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A rigorous solution of the quantum Einstein equations

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We show that the second coefficient of the Conway knot polynomial is annihilated by the Hamiltonian constraint of canonically quantized general relativity in the loop representation. The calculations are carried out in a fully regularized lattice framework. Crucial to the calculation is the explicit form of the skein relations of the second coefficient, which relate it to the Gauss linking number. Contrary to the lengthy formal continuum calculation, the rigorous lattice version can be summarized in a few pictures.

The introduction of the Ashtekar variables [1] and the loop representation have opened new perspectives of finding exact quantum states of the gravitational field. In the loop representation, the diffeomorphism constraint implies that wavefunctions have to be knot invariants [2]. A few years ago it was noticed [3] that a_2 , the second coefficient of a particular knot polynomial—the Conway polynomial—was formally annihilated by the Hamiltonian constraint of quantum gravity in the loop representation. The calculations were unregularized and involved divergent factors, so the result was really of the form “zero times infinity”. Later, another set of formal calculations [4] showed that this solution was a reflection of the fact that in the quantum theory formulated in terms of the Ashtekar connection, the exponential of the Chern-Simons form was formally annihilated by all the constraints of quantum gravity with a cosmological constant. The expression of this fact in the loop representation is that the Kauffman bracket knot polynomial should be a solution of all the constraints with cosmological constant. While checking this fact [5], again at a formal level, it was found that a_2 , the second coefficient of the Conway polynomial, had to be annihilated by the Hamiltonian constraint with zero cosmological constant. This result was later confirmed in a regularized setting via the extended loop representation [6], but it required lengthy manipulations and the introduction of a counterterm in the Hamiltonian constraint. As a consequence of all this the second coefficient of the Conway polynomial appeared as one of the first very nontrivial exact states of quantum gravity in the loop representation and hinted towards a deep connection between notions of knot theory and quantum gravity, this time at a dynamical level.

In this paper we will show that a_2 is rigorously annihilated by the Hamiltonian constraint of quantum gravity in a fully regularized lattice formulation of the theory. In the lattice theory, which is described in detail in [7], the Hamiltonian constraint has a simple geometric action. As in the continuum, it only has a non-trivial action at points where the loops intersect, and the effect is to produce two terms per each pair of independent tangent vectors entering the intersection: one in which the wavefunction is evaluated at a loop in which one of the lines of the intersection has been shifted forward along two links of the lattice in the direction of the other tangent at the intersection, and one of the two sub-loops determined by the intersection is re-routed; the other term is similar, but the loop is shifted in the opposite direction. The total action of the Hamiltonian is given by the difference of both terms added over all possible pairs of tangents entering the intersection (in the lattice at most three pairs for intersections in which the loops traverse “straight through”). The action is shown in figure 1 for a double self-intersection and in figure 4 for a triple self-intersection. It is easy to check that if one takes the continuum limit by leaving loops fixed and refining the lattice they live in, the Hamiltonian has as leading contribution the usual Hamiltonian constraint in terms of Ashtekar new variables. In order to do it, one evaluates the action of the Hamiltonian on a holonomy of a connection A on the lattice and shows that it reduces to the usual formal action of the Hamiltonian constraint in the connection representation $\hat{H} = \epsilon_{ijk} F_{ab}^i \hat{E}_j^a \hat{E}_k^b$ [7] acting on a holonomy.

One can also introduce a diffeomorphism constraint on the lattice, and show that it satisfies the correct diffeomorphism algebra in the continuum limit, but we will not discuss it here for reasons of space. Details are given in [7].

An important property of the Hamiltonian constraint is already apparent from figure 1: the action of the constraint at points with double intersections in the loop produces two contributions that are deformable to each other, if the state one is considering is diffeomorphism invariant in the continuum. Therefore the Hamiltonian automatically

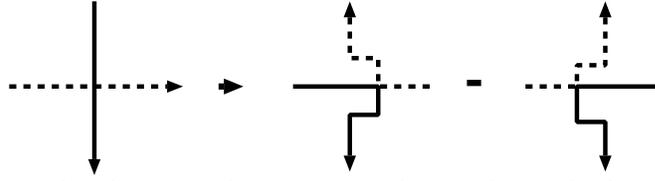


FIG. 1. The action of the Hamiltonian constraint on a double intersection, which is equivalent to a deformation of the loop which can be obtained through a diffeomorphism.

vanishes at double intersections. This fact has a counterpart in the continuum. If one looks at the construction of quantum state in terms of the transform of the Chern-Simons form, all the coefficients of the Jones polynomial were automatically annihilated on double intersections, since the term involving the cosmological constant is proportional to a determinant of the metric, which vanishes on intersections less than triple [5].

Let us now discuss the candidate for solution. The Conway polynomial in the variable z is defined by the following skein relation [8],

$$C(L_+) - C(L_-) = zC(L_0). \quad (1)$$

This relation is to be read in the following way: given a concrete knot take a planar projection and focus at a given crossing in the knot diagram; the value of the polynomial when the crossing is replaced by an L_+ minus the value of the polynomial when the crossing is replaced by L_- is equal to the value of the polynomial where the crossing is replaced by an L_0 . The definition of the L 's is given in figure 2. This relation, together with the normalization condition that the polynomial is equal to one for the unknot completely determines the polynomial for any knot without intersections. In order to consider the polynomial as a state of quantum gravity we have to define its value for loops with intersections. For the calculations at hand we will only need explicitly its definition for double intersections. There are many possible extensions of a given polynomial to intersecting loops. In principle, studying the transform of the Chern-Simons state could provide the appropriate extensions of interest for quantum gravity. At present however, the only available calculations of the transform for intersecting loops are first order perturbation theory ones [9]. These calculations are compatible with the following skein relations,

$$C(L_I) = \frac{1}{2}(C(L_+) + C(L_-)) \quad (2)$$

$$C(L_W) = C(L_0) \quad (3)$$

where the elements L_I and L_W are shown in figure 3 and correspond to a “straight through” intersection and an intersection with a “collision”. Both elements are needed, since they appear in the action of the Hamiltonian constraint.

The skein relations introduced above for the polynomial imply the following relations for a_2 , the second coefficient,

$$a_2(L_+) - a_2(L_-) = a_1(L_0) \quad (4)$$

$$a_2(L_I) = \frac{1}{2}(a_2(L_+) + a_2(L_-)) \quad (5)$$

$$a_2(L_W) = a_2(L_0) \quad (6)$$

$$a_2(\text{unknot}) = 0 \quad (7)$$

Contrary to the relations for the full polynomial, the ones for the second coefficient *are* uniquely determined by perturbative calculations, up to irrelevant factors involving the number of connected components of the loop. The first relation relates a_2 with a_1 , the first coefficient, evaluated on an L_0 . When one replaces a crossing in a knot

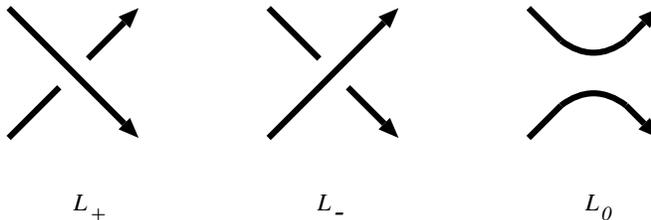


FIG. 2. The elements L_{\pm} and L_0 that intervene in the skein relation of the Conway polynomial without intersections.

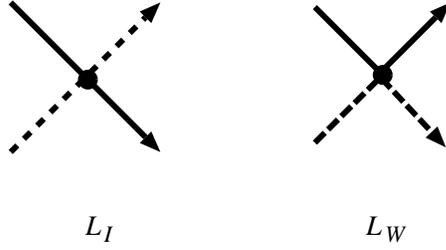


FIG. 3. The elements L_I and L_W that intervene in the skein relation of the Conway polynomial with intersections.

diagram by an L_0 one generically is left with a link composed by two loops. In that case a_1 is identical to the linking number of the two loops in the link. If the resulting knot has a single component a_1 is zero. The linking number of two loops $lk(\gamma_1, \gamma_2)$ has the property of being “Abelian” $lk(\gamma_1 \circ \gamma_2, \gamma_3) = lk(\gamma_1, \gamma_3) + lk(\gamma_2, \gamma_3)$. Also, for rerouted loops, $lk(\gamma^{-1}, \eta) = -lk(\gamma, \eta)$. All these properties will be crucial for the calculations that follow.

Notice also that relations (4,5) imply that one can replace an intersection by an upper or under crossing at the cost of introducing terms involving linking numbers,

$$a_2(L_I) = a_2(L_-) + \frac{1}{2}a_1(L_0) = a_2(L_+) - \frac{1}{2}a_1(L_0) \quad (8)$$

Let us apply the Hamiltonian constraint to the a_2 at a point with a triple “straight through” intersection. The loop in the lattice is shown in figure 4a and the detail of the intersection is shown in figure 4b, the numbers indicating the orientation of the loop. We will only show in detail one half of one of the contributions, that corresponding to when the Hamiltonian deforms “to the right” in the “1256” plane in the notation of figure 4b. The action of the constraint produces a change in the intersection and a reorientation shown in figure 4c. We will now replace the intersection marked as w in the figure, where a collision takes place, with the corresponding skein relation (6). The topology of the resulting loop is shown in figure 4d (we draw it as a smooth loop in the continuum just for easing the visualization process). In the resulting knot, which now has a single intersection, denoted as I in the figure, we replace it using (8). This produces two kinds of terms. One is a_2 evaluated on a loop without any intersection. This term cancels with the corresponding contribution when the Hamiltonian deforms “to the left” in the same plane. The other terms produced are linking numbers of the two subloops determined by the intersection I . This contribution is given by $-lk(\gamma_3^{-1}, \gamma_2^{-1} \circ \gamma_1)$ as can be checked by inspection of figure 4. Using the “Abelian” properties of the linking number we discussed above, the resulting contribution can be rewritten as $-lk(\gamma_1, \gamma_3) + lk(\gamma_2, \gamma_3)$. The action of the Hamiltonian “to the left” in the same “1256” plane yields, after a similar computation, $-lk(\gamma_2^{-1}, \gamma_1 \circ \gamma_3^{-1}) = -lk(\gamma_2, \gamma_3) + lk(\gamma_1, \gamma_2)$. The total contribution of the action of the Hamiltonian along the plane “1256” is therefore $-lk(\gamma_1, \gamma_3) + lk(\gamma_1, \gamma_2)$.

A similar calculation can be straightforwardly performed for the contributions stemming from the action along the other two other planes at the intersection. The result is that all the contributions cancel each other. This completes the proof.

In spite of the great differences with the formal calculation in the continuum, there are some remarkable similarities.

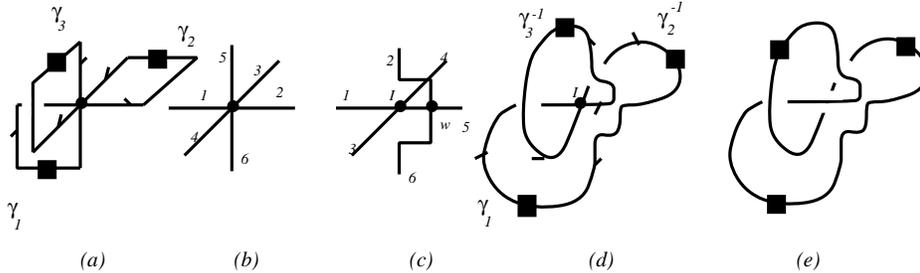


FIG. 4. The action of the Hamiltonian. Given a generic loop with a triple intersection (a), the Hamiltonian splits the triple intersection (b) into two double ones (c). For the case of the second coefficient of the Conway polynomial the resulting loop can be rearranged into a loop without intersections using the skein relations. First one uses the skein relation for an intersection with a kink and obtains loop (d). Then one uses the skein relation for regular intersections to convert the loop to a loop without intersections. The result is the second coefficient evaluated on a loop without intersections plus contributions of linking numbers. The black squares indicate that the different lobes of the loop could have arbitrary knottings and interlinkings. The result only depends on the local connectivity of the intersection.

In the continuum, in terms of either extended loop coordinates or multitanents [3,6] the second coefficient of the Conway polynomial consisted in two terms one involving multitanents of order four and one of order three. When the Hamiltonian acted it produced terms of order three, four and five. The terms of order three and four cancelled among themselves given the resulting topologies of the involved loops (as is the case here for the contributions proportional to a_2). The terms of rank five in the multitanents combined into a series of linking numbers that cancelled among themselves due to the Abelian nature of the linking numbers. This is exactly what we are seeing here in the lattice calculation.

The lattice calculation is simple enough as to consider what would happen for the next coefficient, the a_3 . In that case, one could also arrange several cancellations given the topologies of the loops, but then one would be left not only with linking numbers but also with combinations of a_2 's. Because the a_2 does not share the Abelian character with the linking number it is unlikely that the resulting terms will cancel. A related formal calculation in terms of extended loops shows precisely that behavior [10].

We have performed the calculation for a “straight through” intersection. What happens if the original intersection has “kinks” or “collisions”? A detailed calculation shows the action of the Hamiltonian also vanishes. Crucial for this calculation is to use the appropriate skein relations for intersections with kinks, as derived from the expectation value of a Wilson loop in Chern-Simons theory. It turns out that these relations are quite nontrivial and different from the case of intersections that are “straight through”. In particular, they distinguish intersections in which the kinks are coplanar from those in which they are not. This kind of result is only manifest in the lattice and suggests that “skein relations” for intersecting knots involve more than simply considering the planar projection of a knot, as happened in the case without intersections.

The main objection to the result we have found is that it is tied to a particular Hamiltonian. This Hamiltonian is constructed using the same ideas that led to a lattice framework in which the diffeomorphism constraint has desirable properties [7] and therefore might suggest it is the appropriate one to deal with quantum gravity. It is yet to be checked, however, if the constraint proposed closes the appropriate algebra of constraints (at least in the continuum limit). However, this is a problem that up to the moment has escaped solution in all frameworks in which solutions to the constraints have been found. Yet, it is a decidable matter in the framework presented here.

An important lesson to be learnt from the use of the lattice techniques is the much richer structure that skein relations for intersecting knots have with respect to usual non intersecting ones. The intersecting knot invariants considered here were specially tailored to be compatible with the Mandelstam identities of wavefunctions in the loop representation since they were obtained via the loop transform in a perturbative framework. For coefficients higher than a_2 it is clear that a more rigorous approach is needed and lattice calculations might be a viable scenario for investigating these issues.

We therefore see that the lattice formulation of the loop representation is a powerful tool, where one can prove in a natural, simple and yet rigorous way, formal results of the continuum. It leads also to insights into the details of the construction of intersecting knot theory, which is the kinematical arena of quantum gravity in the loop representation after diffeomorphism invariant has been imposed.

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