of all paths. Then the problem of characterizing closed classes $\mathcal{G} \not\supseteq \mathcal{P}$ is the problem of characterizing graphs that do not have a long path. Roberston and Seymour [7] solved this problem by proving the following theorem.

Theorem 1.4.1. *The following are equivalent for any class $\mathcal{G}$ of graphs.*

(1) There exists $n \in \mathbb{N}$ such that no graph in $\mathcal{G}$ has a path of length n;

(2) there exists $h \in \mathbb{N}$ such that every graph in $\mathcal{G}$ can be constructed from trivial graphs within h iterations by the following construction: from any graphs $G_1, G_2, \cdots, G_t$, a new graph is obtained from their disjoint union by adding a new vertex x and joining x to other vertices arbitrarily.

Now we see how this result solves the above problem. Let $\mathcal{P}_i$ be the class of graphs that do not contain a P_i subgraph; let $\mathcal{Q}_i$ be the class of graphs that can be constructed within at most i iterations using the operation described in (2),

starting from trivial graphs. Then the above theorem implies that the following are equivalent, for any class $\mathcal{G}$ of graphs that is closed under taking subgraphs:

(i) $\mathcal{G} \not\supseteq \mathcal{P}$;

(ii) $\mathcal{G} \subseteq \mathcal{P}_i$ for at least one i;

(iii) $\mathcal{G} \subseteq \mathcal{Q}_i$ for at least one i.

This equivalence is often loosely state as: graphs that do not contain a long path are precisely those that can be constructed using the operation from (2) within a bounded number of iterations, starting from the trivial graph. In this sense, we consider Theorem 1.4.1 as a characterization of graphs that do not contain a long path.

There are many results of the same type on minors. In the following, we discuss some of them.

When we say that $\mathcal{G}$ is *minor-closed*, it means that $H \leq_m G \in \mathcal{G}$ implies $H \in \mathcal{G}$. Let $\mathcal{S} = \{K_{1,1}, K_{1,2}, \cdots\}$ be the set of all subdivisions of stars. The problem of characterizing minor-closed classes $\mathcal{G} \not\supseteq \mathcal{S}$ is the problem of characterizing graphs that do not have a large star as a minor. The following theorem solves this problem. Loosely speaking, it says that graphs without a large star minor are those that are subdivisions of graphs with bounded number vertices.

Theorem 1.4.2. *The following are equivalent for any class $\mathcal{G}$ of connected graphs.*

(1) There exists $n \in \mathbb{N}$ such that no graph in $\mathcal{G}$ has a star of size n as a minor;

(2) there exists $h \in \mathbb{N}$ such that every graph in $\mathcal{G}$ can be constructed from graph with at most h vertices by a sequence of edge subdivisions.

Let $\mathcal{F}$ be the set of all forests. Robertson and Seymour [6] characterized minor-closed classes $\mathcal{G} \not\supseteq \mathcal{F}$ in 1983. They proved that excluding a forest from a graph

amounts to bounding its path-width. Let G be a graph. A *path-decomposition* of G is a sequence $(G_i : i = 1, 2, \cdots, m)$ of subgraphs of G such that:

(i) $G_1 \cup G_2 \cup \cdots \cup G_m = G$;

(ii) if $x \in V(G_i \cap G_j)$ and $i < j$ then x belongs to every G_k in between (i.e. $x \in V(G_k)$ for all k with $i < k < j$).

The *width* of this decomposition is $max_i|G_i| - 1$. The *path-width* of G, denoted $pw(G)$, is the minimum width over all its path decompositions.

Theorem 1.4.3. *The following are equivalent for any class $\mathcal{G}$ of graphs.*

(1) There exists a forest F such that no member of $\mathcal{G}$ contains F as a minor;

(2) there exists $w \in \mathbb{N}$ such that $pw(G) \leq w$ for all graphs $G \in \mathcal{G}$.

A graph is *planar* if it can be drawn on the plane so that no two edges cross each other. Let $\mathcal{H}$ be the class of all planar graphs. The problem of characterizing minor-closed classes $\mathcal{G} \not\supseteq \mathcal{H}$ is the problem of characterizing graphs excluding a general planar graph as a minor. Robertson and Seymour [8] in 1986, proved that graphs of bounded tree-width are precisely graphs that do not contain a large planar graph. In other words, a graph without a fixed planar minor can be constructed from bounded size graphs by sticking them together in a tree-like structure.

To describe graphs that do not a certain large fixed graph minor, we need to introduce tree-width. Let G be a graph. A *tree decomposition* of G is a pair $(T, \{G_t : t \in V(T)\})$, where T is a tree and each G_t is a subgraph of G such that

(i) $\cup G_t = G$, and

(ii) for every vertex x of G, if $x \in V(G_{t'} \cap G_{t''})$ then $x \in V(G_t)$ for all t in the unique $t't''$-path of T.

The width of this decomposition is $max_t|G_t| - 1$. The *tree-width* of G is the minimum width over all tree decompositions of G. We denote it $tw(G)$.

Theorem 1.4.4. *The following are equivalent for any class $\mathcal{G}$ of graphs.*

(1) There exists a planar graph H such that no member of $\mathcal{G}$ contains H as a minor;

(2) there exists $w \in \mathbb{N}$ such that $tw(G) \leq w$ for all graphs $G \in \mathcal{G}$.

In this thesis, we consider similar problems for rooted graphs. In particular, we will generalize Theorem 1.4.1 and Theorem 1.4.2.

1.5 Basic lemmas

We introduced some basic results on unavoidable large graphs in various families in the previous section. In this section, we provide some basic lemmas on unavoidable large graphs which we use in the rest of this thesis.

We begin with a simple but useful lemma, which says that every large connected graph has either a vertex of high degree or a long induced path.

Lemma 1.5.1. *If G is a connected graph with $|G| > 1+d+d(d-1)+\cdots+d(d-1)^{l-1}$, then either $\triangle(G) > d$ or G has an induced path of length $l+1$ starting from any vertex.*

Proof. Suppose $\triangle(G) \leq d$. Let $v \in V$ and let n_k be the number of vertices of distance k away from v. Then $n_0 = 1$, $n_1 = d_G(v)$, and $n_k \leq n_{k-1}(d-1)$ for all $k \geq 2$. It follows that $|G| > n_0 + n_1 + \cdots + n_l$ and thus $n_{l+1} \neq 0$, which means that there is a vertex of distance $l+1$ away from v. □

From this lemma, we can obtain a similar result on unavoidable large connected induced subgraphs.

Lemma 1.5.2. *There exists a function $f_{1.5.2}(n)$ such that every connected graph of order at least $f_{1.5.2}(n)$ contains K_n, $K_{1,n}$, or P_n as an induced subgraph.*

Proof. Let $f_{1.5.2}(n) = 1+1+d+d(d-1)+\cdots+d(d-1)^{n-2}$ where $d = f_{1.3.1}(n)$. We prove that $f_{1.5.2}(n)$ satisfies the requirement of the Lemma. Let G be a connected graph of order at least $f_{1.5.2}(n)$. If G has an induced path of length n, then we are done. So we assume that there is no induced path of length n in G. By Lemma 1.5.1, $\triangle(G) > d$. Let v be a vertex with the maximal degree in G. By Theorem 1.3.1, $N_G(v)$ contains a subset X such that $G[X]$ is either K_n or $\overline{K}_n$. In the first case, G contains an induced K_n and in the second case G contains an induced $G[X+v]$, which is $K_{1,n}$. $\square$

A *comb* is a tree with maximum degree three and such that all its degree three vertices are contained in a path (see Figure 1.1). The *length* of a comb is its number of leaves. A comb of length n is denoted by Z_n. The minimal path containing all degree three vertices in a comb is called the *shaft* of a comb. For each leaf v of a comb, there is a unique path from v to its shaft; we call such path a *tooth* of a comb. The next lemma determines unavoidable trees with many leaves.

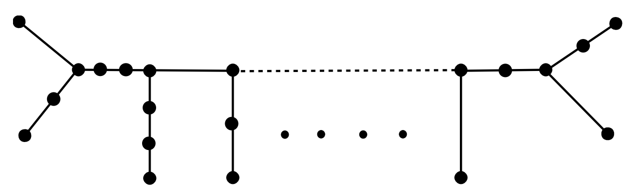

FIGURE 1.1. A comb

Lemma 1.5.3. *If T is a tree with at least d^t leaves, where $d, t \geq 2$ are integers, then T contains either $K_{1,d}$ or Z_t as a subgraph.*

Proof. Since contracting series edges do not change the problem, suppose that T has no vertex of degree two. Since $d^t \geq 4$, T has a vertex v of degree greater than two. If T has a path of length t starting from v, then Z_t subgraph be obtained by extending this path. So we assume that there is no such a path. Since $1+d+d(d-$

$1) + \cdots + d(d-1)^{t-2} < 1 + d + d^2 + \cdots + d^{t-1} < d^t$, for $d \geq 2$, we deduce from Lemma 1.5.1, that $\triangle(T) > d$, which means T contains $K_{1,d}$ as a subgraph. $\square$

Let P and Q be two paths. We say that P and Q are *comparable* if one is a subgraph of the other; we say that they are *overlapping* if they are not comparable but $E(P \cap Q) \neq \emptyset$. The following result identifies the unavoidable patterns of many subpaths from a path.

Lemma 1.5.4. *Let $Q_1, Q_2, \cdots, Q_m$ be non-trivial subpaths of a path Q. If $m > n^3$, then there are n of these subgraphs such that they are either pairwise disjoint, or pairwise comparable, or pairwise overlapping.*

This lemma can be proved using Ramsey theorem. But for a better bound, we choose to use the following Dilworth theorem, which is Corollary 2.3.2 of [2]. Let $(X, \leq)$ be a partially ordered set (poset). A subset C of X is a *chain* if for any $e, f \in C$ either $e \leq f$ or $f \leq e$ holds. A subset A of X is an *antichain* if for any two distinct $e, f \in A$ neither $e \leq f$ nor $f \leq e$ holds.

Lemma 1.5.5 (Dilworth theorem)**.** *In every finite partially ordered set $(X, \leq)$, the minimum number of chains covering X is equal to the maximum cardinality of an antichain in X.*

Proof of Lemma 1.5.4. In this proof we will use the following obvious corollary of Dilworth theorem.

Claim. Let $c, d \in \mathbb{N}$. If $(X, \leq)$ is a poset of $cd + 1$ elements, then it has a chain of $c + 1$ elements or an antichain of $d + 1$ elements.

Let x, y be two vertices of Q. We define Q_{xy} to be the unique subpath of Q between x and y. Let the ends of Q be a, b and let the ends of Q_i be a_i, b_i, where we assume without loss of generality that $|Q_{aa_i}| < |Q_{ab_i}|$. Then $Q_i = Q_{a_ib_i}$ for $1 \leq i \leq m$. Let us define a poset $(\{Q_1, Q_2, \cdots, Q_m\}, \leq)$ such that $Q_i \leq Q_j$ if

and only if either $Q_i = Q_j$ or $|Q_{ab_i}| < |Q_{aa_j}|$. In other words, $Q_i < Q_j$ means Q_i and Q_j are disjoint and Q_i is more close to a than Q_j. It is straightforward to verify that $\leq$ satisfies subgraph transitivity and thus it defines a poset. If this poset has a chain of $n+1$ elements, then elements in this chain, when considered as non-trivial subpaths of Q, are pairwise disjoint. So suppose there is no such a chain in this poset. By the above Claim, this poset has an antichain of n^2+1 elements. Without loss of generality, let this antichain consist of $Q_1, Q_2, \cdots, Q_{n^2+1}$. Let us define a new poset $(\{Q_1, Q_2, \cdots, Q_{n^2+1}\}, \leq')$ such that $Q_i \leq' Q_j$ if and only if $Q_i \leq Q_j$. Again, it is straightforward to verify that $\leq'$ defines a poset. If this new poset has a chain of $n+1$ elements, then elements in this chain, when considered as non-trivial subpaths of Q, are pairwise comparable. Else, by Claim, this new poset has an antichain A of $n+1$ elements. Without loss of generality, let $A = \{Q_1, Q_2, \cdots, Q_{n+1}\}$ and $|Q_{aa_1}| \leq |Q_{aa_2}| \leq \cdots \leq |Q_{aa_{n+1}}|$. Since A is an antichain of the first poset, $|Q_{aa_{n+1}}| \leq |Q_{ab_1}|$. Moreover, since A is also an antichain of the second poset, $|Q_{ab_1}| < |Q_{ab_2}| < \cdots < |Q_{ab_{n+1}}|$. If there exist i and j with $i < j$ such that $|Q_{aa_i}| = |Q_{aa_j}|$ then $Q_i \leq Q_j$. It contradicts the fact that A is an antichain of the second poset. Hence we obtain that $|Q_{aa_1}| < |Q_{aa_2}| < \cdots < |Q_{aa_{n+1}}| \leq |Q_{ab_1}| < |Q_{ab_2}| < \cdots < |Q_{ab_{n+1}}|$, which implies that $|Q_{aa_1}| < |Q_{aa_2}| < \cdots < |Q_{aa_n}| < |Q_{ab_1}| < |Q_{ab_2}| < \cdots < |Q_{ab_n}|$. So in this case we can obtain n pairwise overlapping subpaths $Q_1, Q_2, \cdots, Q_n$ of Q. □

The following lemma identifies the unavoidable subgraphs of a large graph that is not a subdivision of any other graph.

Lemma 1.5.6. *Suppose G is a connected simple graph such that G is not a subdivision of any other simple graph. If $|G| \geq 10t$, then G has a spanning tree with at least t leaves.*

Proof. By Lemma 1.1.1, there exists at least one spanning tree of G. Let T be a spanning tree of G with maximal number of leaves, and let the number of leaves of T be l. Let n_3 be the number of vertices with degree larger than two in T and let n_2 be the number of vertices with degree two in T. Then $|T| = l + n_2 + n_3$.

Claim 1. $n_3 + 2 \leq l$.

Since T is a tree, we know $|E(T)| = |T| - 1$. Hence $2(|T| - 1) = 2|E(T)| = \sum_{v \in V(T)} d_T(v) \geq l + 2n_2 + 3n_3$. Since $|T| = l + n_2 + n_3$, we get the inequality $2(l + n_2 + n_3 - 1) \geq l + 2n_2 + 3n_3$. Then we can get $n_3 + 2 \leq l$. So the claim is proved.

Since T has n_2 degree two vertices, we can obtain a tree T' such that T is a subdivision is T and $|T'| = |T| - n_2 = l + n_3$. By Claim 1, $|T'| < 2l$. If every edge e of T' is subdivided at most four times in obtaining T, then we can obtain $10t \leq |T| \leq 4(|T'| - 1) + |T'| = 5|T'| - 5 < 10l - 5$, which implies that $t \leq l$, we are done since G satisfies the lemma. Suppose there exists an edge e of T' that is subdivided at least five time in obtaining T. In the following we prove that no such an e exists and this would prove the lemma. Let $P = x_0x_1 \cdots x_{k+1}$ be the path of T obtained by subdividing e. It follows that $k \geq 5$ and the degree of each interior vertex of P is two in T.

Claim 2. $d_G(x_3) > 2$.

Suppose $d_G(x_3) = 2$. Then G is a subdivision of G/x_2x_3. Since G is not a subdivision of any simple graph, G/x_2x_3 must have parallel edges, which implies that $x_2x_4 \in E(G)$. Note that x_2, x_4 are non-leaf vertices of T but x_3 is a leaf of $(T + x_2x_4)\backslash x_2x_3$. It follows that $(T + x_2x_4)\backslash x_2x_3$ has more leaves than T, which contradicts the assumption that T has the maximal number of leaves.

By Claim 2, G has an edge x_3y with $y \neq x_2, x_4$. Let C be the unique cycle of $T + x_3y$. Then x_2x_3 or x_3x_4 belongs to C. Without loss of generality, let x_2x_3 belong

to C. If y is a non-leaf vertex of T, then $(T + x_3y)\backslash x_2x_3$ has more leaves than T (since x_2 is a leaf now), which contradicts the assumption that T has the maximal number of leaves. If y is a leaf of T, then $y \neq x_1$, which implies that x_1x_2 belongs to C. Now $(T + x_3y)\backslash x_1x_2$ has more leaves than T, which again contradicts the assumption that T has the maximal number of leaves.

Therefore it is impossible that there exits an edge of G that is subdivided by at least five times, which implies that $l \geq t$ as required. □

The following lemma [7] is a characterization of large connected $K_{1,n}$-free graphs.

Lemma 1.5.7. *There exists a function $f_{1.5.7}(n)$ such that every connected simple $K_{1,n}$-free graph is a subdivision of a connected simple graphs on fewer than $f_{1.5.7}(n)$ vertices.*

Proof. We prove that $f_{1.5.7}(n) = 10n$ satisfies the requirement of the lemma. Let G be a connected simple $K_{1,n}$-free graph. Then G is a subdivision of a simple graph H which is not a subdivision of a connected simple graph. Next we prove that $|H| < f_{1.5.7}(n)$. Suppose $|H| \geq 10n$. By Lemma 1.5.6, H has a spanning tree with at least n vertices, which implies that H contains $K_{1,n}$ as a minor. By Lemma 1.2.1, G contains $K_{1,n}$ as a minor, which contradicts the assumption that G is a $K_{1,n}$-free graph. □

1.6 Rooted graphs

In this section, we provide some basic definitions of a rooted graph, and give a brief summary of our main results in the latter chapters.

A *rooted graph* is a pair (G, R), where G is a graph and $R \subseteq V(G)$. Vertices of R are called *roots* of G. For $X \subseteq V(G)$, we call $(G[X], X \cap R)$ an *induced subgraph* of (G, R). A rooted graph (F, S) is called a *subgraph* of (G, R) if (F, S) satisfies the following conditions:

(i) F is a subgraph of G;

(ii) $S \subseteq R$.

A rooted graph (G, R) is called *connected* if G is connected. A rooted graph (G, R) is *2-connected* if G is 2-connected. Every rooted graph (G, R) can be decomposed into rooted connected graphs, which we call *components* of (G, R). We also say that (G, R) is the disjoint union of these components. A *block* of a rooted graph (G, R) is a maximal non-separable induced subgraph of (G, R). We call (G_1, G_2) a *k-separation of a rooted graph* (G, R) if (G_1, G_2) is a k-separation of G and $R \subseteq V(G_2)$.

Recall Lemma 1.2.2 that a graph H is a minor of a graph G if and only if G has a set $\{G_v : v \in V(H)\}$ of disjoint connected subgraphs and a set $F = \{f_e : e \in E(H)\}$ of edges such that F is disjoint from every $E(G_v)$, and for every edge $e = uv \in E(H)$, the two ends of f_e are in G_u and G_v, respectively.

A rooted graph (H, Q) is a *rooted minor* of a rooted graph (G, R) if H is a minor of G and let $(\{G_v\}, \{f_e\})$ be a model of the minor H of G such that if $v \in V(H) \cap Q$, then $V(G_v) \cap R \neq \emptyset$; if $v \in V(H) \backslash Q$, then $V(G_v) \cap R = \emptyset$.

Next, we define a *generalized rooted graph* $(G, \mathcal{X})$. Let $G = (V, E)$ be a graph and $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$ be a set of mutually disjoint nonempty subsets of V. We assume that there is no edge between any two vertices from each X_i, for $1 \leq i \leq n$. Then $(G, \mathcal{X})$ is called a *generalized rooted graph*.

A generalized rooted graph $(G, \mathcal{X})$ is *connected* if the graph obtained by identifying vertices of each member of $\mathcal{X}$ is connected.

An *induced subgraph* of $(G, \mathcal{X})$ is obtained by the following operations:

(i) $(G \backslash X_i, \mathcal{X} \backslash \{X_i\})$ for any $X_i \in \mathcal{X}$;

(ii) $(G \backslash v, \mathcal{X})$ if $v \notin X_i$ for any $X_i \in \mathcal{X}$.

We call $(H, \mathcal{Y})$ a *subgraph* of $(G, \mathcal{X})$ if $(H, \mathcal{Y})$ satisfies the following conditions:

(i) H is a subgraph of G;

(ii) $\mathcal{X}$ and $\mathcal{Y}$ can be enumerated as $\mathcal{X} = \{X_1, X_2, \cdots, X_n\}$ and $\mathcal{Y} = \{Y_1, Y_2, \cdots, Y_m\}$ such that $n \geq m$ and $Y_i = X_i \cap V(H)$ for all $i \in \{1, 2 \cdots, m\}$, and $X_j \cap V(H) = \emptyset$ for any $j = m+1, m+2, \cdots, n$.

Remark 1.6.1. *Notice that $(H, \mathcal{Y})$ is equivalent to be obtained by the following operations.*

(i) $(G \backslash e, \mathcal{X})$ *for some* $e \in E(G)$;

(ii) $(G \backslash v, (\mathcal{X} \backslash \{X_i\}) \cup \{X_i \backslash v\})$, *for any* $v \in X_i \in \mathcal{X}$; *note that if* $X_i = \{v\}$, *then* $(G \backslash v, \mathcal{X} \backslash \{X_i\})$

(iii) $(G \backslash v, \mathcal{X})$ *if* $v \notin X$ *for any* $X \in \mathcal{X}$.

Given a generalized rooted connected graph $(G, \mathcal{X})$, let $G^{\mathcal{X}}$ be the graph obtained from G by adding all edges of the form $x_i x_j$ where $x_i \neq x_j$ and $x_i, x_j \in X$ for some $X \in \mathcal{X}$. A generalized rooted graph $(G, \mathcal{X})$ is called *2-connected* if $G^{\mathcal{X}}$ is 2-connected.

In this thesis, there are some results on structures of large rooted graphs. There are two research topics in this thesis. The first one is about determining unavoidable substructures in sufficiently large rooted graphs. Chapter 2 and Chapter 3 belong to my first research topic. The second one is about characterizations on rooted graphs excluding certain large graphs. Chapters 4, 5, 6 belong to my second research topic.

In Chapter 2, we identify five families of rooted graphs and prove Theorem 2.1.1 that these are all the unavoidable large connected induced subgraphs of a sufficiently large rooted connected graph. Besides, we identify three families of rooted graphs and prove Theorem 2.1.4 that these are all the unavoidable large connected subgraphs of a sufficiently large rooted connected graph. Further, we

extend these results to generalized rooted connected graphs, we identify seven families of generalized rooted graphs and then prove Theorem 2.2.1 that these are all the unavoidable large generalized connected induced subgraphs; we also identify five families of generalized rooted graphs and provide Theorem 2.2.2 that these are all the unavoidable large generalized connected subgraphs.

In Chapter 3, we try to extend these results from Chapter 2 to rooted graphs of higher connectivity. In particular, we identify four families of rooted 2-connected graphs and then provide the edge version of unavoidable large subgraphs of a sufficiently large rooted 2-connected graph, which is Theorem 3.1.1. We also provide the vertex version of unavoidable large subgraphs of a sufficiently large rooted 2-connected graph, which is Theorem 3.1.5. Furthermore, we generalize these results to generalized rooted 2-connected graphs. We identify eleven families of generalized rooted graphs, and provide Theorem 3.2.1 that these are all the unavoidable large subgraphs of a sufficiently large generalized 2-connected graph.

In Chapter 4, we have some results about characterizing rooted connected graphs excluding a large comb. There are two distinct families of the comb with roots. We say a comb is confined if all leaves of it are roots, and we say a comb is nicely confined if all roots are precisely its leaves. Theorem 4.1.1 says that rooted graphs without a large nicely confined comb subgraph can be constructed from bounded clique-summing graphs without roots to a graph which contains all roots, the graph is obtained by deleting its all roots is connected, and such graph does not contain a long path. Theorem 4.2.1 states that rooted connected graphs without a large confined comb subgraph can be constructed using the operation Φ within a bounded number of iterations, starting from rooted connected graphs (G, R) such that $G = \oplus_2(G_0; G_1, \cdots, G_t)$, where $R \subseteq V(G_0)$ and (G_0, R) is a path with roots; such family of rooted connected graphs can also be constructed from clique-

summing graphs without roots to a graph which contains all roots and it can be obtained by a bounded number of Φ, starting from a class of paths with roots.

In Chapter 5, we have some results about characterizing rooted connected graphs excluding a heavy path. There are two parts of this chapter. One part describes a characterization of rooted connected graphs without a heavy path subgraph. The other part describes a characterization of rooted connected graphs without a heavy path rooted minor. In particular, we define a new operation Ψ, and we provide Theorem 5.2.1, which states that rooted connected graphs without a heavy path subgraph can be constructed using the operation Ψ within a bounded number of iterations, starting from rooted connected graphs (G, R) with $|R| \leq 1$. Besides, we prove Theorem 5.3.1 which says that rooted connected graphs without a heavy path rooted minor can be constructed using the operation Ψ within a bounded number of iterations, starting from connected graphs; such family of rooted connected graphs can also be constructed from clique-summing graphs without roots to a graph which contains all roots and does not contain a heavy path.

In Chapter 6, we prove some theorems on rooted connected graphs excluding a large star. There are three parts in this chapter. The first part is Conjecture 6.1.1, which says that rooted connected graphs without a subdivision of a large nicely confined star subgraph can be constructed from bounded clique-summing graphs to a graph which contains all roots and the degree of its non-root vertex is bounded. The second part provides some base cases of such conjecture. One case is Theorem 6.2.1, which states that rooted connected graphs without a subdivision of a nicely confined $K_{1,4}$ subgraph can be constructed from 3-summing graphs to a graph which contains all roots and the degree of its non-root vertex is no more than 3. The other case is Theorem 6.3.1, which says that rooted graphs without a subdivision of a nicely confined $K_{1,5}$ subgraph can be constructed from 15-summing

graphs to a graph which contains all roots and the degree of non-root vertex in it is no more than 15. The last part of this chapter is a characterization on rooted graphs without a large confined star rooted minor. Theorem 6.4.1 describes rooted connected graphs without a large confined star rooted minor can be constructed from clique-summing graphs to any subdivision of a graph with a bounded number of vertices, whose subdivision contains all roots.

2 Unavoidable Large Rooted Connected Graphs

Recall Theorem 1.3.2 that every sufficiently large connected graph must contain an unavoidable large induced subgraph isomorphic to P_n, $K_{1,n}$, or K_n. In this section, our goal is to determine all the unavoidable large induced subgraphs in a sufficiently large (generalized) rooted connected graph.

2.1 Large rooted graphs

Let (G, R) be a rooted graph and let T be a subtree of G. We will say that T is *confined* if all leaves of T are in R. Note that a non-leaf vertex of a confined tree is allowed to be a root. Moreover, T is *nicely confined* if $V(T) \cap R$ consists of precisely leaves of T.

By *growing a root* in a rooted graph (G, R) we mean the operation of adding a new vertex x and joining it to exactly one $y \in R$ and then replacing R with $(R + x)\backslash y$.

The following theorem says that every sufficiently large rooted connected graph has one of the following five explicitly defined rooted graphs as an induced subgraph (see Figures 2.1 through 2.5).

Theorem 2.1.1. *There exists a function $f_{2.1.1}(n)$ with the following property. For every rooted connected graph (G, R) with $|R| \geq f_{2.1.1}(n)$, there exists $X \subseteq V(G)$ such that one of the following holds. In figures below, large vertices are the ones that belong to R.*

(1) $G[X]$ is a path and $|X \cap R| = n$;

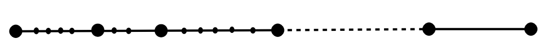

FIGURE 2.1. A path with n roots

(2) $G[X]$ is a subdivision of a confined star of size n and its degree two vertices are not in R;

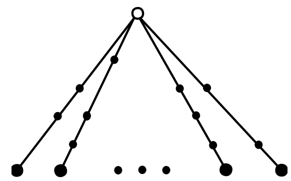

FIGURE 2.2. The center vertex may be a root or may not be a root

(3) $G[X]$ is a nicely confined comb of length n;

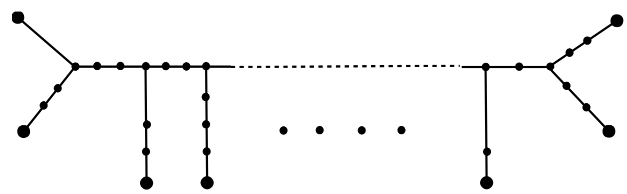

FIGURE 2.3. A nicely confined comb of length n

(4) $G[X]$ is the union of K_n, where $V(K_n) = \{x_1, ..., x_n\}$, and disjoint x_iy_i-paths ($i = 1, ..., n$), where $x_i = y_i$ is allowed. In addition, $X \cap R = \{y_1, ..., y_n\}$;

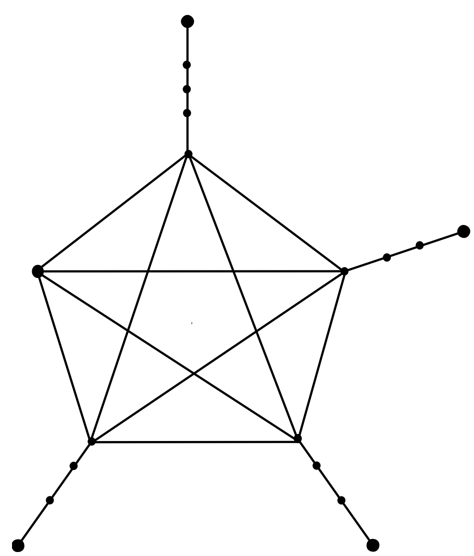

FIGURE 2.4. An example with $n = 5$

(5) $G[X]$ is the union of disjoint x_iy_i-paths ($i = 1, ..., 2n+1$) and triangles $y_{i-1}x_ix_{i+n}$ ($i = 2, ..., n+1$), where $x_i = y_i$ is allowed for $i = n+2, ..., 2n+1$. In addition, $X \cap R = \{x_1, y_{n+1}, ..., y_{2n+1}\}$.

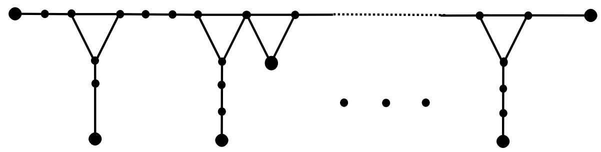

FIGURE 2.5. The last unavoidable induced subgraph

To simplify the proof of Theorem 2.1.1, we first prove the following two lemmas. The first lemma is on how to find an induced path in a connected graph.

Lemma 2.1.2. *Any shortest path between any two vertices in a connected graph must be an induced path.*

Proof. Let G be a connected graph and let $P_{a_0a_n}$ be a shortest path between any two vertices a_0, a_n in G. Let $V(P_{a_0a_n})$=$\{a_0, a_1, \cdots, a_n\}$, for $n \geq 1$. If $n = 1$, then we are done. If $n \geq 2$, then we assume that $P_{a_0a_n}$ is not an induced path, which implies that there exists an edge a_ia_j between two non-adjacent vertices $a_i, a_j \in V(P_{a_0a_n})$ in G. Hence we can find a shorter path between a_0 and a_n in G, which is $(P_{a_0a_n} + a_ia_j)\backslash P_{a_ia_j}$, where $P_{a_ia_j}$ is a subgraph of $P_{a_0a_n}$ with ends a_i, a_j. It contradicts the assumption that $P_{a_0a_n}$ is the shortest path between a_0 and a_n in G. $\square$

The second lemma is about determining minimal connected induced subgraphs that contain three specified vertices.

Lemma 2.1.3. *Let x, y, z be distinct vertices of a connected graph G. Then G contains one of these following as an induced subgraph:*

(1) the union of three independent paths P_{xw}, P_{yw}, P_{zw};

(2) the union of a triangle uvw and three disjoint paths P_{xu}, P_{yv}, P_{zw}.

Proof. Let H be a minimal subgraph of G such that H contains x, y, z. We prove that H satisfies either (1) or (2). Since H is connected, it has a path between y

and z. Let us choose a shortest such path, which we denote by P_{yz}. For the same reason H also has a path between x and P_{yz}. Let P_{xw} be a shortest such path. By Lemma 2.1.2, they are induced paths and $V(P_{yz} \cap P_{xw}) = \{w\}$. If $P_{yz} \cup P_{xw} = H$, then H satisfies (1). So we assume $P_{yz} \cup P_{xw} \neq H$. By the minimality of H, we must have $V(H) = V(P_{yz} \cup P_{xw})$. It follows that H has an edge $uv \notin P_{yz} \cup P_{zw}$. Since P_{yz} and P_{zw} are induced paths, none of them contains both u and v. Without loss of generality, assume $u \in P_{xw}$ and $v \in P_{yw}$, where P_{yw} is the unique subpath of P_{yz} between y and w. If P_{xw} contains a vertex t between u and w, then $H \backslash t$ is connected and contains x, y, z, which contradicts the minimality of H. So u must be the neighbor of w in P_{xw}. Similarly, vw must be an edge of P_{yw}. Since uv was chosen arbitrarily, if H has another edge $e \notin P_{yz} \cup P_{zw}$, then one end of e must be u and the other end of e must be the second neighbor of w in P_{yz}. In this case, $H \backslash w$ is connected and contains x, y, z, which contradicts the minimality of H, so H satisfies (2). $\square$

Proof of Theorem 2.1.1. Let $f_{2.1.1}(n) = ((n-1)f_{1.5.2}(2n))^{8n+8}$. We prove that $f_{2.1.1}(n)$ satisfies the requirement of the theorem. Let (G, R) be a rooted connected graph with $|R| \geq f_{2.1.1}(n)$. We create a new rooted connected graph (G', R') by growing all roots in (G, R). Since deleting all the leaves which are non-root vertices in G' such that every non-root vertex in G' is a cut vertex do not change the problem, suppose that every root is a leaf and every non-root vertex is a cut vertex in G'.

First we observe that for $Y \subseteq V(G')$, if L is the set of leaves of $G'[Y]$, then $G'[Y \backslash L] = G[Y \backslash L]$ and the set of roots in $G[Y \backslash L]$ is the set of neighbors of L in $G'[Y]$. Next we start proving the theorem using the above observation.

By Lemma 1.1.3, there exists a block tree $T_{G'} = V(G) \cup \mathcal{B}$ of G', which is a graph $V(G) \cup \mathcal{B}$ formed by the edges sB with $s \in B$, where $V(G)$ is the set of cut vertices of G' and $\mathcal{B}$ is the set of all blocks of G'. Since every leaf v of G' is not only a root but also an end of a pendant edge, and the unique neighbor vertex of v is a cut vertex, then every pendent edge in G' is a block and such block is a leaf of $T_{G'}$. Hence the number of leaves in $T_{G'}$ is $|R'| = |R| \geq f_{2.1.1}(n)$.

By Lemma 1.5.3, if $T_{G'}$ is a tree with at least $((n-1)f_{1.5.2}(2n))^{8n+8}$ leaves, where $(n-1)f_{1.5.2}(2n), 8n+8 \geq 2$ are integers, then either $\triangle(T_{G'}) > (n-1)f_{1.5.2}(2n)$ or $T_{G'}$ contains a comb of length $8n+8$ as a subgraph.

Firstly, we consider $\triangle(T_{G'}) > max\{n-1, (n-1)f_{1.5.2}(2n)\} = (n-1)f_{1.5.2}(2n) > (n-1)$, since $n \geq 2$. Then there are two cases on the vertex with the maximal degree in $T_{G'}$.

Case 1.1. There exists a cut vertex v in $T_{G'}$ such that $d_{T_{G'}}(v) > n-1$.

There are at least n components $H_1, H_2, \cdots, H_n$ of $G' \backslash v$ such that each H_i contains at least one root r_i, for $1 \leq i \leq n$. Let us choose the shortest path P^i between v and r_i in $G'[V(H_i)+v]$. The length of each P^i is at least one. By Lemma 2.1.2, P^i is an induced path in $G'[V(H_i)+v]$. Let $Y = V(P^1 \cup \cdots \cup P^n)$ and $L = \{r_1, r_2, \cdots, r_n\}$. By the observation, there exists $X = Y \backslash L$ such that $G[X]$ satisfies (2).

Case 1.2. There is no cut vertex v in $T_{G'}$ such that $d_{T_{G'}}(v) > n-1$, which means there exists a block point b in $T_{G'}$ such that $d_{T_{G'}}(b) > (n-1)f_{1.5.2}(2n)$.

Let B be the corresponding block of b in G'. By our assumption, every non-root vertex is a cut vertex in B. Hence $V(B) \geq f_{1.5.2}(2n)$. By Theorem 1.5.2, B contains K_{2n}, $K_{1,2n}$ or P_{2n} as an induced subgraph.

Suppose B contains K_{2n} as an induced subgraph. Then B must contain K_n as an induced subgraph. Let $V(K_n) = \{v_1, \cdots, v_n\}$. There are at least 2 components

$H_{i,1}, H_{i,2}$ of $G'\backslash v_i$ such that $H_{i,1}$ contains $K_n\backslash v_i$ as an induced subgraph and $H_{i,2}$ does not contain $K_n\backslash v_i$ as an induced subgraph, for $1 \leq i \leq n$. By the construction of (G', R'), each $H_{i,2}$ has at least one root $r_{i,2}$. Let us choose the shortest path P^i between v_i and $r_{i,2}$ in $G'[V(H_{i,2}) + v_i]$. The length of each P^i is at least one. By Lemma 2.1.2, P^i is an induced path between v_i and $r_{i,2}$ in G'. Let $Y = V(P^1 \cup \cdots \cup P^n)$ and $L = \{r_{1,2}, r_{2,2}, \cdots, r_{n,2}\}$. By the observation, there exists $X = Y\backslash L$ such that $G[X]$ satisfies (4).

Suppose B contains $K_{1,2n}$ as an induced subgraph. Then B must contain $K_{1,n}$ as an induced subgraph. Let $V(K_{1,n}) = \{u_0, u_1, \cdots, u_n\}$, where $d_{K_{1,n}}(u_0) \neq 1$. Similar to the above situation, there are $n+1$ mutually disjoint induced paths $P^0, P^1, \cdots, P^n$ in G' such that P^j is the shortest path between u_j and r_j, for $0 \leq j \leq n$. The length of each P^j is at least one. Let $Y = V(P^0 \cup \cdots \cup P^n)$ and $L = \{r_0, \cdots, r_n\}$. By the observation, there exists $X = Y\backslash L$ such that $G[X]$ satisfies (2).

Suppose B contains P_{2n} as an induced subgraph. Let $P_{2n} = a_0a_1 \cdots a_{2n}$. Similar to the above situation, there exist $2n$ mutually disjoint induced paths $P^1, P^2, \cdots, P^{2n}$ in G' such that each such P^t is from a_t to one root r_t in G', and the length of P^t is at least one, for $1 \leq t \leq 2n$. There are either n such paths $P^1, \cdots, P^n$ of length one or n such paths $P^{n+1}, \cdots, P^{2n}$ of length at least two. Let $Y = V(P^1 \cup \cdots \cup P^n) \cup V(P_{2n})$, and $L = \{r_1, r_2, \cdots, r_n\}$. By the observation, there exists $X = Y\backslash L$ such that $G[X]$ satisfies (1). Let $Y = V(P^{n+1} \cup \cdots \cup P^{2n}) \cup V(P_{2n})$, and $L = \{r_{n+1}, \cdots, r_{2n}\}$. By the observation, there exists $X = Y\backslash L$ such that $G[X]$ satisfies (3).

Secondly, we consider that $T_{G'}$ contains a comb of length $8n + 8$ as an induced subgraph. There are two cases on the vertex with the degree three in such comb.

Case 2.1. There exists a comb Z_{2n+2} in $T_{G'}$ such that every vertex with degree three in Z_{2n+2} is a cut vertex as an induced subgraph in $T_{G'}$, including $s_1, s_2, \cdots, s_{2n}$. Similar to the above situation, there exist at least $2n+2$ mutually disjoint induced paths in G' such that P^t is from s_t to one root r_t in G', and the length of P^t is at least one, for $1 \leq t \leq 2n$ and a path $P^0 = \bigcup_{t=1}^{2n-1} P^{0,t}$, where $P^{0,t}$ is the shortest path from s_t to s_{t+1} in G'. There are either n such paths $P^1, \cdots, P^n$ of length one or n such paths $P^{n+1}, \cdots, P^{2n}$ of length at least two. Let $Y = V(P^0 \cup P^1 \cup \cdots \cup P^n)$ and $L = \{r_1, \cdots, r_n\}$. By the observation, there exists $X = Y \backslash L$ such that $G[X]$ satisfies (1). Let $Y = V(P^0 \cup P^{n+1} \cup \cdots \cup P^{2n})$, and $L = \{r_{n+1}, \cdots, r_{2n}\}$. By the observation, there exists $X = Y \backslash L$ such that $G[X]$ satisfies (3).

Case 2.2. There exists a comb Z'_{4n+4} in $T_{G'}$ such that every vertex with the degree three of Z'_{4n+4} is a block point as an induced subgraph in T'_G, including $b_0, b_1, \cdots, b_{4n+1}$. Let P be the shaft of Z'_{4n+4} in $T_{G'}$ and let $c_{i,1}, c_{i,2}, c_{i,3}$ be neighbors of b_i in $T_{G'}$. Without loss of generality, let $c_{i,1}, c_{i,2} \in V(P)$ for $1 \leq i \leq 4n$. Let B_i be the corresponding block of b_i in G'. There are three distinct vertices $c_{i,1}, c_{i,2}, c_{i,3}$ in B_i. Similar to the above situation, there are at least $4n$ mutually disjoint induced path P^i between $c_{i,3}$ and a root r_i in G', and one induced path $P^0 = \bigcup_{i=1}^{4n} P^{0i}$, where P^{0i} is the shortest path between $c_{i,2}$ and $c_{i+1,1}$ in G', for $1 \leq i \leq 4n$. Note that $c_{i,2}$ maybe $c_{i+1,1}$ in G'. The length of P^i is at least one. By Lemma 2.1.3, there are two subcases on B_i in G'.

Subcase 2.2.1. Z'_{4n+4} contains a comb Z^1_{2n+2} as an induced subgraph, without loss of generality, $b_1, b_2, \cdots, b_{2n}$ are vertices with the degree three in Z^1_{2n+2}, which satisfies that each B_i contains an induced subgraph J^i, which is the union of three independent paths, including $c_{i,1}w_i$-path, $c_{i,2}w_i$-path and $c_{i,3}w_i$-path, for $1 \leq i \leq 2n$, and the length of such independent path is at least one. Since w_i maybe

$c_{i,1}$ or $c_{i,2}$ in B_i. Without loss of generality, we assume that $w_{2j+1} = c_{2j+1,1}$ and $w_{2j} = c_{2j-1,2}$, for $0 \leq j \leq n-1$. Let $Y = V(P^1 \cup P^3 \cdots \cup P^{2n-1}) \cup V(J^1 \cup J^3 \cup \cdots \cup J^{2n-1}) \cup V(P^0)$ and $L = \{r_1, r_3 \cdots, r_{2n-1}\}$. By the observation, there exists $X = Y \backslash L$ such that $G[X]$ satisfies (3).

Subcase 2.2.2. Z'_{4n+4} contains a comb Z^2_{n+2} as an induced subgraph, without loss of generality, $b_{2n+1}, b_{2n+2}, \cdots, b_{3n}$ are vertices with the degree three in Z^2_{n+2}, which satisfies that each B_j has an induced subgraph I^j, which is the union of a triangle $w_j u_j v_j$ and three independent paths, including $c_{j,1}w_j$-path, $c_{j,2}v_j$-path and $c_{j,3}u_j$-path, for $2n+1 \leq j \leq 3n$. The length of such independent path is at least one. Let $Y = V(P^{2n+1} \cup P^{2n+2} \cup \cdots \cup P^{3n}) \cup V(I^{2n+1} \cup \cdots \cup I^{3n}) \cup V(P^0)$ and $L = \{r_{2n+1}, \cdots, r_{3n}\}$. By the observation, there exists $X = Y \backslash L$ such that $G[X]$ satisfies (5). □

From this theorem, we can obtain a similar result on unavoidable large subgraphs of every sufficiently large rooted connected graph.

Theorem 2.1.4. *There exists a function $f_{2.1.4}(n)$ with the following property. For every rooted connected graph (G, R) with $|R| \geq f_{2.1.4}(n)$, there exists a subgraph H such that H is a path containing n roots, or a nicely confined comb of length n, or a subdivision of a confined star of size n.*

Proof. Let $f_{2.1.4}(n) = f_{2.1.1}(2n)$. We prove that $f_{2.1.4}(n)$ satisfies the requirement of the theorem.

By Theorem 2.1.1, there exists $X \subseteq V(G)$ such that $G[X]$ is one of five explicitly defined induced subgraphs of (G, R). Since an induced subgraph of a rooted connected graph is a subgraph of it, then $H' = G[X]$ such that H' is a path containing $2n$ roots if Theorem 2.1.1 (1) holds. Similarly, $H' = G[X]$ is a subdivision of a confined star of size $2n$ if Theorem 2.1.1 (2) holds, and $H' = G[X]$

is a nicely confined comb of length $2n$ if Theorem 2.1.1 (3) holds. We can obtain $H' = G[X] \backslash x_1x_{2n} \backslash \{x_ix_j : 1 \leq i, j \leq 2n, \text{ and } |i-j| \geq 2\}$ if Theorem 2.1.1 (4) holds. There exists a subgraph H of H' such that H is either a nicely confined comb of length n or a path with n roots. We can obtain $H' = G[X] \backslash \{x_ix_{i+n} : 2 \leq i \leq n\}$ as a nicely confined comb of length $2n$ if Theorem 2.1.1 (5) holds.

Let H be a subgraph of H'. By the definition of a subgraph of a rooted graph, it is obvious to obtain a subgraph H of G such that H is a path containing n roots, or a nicely confined comb of length n, or a subdivision of a confined star of size n. □

2.2 Large generalized rooted graphs

In this section, our goal is to obtain a generalized rooted connected graph version of unavoidable large induced subgraphs and that of unavoidable large subgraphs.

Next, we identify seven special generalized rooted connected graphs (see Figures 2.6 through 2.12), which are unavoidable subgraphs of a sufficiently large generalized rooted connected graph.

Let $\Gamma_1(n) = (G, \mathcal{X})$ be a generalized rooted connected graph (see Figure 2.6) in which $\mathcal{X} = \{X_1, \cdots, X_n\}$, $V(G) = X_1 \cup X_2 \cup \cdots \cup X_n \cup Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1}$ and $E(G) = E_1 \cup E_2 \cup \cdots \cup E_{n-1}$, where Y_i and E_i are defined as follows:

If $Y_i = \emptyset$, then E_i is a non-empty set of edges between X_i and X_{i+1}.

If $Y_i \neq \emptyset$, then Y_i is a set of vertices of a path P_i with ends u_i, v_i (u_i could be equal to v_i). In this case, E_i is the union of $E(P_i)$, a set of at least one edge between X_i and u_i, and a set of at least one edge between X_{i+1} and v_i.

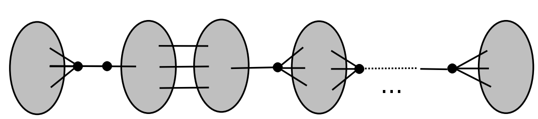

FIGURE 2.6. $\Gamma_1(n)$

Let $\Gamma_2(n) = (G, \mathcal{X})$ be a generalized rooted connected graph (see Figure 2.7) in which $\mathcal{X} = \{X_1, \cdots, X_n\}$, $V(G) = X_1 \cup X_2 \cup \cdots \cup X_n \cup Y_1 \cup Y_2 \cup \cdots \cup Y_n$ and $E(G) = E_1 \cup E_2 \cup \cdots \cup E_n$, where v, Y_i and E_i are defined as follows:

Let $v = Y_1 \cap Y_2 \cap \cdots \cap Y_n$, let Y_i be a set of vertices of a path P_i with ends u_i, v (u_i could be equal to v), and let E_i consist of $E(P_i)$ and a set of at least one edge between X_i and u_i.

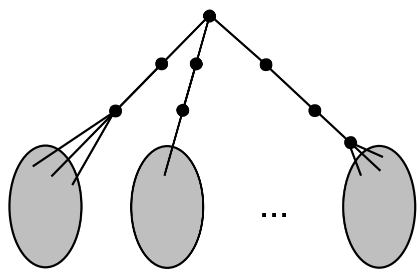

FIGURE 2.7. $\Gamma_2(n)$

Let $\Gamma_3(n) = (G, \mathcal{X})$ be a generalized rooted connected graph (see Figure 2.8) in which $\mathcal{X} = \{X_0, X_1, \cdots, X_n\}$, and $V(G) = X_0 \cup X_1 \cup \cdots \cup X_n \cup Y_1 \cup \cdots \cup Y_n$ and $E(G) = E_1 \cup E_2 \cup \cdots \cup E_n$ where Y_i and E_i are defined as follows.

Let Y_i be a set of vertices of a path P_i with ends u_i, v_i (u_i could be equal to v_i) such that any $v_i \in X_0$, for any $1 \leq i \neq j \leq n$, $Y_i \cap Y_j = \emptyset$. Let E_i be the union of $E(P_i)$, a set of at least one edge between X_i and u_i and a set of at least one edge between X_0 and v_i for $1 \leq i \leq n$.

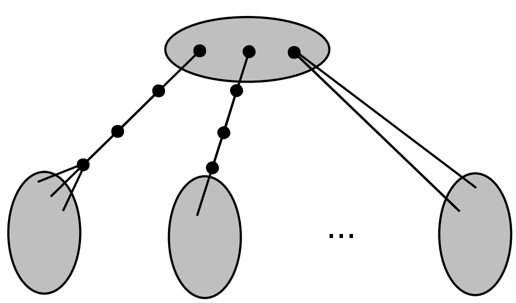

FIGURE 2.8. $\Gamma_3(n)$

Let $\Gamma_4(n) = (G, \mathcal{X})$ be a generalized rooted connected graph (see Figure 2.9) in which $\mathcal{X} = \{X_0, X_1, \cdots, X_n\}$, and $V(G) = X_0 \cup X_1 \cup \cdots \cup X_n \cup Y_1 \cup \cdots \cup Y_n$ and $E(G) = E_1 \cup E_2 \cup \cdots \cup E_n$ where v, Y_i and E_i are defined as follows.

Let $v = Y_1 \cap Y_2 \cap \cdots \cap Y_n \in X_0$, and let Y_i be a set of vertices of a path P_i with ends u_i, v (u_i could be equal to v). Let E_i be the union of $E(P_i)$, a set of at least one edge between X_i and u_i, and a set of at least one edge between X_0 and v_i, for $1 \leq i \leq n$.

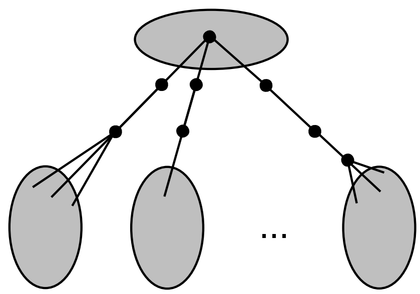

FIGURE 2.9. $\Gamma_4(n)$

Let $\Gamma_5(n) = (G, \mathcal{X})$ be a generalized rooted connected graph (see Figure 2.10) in which $\mathcal{X} = \{X_1, \cdots, X_n\}$, and $V(G) = X_1 \cup \cdots \cup X_n \cup Y_1 \cup \cdots \cup Y_n \cup Z_0$ where $Y_i \cap Y_j = \emptyset$ for $1 \leq i \neq j \leq n$, and $E(G) = E_0 \cup E_1 \cup \cdots \cup E_n$, where Y_i and E_i are defined as follows.

Let Y_i be a set of vertices of a path P_i with ends u_i, v_i (u_i could be equal to v_i) and let Z_0 be a set of vertices of a path P with any $v_i \in V(P)$ and $v_i \neq v_j$ for $1 \leq i \neq j \leq n$. Let E_0 be $E(P)$ and let E_i be the union of $E(P_i)$ and a set of at least one edge between u_i and X_i.

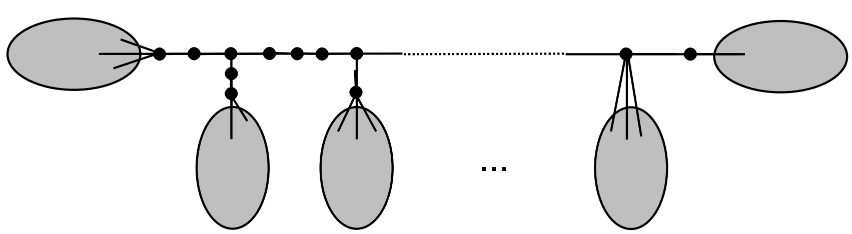

FIGURE 2.10. $\Gamma_5(n)$

Let $\Gamma_6(n) = (G, \mathcal{X})$ be a generalized rooted connected graph (see FIGURE 2.11) in which $\mathcal{X} = \{X_1, \cdots, X_n\}$, and $V(G) = X_1 \cup \cdots \cup X_n \cup Y_1 \cdots \cup Y_n$ and $E = E_0 \cup E_1 \cup \cdots \cup E_n$, where Y_i and E_i are defined as follows.

Let Y_i be a set of vertices of a path P_i with ends u_i, v_i (u_i could be equal to v_i). Let E_0 be a set of K_n, where $V(K_n) = \{v_1, \cdots, v_n\}$, and either E_i is a set of at least one edge between X_i and $V(K_n) - v_i$ if $Y_i = \emptyset$, or E_i consists of $E(P_i)$ and a set of at least one edge between X_i and u_i if $Y_i \neq \emptyset$.

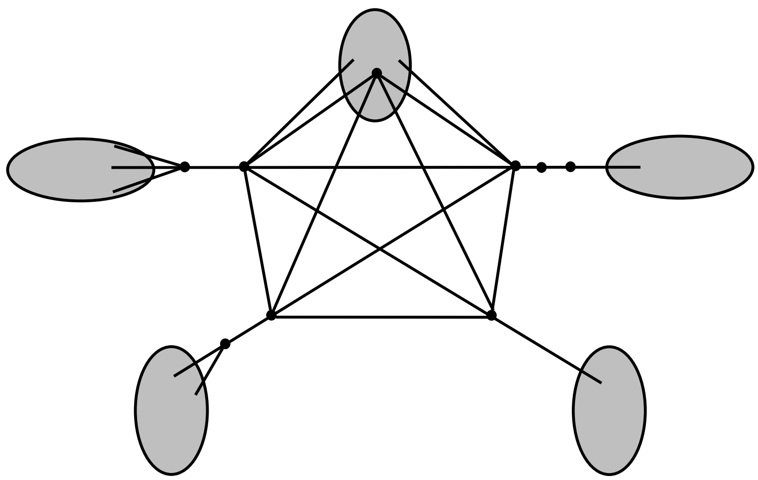

FIGURE 2.11. $\Gamma_6(n)$

Let $\Gamma_7(n) = (G, \mathcal{X})$ be a generalized rooted connected graph (see Figure 2.12) in which $\mathcal{X} = \{X_1, \cdots, X_n\}$, and $V(G) = X_1 \cup \cdots \cup X_n \cup Y_1 \cdots \cup Y_n \cup Z_0$ and $E(G) = E_0 \cup E_1 \cup \cdots \cup E_n$, where Y_i and E_i are defined as follows.

Let Y_i be a set of vertices of a path P_i with ends u_i, v_i (u_i could be equal to v_i) and w_i is the neighbor of v_i in P_i and let Z_0 be a set of vertices of a path P with any $v_i \in V(P)$ and $v_i \neq v_j$ for $1 \leq i \neq j \leq n$. Let E_0 be a nonempty set of a path $E(P)$, and let either E_i consist of $E(P_i)$ and a set of at least one edge between u_i and X_i, and a set of at least one edge between w_i and t_i, where t_i is the neighbor vertex of v_i in P if $|Y_i| \geq 2$, or E_i consist of a set of at least one edge between u_i and X_i, and a set of at least one edge between t_i and X_i if $|Y_i| = 1$.

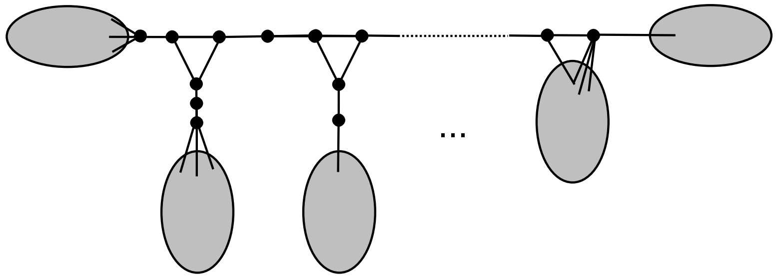

FIGURE 2.12. $\Gamma_7(n)$

Let $(G, \mathcal{X})$ be a generalized rooted connected graph, where $\mathcal{X} = \{X_1, X_2, \cdots, X_t\}$. By *shrinking* $(G, \mathcal{X})$ we end up with a rooted graph (H, Y), where a simple graph H is obtained by identifying all vertices from each X_i, and Y is the set of identified vertices $\{y_1, \cdots, y_t\}$. Observe that for any induced subgraph (H', Y') of (H, Y), $(G, \mathcal{X})$ has an induced subgraph $(G', \mathcal{X}')$ such that (H', Y') is the result of shrinking $(G', \mathcal{X}')$. In the following proof, we will repeatedly use this observation.

The following theorem says that every sufficiently large generalized rooted connected graph has one of the following seven explicitly defined generalized rooted connected graphs as an induced subgraph (see Figures 2.6 through 2.12).

Theorem 2.2.1. *There exists a function $f_{2.2.1}(n)$ such that every generalized rooted connected graph $(G, \mathcal{X})$ with $|\mathcal{X}| \geq f_{2.2.1}(n)$ must contain some $\Gamma_i(n)$ ($i = 1, 2, 3, 4, 5, 6, 7$) as an induced subgraph.*

Proof. Let $f_{2.2.1}(n) = f_{2.1.1}(n^2)$. We prove that $f_{2.2.1}(n)$ satisfies the requirement of the theorem. By shrinking $(G, \mathcal{X})$, we can obtain a rooted connected graph (H, Y) and $|Y| \geq f_{2.1.1}(n^2)$. By Theorem 2.1.1, there exists $X \subseteq V(H)$ such that $H[X]$ is one of induced subgraphs described in Theorem 2.1.1.

Suppose $H[X]$ is a path and $|X \cap Y| = n$, there exists an induced subgraph $(G', \mathcal{X}')$ of $(G, \mathcal{X})$ such that $(H[X], X \cap Y)$ is obtained by shrinking $(G', \mathcal{X}')$. Let $\Gamma_1(n) = (G', \mathcal{X}')$.

Suppose $H[X]$ is a subdivision of a confined star of size n^2. If the center vertex of $H[X]$ is not a root, then there exists an induced subgraph $(G', \mathcal{X}')$ of $(G, \mathcal{X})$ such that $(H[X], X \cap Y)$ is obtained by shrinking $(G', \mathcal{X}')$. Let $\Gamma_2(n) = (G', \mathcal{X}')$. If the center vertex of $H[X]$ is a root, then there exists an induced subgraph $(G', \mathcal{X}')$ of $(G, \mathcal{X})$ such that $(H[X], X \cap Y)$ is obtained by shrinking $(G', \mathcal{X}')$. Then $(G', \mathcal{X}')$ is either $\Gamma_3(n)$ or $\Gamma_4(n)$.

Suppose $H[X]$ is a nicely confined comb of length n, then there exists an induced subgraph $(G', \mathcal{X}')$ of $(G, \mathcal{X})$ such that $(H[X], X \cap Y)$ is obtained by shrinking $(G', \mathcal{X}')$. Let $\Gamma_5(n) = (G', \mathcal{X}')$.

Suppose $H[X]$ is the union of K_n, where $V(K_n) = \{x_1, ..., x_n\}$, and disjoint $x_i y_i$-paths ($i = 1, ..., n$), where $x_i = y_i$ is allowed. In addition, $X \cap Y = \{y_1, ..., y_n\}$, then there exists an induced subgraph $(G', \mathcal{X}')$ of $(G, \mathcal{X})$ such that $(H[X], X \cap Y)$ is obtained by shrinking $(G', \mathcal{X}')$. Let $\Gamma_6(n) = (G', \mathcal{X}')$.

Suppose $H[X]$ is the union of disjoint $x_i y_i$-paths ($i = 1, ..., 2n+1$) and triangles $y_{i-1} x_i x_{i+n}$ ($i = 2, ..., n+1$), where $x_i = y_i$ is allowed for $i = n+2, ..., 2n+1$. In addition, $X \cap R = \{x_1, y_{n+1}, ..., y_{2n+1}\}$, there exists an induced subgraph $(G', \mathcal{X}')$ of $(G, \mathcal{X})$ such that $(H[X], X \cap Y)$ is obtained by shrinking $(G', \mathcal{X}')$. Let $\Gamma_7(n) = (G', \mathcal{X}')$. $\square$

We define four families of generalized rooted connected graphs $\Gamma'_1(n)$, $\Gamma'_2(n)$, $\Gamma'_4(n)$, $\Gamma'_5(n)$ as below, for any positive integer n, which play an important role on determining unavoidable subgraphs of a sufficiently large generalized rooted connected graph.

Let $\Gamma'_1(n) = (G, \mathcal{X})$ be a special case of $\Gamma_1(n)$ (see FIGURE 2.13) if every E_i consists of $E(P_i)$, one edge between X_i and u_i and one edge between X_{i+1} and v_i.

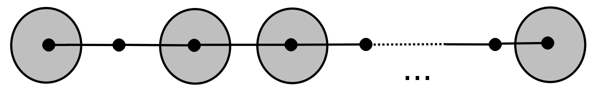

FIGURE 2.13. $\Gamma_1'(n)$

Let $\Gamma_2'(n) = (G, \mathcal{X})$ be a special case of $\Gamma_2(n)$ (see Figure 2.14) if E_i consists of $E(P_i)$ and a set of one edge between X_i and u_i.

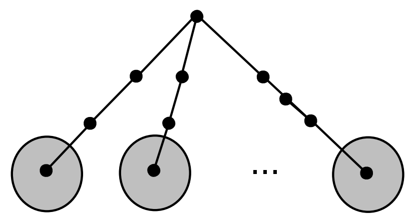

FIGURE 2.14. $\Gamma_2'(n)$

Let $\Gamma_3'(n) = (G, \mathcal{X})$ be a special case of $\Gamma_3(n)$ (see Figure 2.15) if E_i consists of $E(P_i)$ and a set of unique one edge between each X_i and each u_i.

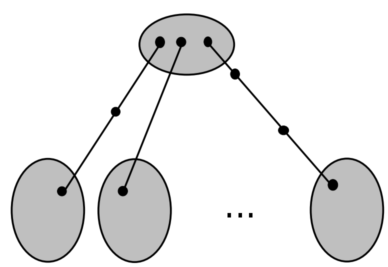

FIGURE 2.15. $\Gamma_3'(n)$

Let $\Gamma_4'(n) = (G, \mathcal{X})$ be a special case of $\Gamma_4(n)$ (see Figure 2.16) if E_i consists of $E(P_i)$ and a set of unique one edge between each X_i and each u_i.

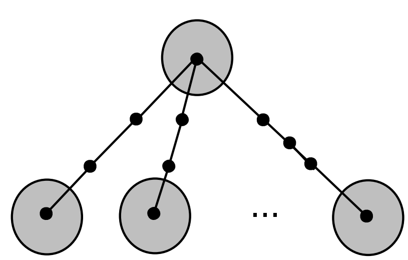

FIGURE 2.16. $\Gamma_4'(n)$

Let $\Gamma'_5(n) = (G, \mathcal{X})$ be a special case of $\Gamma_5(n)$ (see Figure 2.17) if E_i consists of $E(P_i)$ and a set of unique one edge between each u_i and each X_i.

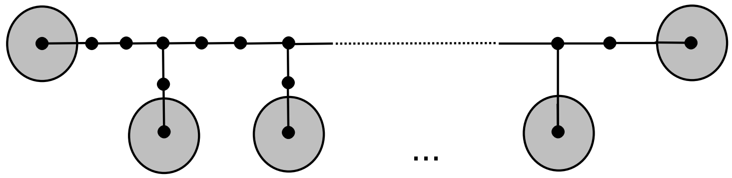

FIGURE 2.17. $\Gamma'_5(n)$

The following theorem is the result on determining unavoidable subgraphs of a sufficiently large generalized rooted connected graph.

Theorem 2.2.2. *There exists a function $f_{2.2.2}(n)$ such that every generalized rooted connected graph $(G, \mathcal{X})$ with $|\mathcal{X}| \geq f_{2.2.2}(n)$ must contain one of $\Gamma'_1(n)$,$\Gamma'_2(n)$, $\Gamma'_3(n)$, $\Gamma'_4(n)$ and $\Gamma'_5(n)$ as a subgraph.*

Proof. Let $f_{2.2.2}(n) = f_{2.1.4}(n^2)$. We prove that $f_{2.2.2}(n)$ satisfies the requirement of the theorem. Let (H, Y) be a rooted connected graph by shrinking $(G, \mathcal{X})$. By Theorem 2.1.4, there exists a subgraph $(H', V(H') \cap Y)$ of (H, Y) such that $(H', V(H') \cap Y)$ is a path containing n roots, or a nicely confined comb of length n, or a subdivision of a confined star of size n^2.

Suppose $(H', V(H') \cap Y)$ is a path containing n roots, then there exists a subgraph $\Gamma'_1(n)$ such that $(H', V(H') \cap Y)$ is obtained by shrinking $\Gamma'_1(n)$.

Suppose $(H', V(H') \cap Y)$ is a subdivision of a confined star of size n^2, then there exists either a subdivision of a confined star $(H'_1, V(H'_1) \cap Y)$, $(H'_2, V(H'_2) \cap Y)$ of size n whose center vertex is a root or a subdivision of a nicely confined star $(H'_3, V(H'_3) \cap Y)$ of size n as a subgraph of (H, Y). Hence there exists either a $\Gamma'_2(n)$, a $\Gamma'_3(n)$ or a $\Gamma'_4(n)$ as a subgraph of $(G, \mathcal{X})$ such that $(H'_1, V(H'_1) \cap Y)$ is obtained by shrinking $\Gamma'_2(n)$, $(H'_2, V(H'_2) \cap Y)$ is obtained by shrinking $\Gamma'_3(n)$, or $(H'_3, V(H'_3) \cap Y)$ is obtained by shrinking $\Gamma'_4(n)$.

Suppose $(H', V(H') \cap Y)$ is a confined comb of length n, then there exists a $\Gamma_5'(n)$ as a subgraph of $(G, \mathcal{X})$ such that $(H', V(H') \cap Y)$ is obtained by shrinking $\Gamma_5'(n)$. $\square$

3 Unavoidable Large Rooted 2-connected Graphs

In the last chapter, we determined unavoidable large (induced) subgraphs in a sufficiently large (generalized) rooted connected graph. In this chapter, we will extend these results to 2-connected cases. Recall Theorem 1.3.3 that every sufficiently large 2-connected graph must contain either a long cycle or a subdivision of a large $K_{2,n}$. We strengthen this result to (generalized) rooted 2-connected graphs.

3.1 Large rooted graphs

In this section, we provide a theorem on the edge version of unavoidable large subgraphs of a sufficiently large rooted 2-connected graph, and also provide a theorem on the vertex version of unavoidable large subgraphs of a sufficiently large rooted 2-connected graph. We remark that, in this section, we allow parallel edges in a graph. In addition, we consider a graph that consists of two or more parallel edges as 2-connected.

Let $n \geq 2$ be an integer. We first define the following three graphs.

A *fan* F is obtained from a path P by adding a vertex x and joining paths between x and P such that these paths are mutually disjoint except for x (see Figure 3.1). We call x the *center* of F. We call paths between x and P *ribs* of F. The number of ribs of F is called the *length* of F. A fan of length n is denoted by F_n.

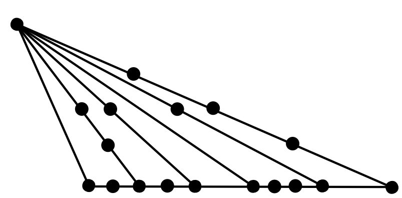

FIGURE 3.1. A fan

A *theta* θ is a graph with independent paths between two vertices (see Figure 3.2). We call each of these paths a *branch* of θ. The number of branches of θ is called the *size* of θ. A θ of size n is denoted by θ_n. If all branches of θ_n are edges, then we call it *trivial theta*, which is denoted by θ_n^0.

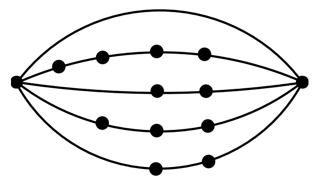

FIGURE 3.2. A theta

A *ladder* L' is a graph with two disjoint paths $P_{x_1x_n}$, $P_{x_1'x_n'}$ and mutually disjoint paths $P_{x_1x_1'}, P_{x_ix_i'}, \cdots, P_{x_nx_n'}$ for some $1 \leq i \leq n$ (see Figure 3.3). We call each $P_{x_ix_i'}$ a *rung* of L'. We call $P_{x_1x_n}, P_{x_1'x_n'}$ *rails* of L'. The number of rungs of L' is called the *length* of L'. A ladder of length n is denoted by L_n'.

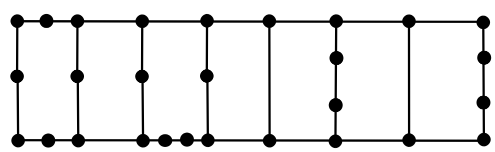

FIGURE 3.3. A ladder

The next theorem is the edge version of determining unavoidable large subgraphs of a sufficiently large rooted 2-connected graph.

Theorem 3.1.1. *There exists a function $f_{3.1.1}(n)$ with the following property. For every 2-connected loopless graph $G = (V, E)$ and every $E' \subseteq E$, if $|E'| \geq f_{3.1.1}(n)$, then G contains a subgraph H such that*

(1) H is a cycle containing at least n edges from E', or

(2) H is a theta of size n in which each branch contains at least one edge from E', or

(3) H is a fan of length n in which each rib contains at least one edge from E' while the rim contains no edge from E', or

(4) H is a ladder of length n in which each rung contains at least one edge from E' while the rails contains no edge from E'.

To simplify the proof of Theorem 3.1.1, we first need to prove the following three lemmas. The first lemma is to say that any large enough 2-connected graph can be contracted or deleted one edge into a smaller 2-connected graph.

Lemma 3.1.2. *If e is an edge of a 2-connected loopless graph G with $|E| \geq 3$, then at least one of $G\backslash e$ and G/e is 2-connected.*

Proof. Let $e = xy$. Since G is 2-connected, there exists a 2-separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = \{x, y\}$. Hence there are two independent paths P_{ax}, P_{ay} from any vertex $a \in V(G_1)$ to x, y in G_1, and there are two independent paths P_{bx}, P_{by} from any vertex $b \in V(G_2)$ to x, y in G_2. If G/e is 2-connected, then we are done. If G/e is not 2-connected, then there exists a cycle which is the union of $P_{ax}, P_{ay}, P_{bx}, P_{by}$ containing x and y in $G\backslash e$. Next we need to prove that $G\backslash e$ is 2-connected. Otherwise, there is a 1-separation (H_1, H_2) of $G\backslash e$. Suppose either $x, y \in V(H_1)$ or $x, y \in V(H_2)$, which contradicts the assumption that G is 2-connected. Suppose $x \in V(H_1\backslash H_2)$ and $y \in V(H_2\backslash H_1)$, which contradicts the assumption that there is a cycle containing x, y in $G\backslash e$. $\square$

The second lemma is on unavoidable minors of a sufficiently large 2-connected graph.

Lemma 3.1.3. *There exists a function $f_{3.1.3}(n)$ such that every 2-connected loopless graph with at least $f_{3.1.3}(n)$ edges contains either a cycle C_n or θ_n^0 as a minor.*

Proof. Let $f_{3.1.3}(n) = n\binom{f_{1.3.3}(n)}{2}$. We want to prove that $f_{3.1.3}(n)$ satisfies the requirement of the lemma. Let G be a 2-connected loopless graph with at least $f_{3.1.3}(n)$ edges. There are two cases on the number of vertices of G.

Case 1. G has at least $f_{1.3.3}(n)$ vertices. By Lemma 1.3.3, it contains either C_n or $K_{2,n}$ as a minor, where θ_n^0 is a minor of $K_{2,n}$. By Lemma 1.2.1, either θ_n^0 or C_n is a minor of G.

Case 2. G has at most $f_{1.3.3}(n) - 1$ vertices. There are at least n parallel edges between any two vertices in G, we can find θ_n^0 as a subgraph of G, which implies there exists a θ_n^0 of G as a minor. □

We use above two lemmas to prove the following lemma, which plays an important role on proving Theorem 3.1.1. It says that any sufficiently large 2-connected graph with specified edges must contain either a large cycle or a trivial theta with specified edges as a minor.

Lemma 3.1.4. *There exists a function $f_{3.1.4}(n)$ such that for every 2-connected $G = (V, E)$ and for every $E' \subseteq E$, if $|E'| \geq f_{3.1.4}(n)$, then G must contain either a cycle C_n or a trivial theta θ_n^0 as a minor in which all edges of the minor are in E'.*

Proof. Let $f_{3.1.4}(n) = f_{3.1.3}(n)$. Proof by the induction on $|E \backslash E'|$.

For $|E \backslash E'| = 0$, the assertion is true, by Lemma 3.1.3.

Now let $|E \backslash E'| > 0$, and assume that the assertion holds for every 2-connected $G = (V, E)$ and for every $E' \subseteq E$ such that $|E \backslash E'| \leq k - 1$.

Suppose $|E \backslash E'| = k$. Let $e \in E \backslash E'$. Let H be one of $G \backslash e$ and G/e. By Lemma 3.1.2, H is a 2-connected graph. Then $|E(H) \backslash E'| = k-1$. By induction, H contains either C_n or θ_n^0 satisfying all edges in E' as a minor of G. By Lemma 1.2.1, G must contain either C_n or θ_n^0 as a minor in which all edges of the minor are in E'. □

Proof of Theorem 3.1.1. Let $a = f_{2.1.4}(n) * f_{2.1.4}(f_{2.1.4}(n))$, and let $f_{3.1.1}(n) = f_{3.1.4}(a)$. We want to prove that $f_{3.1.1}(n)$ satisfies the requirement of the theorem. Let $G = (V, E)$ be a 2-connected loopless graph and every $E' \subseteq E$ such that $|E'| \geq f_{3.1.1}(n)$.

By Lemma 3.1.4, G has a cycle C_a or a trivial theta θ_a^0 as a minor in which all edges of the minor are in E', where $a \geq n$.

Suppose G has C_a as a minor in which there are at least a edges from E'. In this case, there exists a cycle C_n which has n edges from E' is a minor of C_a, since $a \geq n$. By Lemma 1.2.1, G has a minor C_n which has at least n edges from E'. By Lemma 1.2.4 (2), G has such cycle as a topological minor. By Lemma 1.2.3, there exists a cycle H with at least n edges from E' as a subgraph of G. We are done with (1).

Suppose G has a trivial theta θ_a^0 as a minor in which there are at least a edges from E', take $V(\theta_a^0) = \{u, v\}$. By Lemma 1.2.2, G has a set $\{G_v, G_u\}$ of disjoint connected subgraphs and a set $F = \{f_e : e \in E(\theta_a^0)\}$ of edges such that F is disjoint from $E(G_v)$ and $E(G_u)$, and for every edge $e = uv \in E(H)$, the two ends of f_e are in G_v and G_u separately. There are at least a edges between G_v and G_u. There are two cases on the number of incident such edges of vertices in G_u.

Case 1. There exists one vertex v in G_u such that v is incident with at least $f_{2.1.4}(n)$ edges in F.

Let the set of ends of such $f_{2.1.4}(n)$ edges in G_v be R. By Theorem 2.1.4, (G_v, R) contains a subgraph H such that H is a path containing n roots, or a nicely confined comb of length n, or a subdivision of a confined star of size n. If H is a path containing n roots, then the union of H and a set of edges in F such that its one end is v and its other end is a root in H. We are done with (3). If H is a nicely confined comb of length n, then the union of H and a set of edges in F such that

its one end is v and its other end is a root in H. We are done with (3). If H is a subdivision of a confined star of size n, then the union of H and a set of edges in F such that its one end is v and its other end is a root in H. We are done with (2).

Case 2. There is no vertex in G_u such that it is incident with at least $f_{2.1.4}(n)$ edges in F.

That means that there are at least $f_{2.1.4}(f_{2.1.4}(n))$ vertices in G_u such that every vertex is incident with one such edge from F. Let the set of these vertices be U and then the set of edges from F and incident with vertices in U be F_1. Let us consider a rooted connected graph $(G_v \cup F_1, U)$. By Theorem 2.1.4, $(G_v \cup F_1, U)$ contains a subgraph H' such that H' is either a nicely confined comb of length $f_{2.1.4}(n)$, or a subdivision of a confined star of size $f_{2.1.4}(n)$.

If H' is a nicely confined comb of $f_{2.1.4}(n)$ roots, let the set of those roots be R_1. Then we consider a rooted connected graph (G_u, R_1). By Theorem 2.1.4, (G_u, R_1) contains a subgraph H'' such that H'' is a path containing n roots, or a nicely confined comb of length n, or a subdivision of a confined star of size n. If H'' is a path containing n roots, then the union of $H'' \cup H'$ and edges in F such that each edge with one end in H'' and the other is a root in H''. We are done with (4). If H'' is a nicely confined comb of length n, then the union of $H'' \cup H'$ and edges in F such that each edge with one end in H'' and the other is a root in H''. We are done with (4). If H'' is a subdivision of a confined star of size n, then the union of $H'' \cup H'$ and edges in F such that each edge with one end in H'' and the other is a root in H''. We are done with (3).

If H' is a subdivision of a confined star of size $f_{2.1.4}(n)$, let the set of those roots be R_2. Then we consider a rooted connected graph (G_u, R_2). By Theorem 2.1.4, (G_u, R_2) contains a subgraph H''' such that H''' is a path containing n roots, or

a nicely confined comb of length n, or a subdivision of a confined star of size n. Similarly, if H''' is a path containing n roots, then we are done with (3); if H''' is a nicely confined comb of length n, then we are done with (3); if H''' is a subdivision of a confined star of size n, then we are done with (2). $\square$

Let $G = (V, E)$ be a graph and $X \subseteq V$. We call a set $E' \subseteq E$ an *edge cover* of X if every vertex of X is incident with at least one edge in E'.

The following theorem is the vertex version of determining unavoidable large subgraphs of every sufficiently large rooted 2-connected graph.

Theorem 3.1.5. *There exists a function $f_{3.1.5}(n)$ with the following property. For every 2-connected graph $G = (V, E)$ and every $X \subseteq V$, if $|X| \geq f_{3.1.5}(n)$, then G contains a subgraph H such that*

(1) H is a cycle containing at least n vertices from X, or

(2) H is a theta of size n in which the interior of each branch contains at least one vertex from X, or

(3) H is a fan of length n in which the interior of each rib contains at least one vertex from X while the rim contains no vertex from X, or

(4) H is a ladder of length n in which each rung contains at least one vertex from X while the rails contain no vertex from X.

Proof. Let $f_{3.1.5}(n) = 2f_{3.1.1}(2n)$. We prove that $f_{3.1.5}(n)$ satisfies the requirement of the theorem. Let $E' \subseteq E$ be the minimal edge cover of X. By the definition of an edge cover, $|E'| \geq |X|/2$. By Theorem 3.1.1, every 2-connected graph $G = (V, E)$ satisfying $E' \subseteq E$ and $|E'| \geq f_{3.1.1}(2n)$ must contain one of subgraphs described in Theorem 3.1.1.

Suppose a cycle C containing at least $2n$ edges from E' as a subgraph of G can cover at least $2n > n$ vertices in X. Let $H = C$, we are done with (1).

Suppose a θ_{2n} in which each branch contains at least one edge from E'. Let independent paths $P^1, P^2, \cdots, P^{2n}$ be branches of θ_{2n}. There is at least one edge from E' in each P^i, for $1 \leq i \leq 2n$. Then there exists a θ_{n+2} in which has at least one vertex from X as the interior of each branch. Otherwise, it contradicts the assumption that the minimality of E' covering X. Let $H = \theta_n$, we are done with (2).

Suppose a F_{2n} in which each rib contains at least one edge from E'. Let independent paths $P_{x_0x_1}, P_{x_0x_2}, \cdots, P_{x_0x_{2n}}$ be ribs of F_{2n}, where $n \geq 2$. Let $P_{x_1x_n} = P_{x_1x_2} \cup P_{x_2x_3} \cup \cdots \cup P_{x_{2n-1}x_{2n}}$ be the rim of F_{2n}. By the minimality of E', there exists at most one blade such that one edge from E' covering x_0. There exist at most $n-1$ ribs of F_{2n} such that the edge from E' just covers x_i. Otherwise, we can find a cycle which is the union of $P_{x_0x_1}, P_{x_1x_{2n}}, P_{x_0x_{2n}}$ and it contains at least n vertices from X. Let $H = F_n$, we are done with (3).

Suppose a L'_{2n} in which each rung contains at least one edge from E' while the rail of L'_{2n} contains no edge from E'. Let two disjoint paths $P_{x_1x_{2n}}$, $P_{x'_1x'_{2n}}$ be rails of L'_{2n}. Let $2n$ mutually disjoint paths $P_{x_1x'_1}, \cdots, P_{x_{2n}x'_{2n}}$ be rungs of L'_{2n}. By Theorem 3.1.1, each $P_{x_tx'_t}$ has at least one edge from E', for $1 \leq t \leq 2n$. Then there are at most n rungs containing one edge from E' to cover x_i or x'_i, without loss of generality, $P_{x_1x'_1}, P_{x_2x'_2}, \cdots, P_{x_nx'_n}$. Otherwise, we can find a cycle which is the union of $P_{x_1x_{2n}}, P_{x'_1x'_{2n}}, P_{x_1x'_1}, P_{x_{2n}x'_{2n}}$ such that there are at least n vertices in X. Let P^1 be a subgraph of $P_{x_1x_{2n}}$ with ends x_1, x_n, and let P^2 be a subgraph of $P_{x'_1x'_{2n}}$ with ends x'_1, x'_n. Let $H = P^1 \cup P^2 \cup P_{x_1x'_1} \cup P_{x_2x'_2} \cup \cdots \cup P_{x_nx'_n}$, we are done with (4). $\square$

3.2 Large generalized rooted graphs

In the last section, we determine unavoidable subgraphs of a sufficiently large rooted 2-connected graph. In this section, we extend Theorem 3.1.5 to any sufficiently large generalized rooted 2-connected graph.

Let $(G, \mathcal{X})$ be a generalized rooted connected graph. Suppose vertices of G can be enumerated as $v_1, v_2, \cdots, v_k$ such that (see Figure 3.4) v_1, v_k are outside every $X \in \mathcal{X}$, every edge of G is between two consecutive vertices, and every $X \in \mathcal{X}$ consists of either only one vertex or two consecutive vertices. Then we call $(G, \mathcal{X})$ a *rooted path* between v_1, v_k. We also call $|\mathcal{X}|$ the length of this path. Note that a rooted path of length zero is an ordinary path.

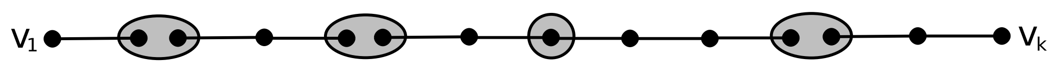

FIGURE 3.4. A rooted path

In the following we describe the unavoidable graphs (see Figures 3.5 through 3.15), (in figures below, each line represent a rooted path of length ≥ 0).

(1) $(G, \mathcal{X})$ is a *rooted cycle* if it is connected and its vertices can be cyclically ordered such that every edge is between two consecutive vertices and every $X \in \mathcal{X}$ consists of either a single vertex or two consecutive vertices. Again, the *length* of this cycle is $|\mathcal{X}|$.

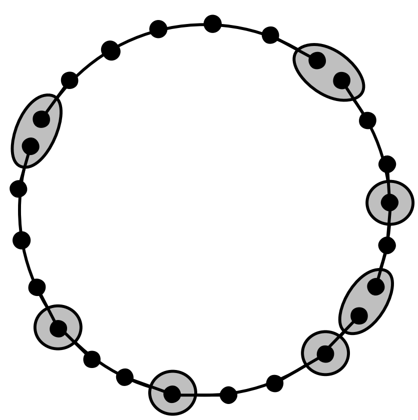

FIGURE 3.5. A rooted cycle

(2) $(G, \mathcal{X})$ is a *rooted theta graph of size* n if it is obtained from θ_n by replaying each branch with a rooted path of length ≥ 1.

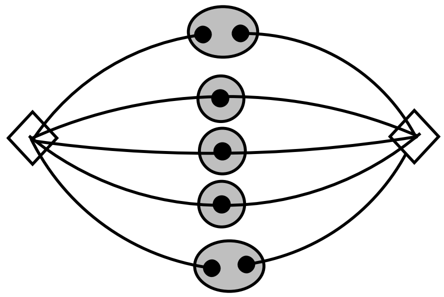

FIGURE 3.6. A rooted theta

(3) $(G, \mathcal{X})$ is a *type-II rooted theta of size* n if there exists $X_0 = \{x_0, x_0'\} \in \mathcal{X}$ such that $(G, \mathcal{X})$ is obtained from X_0 by adding rooted paths of length ≥ 1 between x_0, x_0', where the paths are disjoint except for x_0, x_0'.

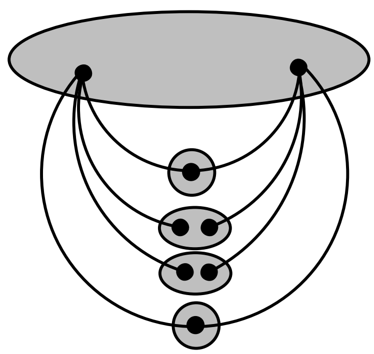

FIGURE 3.7. A type-II rooted theta

(4) $(G, \mathcal{X})$ is a *rooted fan of length* n if it is obtained from F_n by replacing each rib with a rooted paths of length ≥ 1.

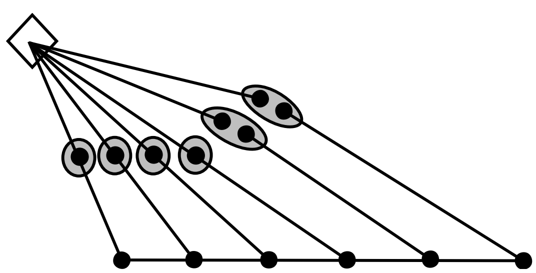

FIGURE 3.8. A rooted fan

(5) $(G, \mathcal{X})$ is a *type-II rooted fan of length* n if there exist $X_0 = \{x_1, \cdots, x_n\} \in \mathcal{X}$ ad $x_0 \in V(G) \backslash X_0$ such that $(G, \mathcal{X})$ is obtained from x_0 and X_0 by adding rooted paths of length ≥ 1 between x_0 and x_i for $i = 1, \cdots, n$, where the paths are disjoint except for x_0.

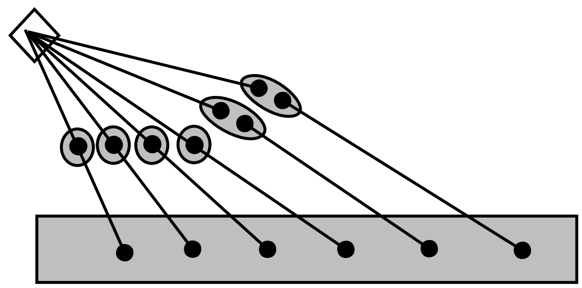

FIGURE 3.9. A type-II rooted fan

(6) $(G, \mathcal{X})$ is a *type-III rooted fan of length* n if there exist $X_0 = \{x_0, x_1, \cdots, x_n\} \in \mathcal{X}$ such that $(G, \mathcal{X})$ is obtained from X_0 by adding rooted paths of length ≥ 1 between x_0 and x_i $(i = 1, \cdots, n)$, where the paths are disjoint except for x_0.

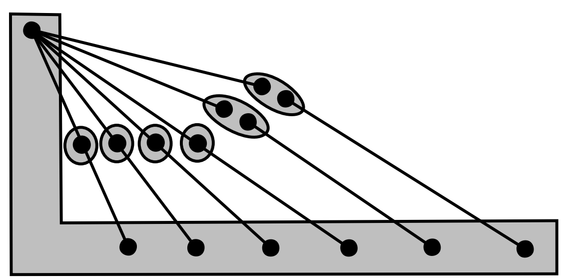

FIGURE 3.10. A type-III rooted fan

(7) $(G, \mathcal{X})$ is a *rooted ladder of length* n if it is obtained from L_n by replacing each rung with a rooted paths of length ≥ 1.

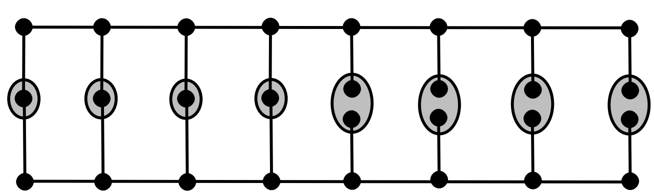

FIGURE 3.11. A rooted ladder

(8) $(G, \mathcal{X})$ is a *type-II rooted ladder of length* n if it is obtained from $X = \{x_1, \cdots, x_n\} \in \mathcal{X}$ and a path P as follows. Let $\{y_1, \cdots, y_n\} \subseteq V(P)$, which contains both ends of P. Then $(G, \mathcal{X})$ is obtained from X and P by adding disjoint rooted paths of length ≥ 1 between x_i and y_i $(i = 1, \cdots, n)$.

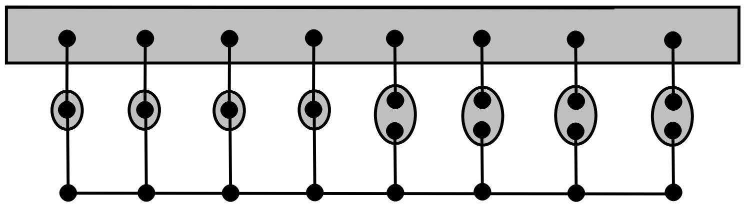

FIGURE 3.12. A type-II rooted ladder

(9) $(G, \mathcal{X})$ is a *type-III rooted ladder of length* n if it is obtained from $\{x_{i,1}, \cdots, x_{i,n}\} \in \mathcal{X}$ $(i = 1, 2)$ by adding disjoint rooted paths of length ≥ 1 between $x_{1,j}$ and $x_{2,j}$ $(j = 1, \cdots, n)$.

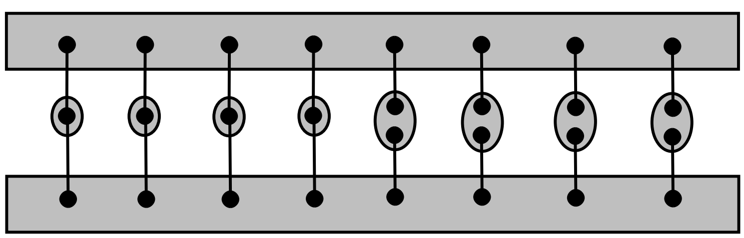

FIGURE 3.13. A type-III rooted ladder

(10) $(G, \mathcal{X})$ is a *type-IV rooted ladder of length* n if it is obtained from $X_1, \cdots, X_n \in \mathcal{X}$ as follows. Let each X_i be the union of two subsets $\{x_{i,1}, x_{i,2}\}$ and $\{y_{i,1}, y_{i,2}\}$, where $x_{i,1} \neq x_{i,2}$ and $y_{i,1} \neq y_{i,2}$, but $\{x_{i,1}, x_{i,2}\} \cap \{y_{i,1}, y_{i,2}\}$ could be nonempty. Then $(G, \mathcal{X})$ is obtained from $X_1, \cdots, X_n$ by adding rooted paths of length ≥ 0 between $x_{i,j}$ and $y_{i+1,j}$ $(i = 1, \cdots, n-1$ and $j = 1, 2)$, where the paths are disjoint except for their ends.

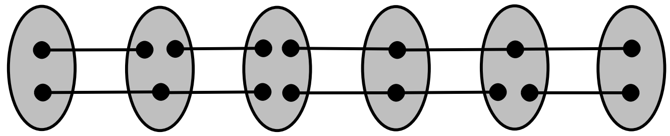

FIGURE 3.14. A type-IV rooted ladder

(11) $(G, \mathcal{X})$ is a *rooted flower of size* n if there exists $X_0 = \{x_1, \cdots, x_{2n}\} \in \mathcal{X}$ such that $(G, \mathcal{X})$ is obtained from X_0 by adding disjoint rooted paths of length ≥ 1 between x_{2i-1} and x_{2i} $(i = 1, \cdots, n)$.

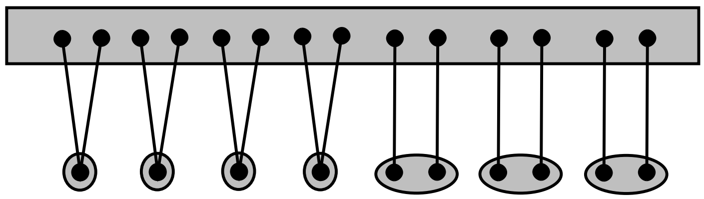

FIGURE 3.15. A rooted flower

The following theorem is determining unavoidable large subgraphs of any sufficiently large generalized rooted 2-connected graph.

Theorem 3.2.1. *There exists a function $f_{3.2.1}(n)$ such that every generalized rooted 2-connected graph $(G, \mathcal{X})$ with $|\mathcal{X}| \geq f_{3.2.1}(n)$ must contain one of generalized rooted graphs (1)-(11) as a subgraph.*

We first prove the following lemma which determines unavoidable subgraphs in every sufficiently large loopless graph before proving Theorem 3.2.1.

Lemma 3.2.2. *There exists a function $f_{3.2.2}(n)$ such that every loopless graph $G = (V, E)$ with $|E| \geq f_{3.2.2}(n)$ must contain n edges parallel to each other, or n edges that are mutually disjoint, or n edges sharing the unique vertex.*

Proof. Let $f_{3.2.2}(n) = 2(n-1)^3 + 1$. We prove that $f_{3.2.2}(n)$ satisfies the requirement of the lemma. If there are n edges parallel to each other in G, we are done. If not, then it means any pair of vertices in V has at most $n - 1$ parallel edges.

Since we can delete all parallel edges to obtain a simple graph $H = (V, E')$ where $|E'| \geq |E|/(n-1)$. If there exists a vertex with degree at least n in H, we are done. If not, then it means any vertex in V has at most $n-1$ degree. If there are n mutually disjoint edges in H, we are done. If not, then there are at most $n-1$ mutually disjoint edges in H, which means there are at most $2(n-1)^2$ edges, which contradicts the assumption that $|E| \geq 2(n-1)^3 + 1$. $\square$

Proof of Theorem 3.2.1. Let $f_{3.2.1}(n) = g(n)h(n)$, where $g(n) = f_{3.1.5}(f_{3.2.2}(n))$ and $h(n) = 1 + f_{3.2.2}(n) + f_{3.2.2}(n)(f_{3.2.2}(n)-1) + \cdots + f_{3.2.2}(n)(f_{3.2.2}(n)-1)^{2n}$. We prove that $f_{3.2.1}(n)$ satisfies the requirement of the theorem. Let $\mathcal{X} = \{X_1, X_2, \cdots, X_t\}$, for $t \geq f_{3.2.1}(n)$.

Let (H, Y) be a rooted graph which is obtained by shrinking $(G, \mathcal{X})$. Let $Y = \{y_1, \cdots, y_t\}$. By the definition of generalized rooted 2-connected graph and shrinking, since $(G, \mathcal{X})$ is 2-connected, then (H, Y) is connected. By Lemma 1.1.1, H has a block tree T_H. There are two cases on the number of roots in each block.

Case 1. Every block has no more than $g(n)$ roots in (H, Y).

For this case, since $|\mathcal{X}| \geq g(n)h(n)$, then $|T_H| \geq h(n)$. By Lemma 1.5.1, there exists either $\triangle(T_H) > f_{3.2.2}(n)$ or T_H has an induced path of length $2n+1$ starting from any vertex. There are two subcases.

Subcase 1.1. There exists a long sequence of blocks $B_1, B_2, \cdots, B_{n+1}$ of (H, Y), $y_i = V(B_i) \cap V(B_{i+1})$ for $1 \leq i \leq n$. Let $(G_i, \mathcal{X}_i)$ be an induced subgraph of $(G, \mathcal{X})$ such that (B_i, Y_i) is the result of shrinking $(G_i, \mathcal{X}_i)$. To be more precise, $(G_i, \mathcal{X}_i)$ is obtained from $(G, \mathcal{X})$ by deleting vertices of G that belong to $H \backslash B_i$ and deleting X_j for which y_j is not a vertex of B_i. Let G_i' be obtained from G_i by identifying vertices from each X_j, where $X_j \in \mathcal{X}_i$ and $j \neq i, i-1$. Since $(G, \mathcal{X})$ is 2-connected and B_i is a block of H, G_i' has no separation of order < 2 that separates

X_i from X_{i-1}. So G_i' has two disjoint paths between X_i and X_{i-1}. Since there are two disjoint paths between X_1 and X_n which is the union of two disjoint paths in every G_i' and X_i satisfies the description in (10). Hence we can find type-IV rooted ladder of length n as a subgraph of $(G, \mathcal{X})$.

Subcase 1.2. $\triangle(T_H) > f_{3.2.2}(n)$ and the vertex with the maximal degree in T_H is a cut vertex y_0

Then the corresponding of y_0 is $X_0 \in \mathcal{X}$ in $(G, \mathcal{X})$. Let neighbors of y_0 be $b_1, b_2, \cdots, b_{f_{3.2.2}(n)}$ in T_H. Their corresponding blocks are $B_1, \cdots, B_{f_{3.2.2}(n)}$ in H. We assume that every block B_i meeting y_0 also contains another root y_i. Then such a block contains a rooted path of length ≥ 1 with both ends in X_0. We create a new graph $D = (V, E)$ where $V = X_0$ and $E = \{e_1, e_2, \cdots, e_{f_{3.2.2}(n)}\}$ such that e_i in B_i for $1 \leq i \leq f_{3.2.2}(n)$. Hence $|E| \geq f_{3.2.2}(n)$. By Lemma 3.2.2, we can obtain either a type-III rooted fan of length n, or type-II rooted theta of size n, or a rooted flower of size n as a subgraph of $(G, \mathcal{X})$. Then (3), (6), or (11) holds.

If the vertex with the maximal degree in T_H is a block vertex b_0, then there exists at least $f_{3.2.2}(n)$ roots in B_0, which is similar to Case 2.

Case 2. There exists a block B_0 with at least $g(n)$ roots in (H, Y). By Theorem 3.1.5, we have four subcases.

Subcase 2.1. Suppose there exists a cycle containing at least $f_{3.2.2}(n) > n$ vertices from Y in B_0. Then it is obvious to obtain a rooted cycle of length n in $(G, \mathcal{X})$, which means that (1) holds.

Subcase 2.2. Suppose there exists a theta of size $f_{3.2.2}(n)$ in which the interior of each branch contains at least one vertex from Y in B_0. Let ends of such theta be y_0, y_0'. If y_0, y_0' are not roots, then we can obtain a rooted theta graph of size n as a subgraph of $(G, \mathcal{X})$. If at least one of y_0, y_0' is a root, then we consider a graph $D_2 = (V_2, E_2)$, where $V_2 = Y_0 \cup Y_0'$ and E_2 is a set such that every element replaces

each branch in B_0. Hence $|E_2| \geq f_{3.2.2}(n)$. By Lemma 3.2.2, we can obtain either a rooted theta of size n, or a type-II rooted fan of length n, or type-III rooted ladder of length n. Then (2), (5), or (9) holds.

Subcase 2.3. Suppose there exists a fan of length $f_{3.2.2}(n) > n^2$ in which the interior of each rib contains at least one vertex from Y while the rim contains no vertex from Y in B_0. If the center vertex of such fan is not a root, then we can obtain a rooted fan of length n as a subgraph of $(G, \mathcal{X})$. If the center vertex of such fan is a root y_0, its corresponding root X_0, then we can obtain either a rooted fan of length n or a type-II rooted ladder of length n. Then (4), or (8) holds.

Subcase 2.4. Suppose there exists a ladder of length $f_{3.2.2}(n)$ in which each rung contains at least one vertex from Y while the rails contain no vertex from Y. Because the definition of shrinking a generalized rooted graph, it is obvious to obtain a rooted ladder of length n as a subgraph of $(G, \mathcal{X})$. Then (7) holds. $\square$

4 Excluding a Large Comb

In the first three chapters, we have seen several results involving large combs. In particular, Lemma 1.5.3 says that a large comb is one of the unavoidable subgraphs in a tree with many leaves. Also, Theorem 2.1.1 says that a large nicely confined comb is one of the unavoidable induced subgraphs in a sufficiently large rooted graph. Besides, Theorem 2.1.4 says that a large nicely confined comb is one of the unavoidable subgraphs in a sufficiently large rooted graph. In this chapter, we characterize large rooted graphs that do not contain a large (nicely) confined comb. We remark that, in this chapter, when we say that a rooted graph contains a (nicely) confined comb we mean that a rooted graph contains a (nicely) confined comb as a subgraph.

4.1 Excluding a nicely confined comb

For any series of mutually disjoint graphs $G_0, G_1, \cdots, G_t$, where $t \geq 0$, let $\oplus_k(G_0; G_1, \cdots, G_t)$ be a graph obtained by k_i-summing G_i $(i = 1, 2, \cdots, t)$ to G_0, where $k_i \leq k$. The following is a characterization of rooted graphs that do not contain a large nicely confined comb.

Theorem 4.1.1. *Let $\mathcal{G}$ be a class of rooted graphs. Then the following are equivalent.*

(1) there exists an integer $n \geq 3$ such that no member of $\mathcal{G}$ contains a nicely confined comb of length n;

(2) there exist $p, s \in \mathbb{N}$ such that every $(G, R) \in \mathcal{G}$ can be expressed as $\oplus_s(G_0; G_1, G_2, \cdots, G_t)$, for $t \geq 0$, where G_0 contains R, $G_0 \backslash R$ is connected, and $G_0 \backslash R$ has no path of length p.

To simplify the proof of Theorem 4.1.1, we first need to prove the following two lemmas. The first lemma proves the base case of the theorem.

Lemma 4.1.2. *A rooted graph (G, R) does not contain a nicely confined comb of length three if and only if $G = \oplus_2(G_0; G_1, \cdots, G_t)$, where $V(G_0) = R$ and $t \geq 0$.*

Proof. The backward implication easily follows from the definition of a nicely confined comb and $\oplus_2$. Next we consider the forward implication.

Let $H = G[R]$. We need to show that every H-bridge has at most two feet. Otherwise, there exists a H-bridge B with at least three feet. By the definition of H-bridges, there exists a connected component C of $G \backslash H$ such that $E(B)$ consists of all edges incident with at least one vertex of C. Since B has at least three feet, so $|B \cap H| \geq 3$. Let $V(B \cap H) = \{l_1, l_2, \cdots, l_i\}$. Then $i \geq 3$. Since C is connected, and B is the union of C and all edges with exactly one end in C, so B is connected and $d_B(l_j) = 1$ for $1 \leq j \leq i$. Since B is connected, so there exists a path P^1 between l_1 and l_2 in B, and then there exists a path P^2 between $P^1 \backslash l_1 \backslash l_2$ and l_3 in B. Then $P^1 \cup P^2$ is a nicely confined comb of three, since $(P^1 \cup P^2) \backslash l_1 \backslash l_2 \backslash l_3$ is a shaft of $P^1 \cup P^2$, and l_1, l_2, l_3 are leaves of $P^1 \cup P^2$. It contradicts the assumption that (G, R) does not contain a nicely confined comb of length three. So it means that every H-bridge has at most two feet. Let $G_0 = H$, and let H-bridges of $G \backslash H$ be $G_1, G_2, \cdots, G_t$. By the definition of $\oplus_2$, $G = \oplus_2(G_0; G_1, \cdots, G_t)$, where $V(G_0) = R$ and $t \geq 0$. □

In the next lemma, we show that any two maximum nicely confined combs must meet in a rooted graph (G, R) for which $G \backslash R$ is connected.

Lemma 4.1.3. *Let (G, R) be a rooted graph such that $G \backslash R$ is connected. If (G_1, R_1), (G_2, R_2) are nicely confined combs of length n and there is no nicely confined comb of length $n + 1$, where $n \geq 3$, then $V(G_1 \cap G_2) \neq \emptyset$.*

Proof. Suppose $V(G_1 \cap G_2) = \emptyset$. Since $G \backslash R$ is connected, there exists a path P between $G_1 \backslash R_1$ and $G_2 \backslash R_2$. Let $P^{i,1}, P^{i,2}, \cdots, P^{i,n}$ be all the teeth of G_i, listed in the order that they appear in G_i, let the shaft of G_i be S_i, and $P_{i,m} \cap S_i = x_{i,m}$ for $i = 1, 2$ and $2 \leq m \leq n-1$. We denote $S_i[x_{i,l}, x_{i,m}]$ as a path of S_i with ends $x_{i,l}, x_{i,m}$ for $2 \leq l \leq m \leq n-1$. Let ends of P be p_1, p_2. We define $G_{i,1}, G_{i,2}$ which are the subgraphs of G_i and satisfy the following conditions:

(i) $G_{i,1} \cup G_{i,2} = G_i$;

(ii) If $p_i \in P^{i,m}$, for some m and $2 \leq m \leq n-1$. We define $G_{i,1}$ as the union of $P^{i,1}, P^{i,2}, \cdots, P^{i,m}, S_i[x_{i,2}, x_{i,m+1}]$ and $G_{i,2}$ as the union of $P^{i,m}, P^{i,m+1}, \cdots,$ $P^{i,n}, S_i[x_{i,m-1}, x_{i,n-1}]$, or

if $p_i \in P^{i,1}$ or $P^{i,n}$. We define $G_{i,1}$ as G_i and $G_{i,2}$ as G_i, or

if $p_i \in S_i[x_{i,l}, x_{i,l+1}]$ for $2 \leq l \leq n-2$. We define $G_{i,1}$ as the union of $P^{i,1}, \cdots, P^{i,l+1}, S_i[x_{i,1}, x_{i,l+1}]$ and $G_{i,2}$ as the union of $P^{i,l-1}, \cdots, P^{i,n}$, $S_i[x_{i,l}, x_{i,n-1}]$.

By the construction of $G_{i,1}, G_{i,2}$, the length of $G_{i,1} \cup G_{i,2}$ are larger than the length of G_i. Without loss of generality, we set $G_{i,1}$ as a comb with larger length between $G_{i,1}$ and $G_{i,2}$. Then $(G_{1,1} \cup G_{2,1} \cup P, V(G_{1,1} \cup G_{2,1}) \cap R_1 \cap R_2)$ is a nicely confined comb, whose shaft is $S_1[x_{1,1}, x_{1,p_1}] \cup S_2[x_{2,1}, x_{2,p_2}] \cup P$ and the set of teeth is $\{P^{1,1}, \cdots, P^{1,l_1}, P^{1,l_1+1} \cup S_1[x_{1,p_1}, x_{1,l_1+1}], P^{2,1}, \cdots, P^{2,l_2}, P^{2,l_2+1} \cup S_2[x_{2,p_2}, x_{2,l_2+1}]\}$ for $\lfloor \frac{n}{2} \rfloor \leq l_1, l_2 \leq n-1$. Therefore the length of it is larger than n, which contradicts the assumption that there is no a nicely confined comb of length $n+1$ in (G, R). $\square$

We provide a counterexample of the above lemma to show that it is false for $n < 3$. The counterexample of the above lemma for $n = 2$, let G be a path and G_1, G_2 be two subgraphs of G such that G_1, G_2 are both nicely confined comb of

length two, and $V(G_1 \cap G_2) = \emptyset$. There is no a nicely confined comb of length three in G, however, $V(G_1 \cap G_2) = \emptyset$.

Proof of Theorem 4.1.1. From (2) to (1). Let $n = (p+2)(p+1+s)+1$.

Suppose (G, R) contains a nicely confined comb of length n. Let P be the shaft of such comb, then P is contained in $G \backslash R$. Path P is divided into subgraphs by $V(P \cap G_0)$. There are two cases on such subgraphs. Case 1. Unique one end of each such subgraph is in $V(P \cap G_0)$. Case 2. Unique both ends of each such subgraph are in $V(P \cap G_0)$. It is easy to see there are at most two such subgraphs in Case 1. Each of these subpaths is contained in some G_i for $1 \leq i \leq t$. Firstly, we need to prove that $|P \cap G_0| \leq p+1$, which means the number of subpaths is at most $p+2$. Otherwise there exists a path P' with length at least $p+1$ in $G_0 \backslash R$, since every such subpath of P can be replaced by an edge in $G_0 \backslash R$ such that they share the common ends, which contradicts the assumption that $G \backslash R$ has no path of length p. Hence the definition of n implies that one of these paths (call it Q) contains at least $p+1+s$ cubic vertices of such comb.Then there exists some G_i for $i \geq 0$ contains Q. By the above proof, we know that $|P \cap G_0| \leq p+1$, thus the case $Q \subseteq G_0$ can be eliminated. Let X_i be the common clique of G_i and G_0. Since $|V(X_i) \cap R| \leq s$, which means there is a separation (H_1, H_2) separating Q to R such that Q is a subgraph of H_1 and $R \subseteq V(H_2)$, where $|H_1 \cap H_2| \leq s$. By Menger's Theorem, there are at most s mutually disjoint paths between Q and R, which implies that there are at most s cubic vertices in Q. It contradicts the assumption that Q has at least $p+1+s$ cubic vertices.

From (1) to (2), let $p = g(n) = (n-3)^3/3 + (n-3)^2/2 + (n-3)/6 + 1$, $s = f(n) = (n-3)^3/3 + (n-3)^2/2 + (n-3)/6 + 2$. For any (G, R) does not contain a nicely confined comb of length n, we prove that G admits an expression

as described in (2). Without loss of generality, assume that G is connected. We proceed by induction on n.

For $n = 3$, we can obtain that $f(3) = 2, g(3) = 1$. By Lemma 4.1.2, it means the assertion is true. Now let $n > 3$, and assume the assertion holds for rooted graphs that do not contain a nicely confined comb of length less than n.

Let H be a nicely confined comb of maximum length in (G, R). If the length of H is less than $n-1$, then by induction (G, R) can be expressed as $\oplus_{s'}(G_0; G_1, \cdots, G_t)$, where $s' = f(n-1)$, $R \subseteq V(G_0)$ and $G_0 \backslash R$ has no path of length p', where $p' = g(n-1)$. Since $f(n-1) < f(n)$ and $g(n-1) < g(n)$ for $n > 3$, so $s' < s$ and $p' < p$. By the definition of $\oplus_s$, it is obviously to obtain $\oplus_{s'}(G_0; G_1, \cdots, G_t) = \oplus_s(G_0; G_1, \cdots, G_t)$. Since $G_0 \backslash R$ has no path of length p', so $G_0 \backslash R$ has no path of length p, since $p' < p$. That is to say, the assertion holds for the length of H with less than $n-1$. So we assume that the length of H is $n-1$ in (G, R). Let $H \cap R = R_0$.

Let $R_1 = R \backslash R_0$. If G has at least $(n-1)^2 + 1$ disjoint paths between R_1 and H, then there exists at least one tooth T of H, which has at least $n-1$ disjoint paths between T and R_1. The union of such $n-1$ disjoint paths, the shaft of H and one another tooth in H is a nicely confined comb of length n, which contradicts the assumption that (G, R) does not contain a nicely confined comb of length n. By Menger's Theorem, G has a separation (J_1, J_2) of order no more than $(n-1)^2$ such that $R_1 \subseteq V(J_2)$ and H is a subgraph of J_1. Suppose $(J_2 \backslash J_1, R_1)$ contains a nicely confined comb H' of length $n-1$, which means H' is a subgraph of $(J_2 \backslash J_1, R_1)$. Since H is a subgraph of $(J_1, R \cap V(J_1))$, so $V(H \cap H') = \emptyset$, which contracts that any two maximum nicely confined combs must meet, by Lemma 4.1.3. So $(J_2 \backslash J_1, R_1)$ does not contain a nicely confined comb of length $n-1$. By the induction hypothesis, therefore $(J_2 \backslash J_1, R_1)$ can be expressed as $\oplus_{s'}(G'_0; G'_1, \cdots, G'_{t'})$, where $s' = f(n-1)$,

$R_1 \subseteq V(G'_0)$ and $G'_0 \backslash R_1$ has no path of length p', where $p' = g(n-1)$. Let S'_i be the common clique of G'_i and G'_0, for $1 \leq i \leq t'$.

We add missing edges between vertices in $V(J_1 \cap J_2) \cup (V(H) \cap R)$ to obtain a clique M_0 with $V(M_0) = V(J_1 \cap J_2) \cup (V(H) \cap R)$. We add edges between vertices of $V(J_1 \cap J_2)$ and vertices of $V(S'_i)$ to obtain a clique M_i with $V(M_i) = V(J_1 \cap J_2) \cup V(S'_i)$.

We create G_0 as $G[V(G'_0 \cup (J_1 \cap J_2) \cup (V(H) \cap R))] \cup M_0 \cup M_1 \cup \cdots \cup M_{t'}$. We create $G_i = G[V(G'_i \cup (J_1 \cap J_2))]$ for $1 \leq i \leq t'$, and create $G_{t'+1} = G[V(J_1)]$. Let $s = \max\{s' + |J_1 \cap J_2|, |J_1 \cap J_2| + n - 1\}$ and $p = 2^{n^2+1}p'$. Since $R_1 \subseteq V(G'_0) \subseteq V(G_0)$ and $V(H) \cap R \subseteq V(M_0) \subseteq V(G_0)$, so $R = R_1 \cup (V(H_0) \cap R) \subseteq V(G_0)$. So $G[R \cup V(J_2)] = \oplus_s(G_0; G_1, \cdots, G_{t'})$. Therefore G can be expressed as $\oplus_s(G_0; G_1, \cdots, G_{t'}, G_{t'+1})$.

Let $p = 2^{n^2+1}p'$. We show by induction on $|J_1 \cap J_2|$ that $G_0 \backslash R$ has no path of length p. If $|J_1 \cap J_2| = 1$, then $p = 4p' \geq 2p' + 2$. Suppose $|J_1 \cap J_2| \geq 1$, and assume that $G_0 \backslash R$ has no path of length $2^{(n-1)^2+1}p'$ if $|J_1 \cap J_2| = n^2 - 1$. Suppose $|J_1 \cap J_2| = n^2$, we show that $G_0 \backslash R$ has no path of length $2^{n^2+1}p'$. By the structure of G_0, $G_0 \backslash R$ has no path of length $2^{n^2+1}p' > 2(2^{(n-1)^2+1}p') + 2$. That is to say $p = 2^{n^2+1}p'$, which means that $g(n) = 2^{n^2+1}g(n-1)$. The recursive formula implies that $g(n) = (n-3)^3/3 + (n-3)^2/2 + (n-3)/6 + 1$.

By the definition of s, we obtain $f(n) = f(n-1) + n^2$. The recursive formula implies that $f(n) = (n-3)^3/3 + (n-3)^2/2 + (n-3)/6 + 2$. □

4.2 Excluding a confined comb

In the previous section, we prove Theorem 4.1.1, which solved the problem of a characterization on rooted connected graphs that do not contain a large nicely confined comb. In this section, our goal is to provide a characterization on rooted connected graphs that do not a large confined comb.

Firstly, we state with the definition of a construction of Φ.

Let $H_1, H_2, \cdots, H_k$ $(k \geq 1)$ be mutually disjoint rooted connected graphs, where $H_i = (G_i, R_i)$, for $1 \leq i \leq k$. We define $\Phi(H_1, H_2, \cdots, H_k)$ to be a rooted graph (G, R) constructed as below:

(i) G is obtained from the disjoint union of $G_1, G_2, \cdots, G_k$ by adding a new vertex v and linking v to at least one vertex from each of $G_1, G_2, \cdots, G_k$;

(ii) either $R = R_1 \cup R_2 \cup \cdots \cup R_k$, or $R = \{v\} \cup R_1 \cup R_2 \cup \cdots \cup R_k$.

Let $\mathcal{G}$ be a class of rooted graphs. Let $\Phi^0(\mathcal{G}) = \mathcal{G}$ and inductively, let $\Phi^t(\mathcal{G}) = \Phi^{t-1}(\mathcal{G}) \cup \{\Phi(H_1, \cdots, H_k) : H_1, \cdots, H_k \in \Phi^{t-1}(\mathcal{G}) \text{ and } k \geq 1\}$ for each $t \geq 1$. Let $\mathcal{P}$ be the class of rooted graphs (G, R) such that G is a path.

We remark that $\Phi^{t-1}(\mathcal{G}) \subseteq \Phi^t(\mathcal{G})$.

Theorem 4.2.1. *The following are equivalent for any class $\mathcal{G}$ of rooted connected graphs.*

(1) there exists an integer $n \geq 3$ such that no member of $\mathcal{G}$ contains a confined comb of length n;

(2) there exists $k \in \mathbb{N}$ such that $\mathcal{G} \subseteq \Phi^k(\mathcal{G}_0)$, where $\mathcal{G}_0$ consists of rooted connected graphs (G, R) such that $G{=}\oplus_2(G_0; G_1, \cdots, G_t)$, where $R \subseteq V(G_0)$ and $(G_0, R) \in \mathcal{P}$;

(3) there exists $m \in \mathbb{N}$ such that for every $(G, R) \in \mathcal{G}$, $G{=}\oplus(G_0; G_1, \cdots, G_t)$, for some $t \in \mathbb{N}$, where $R \subseteq V(G_0)$ and $(G_0, R) \in \Phi^m(\mathcal{P})$.

To simplify the proof of Theorem 4.2.1, we first need to prove the following three lemmas. The first lemma proves that two operations Φ and $\oplus$ commute.

Lemma 4.2.2. *For $i = 1, \cdots, k$, let $H_i = (G_i, R_i)$ and $G_i = \oplus(G_i^0; G_i^1, \cdots, G_i^{t_i})$, where $t_i \geq 0$ and $R_i \subseteq V(G_i^0)$. Suppose $(G, R) = \Phi(H_1, \cdots, H_k)$. Then there exist graphs $G^1, \cdots, G^m$ such that $G = \oplus(G^0; G^1, \cdots, G^m)$, where $(G^0, R) = \Phi((G_1^0, R_1), \cdots, (G_k^0, R_k))$.*

Proof. Let v be a new vertex in $\Phi(H_1, \cdots, H_k)$. Let (G_0, R) be one of $\Phi((G_1^0, R_1), \cdots, (G_k^0, R_k))$. Let G^l be a graph such that $V(G^l) = V(G_i^j \cup v)$ and $E(G^l) = E(G_i^j) \cup E_{s_1}^j \cup E_{s_2}^j$, for $1 \leq l \leq m$, $1 \leq s \leq k$, and $1 \leq j \leq t_i$. There are two cases on v. Case 1. If there exists at least one edge between v and $V(G_i^j)$ in G, then $E_{s_1}^j$ is an edge set of edges between v and all vertices in the common clique of G_0^j and G_i^j in G_0^j, and $E_{s_2}^j$ is an edge set of edges between v and $V(G_i^j)$. Case 2. Else, $E_{s_1}^j$ and $E_{s_2}^j$ are both empty edge sets. Next, we want to prove that $G = \oplus(G^0; G^1, \cdots, G^m)$, where $E(G^0 \cap G^l) = E(G[V(G_i^0 \cap G_i^j) \cup v])$.

Firstly, we want to prove $V(G) = V(\oplus(G^0; G^1, \cdots, G^m))$. Since $V(G) = V(\Phi(H_1, H_2, \cdots, H_k)) = V(G_1 \cup \cdots \cup G_k \cup v) = V(G_1^0 \cup G_1^1 \cup \cdots \cup G_1^{t_1} \cup G_2^0 \cup \cdots \cup G_k^{t_k} \cup v) = V(G_1^0 \cup G_2^0 \cup \cdots \cup G_k^0 \cup G^1 \cup G^2 \cup \cdots \cup G^m)) = V(\oplus(G^0; G^1, \cdots, G^m))$.

Next, we want to prove that $E(G) = E(\oplus(G^0; G^1, \cdots, G^m))$, since $E(\oplus(\Phi(G_1^0 \cup \cdots \cup G_k^0); G^1, \cdots, G^m)) = E(G_1^0 \cup G_2^0 \cup \cdots \cup G_k^0) \cup E(G_1^1 \cup G_1^2 \cup \cdots \cup G_1^{t_1} \cup G_2^1 \cup \cdots \cup G_2^{t_2} \cup \cdots \cup G_1^k \cup \cdots \cup G_k^{t_k}) \cup E_{1_1}^1 \cup E_{1_2}^1 \cup \cdots \cup E_{1_1}^{t_1} \cup E_{1_2}^{t_1} \cup E_{2_1}^1 \cup \cdots \cup E_{2_2}^{t_2} \cup \cdots \cup E_{k_1}^{t_k} \cup E_{k_2}^{t_k} = \bigcup_{j=1}^{t_i} E_{s_2}^j \cup E(G_1 \cup G_2 \cup \cdots \cup G_k)$. By the construction of Φ, $E(\oplus(\Phi(G_1^0 \cup \cdots \cup G_k^0); G^1, \cdots, G^m)) = E(\Phi(H_1, \cdots, H_k))$.

Finally, we want to prove that $\oplus(G^0; G^1, \cdots, G^m)$ and $\Phi(H_1, \cdots, H_k)$ have same degrees of each vertex. By the construction and $\oplus$, the assertion is true.

So G can be obtained from $\oplus(G^0; G^1, \cdots, G^m)$, where $G^0 = \Phi(G_1^0, \cdots, G_k^0)$. $\square$

The following lemma shows the base case of the above theorem is correct.

Lemma 4.2.3. *A rooted connected graph (G, R) does not contain a confined comb of length three if and only if $G{=}\oplus_2(G_0; G_1, \cdots, G_t)$, where G_0 is a path or a cycle and it contains R.*

Proof. For the forward implication, there are two cases on the number of R if (G, R) does not contain a confined comb of length three.

Case 1. $|R| \leq 2$. If $|R| = 2$, then let G_0 be a path P with ends as roots, and let G_i be the component of $G \backslash P$. Hence $G = \oplus_2(G_0; G_1, \cdots, G_t)$, Since G is connected, there exists a path between any two vertices from G. If $|R| = 1$, then let G_0 be a root, and let G_i be the component of $G \backslash G_0$. Hence $G = \oplus_2(G_0; G_1, \cdots, G_t)$.

Case 2. $|R| \geq 3$. Let $H = G[R]$. Let us consider H-bridge.

Claim 1. Any H-bridge has at most two feet. Otherwise there exists a H-bridge with at least three feet. By the definition of H-bridge, there exists a subgraph B of $G \backslash E(H)$ that satisfies there exists a connected component C of $G \backslash R$ such that $E(B)$ consists of all edges incident with at least one vertex of C. Since H-bridge has at least three feet, so $|B \cap H| > 3$. Let $V(B \cap R) = \{l_1, l_2, \cdots, l_t\}$ for $t \geq 3$. Since C is connected, and B is the union of C and all edges incident with unique one vertex of C, so B is connected and $d_B(l_i) = 1$ for $1 \leq i \leq 3$. Since B is connected, so there exists a path P^1 between l_1 and l_2 in B, and then there exists a path P^2 between P^1 and l_3 in B. Then $P^1 \cup P^2$ is a confined comb of length three, which contradicts the assumption that (G, R) does not contain a confined comb of length three.

Let $V(G_0) = R$ and $E(G_0) = \{xy : x$ and y are the two different feet of one R-bridge$\}$.

Claim 2. The maximal degree of G_0 is two. Otherwise there exists a vertex v of G_0 with at least three degree, and let neighbor vertices of v be $\{u_1, \cdots, u_k\}$, for $k \geq 3$. By Claim 1, every edge in G_0 belongs to one unique R-bridge. Then there exists a vu_i-path between v and each u_i in G, where vu_1-path, vu_2-path, vu_3-path are mutually disjoint paths except v, we can find a confined comb which is a union of vu_1-path, vu_2-path, vu_3-path. It implies there exists a confined comb of length three, which contradicts the assumption that there is no a confined comb of length three. Hence the maximal degree of G_0 is two.

By Claim 2, hence G_0 is a cycle or a path. Let G_i be H-bridge of G. By Claim 1, $G = \oplus_2(G_0; G_1, \cdots, G_t)$, for $t \geq 1$.

The forward implication was shown above. Consequently, for any vertex v in $G = \oplus_2(G_0; G_1, \cdots, G_t)$, where G_0 is a path or a cycle and it contains R. There are two cases on v.

Case 1. $v \in V(G_i \backslash G_0)$, for some i, where $1 \leq i \leq t$. Let $V(G_i \cap G_0)$ be a cut vertex set to separate v to R. By the construction of G, we know that $|G_i \cap G_0| \leq 2$.

Case 2. $v \in V(G_0)$. Let the neighbor vertices r_1, r_2 of v in G_0 be a cut vertex set to separate v to the rest of R.

By Menger's Theorem, there are at most two mutually disjoint paths from v to R, that's why G does not contain a confined comb of length three. $\square$

We remark that for any $(G, R) \in \mathcal{G}$ satisfies that G is constructed from a cycle with 2-summing connected graphs and R is the subset of the vertex set of cycle must belong to $\Phi((G_1, R_1))$, where G_1 is constructed from a path with 2-sum connected graphs and R_1 is the subset of the vertex set of path. That's why we say that $(G, R) \in \Phi(\mathcal{P})$, if G is constructed from a cycle with 2-sum connected graphs and R is the subset of the vertex set of cycle.

In the next lemma, we show that any two maximum confined combs must meet in a rooted connected graph.

Lemma 4.2.4. *Let (G, R) be a rooted connected graph. If $(G_1, R_1), (G_2, R_2)$ are confined combs of length n and there is no confined comb of length $n+1$, where $n \geq 3$, then $V(G_1 \cap G_2) \neq \emptyset$.*

Proof. Suppose $V(G_1 \cap G_2) = \emptyset$. Since G is connected, there exists a path P between G_1 and G_2. Let $P^{i,1}, P^{i,2}, \cdots, P^{i,n}$ be all the teeth of G_i, listed in the order that they appear in G_i, let the shaft of G_i be S_i, and $P_{i,m} \cap S_i = x_{i,m}$ for

$i = 1, 2$ and $2 \leq m \leq n - 1$. We denote $S_i[x_{i,l}, x_{i,m}]$ as a path of S_i with ends $x_{i,l}, x_{i,m}$ for $2 \leq l \leq m \leq n - 1$. Let ends of P be p_1, p_2. We define $G_{i,1}, G_{i,2}$ which are the subgraphs of G_i and satisfy the following conditions:

(i) $G_{i,1} \cup G_{i,2} = G_i$;

(ii) If $p_i \in P^{i,m}$, for some m and $2 \leq m \leq n - 1$. We define $G_{i,1}$ as the union of $P^{i,1}, P^{i,2}, \cdots, P^{i,m}, S_i[x_{i,2}, x_{i,m+1}]$ and $G_{i,2}$ as the union of $P^{i,m}, P^{i,m+1}, \cdots,$ $P^{i,n}, S_i[x_{i,m-1}, x_{i,n-1}]$, or

if $p_i \in P^{i,1}$ or $P^{i,n}$. We define $G_{i,1}$ as G_i and $G_{i,2}$ as G_i, or

if $p_i \in S_i[x_{i,l}, x_{i,l+1}]$ for $2 \leq l \leq n - 2$. We define $G_{i,1}$ as the union of $P^{i,1}, \cdots, P^{i,l+1}, S_i[x_{i,1}, x_{i,l+1}]$ and $G_{i,2}$ as the union of $P^{i,l-1}, \cdots, P^{i,n}$, $S_i[x_{i,l}, x_{i,n-1}]$.

By the construction of $G_{i,1}, G_{i,2}$, the length of $G_{i,1} \cup G_{i,2}$ are larger than the length of G_i. Without loss of generality, we set $G_{i,1}$ as a comb with larger length between $G_{i,1}$ and $G_{i,2}$. Then $(G_{1,1} \cup G_{2,1} \cup P, V(G_{1,1} \cup G_{2,1}) \cap R_1 \cap R_2)$ is a confined comb, whose shaft is $S_1[x_{1,1}, x_{1,p_1}] \cup S_2[x_{2,1}, x_{2,p_2}] \cup P$ and the set of teeth is $\{P^{1,1}, \cdots, P^{1,l_1}, P^{1,l_1+1} \cup S_1[x_{1,p_1}, x_{1,l_1+1}], P^{2,1}, \cdots, P^{2,l_2}, P^{2,l_2+1} \cup S_2[x_{2,p_2}, x_{2,l_2+1}]\}$ for $\lfloor \frac{n}{2} \rfloor \leq l_1, l_2 \leq n-1$. Therefore the length of it is larger than n, which contradicts the assumption that there is no a confined comb of length $n + 1$ in (G, R). $\square$

We provide a counterexample of the above lemma to show that it is false for $n < 3$. The counterexample of the above lemma for $n = 2$, let G be a path and G_1, G_2 are two subgraphs of G such that G_1, G_2 are both confined combs of length two, and $V(G_1 \cap G_2) = \emptyset$. There is no a confined comb of length three in G, however, $V(G_1 \cap G_2) = \emptyset$.

Proof of Theorem 4.2.1. From (2) to (3), let $m = k$. Proof by induction on k. For $k = 0$, we can obtain $m = 0$. For any $(G, R) \in \mathcal{G} \subseteq \Phi^0(\mathcal{G}_0) = \mathcal{G}_0$,

$G = \oplus_2(G_0; G_1, \cdots, G_t)$ where $R \subseteq V(G_0)$ and $(G_0, R) \in \mathcal{P} = \Phi^0(\mathcal{P})$. It means that the assertion is true. Now let $k > 0$, and assume that the assertion holds for rooted connected graphs belong to $\Phi^{k-1}(\mathcal{G}_0)$.

We want to prove that any $(G, R) \in \Phi(\Phi^{k-1}(\mathcal{G}_0))$ admits an expression as described in (3). Let $(G, R) = \Phi(H_1, \cdots, H_t)$ such that $H_i \in \Phi^{k-1}(\mathcal{G}_0)$ for $1 \leq i \leq t$. By induction, each $H_i = \oplus(H_0^i; H_1^i, \cdots, H_{t_i}^i)$ where $R_i \subseteq V(H_0^i)$ and $(H_0^i, R_i) \in \Phi^{k-1}(\mathcal{P})$. By Lemma 4.2.2, $G = \oplus(G^0; G^1, \cdots, G^l)$ where $(G^0, R) = \Phi((H_0^i, R_i), \cdots, (H_0^t, R_t)) = \Phi^k(\mathcal{P})$ and $G^i = H_j^i$ for $1 \leq j \leq t_i$.

From (3) to (1), let $n = g(m) = 2^{m+2}$.

Proof by induction on m. For $m = 0$, we obtain $g(0) = 4$. There is no confined comb of length four in $\mathcal{P}$. It means that the assertion is true. Now let $m > 0$, and assume that assertion holds for every $(G, R) \in \mathcal{G}$, $G = \oplus(G_0; G_1, \cdots, G_t)$, for some $t \in \mathbb{N}$, where $R \subseteq V(G_0)$ and $(G_0, R) \in \Phi^{m-1}(\mathcal{P})$. There exists one vertex $v \in V(G_0)$ such that each component of $(G_0 \backslash v, R \backslash v) \in \Phi^{m-1}(\mathcal{P})$. Then each component of $(G \backslash v, R \backslash v)$ can be expressed $\oplus(G_j^0; G_j^1, \cdots, G_j^{t_i})$ where G_j^0 is one component of $G_0 \backslash v$. and G_j^i is one component of $G_i \backslash v$ for $1 \leq i \leq t$ and $j \geq 1$, since every two components of $G_0 \backslash v$ do not share a clique to be clique-sum one component of $G_i \backslash v$. By induction, there is no confined comb of length $g(m-1)$ in each component of $(G \backslash v, R \backslash v)$. By the construction of Φ and the definition of a confined comb, it is easy to obtain $2(g(m-1) - 1) + 1 = g(m) - 1$. We can get $n = 2^{m+2}$.

From (1) to (2), let $k = f(n) = n^3/3 + n^2/2 + n/6$. For any (G, R) does not contain a confined comb of length n, we prove that (G, R) admits an expression as described in (2).

For $n = 3$, we can obtain that $f(3) = 14$. By Lemma 4.2.3, $0 < k = 14$. Since $\mathcal{G}_0 = \Phi^0(\mathcal{G}_0) \subseteq \Phi^{14}(\mathcal{G}_0)$. It means that the assertion is true. Now let $n > 3$, and

assume that the assertion holds for rooted graphs that do not contain a confined comb of length less than n.

Let H be a confined comb of maximum length in (G, R). In addition, we choose H among all the maximum confined combs such that $|R_1| = |R \cap H|$ is minimum. If the length of H is less than $n-1$, then the result follows from induction, so we assume the length of H is $n-1$. Let $R_2 = R \backslash R_1$. If G has at least $(n-3)^2+1$ disjoint paths between R_2 and H. There exists at least one tooth T of H such that there are at least n disjoint paths between T and R_2, (T, R_2) is a confined comb of length n in (G, R), which contradicts the assumption that there is no confined comb of length n in (G, R).

By Menger's Theorem, (G, R) has a separation (J_1, J_2) such that $|J_1 \cap J_2| \leq (n-3)^2+1$ where H is a subgraph of J_1 and $R_2 \subseteq V(J_2)$. Let $r \in R_1$ and $S = \{r\} \cup V(J_1 \cap J_2)$. We only need to show that $(G \backslash S, R \backslash S)$ does not contain a confined comb of length $n-1$, because by induction we will have $(G, R) \in \mathcal{G} \subseteq \Phi^k(\mathcal{G}_1)$ where $((n-3)n+1)+((n-4)(n-1)+1)+\cdots+1 < n^3/3+n^2/2+n/6 = k$ such that $\mathcal{G} \subseteq \Phi^k(\mathcal{G}_1)$, where $\mathcal{G}_1$ is a class of rooted connected graphs does not contain a confined comb of length three. Hence $k = n^3/3+n^2/2+n/6$ such that $\mathcal{G} \subseteq \Phi^k(\mathcal{G}_0)$.

To prove that $(G \backslash S, R \backslash S)$ does not contain a confined comb of length $n-1$. We assume that $(G \backslash S, R \backslash S)$ contains a confined comb A of length $n-1$. If A is contained in $J_1 \backslash S$, since $J_1 \backslash S$ does not contain roots rather than R_1, A has at most $|R_1|-1$ roots. It contradicts the assumption that the minimality of $|R_1|$. Else A is contained $J_2 \backslash S$. Since $J_1 \cap (J_2 \backslash J_1) = \emptyset$, $V(A) \subseteq V(J_2 \backslash S) \subseteq V(J_2)$ and H is a subgraph of J_1, so $V(A \cap H) = \emptyset$. It contradicts Lemma 4.2.4 that any two maximum confined combs must meet.

Finally, we prove that (G, R) can be constructed $|S|$ steps of Φ from $(G\backslash S, R\backslash S)$.

$\square$

Let $R = V(G)$, we can apply Theorem 4.2.1 to obtain Corollary 4.2.5.

Corollary 4.2.5. *The following are equivalent for any class $\mathcal{G}$ of connected graphs.*

(1) there exists an integer $n \geq 3$ such that no member of $\mathcal{G}$ has a comb of length n;

(2) there exists an integer $k \geq 0$ such that $\mathcal{G} \subseteq \Phi^k(\mathcal{P})$, where $\mathcal{P}$ is the class of all paths.

5 Excluding a Heavy Path

In front chapters, we have seen several results involving heavy paths. In particular, Theorem 1.4.1 says that a characterization on excluding a heavy path. Also, Theorem 1.5.1 says that a heavy path is one of the unavoidable subgraphs in a sufficiently large connected graph. Besides, Theorem 1.5.2 says that a heavy path is one of the unavoidable induced subgraphs in a sufficiently large connected graph. Furthermore, Theorem 2.1.1 says that a path with many roots is one of the unavoidable induced subgraphs in a sufficiently large rooted connected graph. Further, Theorem 2.1.4 says that a path containing many roots is one of the unavoidable subgraphs in a sufficiently large rooted connected graph. In this chapter, we characterize that large rooted graphs do not contain a path containing many roots. We remark that, in this chapter, when we say that a rooted graph contains a path with many roots, we mean that a rooted graph contains a path with many roots as a subgraph.

5.1 Packing ABA-paths

If T is a path, then we call $|V(T) \cap R|$ the *weight* of T.

For a directed graph $G = (V, E)$ whose edges are labelled by the elements of a group Γ and $A \subseteq V$, Chudnovsky, Geelen, Gerards, Goddyn, Lohman and Seymour [1] define an *A-path* as a path with both ends in A in the underlying graph of G, as well as define the *weight* of a path P in G is the sum of the group values on forward oriented arcs minus the sum of the backward oriented arcs in P.

Let G be a graph and $A, B \subseteq V(G)$. An *ABA-path* of G is a path with two ends from A and at least one vertex from B.

For the definition of an A-path [1], an A-path may have more than two vertices from A. However, we want to use the definition of an ABA-path which only contains two vertices from A in my thesis.

Remark 5.1.1. *There exist k vertex-disjoint A-paths [1] if and only if there exist k vertex-disjoint ABA-paths in my thesis.*

The backward implication follows that an ABA-path from our paper is a special A-path [1], and the forward implication follows that we can choose a subpath of an A-path [1], which only contains two vertices from A as ends.

Theorem 5.1.2. *Let A, B be disjoint vertex sets of a graph G. For any $k \in \mathbb{N}$, if G does not have k disjoint ABA-paths then $V(G)$ admits a partition $(X, Y_0, Y_1, ..., Y_n)$, where $n \geq 0$, such that*

(1) $|X| \leq 2k - 2$,

(2) $A \subseteq X \cup Y_0$ and $B \cap Y_0 = \emptyset$, and

(3) for $i = 1, ..., n$, $N_G(Y_i) \subseteq X \cup Y_0$ and $|N_G(Y_i) \cap Y_0| \leq 1$.

To simplify the proof of Theorem 5.1.2, we first need to prove the following two lemmas.

Lemma 5.1.3 ([1]). *Let Γ be a group, let $G = (V, E)$ be an oriented graph with edge-labels from Γ, and let $A \subseteq V$. Then, for any $k \in \mathbb{N}$, either there are k vertex-disjoint A-paths each of non-zero weight, or there is a set of at most $2k-2$ vertices that meets each non-zero A-path.*

Lemma 5.1.4. *Let $G = (V, E)$ be a graph with two specified disjoint vertex sets A, B. Then G either has k vertex disjoint ABA-paths, or has a set of at most $2k - 2$ vertices meeting all such paths.*

Proof. We assume that an arbitrary orientation of G. We assign labels γ_e to edges e whose at least one end in Y and assign labels 0 to all other edges. Let Γ be a free group generated by $\{\gamma_e : V(e) \cap Y \neq \emptyset\}$. Let $A = X \subseteq V$.

Claim. An A-path is an ABA-path if and only if it is non-zero weight.

Firstly, we consider the forward implication. By the definition of an A-path in our paper, there are two cases on an A-path. Case 1. A path with just both ends in A. By the labels for E, the weight of such path is zero. Case 2. A path with just both ends in A and containing a vertex in B, which is an ABA-path. Similarly, the weight of such path is non-zero.

Next, we consider the backward implication. By the definition of an A-path in our paper and $A = A$, an A-path is a path with only both ends in A. By the definition of free group, if the weight of an A-path is non-zero, then there exist some edges in such path and meet B.

By Lemma 5.1.3, an arbitrary orientation of G either has k vertex-disjoint A-paths each of non-zero weight, or has a set of at most $2k-2$ vertices meeting all such paths.

By Claim, G either has k vertex-disjoint ABA-paths, or has a set of at most $2k-2$ vertices meeting all such paths. □

Proof of Theorem 5.1.2. By Lemma 5.1.3, it is easy to get (1). Since $G \backslash X$ has no ABA-paths, we can know that $A = \emptyset$ or $G \backslash X {=} \oplus_1(G_0; G_1, \cdots, G_n)$ such that $A \subseteq V(G_0)$ and $B \cap V(G_0) = \emptyset$, by Lemma 5.1.4. Let $V(G_0) = Y_0, V(G_i) = Y_i$, we are done. □

Let $G_0, G_1, \cdots, G_k$ be mutually vertex disjoint graphs. A $\oplus_1(G_0; G_1, \cdots, G_k)$ is a graph obtained from $\cup_{i=1}^{k} G_0 \oplus_1 G_i$.

Lemma 5.1.5. *Let $G = (V, E)$ be a graph with two specified disjoint vertex sets A, B. If G does not contain ABA-path, then either $A = \emptyset$ or $G = \oplus_1(G_0; G_1, G_2, \cdots, G_k)$ such that $A \subseteq V(G_0)$ and $B \cap V(G_0) = \emptyset$.*

Proof. There are three cases on the ABA-path.

Case 1. $A = \emptyset$.

Case 2. $B = \emptyset$. Let $G_0 = G$ and $G_i = \emptyset$ such that $A \subseteq V(G_0)$ and $B \cap V(G_0) = \emptyset \cap V(G_0) = \emptyset$.

Case 3. $A \neq \emptyset$ and $B \neq \emptyset$. For any vertex $b \in B$, there exists exactly one path from b to A in G, since G does not contain ABA-path. By Menger's Theorem, there is a separation (H_1, H_2) such that $A \in V(H_1)$ and $b \in V(H_2)$ and $V(H_1) \cap V(H_2) = \{c\}$.

By induction, $H_1 = \oplus_1(G_0; G_1, \cdots, G_{k-1})$. $A \subseteq V(G_0)$ and $B \cap V(G_0) = \emptyset$. Let $H_2 = G_k$ such that $b \in V(G_k)$. There are three cases on c.

Case 1. c is not in $V(G_0)$, then $c \in G_s$ for some $1 \leq s \leq k-1$. Hence there exists $G_s \cup H_2$ containing b and $V(G_s) \cap V(G_0)$ is a cut vertex to separate A to b, we are done.

Case 2. $c \in V(G_0) \cap V(G_s)$, for some $1 \leq s \leq k-1$, then there exists $G_s \cup H_2$ containing b and c is a cut vertex to separate A to b, we are done.

Case 3. c is in $V(G_0)$ and $c \notin G_1, G_2, \cdots, G_{k-1}$, then $G = \oplus_1(G_0; G_1, \cdots, G_k)$, where $B \cap V(G_0) = \emptyset$, we are done. □

5.2 Excluding a heavy path

In front chapter, we prove Theorem 4.1.1, which solved the problem of a characterization on rooted graphs that do not contain a large nicely confined comb. Alse, Theorem 4.2.1 solved the problem of a characterization on rooted connected graphs that do not contain a large confined comb. In this section, our goal is to

provide a characterization on rooted connoted graphs that do not contain a path with many roots.

Firstly, we state with the definition of a construction of Ψ.

Let $H_0, H_1, \cdots, H_k$ be mutually vertex disjoint rooted connected graphs, where $H_i = (G_i, R_i)$, for $0 \le i \le k$. We define $\Psi(H_0; H_1, \cdots, H_k)$ to be a rooted graph (G, R) constructed as below:

(i) G is obtained from the disjoint union of $G_0, G_1, \cdots, G_k$ by first choosing (not necessarily distinct) vertices $v_1, v_2, \cdots, v_t$ from G_0 and then, for each $i = 1, 2, \cdots, k$, adding at least one edge from v_i to G_i;

(ii) $R = R_0 \cup R_1 \cup \cdots \cup R_k$.

Let $\mathcal{G}$ be a class of rooted connected graphs. Let $\Psi^0(\mathcal{G}) = \mathcal{G}$ and inductively, $\Psi^t(\mathcal{G}) = \{\Psi(H_0; H_1, \cdots, H_k)\ H_0 \in \mathcal{G}$ and $H_1, \cdots, H_k \in \Psi^{t-1}(\mathcal{G})$ and $t \ge 1\}$ for each $k \ge 1$.

We remark that $\Psi^{t-1}(\mathcal{G}) \subseteq \Psi^t(\mathcal{G})$.

We denote $P[a, b]$ as a subgraph of P with ends a, b.

Theorem 5.2.1. *The following are equivalent for any class $\mathcal{G}$ of rooted connected graphs.*

(1) there exists an integer $n \ge 0$ such that no member of $\mathcal{G}$ contains a path with n roots;

(2) there exists an integer $k \ge 0$ such that every member of $\mathcal{G}$ can be constructed within k steps, starting from rooted connected graphs (G, R) with $|R| \le 1$, by operation Ψ.

To simplify the proof of Theorem 5.2.1, we first need to prove the following one lemma. It proves that any two path containing maximum number roots must meet in a rooted 2-connected graph (G, R).

Lemma 5.2.2. *Let (G, R) be a rooted 2-connected graph. If P^1, P^2 are paths with n roots and there is no path with $n+1$ roots, then $V(P^1 \cap P^2) \neq \emptyset$.*

Proof. Assume that $V(P^1 \cap P^2) = \emptyset$. Since (G, R) is a rooted 2-connected graph, so there are at least two paths L^1, L^2 between P^1 and P^2. Let ends of P^i be a_i, b_i, and let ends of L^i be v_i, u_i for $i = 1, 2$.

There are two cases on n.

Case 1. n is an odd number. Without loss of generality, we set $P^i[a_i, v_i]$ with larger number of roots than $P^i[u_i, b_i]$. So the number of roots of rooted path $P^1[a_1, u_1] \cup P^2[a_2, u_2] \cup L^2$ is at least $(n+1)/2 + (n+1)/2 = n+1$, which contradicts that there is no path with $n+1$ roots.

Case 2. n is an even number.

Claim 1. $P^i[a_i, v_i], P^i[u_i, b_i]$ both have $n/2$ roots.

Otherwise, without loss of generality, we set $P^1[a_1, v_1]$ has at most $n/2 - 1$ roots. Then there exists a path with at least $n+1$, which is $P^1[v_1, b_1] \cup L^1 \cup P^2[v_2, b_2]$. It contradicts the assumption that there is no path with $n+1$ roots in (G, R).

Let $r_{i,1}$ be a root in $P^i[a_i, v_i]$ such that there is no another root in $P^i[r_{i,1}, v_i]$. Let $r_{i,2}$ be a root in $P^i[u_i, b_i]$ such that there is no another root in $P^i[u_i, r_{i,2}]$. We choose the minimal length among $P^1[r_{1,1}, v_1]$, $P^1[u_1, r_{1,2}]$, $P^2[r_{2,1}, v_2]$ and $P^2[r_{2,2}, u_2]$. Without loss of generality, the length of $P^2[r_{2,2}, u_2]$ is minimum. And subject to that, we assume that the length of $P^2[r_{2,2}, u_2]$ is as small as possible. Since (G, R) is a rooted 2-connected graph, there exists one path L^3 connecting $P^2[u_2, b_2] \backslash u_2$ to $P^1 \cup (P^2[a_2, u_2] \backslash u_2) \cup L^1 \cup (L^2 \backslash u_2)$. Let ends of L^3 be t_1, t_2, where $t_1 \in P^2$. There are cases on the position of t_i.

Case 1. $t_1 \in V(P^2[r_{2,2}, u_2])$. Then there are three cases on the position of t_2.

Subcase 1.1. $t_2 \in V(P^1)$.

Without loss of generality, we set $P^1[a_1, t_2]$ as larger number roots than $P^1[t_2, b_2]$. By Claim 1, the number of roots of the path $P^1[a_1, t_2] \cup L^3 \cup P^2[t_1, b_2]$ is at least $n/2 + n/2 + 1 = n + 1$, which contradicts that there is no path with $n + 1$ roots.

Subcase 1.2. $t_2 \in V(L^1)$ or $V(L^2)$.

Without loss of generality, $t_2 \in V(L^1)$. We replace L^1 with $L^1[v_1, t_2] \cup L^3$, the length of $P^2[r_{2,1}, t_1]$ is longer than that of $P^2[r_{2,1}, v_2]$, which contradicts the assumption that the length of $P^2[r_{2,1}, v_2]$ is minimum between one end L^1 and $r_{1,2}$.

Subcase 1.3. $t_2 \in V(P^2)$. We replace L^2 with $L^2 \cup P^2[u_2, t_1]$. Since the length of $P^2[t_1, r_{2,2}]$ is smaller than that of $P^2[u_2, r_{2,2}]$, which contradicts the assumption that the length of $P^2[u_2, r_{2,2}]$ is minimum between one end L^2 and $r_{2,2}$.

Case 2. $t_1 \in V(P^2[r_{2,2}, b_2]\backslash r_{2,2}$. Then there are four cases on the position of t_2.

Subcase 2.1. $t_2 \in V(P^1)$.

By Claim 1, $t_2 \in P^1[r_{1,1}, r_{1,2}]\backslash r_{1,1}\backslash r_{1,2}$. Then the number of roots of the path $P^1[a_1, t_2] \cup L^3 \cup P^2[a_2, t_1]$ is at least $n + 1$, which contradicts the assumption that there is no path with $n + 1$ roots in (G, R).

Subcase 2.2. $t_2 \in V(L^1)$ or $V(L^2)$.

Then there exists one path $P^1[a_1, v_1] \cup L^1[v_1, t_2] \cup L^3 \cup P^2[a_2, t_1]$ with at least $n + 1$ roots, which contradicts the assumption that there is no path with $n + 1$ roots in (G, R).

Subcase 2.3. $t_2 \in V(P^2[a_2, u_2])$.

Then there exists one path $P^1[a_1, v_1] \cup L^1 \cup P^2[v_2, t_2] \cup L^2 \cup L^3 \cup P^2[t_1, u_2]$ with at least $n + 1$ roots, which contradicts that there is no path with $n + 1$ roots in (G, R).

Subcase 2.4. $t_2 \in V(P^2[u_2, r_{2,2}])$.

We replace L^2 with $L^2 \cup P^2[u_2, t_2]$. Then the length of $P^2[t_2, r_{2,2}]$ is smaller than that of $P^2[u_2, r_{2,2}]$, which contradicts the assumption that the length of $P^2[u_2, r_{2,2}]$ is minimum between one end L^2 and $r_{2,2}$. $\square$

Proof of Theorem 5.2.1. From (1) to (2), let $k = f(n) = (n-1) + (n-1)(1/4 \times n^2(n+1)^2 - 9 + ((n-2)(n+3))/2$. For any $(G, R) \in \mathcal{G}$ does not contain a path with n roots. We prove that (G, R) admits an expression as described in (2).

Firstly, we consider a rooted 2-connected graph $(G, R) \in \mathcal{G}$. Proof by induction. For $n = 2$, we can obtain that $0 < f(2) = 1$. Since $\mathcal{G} = \Psi^0(\mathcal{G}) \subseteq \Psi^1(\mathcal{G})$. It means that the assertion is true. Now let $n > 2$, and assume that the assertion holds for rooted graphs that do not contain a path with $n-1$ roots.

By Lemma 5.2.2, we notice that any two paths with maximum number of roots must meet in a rooted 2-connected graph (G, R). Let P be a path with maximum number of roots in (G, R), and then P has at most $n-1$ roots. If P has less than $n-1$ roots, then the result follows from induction. So we assume that P has $n-1$ roots. Let $A = V(P)$ and let $B = R \backslash V(P)$. Consider that ABA-paths, notice that each ABA-path contains at least one root.

Claim 1. There are at most n^3 vertex-disjoint ABA-paths in G.

By Lemma 1.5.4, if there are at least $n^3 + 1$ vertex-disjoint ABA-paths, then there n of them such that they are either pairwise disjoint, or pairwise comparable, or pairwise overlapping.

Suppose that there are n pairwise disjoint ABA-paths, let such pairwise disjoint ABA-paths be $P^{1,1}, P^{1,2}, \cdots, P^{1,n}$. Let ends of each $P^{1,i}$ be p_{1,i_1}, p_{1,i_2} for $1 \leq i \leq n$. Then $P \backslash \bigcup_{i=1}^{n} P[p_{1,i_1}, p_{1,i_2}]$ is a path with at least n roots, which contradicts that P is the path with maximum number of roots in (G, R).

Suppose that there are n pairwise comparable ABA-paths. Let such pairwise comparable ABA-paths be $P^{2,1}, P^{2,2}, \cdots, P^{2,n}$. Let ends of each $P^{2,i}$ be p_{2,i_1}, p_{2,i_2}. Without loss of generality, $P[p_{2,i_1}, p_{2,i_2}]$ is a subgraph of $P[p_{2,j_1}, p_{2,j_2}]$ for $1 \leq j \leq i \leq n$. Then there exists a path in $P \backslash P[p_{2,n_1}, p_{2,n_2}]$, which is $\bigcup_{i=1}^{n} P^{2,i} \cup P[p_{2,n-1_1}, p_{2,n-2_1}] \cup P[p_{2,n-2_2}, p_{2,n-3_2}] \cup \cdots \cup P[p_{2,2_1}, p_{2,1_1}]$. Such path has at least n roots in (G, R), which contradicts the assumption that P is the path with the maximal number of roots in (G, R).

Suppose that there are n pairwise overlapping ABA-paths Let such pairwise overlapping ABA-paths be $P^{3,1}, P^{3,2}, \cdots, P^{3,n}$. Let ends of each $P^{3,i}$ be p_{3,i_1}, p_{3,i_2}. Then $P \backslash P[p_{3,n_1}, p_{3,1_2}]$ is the same as $P \backslash P[p_{2,n_1}, p_{2,n_2}]$. Similarly, there exists one path with at least n roots in (G, R), which contradicts the assumption that P is the path with maximum number of roots in (G, R).

By Claim 1, any rooted 2-connected graph (G, R) which does not contain a path with n roots has at most n^3 vertex-disjoint ABA-paths.

By Theorem 5.1.2, $V(G)$ admits a partition $(X, Y_0, Y_1, \cdots, Y_t)$ where $t \geq 0$, such that X of at most $2(n^3+1)-2$ vertices meeting all ABA-paths; $A \subseteq X \cup Y_0$ and $B \cap Y_0 = \emptyset$, and for $i = 1, \cdots, t$, $N(Y_i) \subseteq X \cup Y_0$ and $|N(Y_i) \cap Y_0| \leq 1$. There are components $C^1, C^2, \cdots$ of $G \backslash X \backslash (V(P) \cap R)$. Every such component is the union of one component of $Y_0 \backslash R$ and some Y_j, for $1 \leq j \leq t$. By Lemma 5.2.2, there is no path with $n-1$ roots in such component. Let $a_1 \in V(P) \cap R$ be $H_0^{a_1}$ such that a_1 is adjacent with $C^{1_1}, C^{1_2}, \cdots$ in G. Then $G[a_1 \cup V(C^{1_1} \cup C^{1_2} \cup \cdots)]$ is obtained from $\Psi(H_0^1; C^{1_1}, C^{1_2}, \cdots)$, where $C^{1_1}, C^{1_2}, \cdots, \in \Psi^{f(n-1)}(\mathcal{G})$, by induction hypothesis. After adding all vertices in $V(P) \cap R$ to such components, let one vertex b_1 in X be $H_0^{b_1}$ such that $G[b_1 \cup G^1 \cup \cdots]$ is obtained from $\Psi(H_0^{b_1}; G^1, \cdots)$, where $G^1, \cdots, \in \Psi^{(n-1)+f(n-1)}(\mathcal{G})$. Then $|X| + n - 1 + f(n-1) = f(n)$. The recursive formula implies that $f(n) = 1/4 \times n^2(n+1)^2 - 9 + ((n-2)(n+3))/2$.

Secondly, we consider a rooted connected graph $(G, R) \in \mathcal{G}$. By Lemma 1.1.3, there is a block tree T_G of G. For any two neighbor blocks without roots in T_G, we can consider such two blocks as one block in T_G. Hence we create a new block tree T'_G of G, we choose any block b_0 of T'_G. Then every path from b_0 to a leave in T'_G with at most $n-1$ $b_1, \cdots, b_{n-1}$ and at most $n-1$ $b'_1, \cdots, b'_{n-1}$, where the corresponding block B_i of b_i which has no root, and the corresponding block B'_i of b'_i which has at least one root, and $V(B_i) \cap V(B'_i) = c_i$. It is easy to show $b_0 = b_1$ or $b_0 = b'_1$. Let us choose any one vertex c_0 in B_0, there exists a c_0c_1-path between c_0 and c_1 in B_0. There must exist c_jc_{j+1}-path passing at least one root in B'_j, where $1 \leq j \leq n-1$ and j is an odd number. Let us choose one root r_j of B'_j. Then there are two disjoint paths c_jr_j-path, $c_{j+1}r_j$-path except r_j to c_j, c_{j+1} separately. Otherwise, it contradicts that any block is 2-connected graph. By the above proof, we know if B'_i does not contain a path with n roots, then B'_i can be constructed within $f(n)$ steps, starting from a connected rooted graph with less than 1 root, by operation Ψ. For every rooted connected graph which has no path with n roots, it can be constructed within $k = n - 1 + (n-1)f(n)$ steps Ψ.

From (2) to (1), let $n = g(k) = 2^{k+1}$. Proof by induction. For $k = 0$, we can obtain $g(0) = 2$. Since $(G, R) \in \mathcal{G}$ satisfying $|R| \leq 1$, then (G, R) does not contain a path with 2 roots. It means the assertion is true. Now, let $k > 0$, and assume that the assertion holds for rooted graphs that are constructed by $k-1$ steps Ψ. By induction hypothesis, any rooted connected graph in $\Psi^{k-1}(\mathcal{G})$ does not contain a path with $g(k-1)$ roots, where $\mathcal{G}$ is a class of rooted connected graphs with at most one root. By the construction of the operation Ψ, we can get that $2(g(k-1)-1)+1 = g(k)-1$. The recursive formula implies that $n = 2^{k+1}$. $\square$

Let $R = V(G)$, we can apply Theorem 5.2.1 to obtain Corollary 5.2.3. By the definition of $\Psi(\mathcal{G})$, every member of $\mathcal{G}$ can be constructed within k steps, starting from K_1, by operation $\Psi = \Phi$ if $R = V(G)$.

Corollary 5.2.3. *The following are equivalent for any class $\mathcal{G}$ of connected graphs.*

(1) there exists an integer $n \geq 0$ such that no member of $\mathcal{G}$ has a path of length n as a subgraph;

(2) there exists an integer $k \geq 0$ such that $\mathcal{G} \subseteq \Phi^k(\{K_1\})$;

5.3 Excluding a heavy path rooted minor

In this section, our goal is to provide a characterization on rooted connected graphs that do not contain a path with many roots as a rooted minor.

Theorem 5.3.1. *The following are equivalent for any class $\mathcal{G}$ of rooted connected graphs.*

(1) there exists an integer $n > 0$ such that no member of $\mathcal{G}$ has a path with n roots as a rooted minor;

(2) there exists an integer $s > 0$ such that $\mathcal{G} \subseteq \Phi^s(\mathcal{G}_0)$, where $\mathcal{G}_0 = \{(G, \emptyset) : G$ is connected$\}$;

(3) there exists an integer $h > 0$ such that for every $(G, R) \in \mathcal{G}$, $G = \oplus(G_0; G_1, \cdots, G_k)$, for some $k \in \mathbb{N}$ where $R \subseteq V(G_0)$ and G_0 has no path of length h.

To simplify the proof Theorem 5.3.1, we first need to prove the following lemma. It proves that any rooted graph contains a heavy path as a rooted minor if and only if it contains either a heavy path or a large confined comb as a subgraph.

Lemma 5.3.2. *Let (G, R) be a rooted graph. For any integer $n > 0$.*

(1) If (G, R) contains a path with $2n$ roots as a rooted minor, then (G, R) contains either a path with n roots or a confined comb of length n as a subgraph;

(2) if (G, R) contains either a path with n roots or a confined comb of length n as a subgraph, then (G, R) contains a path with n roots as a rooted minor.

Proof. (2) If (G, R) contains a path with n roots as a subgraph, then such path is a rooted minor of (G, R); if (G, R) contains a confined comb of length n, then such confined comb after contracting series edges in all its teeth is a path with n roots, which is a rooted minor of (G, R).

(1) If (G, R) contains a path P with $2n$ roots as a rooted minor. Suppose that (G, R) contains neither a path with n roots nor a confined comb of length n as a subgraph.

Let $(P, V(P))$ be a rooted minor of (G, R). There are at least n disjoint connected subgraphs $G_{v_1}, G_{v_2}, \cdots, G_{v_{2n}}$ such that each G_{v_i} contains at least one root. Let $V(G_{v_i} \cap P)$ be $\{u_i, v_i\}$, and let r_i be one root of G_{v_i}. Since G_{v_i} is connected, so there exists one u_iv_i-path in G_{v_i}. If r_i is in u_iv_i-path, then there are at most $n-1$ disjoint connected subgraphs in G such that every u_iv_i-path containing r_i. Without loss of generality, such connected subgraphs $G_{v_1}, G_{v_2}, \cdots, G_{n-1}$. If r_i is not in the u_iv_i-path, then there exists another path L^i between r_i and u_iv_i-path in G_{v_i}. Therefore, there are at least n such disjoint connected subgraphs $G_{v_n}, G_{v_{n+1}}, \cdots, G_{v_{2n}}$ in G. There exists a confined comb Z in (G, R) such that the shaft of Z is $\bigcup_{j=n}^{2n} v_ju_j$-path and teeth set of Z is $\{L^1, L^2, \cdots, L^{n+1}\}$, which contradicts the assumption that (G, R) does not contain a confined comb of length n. □

Proof of Theorem 5.3.1. From (1) to (2), by Lemma 5.3.2 (2), if any $(G, R) \in \mathcal{G}$ does not contain a path with n roots as a rooted minor, then (G, R) does neither contain a path with n roots nor contain a confined comb of length n as a subgraph.

By Theorem 4.2.1, if no member of $\mathcal{G}$ contains a confined comb of length n, then there exists $k \in \mathbb{N}$ such that for every $\mathcal{G} \subseteq \Phi^k(\mathcal{G}_0)$, where $\mathcal{G}_0$ consists of rooted

connected graphs (G, R) such that $G = \oplus_2(G_0; G_1, \cdots, G_t)$, where $R \subseteq V(\mathcal{G}_0)$ and $(G_0, R) \in \mathcal{P}$, for $t \geq 0$. For every member in $\mathcal{G}$ does not contain a path with n roots as a subgraph , it means that $(G_0, R) \in \mathcal{P}$ contains a path with less than $n-1$ roots. By the construction of the operation Φ, every $(G_0, R) \in \Phi^{n-1}(\mathcal{G}_0)$ and then every $\mathcal{G} \subseteq \Phi^{k+n-1}(\mathcal{G}_0)$, where $\mathcal{G}_0$ is a class of connected graphs. Since $2(k+n-1)-1 > n$, then $s = n-1$.

From (2) to (3), for every $(G, R) \in \Phi^s(\mathcal{G}_0)$, where $\mathcal{G}_0 = \{(G, \emptyset) : G \text{ is connected}\}$, let mutually disjoint subgraphs $H_1, \cdots, H_t$ of G such that $H_i \in \mathcal{G}_0$. Let the neighbor vertex set of H_i in G be A_i. We add missing edges between vertices in A_i to obtain a clique M_i. Let G_i be $G[V(H_i) \cup A_i] \cup M_i$. Let $G_0 = G \cup \bigcup_{i=1}^{t} M_i \setminus \bigcup_{i=1}^{t} H_i$. By the definition of $\oplus$, $G = \oplus(G_0; G_1, \cdots, G_t)$. By the definition of operation Φ, $R \subseteq V(G_0)$, and G_0 has no path with $2s$ roots, which means $h = 2s$.

From (3) to (1), let $n = 2f_{4.2.1}(f_{5.2.3}(h))$. By Lemma 5.3.2 (1), we prove that (G, R) neither contains a path with $f_{4.2.1}(f_{5.2.3}(h))$ roots nor contains a confined comb of length $f_{4.2.1}(f_{5.2.3}(h))$ as a subgraph. Firstly, we prove that (G, R) does not contain a path with $f_{4.2.1}(f_{5.2.3}(h))$ roots as a subgraph. Since $R \subseteq V(G_0)$ and G_0 does not contain a path of length $h < f_{4.2.1}(f_{5.2.3}(h))$. By Corollary 5.2.3, $G_0 \in \Phi^{f_{5.2.3}(h)}(K_1)$ and then $(G_0, V(G_0)) \in \Phi^{f_{5.2.3}(h)}(K_1)$ where K_1 is a root. Since $(G_0, R) \in \Phi^{f_{5.2.3}(h)}(\mathcal{P})$. By Theorem 4.2.1, (G, R) does not contain a confined comb of length $f_{4.2.1}(f_{5.2.3}(h))$. $\square$

Let $R = V(G)$, we apply Theorem 5.3.1 to obtain Corollary 5.3.3.

Corollary 5.3.3. *The following are equivalent for any class $\mathcal{G}$ of connected graphs.*

(1) there exists an integer $n > 0$ such that no member of $\mathcal{G}$ has a path of length n as a minor;

(2) there exists an integer $s > 0$ such that $\mathcal{G} \subseteq \Phi^s(\{K_1\})$;

(3) there exists an integer $h > 0$ such that for every $G \in \mathcal{G}$ has a spanning normal tree of height at most h.

6 Excluding a Large Star

In previous chapters, we have seen several results involving large stars. In particular, Theorem 1.4.2 says that a characterization on excluding a large star as a minor. Also, Theorem 1.5.2 says that a large star is one of the unavoidable induced subgraphs in a sufficiently large connected graph. Besides, Theorem 1.5.3 says that a large star is one of the unavoidable subgraphs in a tree with many leaves. Further, Theorem 2.1.1 says that a large confined star is one of the unavoidable induced subgraphs in a sufficiently large rooted connected graph. In this chapter, we keep on trying to find a characterization of rooted connected graphs excluding a subdivision of a large nicely confined star as a subgraph. Unfortunately, we just provide one conjecture on such characterization. Yet we provide the characterizations of rooted connected graphs excluding a subdivision of a large nicely confined $K_{1,4}$, $K_{1,5}$ as a subgraph. Also, we state the characterization of rooted connected graphs excluding a large confined star as a rooted minor.

6.1 One conjecture

We provide Conjecture 6.1.1, which characterizes rooted connected graphs that do not have a subdivision of a large nicely confined star as a subgraph.

Conjecture 6.1.1. *The following are equivalent for any class $\mathcal{G}$ of rooted connected graphs.*

(1) there exists an integer $n > 2$ such that no member of $\mathcal{G}$ contains a subdivision of a nicely confined $K_{1,n}$ as a subgraph;

(2) there exist integers $d, s > 0$ such that, for every $(G, R) \in \mathcal{G}$, G can be expressed as $\oplus_s(G_0; G_1, \cdots, G_t)$, where G_0 contains R and $d_{G_0}(x) \leq d$ for every vertex x of $G_0 \backslash R$.

For $n = 3$, a characterization of rooted connected graphs excluding a subdivision of a large nicely confined $K_{1,3}$ as a subgraph is Lemma 4.1.2. Let $s = 2$ and $d = 1$, for every $(G, R) \in \mathcal{G}$, $G = \oplus_2(G_0; G_1, \cdots, G_t)$, where $R \subseteq V(G_0)$ and $d_{G_0}(x) = 0 < 1$ for every vertex $x \in V(G_0 \backslash R)$. It means that Conjecture 6.1.1 is true for $n = 3$.

In the following sections, we provide characterizations of rooted connected graphs excluding a subdivision of a nicely confined $K_{1,4}$ as a subgraph, and we provide the structure of rooted connected graphs without a subdivision of a nicely confined $K_{1,5}$ subgraph.

6.2 Excluding a nicely confined $K_{1,4}$

In this section, our goal is to provide a characterization on rooted connected graphs that do not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph.

Let (G, R) be a rooted connected graph. We call k-separation (G_1, G_2) of (G, R) *rich* if $d_{G_1}(x) \geq 2$ for every $x \in V(G_1 \cap G_2)$. Let $V(G_1 \cap G_2) = \{x_1, x_2, \cdots, x_k\}$, we denote $G_i^+ = G_i \cup_{i \neq j}^k x_i x_j$, for an integer $k > 0$ and $i = 1, 2$.

Theorem 6.2.1. *The following are equivalent for any class $\mathcal{G}$ of rooted connected graphs.*

(1) no member of $\mathcal{G}$ contains a subdivision of a nicely confined $K_{1,4}$ as a subgraph;

(2) for every $(G, R) \in \mathcal{G}$, G can be expressed as $\oplus_3(G_0; G_1, \cdots, G_t)$, where $R \subseteq V(G_0)$ and $d_{G_0}(x) \leq 3$ for every vertex x in $G_0 \backslash R$.

To simplify the proof of Theorem 6.2.1, we first prove the following lemmas. The first lemma proves that for any rich 3-separation (G_1, G_2) of a rooted 3-connected graph (G, R) excluding a subdivision of a nicely confined $K_{1,4}$ as a subgraph, (G_2^+, R) does not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph.

Lemma 6.2.2. *Let (G, R) be a rooted 3-connected graph. If (G, R) does not contain a subdivision of a nicely confined $K_{1,t}$ as a subgraph, then for any rich 3-separation (G_1, G_2) of (G, R), (G_2^+, R) does not contain a subdivision of a nicely confined $K_{1,t}$ as a subgraph, for $t \geq 4$.*

Proof. Assume that (G_2^+, R) contains a subdivision of a nicely confined $K_{1,t}$ as a subgraph. Let the center vertex of such nicely confined star be v and $v \notin R$. Since $(G_1^+ \oplus G_2^+, R)$ does not contain a subdivision of a nicely confined $K_{1,t}$, which means there exists at least one another independent path from v to R in G_2^+. Hence there are at most two such independent paths and each path has one edge of xy, yz, xz in G_2^+. There are two cases on the number of such independent paths.

Case 1. There exists unique one such independent path P^1 from v to R in G_2^+. Without loss of generality, P^1 contains an edge xy. Since (G, R) is a rooted connected graph, so there exists one path P^2 between x, y in G_1. Hence there exists an independent path $P^1 \cup P^2 \backslash xy$ from v to R in $G_1 \cup G_2$, which contradicts that there is no a subdivision of a nicely confined $K_{1,t}$ in (G, R).

Case 2. There exist two such independent paths P'^1, P'^2 from v to R in G_2^+. Without lost of generality, P'^1 has an edge xy and P'^2 has an edge yz. By the definition of the independent path, $y = v$. Since (G, R) is a rooted connected graph, so there exist two paths P'^3, P'^4 between x, v, v, z in G_1.

Claim. P'^3 and P'^4 are independent paths from v to x, z in G_1.

Otherwise, $|P'^3 \cap P'^4| \geq 2$. Let $\{v, u\} \subseteq V(P'^3 \cap V'^4)$. Let P'^{31} be a subgraph of P'^3 with ends v, u, and P'^{41} be a subgraph of P'^4 with ends v, u. If $|P'^{31}| \geq 3$ or $|P'^{41}| \geq 3$, without loss of generality, let $w \in V(P'^{31})$ and $w \neq v, u$, then we can separate w from R by deleting v, u in (G, R), which contradicts that (G, R) is a rooted 3-connected graph. Hence $|P'^{31}| = |P'^{41}| = 2$. Since (G, R) is a simple

rooted connected graph, so $P'^{31} = P'^{41} = vu$. Then $d_{G_1}(v) = 1$, which contradicts that (G_1, G_2) is a rich separation of (G, R).

By Claim, P'^3, P'^4 are two independent paths from y to x, z in G_1, respectively. Similar to Case 1, there exist two independent paths $P'^1 \cup P'^3 \backslash xy$, $P'^2 \cup P'^4 \backslash yz$ from v to R in $G_1 \cup G_2$, which contradicts that there is no a subdivision of a nicely confined $K_{1,t}$ in (G, R). $\square$

The second lemma is about determining non-rich 3-separation of a rooted 3-connected graph.

Lemma 6.2.3. *Let (G, R) be a rooted 3-connected graph. If there is no rich 3-separation (G_1, G_2) of (G, R), then G_1 must be one of these following graphs:*

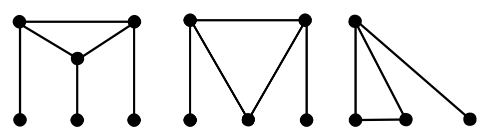

FIGURE 6.1. $G_{1,1}, G_{1,2}, G_{1,3}$

Proof. Let neighbors of x, y, z in G_1 be x', y', z'. Since any 3-separation (G_1, G_2) is not rich, so there exists at least one vertex of $V(G_1 \cap G_2)$ such that its degree is one in G_1. Without loss of generality, we assume that $d_{G_1}(x) = 1$.

Claim. $d_{G_1}(x') = d_{G_1}(y') = d_{G_1}(z') = 3$.

Suppose that there exists at least one vertex of x', y', z' whose degree is not three in G_1, without loss of generality, let $d_{G_1}(x') \neq 3$.

Suppose $d_{G_1}(x') \leq 2$, which means there are at most two independents paths from x' to R in (G, R), which contradicts that (G, R) is rooted 3-connected graph.

Suppose $d_{G_1}(x') \geq 4$.

If there exists another vertex v rather than x, y, z, x', y', z' in G_1, there are three cases on the degree of y, z in G_1.

Case 1.1. $d_{G_1}(y) \geq 2$ and $d_{G_1}(z) \geq 2$.

There exists a rich 3-separation $(G_1\backslash xx', G_2 + xx')$ to separate v from R, which contradicts that there is no rich 3-separation in (G, R).

Case 1.2. $d_{G_1}(y) = 1$ and $d_{G_1}(z) \geq 2$.

Then $d_{G_1}(y') \geq 3$, there exists a rich 3-separation $(G_1\backslash xx'\backslash yy', G_2+xx'+yy')$ to separate v from R, which contradicts that there is no rich 3-separation in (G, R).

Case 1.3. $d_{G_1}(y) = 1$ and $d_{G_1}(z) = 1$.

Then $d_{G_1}(y') \geq 3$ and $d_{G_1}(z') \geq 3$, we can find a rich 3-separation $(G_1\backslash xx'\backslash yy'\backslash zz'$, $G_2 + xx' + yy' + zz')$ to separate v from R, which contradicts with the assumption that there is no rich 3-separation in (G, R).

If there exists no vertex v rather than x, y, z, x', y', z' in G_1, then there are two cases on neighbor vertices of x' in G_1.

Case 2.1. The neighbor vertex set of x' in G_1 is $\{x, y, z, y'\}$.

Then $d_{G_1}(y) \geq 2$ and $d_{G_1}(z) \geq 2$, there exits a rich 3-separation $(G_1\backslash xx', G_2 + xx')$ to separate y' to R in (G, R), which contradicts that there is no rich 3-separation in (G, R).

Case 2.2. The neighbor vertex set of x' in G_1 is $\{x, y, y', z'\}$.

Then $d_{G_1}(y) \geq 2$ and $d_{G_1}(z') \geq 2$, there exists a rich 3-separation $(G_1\backslash xx'\backslash zz', G_2+ xx' + zz')$ to separate y' to R in (G, R), which contradicts that there is no rich 3-separation in (G, R). We are done with Claim.

By Claim, there are three cases on the degree of x, y, z in G_1, since (G_1, G_2) is a non-rich 3-separation of (G, R).

Case 1. $d_{G_1}(x) = d_{G_1}(y) = d_{G_1}(z) = 1$. By Claim, we can get the graph $G_1 = G_{11}$ described in the lemma.

Case 2. $d_{G_1}(x) = d_{G_1}(y) = 1$. By Claim, we can get the graph $G_1 = G_{12}$ described in the lemma.

Case 3. $d_{G_1}(x) = 1$. By Claim, we can get the graph $G_1 = G_{13}$ described in the lemma. $\square$

The third lemma is about determining the degree of non-root vertex in a rooted 3-connected graph, which excludes a subdivision of a nicely confined $K_{1,4}$ as a subgraph, and does not contain a rich 3-separation.

Lemma 6.2.4. *Let (G, R) be a rooted 3-connected graph. If there is no rich 3-separation of (G, R), then $d_G(v) = 3$ for all $v \notin R$.*

Proof. Since (G, R) is a rooted 3-connected graph, so $d_G(v) \leq 3$ for all $v \notin R$. By Lemma 6.2.3, $d_{G_1}(v) = 3$. $\square$

Proof of Theorem 6.2.1. From (1) to (2), for any (G, R) does not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph, we prove that (G, R) admits an expression as described in (2).

Claim 1. If any 1-separation (G_1, G_2) of (G, R) does not contain a subdivision of a nicely confined $K_{1,t}$ as a subgraph, then (G_2, R) does not contain a subdivision of a nicely confined $K_{1,t}$ as a subgraph, for $t \geq 4$.

By Menger's Theorem, if there exists at most one path from $v \notin R$ to R in G, then there exits 1-separation (G_1, G_2) to separate v from R, where $R \subseteq V(G_2)$.

By Claim 1, we can separate all these kind of v from R successively, including $G_{1,1}, G_{1,2}, \cdots, G_{1,t_1}$, to obtain (G_1^*, R) such that there is no 1-separation of (G_1^*, R) to separate non-root vertex from R in (G_1^*, R), for $t_1 \geq 1$. Then $G = \oplus_1(G_1^*; G_{1,1}, G_{1,2}, \cdots, G_{1,t_1})$, where $R \subseteq V(G_1^*)$ and $d_{G_1^*}(x) \leq 3$ for every vertex $x \in V(G_1^* \backslash R)$, for $1 \leq i \leq t_1$.

Claim 2. If any 2-separation (G_1, G_2) of (G_1^*, R) does not contain a subdivision of a nicely confined $K_{1,t}$ as a subgraph, then (G_2^+, R) does not contain a subdivision of a nicely confined $K_{1,t}$ as a subgraph, for $t \geq 4$.

Assume that (G_2^+, R) contains a subdivision of a nicely confined $K_{1,t}$ as a subgraph. Let the center vertex of such subdivision of a nicely confined $K_{1,t}$ be v and $v \notin R$. Since (G_1^*, R) does not contain a subdivision of a nicely confined $K_{1,t}$, which means there exists another independent path P^1 from v to R in G_2^+. There exists one edge xy of P^1 such that xy in G_2^+. Since (G_1^*, R) is a rooted connected graph, so there exists one path P^2 between x and y in G_1. Hence there exists an independent path $P^1 \cup P^2 \backslash xy$ from v to R in (G_1^*, R), which contradicts that P^1 is another independent path from v to R in G_2^+ than in (G_1^*, R).

By Claim 2, we can separate all non-root vertices in (G_1^{*+}, R) which has at most two independent paths from it to R successively, including $G_{2,1}^+, G_{2,2}^+, \cdots, G_{2,t_2}^+$, to obtain (G_2^{*+}, R) such that there is no 2-separation of (G_2^{*+}, R) to separate any non-root vertex to R in (G_2^{*+}, R), for $t_2 \geq 1$. Then $G_1^* = \oplus_2(G_2^{*+}; G_{2,1}^+, G_{2,2}^+, \cdots, G_{2,t_2}^+)$, where $R \subseteq V(G_2^{*+})$ and $d_{G_2^{*+}}(x) \leq 3$ for every vertex $x \in V(G_2^{*+} \backslash R)$, for $1 \leq i \leq t_2$.

By Lemma 6.2.2, (G_2^{*+}, R) does not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph, which means that every non-root vertex in (G_2^{*+}, R) has at most three independent paths to R in G_2^{*+}. We can separate all non-root vertices in (G_2^{*+}, R) to R successively, including $G_{3,1}^+, G_{3,2}^+, \cdots, G_{3,t_3}^+$, to obtain (G_3^{*+}, R) such that there is no rich 3-separation in (G_3^{*+}, R), for $t_3 \geq 1$. Then $G_2^{*+} = \oplus_3(G_3^{*+}; G_{3,1}^+, G_{3,2}^+, \cdots, G_{3,t_3}^+)$.

By Lemma 6.2.4, $d_{G_3^{*+}}(x) = 3$ for every $x \in V(G_3^* \backslash R)$. Let $G_0 = G_3^{*+}$. Then $G = \oplus_3(G_0; G_{1,1}, \cdots, G_{1,t_1}, G_{2,1}, \cdots, G_{2,t_2}, G_{3,1}, \cdots, G_{3,t_3})$ where $R \subseteq V(G_0)$, and $d_{G_0}(v) \leq 3$ for every $v \in V(G_0 \backslash R)$.

From (2) to (1), since $d_{G_0}(v) \leq 3$ for all $v \notin R$, which implies (G_0, R) does not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph. Next we prove that (G, R) does not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph.

Suppose that (G, R) contains a subdivision of a nicely confined $K_{1,4}$ as a subgraph, let the center vertex of it be v, which means there exists at least another independent path from v to R in $G \backslash G_0$ than G_0. There are three cases on k-sums of connected graph and G_0 to obtain G having another independent path from v to R.

Case 1. There exists one 1-sum of G_1 and G_0 such that G_1 contains another independent path P from v to R. It is impossible to find P, since v and R are both in G_0, which contradicts that $d_{G_0}(v) = 3$.

Case 2. There exists one 2-sum of G_2 and G_0 such that $G_0 \oplus_2 G_2$ contains another independent path P' from v to R, where $P'^1 = G_2[V(P')]$. Let $V(G_2 \cap G_0) = \{v_1, u_1\}$, so $v_1, u_1 \in V(P'^1)$. Hence there exists another independent path $P' \backslash P'^1 + v_1u_1$ from v to R in G_0, which contradicts that $d_{G_0}(v) = 3$.

Case 3. There exists one 3-sum of G_3 and G_0 such that $G_3 \cup G_0$ contains another independent path P'' from v to R, where $P''^1 = G_3[V(P'')]$. Let $V(G_3 \cap G_0) = \{v_2, u_2, t_2\}$, so there are at least two of $\{v_2, u_2, t_2\}$ in P''^1. If P''^1 exactly passes two vertices, without loss of generality, such as v_2, u_2. Then there exists another independent path $P'' \backslash P''^1 + v_2u_2$ from v to R in G_0, which contradicts there is another independent path from v to R in G_0. If P'' passes three vertices v_2, u_2, t_2, without loss of generality, $P''^1 + v_2t_2$ is a cycle in $G_3 \cup G_0$, then there exists one other independent path $P'' \backslash P''^1 + v_2t_2$ in G_0, which contradicts the assumption that $d_{G_0}(v) = 3$.

Hence $G = \oplus_3(G_0; G_1, G_2, \cdots, G_t)$ does not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph, where $R \subseteq V(G_0)$ and $d_{G_0}(x) \leq 3$ for every $x \in V(G_0 \backslash R)$ if (G_0, R) does not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph. $\square$

6.3 Excluding a nicely confined star $K_{1,5}$

In the last section, we prove Theorem 6.2.1, which solved the problem of a characterization on rooted connected graphs that do not contain a subdivision of a nicely confined $K_{1,4}$ as a subgraph. In this section, our goal is to provide the structure of rooted connected graphs that do not contain a subdivision of a nicely confined $K_{1,5}$ as a subgraph.

We say that two separations (G_1, G_2) and (H_1, H_2) of G *cross* each other if $V(G_i \cap H_j)\backslash V_0 \neq \emptyset$ for all i, j, where $V_0 = V(G_1 \cap G_2 \cap H_1 \cap H_2)$.

Let (G, R) be a rooted connected graph, and let $R \subseteq V(G)$ with $|R| > 5$ and let $V_{16} = \{v \in V(G)\backslash R : d_G(v) \geq 16\}$. We assume that $V_{16} \neq \emptyset$.

A *carving* is a set $\mathcal{C}$ of 4-separations (G_1, G_2) of G with $R \subseteq V(G_2)$. We call $\mathcal{C}$ *complete* if for every $v \in V_{16}$, there exists $(G_1, G_2) \in \mathcal{C}$ with $v \in V(G_1)$ and $|N_G(v)\backslash V(G_1)| \leq 2$.

For $n = 5$, we can get the following theorem.

Theorem 6.3.1. *Let any class $\mathcal{G}$ of rooted connected graphs, no member of $\mathcal{G}$ contains a subdivision of a nicely confined $K_{1,5}$ as a subgraph. Then for $(G, R) \in \mathcal{G}$, G can be expressed as $\oplus_{15}(G_0; G_1, \cdots, G_t)$, where $R \subseteq V(G_0)$ and $d_{G_0}(x) \leq 15$ for every vertex x of $G_0\backslash R$.*

Proof. By Claim 1 from Theorem 6.2.1, there exist $G_{1,1}, \cdots, G_{1,s_1}$ such that $G = \oplus_1(G_1^*; G_{1,1}, G_{1,2}, \cdots, G_{1,s_1})$ such that G_1^* does not contain any subdivision of a nicely confined $K_{1,5}$ as a subgraph and $R \subseteq G_1^*$. By Claim 2 from Theorem 6.2.1, there exist $G_{2,1}^+, \cdots, G_{2,s_2}^+$ such that $G_1^* = \oplus_2(G_2^{*+}; G_{2,1}^+, G_{2,2}^+, \cdots, G_{2,s_2}^+)$ such that G_2^{*+} does not contain a subdivision of a nicely confined $K_{1,5}$ as a subgraph and $R \subseteq G_2^*$. By Lemma 6.2.3, there exist $G_{3,1}^+, \cdots, G_{3,s_3}^+$ such that

$G_2^{*+} = \oplus_3(G_3^{*+}; G_{3,1}^+, G_{3,2}^+, \cdots, G_{3,s_3}^+)$ such that G_3^{*+} does not contain any subdivision of a nicely confined $K_{1,5}$ as a subgraph.

Suppose there is no $v \in V_{16}$ such that G_3^{*+} has five independent paths from v to R. Then there exists a complete carving $\mathcal{C}$. Let us choose $\mathcal{C}$ such that:

(i) $|\mathcal{C}|$ is minimized;

(ii) subject to (i), the sum of $|G_1|$ over all $(G_1, G_2) \in \mathcal{C}$ is minimized.

For any two distinct members $(G_1, G_2), (H_1, H_2)$ of $\mathcal{C}$, their relative positions must be the following:

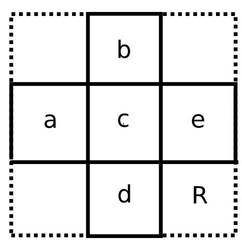

FIGURE 6.2. two distinct (G_1, G_2), $(H_1, H_2) \in \mathcal{C}$

Let $|V(G_1 \cap G_2)| = a + c + e$ and $|V(H_1 \cap H_2)| = b + c + d$, where $R \subseteq V(G_2 \cap H_2)$. Since (G_1, G_2), (H_1, H_2) are 4-separations of (G, R), which means $a + c + e = b + c + d = 4$.

By (i), for any distinct (G_1, G_2), $(H_1, H_2) \in \mathcal{C}$, there exists one separation $(G_1 \cup H_1, G_2 \cap H_2)$ has order at least 5, which implies $c + d + e \geq 5$ and $a + b + c \leq 3$. Otherwise, we can replace (G_1, G_2), (H_1, H_2) with $(G_1 \cup H_1, G_2 \cap H_2)$ to separate any member of V_{15} from R, which contradicts that $|\mathcal{C}|$ is minimized.

Without loss of generality, we assume that $a \geq b$. There are five cases on a, b, c

Case 1. $a \neq b$. It means $b + c + e \leq 3$, since $b + c + e = 4 - a + b$ and $a > b$. By the above proof, we know that $a + b + c \leq 3$ and $b + c + e \leq 3$. By Lemma 6.2.2,

Case 1 does not happen since the horizontal cut does not separate any vertex of V_{16} from R.

Case 2. $a = b = 0$ and $c \leq 2$. Case 2 does not happen since the two separations do not cross.

Case 3. $a = b = 1$ and $c = 0$. We can replace the vertical cut to improve (ii).

Case 4. $a = b = 1$ and $c = 1$. We can replace either the vertical cut or the horizontal cut to improve (ii).

Case 5. $a = b = 0$ and $c = 3$. Then these two separations cross. So modulo these 3-separations $\mathcal{C}$ is cross-free. So we created a new $\mathcal{C}'$ such that if there exist three members of the cut of every separation $(G_1, G_2) \in \mathcal{C}$ is the cut of 3-separation $(G_{1,1}, G_{2,1})$ of (G, R), then we create a new separation $(G_1 + G_{1,1}, G_2 + G_{1,1})$ sequently. By Lemma 6.2.2, any separation in $\mathcal{C}'$ has order at most $4 + 3 \times 4 = 16$.

Claim 1. $\mathcal{C}'$ is cross-free.

There are two cases on the relationship between two distinct $(G_1, G_2), (H_1, H_2) \in \mathcal{C}$.

Case 1. (G_1, G_2) and (H_1, H_2) are crossing. By Case 5, we know that the relative positions of them are $a = 0, b = 0, c = 3$.

If any 4-separation (G_1, G_2) of (G, R) satisfying $a = 0, b = 0, c = 3$, and we can obtain a new separation $(G_1 \cup G_{1,1}, G_2 \cup G_{1,1})$ if there exists three members of $G_1 \cap G_2$ satisfying those are the cut of 3-separation $(G_{1,1}, G_{2,1})$ of (G, R).

We can obtain a new separation (G_1', G_2') such that $G_1' = G_1 + G_{1,1} + G_{1,2} + \cdots + G_{1,t}$, $G_2' = G_2 \cup G_{1,1} \cup G_{1,2} \cup \cdots \cup G_{1,t}$ if there exists three numbers of $G_1 \cap G_2$ satisfying those are the cut of 3-separation $(G_{1,t}, G_{2,t})$ of (G, R), for $t \leq 4$. By Lemma 6.2.2, $|G_1' \cap G_2'|$ is at most $4 + 3 \times 4 = 16$.

Similarly, we can obtain a new separation (H_1', H_2') from (H_1, H_2).

Then (G_1', G_2') and (H_1', H_2') are uncrossing. By Lemma 6.2.2, the new horizontal cut and the new vertical cut do not separate any vertex of V_{15} from R, since the degree of any vertex in new horizontal cut or vertical cut is at most $2 \times (3+3-1)+4 = 14$.

Case 2. (G_1, G_2) and (H_1', H_2') are uncrossing. Similarly to Case 1.

By Claim 1, we can separate any vertex of V_{15} from R by any member of $\mathcal{C}'$, we are done. □

6.4 Excluding a large confined star rooted minor

In the last chapter, Theorem 5.3.1 says that a characterization of rooted connected graphs that do not contain a heavy path as a rooted minor. In this section, our goal is to provide a characterization on rooted connected graphs that do not contain a large confined star as a rooted minor.

Theorem 6.4.1. *The following are equivalent for any class $\mathcal{G}$ of rooted connected graphs.*

(1) there exists an integer $n > 2$ such that every $(G, R) \in \mathcal{G}$ does not contain a confined $K_{1,n}$ as a rooted minor;

(2) there exists an integer $k > 0$ such that every $(G, R) \in \mathcal{G}$ can be constructed from a graph G_0 on at most k edges by subdividing some edges to obtain G_1, and then clique-summing graphs to G_1 to obtain G and finally setting $R \subseteq V(G_1)$.

Proof. From (1) to (2), let $k = f_{1.4.2}(n)$. Let H be the graph with $V(H) = R$ and $E(H) = \{uv$: if there exists a R-bridge containing u, v in $G\}$. By the connectivity of G, H is connected. By Lemma 1.1.1, there exists a spanning tree T of H.

Claim 1. T has at most $n-1$ leaves.

Otherwise, there are at least n leaves in T, let the leaves set be $\{x_1, x_2, \cdots, x_t\}$, for $t \geq n$. Hence there exists a rooted minor (F, Q) of (G, R), where F is the union of R-bridges of all edges from T and $Q = \{x_1, x_2, \cdots, x_t\}$. Since the connectivity of tree, it is obviously to know that F is connected and $|Q| \geq n$. Since (F, Q) has $(K_{1,t}, Q)$ as a rooted minor. By Lemma 1.2.1, $K_{1,t}$ is a minor of G and $(K_{1,t}, Q)$ is a rooted minor of (G, R), which contradicts the assumption that there is no a confined $K_{1,n}$ as a rooted minor of (G, R).

Claim 2. H is a connected simple $K_{1,n}$-free graph.

By Claim 1, there is no n leaves in any spanning tree of H and H is a simple graph, since G is a simple graph.

By Theorem 1.4.2 and Claim 2, there exists a function $f_{1.4.2}(n)$ such that H is a subdivision of a connected simple graph G_0 on fewer than $f_{1.4.2}(n)$ vertices. And $G_1 = H$, and $G = \oplus(G_1; H_1, \cdots, H_t)$ where each H_i is an H-bridge and $R = V(G_1)$.

From (2) to (1), let $n = g(k) = k^2 + 1$. Since every connected simple graph with k vertices has no more than $k(k-1)/2$ edges. G_1 is obtained by subdividing edges of G_0, G is obtained by clique-summing graphs to G_1. Let T_G be a tree in G and L be the set of leaves of T_G such that $L \subseteq R$. Let $x_i y_i$-path in G_1 by subdividing each edge $x_i y_i$ of G_0.

Claim 3. There are at most 2 leaves of T_G on each path $x_i r_{i,1} r_{i,2} \cdots r_{i,t_i} y_i$ rather than x_i, y_i in G.

Suppose that there are at least 3 leaves r_a, r_b, r_c of T_G on some path $x_1 r_1 r_2 \cdots r_t y_1$ in G, where $1 \leq a < b < c \leq t$. Since G is obtained by clique-summing $G_1, \cdots, G_s$ to G_1, then $G \backslash r_a \backslash r_c$ is splitted G into two disjoint subgraphs H_1, H_2, H_1 is a path $r_{a+1} r_{a+2} \cdots r_{c-1}$ by clique-summing graphs $G_1, \cdots$, and $H_2 = G \backslash H_1 \backslash r_a \backslash r_c$. Since any tree deleting leaves is connected, so $T_G \backslash r_a \backslash r_c$ is a subgraph of H_1. By the

construction of H_1, $H_1 \backslash r_b$ is not connected. If $T_G \backslash r_a \backslash r_b \backslash r_c$ is a subgraph of one of disjoint subgraphs $H_{1,1}, H_{1,2}$ of $H_1 \backslash r_b$, which implies that T_G is the subgraph of $G[V(H_{1,1}) \cup r_a \cup r_b]$, then r_c is not a leaf of T_G. It contradicts that r_c is a leaf of T_G.

There are at most $k(k-1)/2$ edges and k vertices in G_0, by Claim 3, any tree in G with at most $k(k-1)+k$ leaves in R. If $n = k^2+1$, then (G, R) does not contain a confined $K_{1,n}$ as a rooted minor. $\square$

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Vita

Shilin Wang was born on August 27 1986, in Jilin City, Jilin. She finished her undergraduate studies at South China Normal University June 2009. She earned a master of science degree in mathematics from South China Normal University in June 2012. She earned a master of science degree in mathematics from Louisiana State University in May 2014. She also earned a master of applied statistics degree in experimental statistics department from Louisiana State University in Dec 2017. In August 2012 she came to Louisiana State University to pursue graduate studies in mathematics. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May 2018.