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JONES POLYNOMIALS FOR INTERSECTING KNOTS AS PHYSICAL STATES OF QUANTUM GRAVITY

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We find a consistent formulation of the constraints of Quantum Gravity with a cosmological constant in terms of the Ashtekar new variables in the connection representation, including the existence of a state that is a solution to all the constraints. This state is related to the Chern-Simons form constructed from the Ashtekar connection and has an associated metric in spacetime that is everywhere nondegenerate. We then transform this state to the loop representation and find solutions to all the constraint equations for intersecting loops. These states are given by suitable generalizations of the Jones knot polynomial for the case of intersecting knots. These are the first physical states of Quantum Gravity for which an explicit form is known both in the connection and loop representations. Implications of this result are also discussed.

I. INTRODUCTION

The introduction of a new set of canonical variables for the Hamiltonian treatment of General Relativity by Ashtekar [1] has opened new possibilities of achieving a Dirac canonical quantization of the gravitational field. The new variables cast the dynamics of General Relativity in terms of a connection rather than in terms of a metric, making the phase space appear as imbedded in that of a Yang Mills theory. If one quantizes the theory taking a polarization in which wavefunctions are functions of the Ashtekar Connection $\Psi[A]$, one obtains the so called connection representation [1,2]. Another representation [3], similar to the ones previously introduced for Yang Mills theories [4], is the Loop Representation. In it, wavefunctions are functionals on loop space $\Psi[\gamma]$.

The classical canonical variables of Ashtekar are a set of triads or frame fields on the three manifold Σ of a foliation of space time, and are usually denoted by $\tilde{E}_i^a(x)$, where a is a spatial index on Σ and i is a flat Euclidean index, which can be thought of as an $SO(3)$ index (the tilde denotes a density weight). Canonically conjugate to these variables is an $SO(3)$ connection, obtained by pulling back to Σ the self dual part of the spin connection of spacetime. It is usually denoted as $A_a^i(x)$ and again a is a tensor index on Σ and i is an $SO(3)$ index. The constraints equations of General Relativity, with a cosmological constant Λ are:

$$\mathcal{G}^i = D_a \tilde{E}^{ai} \tag{1}$$

$$\mathcal{C}_b = \tilde{E}^{ai} F_{ab}^i = 0 \tag{2}$$

$$\mathcal{H} = \epsilon_{ijk} \tilde{E}^{ai} \tilde{E}^{bj} F_{ab}^k - \frac{\Lambda}{6} \eta_{abc} \epsilon^{ijk} \tilde{E}^{ai} \tilde{E}^{bj} \tilde{E}^{ck} \tag{3}$$

where F_{ab}^i is the curvature of A_a^i .

When quantizing the theory in the connection representation, the wavefunction are holomorphic functionals of the connection $\Psi[A]$ so that the connection is a multiplicative operator $\hat{A}_a^i \Psi[A] = A_a^i \Psi[A]$ and the triad is a functional derivative $\hat{E}^{ai} \Psi[A] = \frac{\delta}{\delta A_a^i} \Psi[A]$. To promote the constraints to quantum wave equations one is faced with a regularization and a factor

ordering problem.

Two main factor orderings have been considered in the literature, which basically consist in ordering the functional derivatives all to the left or all to the right. We will call these two choices I and II respectively and the explicit expressions for the constraints are shown in table I. The first one was considered by Ashtekar [1] and the second one by Jacobson and Smolin [2].

Factor ordering I has the following features:

- a) The algebra of constraints formally closes. (This result is only formal, the factor ordering probably cannot be properly addressed without the introduction of a regularization [5,6]).
- b) The diffeomorphism constraint *fails* to generate diffeomorphisms on the wavefunctions, at least formally.
- c) There exists a solution to the Hamiltonian constraint with a cosmological constant, which is also diffeomorphism invariant, given by [7,8]:

$$\Psi_{\Lambda}[A] = \exp\left(-\frac{6}{\Lambda} \int \tilde{\eta}^{abc} \text{Tr}[A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c]\right) \quad (4)$$

That is, the exponential of the Chern Simons form is a solution to the Hamiltonian constraint, which can be very easily checked using the relation:

$$\frac{\delta}{\delta A_a^i} \Psi_{\Lambda}[A] = \frac{3}{\Lambda} \tilde{\eta}^{abc} F_{bc}^i \Psi_{\Lambda}[A] \quad (5)$$

Moreover this function is invariant under diffeomorphisms, but at least formally it is not annihilated by the diffeomorphism constraint, which in this factor ordering fails to generate diffeomorphisms on the wavefunctions. It is also a nondegenerate solution in the sense of [9].

- d) It is from this factor ordering that one can obtain the Loop Representation via the introduction of the transform [3,4]

$$\Psi[\gamma] = \int dA \quad \text{Tr}(\text{Pexp} \oint \dot{\gamma}^a A_a) \Psi[A] \quad (6)$$

Consider how an operator \hat{O}_{γ} in the loop representation is obtained as the transform of \hat{O}_A in the connection representation:

$$\hat{O}_\gamma \Psi[\gamma] \equiv \int dA \text{Tr}(P \exp \oint \dot{\gamma}^a A_a) \hat{O}_A \Psi[A] \quad (7)$$

$$= \int dA (\hat{O}_A^\dagger \text{Tr}(P \exp \oint \dot{\gamma}^a A_a)) \Psi[A] \quad (8)$$

Assuming that \hat{O}_A is self-adjoint with respect to the measure in the transform, \hat{O}_γ is the operator which acting on the kernel of the transform has the same action as \hat{O}_A . The loop representation [3,10] is based on factor ordering I and on the assumption that \hat{A}_a^i and \hat{E}^{ai} are self-adjoint. But when finding the transform of products, the factor ordering is reversed, i.e. the constraints in the loop representation are based on the action of the constraints on the kernel with factor ordering II. The opposite factor ordering does not lead to the loop representation.

e) In particular, the proposed wavefunction can be transformed to the loop representation, as was considered, for instance (for the restricted case of nonintersecting loops) in [11–13], its transform being related to the Jones Polynomial. These wavefunctions belong to the well known -and degenerate [9]- set of solutions to the constraints of quantum gravity based on non-intersecting loops [2,3,10]

Factor ordering II has been considered elsewhere [2,9]. In this factor ordering it was found that the holonomy of the Ashtekar connection (for the case of a smooth loop) was a solution to the Hamiltonian constraint of the theory. Moreover the diffeomorphism constraint coincides with the generator of diffeomorphisms on the wavefunctions. However, this factor ordering is in principle endowed with a difficulty: formally the algebra of constraints fails to close. (This was already noticed in [2]). To be more precise the commutator of two Hamiltonians is "proportional" to a diffeomorphism, as it should be, but due to the factor ordering the proportionality factor appears *to the right*. This means that if one has a physical state $\Psi_{\text{ph}}[A]$ (annihilated by both diffeomorphism and hamiltonian constraints) one could get a potential inconsistency in that the equation:

$$[\hat{\mathcal{H}}(M), \hat{\mathcal{H}}(N)] \Psi_{\text{ph}}[A] = 2\hat{\mathcal{C}}_a (M\partial_b N - N\partial_b M) \hat{q}^{ab} \Psi_{\text{ph}}[A] \quad (9)$$

would have a left member identically zero and a nonvanishing right member. This, plus other reasons lead us to consider in this paper factor ordering I.

The main objective of this paper is to show that the following points can be accomplished:

a) The construction of a regularized, factor ordered version of the connection representation which is consistent in the sense expounded above.

b) The construction of nondegenerate physical states in the connection representation so developed.

c) An explicit calculation of the transform to the loop representation of the solutions so constructed.

d) The discussion of up to what extent the loop and connection representations are consistent with each other.

The organization of this article is as follows: in section 2 we will show how a regularized version of the diffeomorphism constraint in factor ordering I generates diffeomorphisms on the wavefunctions and therefore $\Psi_\Lambda[A]$ becomes now a solution to all the constraints of Quantum Gravity and therefore a physical state of the theory. In section 3 we introduce the generalization of the Jones Polynomials to the case of intersecting loops. The idea is to compute the transform of the physical state $\Psi_\Lambda[A]$ to the Loop Representation, including the nontrivial case of intersecting loops. This is done in section 4. The resulting polynomials, in spite of having an arbitrary number of intersections, will automatically solve the complicated hamiltonian constraint of Quantum Gravity in the loop representation. In section 5 we discuss the issue of diffeomorphism invariance of the solutions found and how the use of loops requires a regularization that breaks diffeomorphism invariance.

II. REGULATING THE DIFFEOMORPHISM CONSTRAINT

If we consider factor ordering I we have an almost satisfactory situation at the formal level. The algebra of constraints closes at the quantum level, there exists a solution to the Hamiltonian constraint (which in addition is "nondegenerate" in the sense exposed in [9]). However, a very unsatisfactory point arises when one notices that (formally) the diffeomorphism constraint fails to generate diffeomorphisms on the wavefunctions. At this point it

is convenient to take into account the analysis of Tsamis and Woodard [5] and Friedman and Jack [6] concerning the need of a regularization for a consistent treatment of the factor ordering problem and the constraint structure of Quantum Gravity. We will therefore regulate the diffeomorphism constraint in the same way the hamiltonian constraint is usually [2,3,10] regulated: by point splitting. If we point split the diffeomorphism constraint:

$$\hat{\mathcal{C}}_\epsilon(\vec{N}) = \int d^3x \int d^3y N^a(x) f_\epsilon(x, y) \frac{\delta}{\delta A_b^i(x)} F_{ab}^i(y) \quad (10)$$

where $f_\epsilon(x, y)$ is any even ($f_\epsilon(x, y) = f_\epsilon(y, x)$) regulator that in the limit $\epsilon \rightarrow 0$ goes to $\delta(x, y)$. One can immediately check that,

$$\hat{\mathcal{C}}_\epsilon(\vec{N}) = \int d^3x \int d^3y N^a(x) f_\epsilon(x, y) F_{ab}^i(y) \frac{\delta}{\delta A_b^i(x)} \quad (11)$$

since due to the symmetry of the regulator the extra term:

$$\int d^3x N^a(x) \int d^3y f_\epsilon(x, y) \partial_{[a} \delta(x, y) \delta_{b]}^b \delta^i \quad (12)$$

vanishes upon integration by parts.

That is, in the regularized version, the diffeomorphism constraint generates diffeomorphisms in the factor ordering prescribed. This simple calculation shows that factor ordering I is consistent for quantum gravity. It respects the symmetry of the theory under diffeomorphisms, gives a correct closure (at least formally) to the algebra of constraints and allows the construction of a simple and nondegenerate physical state. This state is given by expression (4), which is now annihilated by all the regularized constraints of Quantum Gravity with cosmological constant.

It should be noticed that the solutions to the Hamiltonian constraint known in the connection representation up to present [2,14,9] correspond to the factor ordering II. Therefore they cannot be used as a starting point to construct solutions to the Hamiltonian constraint of Quantum Gravity in loop space.

III. KNOT POLYNOMIALS FOR INTERSECTING LOOPS

It is well known that the transform of the state (4) into the loop representation is given by a knot polynomial closely related to the Jones Polynomial. This calculation has been performed by several different techniques. Witten [11] considered a nonperturbative approach whereas Smolin [13] and Cotta-Ramusino et al [12] checked perturbatively Witten's claim (this latter method is the one to be considered later on in this paper). In all these calculations however, only smooth nonintersecting loops have been considered.

In Quantum Gravity however, one needs more generality. The Hamiltonian constraint of Quantum Gravity in the Loop Representation trivially annihilates all states with support only on smooth nonintersecting loops. Unfortunately, since states with support on smooth nonintersecting loops are also annihilated by the determinant of the spatial metric, they become states of the gravitational field for an arbitrary value of the cosmological constant (to see this notice that the only difference between the Hamiltonian formalism with and without a cosmological constant is a term proportional to the determinant of the three metric). Therefore all these states are physical in the sense that they are annihilated by all the constraints, but basically correspond to spatially degenerate metrics [9]. Moreover, the determinant of the three metric appears not only in the cosmological constant term but in matter couplings in general [15].

One can, however, perform the loop transform of the state (4) for the case of loops with intersections. This will be one of the main points of this paper. The result will be knot polynomials (appropriately generalized to include intersections). These polynomials, in spite of including an arbitrary number of arbitrary-order intersections, will however, manage to solve the complicated hamiltonian constraint of quantum gravity. The reason for this is that we will obtain them as loop transforms of a functional that solves the hamiltonian constraint in the connection representation!

We are therefore interested in obtaining the loop transform allowing the loops to have intersections. For this we will first need to generalize notions of knot polynomials to the

intersecting case. This has received some attention in the past [16,17]. We now briefly sketch how to generate knot polynomials with intersections from notions of the Braid Group.

A standard technique for constructing knot polynomials is to start from the Braid Group. The Braid Algebra B_n is composed of elements g_i , with $0 < i < n$ (as depicted in figure 1) that satisfy:

$$g_i g_j = g_j g_i \quad \text{for } |i - j| > 1 \quad (13)$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad (14)$$

Each element of B_n represents a braid diagram composed by lines, called strands lying on a plane and moving each around the other. If g_i represents an over crossing of the line i and $i + 1$ the corresponding undercrossing is represented by g_i^{-1} . Two braids are equivalent if they may be transformed into each other by smooth deformations of the strands in R^3 , leaving their endpoints fixed. To proceed from braids to knots one identifies the top and bottom ends of the braid. Two knots are equivalent if they differ by a finite sequence of moves known as Markov moves [18]. The Braid Algebra can be enlarged to consider the case of Braids with intersections [16,13]. One just introduces a new generator a_i representing a 4-valent rigid vertex . If one wishes to consider intersections of more than two braids at each point the algebra has to be enlarged further [17].

The extended braids are also subject to relations following from equivalence under smooth deformations in R^3 . They are:

$$a_i g_i = g_i a_i \quad (15)$$

$$g_i^{-1} a_{i+1} g_i = g_{i+1} a_i g_{i+1}^{-1} \quad (16)$$

and:

$$[g_i, a_j] = 0 \quad [a_i, a_j] = 0 \quad |i - j| > 1 \quad (17)$$

One can find matrix representations for the Braid Algebras [19–21]. From these representations one can derive skein relations for the knot polynomials. The one that yields the

Jones Polynomial (in the nonintersecting case) is related with associating to each generator g_i a $2^n \times 2^n$ matrix acting on the linear space $V(n) = V_1 \otimes V_2 \otimes \dots \otimes V_n$ where V_i is a two dimensional space corresponding to the strand i . More precisely G_i will be represented by:

$$G_i = q^{1/4}(I \otimes \dots \otimes K \otimes \dots \otimes I) \quad (18)$$

where q is an arbitrary complex number, I is the 2×2 identity matrix and the matrix K , which acts on $V_i \otimes V_{i+1}$, is given by:

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - q^{-1} & q^{-1/2} & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (19)$$

An extended representation including the vertex generators is given by the $2^n \times 2^n$ matrices [17]:

$$A_i = I \otimes \dots \otimes A \otimes \dots \otimes I \quad (20)$$

where A is given by the matrix acting on $V_i \otimes V_{i+1}$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & (1-a)q^{1/2} & 0 \\ 0 & (1-a)q^{1/2} & 1 - (1-a)q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (21)$$

where a is another complex parameter.

The skein relations result from the following identities satisfied by the G_i and A_i matrices in this representation:

$$q^{1/4}G_i - q^{-1/4}G_i^{-1} - (q^{1/2} - q^{-1/2})I_i = 0 \quad (22)$$

$$A_i = q^{1/4}(1-a)G_i^{-1} + aI_i \quad (23)$$

To construct a regular isotopy invariant link polynomial one defines the enhancement matrix:

$$M_n = \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_n \quad (24)$$

with:

$$\mu_i = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \quad (25)$$

Then one can show that:

$$F(B) = Tr[BM_n] \quad (26)$$

is a regular isotopy link invariant, where the trace is taken in the vector space $V(n)$, and B is a matrix representing an arbitrary element of B_n .

Given the diagrams of figure 2, we can now write the skein relations satisfied by $F(q, a)$ as:

$$F_{\hat{L}_+} = q^{3/4} F_{\hat{L}_0} \quad (27)$$

$$F_{\hat{L}_-} = q^{-3/4} F_{\hat{L}_0} \quad (28)$$

$$q^{1/4} F_{L_+} - q^{-1/4} F_{L_-} = (q^{1/2} - q^{-1/2}) F_{L_0} \quad (29)$$

$$F_{L_I} = q^{1/4} (1 - a) F_{L_-} + a F_{L_0} \quad (30)$$

$$F_0 = 1 \quad (31)$$

the last being the standard normalization condition on the unknot.

This ends our discussion of the generalized Jones polynomial for intersecting loops. It is clear that for non intersecting loops it reduces to the standard form of the Jones Polynomial if one multiplies our results by $q^{-3/4w(L)}$ where $w(L)$ is the writhing of L . The reason for this difference is that the Jones Polynomial is ambient isotopic invariant whereas we are more interested in polynomials that are regular isotopic invariant for reasons we will see in the next section.

IV. SKEIN RELATIONS FOR THE PHYSICAL STATE

Let us consider the loop transform of the physical state $\Psi_\Lambda[A]$. For the nonintersecting case similar calculations were performed by Smolin [13] and Cotta-Ramusino et al. [12]. The first calculation was actually performed by Witten [11] but with different techniques.

The loop transform is given by:

$$\Psi[\gamma] = \int dA \quad Tr[U(\gamma)] \exp(-\frac{12}{\Lambda} S_{CS}[A]) \quad (32)$$

where $Tr[U(\gamma)] = Tr[P \exp(\oint A_a \dot{\gamma}^a)]$ and $S_{CS}[A] = \frac{1}{2} \int \tilde{\eta}^{abc} Tr[A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c]$ is the Chern-Simons action.

We now consider the variation of this expression when a small loop of area Σ^{ab} is appended to the loop γ . Let us first consider the case without intersections. We get:

$$\Sigma^{ab} \Delta_{ab}(x) \Psi[\gamma] = \int dA \quad \Sigma^{ab} F_{ab}^i(x) Tr[\tau^i U(\gamma_x^x)] \exp(-\frac{12}{\Lambda} S_{CS}) \quad (33)$$

where Δ_{ab} is the area derivative [22,23], τ^i is one of the generators of $SU(2)$ and we have used $\Delta_{ab}(x) Tr[U(\gamma)] = F_{ab}^i(x) Tr[\tau^i U(\gamma_x^x)]$ and γ_x^x is the loop with origin at the point x .

Using the relation (5) and integrating by parts, one obtains:

$$-\frac{\Lambda}{6} \int dA \quad \Sigma^{ab} \eta_{abc} \int dy^c \delta(x-y) Tr[\tau^i U(\gamma_x^y) \tau^i U(\gamma_y^x)] \exp(-\frac{12}{\Lambda} S_{CS}) \quad (34)$$

The integral depends on the volume factor

$$\Sigma^{ab} \eta_{abc} dy^c \delta(x-y) \quad (35)$$

which depending on the relative orientation of the two-surface Σ^{ab} and the differential dy^c (which is tangent to γ), can lead to ± 1 or zero. (This expression should really be regularized. We have absorbed appropriate extra factors in the definition of the cosmological constant so to normalize the volume to ± 1). Consequently, depending on the value of the volume there are three possibilities:

$$\delta\Psi[\gamma] = 0 \quad (36)$$

$$\delta\Psi[\gamma] = \pm \frac{\Lambda}{8} \Psi[\gamma] \quad (37)$$

These equations can be diagrammatically interpreted in the following way:

$$\Psi[\hat{L}_\pm] - \Psi[\hat{L}_0] = \pm \frac{\Lambda}{8} \Psi[\hat{L}_0] \quad (38)$$

When the volume element vanishes it corresponds to a variation that does not change the topology of the crossing.

Let us now consider the case of a point where there is an intersection. As before, we consider an infinitesimal deformation of the loop consisting in the addition of a small closed loop, in this case at the point of intersection (see figure 3):

$$\Sigma^{ab} \Delta_{ab}(y) \Psi[\gamma] = \frac{\Lambda}{6} \int dA \Sigma^{ab} \eta_{dab} \text{Tr}[\tau^i U_{23}(\gamma_y^y) U_{41}(\gamma_y^y)] \frac{\delta}{\delta A_d^i(y)} \exp(-\frac{12}{\Lambda} S_{CS}) \quad (39)$$

Again integrating by parts and choosing the element of area Σ^{ab} parallel to segment 1-2 so that the contribution of the functional derivative corresponding to the action on the segment 1-2 vanishes (since the volume element is zero) we get:

$$\begin{aligned} \Sigma^{ab} \Delta_{ab} \Psi[\gamma] &= -\frac{\Lambda}{6} \int dA \Sigma^{ab} \eta_{abc} \int dv^c \delta(y-v) \times \\ &\quad \times \text{Tr}[\tau^i U_{23}(\gamma_y^y) \tau^i U_{41}(\gamma_y^y)] \exp(-\frac{12}{\Lambda} S_{CS}) \end{aligned} \quad (40)$$

Making use of the Fierz identity for $SU(2)$:

$$\tau_B^{iA} \tau_D^{iC} = -\frac{1}{2} \delta_D^A \delta_B^C + \frac{1}{4} \delta_B^A \delta_D^C \quad (41)$$

one finally gets:

$$\begin{aligned} \Sigma^{ab} \Delta_{ab} \Psi[\gamma] &= \\ &= \frac{\Lambda}{12} \int dA \Sigma^{ab} \eta_{abc} \int dv^c \delta(y-v) \text{Tr}[U_{23}(\gamma_y^y)] \text{Tr}[U_{41}(\gamma_y^y)] \exp(-\frac{12}{\Lambda} S_{CS}) \\ &\quad - \frac{\Lambda}{24} \int dA \Sigma^{ab} \eta_{abc} \int dv^c \delta(y-v) \text{Tr}[U_{23}(\gamma_y^y) U_{41}(\gamma_y^y)] \exp(-\frac{12}{\Lambda} S_{CS}) \end{aligned} \quad (42)$$

where we have called $U_{ij}(\gamma_{x_1}^{x_2})$ the holonomy from point x_1 to x_2 traversing through lines i and j .

These relations can be interpreted as the following skein relation for the intersection.

$$\Psi[L_{\pm}] = (1 \mp \frac{\Lambda}{24})\Psi[L_I] \pm \frac{\Lambda}{12}\Psi[L_0] \quad (43)$$

$$\Psi[\hat{L}_{\pm}] = (1 \pm \frac{\Lambda}{8})\Psi[\hat{L}_0] \quad (44)$$

In order to compare with the link polynomials we must first notice that the results we have obtained correspond to a linear approximation, since we have only considered an infinitesimal deformation of the link. In order to consider a finite deformation we would have to consider higher order derivatives of the wavefunction.

It is convenient to rewrite the relations obtained in such a way that the correspondence with those of the Jones Polynomials in the intersecting case is manifest. To do this we notice that the factor $(1 + \frac{\Lambda}{8})$ plays the role of $q^{3/4}$ and therefore in the linearized case if we define q as $q = e^k$, then $k = \frac{\Lambda}{6}$. Inverting the relation (43) we get:

$$\Psi[L_I] = (1 \pm \frac{\Lambda}{24})\Psi[L_{\pm}] \mp \frac{\Lambda}{12}\Psi[L_0] \quad (45)$$

which allows us to recognize that the value of the variable a of the Generalized Jones Polynomial is $a = 1 - e^{-\frac{\Lambda}{12}}$ which to first order yields $a = \frac{\Lambda}{12}$.

Expression (29) relating $\Psi[L_+]$ and $\Psi[L_-]$ can be obtained in this case by combining eqs. (45). Previous derivations of these expressions (for the nonintersecting case) [12] are somewhat misleading since the functional derivative in the case where there is no intersection only can act on one side of the crossing and therefore gives a vanishing contribution.

So we see that the Generalized Jones Polynomials, introduced in the last section for loops with double self-intersections from the Braid Group, are actually the loop transforms of a physical nondegenerate quantum state of the gravitational field defined by values of q and a that, to first order in perturbation theory coincide with the ones presented above.

In this sense we can therefore say that they are annihilated by the constraints of quantum gravity in the loop representation and are therefore physical states of the gravitational field. The issue of finding physical states of the gravitational field that were nondegenerate received a lot of attention recently and great effort went into the construction of examples of such states [2,14,9,24]. Since we started from a $\Psi_{\Lambda}[A]$ which was nondegenerate in the connection

representation, its transform to loop space will also be nondegenerate. In this paper we have considered a restricted transform in the subspace of loops with double intersections. A simple extension of the calculations presented in this paper, by including higher order intersecting loops in the transform would allow the construction of a physical state in loop space that is not annihilated by the determinant of the three metric and which would therefore be nondegenerate.

V. DISCUSSION AND CONCLUSIONS:

We have found a solution to all the constraints of Quantum Gravity in both the connection and the loop representations. This allows for the first time to analyze the consistency between both representations, and to some extent clarify the properties of the loop transform. The existence of a transform has received great attention recently due to the attempts to formulate a rigorous definition for it in several theories by Ashtekar and Isham [25].

In connection with this point a surprising fact arises, which was already known to people working in Chern-Simons theories [11]: although the state $\Psi_\Lambda[A]$ in the connection representation is diffeomorphism invariant (and annihilated by the diffeomorphism constraint), the transformed state $\Psi_\Lambda[\gamma]$ is only a *regular* isotopic invariant. In fact the physical state can be written as:

$$\Psi_\Lambda[\gamma] = \left(1 - \frac{\Lambda}{8}\right)^{w(\gamma)} P(\gamma) \tag{46}$$

where $w(\gamma)$ is the writhing of γ and $P(\gamma)$ is the Jones Polynomial. The Jones Polynomial *is* diffeomorphism invariant (it is ambient isotopic invariant in the knot theory language), however the writhing number is not. Therefore we see that the failure to obtain a diffeomorphism invariant solution is concentrated in the first factor. One could try to fix this problem by eliminating this first factor and directly try to propose the Jones Polynomial as a solution to the constraints. This however, fails. The Hamiltonian constraint has a nontrivial action on the writhing number. What is a solution of all the constraints is $\Psi_\Lambda[\gamma]$ and not $P(\gamma)$.

A reasonable question therefore is to ask what happened to the consistency of both representations. One started from a state that was diffeomorphism invariant and annihilated by the diffeomorphism constraint and by transforming to the loop representation diffeomorphism invariance was lost. This occurs since the writhing depends on how the three dimensional knot is projected into a plane. For a fixed projection, a diffeomorphism of the knot in three dimensions can change the writhing since \hat{L}_\pm and \hat{L}_0 contribute differently. The ambiguity is not present if one considers bands instead of loops (see figure 4). A prescription which given a loop produces a band is called framing. Therefore in order to obtain a diffeomorphism invariant state one needs a diffeomorphism invariant framing. Evidently, the choice of framing was hidden in the implicit measure dA in the space of connections used in the transform and this is how diffeomorphism invariance was lost.

At this point it is clear that the very use of loops is generating problems with invariance under diffeomorphisms. This can be seen with greater clarity if one considers a simpler example, the case of an Abelian theory. If one considers the loop transform of an Abelian Chern-Simons functional:

$$\int dA \exp(k \int d^3x \tilde{\eta}^{abc} A_a \partial_b A_c) \times \exp(i \oint \dot{\gamma}^a A_a) \quad (47)$$

the integral can be explicitly computed since it is a gaussian and corresponds to the exponential of the Gauss Self Linking number:

$$\oint ds \oint dt \dot{\gamma}^a(s) \dot{\gamma}^b(t) \epsilon_{abc} \frac{(\gamma(s) - \gamma(t))^c}{|\gamma(s) - \gamma(t)|^3} \quad (48)$$

This quantity is ill defined when $\gamma(s) = \gamma(t)$. Therefore it has to be regularized (These issues have extensively been discussed in [21] and references therein). This is again accomplished by means of a framing and leads to a functional of bands rather than of knots. This problem evidently comes from the singular (distributional) character of loops as functions of the three manifold and the associated need of a regularization.

We would like to propose a possible solution to this problem based on the replacement of the loops by smooth nonsingular objects. We will only illustrate the point with a brief

discussion of the abelian case, the nonabelian generalization and a detailed discussion of the point in general being beyond the scope of this article. The key point is to notice that the only way "loops" have entered the formalism is via the holonomy, in which they are represented by divergence-free vector densities $\tilde{X}^a(x)$, defined by $\tilde{X}^a(x) = \oint ds \dot{\gamma}^a(s) \delta^3(x - \gamma(s))$ which we will call "loop coordinates" (they have also been called "form factors" [26]). In terms of them, the holonomy is a three dimensional integral $h[\gamma, A] = \int d^3x A_a(x) \tilde{X}^a(x)$. Now one could consider a representation based, instead of on loops, *on smooth divergence free vector densities* $\tilde{X}^a(x)$, which is inherently free of the mentioned singularities. For instance, the transform to this representation of the Chern-Simons wavefunction would be the exponential of:

$$\int d^3x \int d^3y \tilde{X}^a(x) X_a(y) \quad (49)$$

where $X_a(y)$ is the "potential" defined by $\partial_{[b} X_{a]}(y) = \eta_{bac} \tilde{X}^c(y)$ (whose existence is guaranteed due to the fact that the \tilde{X}^a are divergence free; one can also check that if one considers the $\tilde{X}^a(x)$ as given by the expression in terms of the loops this expression yields the linking number). This expression is completely nonsingular, well defined and diffeomorphism invariant.

This strongly suggests that a nonabelian generalization of this construction, based on coordinates on loop space formed by multivector densities [27,28,24] could analogously solve the problem of the loss of diffeomorphism invariance of the loop transformed expressions of the physical states here introduced.

A natural calculation that arises from the issues discussed here is to attempt to show in an explicit way if the wavefunctions presented here in the loop representation are annihilated by the Hamiltonian constraint in that representation directly. The calculation is long and exceeds the scope of this paper. However one can immediately draw some qualitative conclusions from it by looking at the first terms in the expansion. They correspond to known knot invariants [29], on which one can explicitly compute the action of the Hamiltonian constraint in the loop representation. This allows, by requiring consistency at each

order, to draw conclusions about the behaviour of each knot invariant under the action of the hamiltonian constraint. The results of this analysis will be presented elsewhere. In particular, to zeroth order in the cosmological constant, the polynomial becomes a function of the number of connected parts of the link. This term is immediately a solution of the Hamiltonian constraint (in vacuum) since it is annihilated by the area derivative. It can also be checked that it is not annihilated by the determinant of the three metric (for instance for three loops, applying formula (12) of reference [24]), and therefore it is the simplest example of a nondegenerate physical state of the gravitational field.

In a forthcoming paper we will analyze how, by looking order by order in the expansion of $\Psi_\Lambda[\gamma]$ we can actually find an infinite set of solutions to all the constraints of quantum gravity (with vanishing cosmological constant) in the loop representation, including as a particular case a remarkably simplified derivation of the results of reference [24].

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TABLES

TABLE I. Two factor orderings for the constraints of Quantum Gravity in terms of Ashtekar's variables

Factor Ordering I	Factor Ordering II
$\hat{\mathcal{G}}^i = D_a \frac{\delta}{\delta A_a^i}$	$\hat{\mathcal{G}}^i = D_a \frac{\delta}{\delta A_a^i}$
$\hat{\mathcal{C}}_b = \frac{\delta}{\delta A_a^i} F_{ab}^i$	$\hat{\mathcal{C}}_b = F_{ab}^i \frac{\delta}{\delta A_a^i}$
$\hat{\mathcal{H}} = \epsilon_{ijk} \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^j} F_{ab}^k - \frac{\Lambda}{6} \epsilon_{ijk} \mathcal{L}^{abc} \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^j} \frac{\delta}{\delta A_c^k}$	$\hat{\mathcal{H}} = \epsilon_{ijk} F_{ab}^k \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^j} - \frac{\Lambda}{6} \epsilon_{ijk} \mathcal{L}^{abc} \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^j} \frac{\delta}{\delta A_c^k}$

FIGURES

FIG. 1. Graphic picture of the Braid Group relations (12-16) for the case of braids without and with intersections.

FIG. 2. Knot configurations involved the skein relations (43-45).

FIG. 3. Addition of a small closed loop for the calculation of the skein relations in the intersecting case.

FIG. 4. A depiction of regular isotopic invariance. Both bands can be associated to loops that are related by a diffeomorphism. However the corresponding bands are not diffeomorphically related.