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Extended loops: a new arena for nonperturbative quantum gravity

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Abstract

We propose a new representation for gauge theories and quantum gravity. It can be viewed as a generalization of the loop representation. We make use of a recently introduced extension of the group of loops into a Lie Group. This extension allows the use of functional methods to solve the constraint equations. It puts in a precise framework the regularization problems of the loop representation. It has practical advantages in the search for quantum states. We present new solutions to the Wheeler-DeWitt equation that reinforce the conjecture that the Jones Polynomial is a state of nonperturbative quantum gravity.

The introduction of the loop representation has opened a new avenue for the nonperturbative canonical quantization of general relativity. In particular it allows to immediately code the invariance under spatial diffeomorphisms of wavefunctions in the requirement of knot invariance [1]. Also for the first time a large class of solutions to the Wheeler-DeWitt equation has been found in terms of nonintersecting knot invariants. It turns out however, that these solutions correspond to degenerate metrics [2]. If one wants nondegenerate metrics one needs to consider knot invariants of intersecting loops and solve the Wheeler-DeWitt equation in loop space [3]. Solutions of this kind have actually been found [4] and they are related with the existence of a Chern-Simons state in terms of Ashtekar variables [5]. One of these solutions turns out to be the first nontrivial coefficient of a certain expansion of the Jones polynomial and this led to the conjecture that the Jones polynomial may be a state of quantum gravity [6].

These results were plagued by regularization ambiguities. Since loops are one dimensional objects living in a three dimensional manifold, they naturally lead to the appearance of distributional quantities. In particular the few knot invariants for which we have analytic expressions require the introduction of regularizations (framings) in the case of intersecting knots. Some of them even require them for smooth loops [7,8].

This difficulty not only arises for the gravitational case. In non Abelian gauge theories [9,10], and even in the simple case of a free Maxwell field [11], it is known that the quantum states in the loop representation are ill defined and a regularization is needed.

Loops are classes of closed curves that give the same holonomy for any gauge connection. They form a group under composition, called the group of loops [12]. This group is not a Lie group, since the composition of loops is only defined for an integer number of loops. Recently, a completion of this group into a Lie group was introduced [13]. The group of loops is included in this Lie group and can be thought as a discrete subgroup. The elements of the Lie groups are called “extended loops” and are the mathematical basis of the new representation we propose.

Let us discuss for a moment these ideas in the more familiar context of Maxwell theory.

The point of departure to construct a usual loop representation is to consider the Wilson loop functional, $W_A(\gamma) = \exp \oint_\gamma dy^a A_a(y)$, where γ is a loop.

Since Wilson loops form an (over)complete basis of gauge invariant functions (modulo subtleties we will discuss later) one can express the wavefunctions $\Psi[A]$ in terms of Wilson loops and go to a representation purely in terms of loops via the loop transform,

$$\Psi(\gamma) = \int DAW_A(\gamma)\Psi[A]. \quad (1)$$

The wavefunctions $\Psi(\gamma)$ are in the loop representation and in this representation one can realize the gauge invariant operators of physical interest of the theory, for instance the Hamiltonian, as was discussed in reference [11]. It needs to be regularized. If one computes the vacuum, one gets,

$$\Psi_0(\gamma) = \exp(\oint dx^a \oint dy^b D_{ab}(x-y)) \quad (2)$$

where $D_{ab}(x-y)$ is the spatial restriction of the Feynman propagator. This quantity is also ill-defined due to the divergence of the propagator where $x = y$. So we have a representation where both operators and wavefunctions have to be regularized (if one introduces a regularized propagator, the expression (2) is the vacuum of the regularized theory).

Consider now the quantities $W_A[X] = \exp \int d^3y X^a(y) A_a(y)$. If $X^a(y)$ is a divergence-free vector density the $W_A[X]$'s are gauge invariant. The X 's are the Abelian analogues of the extended loops we will consider later on. If one uses $W_A[X]$ instead of the Wilson loop functional in the transform (1) one ends with a representation in which wavefunctions are functionals of transverse vector densities. It is easy to check that this just corresponds to the electric field representation of the canonically quantized Maxwell field. The Hamiltonian is well known. The vacuum of the theory is simply given by,

$$\Psi_0(X) = \exp\left(\int d^3y \int d^3z X^a(y) X^b(z) D_{ab}(y-z)\right). \quad (3)$$

and is well defined without the need of a regularization.

We therefore see that by extending the idea of loop we have several advantages: on the one hand we end up with a usual representation in terms of fields (it is just the electric field

representation); on the other hand because we are using fields instead of distributional objects (loops) the regularization difficulties associated with the loop representation are solved.

Another issue to be considered to complete the quantization is the introduction of the inner product. There is no clear idea of how to perform an integration in loop space and therefore there are no natural candidates for inner products in the loop representation. By going to the extended loops one can use usual functional integrals and an inner product for Maxwell theory have been introduced this way [14,11]. With this inner product the usual Fock structure and the interpretation of the excited states in terms of photons can be recovered completely.

The intention of this letter is to outline the generalization of this procedure to the nonabelian case, concentrating on the case of gravity.

Let us start by proposing an extension of the notion of holonomy for a non Abelian field similar to the one we introduced for Maxwell theory. To this aim we rewrite the usual expression for the holonomy as,

$$U_A(\gamma) = P \exp\left(\oint_{\gamma} A_a dy^a\right) = 1 + \sum_{n=1}^{\infty} \int dx_1^3 \cdots dx_n^3 A_{a_1}(x_1) \cdots A_{a_n}(x_n) X^{a_1 x_1 \dots a_n x_n}(\gamma) \quad (4)$$

where γ is a loop and the “multitangents” X are defined by,

$$X^{a_1 x_1 \dots a_n x_n}(\gamma) = \oint_{\gamma} dy_n^{a_n} \int_0^{y_n} dy_{n-1}^{a_{n-1}} \cdots \int_0^{y_2} dy_1^{a_1} \delta(x_n - y_n) \cdots \delta(x_1 - y_1). \quad (5)$$

The advantage of rewriting the holonomy in this way is that we have captured all the loop dependent information in the multitangents, which behave as multitenor densities on the spatial manifold. The holonomy therefore can be written in a very economical fashion as the contraction $U_A(\gamma) = A_{\tilde{\mu}} X^{\tilde{\mu}}$ where the indices $\tilde{\mu}$ are a shorthand for $a_1 x_1 \dots a_n x_n$ and we assume a “generalized Einstein convention” in which we sum from one to three for each repeated index a_i and we integrate over the three manifold for each repeated x_i . A repeated index with a tilde also involves a summation from $n = 0$ to infinity.

The key observation is to notice that if one substitutes in (4) a multitenor density $X^{a_1 x_1 \dots a_n x_n}$ (not necessarily associated with a loop) such that,

$$\partial_{a_i} X^{a_1 x_1 \dots a_i x_i \dots a_n x_n} = (\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1})) X^{a_1 x_1 \dots a_{i-1} x_{i-1} a_{i+1} x_{i+1} \dots a_n x_n} \quad (6)$$

the resulting “extended holonomy” $U_A[X]$ is gauge covariant under gauge transformations. In other words, its trace is invariant under gauge transformations connected with the identity. We will call it the extended Wilson functional $W_A[X] = Tr(U_A[X])$. An important difference is that the use of extended loops does not necessarily lead to the construction of quantities that are invariant under gauge transformations not connected with the identity. Therefore the variables X are able to capture more information than usual loop variables. In particular, information of topological nature. This can also be seen in 2+1 gravity, where usual loops fail to capture all the information of the theory [15]. As we mentioned before, loops form a group. The quantities X can also be endowed with a group structure, and the presence of the extra elements (the ones not associated with loops) allow to extend the group of loops into a Lie group called the “Extended group of loops” [13]. The usual group of loops is a subgroup naturally associated with the multitangents $X(\gamma)$. The group product between multitensors is $(X_1 \times X_2)^{a_1 x_1 \dots a_n x_n} = \sum_{k=0}^n X_1^{a_1 x_1 \dots a_k x_k} X_2^{a_{k+1} x_{k+1} \dots a_n x_n}$ and it naturally reproduces loop composition among multitangents $X(\gamma_1 \circ \gamma_2) = X(\gamma_1) \times X(\gamma_2)$.

The extended Wilson functionals satisfy a series of identities associated with the fact that the gauge theory is associated with a particular gauge group (in the case of gravity $SU(2)$), which can be explicitly written. In the case of ordinary loops these are the well known Mandelstam identities

The extended loop representation can be constructed following the same steps that led to the loop representation. Given a wavefunction in the connection representation $\Psi[A]$ it can be transformed to the extended loop representation via the “extended loop transform”, obtained by replacing $W_A(\gamma)$ by $W_A[X]$ in equation (1). One can also construct the extended representation without invoking a transform directly by quantizing a noncanonical algebra of quantities dependent on the extended coordinates, very much in the same fashion as in the usual loop representation [10,1], but we will not discuss it here for reasons of space.

Some observations should be made about the resulting space of wavefunctionals. First

of all, they are functionals of the “infinite tower” of multitensors of all orders. Moreover they are *linear* functionals (since the extended Wilson loop is linear in the multitensors). Wavefunctions must also satisfy Mandelstam identities.

Up to the moment the discussion has been generic, in the sense that it could as well apply to any gauge theory. We will now particularize to the case of quantum gravity written in terms of Ashtekar’s new variables [16],

$$\hat{\mathcal{C}}_b \Psi[A] = \hat{E}_i^a \hat{F}_{ab}^i \Psi[A] = 0 \quad (7)$$

$$\hat{\mathcal{H}} \Psi[A] = \epsilon_{ijk} \hat{E}_i^a \hat{E}_j^b \hat{F}_{ab}^k \Psi[A] = 0 \quad (8)$$

where E_i^a is a triad, A_a^j is the Sen connection and F_{ab}^i is the curvature constructed from the Sen connection. The first set of equations is the diffeomorphism constraint which says that the theory is invariant under diffeomorphisms of the three manifold and the last equation is the Hamiltonian constraint, which corresponds to the Wheeler-DeWitt equation of the usual canonical formulation.

In order to write these equations in the extended representation we choose a polarization in which \hat{A} is multiplicative and \hat{E} a functional derivative. We then consider the action of the elementary operators on extended holonomies and rewrite them in terms of the X ’s, exactly as one proceeded with loops [1]. From there one can obtain the action of any gauge invariant operator in the extended representation. An important fact is that due to the linearity of *all* wavefunctions in the extended representation, any gauge invariant operator can only be a *first order* differential operator. In particular the form of the constraints is [21],

$$\hat{\mathcal{C}}_a(x) \Psi[X] = (\mathcal{R}_{ab}(x) \times R^{bx})^{\bar{\mu}} \frac{\delta}{\delta R^{\bar{\mu}}} \Psi[X] \quad (9)$$

$$\hat{\mathcal{H}} \Psi[X] = (\mathcal{R}_{ab}(x) \times R^{ax \bar{b}x})^{\bar{\mu}} \frac{\delta}{\delta R^{\bar{\mu}}} \Psi[X] \quad (10)$$

where $R^{\bar{\mu}} = \frac{1}{2}(X^{\bar{\mu}} + (-1)^n X^{a_n x_n \dots a_1 x_1})$, $(R^{bx})^{\bar{\mu}} = \sum_{k=0}^n R^{a_{k+1} x_{k+1} \dots a_n x_n b x a_1 x_1 \dots a_k x_k}$ and \mathcal{R}_{ab} is an element of the loop algebra such that the field tensor is given in terms of the connection as $F_{ab} = \mathcal{R}_{ab}^{\bar{\mu}} A_{\bar{\mu}}$. An explicit form for \mathcal{R}_{ab} can be easily written

and the only nonvanishing components are of the first and second rank. $R^{ax\bar{b}x}\tilde{\mu} = \sum_{k=0}^n (-1)^{n-k} R_c^{a x a_1 x_1 \dots a_k x_k b x a_n x_n \dots a_{k+1} x_{k+1}}$ and the inversion in the order of the last set of indices in this definition has a role totally analogous to the “reroutings” at the intersections of loops of the traditional Hamiltonian constraint in the loop representation. The subindex c means take the cyclic combination in upper indices. \times is the product in the extended group of loops.

One can particularize the above expressions to the case when the multitensors are the multitangents to a loop. The resulting expressions *correspond to the usual constraints of quantum gravity in the loop representation* [3,17].

What about solutions to the constraints? It should be pointed out that since loops are a particular case of multitensors, *any* solution found in terms of multitensors can be particularized to loops and would yield in the limit a solution to the usual constraints of quantum gravity in the loop representation (the limit could be singular in some cases, for instance when loop expressions need to be framed, as in the case of regular isotopic knot invariants [7,8]). The converse is not necessarily true: given a solution in the loop representation, it may not generalize to a solution in the extended representation. An immediate example are the unphysical solutions to the Hamiltonian based on smooth nonintersecting loops, which find no analogue in the extended representation.

There exists a particular family of solutions in the loop representation which *do* generalize to the extended representation. In the connection representation based on Ashtekar variables, the exponential of the Chern-Simons form built with the Ashtekar connection is a solution of all the constraints of quantum gravity with a cosmological constant [18,5]. When transformed into the loop representation, the resulting wavefunction is the Kauffman Bracket knot polynomial, which is a phase factor times the Jones Polynomial. Through a close examination, it was conjectured that the coefficients of the Jones Polynomial are solutions of the Hamiltonian constraint without cosmological constant. Evaluating the loop transform using perturbative techniques of Chern-Simons theory [8], explicit expressions for

the Kauffman Bracket (KB) coefficients can be found,

$$\text{KB}_\Lambda(\gamma) = e^{\Lambda GL(\gamma)}(1 + a_2(\gamma)\Lambda + a_3(\gamma)\Lambda^3 + \dots) \quad (11)$$

where $GL(\gamma) = g_{axby}X^{ax}(\gamma)X^{by}(\gamma)$ is the Gauss self linking number ($g_{axby} = \epsilon_{abc}(x - y)^c/|x - y|^3$ is the free propagator of Chern-Simons theory). $a_2(\gamma)$ and $a_3(\gamma)$ are coefficients of an expansion of the Jones polynomial evaluated in $\exp(\Lambda)$ that can be explicitly written as linear functions of the multitangents with coefficients constructed from g_{axby} . Through a laborious computation it was shown that $a_2(\gamma)$ satisfied the Hamiltonian constraint of quantum gravity [4] and it was later conjectured [6] that similar results may hold for $a_3(\gamma)$ and higher coefficients.

These solutions can be easily generalized to the extended representation simply by replacing the multitangents that appear in the definition of the coefficients with arbitrary multitenors. It can be checked that the resulting expressions are diffeomorphism invariant and no framing problem arise. The remarkable fact is that due to the simplification of the constraints that appears in the extended representation, one can actually check in a straightforward manner that a_2 is a solution as was already known. Moreover, the recently obtained expression [19] for the third coefficient of the Jones Polynomial, a_3 , *can also be checked to be a solution of the Wheeler-DeWitt equation* [20] in the extended representation. This adds more credibility to the conjecture that the Jones Polynomial (in this case its extension to arbitrary multitangents) could be a state of quantum gravity in the extended loop representation. Notice that in all these solutions *no assumption has been made on the domain of dependence of the functions*. All solutions that were known in the loop representation required restricting the domain of dependence of the wavefunctions (say, to smooth loops or to loops with a certain number of intersections).

Generically, the multitenors are distributional, as can be immediately seen from the equation they satisfy (6). However their distributional character is under control. As was shown in reference [13] a generic multitenor satisfying (6) can be written as a linear combination of transverse multitenors (which one can restrict to be smooth) times some well-defined

distributional coefficients. This has important consequences. In particular the solutions considered above are such that the dependence on the distributional coefficients drop off and they are only functions of the smooth part of the multitensors. They are therefore well defined. This was not the case in the loop representation, where they needed to be framed to be well defined. This is because in order to recover the usual loop representation one needs to choose the transverse part of the multitensors to be distributional.

The fact that the wavefunctions are well defined without regularization ambiguities does not directly imply that the operators in this representation are well defined. As in any functional representation, the presence of functional derivatives may introduce singularities that need to be regularized and renormalized. In the extended representation, an obvious regularization problem is present in the Hamiltonian constraint which involves a multitensor with a repeated spatial dependence (as can be seen from (6) repeating a spatial dependence involves a singularity). However, because the singular nature of the multitensors is under control one can perform a precise point-splitting regularization and it can be checked detail if the coefficients of the extended Jones polynomial presented above are annihilated by the regularized constraints or not. This issue is currently being studied.

One can view the role of the multitensors in the extended representation as configuration space variables of a canonical theory. The conjugate momenta are represented by functional derivatives. This suggests that there exists an underlying classical Hamiltonian theory that under canonical quantization yields directly the extended loop representation. This was unclear with loops, where the loop representation could only be introduced through a non-canonical quantization. For the Maxwell case this theory was studied [22] and found to be equivalent to the usual Maxwell theory. For nonabelian cases it is yet to be studied.

Summarizing, the extended representation presents practical calculational advantages and offers new possibilities to precisely regularize the theory and set it in a more rigorous framework without losing some of the topological and geometric insights of the loop representation.

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