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**QUANTUM EINSTEIN-MAXWELL FIELDS:
A UNIFIED VIEWPOINT
FROM THE LOOP REPRESENTATION**

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We propose a naive unification of Electromagnetism and General Relativity based on enlarging the gauge group of Ashtekar's new variables. We construct the connection and loop representations and analyze the space of states. In the loop representation, the wavefunctions depend on two loops, each of them carrying information about both gravitation and electromagnetism. We find that the Chern-Simons form and the Jones Polynomial play a role in the model.

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I. INTRODUCTION

The introduction of the Ashtekar New Variables [1] for the treatment of canonical gravity has opened new hopes that General Relativity may be nonperturbatively quantized. In particular the Loop Representation [2] allows us to construct for the first time physical states of quantum gravity without recurring to minisuperspace approximations [3]. Knot theory and in particular the Jones Polynomial play a crucial role in the theory [4].

In spite of these successes it is still not obvious up to which extent this program can allow us to understand the physics at the energies of relevance for quantum gravity. In particular, the main results of the program at the moment seem quite tailored to vacuum General Relativity in four space time dimensions. This raises the question of up to what extent is this program a suitable approach for the incorporation of other interactions.

In this work we will suggest that the idea of a unified theory described in terms of Ashtekar's new variables is possible and that several appealing results of the vacuum theory find very naturally their counterpart in the unified model. We will show, for instance, that knot theory still plays a crucial role and that the techniques used to find states for the theory in the vacuum case are still applicable. In summary, these ideas for quantizing gravity can lead to interesting new insights also in the case where matter fields are present and therefore are well suited for understanding the physics of particles at the energies of unification and not just pure gravity.

The idea of unifying gravity with other forces enlarging the group of Ashtekar variables is not new [5]. However the program outlined in this paper is less ambitious than others. We do not pretend to recover the same form of the constraints as the vacuum ones just with an enlarged group. We will see, however, that the "kinematic" constraints can actually be rewritten in this way. This will be enough to find several connections between results of the vacuum theory and the unified theory. The Hamiltonian constraint will be quite different and we will briefly outline the consequences of this. Although we will only give details for the Einstein-Maxwell case, the same ideas can be straightforwardly generalized

to Einstein-Yang-Mills theories for $SU(N)$.

II. EINSTEIN-MAXWELL THEORY IN TERMS OF A $U(2)$ CONNECTION

We begin with a brief summary of the Einstein-Maxwell theory in terms of Ashtekar's new variables. We will assume a 3+1 decomposition of spacetime has been performed, slicing it into spatial three- surfaces Σ on which all variables are defined. The variables for the gravitational part are a (densitized) triad \tilde{E}_i^a , which determines the spatial metric by $\tilde{q}^{ab} = \tilde{E}_i^a \tilde{E}_i^b$ and an $SU(2)$ connection A_a^i as conjugate momentum. For the Maxwell field the variables are the electric field $\tilde{\mathbf{e}}^a$ and the vector potential \mathbf{a}_a . The dynamics of the theory is pure constraint, and the constraints (including a cosmological constant Λ) are [6],

$$\partial_a \tilde{\mathbf{e}}^a = 0 \tag{1}$$

$$D_a \tilde{E}^a = 0 \tag{2}$$

$$i\sqrt{2}\tilde{E}_i^a F_{ab}^i - \frac{1}{2}\tilde{\mathbf{e}}^a \mathbf{f}_{ab} = 0 \tag{3}$$

$$\epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{ab}^k + \Lambda \det(E)^2 - \tag{4}$$

$$-\frac{1}{8}\det(E)^{-2} \{ \tilde{E}_i^a \tilde{E}_i^b \tilde{E}_j^c \tilde{E}_j^d \eta_{acf} \eta_{bdg} [\tilde{\mathbf{e}}^f \tilde{\mathbf{e}}^g + \tilde{\mathbf{b}}^f \tilde{\mathbf{b}}^g] \} = 0$$

where $\det(E)^2 = \frac{-1}{3\sqrt{2}}\eta_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \tilde{E}_k^c$

Equation (1) is the Gauss Law of the $U(1)$ symmetry. Equation (2) is the Gauss Law of Ashtekar's formalism, stemming from the invariance under triad rotations. Equation (3) is the diffeomorphism constraint and equation (5) is the Hamiltonian constraint. Notice that this constraint can be made polynomial by multiplying by $\det(E)^2$.

We will now show how to write these equations in terms of a single set of variables. We introduce a $U(2)$ connection in the following way,

$$\mathcal{A}_a = A_a^i \sigma_i + i \mathbf{a}_a \mathbf{1}, \quad (5)$$

where e is the electric charge, and in a similar fashion a $U(2)$ electric field,

$$\tilde{\mathcal{E}}_a = \tilde{E}_a^i \sigma_i + i \tilde{\mathbf{e}}_a \mathbf{1}. \quad (6)$$

That is we are taking the direct product of $U(1)$ and $SU(2)$ to form a $U(2)$ symmetry. We can similarly introduce a field tensor \mathcal{F}_{ab} and a magnetic field. From these one can recover the original quantities by taking traces,

$$\tilde{E}^a = \tilde{\mathcal{E}}^a - \frac{1}{2} \text{Tr}(\tilde{\mathcal{E}}^a) \quad (7)$$

$$\tilde{e}^a = \frac{1}{2} \text{Tr}(\tilde{\mathcal{E}}^a) \quad (8)$$

and so on.

The introduction of these quantities allows us to rewrite the constraint equations as,

$$D_a \tilde{\mathcal{E}}^a = 0 \quad (9)$$

$$\text{Tr}(\tilde{\mathcal{E}}^a \mathcal{F}_{ab}) = 0 \quad (10)$$

$$\begin{aligned} & \frac{1}{6} \eta_{abc} \eta_{edf} \text{Tr}(\tilde{\mathcal{E}}^a \tilde{\mathcal{E}}^b \tilde{\mathcal{E}}^c) \text{Tr}(\tilde{\mathcal{E}}^e \tilde{\mathcal{E}}^d \tilde{\mathcal{B}}^f) + \\ & + \eta_{abc} \eta_{edf} \text{Tr}(\tilde{\mathcal{E}}^a \tilde{\mathcal{E}}^e) \text{Tr}(\tilde{\mathcal{E}}^b \tilde{\mathcal{B}}^c) \text{Tr}(\tilde{\mathcal{E}}^d \tilde{\mathcal{B}}^f) - \\ & - \eta_{abc} \eta_{edf} \text{Tr}(\tilde{\mathcal{E}}^a \tilde{\mathcal{E}}^e) (\text{Tr}(\tilde{\mathcal{E}}^c) \text{Tr}(\tilde{\mathcal{E}}^f) + \\ & + \text{Tr}(\tilde{\mathcal{B}}^c) \text{Tr}(\tilde{\mathcal{B}}^f)) + \\ & + \frac{\Lambda}{36} \eta_{abc} \eta_{edf} \text{Tr}(\tilde{\mathcal{E}}^a \tilde{\mathcal{E}}^b \tilde{\mathcal{E}}^c) \text{Tr}(\tilde{\mathcal{E}}^d \tilde{\mathcal{E}}^e \tilde{\mathcal{E}}^f) = 0 \end{aligned} \quad (11)$$

Notice that we have rescaled the Hamiltonian constraint with a factor $\det(E)^2$ in order to make it polynomial.

It is worthwhile noticing that this is just a *rewriting* of the equations, that is, the theory remains exactly the same. Therefore, for instance, the constraint algebra and the consistency of the theory with the reality conditions [6] are automatically preserved.

A remarkable fact of this construction is that the “kinematic” constraints —the Gauss Law and the Diffeomorphism constraint— look *exactly* the same as those of the vacuum theory, only evaluated for a different group, $U(2)$. This will allow us to exploit several results found for the vacuum theory for the unified model.

III. CONNECTION REPRESENTATION AND THE CHERN-SIMONS FORM

We now attempt to quantize the theory in the “Connection Representation”, that is, we take a polarization in which wavefunctions are functionals of the connection $\Psi[\mathcal{A}]$ and $\hat{\mathcal{A}}_a \Psi[\mathcal{A}] = \mathcal{A}_a \Psi[\mathcal{A}]$, $\hat{\mathcal{E}}^a \Psi[\mathcal{A}] = \frac{\delta}{\delta \mathcal{A}_a} \Psi[\mathcal{A}]$. In the vacuum theory two main results have been achieved with this representation: a) The result of Jacobson and Smolin [7] that showed that Wilson loops constructed with Ashtekar’s connection were solutions to the Hamiltonian constraint and b) The result of Kodama [8], later extended in ref. [4], that showed that the exponential of the Chern-Simons form constructed with Ashtekar’s connection was a solution to all the constraints with a cosmological constant.

Let us examine these facts for our model. Start by considering Wilson loops $W_L(\mathcal{A}) = \text{Tr}(\text{P exp} \oint dy^a \mathcal{A}_a(y))$. For the vacuum case these quantities solve the Hamiltonian Constraint. For our case, however, they fail to be solutions. This is due to two facts. First of all, since we rescaled our constraint by the determinant of \mathcal{E} , this quantity should be nonvanishing for the state of interest. This fact never occurs for the solutions of the vacuum theory [9]. Even if one considered a loop with a generic triple intersection, in order to make the determinant nonvanishing, it is unlikely that finite combinations of these Wilson loops will be able to solve the constraint. This is basically due to the fact that the Maxwellian part of the Hamiltonian constraint has a term $\tilde{\mathbf{b}}^2$ which is purely multiplicative in this representation and therefore cannot be cancelled by any of the other terms.

Let us now consider the state,

$$\Psi_\Lambda[\mathcal{A}] = \exp\left(-\frac{6}{\Lambda} \text{Tr} \int \mathcal{A} \wedge d \wedge \mathcal{A} - \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right) \quad (12)$$

It can be readily seen that this state can be decomposed like $\Psi_\Lambda[\mathcal{A}] = \Psi_\Lambda[A]\Psi_\Lambda[\mathbf{a}]$ in terms of the usual Ashtekar variables. It is a remarkable fact that this state actually manages to solve all the constraints of the theory with a cosmological constant. This is quite easy to see. The portion $\Psi_\Lambda[A]$ is a solution of the constraints of vacuum gravity with a cosmological constant [8]. Besides, the portion $\Psi_\Lambda[\mathbf{a}]$ is an eigenstate of zero eigenvalue for the Hamiltonian of Maxwell theory. Therefore the product annihilates separately the gravitational and electromagnetic part of the Hamiltonian constraint. This state will have important consequences in the loop representation.

What can one say about the physical relevance of this state? The Chern-Simons form is not a physically relevant state for Maxwell theory, since it is not normalizable. Evidently there is potential here for such a problem. Moreover Maxwell theory has a vacuum. Can one conceive of a state for the combined theory that would bear some resemblance to the vacuum? These questions will go largely unanswered. Clearly the Chern-Simons form does not decay fast enough as to be normalizable in pure Maxwell theory. On the other hand, the notion of distance that is used to define the vacuum (and to make it fall off fast enough to assure normalizability) is absent in the combined case, since the theory is invariant under diffeomorphisms. This point clearly requires a more careful study before we can reach a conclusion. Since at the moment there is no candidate for an inner product for the combined theory we simply cannot say anything about normalizability.

IV. LOOP REPRESENTATION

The construction of the loop representation for this theory follows the same steps as those for the vacuum theory so we will only highlight some points. The reader interested in details of the construction of loop representations is referred to [2]. The main difference with the usual case is that the group is $U(2)$ instead of being $SU(2)$. This changes the form of the Mandelstam identities and therefore the kinematics of the loop representation is different. As usual we identify,

$$\Psi[L_1 \circ L_2] = \Psi[L_2 \circ L_1]. \quad (13)$$

but there is no relation between wavefunctions of retraced loops, i.e. $\Psi[L] \neq \Psi[L^{-1}]$. In the vacuum theory one also has the Mandelstam identity, which states,

$$\Psi[L_1, L_2] = \Psi[L_1 \circ L_2] + \Psi[L_1 \circ L_2^{-1}]. \quad (14)$$

That is, it allows to express any wavefunction of n loops as a function of $n-1$ loops. This can be used recursively to reduce all wavefunctions to functions of only one loop.

For our case the Mandelstam identity now reads,

$$\begin{aligned} \Psi[L_1, L_2, L_3] &= \Psi[L_1 \circ L_2, L_3] + \Psi[L_2 \circ L_3, L_1] + \\ &+ \Psi[L_3 \circ L_1, L_2] - \Psi[L_1 \circ L_2 \circ L_3] - \Psi[L_1 \circ L_3 \circ L_2]. \end{aligned} \quad (15)$$

Where we see that it only allows us to reduce a function of n loops to a function of $n-1$ and $n-2$ loops. Therefore we cannot reduce all wavefunctions to functions of only one loop, we need at least two loops to represent a generic wavefunction. Therefore the kinematics of the theory is different from the vacuum one, as one may expect from the fact that the constraint that defines this kinematics, the Gauss Law, is different in both theories.

An interesting point is that one could proceed in the traditional fashion (without combining both connections into a $U(2)$ and construct two loop representations, one for the Ashtekar connection and another one for the Maxwell one (as was done in [10] for the 2+1 case). One would then have wavefunctions depending of two loops, an “electromagnetic” and a “gravitational” one. Here we also encounter two loops, but each of them carry information about both gravitation and electromagnetism. There are other important differences in the construction of the loop representation in both cases but we will not discuss them here.

The diffeomorphism constraint still works as a generator of infinitesimal diffeomorphisms in loop space, and we can represent it in terms of the area derivative,

$$\hat{C}(\vec{N})\Psi[L] = \int d^3x N^a(x) \oint_L dy^b \delta^3(x-y) \Delta_{ab}(L_0^y) \Psi[L]. \quad (16)$$

The reader is referred to ref. [11] for details of this expression. The important point here is that the space of physical states of the theory will still be represented by functionals of loops that are invariant under diffeomorphisms, i.e. they will be functionals of the link class of the loop rather than of the loop itself, exactly as in the vacuum case.

We now turn our attention to the Hamiltonian constraint. Again, it could be realized in loop space in terms of the area derivative, as one does for the vacuum theory [12]. The calculation is lengthy and at the moment we do not need the specific form so we will not exhibit it here. In this case, the constraint is very different from the vacuum theory, as expected.

One of the main achievements of the loop representation is to make possible the construction of states that solve *all* the constraints of the theory. We will see that this is the case also for the model of interest. First of all it is quite simple to see, from the structure of the Hamiltonian constraint [13], that wavefunctionals with support on smooth nonintersecting loops are *not* solutions to the constraint. As in the connection representation this stems from the fact that there is present a purely multiplicative term in the constraint that fails to annihilate these functionals. Moreover we should remember that the constraint was rescaled by a factor that vanishes for loops with less than a triple intersection.

In spite of this we can construct a solution to all the constraints in the loop representation following the same reasonings that also allowed to construct solutions in the vacuum case: since the Chern-Simons form is a solution in the Connection Representation, it should also be a solution in the loop representation. It turns out that this transform is known. For our case it turns out to be,

$$\Psi_{\Lambda}^{CS}[L] = e^{-\frac{\Lambda}{8}w(L)} J[L](\Lambda). \quad (17)$$

Where $w(L)$ is the writhing number of the (framed) loop L and $J[L](\Lambda)$ is the Jones Polynomial for the loop L in the variable Λ ; e is the electric charge. This state mimics the similar one in the vacuum case [4]. Notice also that in order to be a generic state of the theory we need at least two loops. This presents no difficulty, since the transform of the Chern-Simons

state is well understood for more than one loop [14].

In the loop formulation of gauge theories, it is usual to introduce charges by opening up the loops. The open path formalism describes lines of flux with charges at their ends. This has been studied for the Maxwell theory [15]. It is interesting to note what happens if one attempts to construct such a formalism for our unified model. If one opens up one of the loops in question, one not only fails to satisfy the Gauss Law of the Maxwell theory (which introduces electric charges) but also one fails to satisfy the Gauss Law of the Ashtekar formalism. This latter fact only occurs if one couples the theory to fermions. That is, the loop representation requires that charged objects should be fermionic.

Another point is that many authors [16] have argued that only by taking into account the quantum properties of the matter that form the reference frames, physical quantum observables can be defined in quantum gravity. In that sense, the description of electromagnetic fields in interaction with gravity could allow to explore an alternative way for getting physical observables.

We end by mentioning that in the vacuum theory one can perform an analysis order by order in the Jones Polynomial and retrieve physical states for the theory without cosmological constant [17]. It would be interesting to try to carry out a similar analysis for the model we are considering.

V. CONCLUSIONS

We have studied the Ashtekar formulation of the Einstein-Maxwell theory. We show how one can rewrite the equations in terms of a single $U(2)$ connection. The kinematic structure of the theory is quite similar to that of pure General Relativity and allows the generalization of several results of that case to the combined Einstein-Maxwell theory. In particular, the loop representation is quite natural and the Jones Polynomial turns out to be a physical state of the theory, as happens in pure General Relativity. Summarizing, we can see that the Ashtekar variables/Loop Representation approach to the quantization of Gravity can lead

to quite appealing results when one incorporates other interactions in an unified fashion.

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REFERENCES

- [1] A. Ashtekar, Phys. Rev. Lett. **57** 2244, (1986); Phys. Rev. **D36**, 1587 (1987).
- [2] C. Rovelli, L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nuc. Phys. **B331**, 80 (1990).
- [3] B. Brüggmann, R. Gambini, J. Pullin, Phys. Rev. Lett. **68**, 431 (1992); see also R. Gambini, B. Brüggmann, J. Pullin, "Knot invariants as nondegenerate states of four dimensional quantum gravity", in "Proceedings of the XXth. DGM", S. Catto, A. Rocha, eds., World Scientific, Singapore (1991).
- [4] B. Brüggmann, R. Gambini, J. Pullin, "Jones Polynomials for intersecting knots as physical states of quantum gravity", to appear in Nuc. Phys. **B**.
- [5] P. Peldan, Phys. Rev. **D46**, R2279 (1992). (1992).
- [6] A. Ashtekar, J. Romano, R. Tate, Phys. Rev. **D40**, 2572 (1989).
- [7] T. Jacobson, L. Smolin, Nuc. Phys. **B299**, 295 (1988).
- [8] H. Kodama, Phys. Rev. **D42**, 2548 (1990).
- [9] B. Brüggmann, J. Pullin, Nuc. Phys. **B363**, 221 (1991).
- [10] C. Nayak, Gen. Rel. Grav. **23**, 931 (1991).
- [11] R. Gambini, A. Trias, Phys. Rev. **D27**, 2935 (1983).
- [12] R. Gambini, Phys. Lett. **B255**, 180 (1991); B. Brüggmann, J. Pullin, Nucl. Phys. **B** (in press).
- [13] Although we have not written it in the loop representation, this conclusion follows also immediately from the form in the connection representation
- [14] E. Guadagnini, M. Martellini, M. Mintchev, Nuc. Phys. **B330**, 575 (1990).
- [15] R. Gambini, J. Griego, Phys. Lett. **B256**, 437 (1991).

[16] See for instance, C. Rovelli, *Class. Quan. Grav.* **8**, 297 (1991); **8**, 317 (1991) and references therein.

[17] B. Brügmann, R. Gambini, J. Pullin, *Gen. Rel. Grav.* (in press).