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The escape of gravitational radiation from the field of massive bodies

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We consider a compact source of gravitational waves of frequency ω, in or near a massive spherically symmetric distribution of matter or a black hole. Recent calculations have led to apparently contradictory results for the influence of the massive body on the propagation of the waves. We show here that the results are in fact consistent and in agreement with the "standard" viewpoint in which the high frequency compact source produces the radiation as if in a flat background, and the background curvature affects the propagation of these waves.

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Some of the most interesting potential sources of gravitational radiation consist of relatively compact astrophysical configurations, in particular binary neutron stars, embedded in much larger and massive galaxies. The standard viewpoint for radiation from such arrangements is to separate the problem into that of the generation of the radiation by the compact source, and that of the propagation of the radiation through the galaxy [1]. The radiation generated by the source is calculated as if the source were in a flat background. For orbiting binary neutron stars the standard quadrupole formalism would be a good approximation. The effect of the spacetime curvature created by the host galaxy is then understood in terms of its influence on the propagation of the waves to a distant observer. In the standard viewpoint the major propagation effects are the gravitational redshift and gravitational lensing. For a galaxy of mass M and radius R these effects are of order M/R and for ordinary galaxies, and for most purposes, are very small. (We use here, and throughout this paper, units in which $c = G = 1$.)

Despite the apparent simplicity of this prevalent viewpoint, there are some unclear issues. One of us (PK) has found mathematical relations suggesting that the gravitational field of the galaxy might suppress, by many orders of magnitude, the emergence of quadrupole gravitational waves generated inside it or nearby [2]. Two of us (RP and JP) have studied the same problem and have found that the galactic gravitational background has a minimal effect on the emergence of the waves, and that the "standard viewpoint" is valid [3]. It is now clear how the specific results of the two studies can be compatible, and what the implication is for astrophysical sources of gravitational radiation.

The mathematics which gave rise to the appearance of suppression was framed in the language of the Newman-Penrose [4] (hereafter NP) formalism, and is based on the Weyl projection Ψ_0 in that formalism. For an outgoing solution, Ψ_0 takes the form $\Psi_0 = \psi_0^0(u, \theta, \phi) r^{-5} + \mathcal{O}(r^{-6})$ where u is retarded time. It is well accepted that information about outgoing gravitational waves is encoded in the shear $\sigma = \sigma_0(u, \theta, \phi) r^{-2} + \mathcal{O}(r^{-4})$ and in the Bondi news function [5] $d\sigma_0/du$.

In spherically symmetric backgrounds it is convenient to consider a multipole decomposition and to treat separately each multipole mode. For modes of multipole index ℓ we can write $\Psi_0 = \tilde{\Psi}_0 \; {}_2Y_{\ell m} \quad \psi_0^0 = \tilde{\psi}_0^0 \; {}_2Y_{\ell m} \quad \sigma_0 = \tilde{\sigma}_0 \; {}_2Y_{\ell m}$, where ${}_2Y_{\ell m}$ are the spin-weight 2 spherical harmonics. A useful feature of the quantities $\tilde{\Psi}_0, \tilde{\psi}_0^0, \tilde{\sigma}_0$, with angular variables removed, is that their real and imaginary parts correspond respectively to even- and odd-parity modes. If the background spacetime is a Schwarzschild spacetime of mass M, and if time dependence $e^{i\omega t}$ is assumed, the NP equations lead to the relations

$$\tilde{\sigma_0} = \pm \frac{r^2\omega^2}{6(1 \pm i\omega M/2)} \tilde{\psi}_0^0 \ , \quad \tilde{\sigma_0} = \pm \frac{(\ell-2)!}{(\ell+2)!} \frac{r^2\omega^2}{\left[1 \pm \frac{(\ell-2)!}{(\ell+2)!} 12 i\omega M\right]} \tilde{\psi}_0^0 \ , \tag{1}$$

for the quadrupole and general ℓ cases. Here the + signs apply for even-parity perturbations, and the - signs for odd.

It is eq. (1) that suggests suppression of radiation. The intensity of the gravitational radiation is represented by $\tilde{\sigma_0}$. If $\tilde{\psi}_0^0$ is taken to represent the quadrupole moment of a source, it follows that for a given quadrupole moment oscillating at frequency ω, the radiation is reduced due to the mass of the Schwarzschild background by the factor $(1 \pm i\omega M/2)^{-1}$, so that the radiation power flux (proportional to $|d\sigma_0/du|^2$) is reduced by the factor $(1+\omega^2 M^2/4)^{-1}$. For a typical galaxy $M \approx 10^{16}$ cm, and for the radiation sources of greatest interest $\omega \approx 10^{-7}\mathrm{cm}^{-1}$. Equation (1) then can be interpreted as imposing a suppression of the radiation flux by more than 17 orders of magnitude!

The mathematics, and most of the issues of physical interpretation, leading to eq. (1) are not controversial. From the beginning of the debate about the physical reality of the suppression, the crucial question has been whether ψ_0^0 could be interpreted as the quadrupole moment of the source, as is done in flat space. What is really needed, of course, is a source calculation clearly showing the relationship between ψ_0^0 and the source quadrupole moment. The first approach to this was a scalar model given by Kozameh, Newman and Rovelli [6]. Two of us (RP and JP) did an explicit calculation [3] for a compact source at the center of a spherical, perfect fluid “galaxy.” A Green function solution gave the relationship between the source and the waves outside the galaxy, and showed no evidence for suppression. This calculation, however, was not done directly in terms of ψ_0^0. It could not, therefore, directly reveal the "enhancement" that must appear in a calculation of ψ_0^0 in order to offset the mathematical suppression present in eq. (1). At about the same time, one of us (PK) wrote down the form of the Green function solution directly in terms of ψ_0^0, and found no evidence for this enhancement [7,8], suggesting that the suppresion may be a real physical effect.

Here we resolve the apparently divergent findings. To describe the unperturbed background spacetime, both inside and outside the galaxy, we follow the notation of [3] and take the form of the metric to be $ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ with ν and λ functions of r only. We define the radial variable r_* by $dr/dr_* \equiv e^{(\nu-\lambda)/2} \equiv e^{\alpha(r)}$ and the retarded time u by $u \equiv t - r_*$.

We treat the source of gravitational waves as a perturbation on the background of the metric given above, and we write $\tilde{\Psi}_0 = \hat{\Psi}_0(r,\omega)e^{i\omega t} \quad \tilde{\psi}_0^0 = \hat{\psi}_0^0(r,\omega)e^{i\omega t}$ for an an ℓ-pole moment with time dependence $e^{i\omega t}$. The perturbed field equations in general relate perturbations of the Weyl projections, the NP spin coefficients, and the stress-energy perturbations. For clarity of description, we will assume that the equations can be combined to give a decoupled equation for $\hat{\Psi}_0$ of the form

$$\mathcal{D}\hat{\Psi}_0 = \hat{\Psi}_0'' + f(r,\omega)\hat{\Psi}_0' + g(r,\omega)\hat{\Psi}_0 = S(r,\omega)\,, \tag{2}$$

in which the source function $S(r,\omega)$ is known, in which the coefficient functions f and g are known functions constructed from the background metric, and in which prime denotes differentiation with respect to r. There are at least two cases for which such an equation can explicitly be found: (i) if the "galaxy" is made of perfect fluid and the waves are odd-parity, as in [3], and (ii) if the source lies in the vacuum exterior of a galaxy or hole, as in [7,8]. The statements below about the radial dependence of the Wronskians, and other functions, refer to calculations made in these two cases.

We choose boundary conditions for eq. (2) appropriate to the galactic center and for outgoing waves far from the galaxy. To construct a Green function satisfying these conditions we define two solutions of the homogeneous equation $\mathcal{D}\mathcal{R} = 0$. The function $\mathcal{R}_{\rm c}$ with the limit $\mathcal{R}_{\rm c} \stackrel{r\to 0}{\longrightarrow} r^{\ell-2}$ has the correct behavior at the center of the galaxy, while $\mathcal{R}_{\rm w}$ with the limit $\mathcal{R}_{\rm w} \stackrel{r\to\infty}{\longrightarrow} r^{-5}e^{-i\omega r_*}$ represents outgoing waves.

In terms of these functions it is straightforward to write the solution to eq. (2), outside the source, as $\hat{\Psi}_0 = \mathcal{R}_{\rm w}\int \frac{S(\tilde{r},\omega)\mathcal{R}_{\rm c}}{W(\mathcal{R}_{\rm c},\mathcal{R}_{\rm w})}d\tilde{r}$ where $W(\mathcal{R}_{\rm c},\mathcal{R}_{\rm w})$ is the Wronskian $\mathcal{R}_{\rm c}\mathcal{R}'_{\rm w} - \mathcal{R}'_{\rm c},\mathcal{R}_{\rm w}$. From these definitions we arrive at a Green function solution

$$\hat{\psi}^0_0 = \int \frac{S(\tilde{r},\omega)\mathcal{R}_{\rm c}}{W(\mathcal{R}_{\rm c},\mathcal{R}_{\rm w})}d\tilde{r} \tag{3}$$

for ψ^0_0.

It will be useful in discussing this result for us to consider a case (like that of a neutron star binary in an ordinary galaxy) for which both the galaxy and the gravitational wave source are nonrelativistic, and for which ωM is enormous. In this case, eq. (1) predicts enormous suppression. If the suppression is a mathematical artifact, not a physical suppression, we must find in eq. (3) a counterbalancing enhancement factor. Since the galaxy is nonrelativistic, we can immediately eliminate several possibile sources of such a factor. The function $\mathcal{R}_{\rm c}$ cannot give rise to the enhancement factor since it retains the same form as in flat space at the center. The source term $S(\tilde{r},\omega)$ is constructed from the source stress-energy (which cannot contain a reference to M outside the galaxy), and from the spacetime geometry at the region of the source (which is only influenced by a negligible M/R contribution.

We conclude that the numerator in the integrand in eq. (3) is negligibly different from what it would be in a flat spacetime background. In particular, it cannot contain a large enhancement factor of the form $\left[1 \pm 12\frac{(\ell-2)!}{(\ell+2)!}iM\omega\right]$. If such an enhancement factor is to appear it must come from the Wronskian, and the Wronskian, unlike the other terms in eq. (3), cannot be ruled out as the source of such a factor. The Wronskian contains solutions normalized both at the center of the galaxy and in the exterior, and hence "knows" what the mass of the galaxy is. Aside from small corrections (such as the central redshift), of order M/R, we find that at $r \to 0$

$$W(\mathcal{R}_{\rm c},\mathcal{R}_{\rm w}) = \kappa_1\left[1 - 12\frac{(\ell-2)!}{(\ell+2)!}iM\omega\right]^{-1} , \tag{4}$$

where κ_1 is a function of ω, r and ℓ which has the same value it would have in flat spacetime.

We conclude that eq. (3), for a compact source at the center of a nonrelativistic galaxy, reduces to

$$\hat{\psi}_0^0 = \kappa_2 \left[1 - 12 \frac{(\ell-2)!}{(\ell+2)!} iM\omega \right] , \tag{5}$$

where κ_2 has the same value it would have in a flat background except for (negligible) corrections of order M/R. The enhancement factor, in the square brackets in eq. (5), cancels the suppression factor in eq. (1), and we conclude that in this situation $\hat{\sigma}_0$ is the same (aside from negligible corrections of order M/R) as it would be in flat spacetime. *Aside from the small redshift effect, the gravitational field of the galaxy has no consequences for the emergence of radiation produced at its center.*

The above argument applies only for a source at a distance from the galactic center small compared to the wavelength of the radiation it produces. As the source location [$\tilde{r}$ in (3)] moves outward, there are significant changes in the forms both of $W(\mathcal{R}_{\rm c}, \mathcal{R}_{\rm w})$ and of $S(\tilde{r}, \omega)$. We can see the trends most clearly if we consider the source to be in the Schwarzschild exterior of the galaxy. In the exterior (4) is replaced by $W(\mathcal{R}_{\rm c}, \mathcal{R}_{\rm w}) = \kappa_1$ where corrections of order M/R have been omitted. The enhancement factor needed to cancel the suppression in (1) is now missing. It should be emphasized that this change in the character of the Wronskian is the key to resolving previous apparently contradictory results. The nontrivial behavior, in (4), of the Wronskian near $r = 0$ was not anticipated when the original arguments for suppression were made.

More than the Wronskian changes when the source is moved to the exterior; there is also an important change in $\mathcal{R}_{\rm c}$. In the Schwarzschild exterior, aside from small corrections, $\mathcal{R}_{\rm c}$ takes the form

$$\mathcal{R}_{\rm c} = \kappa_3 e^{i\omega r_*} + \kappa_4 \left[1 - 12 \frac{(\ell-2)!}{(\ell+2)!} iM\omega \right] e^{-i\omega r_*} , \tag{6}$$

in which the κ_i have the same form as for a flat background. In the exterior, therefore, the form of $\mathcal{R}_{\rm c}$ supplies an enhancement factor for *part* of the source integral in (3). For this part of $\hat{\psi}_0^0$ the enhancement factor will cancel the suppression factor in (1) and the contribution to the radiation will be the same (aside from small corrections) as in a flat background. But the remainder of the source integral for $\hat{\psi}^0_0$, that due to the κ_3 term in (6), will be reduced by the suppression factor.

The situation is somewhat similar for sources in the vicinity of a black hole, but some details must be altered. In the case of galaxies, we used the homogeneous function $\mathcal{R}_{\rm c}$, normalized at $r=0$. For holes we use instead the function $\mathcal{R}_{\rm hole}$ representing waves ingoing at the horizon. The Wronskian in the Green function then has the value

$$W(\mathcal{R}_{\rm hole},\mathcal{R}_{\rm w}) = \frac{1}{T_\ell}\,\frac{K(\ell)}{\omega}\,\frac{8i\omega^3}{r^6(1-2M/r)^3}\,, \tag{7}$$

and at large r the form of $\mathcal{R}_{\rm hole}$ is

$$\mathcal{R}_{\rm hole} \stackrel{r\to\infty}{\longrightarrow} \frac{\kappa_5}{T_\ell(\omega)}\left\{\kappa_3 e^{i\omega r_*} + \kappa_4 R_\ell(\omega)\left[1-12\frac{(\ell-2)!}{(\ell+2)!}iM\omega\right]e^{-i\omega r_*}\right\}\,. \tag{8}$$

Here κ_3 and κ_4 are the same as in (6), and $K(\ell)$ and κ_5 have the same value as in flat spacetime. The factors $T_\ell(\omega)$ and $R_\ell(\omega)$ are the transmission and reflection coefficients for gravitational waves. For high frequencies ($\omega M \gg 1$) the transmission coefficient is negligibly different from unity, and the reflection coefficient is negligibly small, at least for quadrupole and other low ℓ-pole moments. Since the enhancement factor is only present for the κ_4 term in (8) it is the only part that will not be reduced – relative to the flat space value – by the suppression factor in (1). Since the reflection coefficient should be small, the result gives the appearance of significant suppression of radiation for a compact source outside a hole.

The first step in trying to understand these results is an often overlooked point about sources: the emission from a compact "quadrupole" (i.e., nonrelativistic) source, located far from the coordinate center, *will not be dominated by* $\ell = 2$. A compact source, far from the coordinate origin, radiates predominately high ℓ multipoles. *A distant source, even a nonrelativistic "quadrupole" source, radiates a negligible fraction of its power at low ℓ-pole moments.*

Source integrals combine the source stress-energy and a solution of the homogeneous wave equation. Near $r=0$ the homogeneous solution has a power-law form, and the source integrals *if confined to the region near* $r = 0$, take the form of integrals for the various multipole moments of the source mass distribution. For a source located at large r (i.e., not within a small fraction of a wavelength of $r = 0$) the result is very different. In this case the source integral for ℓ-pole radiation is not related to the ℓth multipole moment of the mass distribution of the source. Rather, the radiation will be dominated by the multipoles which couple best to the source distribution. For a nonrelativistic source, of wavelength λ at radius (Schwarzschild radial coordinate value) $R_{\rm source}$, the best coupling will be for $\ell \sim R_{\rm source}/\lambda$.

We must therefore consider large values of ℓ when we consider the suppression factor

$$\left[1 - 12\frac{(\ell-2)!}{(\ell+2)!}iM\omega\right] \approx 1 - 12iM\omega/\ell^4 \ .$$

Typical numbers for a compact source and a galaxy are $\omega \sim 10^{-7}{\rm cm}^{-1}$, $R \approx 10^{23}$cm, $M \approx 10^{16}$cm. The distance to an exterior source must be at least as large as the galaxy radius, and this means that the radiation will characteristically be at $\ell \sim R_{\rm source}/\lambda \geq R/\lambda$, and R/λ is on the order 10^{15}. The suppression factor then differs from unity by a correction $12M\omega/\ell^4$ of order 10^{-37} or smaller.

Thus for "typical" sources and galaxies, the suppression factor does not play a significant role in determining the radiation reaching a distant observer. Can one think of sources —at least in principle— for which the mathematics indicates that there is significant suppression, but intuition demands that there is not? We can easily argue that no such situation can arise. First, for suppression to be important $M\omega$ must be large. Second, the distance to the source $R_{\rm source}$ must be no smaller than the Schwarzschild radius $2M$ of the galaxy. Thus we have

$$\ell \sim \frac{R_{\rm source}}{\lambda} \geq \frac{M}{\lambda} \sim M\omega \ . \tag{9}$$

It follows that $M\omega/\ell^4$ is no larger than order $(M\omega)^{-3}$ and hence cannot be large. The suppression factor can have an important effect only for a source just outside a black hole ($R_{\rm source} \sim M$), with source wavelength on the order of the radius of the hole ($M\omega \sim 1$). But in these circumstances we would certainly expect the curved background to influence the emergence of radiation! One example is a test particle falling radially into a Schwarzschild black hole of mass M. The early work of Davis *et al.* [9] shows that the emitted radiation is predominantly quadrupolar and that the spectrum peaks at $\omega = \omega_{max} \approx 0.32M^{-1}$. At the peak frequency the suppression factor for $\ell = 2$ waves is $1 + 0.16i$, yielding a small but nonnegligible reduction (of about 3%) of the radiated energy per unit frequency interval $dE/d\omega$. This reduction, relative to the calculated radiation for a flat background, is presumably already included in the numerical results, since the calculation took explicit account of the strongly curved background.

We can then conclude that *for any configuration of source and galaxy or black hole, the suppression factor can have a significant effect only in the case ($R_{\rm source} \sim M \sim \lambda$) that a significant effect would be predicted on the basis of the standard viewpoint.*

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