On Matroid Connectivity.

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On matroid connectivity

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by

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This thesis is dedicated to my parents, my wife Linda, my son Blake, and my brothers Haitham and Ghassan.

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Certain classes of 2- and 3-connected matroids are studied in this thesis.

In Chapter 2 we give a characterization of those 2-connected matroids $M$ with the property that, for a given positive integer $m$, the deletion of every non-empty subset of $M$ having at most $m$ elements is disconnected. A bound on the maximum number of elements of such a matroid in terms of its rank is also given, along with a complete description of the matroids attaining this bound. These results extend results of Murty and Oxley for minimally 2-connected matroids.

A characterization of the 3-connected matroids $M$ that have the property that every 2-element deletion of $M$ is disconnected is given in Chapter 3. It is shown that these matroids are exactly the duals of Sylvester matroids having at least four elements.

In Chapter 4 we prove the following result: Let $M$ be a 3-connected matroid other than a wheel of rank greater than three, and let $C$ be a circuit of $M$. If the deletion of every pair of elements of $C$ is disconnected, then every pair of elements of $C$ is contained in a triad of $M$.

For an integer $t$ greater than one, an $n$-element matroid $M$ is $t$-cocyclic if every deletion having at least $n - t + 1$ elements is 2-connected, and every deletion having exactly $n - t$ elements is disconnected. A matroid is $t$-cyclic if its dual is $t$-cocyclic. In Chapter 5 we investigate the matroids that are both $t$-cocyclic and
t-cyclic. It is shown that these matroids are exactly the uniform matroids $U[t, 2t]$ and the Steiner Systems $S(t, t+1, 2t+2)$. 
CHAPTER 1

INTRODUCTORY NOTIONS

The purpose of this chapter is to establish notation and to give some basic definitions and theorems that are referred to in the succeeding chapters. We shall use the ordinary set-theoretic terminology, and we note that all the sets that appear in this work are finite. For graph-theoretic terminology, we refer the reader to Bondy and Murty [2]. The theorems for which we do not cite a reference can be found in Welsh [25].

A matroid \( M \) on a ground set \( E \) is a collection \( \mathcal{C} \) of non-empty subsets of \( E \), called circuits, satisfying (C1) and (C2).

(C1) *Minimality.* No proper subset of a circuit is a circuit.

(C2) *Circuit elimination.* Let \( C, C' \) be distinct circuits and \( f \) be an element of \( C - C' \). If \( e \) is an element of \( C \cap C' \), then there is a circuit \( C'' \) such that \( f \) is an element of \( C'' \) and \( C'' \) is contained in \( (C \cup C') - e \).

A circuit \( C \) of cardinality \( m \) is called an \( m \)-circuit and is denoted by \( C_m \). A 1-circuit is a loop. The elements of a 2-circuit are said to be in parallel. The set consisting of an element \( e \) and all elements parallel to it is called a parallel class. A matroid
is simple if it has no loops or parallel elements. A 3-circuit is called a triangle.

A subset \( I \) of \( E \) is independent if it contains no circuits. A maximal independent subset \( B \) of \( E \) is called a base. All bases of a matroid have the same cardinality, see, for example, Welsh [25].

Let \( A \) be a subset of \( E \) and \( I \) be an independent subset of \( A \) such that \( |I| \) is maximal. Then the rank of \( A \), denoted by \( \text{rk}A \), is the cardinality of \( I \). The rank of a matroid \( M \), denoted by \( \text{rk}M \), is the rank of its ground set \( E \). Thus \( \text{rk}M \) is the common cardinality of the bases of \( M \). We call \( \text{rk} \) the rank function of \( M \).

(1.1) Theorem. Let \( \text{rk} \) be the rank function of a matroid \( M \). Then the function \( \text{rk} \) has the following properties.

(R1) For all subsets \( A \) of \( E(M) \), \( 0 \leq \text{rk}A \leq |A| \).

(R2) If \( A, A' \) are subsets of \( E(M) \) such that \( A \subseteq A' \), then \( \text{rk}A \leq \text{rk}A' \).

(R3) If \( A, A' \) are subsets of \( E(M) \), then \( \text{rk}A + \text{rk}A' \geq \text{rk}(A \cup A') + \text{rk}(A \cap A') \).

Let \( E \) and \( E' \) be the ground sets of the matroids \( M \) and \( M' \), respectively. \( M \) and \( M' \) are isomorphic if there is a bijection \( \Phi : E \to E' \) such that \( C \) is a circuit of \( M \) if and only if \( \Phi(C) \) is a circuit of \( M' \). If no confusion arises, we will write \( M = M' \) when \( M \) and \( M' \) are isomorphic or identical.

If \( A \) is a subset of \( E \) and \( e \) is an element of \( E \), then \( e \) depends on \( A \) if \( \text{rk}A = \text{rk}(A \cup e) \). In other words, \( e \) depends on \( A \) if \( e \) is an
element of $A$ or $e$ is an element of a circuit $C$ which is contained in $A \cup e$. The closure of a set $A$, denoted by $\overline{A}$, is $A \cup \{ e \mid e \text{ depends on } A \}$. A set $A$ is closed or a flat if $A = \overline{A}$. A hyperplane is a flat of rank $\text{rk} M - 1$.

SECTION 2. DUALS, DELETIONS AND CONTRACTIONS

There are several ways of defining new matroids from some given set of matroids. In this section, we will describe how to induce three types of matroids from a given matroid $M$. In a later section, we will describe various ways of obtaining a new matroid from a given collection of matroids.

Let $M$ be a matroid on a set $E$ and $\{ B_i \}$ be the collection of bases of $M$ where $i$ is an element of some index set $I$. Whitney [26] proved the following fundamental result.

(1.2) Theorem. The collection of sets $\{ E - B_i \}$ is the collection of bases of a matroid $M^*$ on $E$.

We call $M^*$ the dual matroid of $M$. The bases, circuits, and hyperplanes of $M^*$ are called cobases, cocircuits and cohyperplanes of $M$, respectively. A coloop of $M$ is a loop of $M^*$. A triad is a 3-cocircuit. The elements of a 2-cocircuit are said to be in series. The set consisting of an element $e$ and all elements in series with it is called a series class. It follows from the
definition of the dual that the rank of $M^*$, or $corkM$, is $|E| - rkM$.

This is a special case of the following theorem.

(1.3) Theorem. The rank functions $rk$, $cork$ of $M$, $M^*$ respectively are related by: $corkA = |A| + rk(E - A) - rkM$ where $A$ is a subset of $E$.

From the last theorem it is easy to derive the following result.

(1.4) Theorem. Let $\{X, Y\}$ be a bipartition of the ground set $E$ of a matroid $M$. Then $rkX + rkY - rkM = corkX + corkY - corkM = rkX + corkX - |X|$.

The following property, called orthogonality, is often used in the later chapters.

(1.5) Theorem. Let $C$ be a circuit and $C^*$ be a cocircuit of a matroid $M$. Then $|C \cap C^*| \neq 1$.

The following is a very useful relationship between hyperplanes and cocircuits.

(1.6) Theorem. A set $H$ is a hyperplane of a matroid $M$ if and only if $E - H$ is a cocircuit of $M$. Dually, a set $H^*$ is a cohyperplane of a matroid $M$ if and only if $E - H^*$ is a circuit of $M$. 
It is clear that every proper flat $A$ of a matroid $M$ is contained in some hyperplane $H$ of $M$. Thus the following result follows from the last theorem.

\[(1.7)\text{ Corollary. Let } A \text{ be a proper flat of a matroid } M. \text{ Then } E - A \text{ contains a cocircuit of } M.\]

In fact, by using Theorem 1, page 38, of Welsh [25], we can easily show that the following theorem holds.

\[(1.8)\text{ Theorem. The set } A \text{ is a flat of a matroid } M \text{ if and only if } E - A \text{ is a union of cocircuits of } M.\]

Let $A$ be a subset of $E$ and suppose $A = \{e_i, e_2, \ldots, e_k\}$. There are two natural ways to obtain matroids on the set $E - A$.

\[(1.9)\text{ Theorem. The circuits of } M \text{ that are contained in } E - A \text{ are the circuits of a matroid } M \setminus A \text{ on } E - A.\]

We say that $M \setminus A$ is the matroid obtained from $M$ by deleting the set $A$. We also write $M \setminus A$ as $M \setminus e_1, e_2, \ldots, e_k$. It is clear that the rank of the matroid $M \setminus A$ is $rk(E - A)$.

\[(1.10)\text{ Theorem. Let } \zeta' \text{ be the collection of minimal non-empty sets of the form } C \cap A \text{ where } C \text{ is a circuit of } M. \text{ Then} \]
\( \zeta' \) is the collection of circuits of a matroid \( M/A \) on \( E - A \).

We say that \( M/A \) is the matroid obtained from \( M \) by contracting the elements of \( A \). We also write \( M/A \) as \( M/e_1, e_2, \ldots, e_K \). From the definition of \( M/A \) we can prove the following theorem.

(1.11) Theorem. Let \( \text{rk}' \) be the rank function of \( M/A \) and \( F \) be a subset of \( E(M/A) \). Then \( \text{rk}'F = \text{rk}(F \cup A) - \text{rk}A \). In particular, \( \text{rk}'(M/A) = \text{rk}M - \text{rk}A \).

In the next theorem we state an important result relating contractions to deletions.

(1.12) Theorem. Let \( M \) be a matroid on \( E \) and \( A \) be a subset of \( E \). Then \( (M\setminus A)^* = M^*/A \) and \( (M/A)^* = M^*\setminus A \).

Next we give a basic property of contractions and deletions that we use implicitly in many theorems of this thesis.

(1.13) Theorem. Let \( M \) be a matroid on a set \( E \) and \( A, A' \) be disjoint subsets of \( E \). Then \( M\setminus A/A' = K/A'\setminus A, M\setminus A\setminus A' = M\setminus A'\setminus A, \) and \( M/A/A' = M/A'/A \), that is, the operations of contractions and deletions commute.
A matroid $N$ is a minor of a matroid $M$ if, for disjoint subsets $A, A'$ of $E(M)$, $N$ is isomorphic to $M \setminus A/A'$.

SECTION 3. CONNECTEDNESS

A matroid $M$ is connected if, for every subset $\{ e, f \}$ of $E(M)$, the set $\{ e, f \}$ is contained in a circuit $C$ of $M$. If $M$ is not connected, it is called disconnected or separable.

(1.14) Theorem. A matroid $M$ is connected if and only if its dual $M^*$ is connected.

A component $K$ of a matroid $M$ is a maximal subset of $E(M)$ having the property that every pair $\{ e, f \}$ of its elements is contained in a circuit of $M$. Thus if $M$ is disconnected, then $M$ has at least two components.

(1.15) Theorem. The components of a matroid $M$ form a partition of $E(M)$ and coincide with the components of the dual matroid $M^*$.

SECTION 4. THE CYCLE MATROID OF A GRAPH

For every graph $G$ there is a matroid associated with its cycles. We state this more precisely in the following theorem.
(1.16) Theorem. Let $G$ be a graph. Then the cycles of $G$ are the circuits of a matroid $M(G)$ on the set of edges $E(G)$.

A matroid is **graphic** if it is isomorphic to the cycle matroid of some graph. A matroid is **cographic** if its dual is isomorphic to the cycle matroid of some graph. We note here that a matroid can be the cycle matroid of two or more non-isomorphic graphs.

Let $e$ be an edge of a graph $G$. The **deletion** of $e$ is the operation of removing this edge while keeping its end vertices. The graph thus obtained is denoted by $G\setminus e$. The **contraction** of $e$ is the operation of deleting $e$ if it is a loop; or the operation of deleting $e$ and identifying its end vertices if it is not a loop. The graph thus obtained is denoted by $G/e$.

It is not difficult to verify the following theorem.

(1.17) Theorem. Let $A$ be a set of edges of a graph $G$. Then

$$M(G\setminus A) = M(G)\setminus A,$$

and

$$M(G/A) = M(G)/A.$$  

A graph $G$ is **n-connected** if the minimum number of vertices whose removal results in a disconnected or trivial graph is at least $n$. Thus a 0-connected graph is a disconnected or trivial graph. A 1-connected graph is a connected graph.

The definition of n-connectedness of a graph is stated in terms of its vertices. However, Harary [8] gave the following characterization of 2-connected graphs.
(1.18) Theorem. Let $G$ be a loopless graph. Then $G$ is 2-connected if and only if every pair $e, f$ of edges is contained in a cycle of $G$.

It follows from the last theorem that a loopless graph $G$ is 2-connected if and only if its cycle matroid $M(G)$ is connected.

SECTION 5. CONNECTIVITY

In this section, we discuss the theory of $n$-connection of matroids developed by Tutte [24]. We also consider some of the consequences of this theory.

Let $M$ be a matroid on a set $E$ and $k$ be a positive integer. $M$ is $k$-separated if there is a bipartition $(X, Y)$ of $E$ such that each of $X$ and $Y$ has at least $k$ elements and $rkX + rkY \leq rkM + k - 1$. For an integer $n$ greater than one, $M$ is $n$-connected if it has no $k$-separation for $k < n$. A $k$-separation $(X, Y)$ is minimal if $|X| = k$ or $|Y| = k$. If there is a least positive integer $j$ such that $M$ is $j$-separated, it is called the connectivity $\lambda(M)$ of $M$. If there is no such integer, we say that $\lambda(M) = \infty$.

The next theorem follows easily from Theorem 1.4.

(1.19) Theorem. A matroid $M$ is $n$-connected if and only if its dual $M^*$ is $n$-connected.
We now relate the notions of connectedness, as defined in Section 3, and n-connectedness.

(1.20) Theorem. A matroid $M$ is connected if and only if it is 2-connected.

In Section 4, we defined the notion of n-connectedness for a graph $G$. This notion does not, in general, coincide with the notion of n-connectedness of the cycle matroid $M(G)$. However, we have the following results (see Welsh [25, pp. 78]).

(1.21) Theorem. Let $G$ be a loopless graph with at least three vertices. $G$ is 2-connected if and only if $M(G)$ is 2-connected.

(1.22) Theorem. Let $G$ be a simple graph with at least four vertices. Then $G$ is 3-connected if and only if $M(G)$ is 3-connected.

Now we state some theorems that we use frequently in the following chapters. The first two of these are results of Oxley [19, Lemmas 2.2, 1.6].

(1.23) Theorem. If $M$ is an n-connected matroid and $E(M)$ has at least $2(n - 1)$ elements, then every circuit and every cocircuit of $M$ contains at least $n$ elements.
(1.24) Theorem. If \( M \) is an \( n \)-connected matroid and \( E(M) \) has at least \( 2n - 1 \) elements, then \( M \) has no \( n \)-circuit which is also a cocircuit. In particular, a 3-connected matroid with at least five elements has no triangle which is also a triad.

The proof of the next theorem can be found in Coullard [6, 2.4].

(1.25) Theorem. Let \( M \) be a 3-connected matroid and \( e \) be an element of \( E(M) \). Suppose that \( E(M) \) has at least seven elements and that \( M/e \) has no non-minimal 2-separations. If \( T, T' \) are triangles of \( M \), each containing \( e \), then \( T \cap T' = \{e\} \).

One of the most useful results in the theory of connectivity is the following result of Tutte [24, 6.5].

(1.26) Theorem. Let \( M \) be a 2-connected matroid and \( e \) be an element of \( E(M) \). If \( M\setminus e \) is not 2-connected, then \( M/e \) is 2-connected.

The simplification of a matroid \( M \) is the matroid obtained from it by deleting all its loops and all but one element from each of its parallels classes. The cosimplification of a matroid \( M \) is the matroid obtained from \( M \) by contracting all its coloops and all but one element from each of its series classes.
Although there is no exact analogue of Theorem 1.26 for 3-connected matroids, we have the following similar result of Bixby [1].

(1.27) Theorem. Let $M$ be a 3-connected matroid and $e$ be an element of $E(M)$. Then $M\setminus e$ or $M/e$ has no non-minimal 2-separations. Moreover, $(M\setminus e)$ or $(M/e)$ is 3-connected.

We end this section by giving the following definition. A graph or a matroid $M$ is minimally $n$-connected if it is $n$-connected and, for all elements $e$ of $E(M)$, $M\setminus e$ is not $n$-connected.

SECTION 6. SOME IMPORTANT MINOR THEOREMS

In this section, we will give some properties of certain minors of a matroid $M$. Also we give some properties of $M$ knowing that it has certain minors.

The proofs of Theorem 1.28 - Theorem 1.30 can be found in [6].

(1.28) Theorem. An element $e$ of a matroid $M$ is a loop or a coloop if and only if $M\setminus e = M/e$.

(1.29) Theorem. If $\{e,f\}$ is a circuit or a cocircuit of a matroid $M$, then $M/e\setminus f = M\setminus e/f$. 
(1.30) Theorem. If \{ e, f, g \} is a triangle of a matroid \( M \), then 
\( M/e,f\setminus g = M/e,f,g \). Dually, if \{ e, f, g \} is a triad 
of a matroid \( M \), then \( M\setminus e,f/g = M\setminus e,f,g \).

Theorems 1.31 and 1.32 were proved by Oxley in [18] and [19] respectively.

(1.31) Theorem. Let \( M \) be a matroid and \( e \) be an element of \( E(M) \). If 
\( M/e \) is \( n \)-connected but \( M \) is not, then either \( e \) is a 
loop of \( M \) or \( M \) has a cocircuit containing \( e \) and 
having fewer than \( n \) elements.

(1.32) Theorem. Let \( M \) be an \( n \)-connected matroid having at least 
\( 2(n - 1) \) elements. If, for distinct elements \( e,f \) of 
\( E(M) \), the matroid \( M\setminus e \) is not \( n \)-connected and \( M\setminus e/f \) is 
\( n \)-connected, then \( M \) has an \( n \)-cocircuit containing \( e \) and \( f \).

SECTION 7. SUMS, SERIES AND PARALLEL CONNECTIONS

In this section, we will describe four methods of obtaining a 
new matroid from certain known matroids.

Let \( M',M'' \) be matroids on disjoint sets \( E',E'' \). Let 
\( \zeta = \{ C \mid C \text{ is a circuit of } M' \text{ or of } M'' \} \). Then \( \zeta \) is the collection
of circuits of a matroid $M$ on $E' \cup E''$. We call $M$ the direct sum or 1-sum of $M'$ and $M''$.

Let $M', M''$ be matroids on $E', E''$ such that each of $E'$ and $E''$ has at least three elements. Suppose that $E' \cap E'' = \{p\}$ and that $p$ is not a loop or coloop of either $M'$ or $M''$. Seymour [23] defined the 2-sum $M' \Delta M''$ of $M'$ and $M''$ to be the matroid on $E' \cup E'' - p$ whose circuits are the following:

(i) all the circuits of $M'$ and $M''$ that do not contain $p$, and

(ii) all the sets of the form $C' \cup C'' - p$ where $C'$ is a circuit of $M'$ containing $p$ and $C''$ is a circuit of $M''$ containing $p$.

The next theorem was proved by Seymour [23].

(1.33) Theorem. If $M'$ and $M''$ are 2-connected, then $M' \Delta M''$ is 2-connected with 2-separation $\{E' - p, E'' - p\}$.

He also showed that the converse of the above theorem holds.

(1.34) Theorem. Let $M$ be a 2-connected matroid with a 2-separation $\{X, Y\}$. Let $p$ be an element which is not in $E(M)$. Then there are two matroids $M', M''$ on $X \cup p, Y \cup p$ respectively, both isomorphic to minors of $M$, such that $M = M' \Delta M''$. 
Next we want to define the series connection of matroids and state some properties of such a connection. For a thorough treatment of this topic we refer the reader to Brylawski [4].

Let $M', M''$ be matroids on disjoint sets $E', E''$. Let $p'$ be an element of $E'$, $p''$ be an element of $E''$ and $p$ be an element which is not in $E' \cup E''$. Then the series connection $S((M'; p'), (M''; p''))$ of $M'$ and $M''$ with respect to the basepoints $p'$ and $p''$ is the matroid on $(E' - p') \cup (E'' - p'') \cup p$ whose circuits are the following:

(i) all the circuits of $M'$ not containing $p'$ and all the circuits of $M''$ not containing $p''$; and

(ii) all the sets $(C' - p') \cup (C'' - p'') \cup p$ where $C'$ is a circuit of $M'$ containing $p'$ and $C''$ is a circuit of $M''$ containing $p''$.

When the basepoints are clear we will sometimes denote the series connection by $S(M', M'')$. We note that the series connection of two matroids $M', M''$ can be defined when their ground sets $E', E''$ have exactly one element $p$ in common. This is done by replacing $p'$ and $p''$ by $p$ in the above definition of the series connection.

The next seven theorems list some basic properties of series connection.

(1.35) Theorem. Let $M = S((M'; p'), (M''; p''))$ and suppose that $p', p''$ are not loops or coloops of $M', M''$. Then $rKM = rKM' + rKM''$.

(1.36) Theorem. If $M = S(M', M'')$ and $M''$ is a loop, then $M = M'$. 
(1.37) Theorem. If the basepoint p' is a coloop of M', then
\[ S(M',M'') = p' + M' \setminus p' + M'' \setminus p''. \]

(1.38) Theorem. Suppose that each of M' and M'' has at least two elements. Then \( S(M',M'') \) is 2-connected if and only if both M' and M'' are 2-connected.

(1.39) Theorem. Let M be a 2-connected matroid and p be an element of E(M). Suppose that \( M \setminus p = N' + N'' \). Then
\[ M = S((M/N';p), (M/N'';p)). \]

(1.40) Theorem. If \( M = S((M';p'), (M'';p'')) \), then \( M \setminus p = M' \setminus p' + M'' \setminus p'' \).

(1.41) Theorem. Let e be an element of E(M') - p'. Then
\[ M \setminus e = S(M' \setminus e, M'') \text{ and } M/e = S(M'/e, M''). \]

We now want to discuss the dual operation of the series connection. Let M', M'' be matroids on disjoint sets E', E''. Let p' be an element of E', p'' be an element of E'', and let p be an element which is not in \( E' \cup E'' \). Then the parallel connection \( P((M';p'), (M'',p'')) \) of M' and M'' with respect to the basepoints p' and p'' is the matroid on \( (E' - p') \cup (E'' - p'') \cup p \) whose circuits are the following:

(i) all the circuits of M' and M'' with the understanding that p' and p'' are replaced by p, and
(ii) all the sets of the form \((C' - p') \cup (C'' - p'')\) where \(C'\) is a circuit of \(M'\) containing \(p'\) and \(C''\) is a circuit of \(M''\) containing \(p''\).

Here we state a few properties of the parallel connection. For fuller discussion of this operation we again refer the reader to Brylawski [4]. We note that \(P((M';p'), (M'';p''))\) will often be written as \(P(M',M'')\).

(1.42) Theorem. Let \(M = P((M';p'), (M'';p''))\). If \(p\) is neither a loop nor a coloop of \(M'\) or \(M''\), then \(\text{rk} M = \text{rk} M' + \text{rk} M'' - 1\).

(1.43) Theorem. If \(M = P(M',M'')\) and \(E(M)\) has at least two elements, then \(M\) is 2-connected if and only if both \(M'\) and \(M''\) are 2-connected.

(1.44) Theorem. Let \(M = P((M';p), (M'';p))\). Then the following statements hold.
(i) \(M/p = M'/p + M''/p\).
(ii) If \(e\) is an element of \(E(M') - p\), then \(M/e = P((M'/e;p), (M'';p))\) and \(M\setminus e = P((M\setminus e;p), (M'';p))\).

(1.45) Theorem. Let \(M\) be a 2-connected matroid and \(p\) be an element of \(M\). If \(M/p = N' + N''\), then \(M = P((M\setminus E(N'));p), (M\setminus E(N''));p))\).
The last theorem of this section shows that the 2-sum and the series and parallel connections are closely related. It extends Theorem 1.34.

(1.46) Theorem. Let $M$ be a 2-connected matroid with 2-separation $\{X, Y\}$. Let $p$ be an element which is not in $E(M)$. Then there are matroids $M'$ and $M''$ on $X \cup p$ and $Y \cup p$ respectively, both isomorphic to connected minors of $M$, such that $M = M' \Delta M'' = S((M'; p), (M''; p))/p = P((M'p); M''; p)) \setminus p$.

SECTION 8. COORDINATIZATIONS

Let $N$ be a matrix over a field $F$ and $E$ be the set of columns of $N$. The dependence matroid $D(N)$ on $E$ is the matroid whose circuits are the minimal linearly dependent subsets of $E$. A matroid $M$ is coordinatizable over a field $F$ if there is a matrix $N$ over $F$ with $M = D(N)$. The matroids that are coordinatizable over $GF(2)$ are called binary matroids.

Some of the most difficult unsolved problems in matroid theory lie in the area of coordinatizations. We will only give two well-known results that we will use later.

(1.47) Theorem. A graphic matroid is coordinatizable over every field
(1.48) Theorem. A matroid $M$ is binary if and only if $|C \cap C^*|$ is even for every circuit $C$ and every cocircuit $C^*$ of $M$.

SECTION 9. SOME PARTICULAR MATROIDS

Let $E$ be a set having $k$ elements. The matroid on $E$ whose circuits are exactly the $(r+1)$-subsets of $E$ is the uniform matroid $U[r,k]$.

The following is a useful characterization of uniform matroids.

(1.49) Theorem. A matroid $M$ is uniform if and only if $C \cap C^*$ is not empty for every circuit $C$ and every cocircuit $C^*$ of $M$.

The next theorem was proved by Inukai and Weinberg [9].

(1.50) Theorem. The connectivity of the uniform matroid $U[r,k]$ is as follows:

$$
\lambda(U[r,k]) = \begin{cases} 
  r + 1 & \text{if } k \geq 2r + 2 \\
  \infty & \text{if } 2r - 1 \leq k \leq 2r + 1 \\
  k - r + 1 & \text{if } r \geq 1 \text{ and } k \leq 2r - 2 
\end{cases}
$$
Let $n$ be an integer greater than two. The wheel $W_n$ of rank $n$ is the graph obtained from an $n$-cycle, called the rim, by adding a new vertex, called the hub, and then joining this new vertex to each vertex of the rim by an edge, called a spoke. $W_3$ and $W_4$ are shown in the figure below.

![Figure 1](image)

**SECTION 10. SOME NEW DEFINITIONS**

Most of the work in this thesis is concerned with certain classes of $n$-connected matroid $M$ that have the property that, for every subset $F$ of $E(M)$ having at most $m$ elements, the connectivity of $M \setminus F$ is $n - |F|$. We now give the precise definition of these matroids that we call $m$-element minimally $n$-connected matroids.

Let $m$ be a non-negative integer. We define recursively what is meant by saying that $M$ is an $(m,n)$-matroid or that $M$ is $(m,n)$. 
(1) \( M \) is a \((0,n)\)-matroid if it is \( n \)-connected;

(2) \( M \) is a \((1,n)\)-matroid if it is minimally \( n \)-connected;

(3) \( M \) is an \((m,2)\)-matroid, for \( m \geq 1 \), if, for every non-empty subset \( F \) of \( E(M) \) having at most \( m \) elements, \( M \setminus F \) is not \( 2 \)-connected;

(4) \( M \) is an \((m,n)\) matroid, for \( m > 1 \), \( m > 2 \), if, for all elements \( e \) of \( E(M) \), the matroid \( M \setminus e \) is an \((m-1, n-1)\) -matroid.

Now suppose that \( A \) is a subset of \( E(M) \) having at least \( m \) elements. Next we define recursively what is meant by saying that \( M \) is an \((m,n)\)-matroid relative to \( A \).

(1) \( M \) is a \((1,n)\)-matroid relative to \( A \) if, for all elements \( e \) of \( A \), the matroid \( M \setminus e \) is not \( n \)-connected;

(2) \( M \) is an \((m,2)\)-matroid relative to \( A \) if, for every non-empty subset \( F \) of \( A \) having at most \( m \)-elements, \( M \setminus F \) is not \( 2 \)-connected;

(3) \( M \) is an \((m,n)\)-matroid relative to \( A \), for \( m > 1 \) and \( n > 2 \), if, for every element \( e \) of \( A \), the matroid \( M \setminus e \) is an \((m-1, n-1)\) -matroid relative to \( A - e \).
A graph $G$ is $(m,n)$ if its cycle matroid $M(G)$ is $(m,n)$. Similarly, a graph $G$ is $(m,n)$ relative to a set of edges $A$ if $M(G)$ is $(m,n)$ relative to $A$. 
CHAPTER 2

(m,2)-MATROIDS

In this chapter, we shall investigate the structure of the matroids that are (m,2). Recall that a 2-connected matroid $M$ is (m,2) if, for every non-empty subset $F$ of $E(M)$ having at most $m$ elements, $M\setminus F$ is not 2-connected. In particular, a (0,2)-matroid is a 2-connected matroid, and a (1,2)-matroid is a minimally 2-connected matroid.

In [20], Oxley showed that a (1,2)-matroid having at least four elements contains no triangles. The aforementioned result is a generalization of a graph-theoretic result of Dirac [7] and Plummer [21]. We will show that an (m,2)-matroid having more than $2m + 1$ elements contains no circuits with fewer than $2m + 2$ elements. We then use this result to give a characterization of (m,2)-matroids. We show that an (m,2)-matroid $M$ can be obtained as a series connection of two matroids that are close to being (m,2), and have fewer elements than $M$. This characterization, which extends the one given by Oxley [20] for minimally 2-connected matroids, is then used to obtain an upper bound on the number of elements of an (m,2)-matroid. This bound generalizes the bound given by Murty [17] for minimally 2-connected matroids.

Let $M$ be a 2-connected matroid and $m$ be a positive integer. Let $A$ be a subset of $E(M)$ having at least $m$ elements. Recall that $M$
(m,2) relative to A if, for every non-empty subset F of A having at most m elements, M\F is not 2-connected.

Mader [11] proved that if C is a cycle of a simple n-connected graph G having the property that, for all edges e of C, the graph G\e is not n-connected, then C meets a vertex which has degree n in G. In [20], Oxley strengthened this result for (1,2)-graphs relative to a cycle C. He showed that if G is not the cycle C, then C meets two vertices of degree two in G. Moreover, these two vertices are separated on C be vertices of degree greater than two. We will generalize this result to the graphs that are (m,2) relative to a cycle C.

SECTION 1. (m,2)-MATROIDS

We will show that an (m,2)-matroid has no small circuits, and that it is the series connection of two matroids that are close to being (m,2). Then using these results we show that an (m,2)-matroid M of rank greater than 2m has at most \( \frac{m+1}{m} (\text{rk} M - 1) \) elements. We also give a complete description of the matroids attaining this bound.

We now give several lemmas which are used to prove the main results of this chapter. The first of these is a special case of Theorem 1.32.

(2.1) Lemma. Let M be a 2-connected matroid having at least two elements. If, for distinct elements e,f of E(M),
the matroid $M\setminus e$ is not 2-connected and $M\setminus e/f$ is 2-connected, then $\{e,f\}$ is a 2-cocircuit of $M$.

The next lemma follows from the dual of (1.31) and the fact that a 2-connected matroid having at least two elements has no loops.

(2.2) Lemma. Let $M$ be a 2-connected matroid and $e,f$ be distinct elements of $E(M)$. If $M\setminus e$ is not 2-connected and $M\setminus e,f$ is 2-connected, then $\{e,f\}$ is a 2-cocircuit of $M$.

In the course of proving Theorem 2.4 of [20], Oxley showed that the following result holds.

(2.3) Lemma. Let $C$ be a circuit of a 2-connected matroid $M$. If $C$ is contained in a series class of $M$, then $M = C$.

The next fact follows from orthogonality.

(2.4) Lemma. Let $C$ be a circuit of a matroid $M$ and $S$ be a series class of $M$. If $S$ and $C$ have a non-empty intersection, then $S$ is contained in $C$.

The following is a generalization of a well-known result of Tutte, see Theorem 1.26.
(2.5) Lemma. Let $A$ be a non-empty subset of a 2-connected matroid $M$. If, for all non-empty subsets $F$ of $A$, the matroid $M \setminus F$ is not 2-connected, then $M/A$ is 2-connected and $A$ is independent.

Proof. We argue by induction on $|A|$. The case $|A| = 1$ is Theorem 1.26, so suppose $|A| > 1$ and let $e$ be an element of $A$. If, for some non-empty subset $F$ of $A - e$, the matroid $M/e \setminus F$ is 2-connected, then, since $M \setminus F$ is not 2-connected and has no loops, $e$ is a coloop of $M \setminus F$, by Theorem 1.31. So, by (1.28), $M/e \setminus F = M \setminus F \setminus e$. But $M \setminus F \setminus e$ is not 2-connected contradicting the assumption that $M/e \setminus F$ is 2-connected. Thus $M/e \setminus F$ is not 2-connected for all non-empty subsets $F$ of $A - e$. So, by the induction hypothesis, $M/e/(A - e)$ is 2-connected, that is, $M/A$ is 2-connected.

Now assume $C$ is a circuit of $M$ contained in $A$ and $e$ is an element of $C$. The set $C - e$ satisfies the hypotheses of this lemma and so $M/(C - e)$ is 2-connected. But the last matroid has $e$ as a loop. Therefore $M$ is the circuit $C$ and $C = A$. This is a contradiction since it implies that for a subset $F$ of $A$ having $|A| - 1$ elements, $M \setminus F$ is 2-connected. \(\checkmark\)

The next lemma is a generalization of (2.2).

(2.6) Lemma. Let $A$ be a non-empty subset of 2-connected matroid $M$ such that $M \setminus A$ is 2-connected. If, for every non-empty
proper subset $F$ of $A$, the matroid $M \setminus F$ is not 2-connected, then $A$ is contained in a series class of $M$. Actually, $A$ is a series class of $M$ unless $M$ is the circuit $C_{|A| + 1}$. Moreover, for any element $e$ of $A$, $M \setminus A = M/(A - e) \setminus e$.

Proof. We argue by induction on $|A|$. If $|A| = 1$, the conclusion vacuously holds, so suppose $|A| \geq 2$. Consider the 2-connected matroid $M/e$, where $e$ is an element of $A$. If, for some non-empty proper subset $F$ of $A - e$, the matroid $M/e \setminus F$ is 2-connected, then $e$ is a coloop of $M \setminus F$. Thus, by Theorem 1.28, $M/e \setminus F = M \setminus F, e$. But the latter is not 2-connected, a contradiction. Thus, for every non-empty proper subset $F$ of $A - e$, the matroid $M/e \setminus F$ is not 2-connected. So, by the induction hypothesis, $A - e$ is contained in a series of class of $M/e$ and consequently, of $M$. Similarly, for an element $f$ of $A$ distinct from $e$, the set $A - f$ is contained in a series class of $M$. Therefore, $A$ is contained in a series class of $M$.

Now suppose $M$ is not $C_{|A| + 1}$ and $g$ is an element of $E(M)$ contained in the series class containing $A$. Since $M \setminus A$ is 2-connected and has more than one element, $g$ is an element of $A$ and thus $A$ is a series class of $M$.

To verify the last statement, note that $A - e$ consists of coloops of $M \setminus e$ and so, by (1.28), $M \setminus e (A - e) = M \setminus e/(A - e)$. $\n$
The next lemma shows that if $M$ is a disconnected matroid and has a 2-connected minor whose ground set is the complement of an independent set of $M$, then $M$ has a coloop.

(2.7) Lemma. Let $M$ be a disconnected matroid and $A,A'$ be disjoint subsets of $E(M)$. Suppose that $A \cup A'$ is independent. If $M/A\setminus A'$ is 2-connected, then $A \cup A'$ contains a coloop of $M$.

Proof. Since $M$ is not 2-connected but $M/A\setminus A'$ is, $A \cup A'$ contains a component $K$ of $M$. Therefore, since $A \cup A'$ is independent, $K$ contains exactly one element. Moreover, this element is not a loop of $M$. Thus $K$ consists of a coloop of $M$. \( \Box \)

The next four lemmas give some properties of certain minors of an $(m,2)$-matroid.

(2.8) Lemma. Let $m$ be a positive integer. If $M$ is an $(m,2)$-matroid, then, for all elements $e$ of $E(M)$, the matroid $M/e$ is $(m-1,2)$.

Proof. If $m = 1$, the conclusion holds by Theorem 1.26, so suppose $m > 1$. If, for some element $e$ of $E(M)$, $M/e$ is not $(m-1,2)$, then there exists a non-empty subset $F$ of $E(M/e)$ having at most $m - 1$ elements such that $M/e\setminus F$ is 2-connected. Since $M$ is $(m,2)$, $M\setminus F$ is not 2-connected, and so, by (1.31), $e$ is a
coloop of $M \setminus F$. By Theorem 1.28, $M/e \setminus F = M \setminus F, e$. But since $F \cup e$ has at most $m$ elements, $M \setminus F, e$ is not 2-connected. This contradicts the assumption that $M/e \setminus F$ is 2-connected. $\nabla$

(2.9) Lemma. Let $M$ be an $(m,2)$-matroid, for some positive integer $m$, and $A$ be a $k$-subset of $E(M)$, for some $k$ in $\{0,1,...,m\}$. Then $M/A$ is an $(m-k,2)$-matroid.

Proof. We argue by induction on $k$. If $k = 0$, the conclusion is immediate, so suppose $0 < k < n$ and $A$ is a $k$-subset of $E(M)$. Now, for an element $e$ of $A$, the matroid $M/(A - e)$ is $(m-k+1,2)$, by the induction hypothesis. Therefore, since $M/(A - e)/e = M/A$, the matroid $M/A$ is $(m-k,2)$, by (2.8). $\nabla$

(2.10) Lemma. Let $M$ be an $(m,2)$-matroid other than a circuit. If $M$ has an element $e$ contained in a series class of more than $m + 1$ elements, then $M/e$ is $(m,2)$.

Proof. We argue by induction on $m$. If $m = 0$, the conclusion follows from Theorem 1.26, so suppose $m > 1$. If $M/e$ is not $(m,2)$, there exists a non-empty subset $F$ of $E(M/e)$ having at most $m$ elements such that $M/e \setminus F$ is 2-connected. Since, by Lemma 2.8, $M/e$ is $(m-1,2)$, $|F| = m$. So, since $M \setminus F$ is not 2-connected, $e$ is a coloop of $M \setminus F$. But then, by Theorem 1.28, $M/e \setminus F = M \setminus F, e$. Therefore, by Lemma 2.6, $F \cup e$ is a
series class of $M$ having exactly $m + 1$ elements, a contradiction. $\nabla$

(2.11) Lemma. Let $M$ be an $(m,2)$-matroid, for some positive integer $m$, and let $A$ be an $m$-subset of $E(M)$. If $M/A\setminus f$ is 2-connected for some element $f$ of $E(M) - A$, then $A \cup f$ is contained in a series class of $M$. Actually, $A \cup f$ is a series class of $M$ unless $M$ is the circuit $C_{m+2}$.

Proof. We argue by induction on $m$. If $m = 1$, the conclusion follows from Lemma 2.2, so suppose $m > 1$. For an element $e$ of $A$, $M/A\setminus f = M/e/(A - e)\setminus f$, and, by Lemma 3.3, $M/e$ is $(m-1,2)$. So, by the induction hypothesis, $(A - e) \cup f$ is contained in a series class of $M/e$ and therefore of $M$. Similarly, for an element of $g$ of $A$ distinct from $e$, $(A - g) \cup f$ is contained in a series class of $M$. Therefore, $A \cup f$ is contained in a series class of $M$.

Now suppose $M$ is not the circuit $C_{m+2}$ and $g$ is an element of $E(M) - f$ in series with $f$. Since $g$ is a coloop of $M\setminus f$ and $M\setminus f/A$ has more than two elements and is 2-connected, $g$ is contained in $A$. Thus $A$ contains all the elements of $E(M) - f$ that are in series with $f$ and so, $A \cup f$ is a series class of $M$. $\nabla$

We now state and prove the first main result of this section.
(2.12) Theorem. Let \( M \) be an \((m,2)\)-matroid such that \( |E(M)| \geq 2m+2 \).

Then \( M \) has no circuits having fewer than \( 2m + 2 \) elements. Hence \( \text{rk} M \) is at least \( 2m + 1 \).

Proof. We argue by induction on \( m \). If \( m = 0 \), \( M \) has no loops. If \( m = 1 \), \( M \) has no 2-circuits because it is minimally 2-connected. Suppose \( C = \{e_1, e_2, e_3\} \) is a triangle of \( M \) and consider a subset \( \{e_i, e_j\} \) of \( C \). By Lemma 1.26, \( M/e_i \) is 2-connected, and since \( M/e_i, e_j \) is not 2-connected, \( M/e_i, e_j \) is 2-connected. Therefore, by Lemma 2.1, \( \{e_i, e_j\} \) is a 2-cocircuit of \( M \). Thus, every two elements of \( C \) form a 2-cocircuit of \( M \) and so, by Lemma 2.3, \( M = C \), a contradiction to the assumption that \( |E(M)| \geq 4 \).

Suppose \( m > 1 \) and the conclusion holds for all non-negative integers less than \( m \). If \( C = \{e_1, e_2, \ldots, e_{2m}\} \) is a circuit of \( M \), then \( C - e_1 \) is a circuit of \( M/e_1 \) and \( |C - e_1| = 2(m - 1) + 1 \). This contradicts the induction hypothesis, since, by Lemma 2.8, \( M/e_1 \) is \((m-1,2)\). Now suppose \( C = \{e_1, e_2, \ldots, e_{2m} + 1\} \) is a circuit of \( M \), and let \( \{e_i, e_j\} \) be a subset of \( C \). Since \( C - \{e_i, e_j\} \) is a circuit of \( M/e_i, e_j \) having \( 2m - 1 \) elements, \( M/e_i, e_j \) is not \((m-1,2)\). But \( M/e_i, e_j \) is \((m-2,2)\), by Lemma 2.9. So there exists a subset \( F \) of \( E(M/e_i, e_j) \) having exactly \( m - 1 \) elements such that \( M/e_i, e_j \setminus F \) is 2-connected. Since \( M/e_i \setminus F \) is not 2-connected, \( e_j \) is a coloop of \( M/e_i \setminus F \), and so \( M/e_i \setminus F/e_j = M/e_i \setminus F/e_j \). Thus, by Lemma 2.6 applied to \( M/e_i \), the set \( F \cup e_j \) is contained in a series
class of $M$. Similarly, $F \cup e_i$ is contained in a series class of $M$. Therefore, $\{e_i, e_j\}$ is a 2-cocircuit of $M$. Thus every two elements of $C$ form a 2-cocircuit of $M$ and so, by Lemma 2.3, $M$ is the circuit $C$. This contradicts the assumption that $|E(M)| > 2m + 2$. \(\square\)

Next we describe the matroids that are $(m,2)$ and have fewer than $2m + 3$ elements.

(2.13) Lemma. Let $m$ be a positive integer. If $M$ is an $(m,2)$-matroid such that $m + 2 < |E(M)| < 2m + 2$, then $M$ is a circuit.

Proof. First suppose that $|E(M)| = m + 2$. Let $A$ be an $m$-subset of $E(M)$. Then, by (2.5), $A$ is independent. Thus every $m$-subset of $E(M)$ is independent. Now if $C$ is an $(m+1)$-circuit of $M$, then $M \setminus (E(M) - C)$ is the circuit $C$ and therefore 2-connected, a contradiction.

Now suppose that $m + 2 < |C| < |E(M)|$. Then $|E(M) - C| \leq m$, and so, since $M$ is $(m,2)$, $M \setminus (E(M) - C)$ is not 2-connected. This is a contradiction since this matroid is the circuit $C$. \(\square\)

Now we give a characterization of the matroids that are $(m,2)$.
(2.14) Theorem. Let \( m \) be a positive integer. \( M \) is an \((m,2)\)-matroid if and only if \(|E(M)| \geq m + 2\) and either

(i) \( M \) is 2-connected and every element of \( M \) is in a series class of at least \( m + 1 \) elements; or

(ii) \( M = S\left( (M_1/Q_1;p_1), (M_2/Q_2;p_2) \right) \) where, for \( i = 1,2 \), \( M_i \) is \((m,2)\) having at least \( 3m + 3 \) elements, \( |Q_i| = m \), and \( Q_i \cup p_i \) is contained in a series class of \( M_i \).

Proof. Assume \(|E(M)| \geq m + 2\). If \( M \) is 2-connected and every element is in a series class of at least \( m + 1 \) elements, it is clear that \( M \) is \((m,2)\). So suppose that \( M \) is a series connection as in (ii). Then, for \( i = 1,2 \), \( M_i/Q_i \) is 2-connected, by (2.5). Therefore, \( M \) is 2-connected, by (1.38).

Next we show that \( M \) is \((m,2)\). Let \( F \) be a non-empty subset of \( E(M) \) having at most \( m \) elements, and, for \( i = 1,2 \), let \( F_i = F \cap E(M_i) \). If \( p \) is an element of \( F \), then \( M \setminus F = (M_1/Q_1 \setminus F_1 \setminus p_1) + (M_2/Q_2 \setminus F_2 \setminus p_2) \), by (1.40), and so \( M \setminus F \) is not 2-connected. Now suppose that \( p \) is not in \( F \). Then, by (1.41), \( M \setminus F = S\left( (M_1/Q_1 \setminus F_1; p_1), (M_2/Q_2 \setminus F_2; p_2) \right) \). Since at least one of \( F_1 \) and \( F_2 \), say \( F_1 \), is non-empty, we must have that \( M_1/Q_1 \setminus F_1 \) is not 2-connected. To see this note that if \( M_1/Q_1 \setminus F_1 \) is 2-connected, then, since \( M_1 \setminus F_1 \) is not 2-connected and \( Q_1 \) is independent, \( Q_1 \) contains a coloop \( q \) of \( M_1 \setminus F_1 \), by (2.7) applied to \( M_1 \setminus F_1 \) with \( A = Q_1 \) and \( A' = \emptyset \). But since \( \{ q, p_1 \} \) is a 2-cocircuit of \( M_1 \), the element \( p_1 \) is a coloop of \( M_1 \setminus F_1 \) and of
$M \setminus F_1 / Q_1$, a contradiction. Thus $M \setminus F_1 / Q_1$ is not 2-connected and so, by (1.38), $M \setminus F$ is not 2-connected.

For the converse, suppose $M$ is $(m,2)$. Then it is clear that $|E(M)| \geq m + 2$. Assume that $M$ has an element $p$ which is not in a series class of at least $m + 1$ elements. Then $M$ is not a circuit, and so, by (2.13), $|E(M)| \geq 2m + 3$. Now $M \setminus p$ is not 2-connected and has at least one component with more than one element, for if every component of $M \setminus p$ has exactly one element, then $p$ is in a series class of $M$ having at least $m + 1$ elements. Also if $K$ is a component of $M \setminus p$ having at least two elements, $K$ must have at least $2m + 2$ elements, by Theorem 2.12. Furthermore, $M \setminus p$ cannot have exactly one component with two or more elements, otherwise $M \setminus p$ would have at most $m - 1$ coloops and deleting these coloops would result in a 2-connected matroid, contradicting the fact that $M$ is $(m,2)$. Thus $M \setminus p = N_1 + N_2$ where $|N_1|, |N_2| \geq 2m + 2$.

Now, by (1.39), $M = S((M/E(N_1)), (M/E(N_2)))$ where $p$ is the basepoint of both $M/E(N_1)$ and $M/E(N_2)$. Let $N_3 = M/E(N_1)$ and $N_4 = M/E(N_2)$. Then each of $N_3$ and $N_4$ has at least $2m + 3$ elements. Also, since $S(N_3, N_4)$ is 2-connected, $N_3$ and $N_4$ are 2-connected.

Now consider a non-empty subset $F$ of $E(N_3)$ not containing $p$ and having at most $m$ elements. Then, since $M \setminus F = S(N_3 \setminus F, N_4)$ and $M \setminus F$ is not 2-connected, $N_3 \setminus F$ is not 2-connected, by (1.38). Similarly, $N_4 \setminus F$ is not 2-connected for all non-empty subsets $F$ of $E(N_4)$ not containing $p$ and having at
most m elements. Now add m elements in series with $p_i$ in $N_i$ to get a new matroid $M_i$, and let $Q_i$ be the set of these elements. Similarly, let $Q_2$ be the set of m elements added in series to $p_2$ in $N_4$ to form a new matroid $M_2$. Clearly, $M_1$ and $M_2$ are 2-connected. Furthermore, each of $M_1$ and $M_2$ has at least $3m + 3$ elements.

To finish the proof we need to show that, for $i = 1, 2$, $M_i$ is $(m, 2)$. Let $F$ be a non-empty subset of $E(M_i)$ having at most m elements. If $F \cap (Q_i \cup p_i) \neq \emptyset$, then $M_i \setminus F$ is not 2-connected because it has a coloop. So suppose that $F \cap (Q_i \cup p_i) = \emptyset$ and that $M_i \setminus F$ is 2-connected. Since $N_i + F = M_i \setminus F / Q_i$ and the latter is not 2-connected, $N_i + F \setminus F$ has at least two components $K_i$ and $H_i$. Assume that $p_i \in K_i$ and consider an element $x$ of $H_i$. Since $M_i \setminus F$ is 2-connected, there is a circuit $C$ of $M_i \setminus F$ such that $\{p_i, x\} \subseteq C$. But, since $Q_i \cup p_i$ is contained in a series class of $M_i \setminus F$, $Q_i \subseteq C$, by (2.4). Thus, since $C \setminus Q_i$ is a circuit of $M_i \setminus F / Q_i$ containing $p$ and $x$, we have a contradiction.

Then we can see that $M_i$ can be obtained by taking the series connection of $M_i/Q_i$ and $M_2/Q_2$ with respect to the basepoint $p$. $\n$

Since $|E(M_i/Q_i)| \geq 2m + 3$ in the last result, we must have that $|E(M)| \geq 4m + 5$. The cycle matroid of the following graph shows that these two bounds are best possible.
The matroid $U[1,k,d]$ is obtained from the uniform matroid $U[1,k]$ by replacing each of its elements by $d$ elements in series.
(2.15) Corollary. Let \( m \) and \( r \) be positive integers with \( r \geq 2m + 1 \).

An \((m,2)\)-matroid \( M \) of rank \( r \) has at most \( \frac{m+1}{m}(r-1) \) elements. Moreover, the upper bound is attained if and only if \( M \) is isomorphic to \( U[1, \frac{r-1}{m}, (m+1)] \).

Proof. We argue by induction on \( |E(M)| \). If \( m + 2 \leq |E(M)| \leq 2m + 1 \), the rank of \( M \) is less than \( 2m + 1 \), and so the conclusion vacuously holds. If \( |E(M)| = 2m + 2 \), then, by (2.13), \( M \) is a circuit of rank \( 2m + 1 \) and the conclusion holds. Suppose \( |E(M)| \geq 2m + 3 \), and note that, by Theorem 2.12, \( \text{rk} M \geq 2m + 1 \). If \( M \) has an element \( p \) which is not in a series class of at least \( m + 1 \) elements, then, by Theorem 2.14, \( M = S((M_1/Q_1;p_1), (M_2/Q_2;p_2)) \) where, for \( i = 1,2 \), \( M_i \) is \((m,2)\) having at least \( 3m + 3 \) elements and therefore having rank at least \( 2m + 1 \). Now, by (1.35), \( \text{rk} M = \text{rk}(M_1/Q_1) + \text{rk}(M_2/Q_2) \) and thus

\[
\text{rk} M = \text{rk} M_1 + \text{rk} M_2 - 2m \cdots \cdots \cdots \cdots \cdots (1)
\]

Moreover,

\[
|E(M)| = |E(M_1)| + |E(M_2)| - 2m - 1 \cdots (2)
\]

From (2) we can see that, for \( i = 1,2 \),

\[
|E(M_i)| < |E(M)| \text{ and so, by the induction hypothesis,}
\]

\[
|E(M_i)| \leq \frac{m+1}{m}(\text{rk} M_i - 1). \text{ Thus } |E(M_1)| + |E(M_2)| \leq \frac{m+1}{m}(\text{rk} M_1 + \text{rk} M_2 - 2), \text{ and, by (1) and (2), } |E(M)| \leq \frac{m+1}{m}(\text{rk} M + 2m - 2) - 2m - 1 < \frac{m+1}{m}(r - 1).
\]
Now if $M$ has an element $e$ which is in a series class of more than $m + 1$ elements, then, by Lemma 2.10, $M/e$ is $(m,2)$ and so, by the induction hypothesis,

$$|E(M)| - 1 = |E(M/e)| < \frac{m + 1}{m} (\text{rk}(M/e) - 1) = \frac{m + 1}{m} (r - 2) < \frac{m + 1}{m} (r - 1).$$

Now we may assume that every element of $M$ is in a series class of exactly $m + 1$ elements. Let \{ $S(e_1), \ldots, S(e_k)$ \} be the set of series classes of $M$. Since \{ $S(e_1), \ldots, S(e_k)$ \} is a partition of $E(M)$, $|E(M)| = k(m + 1)$. Moreover, $S(e_1) \cup (\bigcup_{j=2}^{k} (S(e_j) - e_j))$ is independent, and so $\text{rk} M > km + 1$. Therefore $|E(M)| < \frac{m + 1}{m} (r - 1)$ with equality being attained if and only if $\text{rk} M = km + 1$. But if $\text{rk} M = km + 1$, then, for \{ $i,j$ \} a subset of \{ $1, 2, \ldots, k$ \}, $S(e_i) \cup S(e_j)$ is a circuit. To see this note that if $S(e_i) \cup S(e_j)$ is independent, then $S(e_i) \cup S(e_j) \cup \bigcup_{i,j \neq t} (S(e_t) - e_t)$ has $km + 2$ elements and so is dependent, a contradiction to Lemma 2.4. Hence if $\text{rk} M = km + 1$, then $M = U[1, \frac{m + 1}{m}, m + 1]$. Evidently the converse of this also holds. ∨

**SECTION 2. (m,2)-MATROIDS RELATIVE TO A SET**

In this section we will show that if $M$ is $(m,2)$ relative to a set $A$, then, unless $A$ is independent, $A$ contains at least $|A| - \text{rk} A + 1$ series classes each containing at least $m + 1$ elements.
elements. Then we use this result to give an application to (m,2)-graphs relative to a cycle C.

(2.16) Lemma. Let M be an (m,2)-matroid relative to a subset A of \( E(M) \), and let \( e \) be an element of A. If \( M/e \setminus F \) is 2-connected for some non-empty subset \( F \) of \( A - e \), then \( |F| \geq m \).

Proof. Suppose \( |F| < m \). Since \( M \setminus F \) is not 2-connected and \( M \setminus F / e \) is 2-connected, \( e \) is a coloop of \( M \setminus F \). So, by Lemma 1.28, \( M \setminus F / e = M \setminus F \setminus e \), but the latter is not 2-connected, a contradiction. \( \square \)

Oxley [20] proved the following result which is a strengthening of a result of Seymour.

(2.17) Lemma. Let \( M \) be a 2-connected matroid having at least two elements and \( \{ e_1, e_2, \ldots, e_k \} \) be a circuit of \( M \) such that \( M \setminus e_i \) is not 2-connected for all \( i \) in \( \{ 1, 2, \ldots, k-1 \} \). Then \( \{ e_1, e_2, \ldots, e_{k-1} \} \) contains a 2-cocircuit of \( M \).

The following is a generalization of this result.

(2.18) Lemma. Let \( C = \{ e_1, e_2, \ldots, e_k \} \) be a circuit of a 2-connected matroid. If \( M \) is an (m,2)-matroid relative
to \{e_1, e_2, \ldots, e_{k-1}\}, then \{e_1, e_2, \ldots, e_{k-1}\} contains a series class having at least \(m + 1\) elements.

Proof. We argue by induction on \(k\). If \(k = m + 1\), then, by Lemma 2.5, \{e_1, e_2, \ldots, e_{k-1}\} does not satisfy the hypotheses of the theorem. Thus we may assume that \(k > m + 1\).

Consider the circuit \{e_2, e_3, \ldots, e_{k-1}\} of the 2-connected matroid \(M/e_1\). If, for every non-empty subset \(F\) of \{e_2, e_3, \ldots, e_{k-1}\} having at most \(m\) elements, \(M/e_1 \setminus F\) is not 2-connected, then, by the induction hypothesis, \{e_2, e_3, \ldots, e_{k-1}\} contains a series class of at least \(m + 1\) elements. So suppose \(M/e_1 \setminus F\) is 2-connected for some non-empty subset \(F\) of \{e_1, e_2, \ldots, e_{k-1}\} having at most \(m\) elements. Then, by Lemma 2.16, \(|F| > m\), and so \(|F| = m\). Since \(M/e_1 \setminus F = M \setminus F \setminus e_1\), the set \(F \cup e_1\) is contained in a series class of \(M\), by Lemma 2.6. Hence \{e_1, e_2, \ldots, e_{k-1}\} contains a series class of at least \(m + 1\) elements. \(\square\)

The following two results are also generalizations of results of Oxley [20].

(2.19) Theorem. Let \(C\) be a circuit of a 2-connected matroid \(M\). If \(M\) is \((m,2)\) relative to \(C\), then either \(M\) is the circuit \(C\) or \(C\) contains at least two series classes of \(M\) each having at least \(m + 1\) elements.
Proof. We argue by induction on $|C|$. If $|C|$ is $m$ or $m+1$, then $C$ does not satisfy the hypotheses, by Lemma 2.5, so suppose $|C| > m+1$. Then, by Lemma 2.18, there exists a series class $S$ of $M$ having at least $m+1$ elements contained in $C$. If every element of $C$ is in $S$, then, by Lemma 2.3, $M = C$. So assume $C$ contains an element $e$ which is not in $S$, and consider the circuit $C - e$ of the 2-connected matroid $M/e$. Now if, for every non-empty subset $F$ of $C - e$ having at most $m$ elements, $M/e\backslash F$ is not 2-connected, it follows, by the induction hypothesis, that $C - e$ contains at least two series classes of $M/e$ each having at least $m+1$ elements. Therefore, $C$ contains the required two series classes of $M$. On the other hand, if $F$ is a non-empty subset of $C - e$ having at most $m$ elements and $M/e\backslash F$ is 2-connected, then, by Lemma 2.16, $|F| = m$. Since $M/e\backslash F = M\backslash F\backslash e$, it follows by Lemma 2.6 that $F \cup e$ is a series class of $M$ which is clearly distinct from $S$. $\Box$

(2.20) Corollary Let $M$ be a 2-connected matroid other than a circuit, and $A$ be a subset of $E(M)$. If $M$ is $(m,2)$ relative to $A$, then either $A$ is independent, or $A$ contains at least $|A| - \text{rk}A + 1$ series classes of $M$ each having at least $m+1$ elements.

Proof. We argue by induction on $|A|$. If $|A|$ is $m$ or $m+1$, then, by Lemma 2.5, $A$ is independent, so suppose $|A| > m+1$ and $A$ is dependent. Let $C$ be a circuit of $M$.
contained in A and e be an element of C. Consider the dependent
set A - e of the 2-connected matroid M/e. If, for every
non-empty subset F of A - e having at most m elements, M/e\F is
not 2-connected, then M/e is (m,2) relative to A - e. So, by
the induction hypothesis, A - e contains at least |A - e| -
rk'(A - e) + 1 series classes of M/e each having at least m + 1
elements, where rk' is the rank function of M/e. But
|A - e| - rk'(A - e) + 1 = |A| - 1 - (rkA - 1) + 1 =
|A| - rkA + 1. Therefore the required conclusion holds.

Suppose M/e\F is 2-connected for some non-empty subset
F of A - e having at most m elements. Then F has exactly m
elements. Since M/e\F = M\F\e, the set F U e is a series class
of M, by Lemma 2.6. Now let A' = A - (F U e). Since, by
(2.19), C contains a series class of M other than F U e, A' has
at least m + 1 elements. Furthermore, M is (m,2) relative to
A'. By Lemma 2.4, rkA' + m = rk(A' U F) ≤ rkA. Now if A' is
dependent, then, by the induction hypothesis, A' contains at
least |A'| - rkA' + 1 series classes of M each having at least
m + 1 elements. But |A'| - rkA' + 1 ≥ |A| - m - 1 -
(rkA - m) + 1 = |A| - rkA. Thus, since F U e is a series
class of M not contained in A', the set A contains at least
|A| - rkA + 1 series classes of M each having at least m + 1
elements. On the other hand, if A' is independent, A' U F is
independent, again by Lemma 2.4, and so, since A = A' U F U e
is independent, A contains exactly one circuit, the circuit C.
But in this case, \(|A| - \text{rk}A + 1 = |A| - (|A| - 1) + 1 = 2\). Hence the conclusion follows from Theorem 2.19. \(\nabla\)

Let \(G\) be a graph and \(A\) be a subset of the set of edges of \(G\). The graph obtained from \(G \setminus A\) by removing all its isolated vertices is denoted by \((G \setminus A)\).

Recall that a graph \(G\) is \((m,2)\) relative to a set of edges \(A\) having at least \(m\) elements if its cycle matroid \(M(G)\) is \((m,2)\) relative to \(A\). In other words, \(G\) is \((m,2)\) relative to \(A\) if, for every non-empty proper subset \(F\) of \(A\), the graph \((G \setminus F)\) is not 2-connected.

In [20], Oxley proved the following lemma for \((1,2)\)-graphs relative to a cycle \(C\). We give a different proof of this result which illustrates the proof of the general case.

\((2.21)\) Lemma. Let \(G\) be a 2-connected graph. If \(G\) is \((1,2)\) relative to a cycle \(C\), then either \(G\) is the cycle \(C\) or the cycle \(C\) meets two vertices of \(G\) of degree two which are separated on \(C\) by vertices of degree greater than two.

Proof. We argue by induction on \(|C|\). If \(|C| = 2\), \(G\) cannot be minimally 2-connected relative to \(C\). Also, if \(|C| = 3\), then, by (2.19), \(G\) is the cycle \(C\). So suppose \(|C| > 3\) and \(G\) is not the cycle \(C\). If \(C\) has an element \(e\) which is in no 2-cocircuits, then, by (2.1), \(M(G)/e\backslash f\) is not 2-connected for all elements \(f\) of the circuit \(C - e\) of the 2-connected matroid \(M(G)/e\). On the other hand, if \(e\) is an
element of $C$ which is in a series class of at least three elements, then for any element $f$ of $C - e$, whether $f$ is in series with $e$ or not, $M(G)/e\backslash f$ is not 2-connected. In either case, the graph $G/e\backslash f$ is not 2-connected, for all elements $f$ of the cycle $C - e$ of $G/e$. But then, by the induction hypothesis, $C - e$ has the required two vertices $u, v$ of $G/e$. Since the contraction of an edge of an loopless graph does not decrease the degree of any vertex of the graph, and since $e$ is not incident with both $u$ and $v$ in $G$, $u$ and $v$ have the required properties of $G$.

We may now suppose that every element of $C$ is in a series class of exactly two elements. Also note that, since $G$ is not the cycle $C$, $C$ meets at least two vertices of degree at least three. Now suppose that, for some 2-cocircuit $\{e_1, e_2\}$ of $M(G)$ contained in $C$, the matroid $M(G)/e_1, e_2$ is 2-connected. Then, for any element $f$ of $C - \{e_1, e_2\}$, the matroid $M(G)/e_1, e_2\backslash f$ is not 2-connected since it has a coloop. Therefore $M(G)/e_1, e_2$ is $(1,2)$ relative to the circuit $C - \{e_1, e_2\}$. Hence, by the induction hypothesis, $G/e_1, e_2$, and consequently $G$, has the required two vertices. Thus, since $M/e_1$ is 2-connected, and by the dual of (1.26), we can assume that, for every 2-cocircuit $\{e_1, e_2\}$ of $M(G)$ contained in $C$, the matroid $M(G)/e_1\backslash e_2$ is 2-connected. But then, since $e_1$ is a coloop of $M(G)\backslash e_2$, we have that $M(G)/e_1\backslash e_2 = M(G)\backslash e_1, e_2$. Thus $M(G)\backslash e_1, e_2$ is 2-connected. Therefore $M(G\backslash e_1, e_2)$ is 2-connected. But since $G\backslash e_1, e_2$ is disconnected, $\{e_1, e_2\}$ is a
vertex cocircuit. Hence, since \( C \) contains at least 2-cocircuits, we have the required conclusion. \( \Box \)

The next lemma is needed in the proof of the general case of the last result.

(2.22) Lemma. Let \( F \) be an independent set of a 2-connected matroid \( M \), and let \( S \) be a series class of \( M \). If \( M \less S \) is not 2-connected, then \( M \less S \less F \) is not 2-connected.

Proof. Suppose \( M \less S \less F \) is 2-connected. Then, since \( M \less S \) is not 2-connected, \( F \) contains a component \( K \) of \( M \less S \). But since \( F \) is independent, \( K \) consists of a single coloop of \( M \less S \), a contradiction since \( S \) is a series class of \( M \). \( \Box \)

(2.23) Theorem. Let \( G \) be a loopless graph which is \((m,2)\) relative to a cycle \( C \). Then either \( G \) is the cycle \( C \) or \( C \) contains two paths \( P_1, P_2 \) each having at least \( m \) consecutive vertices of degree two in \( G \). Moreover, \( P_1, P_2 \) are separated on \( C \) by vertices of degree greater than two.

Proof. We argue by induction on \( m \) and \( |C| \). The case \( m = 1 \) is Lemma 2.21, so suppose \( m > 1 \) and consider the 2-connected matroid \( M(G) \). By Lemma 2.5, \( |C| > m + 1 \). Furthermore, if \( m + 2 \leq |C| \leq 2m + 2 \), then, by Theorem 2.19,
M(G) is the circuit C. So assume |C| > 2m + 2 and M(G) is not the circuit C. If C contains an element e which is in a series class of more than m + 1 elements, consider the circuit C - e of the 2-connected matroid M(G)/e. Let F be a non-empty subset of C - e having at most m elements. Then M(G)/e\F is not 2-connected, for if otherwise, by Lemma 2.6, F U e is a series class of M(G) having at most m + 1 elements, a contradiction. Thus M(G)/e is (m,2) relative to the circuit C - e and consequently, G/e is (m,2) relative to the cycle C - e. Then, by the induction hypothesis on |C|, the cycle C - e has two paths P_1 and P_2 satisfying the required properties of G/e. Therefore, P_1 and P_2 satisfy the required properties in G.

Now suppose that C contains an elements e which is in a series class S having less than m + 1 elements. Consider the circuit C - S of the 2-connected matroid M(G)/S. Now suppose that, for a non-empty subset F of C - S having at most m element, M(G)/S\F is 2-connected. Then, since M(G)\F is not 2-connected, S contains a coloop of M(G)\F. So M(G)/S\F = M(G)\S\F. But then we have a contradiction since, by Lemma 2.22, M(G)\S\F is not 2-connected. Therefore, for every non-empty subset F of C - S having at most m elements, M(G)/S\F is not 2-connected. That is, G/S is (m,2) relative to the cycle C - S. Therefore, by the induction hypothesis on |C|, the cycle C - S contains two paths P_1, P_2 satisfying the required
properties in G/S. Hence \( P_1, P_2 \) are two paths of \( G \) with the required properties.

We may now assume that every element of \( C \) is in a series class of \( M(G) \) having exactly \( m + 1 \) elements. Then, by the induction hypothesis on \( m \), \( C \) contains two paths \( P_1, P_2 \) each having at least \( m - 1 \) consecutive vertices of degree two in \( G \). Furthermore, \( P_1 \) and \( P_2 \) are separated on \( C \) by vertices of degree greater than two. Suppose one of the two paths, say \( P_2 \), has exactly \( m - 1 \) consecutive vertices of degree two in \( G \), and let \( H \) be the set of edges incident with these vertices. Then \( |H| = m \), and \( H \cup e \) is a series class of \( M(G) \) for some edge \( e \) of \( C \). Note that each of the end vertices of \( e \) has a degree of at least three in \( G \). Clearly, \( e \) is a bridge of the connected graph \((G \setminus H)\). Therefore, \((G \setminus H)/e)\) has a cut vertex. That is \( M(G) \setminus H/e \) is not 2-connected. Now consider the circuit \( C - e \) of \( M(G)/e \). It is not difficult to check that, for every non-empty subset \( F \) of \( C - e \) having at most \( m \) elements, \( M(G)/e \setminus F \) is not 2-connected. Therefore, \( G/e \) is \((m,2)\) relative to the cycle \( C - e \). Now the conclusion follows by the induction hypothesis on \( |C| \). \( \nabla \)
In this chapter, we shall examine the structure of the 2-connected matroids \( M \) having at least two elements and satisfying the following two properties:

1. \( M \setminus x, y \) is not 2-connected, for all elements \( x, y \) of \( E(M) \); and
2. \( M \) has at least one element \( e \) such that \( M \setminus e \) is 2-connected.

Let \( M \) be a 2-connected matroid satisfying properties (1) and (2) and \( A \) be the set of elements \( e \) of \( M \) for which \( M \setminus e \) is 2-connected. Here we note the following obvious fact.

**(3.1) Lemma.** There is no element of \( A \) which is in a 2-cocircuit of \( M \). Moreover, there is no element of \( E(M) - A \) which is in a 2-circuit of \( M \).

Now we show that either \( E(M) = A \) or \( |E(M) - A| \) is at least three.

**(3.2) Lemma.** \( |E(M) - A| \neq 1 \).

**Proof.** Suppose \( E(M) - A = \{x\} \). Then \( M \setminus x \) is not 2-connected and, by (3.1) it has no coloops. Therefore \( M = S((M';x), (M'';x)) \) where \( M' \) and \( M'' \) are 2-connected matroids having at least three elements, by (1.39). Now let \( e \) be an
element of $E(M') - x$ and $f$ be an element of $E(M'') - x$. Since $M \backslash e$ is 2-connected and, by (1.41), is equal to $S(M' \backslash e, M'')$, $M' \backslash e$ is 2-connected, by (1.38). Similarly, $M'' \backslash f$ is 2-connected. So, by (1.38), $S((M' \backslash e), (M'' \backslash f))$ is 2-connected. This is a contradiction since $S((M' \backslash e), (M'' \backslash f)) = M \backslash e, f$ and the latter matroid is not 2-connected. □

(3.3) Lemma. Let $M$ be a 2-connected matroid and $p, q$ be distinct elements of $E(M)$. If $M \backslash q$ and $M \backslash p, q$ are not 2-connected, then $M/p \backslash q$ is not 2-connected.

Proof. Suppose that $M/p \backslash q$ is 2-connected. Then, since $M \backslash q$ is not 2-connected, $\{ p, q \}$ is a 2-cocircuit of $M$, by (2.1). Thus, since $p$ is a coloop of $M \backslash q$, we have that $M \backslash q/p = M \backslash q, p$, by (1.8). This is a contradiction since the latter matroid is not 2-connected. □

(3.4) Lemma. $|E(M) - A| \neq 2$.

Proof. Suppose $E(M) - A = \{ x, y \}$. Then, by (3.3), $M/x \backslash y$ is not 2-connected. Now if $z$ is a coloop of $M/x \backslash y$, then, since $M$ has no coloops, $\{ y, z \}$ is a 2-cocircuit of $M$. This is a contradiction since $z$ is an element of $A$. Therefore, since $M/x \backslash y$ is not 2-connected and has no coloops, $M/x = S((M'; y), (M''; y))$ where $M'$ and $M''$ are 2-connected matroids each having at least three elements, by (1.39). Let $e$ be an
element of \( E(M') - y \) and \( f, g \) be elements of \( E(M'') - y \). Since 
\( M/x\{e \} \) and \( M/x\{f \} \) are 2-connected, \( M'\{e \} \) and \( M''\{f \} \) are 2-connected. 
Therefore, since \( M/x\{e, f \} = S((M'\{e \}), (M''\{f \})) \), \( M/x\{e, f \} \) is 
2-connected, by (1.38). So, since \( M\{e, f \} \) is not 2-connected and 
neither \( e \) nor \( f \) is in a 2-cocircuit of \( M \), the set \( \{e, f, x\} \) is a 
triad of \( M \). Similarly, \( \{e, g, x\} \) is a triad of \( M \). Therefore, 
by circuit elimination, \( \{e, f, g\} \) contains a cocircuit of \( M \) 
containing \( g \). But then \( M/x\{e, f \} \) has a coloop, a contradiction. \( \forall \)

We shall consider the following cases:

1. \( |A| = 1 \),
2. \( 2 \leq |A| < |E(M)| - 3 \), and
3. \( E(M) = A \).

**SECTION 1. THE CASE \( A = \{e\} \).**

In Theorem 3.8, we give a characterization of such matroids \( M \) 
showing that either \( M \) has two (2,2) minors each having one element 
fewer than \( M \), or \( M \) is the parallel connection with basepoint \( e \) of two 
matroids that are close to being (2,2). Then this characterization is 
used to obtain an upper bound on the number of elements of \( M \) in terms 
of the rank function. Also the matroids attaining this upper bound 
are completely described.
(3.5) Lemma. $M \setminus e$ is $(2,2)$.

Proof. Let $x, y$ be distinct elements of $E(M) - e$. If $M \setminus e, x, y$ is 2-connected, then, since $M \setminus e, x$ is not 2-connected, $y$ is a coloop of $M \setminus e, x$. So, by (1.28), $M \setminus e, x, y = M/y \setminus x, e$. Now, by (3.3), $M/y \setminus x$ is not 2-connected. Therefore $e$ is a loop or coloop of $M/y \setminus x$. But if $e$ is a loop of $M/y \setminus x$, then, since $M$ is 2-connected, $\{e, y\}$ is a 2-circuit of $M$. This is a contradiction to (3.1). Thus $e$ is a coloop of $M/y \setminus x$. Therefore $\{e, x\}$ a 2-cocircuit of $M$. This is also a contradiction to (3.1). Hence $M \setminus e, x, y$ is not 2-connected. $\nabla$

(3.6) Lemma. $|E(M)| \geq 7$.

Proof. By the last lemma, $M \setminus e$ is $(2,2)$. Therefore $M \setminus e$ has at least four elements. Suppose that $M \setminus e$ has exactly four elements. Then, by (2.13), $M \setminus e$ is the circuit $C_4$. So, since $M$ is 2-connected and $e$ is not in parallel with any other element of $M$, $e$ is in a triangle or 4-circuit of $M$. Let $C$ be a circuit of $M$ containing $e$. Then, since $|E(M)| = 5$, we must have that $1 \leq |E(M) - C| \leq 2$. Therefore $M \setminus (E(M) - C)$ is 2-connected since it is the circuit $C$, a contradiction.

Assume that $M \setminus e$ has exactly five elements. Then, since $M \setminus e$ is $(2,2)$, $M \setminus e = C_5$. Let $C_5 = \{x, y, z, u, v\}$. Note that, by (3.1), $e$ is in no 2-circuit of $M$. Also if $e$ is in a circuit $C$ of $M$ containing four or five elements, then $|E(M) - C|$ is one
or two. Thus \( M \setminus (E(M) - C) \) is 2-connected since it is the circuit \( C \), a contradiction. Therefore, since \( M \) is 2-connected, \( e \) is in a triangle of \( M \). We may assume, without loss of generality, that \( \{e, x, y\} \) is a triangle of \( M \). Consider the circuit \( C_5 \) and the triangle \( \{e, x, y\} \). Then, by circuit elimination, there is a circuit \( C' \) contained in \( \{e, y, z, u, v\} \). Clearly, \( e \) is \( C' \). Moreover, \( C' \) is a triangle of \( M \). Then, by eliminating \( e \) from \( C' \) and \( \{e, x, y\} \), we obtain a circuit \( C'' \) contained in \( C_5 \) and having at most four elements, a contradiction. Hence \( E(M \setminus e) \) contains at least six elements. \( \forall \)

(3.7) Lemma. Suppose that \( M/e \) is 2-connected. Then \( M/e \) is \( (2,2) \).

Proof. Let \( x \) be an element of \( E(M) - e \). Then by (3.3), \( M/e \setminus x \) is not 2-connected. Therefore \( M/e \) is \( (1,2) \).

Now let \( x, y \) be distinct elements of \( E(M) - e \). Suppose that \( M/e \setminus x, y \) is 2-connected. Then, since, by (3.3), \( M/e \setminus x \) is not 2-connected, \( y \) is a coloop of \( M/e \setminus x \). So, \( \{x, y\} \) is a 2-cocircuit of \( M \). Also, since \( M \setminus x, y \) is not 2-connected, \( e \) is a coloop of \( M \setminus x, y \). But then, since \( e \) is not in series with any other element of \( M \), \( \{e, x, y\} \) is a triad of \( M \), a contradiction. Hence, for all elements \( x, y \) of \( E(M) - e \), the matroid \( M/e \setminus x, y \) is not 2-connected. \( \forall \)

(3.8) Theorem. Let \( M \) be a 2-connected matroid having at least two elements. Then the following are equivalent:
(1) \( M \setminus x, y \) is not 2-connected for all elements \( x, y \) of \( E(M) \), and \( M \) has exactly one element \( e \) such that \( M \setminus e \) is 2-connected.

(2) \( M \setminus e \) is \((2,2)\) having at least six elements and either
   (i) \( M / e \) is \((2,2)\); or
   (ii) \( M = P((M'/p',q';e), (M''/p'',q'';e)) \) where \( M' \) and \( M'' \) are \((2,2)\) each having at least six elements, and \( \{ e, p', q' \}, \{ e, p'', q'' \} \) are contained in series classes of \( M', M'' \), respectively.

Proof. Assume that (1) holds. Then, by (3.5) and (3.6), \( M \setminus e \) is \((2,2)\) and has at least six elements.

Now suppose that \( M / e \) is 2-connected. Then by (3.7), \( M / e \) is \((2,2)\). Therefore (i) holds.

Now suppose that \( M / e \) is not 2-connected. Then, since \( M / e \) has no loops, \( M = P((N';e), (N'';e)) \) where \( N' \) and \( N'' \) are 2-connected matroids, by (1.45). Clearly, each of \( N' \) and \( N'' \) has at least three elements. Now if \( E(N') = \{ e,x,y \} \), then \( M \setminus x, y = N'' \), a contradiction since \( M \setminus x, y \) is not 2-connected. Therefore, \( N' \) has at least four elements. Similarly, \( N'' \) has at least four elements.

Let \( x, y \) be distinct elements of \( E(N') - e \). Then, since \( M \setminus x \) and \( M \setminus x, y \) are not 2-connected, \( N' \setminus x \) and \( N' \setminus x, y \) are not 2-connected, by (1.44) (ii) and (1.43). Similarly, if \( x, y \) are distinct elements of \( E(N'') - e \), then \( N'' \setminus x \) and \( N'' \setminus x, y \) are not
2-connected.

Now add two elements $p', q'$ in series with $e$ in $N'$ to get a new matroid $M'$. Also, add two elements $p'', q''$ in series with $e$ in $N''$ to get a new matroid $M''$. Clearly, $M'$ and $M''$ are 2-connected.

Next we verify that $M'$ and $M''$ are $(1,2)$. Let $x$ be an element of $E(M') - \{e, p', q'\}$, and suppose $M' \setminus x$ is 2-connected. Then, since $M' \setminus x, p'$ and $M' \setminus x, p'q''$ are not 2-connected, it follows by applying (2.5) to $M' \setminus x$ that $M' \setminus x/p', q'$ is 2-connected. But then, since $M' \setminus x/p', q' = N' \setminus x$ and the latter is not 2-connected, we have a contradiction. Therefore $M'$ is $(1,2)$. Similarly, $M''$ is $(1,2)$.

In a similar fashion, we can verify that $M'$ and $M''$ are $(2,2)$.

For the converse, suppose that (i) holds. Then, for an element $x$ of $E(M) - e$, the matroid $M \setminus x$ is not 2-connected since neither $M \setminus x, e$ nor $M \setminus x/e$ is 2-connected. Similarly, for any distinct elements $x, y$ of $E(M) - e$, the matroid $M \setminus x, y$ is not 2-connected.

Now suppose that (ii) holds. Then, since $M/e$ is not 2-connected, $M \setminus e$ is 2-connected. Let $x$ be an element of $E(M'/p', q') - e$. Then $M \setminus x$ is not 2-connected. For if $M \setminus x$ is 2-connected, then $M \setminus x/p', q'$ is 2-connected, by (1.43). So, by (2.7), $p'$ or $q'$ is a coloop of $M' \setminus x$. But then, since $\{e, p', q'\}$ is contained in a series class of $M'$, the element $e$ is a coloop of $M' \setminus x$ and of $M' \setminus x/p'q'$, a contradiction.
Similarly, for all elements $x$ of $E(M''/p'',q'') - e$, the matroid $M \setminus x$ is not 2-connected.

To complete the proof that $M$ has the required properties, we need to show that, for all elements $x, y$ of $E(M)$, the matroid $M \setminus x, y$ is not 2-connected. Let $x$ be an element of $E(M'/p',q') - e$ and $y$ be an element of $E(M''/p'',q'') - e$. Then, since $M' \setminus x/p', q'$ and $M'' \setminus y/p'', q''$ are not 2-connected, $M \setminus x, y$ is not 2-connected. Now let $x, y$ be elements of $E(M'/p', q')$. Then $M' \setminus x, y/p', q'$ is not 2-connected. For if $M' \setminus x, y/p', q'$ is 2-connected, then, by (2.7) applied to $M' \setminus x, y$, either $p'$ or $q'$ is a coloop of $M' \setminus x, y$. But then, since $\{e, p', q'\}$ is contained in a series class $M'$, the element $e$ is a coloop of $M' \setminus x, y$ and of $M' \setminus x, y/p', q'$, a contradiction. Thus, for all elements $x, y$ of $E(M'/p', q') - e$, the matroid $M \setminus x, y$ is not 2-connected. Similarly, for all elements $x, y$ of $E(M''/p'', q'') - e$, the matroid $M \setminus x, y$ is not 2-connected. \( \triangledown \)

The matroid $U[l, k, 3; e]$ is the cycle matroid of the following graph.

![Graph](image)

**FIGURE 4**
(3.9) Corollary. Let \( M \) be a 2-connected matroid having at least two elements and \( r \) be the rank of \( M \). Suppose that, for all elements \( x,y \) of \( E(M) \), the matroid \( M\setminus x,y \) is not 2-connected, and that \( M \) has exactly one element \( e \) such that \( M\setminus e \) is 2-connected. Then \( M \) has at most \( \frac{1}{2} (3r - 1) \) elements. Moreover, the upper bound is attained if and only if \( M = U[1, \frac{r-1}{2}, 3; e] \).

Proof. Since \( M\setminus e \) is \((2,2)\) and has at least six elements, \( \text{rk}(M\setminus e) \) is at least five, by (2.12). So, by (2.15), and since \( \text{rk}(M\setminus e) = \text{rk}M \), \( |E(M)| - 1 = |E(M\setminus e)| \leq \frac{3}{2}(\text{rk}(M\setminus e) - 1) = \frac{3}{2}(r - 1) \). Hence \( |E(M)| \leq \frac{1}{2}(3r - 1) \).

Now suppose \( |E(M)| = \frac{1}{2}(3r - 1) \). If \( M/e \) is 2-connected, then, by (3.8)(i), \( M/e \) is \((2,2)\). So, since \( M/e \) has at least six elements, it follows from (2.15) and the equality \( \text{rk}(M/e) = \text{rk}M - 1 \) that
\[
|E(M/e)| \leq \frac{3}{2}(\text{rk}(M/e) - 1) = \frac{3}{2}(r - 2) \ldots (1).
\]

Also,
\[
|E(M/e)| = |E(M)| - 1 = \frac{1}{2}(3r - 1) - 1 = \frac{3}{2}(r - 1) \ldots (2).
\]
So, using (1) and (2), \( \frac{3}{2}(r - 1) \leq \frac{3}{2}(r - 2) \), a contradiction. Therefore, \( M/e \) is not 2-connected. Thus, by (3.8)(ii), \( M/e = P((M'/p',q';e), (M''/p'',q'';e)) \) where \( M' \) and \( M'' \) are \((2,2)\) matroids each having at least six elements and \( \{ e,p',q' \}, \{ e,p'',q'' \} \) are contained in series classes of \( M',M'' \), respectively. Now, by (1.42), \( \text{rk}M = \text{rk}(M'/p'q') + \text{rk}(M''/p'',q'') - 1 \). So,
rkM = rkM' + rkM" - 5 \ldots \ldots \ldots \ldots (3).

Moreover,

$$|E(M)| = |E(M')| + |E(M'')| - 5 \ldots \ldots (4).$$

Now to show that the conclusion holds it is sufficient to prove that

$$|E(M')| = \frac{3}{2}(rkM' - 1) \text{ and } |E(M'')| = \frac{3}{2}(rkM'' - 1),$$

that is, $M' = U[1, \frac{rkM' - 1}{2}, 3]$ and $M'' = U[1, \frac{rkM'' - 1}{2}, 3]$.

Suppose that $$|E(M')| < \frac{3}{2}(rkM' - 1) \text{ or } |E(M'')| < \frac{3}{2}(rkM'' - 1).$$

Then, by (4) and (3),

$$\frac{3}{2}r - \frac{1}{2} = |E(M)| < \frac{3}{2}(rkM' - 1) + \frac{3}{2}(rkM'' - 1) - 5 = \frac{3}{2}(rkM' + rkM'') - 8 = \frac{3}{2}(r + 5) - 8 = \frac{3}{2}r - \frac{1}{2},$$

a contradiction. Therefore $$|E(M')| = \frac{3}{2}(rkM' - 1) \text{ and } |E(M'')| = \frac{3}{2}(rkM'' - 1).$$

Thus, by (2.15), $M' = U[1, \frac{rkM' - 1}{2}, 3]$ and $M'' = U[1, \frac{rkM'' - 1}{2}, 3]$. Hence the required conclusion follows since the parallel connection of the last two matroids is $U[1, \frac{r - 1}{2}, 3; e]$. Evidently, the converse also holds. \(\Box\)

SECTION 2. THE CASE $2 \leq |A| \leq |E(M)| - 3$

We shall show that either $M \setminus e$ is $(2,2)$, for all elements $e$ of $A$, or $M$ has a $(2,2)$ deletion minor having three elements fewer than $M$.

The next lemma shows that, unless $A$ is a 2-circuit, $A$ contains no 2-circuits of $M$.

(3.10) Lemma. If \{e,f\} is a 2-circuit of $M$ contained in $A$, then $A = \{e,f\}$.
Proof. Suppose that $g$ is an element of $A - \{ e,f \}$. Then, since $\{ e,f \}$ is a 2-circuit of the 2-connected matroid $M \setminus g$, the matroid $M \setminus g,e$ is 2-connected, a contradiction. \( \square \)

(3.11) Lemma. Suppose $A = \{ e,f \}$ and $A$ is a 2-circuit of $M$. Then either $M \setminus e$ and $M \setminus f$ are (2,2) or, for some element $x$ such that $\{ e,f,x \}$ is a triad of $M$, the matroid $M \setminus e,f,x$ is (2,2).

Proof. Since $| E(M) - A | \geq 3$, $M \setminus e$ has at least four elements. Suppose $M \setminus e$ is not (2,2). Then there are two distinct elements $x,p$ of $E(M) - e$ such that $M \setminus e,x,p$ is 2-connected. Since $M \setminus x,p$ is not 2-connected, $e$ is a coloop of $M \setminus x,p$. Now, by (3.1), $e$ is contained in no 2-cocircuits of $M$. So, $\{ e,x,p \}$ is a triad of $M$. Now, by orthogonality between the 2-circuit $\{ e,f \}$ and the triad $\{ e,x,p \}$, we must have that $x = f$ or $p = f$. Without loss of generality, we may assume that $p = f$.

Now we show that $M \setminus e,f,x$ is (2,2). First note that, since, by (3.1), $M \setminus e,f,x$ has no circuits with fewer than three elements, $M \setminus e,f,x$ has at least three elements. Also, since $M \setminus e$ is minimally 2-connected, $M \setminus e$ contains no triangles. Therefore, since $\{ e,f \}$ is a 2-circuit of $M$, $M$ contains no triangles.
Now let $y$ be an element of $E(M) - \{e,f,x\}$, and suppose that $M\setminus e, f, x, y$ is 2-connected. Then, since $\{e,f,x\}$ is a triad of $M$, $M\setminus e, f, x, y = M\setminus e/f\setminus x, y$. But then, since $M\setminus e/f\setminus x, y = M\setminus e, y/f\setminus x$ and $M\setminus e, y$ is not 2-connected, $\{e,f,y\}$ or $\{e,x,y\}$ is a triad of $M$, or $\{x,y\}$ is a 2-cocircuit of $M$, by (2.7). In each case, $M\setminus e, f, x$ contains a coloop, a contradiction. Thus, for all elements $y$ of $E(M) - \{e,f,x\}$, the matroid $M\setminus e, f, x, y$ is not 2-connected.

Let $y, z$ be distinct elements of $E(M) - \{e,f,x\}$. Assume that $M\setminus e, f, x, y, z$ is 2-connected. Since $M\setminus y, z$ is not 2-connected, $\{e,f,x\}$ contains a component of $M\setminus y, z$. Now, since $x$ is not in parallel with either $e$ or $f$, the element $x$ is a coloop of $M\setminus y, z$. Also, since $M\setminus e, f, x$ is 2-connected, neither $\{x,y\}$ nor $\{x,z\}$ is a 2-cocircuit of $M$. Therefore, since $M$ is 2-connected, $\{x,y,z\}$ is a triad of $M$. Now, since $M\setminus y\setminus e, f, x, z$ is 2-connected but $M\setminus y$ is not, $\{e,f,x,z\}$ contains a component of $M\setminus y$. Clearly, $\{e,f,x,z\}$ contains no coloops of $M\setminus y$. Therefore, since $\{e,f,x\}$ is a triad and $\{x,z\}$ is a 2-cocircuit of $M\setminus y$, $\{e,f,x,z\}$ is a component of $M\setminus y$. Hence, since $\{e,f\}$ is the only 2-circuit of $\{e,f,x,z\}$, there is a triangle contained in $\{e,f,x,z\}$. This is a contradiction since $M$ contains no triangles. Thus, for all elements $y, z$ of $E(M) - \{e,f,x\}$, the matroid $M\setminus e, f, x, y, z$ is not 2-connected.
Lemma. Suppose \(A\) contains no 2-circuits of \(M\), and let \(\{p, q, r\}\) be a triad of \(M\). If \(M \setminus p, q, r\) is 2-connected then \(\{p, q, r\} \subseteq A\).

Proof. Since \(\{p, q, r\}\) is a triad, \(M \setminus p, q, r = M \setminus p/q/r\). If \(M \setminus p\) is not 2-connected, then, by (2.7) applied to \(M \setminus p\), either \(q\) or \(r\) is a coloop of \(M \setminus p\). Therefore either \(\{p, q\}\) or \(\{p, r\}\) is a 2-cocircuit of \(M\), a contradiction. Thus \(M \setminus p\) is 2-connected, that is, \(p\) is an element of \(A\). Similarly, \(q\) and \(r\) are elements of \(A\). \(\forall\)

Lemma. Suppose \(A\) contains no 2-circuits of \(M\), and let \(\{e, f, g\}\) be a triad of \(M\). If \(M \setminus e, f, g\) is 2-connected, then it is \((2, 2)\).

Proof. By the last lemma, \(\{e, f, g\}\) is contained in \(A\). Also, since \(|A| \leq |E(M)| - 3\), \(M \setminus e, f, g\) has at least three elements. Now let \(p\) be an element of \(E(M) - \{e, f, g\}\). If \(M \setminus e, f, g, p\) is 2-connected, then, since \(M \setminus e, f, g, p = M \setminus e, p/f/g\) is 2-connected and \(M \setminus e, p\) is not 2-connected, either \(f\) or \(g\) is a coloop of \(M \setminus e, p\), by (2.7) applied to \(M \setminus e, p\). Therefore, since, by (3.1), neither \(e\) nor \(f\) is in a 2-cocircuit of \(M\), either \(\{e, f, p\}\) or \(\{e, g, p\}\) is a triad of \(M\). But then \(M \setminus e, f, g\) contains a coloop, a contradiction since \(M \setminus e, f, g\) is 2-connected and has more than two elements. We conclude that for all \(p\) in \(E(M) - \{e, f, g\}\), the matroid \(M \setminus e, f, g \setminus p\) is not 2-connected.
Let \( p, q \) be distinct elements of \( E(M) \setminus \{ e, f, g \} \). Suppose that \( M\setminus e, f, g, p, q \) is 2-connected. Then, since \( M\setminus e, f, g, p \) is not 2-connected, \( q \) is in a cocircuit of \( M \) contained in \( \{ e, f, g, p, q \} \). Moreover, as \( M\setminus e, f, g \) is 2-connected, every such cocircuit containing \( q \) also contains \( p \).

Now suppose that \( \{ p, q \} \) is a 2-cocircuit of \( M \). Then, since \( M\setminus e, f, g, p, q = M\setminus p, q \setminus e/ f/ g \) and \( M\setminus p, q \) is not 2-connected, at least one of \( \{ f, e, p \}, \{ f, e, q \}, \{ g, e, p \} \) and \( \{ g, e, q \} \) is a triad of \( M \). But then \( M\setminus e, f, g \) contains a coloop, a contradiction. Thus \( \{ p, q \} \) is not a 2-cocircuit of \( M \).

Next we show that \( \{ e, f, g, p, q \} \) contains a 4-cocircuit of \( M \) containing \( p \) and \( q \). Suppose that the only cocircuits of \( M \) containing \( \{ p, q \} \) and contained in \( \{ e, f, g, p, q \} \) are triads. Without loss of generality, we may assume that \( \{ e, p, q \} \) is a triad of \( M \). Then, since \( M\setminus e, f, g, p = M\setminus e, f, g, p \), the element \( q \) is a coloop of \( M\setminus e, f, g, p \). So, either \( \{ f, p, q \} \) or \( \{ g, p, q \} \) is a triad of \( M \). Therefore, by circuit elimination, either \( \{ e, f, q \} \) or \( \{ e, g, q \} \) is a triad of \( M \). This is a contradiction to the fact that every cocircuit contained in \( \{ e, f, g, p, q \} \) and meeting \( \{ p, q \} \) contains \( \{ p, q \} \). Hence \( \{ e, f, g, p, q \} \) contains a 4-cocircuit containing \( \{ p, q \} \).

Assume that \( \{ e, f, p, q \} \) is a 4-cocircuit. Then, since \( M\setminus e, p \) is not 2-connected and has no coloops, it has at least two components and every component has at least two elements. Let \( K \) be the component of \( M\setminus e, p \) containing \( f \). Then, since \( \{ f, g, q \} \) is a series class of \( M\setminus e, p \), the set \( \{ f, g, q \} \) is
contained in \( K \). Moreover, since \( M \setminus e, p, f, g, q \) is 2-connected, 
\( \{ f, g, q \} = K \). Now, since \( M \) has no 2-circuits, \( \{ f, g, q \} \) is a triangle of \( M \). Similarly, by considering \( M \setminus e, q \), the set 
\( \{ f, g, p \} \) is a triangle of \( M \). Then, by circuit elimination, 
\( \{ f, p, q \} \) is a triangle of \( M \). But then we have a contradiction, since the triad \( \{ e, f, g \} \) and the triangle 
\( \{ f, p, q \} \) have exactly one element in common. Hence, for all elements \( p, q \) of \( E(M) - \{ e, f, g \} \), the matroid \( M \setminus e, f, g, p, q \) is not 2-connected. \( \nabla \)

(3.14) Theorem. Suppose that \( A \) contains no 2-circuits of \( M \). Then either

(i) \( M \setminus e \) is (2,2) for all elements \( e \) of \( A \); or

(ii) \( M \setminus e, f, g \) is (2,2) where \( \{ e, f, g \} \) is a triad of \( M \) 
    contained in \( A \).

Proof. Suppose that, for some element \( e \) of \( A \), the matroid 
\( M \setminus e \) is not (2,2). Then there exist \( f, g \) elements of \( E(M) - e \) 
such that \( M \setminus e, f, g \) is 2-connected. Then, since \( M \setminus f, g \) is not 
2-connected and \( e \) is in no 2-cocircuits of \( M \), the set \( \{ e, f, g \} \) 
is a triad of \( M \). Then, by (3.12), \( \{ e, f, g \} \) is contained in \( A \) 
and, by (3.13), \( M \setminus e, f, g \) is (2,2). \( \nabla \)

Combining (3.11) and (3.14), we obtain
(3.15) Theorem. Suppose $2 \leq |A| \leq |E(M)| - 3$. then either

(i) $M \setminus e$ is $(2,2)$ for all elements $e$ of $A$; or

(ii) $M \setminus T^*$ is $(2,2)$ where $T^*$ is a triad of $M$ and $|T^* \cap A| \geq 2$.

(3.16) Corollary. Suppose $2 \leq |A| \leq |E(M)| - 3$ and $rkM = r$. If $r \geq 4$, then $|E(M)| \leq \frac{3}{2} r$.

Proof. If $|E(M)| = 5$, then, for some element $e$ of $M$, the matroid $M \setminus e$ is $(2,2)$ and has four elements. So, by (2.13) $M \setminus e = C_4$ and, therefore, $rk(M \setminus e) = 3$. But then, since $rk(M \setminus e) = rkM$, the rank of $M$ is 3. Thus we may assume that $|E(M)| \geq 6$. Suppose, in addition, that $|E(M)| \leq 8$. Then it is not difficult to check that $M$ contains no triad $T^*$ such that $M \setminus T^*$ is $(2,2)$. Also, for some element $e$ of $M$, $M \setminus e$ is $(2,2)$, and using (2.12) and (2.15), we get that $|E(M)| \leq \frac{3}{2} r$.

Now assume that $|E(M)| \geq 9$. Then if $T^*$ is a triad of $M$ and $M \setminus T^*$ is $(2,2)$, then $|E(M \setminus T^*)| \geq 6$. So, by (2.15), $|E(M \setminus T^*)| \leq \frac{3}{2} (rk(M \setminus T^*) - 1)$. Thus, since $rk(M \setminus T^*) = rkM - 1$, $|E(M \setminus T^*)| \leq \frac{3}{2} (r - 2)$. Therefore, $|E(M)| \leq \frac{3}{2} (r - 2) + 3 = \frac{3}{2} r$. On the other hand, if, for some element $e$ of $E(M)$, the matroid $M \setminus e$ is $(2,2)$, then $|E(M \setminus e)| \geq 8$. So, by (2.15), $|E(M \setminus e)| \leq \frac{3}{2} (rk(M \setminus e) - 1)$. Thus, since $rk(M \setminus e) = rkM$, we have that $|E(M \setminus e)| \leq \frac{3}{2} (r - 1)$. Therefore, $|E(M)| \leq \frac{3}{2} (r - 1) + 1 = \frac{3}{2} r - \frac{1}{2}$. \[\square\]
SECTION 3. THE CASE $A = E(M)$

First we show that $M$ is 3-connected and thus, $M$ is $(2,3)$. Then we show that every pair of elements of $M$ is contained in a triad. This shows that the duals of Sylvester matroids are exactly those matroids that are $(2,3)$.

(3.17) Theorem. Let $M$ be a 2-connected matroid having at least two elements. If, for all elements $e, f$ of $E(M)$, the matroid $M\setminus e$ is 2-connected and $M\setminus e, f$ is not 2-connected, then $M$ is 3-connected.

Proof. Evidently $M$ has at least four elements. Moreover, $M$ contains no 2-cocircuits or 2-circuits. Now suppose $M$ is not 2-connected. Then there is a 2-separation $\{X, Y\}$ of $M$ such that $|X|, |Y| > 2$. So, by (1.46), $M = M' \Delta M'' = S((M';p), (M'';p)/p$ where $p$ is not an element of $X \cup Y$ and $M'$ and $M''$ are isomorphic to 2-connected minors of $M$. Furthermore, $E(M') - p = X$ and $E(M'') - p = Y$.

Let $e$ be an element of $X$. Then, since $M\setminus e = S((M'\setminus e; p), (M''; p))/p$ and $M\setminus e$ is 2-connected, $M'\setminus e$ is 2-connected. Similarly, for an element $f$ of $Y$, the matroid $M''\setminus f$ is 2-connected. Therefore, by (1.38), $S((M'\setminus e; p), (M''\setminus f; p))$ is 2-connected. Thus, since $S((M'\setminus e; p), (M''\setminus f; p))/p$ is not 2-connected, $S((M'\setminus e; p), (M''\setminus f; p))/p$ is 2-connected. But then we have a contradiction.
since the last matroid is $M \setminus e, f$ which is not 2-connected. Hence $M$ has no 2-separations, that is, $M$ is 3-connected. $\forall$

(3.8) Lemma. Let $C$ be a circuit of $M$ and $g$ be an element of $E(M)$ not contained in $C$. Then there is a triad $\{g, p, q\}$ of $M$ such that $\{p, q\} \subseteq C$.

Proof. Since $C$ is a circuit of the 2-connected matroid $M \setminus g$ and, for all elements $h$ of $C$, the matroid $M \setminus g, h$ is not 2-connected, $C$ contains a 2-cocircuit $\{p, q\}$ of $M \setminus g$, by (2.17). Hence, since $M$ contains no 2-cocircuits, $\{g, p, q\}$ is a triad of $M$. $\forall$

(3.9) Theorem. Let $e, f$ be distinct elements of $M$. Then $\{e, f\}$ is contained in a triad of $M$.

Proof. Suppose $\{e, f\}$ is contained in no triads of $M$. Then, since $M \setminus e, f$ is not 2-connected and has no coloops, we can find two components $K, K'$ of $M \setminus e, f$ such that each has at least two elements. Let $g$ be an element of $K$ and $C$ be a circuit contained in $K'$. Then, by (3.8), $M$ has a triad $\{g, p, q\}$ with $\{p, q\} \subseteq C$. Therefore, $\{g, p, q\}$ contains a cocircuit of $M \setminus e, f$ containing $g$ and having at least two elements. But then we have a contradiction to (1.15) since this cocircuit meets the distinct components $K$ and $K'$. Hence $\{e, f\}$ is contained in a triad of $M$. $\forall$
Murty [15] investigated the matroids $M$ that have the property that every pair of elements of $M$ is contained in a triangle. He called these Sylvester matroids. Theorem 3.15 shows that the dual of a Sylvester matroid, and therefore, a Sylvester matroid, is 3-connected. Thus combining (3.17) and (3.19), we have the following characterization of $(2,3)$-matroids.

(3.20) Theorem. The following are equivalent for a matroid $M$ having at least four elements.

1. $M$ is 3-connected and for all elements $e,f$, the matroid $M\setminus e,f$ is not 2-connected.
2. Every pair of elements of $M$ is in a triad.
3. $M$ is the dual of a Sylvester matroid.
The main result of this chapter states that if $M$ is a matroid other than a wheel of rank greater than three, and $M$ is $(2,3)$ relative to a circuit $C$, then every pair of elements of $C$ is in a triad of $M$. The proof of this result is rather long, and so, we divide the chapter into three sections.

The first section gives some results about the existence of certain triangles and triads in matroids that are $(1,3)$ or $(2,3)$ relative to a circuit $C$. In the second section, we prove the following weaker version of the main theorem: If $M$ is $(2,3)$ relative to a circuit $C$, then every element of $C$ is in a triad of $M$. Finally, in the third section, we give a proof of the main theorem.

SECTION 1. PRELIMINARY LEMMAS

The following lemma is of fundamental importance in the proof of the wheels and whirls theorem, see Tutte [24].

(4.1) Lemma. Let $M$ be a 3-connected matroid having at least four elements, and let $\{e,f,g\}$ be a triangle of $M$. If $M\backslash e$ and $M\backslash f$ are not 3-connected, then $e$ is in a triad of $M$ with exactly one of $f$ and $g$. 

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Oxley [19] proved that if $M$ is a $(1,3)$-matroid having at least four elements, then every circuit $C$ intersects at least two triads of $M$. In [20], he asked if the corresponding result holds for $(1,3)$-matroids relative to a circuit $C$. Lemos [10] gave the following affirmative answer.

(4.2) Lemma. Let $M$ be a matroid having at least four elements. If $M$ is $(1,3)$ relative to a circuit $C$, then $C$ intersects at least two triads of $M$.

Oxley [19] proved the dual of the following result.

(4.3) Lemma. Let $M$ be an $n$-connected matroid having at least $2(n - 1)$ elements. Suppose that $A$ is a subset of $E(M)$ having fewer than $n$ elements. If $M/A$ is $n$-connected, then, for every subset $A'$ of $A$, the matroid $M/A'$ is $n$-connected.

The following is a strengthening of this result.

(4.4) Lemma. Let $M$ be an $n$-connected matroid having at least $2(n - 1)$ elements. Suppose that $A$ is an independent subset of $E(M)$. If $M/A$ is $n$-connected, then, for every subset $A'$ of $A$, the matroid $M/A'$ is $n$-connected.

Proof. Suppose that $M/A'$ is not $n$-connected for some subset $A'$ of $A$, and assume that $|A'|$ is maximal with respect
to this property. Then, for an element $e$ of $A - A'$, the matroid $M/A'/e$ is $n$-connected. So, by (1.31) applied to $M/A'$, the element $e$ is a loop of $M/A'$ or $M/A'$ has a cocircuit containing $e$ and having fewer than $n$ elements. Now if $e$ is a loop of $M/A'$, then, since $M$ has no loops, $e$ is in a circuit of $M$ contained in $A' \cup e$. This is a contradiction since $A' \cup e$ is independent. On the other hand, if $e$ is in a cocircuit $C^*$ of $M/A'$ having less than $n$ elements, then $C^*$ is a cocircuit of $M$ having less than $n$ elements. This is a contradiction since an $n$-connected matroid has no cocircuits of cardinality less than $n$, by (1.23). Hence, for every subset $A'$ of $A$, the matroid $M/A'$ is $n$-connected. \[\Box\]

The next lemma is used frequently in the proof of the main theorem.

(4.5) Lemma. Let $C$ be a circuit of a 3-connected matroid $M$. Suppose that $e,f,p$ are elements of $C$ such that $p$ is in triads of $M$ with every element of $C - \{e,f\}$. If $M\setminus e,f$ is not 2-connected, then $\{e,f\}$ is contained in a triad of $M$.

Proof. Since $M\setminus e,f$ is not 2-connected, it has a 1-separation $\{X,Y\}$ such that $rkX + rkY = rk(M\setminus e,f)$. Without loss of generality, we may assume that $p$ is in $X$. Now let $T^*$ be a triad of $M$ containing $p$. If the set $T^* - \{e,f\}$ contains a coloop of $M\setminus e,f$, then, since $M$ is 3-connected, $\{e,f\}$ is contained in a triad of $M$. So we may assume that, for every
triad $T^*$ of $M$ containing $p$, the set $T^* - \{e,f\}$ contains no coloops of $M\setminus e,f$. Thus, for every such triad $T^*$, the set $T^* - \{e,f\}$ is either a triad of $M\setminus e,f$, or is contained in a series class of this matroid. So, since every element of $C - \{e,f\}$ is contained in a triad of $M$ containing $p$, every element of $C - \{e,f\}$ is contained in $X$. Thus $\text{rk}(X \cup \{e,f\}) \leq \text{rk}X + 1$. Therefore, $\text{rk}(X \cup \{e,f\}) + \text{rk}Y \leq \text{rk}X + 1 + \text{rk}Y = \text{rk}(M\setminus e,f) + 1$. Since $\text{rk}(M\setminus e,f) = \text{rk}M$, it follows that $\text{rk}(X \cup \{e,f\}) + \text{rk}Y \leq \text{rk}M + 1$. Thus, since $M$ is 3-connected, $|Y| = 1$; that is, $Y$ consists of a coloop of $M\setminus e,f$. Hence $\{e,f\}$ is contained in a triad of $M$. □

(4.6) Lemma. Let $M$ be a 3-connected matroid and $\{e,f\}$ be a subset of $E(M)$. Let $A$ be an independent subset of $M$ contained in $E(M) - \{e,f\}$. If $M\setminus e,f$ is not 2-connected and $M\setminus e,f/A$ is 2-connected, then there is an element $p$ of $A$ such that $\{e,f,p\}$ is a triad of $M$.

Proof. By (2.7) applied to $M\setminus e,f$ with $A' = \emptyset$, the set $A$ contains a coloop $p$ of $M\setminus e,f$. Since $M$ is 3-connected, $M$ contains no cocircuits having less than three elements, by (1.23). Hence $\{e,f,p\}$ is a triad of $M$. □

The next lemma gives a lower bound on the number of elements of a $(2,3)$-matroid relative to a set $\{e,f\}$ in the case when $\{e,f\}$ is contained in no triads of $M$. 


(4.7) Lemma. Let \( M \) be a 3-connected and \( \{ e, f \} \) be a subset of \( E(M) \). If \( \{ e, f \} \) is contained in no triads of \( M \) and \( M \setminus e, f \) is not 2-connected, then \( M \) has at least eight elements.

Proof. Since \( M \setminus e, f \) is not 2-connected and has no coloops, we can find two components of \( M \setminus e, f \) each having at least two elements. But, since \( M \) has no 2-circuits, each of these components has at least three elements. Hence \( M \) has at least eight elements. \( \forall \)

For the rest of this section, we assume that \( M \) is (2,3) relative to a circuit \( C \).

(4.8) Lemma. Let \( e, f \) be distinct elements of \( C \). If \( \{ e, f \} \) is contained in no triads of \( M \), then \( E(M) - C \) contains at least four elements.

Proof. If \( E(M) - C = \{ x \} \), then, since it is a cohyperplane, \( \text{cork} M = 2 \). Therefore, since \( M \) contains no 2-cocircuits, \( \{ e, f, x \} \) is a triad. Thus \( |E(M) - C| \geq 2 \). Now if \( E(M) - C = \{ x, y \} \), then \( \text{cork} M = 3 \). Therefore, since \( \{ e, f \} \) is contained in no triads of \( M \), the set \( \{ e, f, x, y \} \) is a cocircuit of \( M \). So \( \{ x, y \} \) is a 2-cocircuit of \( M \setminus e, f \), and consequently, of \( M \setminus e, f / (C - \{ e, f \}) \). Thus the matroid \( M \setminus e, f / (C - \{ e, f \}) \) is 2-connected. Therefore, by (4.6) applied to \( M \setminus e, f \) and the independent set \( C - \{ e, f \} \), there is an
element p of C - {e,f} such that {e,f,p} is a triad of M, a contradiction.

Now suppose E(M) - C = \{x,y,z\}. Then corkM is three or four. For if corkM = 2, then, for any element p of E(M) - {e,f}, the set \{e,f,p\} is a triad of M, a contradiction. Thus cork(M\{e,f\}) is one or two. Therefore, \{x,y,z\} is contained in a series class of M\{e,f\} or \{x,y,z\} is a triad of M\{e,f\}. In both cases, M\{e,f\}/(C - \{e,f\}) is 2-connected. Therefore, by (4.6), there is an element p of C - \{e,f\} such that \{e,f,p\} is a triad of M, a contradiction. Hence E(M) - C contains at least four elements. □

(4.9) Lemma. Let A be a proper (possibly empty) subset of C such that M/A is 3-connected. Suppose that E(M) - C contains at least four elements. If, for some element g of C - A, the matroid M/A\g has no non-minimal 2-separations, then g is in triads of M with every element of C - A.

Proof. First suppose that M/A\g is 3-connected, and let h be an element of C - A. Then, since E(M/A\g) contains at least four elements, M/A\g,h is 2-connected. Therefore, since M\g,h is not 2-connected, there is an element p of A such that \{g,h,p\} is a triad of M, by (4.6).

Now suppose that M/A\g has a 2-separation. Let T_1^*, T_2^*, ..., T_k^* be the triads of M/A containing g. Since M/A is 3-connected and A is a proper subset of C, the set C - A
contains at least three elements. Thus, since \( E(M) - C \)
contains at least four elements, \( E(M/A) \) contains at least seven
elements. Therefore, for any subset \( \{ i, j \} \) of
\( \{ 1, 2, \ldots, k \} \), \( T_i^* \cap T_j^* = \{ g \} \), by the dual of (1.25).
Moreover, by orthogonality, \( T_i^* \cap (C - A) \) contains at least two
elements for all \( i \) in \( \{ 1, 2, \ldots, k \} \). Now choose exactly one
element from each of the sets
\( (T_i^* - g) \cap C, \ldots, (T_k^* - g) \cap C \), and let \( D \) be the set of
these elements. Then, by (1.27), \( M/A\backslash g/D \) is 3-connected. Now
if \( h \) is an element of \( C - A \) which is not in a triad of \( M/A \) with
g, then, since \( E(M/A\backslash g/D) \) contains at least four elements,
\( M/A\backslash g/D\backslash h \) is 2-connected. Therefore, since \( M\backslash g, h \) is not
2-connected and \( A \cup D \) is independent there is an element \( p \) of
\( A \cup D \) such that \( p \) is a coloop of \( M \), by (2.7). Therefore,
since \( M \) is 3-connected, \( \{ g, h, p \} \) is a triad of \( M \).  

(4.10) Lemma. Suppose \( E(M) - C \) has at least four elements. Let \( p, q, r \)
be elements of \( C \). Then there is no element \( z \) of
\( E(M) - C \) such that \( \{ p, q, z \} \) is a triangle and
\( \{ p, r, z \} \) is a triad of \( M \).

Proof. Suppose that \( z \) is an element of \( E(M) - C \) such that
\( \{ p, q, z \} \) is a triangle and \( \{ p, r, z \} \) is a triad of \( M \). Then,
since \( M\backslash r \) is 2-connected and \( M\backslash r, q \) is not 2-connected, \( M\backslash r/q \) is
2-connected. Now \( M\backslash r/q\backslash p \) has at least four elements, and has \( z \)
as a coloop. Therefore $M \setminus r/q/p$ is 2-connected, by (1.26). This is a contradiction since $z$ is a loop of $M \setminus r/q/p$. \( \Box \)

The next result strengthens the preceding lemma.

**Lemma.** Suppose $E(M) - C$ has at least four elements. Let $p, q$ be elements of $C$. Then there is no element $z$ of $E(M) - C$ such that $\{p, q, z\}$ is a triangle of $M$.

**Proof.** Suppose that, for some element $z$ of $E(M) - C$, the set $\{p, q, z\}$ is a triangle of $M$. By circuit elimination, there is a circuit $C'$ of $M$ that contains $z$ and is contained in $\{q, z\} \cup (C-p)$. Now, since $M/z$ is 2-connected having $\{p, q\}$ as a circuit, $M/z \setminus p$ is 2-connected. Also $C' - z$ is a circuit of $M/z \setminus p$. Now if, for every element $r$ of $C' - \{z, q\}$, the matroid $M/z \setminus p, r$ is not 2-connected, then, by (2.17), $C' - \{z, q\}$ contains a 2-cocircuit of $M/z \setminus p$. But then, since $M$ is 3-connected, $p$ is in a triad of $M$ which is contained in $C - \{q, z\}$. This is a contradiction since this triad meets the circuit $\{p, q, z\}$ in a single element. Hence we may assume that, for some element $r$ of $C' - \{q, z\}$, the matroid $M/z \setminus p, r$ is 2-connected. But then, since $M/p, r$ is not 2-connected, $\{p, r, z\}$ is a triad of $M$, by (4.6). This contradicts the last lemma. \( \Box \)
(4.12) Lemma. Suppose C has at least four elements and E(M) has at least seven elements. Let p, q be elements of C and x be an element of E(M) such that \( \{p, q, x\} \) is a triad of M. Suppose that M/p has a 2-separation. If M/p has only minimal 2-separations, then the following statements hold.

(i) p is in a unique triangle \( \{p, x, y\} \) where y is an element of \( E(M) - C \). Moreover, x is not an element of C.

(ii) \( \{p, q, x\} \) is the unique triad of M containing x.

(iii) M/q is not 3-connected.

Proof. (i) Suppose that T is a triangle of M containing p. Then, since C has at least four elements, T contains an element y of \( E(M) - C \). Also, T contains an element of the triad \( \{p, q, x\} \) other than p. If q is in T, then \( \{p, q, y\} \) is a triangle of M. This is a contradiction to (4.11). Therefore x is in T. Now, by (4.11), x is not an element of C. Thus every triangle of M that contains p also contains x. Now, since M/p has only minimal 2-separations and E(M) contains at least seven elements, \( \{p, x, y\} \) is the unique triangle containing p, by (1.25).

(ii) Suppose \( T^* \) is a triad of M containing x and distinct from \( \{p, q, x\} \). Then, since x is the triangle \( \{p, x, y\} \), the triad \( T^* \) contains x and at least one of p and y. But, since M is a 3-connected matroid having at least seven elements, \( T^* \)
contains exactly one of p and y, by (1.24). Now if
$T^* = \{x, p, z\}$ where z is in $E(M) - \{q, y\}$, then, by circuit
elimination, $\{p, q, z\}$ is a triad of $M$. This is a
contradiction since this set has exactly one element in common
with the triangle $\{p, x, y\}$. On the other hand, if
$T^* = \{x, y, z\}$, then $\{y, z\}$ is a 2-cocircuit of the
3-connected matroid $M/p \setminus x$, a contradiction.

(iii) Suppose $M/q$ is 3-connected. Then $\{p, x, y\}$ is a
triangle and $C - q$ is a circuit of $M/q$. Thus, by circuit
elimination, there is a circuit $C'$ of $M/q$ that contains $x$ and
is contained in $\{x, y\} \cup (C - \{p, q\})$. Now consider the
circuit $C' - x$ of the 2-connected matroid $M/q/x \setminus p$. If, for
every element $r$ of $C' - \{x, y\}$, the matroid $M/q/x \setminus p, r$ is not
2-connected, then, by (2.17), $C' - \{x, y\}$ contains a
2-cocircuit of $M/q/x \setminus p$. But then, since $M/q$ is 3-connected, $p$
is in a triad of $M/q$ contained in $E(M) - \{x, y\}$. This
contradicts orthogonality with the triangle $\{p, x, y\}$. So we
may assume that, for some element $r$ of $C' - \{x, y\}$, the
matroid $M/q/x \setminus p, r$ is 2-connected. But then, since $M \setminus p, r$ is
not 2-connected and $\{q, x\}$ is independent, either $\{p, q, r\}$
or $\{p, r, x\}$ is a triad of $M$, by (4.6). But $\{p, q, r\}$ cannot
be a triad of $M$ because of orthogonality with triangle
$\{p, x, y\}$. Also, $\{p, r, x\}$ cannot be a triad of $M$ because of
uniqueness of the triad $\{p, q, x\}$ containing $x$. Hence $M/q$ is
not 3-connected. ∇
(4.13) Lemma. Suppose $E(M) - C$ has at least four elements. Let $e, f, p$ be elements of $C$ and $x, y$ be elements of $E(M) - C$. If \{ $p, x, y$ \} is a triangle and \{ $e, f, x, y$ \} is a cocircuit of $M$, then either \{ $e, p, x$ \} and \{ $f, p, y$ \} are triads of $M$ or \{ $e, p, y$ \} and \{ $f, p, x$ \} are triads of $M$.

Proof. Since $M \setminus e/p$ is 2-connected and $M \setminus e/p/x$ has $y$ as a loop, $M \setminus e/p/x$ is 2-connected. Also, since $M \setminus e/p/x/y$ has at least three elements and has $f$ as a coloop, $M \setminus e/p/x/y$ is 2-connected. Since \{ $p, x, y$ \} is a triangle of $M \setminus e$, we have $M \setminus e/p/x/y = M \setminus e/p/x/y$, by (1.30). Now, since $M \setminus e/p$ is not 2-connected and \{ $x, y$ \} is independent, either \{ $e, p, x$ \} or \{ $e, p, y$ \} is a triad of $M$, by (4.6). If \{ $e, p, x$ \} and \{ $e, p, y$ \} are triads of $M$, then, by circuit elimination \{ $p, x, y$ \} is a triad of $M$. This is a contradiction to (1.24). Without loss of generality, we may assume that \{ $e, p, x$ \} is a triad of $M$.

Now by replacing $e$ by $f$ in the above argument, we similarly obtain that exactly one of \{ $f, p, x$ \} and \{ $f, p, y$ \} is a triad of $M$. If \{ $f, p, x$ \} is a triad of $M$, then, by circuit elimination applied to the triads \{ $e, p, x$ \} and \{ $f, p, x$ \}, we get that \{ $e, f, p$ \} is a triad of $M$. This is a contradiction since the triangle \{ $p, x, y$ \} has exactly one element in common with \{ $e, f, p$ \}. Therefore \{ $f, p, y$ \} is a triad of $M$. \n
SECTION 2. A WEAKER VERSION OF THE MAIN THEOREM

In this section we show that if $M$ is a $(2,3)$-matroid relative to a circuit $C$, then every element of $C$ is in a triad of $M$. To prove this, we assume that $e$ is an element of $C$ which is in no triads of $M$ and we utilize induction on $|C|$. In the first part of the proof, Lemma 4.14.1 through Lemma 4.14.6, we show that $M/e$ is 3-connected. We also show that, for any element $g$ of $C - e$, the matroid $M/g$ is 3-connected and $g$ is in a triad of $M$ which is contained in $C$.

Definition: Let $C$ be a circuit of a matroid $M$ and $x$ be an element of $E(M) - C$. Then $x$ is a chord of $C$ if $\text{rk}(C \cup x) = \text{rk}C$.

(4.14) Theorem. Let $M$ be a $(2,3)$-matroid relative to a circuit $C$. Then every element of $C$ is in a triad of $M$.

Proof. We argue by induction on $|C|$. If $|C| = 3$, the conclusion follows from (4.1), so suppose $|C| > 4$. Assume that $C$ has an element $e$ which is in no triads of $M$. Since, for any element $f$ of $C - e$, the set $\{e,f\}$ is contained in no triads of $M$, $E(M)$ has at least eight elements, by (4.7). Also, by (4.8), $E(M) - C$ has at least four elements.

Proof. Suppose that x is a chord of C. Then, for some subset F of C, the set F U x is a circuit of M. By circuit elimination, we may assume that e is not in F U x. Now F U x is a circuit of the 2-connected matroid M\e. Moreover, for all elements f of F, the matroid M\e,f is not 2-connected. Therefore, by (2.17), F contains a 2-cocircuit of M\e. Since M is 3-connected, it has no 2-cocircuits. Therefore, e is in a triad of M, a contradiction. √

(4.14.2) Lemma. Let A be a (possibly empty) subset of C such that C - A has at least four elements. If, for an element p of C - A, the set \{ p,x,y \} is a triangle of M/A, then \{ x,y \} is not contained in C - A. Moreover, \{ x,y \} U (C - A - p) is a circuit of M/A.

Proof. Since the circuit C - A of M/A has at least four elements, at least one of x and y, say x, is not in C. If y is an element of C, then x is a chord of the circuit C - A of M/A. Therefore, x is a chord of the circuit C of M, a contradiction to (4.14.1). Hence x and y are not in C.

Now, by circuit elimination, there is a circuit C' of M/A that contains x and is contained in \{ x,y \} U (C - A - p). If y is not C', then x is a chord of C', so x is a chord of C in M. Hence we must have that x is
in $C'$. Now assume that $C'$ is a proper subset of \{x,y\} U (C - A - p). Then by circuit elimination applied to the triangle \{p,x,y\} and the circuit $C'$ of $M/A$, there is a circuit $C''$ of $M/A$ contained in \{(p,y) U C'\} - x. Now $y$ is not in $C''$ otherwise, $y$ is a chord of $C - A$ in $M/A$ and hence is a chord of $C$ in $M$, a contradiction to (4.14.1). Hence $C''$ is a circuit of $M/A$ which is properly contained in the circuit $C - A$ of $M/A$, a contradiction. \(\checkmark\)

(4.14.3) Lemma. $M/e$ is 3-connected.

Proof. Suppose $M/e$ is not 3-connected. Then, since $M\setminus e$ is not 3-connected and $e$ is in no triads of $M$, there is a triangle \{e,x,y\} of $M$, by (1.27). Applying (4.14.2) with $A = \emptyset$, we get that $x$ and $y$ are not in $C$, and that \{x,y\} U (C - e) is a circuit of $M$. Now consider the circuit \{y\} U (C - e) of the 2-connected matroid $M/x\setminus e$. If, for some element $f$ of $C - e$, the matroid $M/x\setminus e,f$ is 2-connected, then, since $M\setminus e,f$ is not 2-connected, \{e,f,x\} is a triad of $M$, a contradiction. So we may assume that, for all elements $f$ of $C - e$, the matroid $M/x\setminus e,f$ is not 2-connected. Therefore, by (2.17), $C - e$ contains a 2-cocircuit of $M/x\setminus e$. This is a contradiction since $M$ contains no 2-cocircuits and $e$ is in no triads of $M$. \(\checkmark\)
(4.14.4) Lemma. Every element of \( C - e \) is in a triad of \( M \).

Proof. By the last lemma, \( M/e \) is 3-connected. Note that \( C - e \) is a circuit of \( M/e \). Now if, for distinct elements \( f, g \) of \( C - e \), the matroid \( M/e \setminus f,g \) is 2-connected, then \( \{ e,f,g \} \) is a triad of \( M \), a contradiction. Therefore, for all elements \( f,g \) of \( C - e \), the matroid \( M/e \setminus f,g \) is not 2-connected. Hence, by the induction hypothesis, every element of \( C - e \) is in a triad of \( M/e \), and consequently, of \( M \).

(4.14.5) Lemma. Let \( g \) be an element of \( C - e \). Then \( M/g \) is 3-connected.

Proof. Suppose that \( M/g \) has a 2-separation. By (4.14.4), \( g \) is in a triad \( \{ g,h,x \} \) of \( M \). Moreover, we may assume that \( h \) is in \( C - e \). Now, by (1.27), either \( M/g \) or \( M \setminus g \) has only minimal 2-separations. But, by (4.9) and the assumption that \( e \) is in no triads of \( M \), the matroid \( M \setminus g \) does not have only minimal 2-separations. Now, by (4.14.1), any triangle of \( M \) containing \( g \) contains two elements of \( E(M) - C \). Thus, since \( g \) is in the triad \( \{ g,h,x \} \), the element \( x \) is in \( E(M) - C \) otherwise, \( M \) has a triangle and a triad meeting only in \( g \). Therefore, by (4.12)(i), \( g \) is in a unique triangle \( \{ g,x,y \} \) of \( M \) and \( x \) and \( y \) are not in \( C \). Hence, as \( M/g \) has no non-minimal 2-separations, \( M/g \setminus y \) is 3-connected, by (1.27).
Consider the circuit $C - g$ of the 3-connected matroid $M/g\setminus y$. If, for distinct elements $p,q$ of $C - f$, the matroid $M\setminus y/g\setminus p,q$ is 2-connected, then, since $M\setminus p,q$ is not 2-connected, either $\{p,q,y\}$ or $\{p,q,g\}$ is a triad of $M$. This is a contradiction since each of $\{p,q,y\}$ and $\{p,q,g\}$ has exactly one element in common with the triangle $\{g,x,y\}$. Therefore, for all elements $p,q$ of $C - g$, the matroid $M\setminus y/g\setminus p,q$ is not 2-connected. Therefore, by the induction hypothesis, every element of $C - g$ is in a triad of $M\setminus y/g$. In particular, $e$ is in a triad $\{e,f,z\}$ of $M\setminus y/g$ where $f$ is an element of $C - g$. Since $e$ is in no triads of $M$, the set $\{e,f,y,z\}$ is a cocircuit of $M$. Also, by orthogonality of the triangle $\{g,x,y\}$ and the cocircuit $\{e,f,y,z\}$, we must have that $z = x$. Thus $\{e,f,x,y\}$ is a cocircuit of $M$. Therefore, by (4.13), $e$ is in a triad of $M$, a contradiction. \(\checkmark\)

(4.14.6) Lemma. Every element $g$ of $C - e$ is in a triad $\{f,g,h\}$ which is contained in $C - e$.

Proof. By (4.14.5), $M/g$ is 3-connected. Consider the circuit $C - g$ of the matroid $M/g$. If, for all distinct elements $f,h$ of $C - g$, the matroid $M/g\setminus f,h$ is not 2-connected, then, by the induction hypothesis, every element of $C - g$ is in a triad of $M/g$, and consequently, of $M$. This contradicts the assumption that $e$ is in no triads of $M$. 


Therefore, for some distinct elements, \( f, h \) of \( C - g \), the matroid \( M/g\f,h \) is 2-connected. Hence, since \( M/f,h \) is not 2-connected, \( \{ f, g, h \} \) is a triad of \( M \).

So far, we have shown that if the element \( e \) is not in a triad of \( M \), then the contraction of each element of \( C \) is 3-connected. Also, we have shown that every element of \( C - e \) is in a triad of \( M \) contained in \( C \). In the next, and final, stage of the proof, we contract as many elements of \( C - e \) as possible in such a way that the resulting contraction minor \( M' \) is 3-connected, and the circuit \( C' \) obtained from \( C \) contains no triads. Then we show that \( C' \) has an element \( g \) which is contained in a unique triangle \( T \) of the minor \( M' \). Then we put back some of the elements we had contracted in such a way that \( T \) remains a triangle of the enlarged minor \( M'' \) of \( M \). Finally, we show that \( e \) is in a triad of \( M'' \), thus obtaining a contradiction.

(4.14.7) Lemma. Let \( A \) be a subset of \( C - e \) and suppose that \( M/A \) is 3-connected. If \( g \) is in a triad of \( M/A \) which is contained in \( C - (A \cup e) \), then \( M/A/g \) is 3-connected.

Proof. Suppose that \( M/A/g \) has a 2-separation. If \( M/A\backslash g \) has no non-minimal 2-separations, then, by (4.9), \( g \) is in triads of \( M \) with each element of \( C - A \), a contradiction since \( e \) is in \( C - A \). Therefore, \( M/A/g \) has only minimal 2-separations, by (1.27). Let \( T \) be a triangle of \( M/A \) containing \( g \). Then, since \( g \) is in a triad of \( M/A \) which is
contained in $C - A$, the triangle $T$ contains an element of $C - A$ other than $g$. Moreover, since $C - A$ has at least four elements, $T \neq C - A$. So $T$ contains an element $x$ which is not in $C - A$. Therefore $x$ is a chord of $C - A$, and consequently, is a chord of $C$ in $M$, a contradiction to (4.14.1). Hence $M/A\backslash g$ is 3-connected. √

Let $A$ be a subset of $C - e$ such that $M/A$ is 3-connected and $|A|$ is maximal with respect to this property. Note that, by (4.14.5), $A$ is non-empty. Also, since $M/A$ is a 3-connected matroid having at least four elements, $C - A$ contains at least three elements.

(4.14.8) Lemma. $C - A$ has at least four elements.

Proof. Suppose that $C - A = \{e,f,g\}$. Note that $M/A\backslash e$ is not 3-connected. For if $M/A\backslash e$ is 3-connected, then, since $E(M/A\backslash e)$ has at least six elements, $M/A\backslash e,f$ is 2-connected. Then, since $M\backslash e,f$ is not 2-connected and $A$ is independent, there is an element $p$ of $A$ such that $\{e,f,p\}$ is a triad of $M$, by (4.6). This is a contradiction since $e$ is in no triads of $M$. Also, by (4.9), $M/A\backslash f$ is not 3-connected. Therefore, by (4.1), $e$ is in a triad of $M$, a contradiction. Hence $C - A$ has at least four elements. √

Now, by (4.9), for every element $p$ of $C - A$, the matroid $M/A\backslash p$ is not 3-connected. Thus $M/A$ is $(1,3)$ relative to the circuit
C - A. Therefore, by (4.2), C - A intersects at least two triads of M/A. Let \{g,h,x\} be one of these triads where \{g,h\} is contained in C - A. Note that x is not in C - A otherwise, by (4.14.7), M/A/g is 3-connected, a contradiction to the maximality of |A|. Now M/A/g and M/A\g have a 2-separation. Since e is in no triads of M/A, the matroid M/A\g does not have only minimal 2-separations, by (4.9). Therefore, M/A/g has only minimal 2-separations.

Let T be a triangle of M/A containing g. Then, since C - A contains at least four elements and C - A has no chords in M/A, we have that (C - A) \cap T = \{g\}. Moreover, since \{g,h,x\} is a triad of M/A and h is in C - A and therefore is not in T, the element x is in T. Thus every triangle of M/A containing g also contains x. Therefore, since M/A has at least eight elements and M/A/g has only minimal 2-separations, g is contained in a unique triangle \{g,x,y\}, by (1.25).

Let A' be a subset of A such that |A'| is minimal with respect to the property that \{g,x,y\} is a triangle of M/A'. Then we have the following lemma.

\[(4.14.9)\quad \text{Lemma.} \quad (i) \quad \text{M/A'} \text{ is 3-connected.} \]

\[(ii) \quad \text{If p,q are distinct elements of } C - (A' U g), \]

then M/A'\p,q is not 2-connected.

\[(iii) \quad \text{M/A'}/g\x \text{ is 3-connected.} \]

Proof. \quad (i) As M/A is 3-connected, Lemma (4.4) implies that M/A' is 3-connected.
(ii) Suppose that, for distinct elements \( p, q \) of \( C - (A' \cup g) \), the matroid \( M/A' \setminus p, q \) is 2-connected. Then, since \( M \setminus p, q \) is not 2-connected and \( A' \) is independent, there is an element \( a \) of \( A' \) such that \( \{ a, p, q \} \) is a triad of \( M \), by (4.6). Consider \( M/(A' - a) \). Since \( \{ a, p, q \} \) is a triad of \( M/(A' - a) \), the set \( \{ g, x, y, a \} \) is not a circuit of \( M/(A' - a) \), by orthogonality. Therefore, as \( \{ g, x, y \} \) is a triangle of \( M/A' \), it is also a triangle of \( M/(A' - a) \), a contradiction to the minimality of \( |A'| \).

(iii) Let \( T \) be a triangle of \( M/A' \) containing \( g \). Then \( T \) is a dependent set in the 3-connected matroid \( M/A \). So \( T \) is a triangle of \( M/A \). Therefore, since \( \{ g, x, y \} \) is the unique triangle of \( M/A \) containing \( g \), the triangle \( T \) is \( \{ g, x, y \} \). Hence \( \{ g, x, y \} \) is the unique triangle of \( M/A' \) containing \( g \). Now, since \( e \) is in no triads of \( M/A' \), the matroid \( M/A' \setminus g \) does not have only minimal 2-separations, by (4.9). Therefore, by (1.27) and the fact that \( M/A'/g \) has a 2-separation, \( M/A'/g \) has only minimal 2-separations. Hence \( M/A'/g \setminus x \) is 3-connected, by (1.27).

\[\text{(4.14.10) Lemma. For some element } f \text{ of } C - (A' \cup g), \text{ the set } \{ e, f, x, y \} \text{ is a cocircuit of } M/A'.\]

Proof. By (4.14.9)(iii), \( M/A'/g \setminus x \) is 3-connected. Suppose that, for distinct elements \( p, q \) of the circuit \( C - (A' \cup g) \) of \( M/A'/g \setminus x \), the matroid \( M/A'/g \setminus x, p, q \) is
2-connected. Then, by (4.14.9)(ii), $M/A'\setminus p,q$ is not 2-connected. Therefore, by (2.7) applied to $M/A'\setminus p,q$, either $g$ or $x$ is a coloop of $M/A'\setminus p,q$. Then, since $M/A'$ is 3-connected, either $\{g,p,q\}$ or $\{x,p,q\}$ is a triad of $M/A'$. This is a contradiction since each of the sets $\{g,p,q\}$ and $\{x,p,q\}$ has exactly one element in common with the triangle $\{g,x,y\}$. Thus, for all elements $p,q$ of $C - (A' \cup g)$, the matroid $M/A'/g\setminus p,q$ is not 2-connected. Therefore, by the induction hypothesis, every element of $C - (A' \cup g)$ is in a triad of $M/A'/g\setminus x$. In particular, for some element $f$ of $C - (A' \cup g)$, the set $\{e,f,z\}$ is a triad of $M/A'/g\setminus x$. Since $e$ is in no triads of $M/A'$, the set $\{e,f,x,z\}$ is a cocircuit of $M/A'$. By orthogonality, we must have $z = y$. Hence $\{e,f,x,y\}$ is a cocircuit of $M/A'$.

Now $M/A'\setminus e,g$ is not 2-connected, for if it is 2-connected, then, since $M\setminus e,g$ is not 2-connected, $\{e,g\}$ is contained in a triad of $M$, by (4.6), a contradiction. Therefore, since $M/A'\setminus e$ is 2-connected, $M/A'\setminus e/g$ is 2-connected. Since $M/A'\setminus e/g\setminus x$ has $y$ as a loop and is therefore not 2-connected, $M/A'\setminus e/g\setminus x$ is 2-connected. Now, since $M/A'\setminus e/g\setminus x,y$ has $f$ as a coloop and is therefore not 2-connected, $M/A'\setminus e/g\setminus x/y$ is 2-connected. So, since $\{g,x,y\}$ is a triangle of $M/A'\setminus e$, we have $M/A'\setminus e/g\setminus x/y = M/A'\setminus e,g/x,y$, by (1.30). Note that $M/A'\setminus e,g$ is not 2-connected. Therefore, by (2.7) applied to $M/A'\setminus e,g$ and the independent set $\{x,y\}$, either $\{e,g,x\}$ or $\{e,g,y\}$ is a triad of $M/A'$, and
therefore, of M. This contradicts the assumption that e is in no triads of M and thereby completes the proof of Theorem 4.14.

SECTION 3. THE MAIN THEOREM

In this section, we will finish the proof of the main theorem.

(4.15) Theorem. Let M be a matroid other than a wheel of rank more than three. If M is (2,3) relative to a circuit C, then every pair of elements of C is contained in a triad of M.

An Outline of the Proof

We utilize induction on the size of C. We first show that if C has three elements, then the conclusion of the theorem holds. Then we assume that C has at least four elements, and that C has a pair of elements \{e,f\} which is not contained in any triads of M. In (4.15.1) through (4.15.3), we show that if M/e is not 3-connected, then we can build a wheel of rank four having C as its rim, thus obtaining a contradiction to the hypotheses of the theorem.

Using the fact that M/e is 3-connected, we show that, for every element g of C - \{e,f\}, the matroid M/g is 3-connected. From this, it can be shown that e is contained in a triad \{e,g,h\} which is contained in C. Moreover, M\{e,g,h\} is 2-connected.
Next, we consider the element $g$. If $g$ is in a triad of $M$ with every element of $C - \{e,f\}$, then, by (4.5), $\{e,f\}$ is contained in a triad of $M$. This contradicts the assumption that $\{e,f\}$ is contained in no triads of $M$.

Then we consider the set $G$ of the elements of $C$ that are not contained in a triad of $M$ with $g$. A maximal number of elements of $G$ is contracted such that the minor obtained is 3-connected. It can easily be shown, using (4.5), that this minor contains an element $a$ of $G$, that is, $M/G$ is not 3-connected. Also, it can be shown that $a$ is in a triangle $T$ of the contraction minor.

Then we put back some of the contracted elements in such a way that $T$ is a triangle of the enlarged minor $M'$. We show that any triangle of $M'$ containing $a$ contains two elements of $E(M) - C$. We also show that $M'\setminus e,g,h$ is 2-connected. Furthermore, we show that, for every triangle $\{a,x,y\}$ of $M'$, the matroids $M'\setminus e,g,h,x$ and $M'\setminus e,g,h,x,y$ are 2-connected.

This sets the stage for the final contradiction. We show that, for some triangle $\{a,x,y\}$ of $M'$, the set $\{e,f,h,x,y\}$ is a cocircuit of $M'$. Therefore, by circuit elimination applied to $\{e,g,h\}$ and $\{e,f,x,y\}$, the set $\{f,g,h,x,y\}$ contains a cocircuit $C^*$ of $M'$. Finally, we show that the existence of $C^*$ contradicts the fact that $M'\setminus e,g,h,x$ and $M'\setminus e,g,h,x,y$ are 2-connected.

Proof of Theorem (4.15) We argue by induction on $|C|$. Suppose $C = \{e,f,g\}$. Since $M\setminus e$, $M\setminus f$ and $M\setminus g$ are not 3-connected, one of the elements, say $g$, is in triads with each of $e$ and $f$, by (4.1). Now we
apply 4.5 to the circuit \( \{ e,f,g \} \) to obtain that \( \{ e,f \} \) is in a triad of \( M \). Thus the theorem holds if \( |C| = 3 \), so suppose \( C \) has at least four elements. Assume that \( C \) has two elements \( e,f \) such that \( \{ e,f \} \) is contained in no triads of \( M \). Notice that \( E(M) - C \) has at least four elements, by (4.8).

(4.15.1) Lemma. Let \( g \) be an element of \( C \) and \( x,z \) be elements of \( E(M) - C \). If \( \{ g,x,z \} \) is a triangle of \( M \) and \( M/g\backslash z \) is 3-connected, then, for all pairs \( \{ p,q \} \) of elements of \( C - g \), the matroid \( M/g\backslash z,p,q \) is not 2-connected.

Proof. Suppose that, for some pair \( \{ p,q \} \) of elements of \( C - g \), the matroid \( M/g\backslash z,p,q \) is 2-connected. Thus \( M/g\backslash p,q \) is 2-connected. Then, since \( M\backslash p,q \) is not 2-connected, \( \{ g,p,q \} \) is a triad of \( M \), by (4.6). This is a contradiction since \( \{ g,p,q \} \) has exactly one element in common with the triangle \( \{ g,x,z \} \). \( \checkmark \)

(4.15.2) Lemma. Let \( g \) be an element of \( C - \{ e,f \} \) and \( x,y,z \) be elements of \( E(M) - C \). Suppose that \( \{ e,x,y \} \) and \( \{ g,x,z \} \) are triangles of \( M \) and that \( \{ g,f,z \} \) is a triad of \( M \). If \( M/g\backslash z \) is 3-connected, then, for some element \( h, C = \{ e,f,g,h \} \) and \( \{ e,h,y \} \) is a triad of \( M \). Moreover, there is an element \( w \) of
E(M) - (C U \{ x,y,z \}) such that \{ f,h,w \} is a triad of M.

Proof.

By (4.15.1), M/g\z is (2,3) relative to the circuit C - g. Therefore, by the induction hypothesis, if h is in C - \{ e,f,g \}, then \{ e,h \} is contained in a triad T* of M/g\z. Since \{ e,x,y \} is a triangle of M/g\z, either T* is \{ e,h,x \} or T* is \{ e,h,y \}. In the first case, \{ e,h,x,z \} is a cocircuit of M, by orthogonality with the triangle \{ g,x,z \}. Now, by (4.13) applied to the triangle \{ g,x,z \} and the cocircuit \{ e,h,x,z \}, either \{ g,h,x \} or \{ g,h,z \} is a triad of M. But \{ g,h,x \} cannot be a triad of M because of orthogonality with the triangle \{ e,x,y \}. So \{ g,h,z \} is a triad of M. But then, by circuit elimination applied to \{ g,h,z \} and \{ f,g,z \}, the set \{ f,g,h \} is a triad of M. This is a contradiction since \{ f,g,h \} has exactly one
element in common with the triangle \( \{ g, x, z \} \). Therefore, \( T^* = \{ e, h, y \} \). Now \( \{ e, h, y, z \} \) cannot be a cocircuit of \( M \) because of orthogonality with the triangle \( \{ g, x, z \} \). Hence, for every element \( h \) of \( C - \{ e, f, g \} \), the set \( \{ e, h, y \} \) is a triad of \( M \). To show that \( C - \{ e, f, g \} \) has exactly one element, suppose that it has distinct element \( h \) and \( h' \). Then, by circuit elimination applied to \( \{ e, h, y \} \) and \( \{ e, h', y \} \), the set \( \{ e, h, h' \} \) is a triad of \( M \). This is a contradiction to the orthogonality with the triangle \( \{ e, x, y \} \). Hence \( C = \{ e, f, g, h \} \).

Next we verify that, for some element \( w \) of \( E(M) - (C \cup \{ x, y, z \}) \), the set \( \{ f, h, w \} \) is a triad of \( M \). Since \( M/g \setminus z \) is \((2, 3)\) relative to the circuit \( \{ e, f, h \} \), the induction hypothesis implies that there is an element \( w \) of \( E(M/g \setminus z) \) such that \( \{ f, h, w \} \) is a triad of \( M/g \setminus z \). By orthogonality, \( w \) is not an element of the triangle \( \{ e, x, y \} \). Moreover, \( \{ f, h, z, w \} \) cannot be a cocircuit of \( M \) because of orthogonality with the triangle \( \{ g, x, z \} \). Therefore, \( \{ f, h, w \} \) is a triad of \( M \). 

\[(4.15.3) \text{Lemma.} \ M/e \text{ and } M/f \text{ are 3-connected.} \]

\textbf{Proof.} We prove that \( M/e \) is 3-connected. Suppose, to the contrary, that \( M/e \) is not 3-connected. Then, since \( M \setminus e \) is not 3-connected, \( M \setminus e \) or \( M/e \) has only minimal 2-separations, by (1.27). If \( M \setminus e \) has only minimal
2-separations, then, by (4.9), e is in triads of M with each element of C. This is a contradiction to the assumption that \{e,f\} is contained in no triads of M. Thus M/e has only minimal 2-separations.

Now, by (4.14), e is in a triad T* of M. By orthogonality, T* contains an element g of C - e. Let T* = \{e,g,x\}. Then, by (4.12), e is in a unique triangle \{e,x,y\} of M and x,y are elements of E(M) - C. Also, M/g is not 3-connected.

Since M\g does not have only minimal 2-separations, by (4.9) and (4.5), M/g does. Therefore, by (4.12) applied to the triad \{g,e,x\}, the element g is in a unique triangle \{g,x,z\} of M. If z = y, then, by circuit elimination applied to \{g,x,y\} and \{e,x,y\}, the set \{e,g,x\} is a triangle of M. This is a contradiction since \{e,g,x\} is both a triangle and a triad of a 3-connected matroid with at least five elements, by (1.24). Therefore z is distinct from y.

Next we show that \{f,g,z\} is a triad of M. By (1.27), M/g\z is 3-connected. Moreover, C - g is a circuit of M/g\z. Now, by (4.15.1), M/g\z is (2,3) relative to C - g. Therefore, there is an element t of M/g\z such that \{e,f,t\} is a triad of M/g\z. Since \{e,f\} is not in any triads of M, the set \{e,f,t,z\} is a cocircuit of M. Now, since z is an element of the triangle \{g,x,z\}, we must have that t = x. Thus \{e,f,x,z\} is a cocircuit of M. Now, by (4.13) applied
to the triangle \( \{ g, x, z \} \) and the cocircuit \( \{ e, f, x, z \} \), either \( \{ f, g, x \} \) or \( \{ f, g, z \} \) is a triad of \( M \). But \( \{ f, g, x \} \) cannot be a triad of \( M \) because of orthogonality with the triangle \( \{ e, x, y \} \). Therefore, \( \{ f, g, z \} \) is a triad of \( M \).

Now, by (4.15.2), \( C = \{ e, f, g, h \} \), the set \( \{ e, h, y \} \) is a triad of \( M \), and, for some element \( w \) of \( E(M) - (C U \{ x, y, z \}) \), the set \( \{ f, h, w \} \) is a triad of \( M \). Moreover, by (4.12)(iii) applied to the triad \( \{ f, g, h \} \) and \( M/g \), the matroid \( M/f \) is not 3-connected. Now, since \( M \setminus f \) does not have only minimal 2-separations, \( M/f \) has only minimal 2-separations. Therefore, by (4.12) applied to the triad \( \{ f, g, z \} \) and then to the triad \( \{ f, h, w \} \), the set \( \{ f, z, w \} \) is the unique triangle of \( M \) containing \( f \). Similarly, \( \{ h, y, w \} \) is the unique triangle of \( M \) containing \( h \).

Let \( W = \{ e, f, g, h, x, y, z, w \} \). It is easy to verify that the circuits of \( M \setminus (E(M) - W) \) are the circuits of \( W_4 \), and that \( \text{rk}^*W = 4 \). Now, by (1.3), \( \text{rk}W + \text{rk}(E(M) - W) - \text{rk}M = \text{rk}W + \text{rk}^*W - |W| \). But \( \text{rk}W + \text{rk}^*W - |W| = 4 + 4 - 8 = 0 \). Therefore, \( \text{rk}W + \text{rk}(E(M) - W) = \text{rk}M \). Thus, since \( M \) is 3-connected, \( E(M) = W \). This is a contradiction since \( M \) is not the wheel \( W_4 \). Hence \( M/e \) is 3-connected. Similarly, by interchanging \( e \) and \( f \) in the argument above we get that \( M/f \) is 3-connected. \( \checkmark \)

(4.15.4) Lemma. Let \( g \) be an element of \( C - \{ e, f \} \). Then \( M/g \) is 3-connected.
Proof. Suppose $M/g$ has a 2-separation. If $M\setminus g$ has no non-minimal 2-separations, then, by (4.9), $g$ is in triads of $M$ with every element of $C - g$. But then, by (4.5), $\{e, f\}$ is contained in a triad of $M$, a contradiction. Therefore $M/g$ has only minimal 2-separations. Now, by (4.14), $g$ is in a triad of $M$. Therefore, by (4.12), $g$ is in a unique triangle $\{g, x, y\}$ of $M$. Now consider the circuit $C - g$ of the 3-connected matroid $M/g\setminus x$. By (4.15.1), $M/g\setminus x$ is $(2, 3)$ relative to $C - g$. Therefore, by the induction hypothesis, $\{e, f, t\}$ is a triad of $M$ where $t$ is an element of $M/g\setminus x$. Therefore, $\{e, f, t, x\}$ is a cocircuit of $M$. Now, by orthogonality with the triangle $\{g, x, y\}$, we must have $t = y$. Thus $\{e, f, x, y\}$ is a cocircuit of $M$. Now, by (4.13) applied to the triangle $\{g, x, y\}$ and the cocircuit $\{e, f, x, y\}$, the set $\{e, g, x\}$ is a triad of $M$. But then, by (4.12)(iii) applied to the triad $\{e, g, x\}$, the matroid $M/e$ is not 3-connected. This is a contradiction to the last lemma. Hence $M/g$ is 3-connected. √

(4.15.5) Lemma. The element $e$ is in a triad $\{e, g, h\}$ of $M$ which is contained in $C$. Moreover, $M\setminus e, g, h$ is 2-connected.

Proof. If, for all elements $g, h$ of $C - e$, the matroid $M/e\setminus g, h$ is not 2-connected, then, by the induction hypothesis, every pair of elements of $C - e$ is contained in a triad of $M/e$. In particular, there is an element $p$ which is
in triads of \( M \) with every element of \( C - e \). Therefore, by (4.5), \( \{ e,f \} \) is contained in a triad of \( M \), a contradiction. Thus, for some elements \( g,h \) of \( C - e \), the matroid \( M/e\setminus g,h \) is 2-connected. Hence, since \( M\setminus g,h \) is not 2-connected, \( \{ e,g,h \} \) is a triad of \( M \). The last conclusion follows from (1.30). √

Now, by the last lemma, there is a triad \( \{ e,g,h \} \) of \( M \) such that \( M\setminus e,g,h \) is 2-connected and \( \{ e,g,h \} \) is contained in \( C \). We may assume, by (4.5), that there is an element \( a \) of \( C - \{ e,f \} \) such that \( \{ g,a \} \) is not contained in any triads of \( M \).

Let \( G \) be the set of elements \( a \) of \( C - \{ e,f \} \) for which \( \{ g,a \} \) is contained in no triads of \( M \). Now if \( M/G \) is 3-connected, then \( g \) is in triads of \( M/G \) with every element of \( C - (G \cup \{ e,f \}) \). Moreover, \( M/G\setminus e,f \) is not 2-connected. Therefore, by (4.5), \( \{ e,f \} \) is contained in a triad of \( M/G \), a contradiction. Hence \( M/G \) is not 3-connected.

Now let \( A \) be a subset of \( G \) such that \( M/A \) is 3-connected, and \( |A| \) is maximal with respect to this property. Then \( A \) is not empty, by (4.15.4). Also, since \( M/G \) is not 3-connected, \( G - A \) is not empty. Let \( a \) be an element of \( G - A \). Then, since \( |A| \) is maximal, \( M/A/a \) is not 3-connected. Also \( M/A\setminus a \) is not 3-connected, otherwise \( M/A\setminus a,g \) is 2-connected and therefore, \( \{ a,g \} \) is in a triad of \( M \), a contradiction. Furthermore, since \( a \) is not in a triad of \( M \) with \( g \), the matroid \( M/A\setminus a \) does not have only minimal 2-separations by (4.9). Therefore, \( M/A/a \) has only minimal 2-separations.
Now let $T$ be a triangle of $M/A$ containing $a$. Let $A'$ be a subset of $A$ such that $T$ is a triangle of $M/A'$ and $|A'|$ is minimal with respect to this property.

\[(4.15.6) \text{Lemma.} \hspace{1cm} \begin{align*} (i) \quad & A' \text{ is non-empty.} \\ (ii) \quad & M/A' \text{ is 3-connected.} \\ (iii) \quad & M/A' \backslash g,a \text{ is not 2-connected.} \\ (iv) \quad & \text{For every triangle } \{a,x,y\} \text{ of } M/A', \text{ neither } M/A'/x \backslash g \text{ nor } M/A'/y \backslash g \text{ is 2-connected.} \\ (v) \quad & \text{For every triangle } \{a,x,y\} \text{ of } M/A', \text{ the set } \{x,y\} \text{ is contained in } E(M) - C. \\ (vi) \quad & \text{Let } p,q \text{ be any elements of } C - (A' \cup a). \text{ Then } M/A' \backslash p,q \text{ is not 2-connected.} \\ (vii) \quad & M/A' \backslash e,g,h \text{ is 2-connected.} \end{align*}\]

\text{Proof.} \hspace{1cm} \begin{align*} (i) \quad & \text{If } A' \text{ is empty, } T \text{ is a triangle of } M. \text{ This is a contradiction since } M/a \text{ is 3-connected, by (4.15.4).} \\ (ii) \quad & M/A' \text{ is 3-connected by (4.4).} \\ (iii) \quad & \text{If } M/A' \backslash g,a \text{ is 2-connected, then since } M \backslash g,a \text{ is not 2-connected and } A' \text{ is independent, } \{g,a\} \text{ is contained in a triad of } M, \text{ by (4.6), a contradiction.} \\ (iv) \quad & \text{Let } \{a,x,y\} \text{ be a triangle of } M/A'. \text{ Suppose that } M/A'/x \backslash g \text{ is 2-connected. Then, since } \{a,y\} \text{ is a 2-circuit of } M/A'/x \text{ and of } M/A'/x \backslash g, \text{ the matroid } M/A'/x \backslash g,a \text{ is 2-connected. Therefore, since } M/A' \backslash g,a \text{ is not 2-connected, } \{g,a,x\} \text{ is a triad of } M/A', \text{ a contradiction. Hence} \\ & \text{\hspace{1cm}} \end{align*}
\( M/A'\setminus x\setminus g \) is not 2-connected. Similarly, \( M/A'\setminus y\setminus g \) is not 2-connected.

(v) Let \( \{a, x, y\} \) be a triangle of \( M \). Since, by (iv), \( M/A'\setminus g\setminus x \) is not 2-connected and \( M/A'\setminus g \) is 2-connected, \( M/A'\setminus g\setminus x \) is 2-connected. Therefore \( x \) is not an element of \( C \) otherwise, \( \{g, x\} \) is contained in a triad of \( M \) with some element of \( A' \), a contradiction. Similarly, \( y \) is not an element of \( C \).

(vi) Suppose that, for distinct elements \( \{p, q\} \) of \( C - (A' \cup a) \), the matroid \( M/A'\setminus p, q \) is 2-connected. Then, by (4.6), \( \{p, q, b\} \) is a triad of \( M \) for some element \( b \) of \( A' \). Consider the 3-connected matroid \( M/(A' - b) \). Since \( \{p, q, b\} \) is a triad of \( M/(A' - b) \), the set \( T \cup b \) is not a circuit of \( M/(A' - b) \), by orthogonality. Therefore, \( T \) is a triangle of \( M/(A' - b) \). This is a contradiction to the minimality of \( |A'| \).

(vii) Suppose \( M/A'\setminus e, g, h \) is not 2-connected. Since \( M\setminus e, g, h \) is 2-connected, we can find a proper subset \( F \) of \( A \) such that \( M\setminus e, g, h/F \) is 2-connected and \( |F| \) is maximal with respect to this property. Let \( b \) be an element of \( A' - F \). Then, by the maximality of \( |F| \), the matroid \( M\setminus e, g, h/F/b \) is not 2-connected. Therefore, \( M\setminus e, g, h/F \setminus b \) is 2-connected. Now, by (1.30), \( M\setminus e, g, h/F \setminus b = M/e\setminus g, h/F \setminus b \). Thus, \( M\setminus g, b/F \setminus e\setminus h \) is 2-connected. Now, by (2.7) applied to \( M\setminus g, b, F \cup e \) and \( \{h\} \), and the fact that \( M \) is 3-connected, \( \{g, b\} \) is
contained in a triad of $M$. This is a contradiction to the fact that $g$ is not in a triad of $M$ with any element of $A'$. 

(4.15.7) Lemma. $M/A'/h$ is 3-connected.

Proof. Suppose $M/A'/h$ is not 3-connected. If $M/A'/h$ has no non-minimal 2-separations, then, by (4.9), $h$ is in triads of $M$ with every element of $C - A'$. Let $B$ be a set of elements $b$ such that $b$ is in $A'$ and $\{b, h\}$ is contained in no triads of $M$. Then $M/B$ is 3-connected and $M/B\setminus e,f$ is not 2-connected. Moreover, by the definition of $B$, the element $h$ is in triads of $M/B$ with every element of the circuit $C - B$. Therefore, by (4.5), $\{e,f\}$ is contained in a triad of $M/B$, a contradiction. Hence $M/A'/h$ has a non-minimal 2-separation.

Now $M/A'/h$ has only minimal 2-separations and $\{e,g,h\}$ is a triad of $M/A'$. So, for some elements $z,w$ of $E(M/A') - C$, the sets $\{h,e,z\}$ and $\{h,g,w\}$ are the only possible triangles of $M/A'$ containing $h$. Suppose both triangles exist. Then, since $M/A'/h\setminus e,g$ is 3-connected, $M/A'/h\setminus e,g,f$ is 2-connected. But since $M/A'/e,f$ is not 2-connected, $\{e,f,h\}$ or $\{e,f,g\}$ is a triad of $M/A'$, a contradiction. Similarly, if $\{h,e,z\}$ is the unique triangle of $M/A'$ containing $h$, then $\{e,f,h\}$ is a triad of $M/A'$, a contradiction. Now assume that $\{h,g,w\}$ is the unique triangle of $M/A'$ containing $h$. Then, since $M/A'/h\setminus g$ is 3-connected, $M/A'/h\setminus g,a$ is 2-connected. But then $\{g,a\}$
is contained in a triad of $M$ with some element of $A' \cup h$, by (4.6), a contradiction. Hence $M/A'/h$ is 3-connected. √

(4.15.8) Lemma. Let $\{ a,x,y \}$ be a triangle of $M/A'$. Then $M/A' \setminus e,g,h,x$ and $M/A' \setminus e,g,h,x,y$ are 2-connected.

Proof. We first prove that $M/A' \setminus e,g,h,x$ is 2-connected. Suppose not. Then since, by (4.15.6)(vii), $M/A' \setminus e,g,h$ is 2-connected, $M/A' \setminus e,g,h/x$ is 2-connected. Now, since $\{ e,g,h \}$ is a triad of $M/A'$, we have $M/A' \setminus e,g,h/x = M/A' \setminus g/e/h/x$. Thus $M/A' \setminus x\setminus g/e/h$ is 2-connected. Now by (4.15.6)(iv), $M/A' \setminus x\setminus g$ is not 2-connected. If $M/A' \setminus x\setminus g/e$ is 2-connected, then either $\{ e,x \}$ is a circuit of $M/A'$, or $\{ e,g \}$ is a 2-cocircuit of $M/A'$. This is a contradiction since $M/A'$ is 3-connected. Therefore $M/A' \setminus x\setminus g/e$ is not 2-connected. Then, since $M/A' \setminus x\setminus g/e/h$ is 2-connected, $h$ is a loop or a coloop of $M/A' \setminus x\setminus g/e$. If $h$ is a loop, then, since $M/A'$ is 3-connected, $\{ h,e,x \}$ is a triangle of $M/A'$. This is a contradiction to (4.15.7). On the other hand, if $h$ is a coloop, then $\{ h,g \}$ is a 2-cocircuit of $M/A'$, again a contradiction since $M/A'$ is 3-connected. Hence $M/A' \setminus e,g,h,x$ is 2-connected.

Suppose that $M/A' \setminus e,g,h,x,y$ is not 2-connected. Then $M/A' \setminus e,g,h,x,y$ is 2-connected and is equal to $M/A' \setminus y\setminus g/e/h,x$. Now, by (4.15.6)(iv), $M/A' \setminus y\setminus g$ is not 2-connected. If $M/A' \setminus y\setminus g/e$ is 2-connected, then $e$ is a loop
or coloop of M/A'/y\g, a contradiction. Therefore M/A'/y\g/e is not 2-connected. If M/A'/y\g/e\h is 2-connected, then 
\{ h,g \} is a 2-cocircuit of M/A' or \{ h,e,y \} is a triangle of M/A'. In both cases, we have a contradiction since M/A' is 3-connected and, by (4.15.7), M/A'/h is 3-connected. Therefore M/A'/y\g/e\h is not 2-connected. Hence, since M/A'/y\g/e\h,x is 2-connected, \{ g,h,x \} is a triad of M/A' or \{ e,x,y \} is a triangle of M/A'. But \{ g,h,x \} cannot be a triad of M/A' because of orthogonality with the triangle \{ a,x,y \}; and \{ e,x,y \} cannot be a triangle of M/A' because of orthogonality with \{ e,g,h \}. Hence M/A'\e,g,h,x,y is 2-connected. √

Let T',T'' be distinct triangles of M/A' containing a. Then, since M/A'/a has only minimal 2-separations and E(M/A') has at least seven elements, T' ∩ T'' = \{ a \}. Also, every triangle of M/A' containing a contains exactly two elements of E(M/A') - C. Now choose exactly one element from each triangle of M/A' containing a such that this element is distinct from a. Let X be the set of these elements. Then, by (1.27), M/A'/a\X is 3-connected. Also C - (A' U a) is a circuit of M/A'/a\X.

(4.15.9) Lemma. M/A'/a\X is (2,3) relative to the circuit C - (A' U a).
Proof. Suppose that, for distinct elements \( p, q \) of \( C - (A' \cup a) \), the matroid \( M/A'/a\backslash X\backslash p,q \) is 2-connected. Then, by (4.15.6)(vi), \( M/A'\backslash p,q \) is not 2-connected. If \( M/A'\backslash p,q/a \) is 2-connected, then \( \{ a,p,q \} \) is a triad of \( M/A' \). This is a contradiction because of orthogonality with the triangle \( T \). Therefore, \( M/A'/a\backslash p,q \) is not 2-connected. Notice that, since every element of \( X \) is in a 2-circuit of \( M/A'/a\backslash p,q \), the set \( X \) is coindependent in \( M/A'/a\backslash p,q \). Then, by the dual of (2.7), \( X \) contains a loop of \( M/A'/a\backslash p,q \), a contradiction since \( M/A'/a \) is 2-connected. Hence, for all elements \( p, q \) of \( C - (A' \cup a) \), the matroid \( M/A'/a\backslash X\backslash p,q \) is not 2-connected. \( \square \)

(4.15.10) Lemma. There is a triangle \( \{ a,x,y \} \) of \( M/A' \) such that \( \{ e,f,x,y \} \) is a cocircuit of \( M/A' \).

Proof. By the last lemma, \( M/A'/a\backslash X \) is (2,3) relative to the circuit \( C - (A' \cup a) \). Therefore, by the induction hypothesis, \( \{ e,f \} \) is contained in a triad \( \{ e,f,y \} \) of \( M/A'/a\backslash X \). Then, since \( \{ e,f \} \) is in no triads of \( M/A' \), there is a non-empty subset \( F \) of \( X \) such that \( \{ e,f,y \} \cup F \) is a cocircuit of \( M/A' \). Now, every element of \( F \) is in a triangle of \( M/A' \). Also, neither \( e \) nor \( f \) is in any of the triangles containing \( a \). Therefore, \( F \) has exactly one element \( x \), that is, \( \{ e,f,y \} \cup F = \{ e,f,x,y \} \). Moreover, by orthogonality, \( \{ a,x,y \} \) is a triangle of \( M/A' \). \( \square \)
To complete the proof of the main theorem, we now obtain a contradiction by showing that the cocircuit \( \{ e,f,x,y \} \) in the last lemma cannot exist.

Consider the triad \( \{ e,g,h \} \) and the cocircuit \( \{ e,f,x,y \} \) of \( M/A' \). By circuit elimination, there is a cocircuit \( C^* \) of \( M/A' \) contained in \( \{ f,g,h,x,y \} \). By orthogonality with the triangle \( \{ a,x,y \} \), either \( C^* \cap \{ x,y \} = \emptyset \), or \( \{ x,y \} \) is contained in \( C^* \).

If \( C^* \cap \{ x,y \} = \emptyset \), then \( C^* = \{ f,g,h \} \). By applying circuit elimination to the triads \( \{ e,g,h \} \) and \( \{ f,g,h \} \), we get that \( \{ e,f,h \} \) is a triad of \( M/A' \), a contradiction. Therefore, \( \{ x,y \} \) is contained in \( C^* \).

Notice that, by orthogonality with the circuit \( C - A' \), the cocircuit \( C^* \) contains at least two elements of \( \{ f,g,h \} \).

Now if \( C^* = \{ g,h,x,y \} \), then \( M/A' \setminus e,g,h,x \) has \( y \) as a coloop. This is a contradiction to (4.15.8). Therefore, \( C^* = \{ f,h,x,y \} \) or \( C^* = \{ f,g,h,x,y \} \). In both cases, \( M/A' \setminus e,g,h,x,y \) has \( f \) as a coloop. Again, we have a contradiction to (4.15.8). Hence \( \{ e,f \} \) is in a triad of \( M \).
We want to discuss some of the consequences of Theorem 4.15. In Section 1 we show that Theorem 3.9 follows from (4.15). We also show that a Sylvester matroid having at least four elements is neither graphic nor cographic. Thus, by (3.20), a (2,3)-matroid in not graphic. Then we give an infinite family of graphs that are (2,3) relative to a triangle. And we show that if a graph G is (2,3) relative to a cycle C, then C is a triangle.

Let M be a 2-connected matroid and t be an integer greater than one. M is t-cocyclic if it has the following two properties:

(i) for every subset F of E(M) having at most t - 1 elements, M \ F is 2-connected; and

(ii) for every subset A of E(M) having exactly t elements, M \ A is not 2-connected.

M is t-cyclic if its dual M* is t-cocyclic. The reason for the usage of the terms t-cocyclic and t-cyclic will be apparent from (5.7) and (5.8).

A matroid M is identically self-dual (ISD) if its sets of circuits and cocircuits coincide.

Let t be an integer greater than one. Let E be any set having v elements and H be a collection of k-subsets of E. The elements of E and H are called points and blocks, respectively. The pair (E,H) is a Steiner system S(t,k,v) if every t-subset is contained
in precisely one block. It is known that the collection of blocks $H$ of a Steiner system $S(t,k,v)$ is the collection of hyperplanes of a matroid $M$ on $E$, [25, Theorem 2.3.1] or [27, Theorem 64].

In Section 2, we give some properties and a characterization of the matroids $M$ that are both $t$-cocyclic and $t$-cyclic. We show that the rank of such a matroid is $t$ or $t + 1$, and that $M$ is identically self-dual. Then we show that $M$ is $t$-cocyclic and $t$-cyclic of rank $t$ if and only if $M$ is isomorphic to $U[t,2t]$, and that $M$ is a $t$-cocyclic and $t$-cyclic of rank $t + 1$ if and only if $M$ is a Steiner system $S(t, t + 1, 2t + 2)$. A theorem of Seymour [22] and a theorem of Mendelsohn [12] follow immediately from this characterization.

SECTION 1. CONSEQUENCES AND EXAMPLES

Theorem 3.9 is a consequence of (4.15).

(5.1) Corollary. Let $M$ be a $(2,3)$-matroid. Then every pair of elements of $E(M)$ is contained in a triad of $M$.

Proof. Let $e,f$ be distinct elements of $M$. Then, since $M$ is 2-connected, $\{e,f\}$ is contained in a circuit $C$ of $M$. Now $M$ is $(2,3)$ relative to $C$. Therefore, by (4.15), $\{e,f\}$ is contained in a triad of $M$. ▽
Next we show that (2,3)-matroids are neither graphic nor cographic. Then we give a family of graphs that are (2,3) relative to a circuit.

(5.2) Theorem. Let M be a Sylvester matroid having at least four elements. Then M is neither graphic nor cographic.

Proof. It is easy to check that a Sylvester matroid having at least four elements is not graphic.

Let M* be the dual of a Sylvester matroid having at least four elements. Assume that M* is the cycle matroid of some graph G. Since M* is 3-connected, G is simple and 3-connected, by (1.22). Now let C be a cycle of G. Since G is 3-connected, G contains an edge e which is not in C.

Consider the 2-connected graph G\e. Then, for every edge f of C, the graph G\e,f is not 2-connected. Therefore, by (2.21), the circuit C has at least two vertices of degree two in G. Thus, since the degree of every vertex of G is at least three, e is a chord of C. Hence there is a bipartition \{F,H\} of the edges of C such that F∪e and H∪e are cycles of G. Further, since G is simple, each of F and H contains at least two edges. Now let g be an element of F. Then G\g is 2-connected. Clearly, H∪e has at most one vertex of degree two in G\g. But, for every element h of H∪e, the graph G\g,h is not 2-connected. This is a contradiction since (2.21)
implies that the cycle $H \cup e$ has at least two vertices of degree two in $G \setminus g$. Hence $M^*$ is not the cycle matroid of any graph $G$. \qed

Although, as the last theorem shows, there are no $(2,3)$-graphs, we can find an infinite number of graphs that are $(2,3)$ relative to a cycle. The following construction of "wheels within wheels" gives such a family of graphs.

Clearly, $W_3$ is $(2,3)$ relative to the cycle formed by its rim elements. Let $u, v, w$ be the rim vertices of $W_3$, and let $u', v', w'$ be three new vertices. Now add the following six edges: $(u, u')$, $(v, v')$, $(w, w')$, $(u', v')$, $(u', w')$ and $(v', w')$. The following figure shows $W_3$ and the graph obtained from it.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure6.png}
\caption{Figure 6}
\end{figure}

It is clear that the second graph is $(2,3)$ relative to the triangle formed by $u', v'$ and $w'$. This construction can be done repeatedly to obtain an infinite family of graphs each of which is
(2,3) relative to a cycle. We note that every graph of this family is (2,3) relative to a triangle. It turns out that this is true for any graph which is (2,3) relative to a cycle. To prove this we need three lemmas. The first of these is the dual of a result of Coullard, [6, Lemma 2.44].

(5.3) Lemma. Let $M$ be a 3-connected matroid having at least seven elements. Suppose that, for an element $e$ of $E(M)$, each of $M\setminus e$ and $M/e$ has a 2-separation, and that $M\setminus e$ has only minimal 2-separations. If $e$ is in at least three triads of $M$, then, for any element $f$ which is in a triad of $M$ containing $e$, the matroid $M/f$ is 3-connected.

(5.4) Lemma. Let $G$ be a graph which is (2,3) relative to a cycle $C$. Then every vertex of $C$ has degree three in $G$.

Proof. Let $v$ be a vertex of $C$ and $e,f$ be the edges of $C$ incident with $v$. Then, by (4.15), there is triad $\{e,f,g\}$ of $G$. Therefore, since $e$ and $f$ are incident with $v$, the edge $g$ is incident with $v$. Moreover, since $\{e,f,g\}$ is a triad of $G$, the only edges of $G$ incident with $v$ are $e,f$ and $g$. Hence $v$ has degree three in $G$. ▽

(5.5) Lemma. Let $G$ be a graph that is (2,3) relative to a cycle $C$, and $p$ be an edge of $C$. Then $M(G)/p$ is not 3-connected.
Proof. Suppose $M(G)/p$ is 3-connected. If, for distinct elements $f, g$ of $C - p$, the matroid $M(G)/p \setminus f, g$ is 2-connected, then, by (4.6), $\{p, f, g\}$ is a triad of $M(G)$. This is a contradiction since $\{p, f, g\}$ is contained in the circuit $C$ and $M(G)$ is binary, see (1.47) and (1.48). Therefore, for all elements $f, g$ of $C - p$, the matroid $M(G)/p \setminus f, g$ is not 2-connected. Therefore, $G/p$ is $(2,3)$ relative to the cycle $C - p$. This contradicts the last lemma since $C - p$ has a vertex of degree greater than three in $G/p$. Hence $M(G)/p$ is not 3-connected. \(\Box\)

(5.6) Theorem. Let $G$ be a $(2,3)$-graph relative to a cycle $C$. Then $C$ is a triangle of $G$.

Proof. The wheel $W_3$ is $(2,3)$ relative to its rim. It is easy to check that this is the only graph of size at most six which is $(2,3)$ relative to a cycle. Suppose that $G$ has more than six edges and that it is $(2,3)$ relative to a cycle $C$ having at least four edges. Consider an edge $e$ of $G$. By the last lemma, $M(G)/e$ has a 2-separation. Thus, since $M(G) \setminus e$ is not 3-connected, $M(G)/e$ or $M(G) \setminus e$ has only minimal 2-separations, by (1.27). Now, since $C$ has at least four elements, $e$ is in at least three triads of $M(G)$, by (4.15). So, by orthogonality and (1.25), $e$ is not in any triangles of $M(G)$. Therefore, $M(G) \setminus e$ has only minimal 2-separations. But then, for an element $f$ of
C - e, the matroid M(G)/f is 3-connected, by (5.3). This is a contradiction to Lemma 5.5. Hence C has exactly three elements. 

SECTION 2. t-CYCLIC AND t-COCYCLIC MATROIDS

Recall that, for an integer t greater than one, a 2-connected matroid M is t-cocyclic if it satisfies the following two properties:

(i) for every subset F of E(M) having at most t - 1 elements, M\F is 2-connected; and

(ii) for every subset A of E(M) having exactly t elements, M\A is not 2-connected.

In Chapter 3, we have seen that if M is a 2-cocyclic matroid, then every pair of its elements is in a triad. Moreover, M is 3-connected. We will show that the first conclusion can be generalized for any t \geq 3.

(5.7) Theorem. Let M be a t-cocyclic matroid. Then every t-subset A of E(M) is contained in a (t+1)-cocircuit of M. Hence every subset of E(M) having at most t elements is coindependent.

Proof. We argue by induction on t. The case t = 2 is (3.9), so suppose t is greater than two. Let A be a t-subset of E(M) and e be an element of A. Consider the 2-connected matroid M\e. It is easy to see that M\e is (t-1)-cocyclic. Therefore,
by the induction hypothesis, every \((t-1)\)-subset of \(E(M \setminus e)\) is contained in a \(t\)-cocircuit of \(M \setminus e\). In particular, \(A - e\) is contained in a \(t\)-cocircuit \(C^*\) of \(M \setminus e\). If \(C^*\) is a cocircuit of \(M\), then, for some element \(f\) of \(C^*\), the matroid \(M \setminus (C^* - f)\) contains \(f\) as a coloop. This is a contradiction since \(M\) is \(t\)-cocyclic. Therefore, \(C^* \cup e\) is a cocircuit of \(M\). Hence \(A\) is contained in the \((t+1)\)-cocircuit \(C^* \cup e\) of \(M\). \(\nabla\)

For future reference, we state the dual of the last theorem.

(5.8) Corollary. Let \(M\) be a \(t\)-cyclic matroid. Then every \(t\)-subset \(A\) of \(E(M)\) is contained in a \((t+1)\)-circuit of \(M\). Hence every subset of \(E(M)\) having at most \(t\) elements is independent.

The next result follows immediately from the last two results.

(5.9) Corollary. A \(t\)-cocyclic matroid of corank \(t\) is isomorphic to \(U[r,t+r]\) where \(r\) is a positive integer greater than one. Dually, a \(t\)-cyclic matroid of rank \(t\) is isomorphic to \(U[t,t+r]\) where \(r\) is a positive integer greater than one.

Since a \(2\)-cocyclic matroid is \(3\)-connected, it is natural to ask whether a \(t\)-cocyclic matroid is \((t+1)\)-connected for any \(t \geq 3\). The following examples show that the answer of this question is negative.
Example. Let $t, r$ be integers such that $t \geq 4$ and $2 \leq r \leq t - 2$. The uniform matroids $U[r, t+r]$ are $t$-cocyclic but are not $(t+1)$-connected. In fact, they are not even $t$-connected, see Theorem 1.50.

Example. The affine space $AG(3,2)$ is $3$-cocyclic. In fact, since it is identically self-dual, it is also $3$-cyclic. But $AG(3,2)$ is not $4$-connected.

However, we can show that if $M$ is $t$-cyclic (dually, $t$-cocyclic) having rank (dually, corank) at least $t + 1$, then $M$ is $t$-connected. First we give a definition, and then we prove three lemmas.

Let $A$ be a $t$-subset of a $t$-cyclic matroid $M$. An element $e$ of $E(M) - A$ is a complement of $A$ if $A \cup e$ is a circuit of $M$. We also say that $A$ is complemented by $e$.

Lemma. Let $M$ be a $t$-cyclic matroid and $B$ be an independent subset of $M$ having at least $t$ elements. If $A, A'$ are distinct $t$-subsets of $B$, then $E(M) - B$ does not contain an element that complements both $A$ and $A'$.

Proof. Suppose that, for distinct $t$-subsets $A, A'$ of $B$ and an element $e$ of $E(M) - B$, the sets $A \cup e$ and $A' \cup e$ are circuits of $M$. Then, by circuit elimination, $A \cup A'$ contains a circuit of $M$. This is a contradiction since $A \cup A'$ is contained in the independent set $B$. \(\n\)
**Lemma.** Let $M$ be a t-cyclic matroid having rank at least $t + 1$. Then $E(M)$ has at least $2t + 2$ elements.

**Proof.** Let $B$ be a base of $M$. Then $B$ contains at least $t + 1$ elements. Now, every $t$-subset of $B$ is complemented by an element of $E(M) - B$. Moreover, by (5.12), no two distinct $t$-subsets of $B$ are complemented by the same element $e$ of $E(M) - B$. Therefore, since $B$ has at least $C(t+1, t)$ $t$-subsets, $E(M) - B$ has at least $t + 1$ elements. Hence $E(M)$ has at least $2t + 2$ elements. 

**Lemma.** Let $M$ be a t-cyclic matroid having rank at least $t + 1$. Then every cocircuit $C^*$ of $M$ has at least $t + 1$ elements.

**Proof.** Let $C^*$ be a cocircuit of $M$. Then, by (1.6), $E(M) - C^*$ is a hyperplane of $M$. Let $e$ be an element of $C^*$ and $B$ be a base of the hyperplane $E(M) - C^*$. Then, since $M$ has a rank at least $t + 1$, the cardinality of $B$ is at least $t$. Now, for every $(t-1)$-subset $F$ of $B$, the set $F U e$ is complemented by some element $f$ of $E(M) - B$. By orthogonality, $f$ is an element of $C^* - e$. Moreover, by (5.12) applied to the independent set $B U e$, if $F, F'$ are distinct $(t-1)$-subsets of $B$, then $F U e$ and $F' U e$ are not complemented by the same element $f$ of $C^* - e$. Therefore, since the number of the $(t-1)$-subsets of $B$ is at
least \( t \), the set \( C^* - e \) has at least \( t \) elements. Hence \( C^* \) contains at least \( t + 1 \) elements. \( \forall \)

\textbf{(5.15) Theorem.} Let \( M \) be a \( t \)-cyclic matroid having rank at least \( t + 1 \). Then \( M \) is \( t \)-connected.

\textbf{Proof.} Suppose that \( M \) is not \( t \)-connected. Then, for some \( k \leq t - 1 \), there is a \( k \)-separation \( \{ X, Y \} \) of \( M \). Thus \( \text{rk}X + \text{rk}Y \leq \text{rk}M + k - 1 \). Without loss of generality, we may assume that \( |X| \geq |Y| \). Now, since every \( t \)-subset is independent, the rank of \( Y \) is at least \( k \). Therefore, since \( \text{rk}X + \text{rk}Y \leq \text{rk}M + k - 1 \), the rank of \( X \) is at most \( \text{rk}M - 1 \). Now consider the sets \( \overline{X} \) and \( E - \overline{X} \). Since \( \overline{X} \) is a flat, \( E - \overline{X} \) contains a cocircuit of \( M \), by \((1.7)\). So, by \((5.14)\), \( E - \overline{X} \) has at least \( t + 1 \) elements and thus \( |E - \overline{X}| > k \) and \( \text{rk}(E - \overline{X}) > t \). Therefore, \( \{ \overline{X}, E - \overline{X} \} \) is a \( k \)-separation of \( M \). Moreover, \( \overline{X} \) is not a hyperplane of \( M \). For if \( \overline{X} \) is a hyperplane of \( M \), then, since \( \text{rk}\overline{X} + \text{rk}(E - \overline{X}) \leq \text{rk}M + k - 1 \), the rank of \( E - \overline{X} \) is at most \( k \). This is a contradiction since \( \text{rk}(E - \overline{X}) \) is at least \( t \) and \( k \) is less than \( t \). Thus, \( E - \overline{X} \) contains a cocircuit \( C^* \) and an element \( e \) which is not in \( C^* \).

Now let \( F \) be a \((t-2)\)-subset of \( \overline{X} \). Then, for an element \( f \) of \( \overline{X} - F \), the \( t \)-set \( F \cup \{ e, f \} \) is complemented by an element \( g \) of \( E(M) \). Since \( \overline{X} \) is a flat and \( e \) is not in \( \overline{X} \), \( g \) is not an element of \( \overline{X} \). Also, by orthogonality, \( g \) is not an element of \( C^* \). Thus \( g \) is an element of \( E - (\overline{X} \cup C^* \cup e) \). Furthermore,
for distinct elements \( f, h \) of \( \overline{X} - F \) the t-sets \( F \cup \{ e, f \} \) and \( F \cup \{ e, h \} \) cannot be complemented by the same element \( g \) of \( E - (\overline{X} \cup C^* \cup e) \), by (5.12) applied to the independent set \( F \cup \{ e, f, h \} \). Therefore, \( E - (\overline{X} \cup C^* \cup e) \) contains at least \( |\overline{X}| - (t - 2) \) elements. Thus, since \( C^* \) contains at least \( t + 1 \) elements, \( E - \overline{X} \) contains at least \( |\overline{X}| + 4 \) elements. This is a contradiction since \( |\overline{X}| \geq |X| \geq |Y| \geq |E - \overline{X}| \).

Hence \( M \) is t-connected. \( \Box \)

In the remaining part of this section, we want to investigate the matroids \( M \) that have the property that \( M \) is t-cocyclic and t-cyclic.

As one would suspect, a matroid \( M \) which is t-cyclic and t-cocyclic has its rank equal to its corank. In fact, as we show in the next theorem, the rank of such a matroid is \( t \) or \( t + 1 \). It is also true, though less obvious, that \( M \) is identically self-dual. We prove this fact in Theorem 5.19.

(5.16) Theorem. Let \( M \) be a t-cyclic and t-cocyclic matroid. Then \( \text{rk} M = \text{cork} M \). Moreover, \( \text{rk} M \) is \( t \) or \( t + 1 \) and therefore \( |\text{E}(M)| \) is \( 2t \) or \( 2t + 2 \).

Proof. Let \( \text{rk} M = r \) and \( \text{cork} M = d \). Let \( B \) be a base of \( M \) and \( B^* = \text{E}(M) - B \). Then \( |B| = r \) and \( |B^*| = d \). We may assume, without loss of generality, that \( r \geq d \). Now, by (5.7), every t-subset of \( \text{E}(M) \) is coindependent. Therefore \( r \geq d > t \).
Since $M$ is $t$-cyclic, every $t$-subset of $B$ is complemented by an element of $B^*$. Moreover, by (5.12), no two distinct $t$-subsets of $B$ are complemented by the same element of $B^*$. Therefore, $C(r,t) \leq d \leq r$. The inequality $C(r,t) \leq r$ holds if and only if $r$ is $t$ or $t+1$. Now if $r = t$, then, since $r \leq d \leq t$, we have $d = t$. And if $r = t+1$, the inequalities $C(t+1, t) \leq d \leq t + 1$ imply that $d = t + 1$. ▽

Using (5.9), we have the following characterization of the matroids $M$ that have the property that $M$ is $t$-cyclic and $t$-cocyclic of rank $t$.

(5.17) Theorem. The following statements are equivalent.

(i) $M$ is a $t$-cyclic and $t$-cocyclic matroid of rank $t$.

(ii) $M$ is the uniform matroid $U[t,2t]$.

(5.18) Lemma. Let $M$ be a $t$-cyclic and $t$-cocyclic matroid having rank $t + 1$. Then, for every circuit $C$ of $M$, $|C| = t + 1$ and $E(M) - C$ is a cocircuit of $M$.

Proof. Suppose that $C$ is a circuit of $M$ having more than $t + 1$ elements. Then $E(M) - C$ has at most $t$ elements, by (5.16). Now, since $rkM = t + 1$, the circuit $C$ contains a base $B$ of $M$. Note that every $t$-subset of $B$ is complemented by an element of $E(M) - C$. Moreover, no two distinct $t$-subsets of $B$ are complemented by the same element of $E(M) - C$, by (5.12).
Therefore, \( C(t+1,t) < |E(M) - C| \). This is a contradiction since \( E(M) - C \) has at most \( t \) elements.

To verify that the last conclusion holds, note that if \( E(M) - C \) is coindependent, then, since \( E(M) - C \) has \( t+1 \) elements, it is a cobase of \( M \), a contradiction. \( \Box \)

(5.19) Theorem. Let \( M \) be a \( t \)-cyclic and \( t \)-cocyclic matroid. Then \( M \) is identically self-dual.

Proof. By (5.16), \( \text{rk} M \) is \( t \) or \( t+1 \). If \( \text{rk} M = t \), then it follows from (5.17) that \( M \) is ISD, so suppose \( \text{rk} M = t+1 \). Let \( A \cup e \) be a circuit of \( M \) and \( C^* = E(M) - (A \cup e) \). Then, by (5.18), \( A \cup e \) contains exactly \( t+1 \) elements and \( C^* \) is a cocircuit of \( M \). Now assume that \( A \cup e \) is not a cocircuit of \( M \). Then there is an element \( f \) of \( C^* \) such that \( A \cup f \) is a cocircuit of \( M \). Let \( g,h \) be elements of \( A \). Since \((A \cup e) - \{ g,h \}\) contains exactly \( t-1 \) elements, \( M/(A - \{ g,h \})/e \) is 2-connected. Moreover, \( \{ g,h \} \) is a 2-circuit and \( C^* \) is a cocircuit of this matroid. Now since \((A - \{ g,h \}) \cup \{ e,f \}\) has cardinality \( t \), the matroid \( M/(A - \{ g,h \})/e/f \) is not 2-connected. Note that \( \{ g,h \} \) is a 2-circuit of \( M/(A - \{ g,h \})/e/f \). For if not, then \( \{ f,g,h \} \) is contained in a parallel class of \( M/(A - \{ g,h \})/e \), a contradiction since \( f \) is in the cocircuit \( C^* \) and \( \{ g,h \} \) does not intersect \( C^* \). Thus \( \{ g,h \} \) is contained in a component \( K \) of \( M/(A - \{ g,h \})/e/f \). Since the ground set of this matroid
has cardinality $t + 1$, the component $K$ contains at most $t$ elements. Therefore, $M/(A - \{ g,h \})/e/f$ has a cocircuit having at most $t$ elements. This is a contradiction since, by (5.7), every $t$-subset of $M$ is co-independent. Hence $A U e$ is a cocircuit of $M$. Dually, if $C^*$ is a cocircuit of $M$, then $C^*$ is also a circuit of $M$. \( \Box \)

(5.20) Lemma. Let $M$ be $t$-cyclic and $t$-cocyclic matroid of rank $t + 1$. Then every $t$-subset $A$ of $E(M)$ is contained in a unique $(t + 1)$-circuit.

Proof. Let $A$ be a $t$-subset of $E(M)$. Then, for some element $e$ of $E(M) - A$, the set $A U e$ is a circuit of $M$. By (5.18), $E(M) - (A U e)$ is a cocircuit of $M$. Therefore, by orthogonality, there is no element $f$ of $E(M) - (A U e)$ such that $A U f$ is a circuit of $M$. Hence $A U e$ is the unique circuit of $M$ containing $A$. \( \Box \)

Combining (5.8), (5.16), (5.18) and (5.20) we obtain:

(5.21) Theorem. Let $M$ be a $t$-cyclic and $t$-cocyclic matroid of rank $t + 1$. Then the collection of circuits of $M$ is the collection of blocks of a Steiner system $S(t, t+1, 2t+2)$. 
Next we show that if \( t \) is even, the only matroids that are \( t \)-cyclic and \( t \)-cocyclic are the uniform matroids \( U[t,2t] \).

(5.22) Theorem. Let \( t \) be an even positive integer. Then there is no \( t \)-cyclic and \( t \)-cocyclic matroid \( M \) of rank \( t + 1 \).

Proof. Suppose that, for an even positive integer \( t \), \( M \) is a \( t \)-cyclic and \( t \)-cocyclic matroid of rank \( t + 1 \). Let \( C \) be a circuit of \( M \) and \( C^* \) be the set \( E(M) - C \). Then, by (5.18), \( C \) contains exactly \( t + 1 \) elements, and \( C^* \) is a cocircuit of \( M \). Let \( e,f \) be elements of \( C \) and consider the \( (t-1) \)-set \( C - \{ e,f \} \). Then, for an element \( g \) of \( C^* \), there is an element \( h \) of \( E(M) - (C - \{ e,f \}) \) such that \( (C - \{ e,f \}) \cup \{ g,h \} \) is a circuit of \( M \). By orthogonality of this circuit with \( C^* \), the element \( h \) is not in \( \{ e,f \} \).

Now let \( \phi \) be a map from \( C^* \) into \( C^* \) defined by \( \phi(g) = h \) where \( h \) is the complement of \( g \) with respect to the set \( (C - \{ e,f \}) \cup g \). Then, by (5.20), \( \phi \) is well-defined. Moreover, it is clear that \( \phi \phi(g) = g \). Thus \( \phi \) is a bijection. Now let \( A \) be a subset of \( C^* \) having exactly \( \frac{t}{2} \) elements. Then \( C - (A \cup \phi(A)) \) consists of one element, say \( b \). Then, since \( b = \phi \phi(b) \) and \( A = \phi \phi(A) \), we have \( b = \phi(b) \), a contradiction. Hence \( t \) is not even. \( \nabla \)

The following result of Seymour [22] is an immediate consequence of (5.16), (5.17) and the last theorem.
(5.23) Corollary. Let $M$ be a 2-connected matroid. If, $M$ is 2-cyclic and 2-cocyclic, then $M$ is isomorphic to $U[2,4]$.

For a proof of the next result, we refer the reader to Welsh [25, pp. 220].

(5.24) Lemma. Let $t$ be an integer greater than two and $S(t, t+1, 2t+2)$ be a Steiner system. Let $M$ be the matroid whose hyperplanes are the blocks of this system. Then $rkM$ is $t + 1$, and every hyperplane is a circuit of $M$.

Now we show that the converse of Theorem 5.21 holds.

(5.25) Theorem. Let $t$ be an integer greater than one and $S(t, t+1, 2t+2)$ be a Steiner system. Let $M$ be the matroid whose hyperplanes are the blocks of this system. Then $M$ is $t$-cyclic and $t$-cocyclic.

Proof. First we note that $M$ is 2-connected and that all its cocircuits have cardinality $t + 1$. Now let $F$ be a non-empty subset of $E(M)$ having at most $t - 1$ elements. Then, since $M$ has no circuits having fewer than $t + 1$ elements, $M/F$ has no loops. Thus if $M/F$ is not 2-connected, then we can find two components $K, K'$ of this matroid each having at least two
elements. But then, since $E(M/F)$ contains at most $2t + 1$ elements, one of these components, say $K$, contains at most $t$ elements. Thus $K$ contains a cocircuit $C^*$ of $M/F$ having at most $t$ elements. This is a contradiction since every cocircuit of $M$ contains exactly $t + 1$ elements. Hence $M/F$ is 2-connected.

Now consider $M\setminus F$. This matroid has no coloops since every cocircuit of $M$ contains exactly $t + 1$ elements. So if $M\setminus F$ is not 2-connected, we can find two components $K,K'$ of $M\setminus F$ each having at least two elements. Moreover, since $E(M\setminus F)$ contains at most $2t + 1$ elements, one of these components, say $K$, has at most $t$ elements. Therefore $K$ contains a circuit $C$ of $M\setminus F$ having at most $t$ elements. This is a contradiction since $C$ is a circuit of $M$ and, by (5.24), $M$ has no circuits having fewer than $t + 1$ elements. Hence $M\setminus F$ is 2-connected.

Now let $A$ be a $t$-subset of $E(M)$. Then, since, by (5.24), $A$ is contained in a $(t+1)$-circuit of $M$, the matroid $M/A$ has a loop and therefore is not 2-connected. Now suppose that $M\setminus A$ is 2-connected. Then, since the circuits of $M\setminus A$ are circuits of $M$, every circuit of $M\setminus A$ contains at least $t + 1$ elements, by (5.24). Thus, since $M\setminus A$ contains $t + 2$ elements, $M\setminus A$ is isomorphic to $U[t, t+2]$ or $U[t+1, t+2]$. Now since, for every subset $F$ of $A$, the matroid $M\setminus F$ is 2-connected, we must have $rk(M\setminus A) = rkM = t + 1$. So $M$ is isomorphic to $U[t+1, t+2]$. Now let $B$ be a base of $M$ contained in $M\setminus A$. Then every $t$-subset of $B$ is complemented by some element of $E(M)-B$. As $M\setminus A$ is isomorphic to $U[t+1, t+2]$, each of these complements must be in $A$. Thus, since the collection
of t-subsets of B has cardinality t + 1 and A has cardinality t, B contains two distinct t-subsets D, D' that are complemented by an element e of A. So, by circuit elimination, D U D' contains a circuit of M, a contradiction since D U D' is contained in B. Hence M\A is not 2-connected. ✓

Combining (5.21) with the last theorem we obtain the following characterization of the Steiner systems S(t, t+1, 2t+2), t > 2.

(5.26) Theorem. Let t be an integer greater than one. Then the following are equivalent:

(i) M is a t-cyclic and t-cocyclic matroid of rank t + 1.

(ii) M is a Steiner system S(t, t+1, 2t+2).

Mendelsohn [12] has shown that if S(t, t+1, 2t+2) exists, then the complement of a block is a block. This follows from the last theorem and (5.19).

From (5.22) we see that if t is even, then there is no S(t, t+1, 2t+2). The following stronger condition is known: S(t, t+1, 2t+2) exists only if t + 2 is prime, see, for example, [27, Theorem 114]. Also, Mendelsohn and Hung [13] showed that S(9, 10, 20) does not exist. If fact, the systems S(t, t+1, 2t+2) are known only for t = 3 and t = 5. In the first case it is the affine space AG(3,2) and in the second case it is the Witt design whose automorphism group is the Mathieu group M_{12}, see, for example, [5].
It would be interesting to see if Theorem 5.26 gives new necessary conditions for the existence of the Steiner systems $S(t, t+1, 2t+2)$. 
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