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The general solution of the quantum Einstein equations?

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We suggest how to interpret the action of the quantum Hamiltonian constraint of general relativity in the loop representation as a skein relation on the space of knots. Therefore, by considering knot polynomials that are compatible with that skein relation, one guarantees that all the quantum Einstein equations are solved. We give a particular example of such invariant and discuss the consistency of the constraint algebra in this approach.

The canonical approach provides a setting to address the nonperturbative aspects of the quantization of gravity. Within this approach, the introduction of the Ashtekar new variables [1] has allowed to formulate the theory in terms of a connection, as opposed to the usual approach in which the metric plays the fundamental role. Having the theory cast in terms of a connection allows to consider traces of holonomies (Wilson loops) as fundamental variables and to build a representation (the loop representation) in which wavefunctions are functions of loops.

Because general relativity is invariant under diffeomorphisms, in the loop representation one deals with functions of loops that have to be invariant under deformations of the loop. Therefore the states of quantum gravity have to be knot invariants. This was the key insight that allowed Rovelli and Smolin in 1988 [2] to show that knot theory was crucially related to quantum gravity.

This connection is only “kinematical”, in the sense that *any* theory of a connection that is invariant under diffeomorphisms is also related to knot theory. No input is needed from the particular Lagrangian or equations of motion of Einstein’s theory to establish the connection. This may raise the following question: up to what point are the structures and ideas of knot theory, which never took into account Einstein’s equations, geared up to this newly discovered connection with gravity?

The point of this essay is to suggest that a growing amount of evidence has accumulated over the last years that indicates that an unsuspected connection may exist between knot theory and the detailed structure of the quantum Einstein equations. The main idea can be summarized in that the action of the Hamiltonian constraint of quantum gravity, when formulated in terms of knot space, is a skein relation. Skein relations are the defining relations for knot polynomials. Therefore it is natural that the solutions of the Hamiltonian constraint should be related to knot polynomials. We will show that a very nontrivial solution that was found through lengthy direct calculations a few years ago, is actually compatible with the skein relation induced by the Hamiltonian and therefore provides a concrete detailed example of the idea we are putting forward.

The departing point is the action of the Hamiltonian constraint of quantum gravity in the loop representation formulated in the lattice that we have recently introduced [4]. The action can be described very simply through the following picture. It is non-zero only at points where there is a triple intersection in the loops, and it acts by converting the triple intersection in a pair of double intersections, one for each pair of strands entering the intersection,

$$\hat{H}\psi(\eta_i) = \sum_{\text{pairs}} [\psi(\eta_1) - \psi(\eta_2)] \quad (1)$$

where the action on one typical pair is depicted in figure 1. For non-straight through intersections the action is different, we will not discuss that case here for reasons of space, but all results claimed for the straight through case go through for the other cases as well.

Although this Hamiltonian was first proposed in the context of a fully regularized lattice framework, it is immediate to take over its action to the continuum theory. Solutions to the constraint in knot space are invariants such that when evaluated on knots like η_1 and η_2 give contributions such that the sum in the right hand side of equation (1) cancels. Let us construct an explicit example of such an invariant. It was shown some time ago [3] that the second coefficient of the Conway [7] polynomial was a solution of the Hamiltonian constraint of quantum gravity in the loop representation. This result was first established in a formal way in the continuum. Later it was shown in a regularized context via the extended loop representation, albeit via the introduction of a delicate counterterm in the Hamiltonian

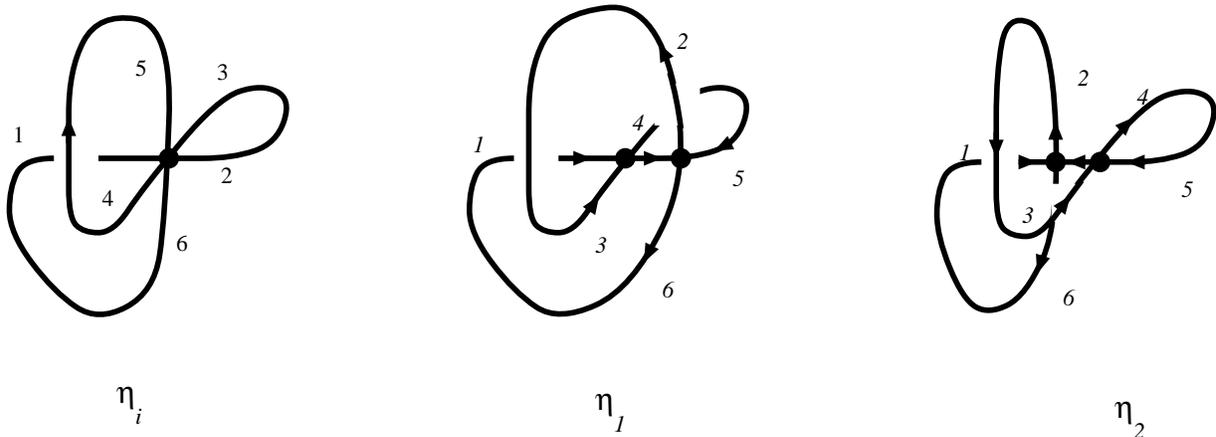


FIG. 1. The action of the Hamiltonian deforms the argument of the wavefunction from η_i to η_{\pm} .

[6]. We will here show that it is actually very easy to see that it satisfies condition (1) and therefore it is a solution. In order to see this we need to consider the skein relations that define the second coefficient of the Conway polynomial. These can be found and suitably generalized to intersecting loops using the recent results of reference [5] by looking at the second coefficient of the expansion in the variable x of the Jones polynomial evaluated for $q = e^x$. The result is,

$$C_2(\times) - C_2(\searrow) = C_1(\smile) \quad (2)$$

$$C_2(\times) = \frac{1}{2}(C_2(\searrow) + C_2(\swarrow)) + \frac{1}{8}(C_0(\times) + C_0(\searrow)) \quad (3)$$

$$C_2(\searrow) = C_2(\smile) \quad (4)$$

$$C_2(\bigcirc) = 0, \quad (5)$$

where $C_0 = 2^{n_c}/2$ where n_c is the number of connected components of the knot ($C_0 = 1$ for a single component knot) and C_1 is an invariant that coincides with the Gauss linking number if the involved loop has two components and zero otherwise. The Gauss linking number $lk(\gamma_1, \gamma_2)$ is a very simple invariant that given two curves γ_1, γ_2 measures how many times one of them pierces a surface that has the other curve as a boundary. It behaves in an ‘‘Abelian’’ manner with respect to composition of curves $lk(\gamma_1 \circ \gamma_2, \gamma_3) = lk(\gamma_1, \gamma_3) + lk(\gamma_2 \circ \gamma_3)$ and also with respect to retracings of curves, $lk(\gamma, \eta^{-1}) = -lk(\gamma, \eta)$.

The above relations imply that for the second coefficient we can turn an intersection into an upper or double crossing ‘‘at the price’’ of a term proportional to the linking number and another one to the number of connected components,

$$C_2(\times) = C_2(\searrow) + \frac{1}{2}C_1(\smile) + \frac{1}{4}C_0(\searrow) = C_2(\swarrow) - \frac{1}{2}C_1(\smile) + \frac{1}{4}C_0(\swarrow). \quad (6)$$

We can then eliminate the double intersection to the left of η_1 and the one to the right in η_2 using (6) and the intersection to the right of η_1 and the one to the left of η_2 using (4) in the action of the Hamiltonian (1) in such a way as to be left with C_2 evaluated in two topologically equivalent knots. Therefore their contributions cancel. The price for lifting and lowering the lines is a collection of terms involving the linking number, that cancel each other by virtue of the Abelian nature of C_1 and C_0 [4]. This proof is remarkably simpler than the original derivation in the continuum, which had to be tackled with computer algebra [3], or even the extended loop version which had several delicate issues of regularization involved [6].

What we accomplished here is to isolate the topological action of the Hamiltonian constraint. The Hamiltonian constraint is an operator that is not diffeomorphism invariant, it depends on a point. Therefore one cannot expect to have a realization of it in knot space as an operator. The best one can hope for is to define its kernel. What we have done to capture this concept is to implement the kernel of the Hamiltonian as a skein relation in knot space. This has an added bonus: one does not need to be concerned about the problem of the constraint algebra. Operators have to satisfy algebras, skein relations do not. An analogy of this in a more familiar context would be given by considering the Gauss law. If one formulates the theory in terms of a connection, the Gauss law is a quantum operator and one has to worry about its constraint algebra. If one goes to the loop representation, one is in the kernel of the Gauss law. The only remnant of the Gauss law are the Mandelstam identities. These one treats as relations and one does not worry about their algebra.

The relations we presented here come about because the regularized action of the Hamiltonian constraint of quantum gravity in loop space always can be written as a non diffeomorphism invariant pre-factor that is point and regularization dependent times a function of loops evaluated in a loop that is deformed from the original loop. By separating the action into a “topological” and “local” part and implementing the “topological” part as a skein relation what we are doing is to construct the physical space of states of the theory. In this space one could represent diffeomorphism invariant observables for the theory that one could use to perform physical measurements.

Viewing quantum gravity in the space of knots brings in a new light the problem of the degrees of freedom of the theory. We are saying that the physical states of gravity are knot invariants (compatible with the Mandelstam identities) that satisfy the skein relation induced by the Hamiltonian. This is not enough to characterize an invariant. There are many invariants that satisfy this (an example was C_2 as discussed above). The resulting freedom corresponds to the degrees of freedom of quantum gravity. This suggests a new way of how to handle the local degrees of freedom when one quantizes a field theory as quantum general relativity that is topological and yet has local degrees of freedom.

The above claims can only be considered as preliminary, but there are a series of consistency checks that one can envisage carrying out in the immediate future. On one hand, the ideas exposed do not depend on the number of dimensions of spacetime. They could therefore be probed carefully in the well-understood domain of gravity in $2 + 1$ dimensions. In that case it is known that the Ashtekar version of the constraints and the Witten one (that simply requires that the curvature vanish) are equivalent only under a certain set of assumptions. A detailed study should show which set of constraints is the skein relation proposed here more closely related to. In the Regge-Ponzano version of $2 + 1$ gravity it is known [8] that the Hamiltonian constraint leads to a relation among states that could be viewed as a skein relation in two dimensions. The solution to it, in the case of a trivial topology, is uniquely determined (as it should, since there are no degrees of freedom in that case). This appears as a direct counterpart in $2 + 1$ dimensions of the ideas we present here in the $3 + 1$ context. In this latter context there are other checks that can be carried out. For instance, it is known that the Kauffman bracket knot polynomial should be a state of quantum gravity with a cosmological constant [9] (it is the loop transform of Kodama’s Chern–Simons state). It is straightforward to write a skein relation corresponding to the Hamiltonian constraint with a cosmological constant (the cosmological constant adds only a rerouting at triple intersections [10]). It would be remarkable if the Kauffman bracket’s skein relations are compatible with this relation. Moreover, calculations in the extended loop representation [11] suggest that the third coefficient of the Jones polynomial is not annihilated by the constraint but its action is proportional to a topological rearrangement of the loops. Again one could check that the skein relation presented here yields the same result.

All of this constitutes a mounting set of evidence that a deep connection at a dynamical level exists between knot theory and quantum gravity. The fact that notions of knot theory that were developed independently of any theory of gravity are now being used to write in a very detailed way the dynamics of quantum general relativity cannot be understressed.

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