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Chern–Simons states in spin-network quantum gravity

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In the context of canonical quantum gravity in terms of Ashtekar's new variables, it is known that there exists a state that is annihilated by all the quantum constraints and that is given by the exponential of the Chern–Simons form constructed with the Asthekar connection. We make a first exploration of the transform of this state into the spin-network representation of quantum gravity. The discussion is limited to trivalent nets with planar intersections. We adapt an invariant of tangles to construct the transform and study the action of the Hamiltonian constraint on it. We show that the first two coefficients of the expansion of the invariant in terms of the inverse cosmological constant are annihilated by the Hamiltonian constraint. We also discuss issues of framing that arise in the construction.

It was noted some time ago [1] that the exponential of the Chern–Simons form

$$\Psi^{CS}[A] = \exp\left(-\frac{6}{\Lambda}\mathrm{Tr}[\int A \wedge dA + \frac{2}{3} A \wedge A \wedge A]\right) \tag{1}$$

was annihilated by all the constraints of canonical quantum gravity (with a cosmological constant) in the connection representation constructed via the Ashtekar [2] formulation of canonical general relativity. Starting with that observation there have been several attempts [1,3–5] to understand the counterpart of this state in the loop representation.

In order to compute the counterpart one has to consider the loop transform of such a state,

$$\Psi(\gamma) = \int DA \exp\left(-\frac{6}{\Lambda}\mathrm{Tr}[\int A \wedge dA + \frac{2}{3} A \wedge A \wedge A]\right) W_\gamma[A]. \tag{2}$$

where $W_\gamma[A]$ is the Wilson loop, ie the trace of the holonomy of the connection along the loop γ.

Wilson loops are constrained by a set of relations known as the Mandelstam identities. The presence of these identities implies that not every function of a loop γ is an admissible wavefunction in the loop representation. It has been shown that a set of gauge invariant quantities that is free of Mandelstam identities can be constructed from loops through the use of the spin-network basis [6]. In this context spin networks consist of lines connected at vertices living in three dimensional space, with each line having associated a holonomy in a given representation of $SU(2)$. One can then construct gauge invariant quantities by contracting the holonomies with appropriate intertwiners provided by the theory of angular momentum recoupling at the intersections. We will call the resulting invariant a "Wilson-net" and denote it $W_\Gamma[A]$ where Γ is a spin network.

One can then introduce a new representation for quantum gravity in which wavefunctions are labelled by spin networks via a transform similar to the one introduced above. In particular, for the Chern–Simons state,

$$\Psi(\Gamma) = \int DA \exp\left(-\frac{6}{\Lambda}\mathrm{Tr}[\int A \wedge dA + \frac{2}{3} A \wedge A \wedge A]\right) W_\Gamma[A]. \tag{3}$$

With the introduction of rigorous measures in the space of connections modulo gauge transformations [7], hopes were raised that the above functional integral could be computed in a rigorous way. It turns out however, that because the Chern–Simons form is only gauge invariant when integrated over the manifold, this makes use of the rigorous measure theory a nontrivial task that has not been accomplished up to present.

On the other hand, the above integral (in the context of ordinary loops) has been explored by a variety of methods, which can be summarized in three groups. One of them considers a two dimensional slice of the Chern–Simons theory and studies the monodromies that result from exchanges of the punctures that the Wilson loops form with the slice [8]. The integral can also be evaluated perturbatively using Feynman diagrammatics [9,10]. This produces a result that is an explicit function given by loop integrals of certain kernels. The last technique (variational calculus) consists of studying infinitesimal deformations of the loop that appears in the integral and computing relations (skein relations) between the values of the wavefunction when one substitutes over and under-crossings in the loop by intersections [11,12,1]. In some cases the infinitesimal results can be exponentiated to yield finite results [13]. On the other hand it was shown in [13] that variational techniques offer a certain amount of regularization freedom in the definition of the transform. Therefore the possibility exists that there are really various possible "transforms" of the Chern–Simons state, essentially corresponding to the introduction of different measures in the computation of the functional integration.

In the context of spin networks, the transform of the Chern–Simons state has been explored by Kauffman and Lins [14] through the use of tangle-theoretic techniques. What we will do in this paper is to reinterpret the tangle results in terms of spin networks and study if the resulting invariant is annihilated by the Hamiltonian constraint of general relativity in the context of spin networks originally proposed by Thiemann [15].

To start setting up the structure we need, we will begin by pointing out certain conventions. To define an invariant of a trivalent spin network, one needs a $3-j$ symbol which is the intertwiner at the intersections, and also an assignment of an "orientation" at the vertices, to determine which incoming holonomy corresponds with each entry of the symbol (there are two cyclic possibilities). The choice we make is to order the three entries by considering the clockwise ordering of the three incoming lines. If one considers the holonomies connected with intertwiners the resulting object will in general be orientation-dependent. For the $SU(2)$ case it can be seen that the orientation dependence boil down to an overall sign of the invariant. We will define our invariants with an additional overall sign in such a way that they are not orientation dependent. The overall factor will be chosen in such a way that if one considers the invariant when the connection is flat its value corresponds to the chromatic evaluation of the graph (see [14]; also [16] for some explicit examples).

Let us show an example of how to construct a gauge invariant function associated with a spin network. We will call these invariant "Wilson network" since it naturally generalizes the idea of Wilson loops to this context. Let us consider a spin network like the one shown in figure 1

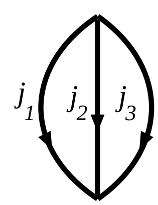


FIG. 1. An example of spin network.

the resulting Wilson network is given by,

$$W_\Gamma = f(\Gamma) \sum_{m,m'} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ -m'_1 & -m'_2 & -m'_3 \end{array} \right\} H^{j_1}_{m_1\, m'_1} H^{j_2}_{m_2\, m'_3} H^{j_3}_{m_3\, m'_3} \tag{4}$$

where $f(\Gamma)$ the factor we discussed above. For all the networks we will consider in this paper, the above factor is given (up to a factor of modulus unity) by the following formula,

$$f(\Gamma) = \pm \prod_{v_i \in \text{vertices}} \sqrt{\Theta_0(v_i)} \tag{5}$$

where

$$\Theta_0(v_i) = \frac{(-1)^{j_1+j_2+j_3}(j_1+j_2+j_3+1)!(j_1+j_2-j_3)!(j_1+j_3-j_2)!(j_2+j_3-j_1)!}{(2j_1)!(2j_2)!(2j_3)!} \tag{6}$$

and j_{1-3} are the spins of the strands associated with the vertex in question.

Ordinary non-intersecting loops are considered in this approach as spin networks composed by two strands joined at "double vertices". If one evaluates (4) for that case, one notices that the result is $(-1)^{2j}$ times the ordinary Wilson loop.

The normalization we have chosen for the spin networks coincides with that of De Pietri and Rovelli [16]. The notation is different from that of Thiemann [15] in the sense that he introduces double vertices along each strand.

We only consider double vertices as the ones we introduced to describe the ordinary Wilson loop, that is, they always appear in pairs.

The invariant we will consider is defined by the following relations:

• Basic symmetries:

If one considers the un-knot in the trivial representation, the Wilson net is equal to one, and the result of the transform is chosen to be

$$E(\text{unknot}^0, k) = 1, \tag{7}$$

this being a normalization condition.

The Wilson nets for $SU(2)$ are real, so $W_\Gamma = \bar{W}_\Gamma$, and therefore,

$$\bar{E}(\Gamma, k) = E(\Gamma, -k). \tag{8}$$

Since the Chern–Simons form changes sign under a parity transformation,

$$E(\tilde{\Gamma}, k) = E(\Gamma, -k), \tag{9}$$

where $\tilde{\Gamma}$ is the mirror-image of the spin-net Γ.

Finally,

$$E(\Gamma, \infty) = W_\Gamma[A = 0] = \text{chromatic evaluation}(\Gamma). \tag{10}$$

This result can be understood remembering that in the case of single loops in the fundamental representation the parameter k is related with the parameter of the Kauffman bracket q via $q = e^{-\frac{4i\pi}{k}}$, so for $k \to \infty$, $q \to 1$ and the invariant is equal to 2, which corresponds (via Giles' [17] reconstruction theorem) to evaluating the Wilson loop for a flat connection. This translates in the case of spin-nets the chromatic evaluation.

• Factorization.

If one considers two spin-nets that are disjoint (by this meaning their planar projections lie in two separate half-planes) then,

$$E(\Gamma_1 \cup \Gamma_2) = < W_{\Gamma_1} W_{\Gamma_2} > = < W_{\Gamma_1} > < W_{\Gamma_2} > = E(\Gamma_1) E(\Gamma_2). \tag{11}$$

This was proven by Witten [8] and can be intuitively seen by remembering that since E is diffeomorphism invariant, one could consider the two spin networks to be as far removed from each other as possible and then all the "interactions" of the Chern–Simons theory would disappear.

• Twists and crossings.

The values of the invariant when one replaces and under-crossing by and over-crossing are related through equations that are generalizations of the usual "skein relations" that are satisfied by the Kauffman bracket (which is the transform of the Chern–Simons state in terms of ordinary loops.) These relations are,

$$E(\hat{L}_\pm^{(j)}, k) = \exp\left(\mp \frac{4i\pi}{k} Q_j\right) E(\hat{L}_0^{(j)}) = q^{\pm Q_j} E(\hat{L}_0^j) \tag{12}$$

where $\hat{L}\pm, \hat{L}_0$ are depicted in figure 2 and $\Lambda = 24i\pi/k$.

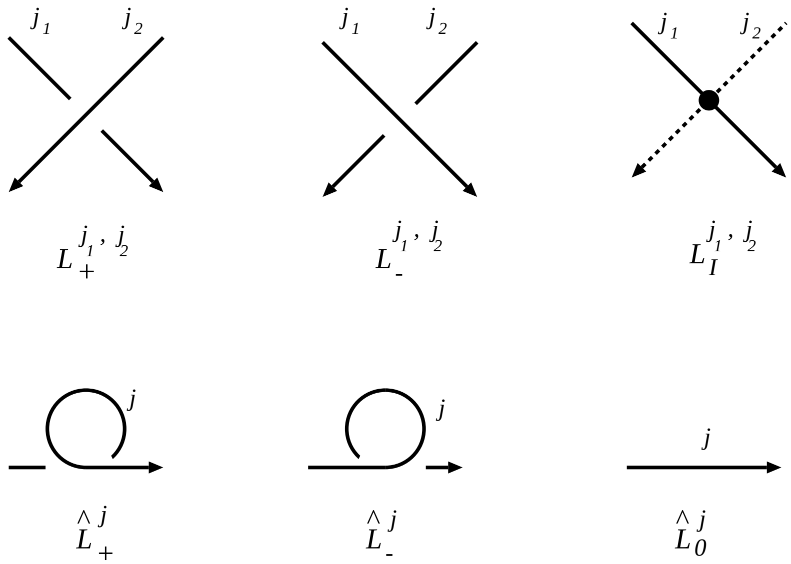


FIG. 2. The different crossings involved in the skein relations for the invariant

For the case of upper and under-crossings,

$$E(L_{\pm}^{j_1,j_2},k) = \sum_{j=|j_1-j_2|}^{j_1+j_2} q^{\mp\frac{1}{2}[Q_{j_1}+Q_{j_2}-Q_j]}(-1)^{j_1+j_2-j}\frac{\Delta_j}{\Theta(j_1,j_2,j)}(-1)^{n_1 j_1+n_2 j_2} E(L_H^{j_1,j_2,j},k) \tag{13}$$

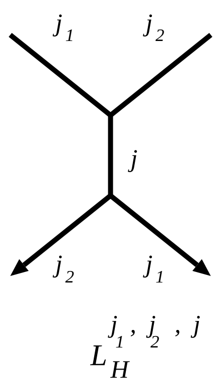
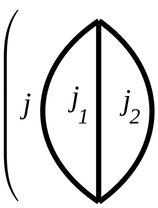


FIG. 3. The double-trivalent element that arises in the computation of the upper and under-crossings.

where n_1 and n_2 are the number of two-valent vertices that have been eliminated by the introduction of the double-trivalent vertex (essentially this means that if one had a single loop with a crossing, it would have had two double-valent crossings, and when the crossing is replaced with the L_H they would disappear, giving rise to an overall factor due to normalization; the situation could also arise from two separate loops that crossed). In the above expression Θ and Δ_j are the evaluations of the invariant for the theta knot and the un-knot,

$\Delta_j = E\Big($  $, k\Big)$ (14)

$\Theta(j_1, j_2, j) = E\Big($  $, k\Big)$ (15)

and we will detail later how to evaluate the invariant for the diagrams shown.

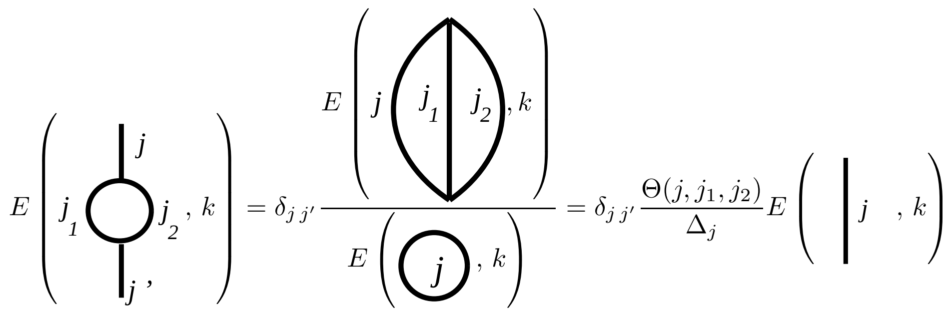
One can obtain these relations using variational techniques [13] and using a generalization of the Fierz identity for higher representations of $SU(2)$,

$$\tau^i_{(j_1)\,A}{}^B \tau^i_{(j_2)\,C}{}^D = \frac{1}{2}\left\{ \sum_{J=|j_1-j_2|}^{j_1+j_2} Q_J \sum_{M=-J}^{J} < j_1\,A\,j_2\,C|J\,M >< J\,M|j_1\,B\,j_2\,D > -(Q_{j_1}+Q_{j_2}))\delta^B_A\delta^D_C \right\}. \tag{16}$$

• Recoupling identities.

The additional identities that will define the invariant cannot be obtained in a simple explicit way using variational calculations. We just impose them at the level of spin nets and assume they define a regularization in terms of which one could perform variational calculations for their evaluation. Together with the above relations they completely characterize the invariant, ie, they allow us to evaluate it for all possible spin networks in three dimensions.

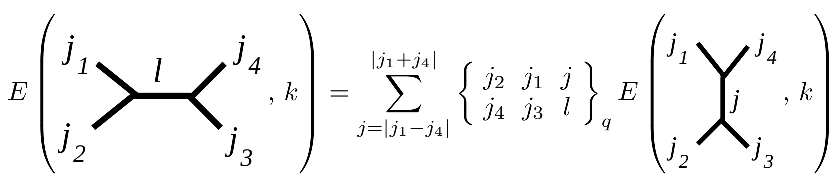
The first relation allows to eliminate a loop from a line,



(17)

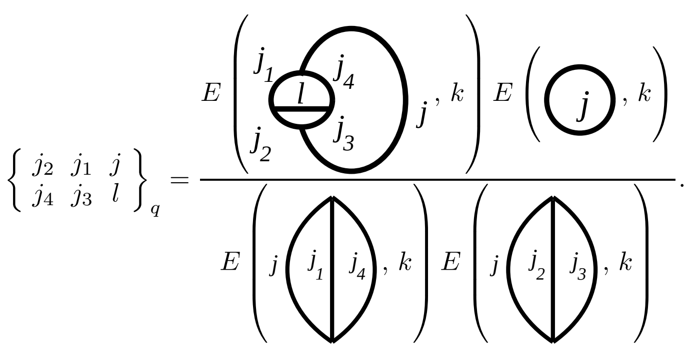
and this relation can be simply understood by "closing up" the upper and lower strand to form the theta and the unknot.

The second relation is,



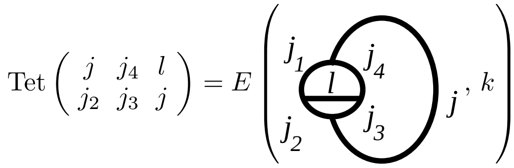
(18)

where the expression in curly braces is the q-deformed Racah symbol, which is defined as,



(19)

The left diagram in the numerator is known as the tetrahedron and denoted,



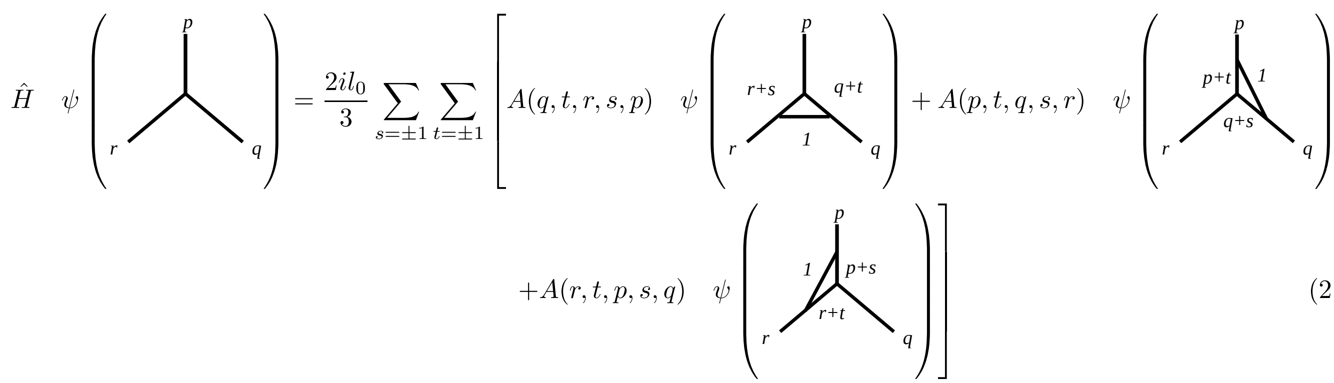
(20)

and an explicit evaluation for it is given in the book by Kauffman and Lins [14], paragraph 8.5.

The above relations completely characterize the invariant. In particular, we can now evaluate its value for the unknot of j valence, Δ_j. In order to do that one considers j parallel loops of valence 1/2 and uses equation (18) with $k = 0$ to convert two parallel lines of valence 1/2 to a line of valence 1. The resulting object (for two loops) is a theta diagram with $j = 1$, which we can transform using (17) to a single loop of higher valence. The reason why we emphasize this construction is to notice that for specifying a loop (or for that matter a single strand) of higher valence one requires to specify a family of "parallel" loops that do not intertwine (if they did, the value assigned to the loop of higher valence would be different). This indicates that the invariant we are constructed is in a very basic nature an invariant of framed loops. One needs to prescribe a framing for each higher valence line, even if no twists of it are present. Of course the invariant is an invariant of framed loops in the ordinary sense, since the addition of a twist changes its value. But there is this deeper, more basic framing associated with a single line. This is a well known effect, it is just a fact that the invariant we are computing is really an invariant of q-deformed spin networks [18] which require such framings. The computation of Θ is also straightforward, we refer the reader to chapter 6.3 of Kauffman and Lins [14].

This weakens considerably the case for studying these invariants in the context of ordinary quantum gravity where the objects in question are non-q-deformed spin networks. However, Major and Smolin [18] have proposed that q-deformed spin networks might play a role in quantum gravity. If this were the case, the invariant considered could be a candidate to quantum state. Based in part on this motivation we will explore the action of Thiemann's [15] Hamiltonian constraint on this state. Of course, Thiemann's Hamiltonian does not necessarily a priori know how to act on framed spin networks. The approach we will take is to assume that the action is the same as on ordinary spin networks. Clearly one could define a Hamiltonian in the context of framed spin networks that would act in such a way.

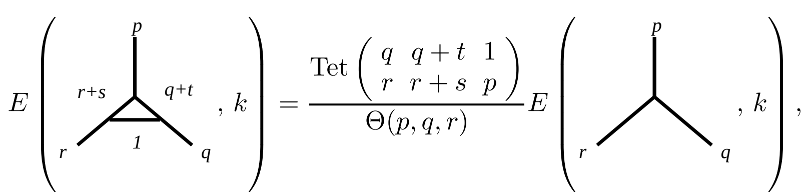
The explicit action of Thiemann's Hamiltonian in terms of spin networks has been worked out by Borissov, De Pietri and Rovelli [19]. The operator only acts at vertices by adding a line forming a triangle with the vertex in the three possible orientations and multiplying times a factor,



(21)

where the coefficients A are explicitly given in [19]. This is the action of the Euclidean part of Thiemann's Hamiltonian constraint. Since the Chern–Simons state is annihilated by the Euclidean Hamiltonian constraint in the connection representation, we will restrict our analysis to the Euclidean part here.

Now, for our invariant $E(\Gamma, k)$, the recoupling relations allow to connect the value of the invariant for the trivalent vertex with the extra line with the value for just a plain trivalent vertex, through the formula,



(22)

let us denote the prefactor in this equation as $\beta(p, q, r, k)$ (it is a function of k).

If one now combines this expression with the action of the Hamiltonian, one basically proves that the action of the Hamiltonian reduces to an overall multiplication times a coefficient that is a linear combination of tetrahedra with the coefficients A. One can examine this expression order by order in the coupling constant k, in effect examining the action of the Hamiltonian constraint on the various powers of the expansion of the invariant E in terms of k. For order k^0 the invariant reduces to the chromatic evaluation of the knot. Such an invariant can be thought of as the counterpart in the spin network representation of a distributional state in the connection representation given by $\psi[A] = \delta[A - \text{flat}]$. That is a state "peaked" around flat connections. Such a state is obviously a solution of the

Hamiltonian constraint in the factor ordering in which the curvature appears to the right (this is the ordering that Thiemann chose for his Hamiltonian). It can actually be checked by explicit computation that the linear combination of $A's$ with the zeroth order expansion of the β prefactor vanishes [20]. This is therefore a solution of the Euclidean Hamiltonian constraint. In fact, the zeroth order coefficient (chromatic evaluation) is framing independent so this is a genuine state of quantum gravity in the spin-network representation.

What happens to higher orders? The computations are quite involved. However, we can immediately reach a conclusion for the first order in k. Because the β coefficient is chiral (a knot and its mirror image give the same result for it) and a chiral operation on the invariant (as we discusses above) is tantamount to a change of sign in k, it is the fact that the β prefactor should have zero first coefficient in its expansion in terms of k. Therefore the action of the Hamiltonian constraint on $E(\Gamma, k)$ to first order in k reduces to its zeroth order action, and is again zero. From this we can conclude that the first coefficient of the expansion of the invariant $E(\Gamma, k)$ in terms of k is also annihilated by the Hamiltonian constraint. Here one should exercise some care since this invariant is not framing independent and therefore is not a genuine state in terms of ordinary spin networks. The only conclusion one can reach is that if one were to consider a framed spin network representation of quantum gravity and define Thiemann's Hamiltonian constraint in a manner analogous to the non-framed case, this state would be a candidate to a solution.

In summary, we have constructed in a rather ad-hoc way an invariant that is a candidate to be the transform of the Chern–Simons state in terms of spin network. The ad-hoc portions of the construction can be ascribed to choices of regularization in the definition of the path integral. The choices made allow the invariant to have the properties of recoupling theory, which in turn reduces the action of the Hamiltonian constraint to a multiplicative operator. As a consequence we are able to find solutions of the Hamiltonian constraint. The price to pay for the choices made in the definition of the invariant is that it is really an invariant of framed spin nets and therefore not giving rise to genuine states (with the exception of the zeroth coefficient) of ordinary (non q-deformed) quantum gravity. This raises the question of if other choices of regularization could not allow the definition of an invariant that is a genuine invariant of spin nets. This issue is currently under consideration. In principle the systematic use of the variational techniques of [13] could in principle allow the construction of such an invariant. It is quite unlikely, however, that the action of the Hamiltonian for such an invariant will reduce to just a multiplication times a factor. On the other hand, the resulting invariant could potentially be made framing independent by the removal of the twist dependences much in the same spirit as the Jones polynomial is obtained by removing the twist dependence of the Kauffman bracket in the context of ordinary loops. Such a result would be of interest in its own right, even if it fails to produce solutions of the Hamiltonian constraint of quantum gravity.

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[1] B. Brügmann, R. Gambini, J. Pullin, Nucl. Phys. **B385**, 587 (1992).
[2] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. **D36**, 1587 (1987).
[3] B. Brügmann, R. Gambini, J. Pullin, Gen. Rel. Grav. **25**, 1 (1993).
[4] C. Di Bartolo, R. Gambini, J. Griego, J. Pullin, Phys. Rev. Lett. **72**, 3638 (1994).
[5] C. Di Bartolo, R. Gambini, J. Griego, Phys. Rev. **D51**, 502 (1995).
[6] C. Rovelli, L. Smolin, Nucl. Phys. **B442**, 593 (1995).
[7] A. Ashtekar, J. Lewandowski, J. Math. Phys. **5**, 2170 (1995).
[8] E. Witten, Commun. Math. Phys **121**, 351 (1989).
[9] P. Cotta-Ramusino, E. Guadagnini, M. Martellini, M. Mintchev Nucl. Phys. **B330**, 557 (1990).
[10] D. Bar-Natan, Topology, **34**, 423 (1995).
[11] L. Smolin, Mod. Phys. Lett. **A4** 1091 (1989).
[12] E. Guadagnini, M. Martellini, M. Mintchev, Nucl. Phys. **B330**, 575 (1990).
[13] R. Gambini, J. Pullin, "Variational derivations of exact skein relations for Chern–Simons theories", Commun. Math. Phys. (to appear).
[14] L. Kauffman, S. Lins, "Temperley–Lieb recoupling theory and invariants of 3-Manifolds", Annals of Mathematics Studies, Princeton University Press, Princeton (1994).

[15] T. Thiemann, Phys. Lett. **B380**, 257 (1996); also preprints gr-qc/9606089-91.
[16] R. De Pietri, C. Rovelli **54** 2664 (1996).
[17] R. Giles, Phys. Rev. **D24**, 2160 (1981).
[18] S. Major, L. Smolin, Nucl. Phys. **B**
[19] R. Borissov, R. De Pietri, C. Rovelli, "Action of Thiemann's Hamiltonian constraint in loop quantum gravity (in preparation).
[20] R. De Pietri, private communication.