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Canonical quantum gravity in the Vassiliev invariants arena: I. Kinematical structure

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We generalize the idea of Vassiliev invariants to the spin network context, with the aim of using these invariants as a kinematical arena for a canonical quantization of gravity. This paper presents a detailed construction of these invariants (both ambient and regular isotopic) requiring a significant elaboration based on the use of Chern-Simons perturbation theory which extends the work of Kauffman, Martin and Witten to four-valent networks. We show that this space of knot invariants has the crucial property —from the point of view of the quantization of gravity— of being loop differentiable in the sense of distributions. This allows the definition of diffeomorphism and Hamiltonian constraints. We show that the invariants are annihilated by the diffeomorphism constraint. In a companion paper we elaborate on the definition of a Hamiltonian constraint, discuss the constraint algebra, and show that the construction leads to a consistent theory of canonical quantum gravity.

I. INTRODUCTION

A. Preliminaries

Since the early attempts in the 60's to construct a canonical quantum theory of general relativity, a persistent problem has been the definition of well-defined quantum operatorial expressions for the constraints of the theory. In particular, the realization of the quantum Hamiltonian constraint (the Wheeler–DeWitt equation), given its nonlinear structure in terms of momenta, has been particularly elusive. In terms of the usual canonical variables (metric and extrinsic curvature), the first impediment found was that the constraint is highly non-polynomial. The introduction of the Ashtekar variables [1] seemed to provide a natural way to construct a polynomial expression for the Hamiltonian constraint. However, the formalism presented two significant complications: a) the variables in question were complex; b) the polynomial Hamiltonian constraint was a density of weight two. The latter fact is crucial at the time of promoting the constraint to a quantum operator. Since in a manifold without a prescribed classical metric (as is the case in quantum gravity where the metric is a quantum operator) the only naturally defined density is of weight one (the Dirac delta), most attempts to regularize [2–5] the Hamiltonian constraint produced operators that exhibited dependence on external, artificial metric structures introduced via the regularization procedure. Such remnant dependences on fiducial metric structures almost inevitably imply that the constraint algebra will not be properly implemented at a quantum level. Since fiducial structures do not transform appropriately under the action of the diffeomorphism constraint, the net effect is to make the Hamiltonian constraint not covariant. At most one could hope for formal consistency, when the algebra was computed ignoring regulators [6].

An important step forward towards solving the two problems we mentioned above was a construction due to Thiemann [7]. He realized that it was indeed possible to construct a polynomial version of the single-densitized Hamiltonian constraint in terms of Ashtekar-like variables. To begin with, Thiemann uses the real connection variables introduced by Barbero [8], thus bypassing the issues associated with the reality conditions of the traditional Ashtekar formulation. Barbero's variables consist of an $SU(2)$ connection A_a^i and a set of triads $\tilde{E}_i^a$ (throughout this paper we denote density weights with tildes). The basic idea to polynomialize the Hamiltonian constraint is the key canonical identity,

$$\frac{2}{G}\{A_a^i, V\} = \frac{\underset{\sim}{\epsilon}_{abc}\tilde{E}_j^b\tilde{E}_k^c\epsilon^{ijk}}{\det q}, \tag{1}$$

where G is Newton's constant, and,

$$V = \int d^3x \sqrt{\underset{\sim}{\epsilon}_{abc}\tilde{E}_i^a\tilde{E}_j^b\tilde{E}_k^c\epsilon^{ijk}}, \tag{2}$$

is the volume element. In terms of these expressions the singly-densitized Hamiltonian constraint can be written as a polynomial operator. To simplify the discussion we focus on the Hamiltonian of Euclidean general relativity

(Thiemann [7] has shown that an analogous construction is possible for the Hamiltonian of the Lorentzian theory, which involves an extra term),

$$\tilde{H} = \frac{\tilde{E}^a_j \tilde{E}^a_k \epsilon^{ijk}}{\det q} F^i_{ab}, \tag{3}$$

which can therefore be written as,

$$\tilde{H} = \frac{2}{G}\{A^i_a, V\} F^i_{bc} \tilde{\epsilon}^{abc}, \tag{4}$$

In [7], Thiemann elaborated a proposal for promoting this expression to a quantum operator acting on diffeomorphism-invariant wavefunctions associated with cylindrical functions of the connection (see [9] for a quick, physical, description). The latter are functions constructed by considering gauge invariant quantities parameterized by multivalently-intersecting graphs embedded in three dimensions, usually referred to in this context as spin networks. On the space of cylindrical functions there are known measures [10] that make states parameterized by non-diffeomorphically-related spin networks orthogonal. The proposed Hamiltonian constraint is strictly speaking only defined on the diffeomorphism-invariant dual of the space of cylindrical functions constructed using the measure we just referred to.

Thiemann's proposal fulfilled several of the usual expected requirements for a quantized Hamiltonian constraint of general relativity. The operator was well defined and finite (even if one coupled the theory to matter), could be made self-adjoint in a controlled fashion under the above mentioned measure, and was covariant under finite diffeomorphisms in a detailed sense. Because the operator was realized in a space of diffeomorphism-invariant wavefunctions, the expected result for the commutator of two Hamiltonian constraints is zero, and indeed this is the result that is obtained. In this approach there is no implementation of the infinitesimal generator of diffeomorphisms. Technically speaking, Thiemann's construction constitutes one of the first-ever consistent theories of quantum gravity. Consistency is a prerequisite for any physically relevant theory, but is not a sufficient condition. It is still to be understood if Thiemann's proposal embodies the correct physics of general relativity at a semiclassical level.

A shadow of doubt was cast on the physical relevance of Thiemann's construction through an argument constructed by Lewandowski and Marolf [11] (see also [12] for further elaboration). Lewandowski and Marolf considered a different space of wavefunctions on which to define a Hamiltonian constraint. In particular, the new "habitat" (to use their terminology) contains the diffeomorphism-invariant dual of cylindrical functions we considered above, but also contains non-diffeomorphism invariant functions. Better yet, the same construction one followed to promote the Hamiltonian constraint to an operator in Thiemann's proposal is applicable in the new habitat. However, a crucial difference that stems from the correspondence with the classical Poisson algebra of constraints, is that the Hamiltonian one constructs should not commute with itself, but yield a commutator that is proportional to a diffeomorphism (in the new habitat the diffeomorphism constraint is a well defined operator yielding the natural geometric action). Yet, the Hamiltonian one obtains applying Thiemann's proposal commutes with itself, even in the habitat of non-diffeomorphism invariant states. In fact, it was also shown that several modifications of Thiemann's construction also produce Hamiltonians with vanishing commutators. In addition, if one artificially modified the Hamiltonians to yield a non-vanishing commutator, the end result did not appear to correspond to the quantization of the right hand side of the classical commutator [12].

Notice that these arguments do not *prove* that Thiemann's construction is incorrect in the diffeomorphism invariant context. It could still be the case that in that context the theory yields the correct semi-classical limit. In fact, Thiemann's construction in $2+1$ dimensions seems to have the difficulties as in $3+1$ dimensions, and still contains within its space of solutions those of the more traditional quantizations of $2+1$ gravity [13]. By carefully choosing an inner product and imposing other restrictions, the approach can be made to yield only the usual solutions of the $2+1$ theory. However, it is disturbing that one does not seem to find a context where this construction reproduces the classical Poisson algebra with a non-vanishing commutator of two Hamiltonians. The commutativity might also have implications for the semiclassical limit of the theory [14].

In a separate set of developments, that has its roots in formal calculations performed some time ago in the language of loops and involving the doubly-densitized Hamiltonian [15–18], another space of wavefunctions has been considered. These are the wavefunctions that arise when one considers in the space of gauge invariant functions of a connection the measure given by the exponential of the Chern–Simons action. The attractiveness of these wavefunctions stems from a large body of results in Chern–Simons theory that relates these states to well studied knot invariants (see [19] for references), including the Vassiliev invariants. In addition, the Chern–Simons measure is in a formal sense a solution of the Hamiltonian constraint of quantum gravity with a cosmological constant [20,16]. Moreover, the space is quite distinct from that of cylindrical functions in the sense that the Vassiliev invariants take non-trivial values on infinite sets of loops (or spin networks) whereas cylindrical functions take non-trivial values on finite sets of spin networks. Most of the literature on these invariants, however, has been focusing on the case of loops, whereas the context that is of interest for quantum gravity requires the use of spin networks. Part of the problems we had to confront were to suitably generalize the Vassiliev invariants to the spin network context. Partial progress along these lines, but with a different motivation, was achieved by Kauffman and Lins [21] and Witten and Martin [22], but we still need to elaborate on this to have an appropriate setting for doing quantum gravity.

Why might it be interesting to consider these states in addressing the difficulties we discussed that appear in relation to Thiemann's Hamiltonian? Essentially because in both the space considered by Thiemann and in the habitat of Lewandowski and Marolf, gauge invariant operators involving $\hat{F}_{ab}$ do not have a natural action. In spaces of these types, the usual way to represent such an operator would be to use the loop derivative, as is customarily in the non-diffeomorphism invariant Yang–Mills context. However, it was soon recognized that a conflict arises between the use of the loop derivative and the diffeomorphism invariance of the spaces in question. To put it naively, the action of the loop derivative "appends a loop of infinitesimal area", and in a diffeomorphism invariant context there is no meaning to "infinitesimal area" in terms of functions of loops or spin nets. In the limit involved in the definition of the loop derivative, the numerator is either zero or finite and the denominator goes to zero. This is what one expects when one is considering what is tantamount to a "derivative of a step function". The result should be a distribution. Finding the correct distribution that represents the operator is the main problem. In the new space we are proposing, it turns out that the loop derivative has a well defined distributional action, which can be derived from the definition of the invariants in terms of the Chern-Simons formulae.

Another advantage of the kind of states we are considering is that generically they have support on an infinite number of spin networks. The loop representation dual of cylindrical functions has support on a finite number of spin nets only. This might have physical consequences. For instance, if one considers the computation of the volume of a finite region, the result will generically vanish for the loop representation non-diffeomorphism-invariant duals of the cylindrical functions (unless the volume in question contains at least one four or higher valent vertex of the finite set of spin networks characterizing the state). For the Vassiliev invariants, the result of the action of the volume operator of a finite region is generically non-vanishing. Thiemann already notices this drawback of cylindrical functions in the $2+1$ context [13]. Essentially, using Vassiliev invariants is tantamount to considering infinite superpositions of cylindrical states. To give an intuitive (and imprecise) analogy, working with cylindrical functions is like working with Dirac deltas in one-dimensional calculus. Working with the Vassiliev invariants is more alike to working with ordinary smooth functions. We will see in the companion paper that these requirements seem to lead to fruitful developments at least in the context of $2+1$ dimensional gravity.

Having at hand a definition for the loop derivative in this context, one is in a much better position to achieve a more realistic quantum operator representing the Hamiltonian constraint. To illustrate this point, let us recall briefly Thiemann's construction. It starts from the classical Hamiltonian (3) which, introducing a simplicial decomposition in space Δ, can be written as,

$$H = \epsilon^{ijk} \mathrm{Tr}(h_{\alpha_{ij}} h^{-1}_{e_k(\Delta)} \{h_{e_k(\Delta)}, V\}). \tag{5}$$

where $h_{\alpha_{ij}}$ is a holonomy along one of the triangles of the tetrahedra of the simplicial decomposition and e_k are edges of the triangles. This expression obviously represents the Hamiltonian (3) in the limit in which the tetrahedra of the discretization become infinitesimal, the holonomy along the closed loop becoming F_{ab} times the tangent vectors of the edges of the triangle. This quantity is later promoted to a quantum operator, essentially by adding hats in all the quantities involved. The resulting operator acts in the space of diffeomorphism invariant cylindrical functions. An important point is that in this context there is no notion of shrinking the simplicial decomposition to infinitesimal size. In fact, this feature allows to bypass in the construction the evaluation of an ill defined limit (the one appearing in the loop derivative) and to simply consider a *finite* holonomy as representing F_{ab}. These steps are technically correct, but one can roughly expect that if one is representing F_{ab} by a finite holonomy instead of an infinitesimal one, one would encounter difficulties at the time of computing commutators. A finite holonomy differs from an infinitesimal one by an infinite number of higher order terms, that in general will have non-vanishing contributions to commutators.

B. Strategy

In this paper we will discuss two main topics: a) the generalization of the notion of Vassiliev invariant to the spin network context; this will give a concrete footing to the wavefunctions we will consider in the companion paper for the action of the Hamiltonian constraint; b) the definition of a loop derivative and an infinitesimal generator of diffeomorhpisms in this arena, which are key to defining the constraints introduced in the companion paper and to the computation of the constraint algebra.

To put this work in context, we should mention that in a previous paper [23] we discussed partially the generalization of Vassiliev invariants to spin networks for the case of trivalent intersections only. This paper expands and supersedes the discussion of the previous one. In particular we will discuss several points which included incorrect interpretations in our previous work. In it, we mostly concentrated on the definition of the expectation value of a Wilson loop based on a spin net in a Chern–Simons theory. We conjectured how one could obtain framing independent invariants assuming factorizability, and using the notion of primitive Vassiliev invariants. Further work showed that one cannot really assume factorizability for intersections of valence four and higher, and therefore a small set [1] of the results of our previous paper do not hold in general, but only for trivalent intersections. We clarify these issues in the current paper. In our previous work we also started analyzing the definition of a diffeomorphism constraint, and, as a warm-up for our current calculations, the definition of a doubly-densitized Hamiltonian constraint. However, the final result was dependent on the regularization and the additional fiducial structures it introduces, and therefore there was no chance that the operator introduced could have the correct commutator algebra. Nevertheless, the operator had some appealing properties, for instance it had the expected [16] solution with a cosmological constant (the suitable generalization of the Kauffman bracket), and annihilated all states based on trivalent intersections [23].

In this paper we will extend the definition of Vassiliev invariants to four-valent spin networks. The four valent case presents considerable technical challenges, since most literature on spin network invariants has heavily concentrated on trivalent vertices [24,22,21]. In particular the whole issue of constructing framing-independent spin network invariants starting from Chern–Simons theory has received virtually no attention (see [25] for first attempts). Notice that in principle these are the invariants of most interest for the gravitational case, where wavefunctions are supposed to be invariant under diffeomorphisms. The crucial need for results involving four valent intersections stems from the fact that the volume operator vanishes on states with trivalent intersections. We will also see, in the companion paper, that the Hamiltonian constraint vanishes on the framing independent Vassiliev invariants based on trivalent intersections (we will, however, be able to do calculations involving the Hamiltonian constraint on related types of wavefunctions on which the action is non-vanishing even for trivalent spin networks). We will also show that one can introduce ambient and regular isotopic invariants in this context, that they are loop differentiable, and that one can define a suitable diffeomorphism constraint (infinitesimal generator of diffeomorphisms) on them. We will also present explicit Feynman-diagrammatic expressions for the Vassiliev invariants and show that they are finite and well defined for spin networks. The latter definition has as a by-product that one can explicitly compute the loop derivative of these invariants without recurring to formal manipulations involving the path integral.

In the companion paper we will further elaborate by defining a single-densitized version of the Hamiltonian constraint in terms of the loop derivative we define here, we will present several non-diffeomorphism invariant "habitats" where the diffeomorphism constraint is non-vanishing and we will discuss in detail the constraint algebra, showing that no anomalies are present. We also discuss a first application of this construction in the $2 + 1$ dimensional context.

The organization of this paper is as follows. In the next section we will briefly introduce the notion of Vassiliev invariants in the context of ordinary loops. In section III we will discuss in detail the generalization of Vassiliev invariants to (up to four-valent) spin-networks and show that one can construct both ambient an regular isotopic invariants. In section IV we will analyze the loop differentiability of the invariants and the definition of an infinitesimal generator of diffeomorphisms on them (diffeomorphism constraint). We will summarize the results and conclusions in section V, and also we include an appendix where the action of the loop derivative over framing independent Vassiliev invariants is studied.

II. VASSILIEV INVARIANTS FOR LOOPS

A. Review of perturbative Chern-Simons theory for ordinary loops

We start by recalling several results we will need from ordinary loops in order to generalize them later to spin networks. It is well known that in the context of loops, the expectation value of a Wilson loop (trace of the holonomy) in a quantum Chern-Simons theory is related to Vassiliev invariants [26,27]. This expectation value can be evaluated in various ways. One possible strategy is to evaluate it perturbatively, taking advantage of the fact that Chern-Simons theories are perturbatively renormalizable [28]. In fact, it is in the perturbative context where Vassiliev invariants arise in the most natural way. Let us therefore consider the expectation value of a Wilson loop $W_A(\gamma) = \mathrm{Tr}(\mathrm{P}\exp \oint_\gamma dy^a A_a(y))$, where γ is a loop in three dimensions [2],

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[1]Specifically, formulas (18) and (30) do not hold.

[2]The constant κ is more commonly written as $\kappa = 4\pi i/k$, where k is usually taken as an integer for the expression to be invariant under large gauge transformations.

$$< W(\gamma) >= \int DAW_A(\gamma)e^{-\frac{1}{\kappa}S_{CS}(A)} \equiv E(\gamma,\kappa,G), \tag{6}$$

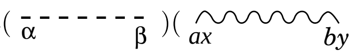
and,

$$S_{CS}(A) = \int d^3x \mathrm{Tr}(A \wedge \partial A + \frac{2}{3} A \wedge A \wedge A), \tag{7}$$

This construction defines a regular isotopic knot invariant [3] $E(\gamma,\kappa,G)$. This invariant is given by a power series in κ and depends on the gauge group G of the Chern–Simons theory in question. For $G = SU(2)$ the invariant that appears is (up to a normalization factor) the Kauffman bracket knot polynomial evaluated for a particular value of its variable $q = \exp(\kappa)$. One recovers the power series if one expands the polynomial in terms of κ. The coefficients of the power series expansion are linear combinations of Vassiliev knot invariants[4].

To evaluate the path integral perturbatively, one needs to fix a gauge and introduce ghosts. It has been established [29] that if one considers up to three-point functions, the contributions of the ghosts produce no radiative corrections. We will therefore ignore them here, since at the order of perturbation we will work they play no role. Explicit calculations of the Feynman diagrams up to higher orders including ghost contributions are available in the literature in the context of loops [29]. The first relevant Feynman diagram for this theory corresponds to the two point Green function,

$$< A^\alpha_a(x) A^\beta_b(y) >= \kappa(\delta^{\alpha\beta})(g_{ax\,by}) = \kappa(\quad)(\quad). \tag{8}$$

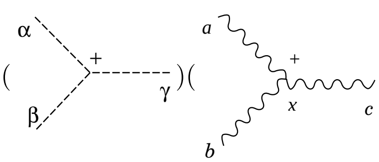


The wavy line represents the propagator,

$$g_{ax\,by} \equiv \frac{1}{4\pi}\epsilon_{abc}\frac{(x-y)^c}{|x-y|^3}, \tag{9}$$

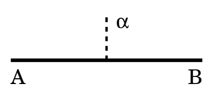
and the dotted line represents the invariant tensor $\delta^{\alpha\beta}$. We also have the vertex,

$$\frac{1}{\kappa}(if_{\alpha\beta\gamma})(-\epsilon^{abc}\int d^3x) = \frac{1}{\kappa}(\quad)(\quad), \tag{10}$$



The Lie algebra generators $T^{(R)}_\alpha$ (in a given representation R associated with the weight of the line going from A to B in the spin net) are represented by the following diagram,

$$(T^{(R)}_\alpha)^A_B = \quad , \tag{11}$$



and from now on we will drop the superscript (R) in the understanding that it is obvious that the T's are representation dependent. In order to evaluate the expectation value of the Wilson loop, we start by considering the expansion of the path-ordered exponential,

$$W_A(\gamma) = \mathrm{Tr}\left[1 + \oint_\gamma dx^a A_a(x) + \oint_\gamma dx^a \int_o^x dy^b A_b(y)A_a(x) \right.$$
$$\left. + \oint_\gamma dx^a \int_o^x dy^b \int_o^y dz^c A_c(z)A_b(y)A_a(x) + \cdots \right]. \tag{12}$$

If one takes the expectation value of the left hand side, one obtains in the right hand side a power series in terms of κ. To zeroth order one gets,

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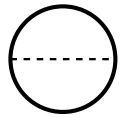
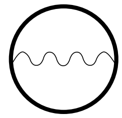
[3]We are using the following normalization for the Lie algebra generators $A_a(x) = A^\alpha_{ax}T^\alpha$: $\mathrm{Tr}[T^\alpha T^\beta] = \delta^{\alpha\beta}/2$, and $[T^\alpha, T^\beta] = if^{\alpha\beta\gamma}T^\gamma$ with $f^{\alpha\beta\gamma}$ the structure constants of the group G.

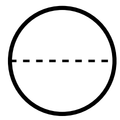
[4]Originally, Vassiliev invariants were only considered for the ambient isotopic case, a straightforward generalization can be done for regular isotopy (framing dependent invariants).

$$< W(\gamma) >^{(0)}= \dim R_G, \tag{13}$$

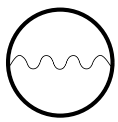
which is the dimension of the representation of the group G we are considering. To first order in κ we get,

$< W(\gamma) >^{(1)}=$   $\equiv r_{11}\alpha^{11}(\gamma)\,,$ (14)

where the bold face line represents the Wilson loop γ, and,

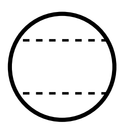
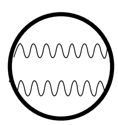
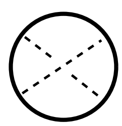
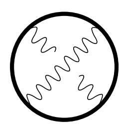
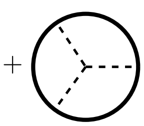
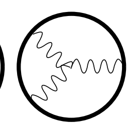
 $= \mathrm{Tr}[T_\alpha T_\beta]\delta^{\alpha\beta},$ (15)

and,

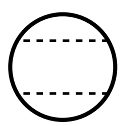
 $= \oint_\gamma dx^a \int_o^x dy^b\, g_{ab}(x,y) = \frac{1}{8\pi}\oint_\gamma dx^a \oint_\gamma dy^b \epsilon_{abc}\frac{(x-y)^c}{|x-y|^3} \equiv \varphi(\gamma).$ (16)

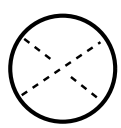
The first order contribution factorizes in the product of two factors[5]: r_{11}, which is group and representation dependent, and $\alpha^{11}(\gamma)$, which depends only on integrals of the propagator along the Wilson line. They are called respectively the "group" and the "geometric" factors. The group factors are given in general by the trace of a product of Lie algebra generators contracted with the invariant tensors of the group. They can be thought of as the insertion at points in the Wilson line of the Lie algebra generators, and the evaluation of the resulting net for a flat connection. This diagrammatic representation of the group factors is usually called a "chord diagram". The geometric factor $\alpha^{11}(\gamma)$ corresponds to the self-linking number $\varphi(\gamma)$ of the loop. The factorization in terms of a group and a geometric factor is a general property of the perturbative expansion of $< W(\gamma) >$.

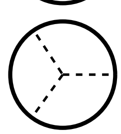
For second order in κ we have the following contributions,

$< W(\gamma) >^{(2)}=$   $+$   $+$   , (17)

with,

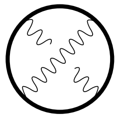
 $= \mathrm{Tr}[T_\alpha T^\alpha T_\beta T^\beta] \equiv r_{21},$ (18)

 $= \mathrm{Tr}[T_\alpha T_\beta T^\alpha T^\beta],$ (19)

 $= \mathrm{Tr}[T_\alpha T_\beta T_\gamma] i f^{\alpha\beta\gamma} \equiv r_{22}.$ (20)

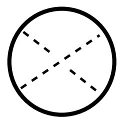
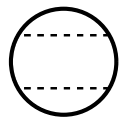
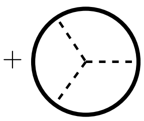
The analytic expression of the geometric factors are obtained directly from the diagrammatic representation. For example,

[5]We use a double index notation because at a given order there usually appear more than one factor. The first index denotes the order in the expansion in κ, the second denotes which factor within a given order.

 $= \frac{1}{4}[\oint_\gamma dx^a \int_o^x du^b \int_o^u dy^c \int_o^y dw^d g_{ac}(x,y) g_{bd}(u,w) + \text{c.p. }],$ (21)

where *c.p.* means to consider terms obtained by cyclic permutations of the first term in the indices $(ax)(bu)(cy)(dw)$.

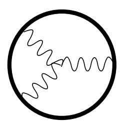
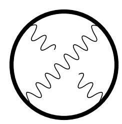
The three chord diagrams included in (17) are not all independent. They are related by the so called "STU" relations [26],

 $=$  $+$  , (22)

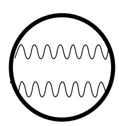
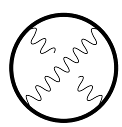
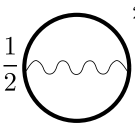
which stems from the commutation relation of the group generators. These relations allow to eliminate all overlapping diagram in terms of a combination of non-overlapping diagrams. Proceeding in this way one can define a canonical basis for the group factors [30]. In terms of this basis, the second order contribution can be written in the following way,

$$< W(\gamma) >^{(2)} = r_{21} \frac{1}{2} [\alpha^{11}(\gamma)]^2 + r_{22} \alpha^{22}(\gamma), \tag{23}$$

where

$\alpha^{22}(\gamma) \equiv$  $+$  . (24)

In the above result two things should be noticed. The first is that, via the application of the STU relations, the geometric factors associated with the independent group factors will include in general a contribution coming from the overlapping diagrams. In particular, the combination of geometric factors associated with r_{21} factorizes in a product of first order contributions,

$\alpha^{21}(\gamma) \equiv$  $+$  $= \frac{1}{2}$  ^{2} (25)

Moreover, applying recoupling [31] identities [6] one can show that

$$r_{21} = \frac{1}{< W(\gamma) >^{(0)}} [r_{11}]^2, \tag{26}$$

so the first term in (23) equals the square of the first order contribution $[W(\gamma) >^{(1)}]^2 / 2 < W(\gamma) >^{(0)}$.

The second observation is that, as r_{ij} depend on the group and representation chosen, this implies that the geometric factors $\alpha^{ij}(\gamma)$ embody the invariance under diffeomorphisms individually. That is, each $\alpha^{ij}(\gamma)$ is an independent knot invariant. We can also reach some conclusions concerning the nature of the constructed invariants. In principle, the integrals involving wavy lines have to be regularized to avoid having propagators with two of their ends evaluated at the same point [28]. In the context of knot theory this regularization corresponds to a framing of the knot. A possible framing consists, for instance, in prescribing a copy of the loop infinitesimally displaced away from the original one along a vector field on the manifold, and evaluating the various integrals along the separate loops. This procedure is not needed for all invariants. For instance, if we in detail look at $\alpha^{22}(\gamma)$ we notice that in the evaluation of the integrals involved, the two ends of each propagator can never coincide. In the first term, one of the ends is always at the vertex, in the second term the ordering of the integrals prevents the ends from coinciding, if they do, one of the accompanying integrals is evaluated along a loop of zero length and vanishes. We refer to this case as saying that there

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[6]For general results about the application of recoupling theory to the factorization property of group factors, see equations (44) and (55) below.

are no “collapsible” propagators. The invariants that do not require the framing procedure are true diffeomorphism invariants of loops, and are called in the knot theory language *ambient* isotopic invariants. The invariants that require of the framing procedure are invariants of “ribbons” (instead of loops), and are called *regular* isotopic invariants.

The explicit expressions for the higher order contributions are progressively more complicated as κ increases. However, at any order one can apply essentially the same procedure we discussed above to isolate the independent group factors, and the geometric factors associated with them. The expectation value can therefore be generically written as,

$$< W(\gamma) >= \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} r_{ij} \alpha^{ij}(\gamma) \kappa^i \,, \tag{27}$$

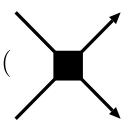
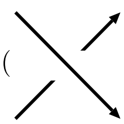
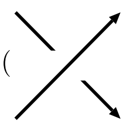
where d_i is the number of independent invariants at order κ^i. The invariant associated with $< W(\gamma) >$ is a regular isotopic invariant, that is, it is framing dependent. One can extract the framing dependence in terms of an overall phase factor [32], and be left with a framing independent invariant $J(\gamma)$ defined by,

$$\exp\left(-\kappa \frac{< W(\gamma) >^{(1)}}{< W(\gamma) >^{(0)}}\right) < W(\gamma) >= J(\gamma). \tag{28}$$

For the case of $SU(2)$, $J(\gamma)$ is the Jones polynomial for a particular value of its variable $q = \exp(\kappa)$ and expanded as a power series in terms of κ. We will return to this issue in terms of spin networks later.

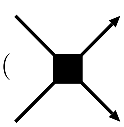
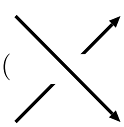
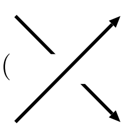
B. Vassiliev invariants

Let us now show that the invariants that appear at order κ^i in the perturbative evaluation of the expectation value of the Wilson loop are Vassiliev invariants of order i. This is a well known result in the case of ordinary loops (see for instance [27]). To do this, we recall the definition of Vassiliev invariants [33]. Given a knot invariant $v^n(\gamma)$, one starts by introducing a “Vassiliev” intersection (which we will illustrate with a black square in diagrams), defined as,

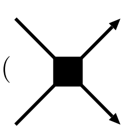
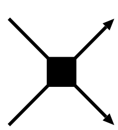
$v^n($  $) \equiv v^n($  $) - v^n($  $).$ (29)

The invariant $v^n(\gamma)$ is called *Vassiliev invariant of order n*, if and only if when evaluated with any knot with $n+1$ Vassiliev intersections, it vanishes. One can immediately see that it also vanishes for all knots containing more than $n+1$ intersections. This is the reason the invariants are sometimes called “of finite type”. This is also related to the observation that the Vassiliev intersection can be viewed as the action of a differential operator. We will see later on that this abstract differential operation ends up remarkably being link to the loop derivative.

To see that the invariants we constructed via perturbation theory are Vassiliev invariants, we need certain properties of the path integral that can be derived by variational techniques [34,35,16,27,36]. For the sake of brevity we will not repeat the whole argument here. We will just recall that one can convert an upper in an under crossing in the expectation value by considering the action of the loop derivative (which appends a small loop). If one does this, one derives the following identity,

$< W($  $) >\equiv< W($  $) > - < W($  $) >= \kappa < h(\gamma_1) T_\alpha h(\gamma_2) T^\alpha > + O(\kappa^2)$ (30)

The meaning of this equation is as follows: any crossing divides a loop γ into two subloops, γ_1 and γ_2, according the connectivity of the loop. The invariant of the right is constructed replacing the holonomy along the whole loop by a holonomy divided into the subloops with insertions of the Lie algebra generators. If one now considers n Vassiliev intersections, one can repeat the above construction for each intersection, i.e.,

$< W($  $, \overset{n\ times}{\cdots},$  $) >= \kappa^n < h(\gamma_1) T_{\alpha_1} h(\gamma_2) T_{\alpha_2} \cdots h(\gamma_k) T^{\alpha_1} h(\gamma_{k+1}) \ldots > + O(\kappa^{n+1}),$ (31)

where now $\gamma_1, \gamma_2, \ldots$ are the arcs that join one crossing with the following. The generators are inserted in each crossing starting from the origin. Notice that each crossing is traversed twice, and that the two generators associated with a given crossing are contracted between themselves. We see that the left hand side of the above equation has vanishing coefficients up to order n, that is, the coefficients of the expansion up to order n behave like Vassiliev invariants of order equal or less than n. A similar behavior is observed for the case of spin networks.

III. VASSILIEV INVARIANTS FOR SPIN NETWORKS

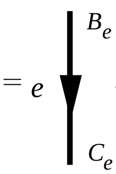
A. Diagrammatic notation for the expectation value of a Wilson net

A spin network [37] is constructed considering a graph Γ (with multivalent intersections) embedded in a three dimensional manifold. The graph is composed by a set of edges $\{e_j\}$, $(j = 1, \ldots, N(\Gamma))$, which are analytic oriented open paths that intersect only in the initial and final points. The intersection point of three (or more) edges defines a tri (or multivalent) vertex v. We denote $N(\Gamma)$ the number of edges, and $V(\Gamma)$ the number of vertices of the graph. Associated with each edge e_i in the graph is a representation labelled by J_i of an element of the gauge group G. For instance, given a connection, one can construct these elements by considering the holonomy along the given edge in a given representation. The group elements are "intertwined" into gauge invariant objects through contraction at the vertices of the network with the invariant tensors of the group. Given a graph Γ and an assignment $\vec{J} = (J_1, \ldots, J_{N(\Gamma)})$ of representations to each edge plus a set of intertwiners $\vec{I} = (I_1, \ldots, I_{V(\Gamma)})$, one can construct starting from a connection an object we call "Wilson net", $W_A(s) \equiv W_A(\Gamma, \vec{I}, \vec{J})$, in the following way,

$$W_A(s) \equiv \left[\prod_v I_v{}^{C_1 \ldots C_{n_v}}_{B_1 \ldots B_{m_v}} \right] \left[\prod_e U(e)^{B_e}_{C_e} \right]. \tag{32}$$

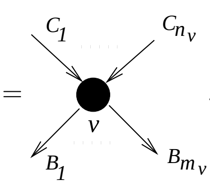
We will use the compact notation s denoting the graph, intertwiners and spin weights together. In the above expression, all the matrix indices of the holonomies $U(e)$ are contracted with the indices of the intertwiners I_v in a way which depends on the connectivity of the net. The edges of the net are oriented according to the following conventions,

$$U(e)^{B_e}_{C_e} = \quad , \tag{33}$$



and,

$$I_v{}^{C_1 \ldots C_{n_v}}_{B_1 \ldots B_{m_v}} = \quad . \tag{34}$$



The invariant tensor $I_v{}^{C_1 \ldots C_{n_v}}_{B_1 \ldots B_{m_v}}$ can be expressed in a basis of $n_v + m_v - 2$ Clebsch-Gordan intertwiners[7]. This construction is feasible for any group, but we will concentrate in the following sections on $SU(2)$ and the representations will be labelled by integers. A more elaborate labeling is needed for other groups. Spin networks were originally introduced in the $SU(2)$ context by Penrose [38].

We now wish to compute the expectation value of a Wilson net in a Chern-Simons theory,

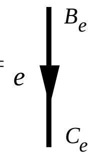
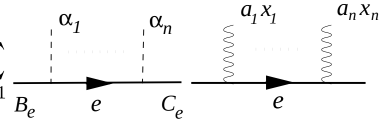
$$< W(s) >= \int DAW_A(s) e^{-\frac{1}{\kappa} S_{CS}(A)} \equiv E(s, \kappa). \tag{35}$$

---

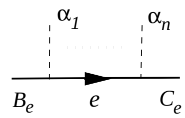
[7]In general, a $n+m$-valent intersection can be thought of as the limit where the intermediate lines are of vanishing length, of $n+m-2$ trivalent intersections connected by additional links, in a choice of basis such that the intermediate links are assigned a definite spin value.

This integral has been studied by Witten and Martin [22] and Kauffman and Lins [21], establishing the crossing relations (skein relations) and recoupling relations for networks including up to trivalent intersections. Their calculations were based on the use of monodromies and the tangle group, respectively. For trivalent vertices we have also shown in a previous paper [25] that one can extract the framing factor and construct ambient isotopic invariants.

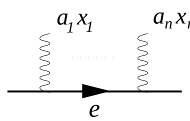
Here we would like to study if analogous results can be found in the case of four-valent intersections, and also perform perturbative studies of the path integral with the aim of constructing explicitly ambient invariants. We therefore proceed as in the case of loops, to consider order by order the perturbative evaluation of expression (35). The general form of the perturbative expansion can be easily deduced from the definition (32). We first introduce a convenient diagrammatic notation. The holonomy associated with a generic edge e is written in the following form, introducing a diagrammatic notation for the explicit expression of the path ordered exponential in terms of the connection,

$$U(e)^{B_e}_{C_e} =$$  $$= \delta^{B_e}_{C_e} + \sum_{n=1}^{\infty}$$  $$A^{\alpha_1}_{a_1 x_1} \dots A^{\alpha_n}_{a_n x_n}, \qquad (36)$$

where,

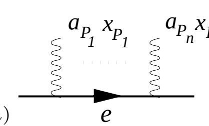
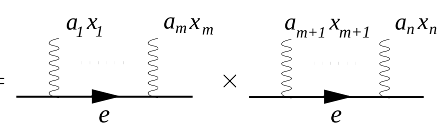
 $$= (T^{(J_e)}_{\alpha_1} \dots T^{(J_e)}_{\alpha_n})^{B_e}_{C_e}, \qquad (37)$$

and,

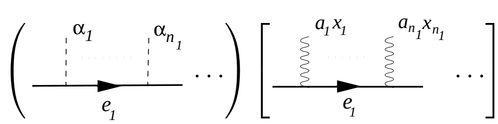
 $$= \int_e dy^{a_n}_n \int_o^{y_n} dy^{a_{n-1}}_{n-1} \dots \int_o^{y_2} dy^{a_1}_1 \delta(x_n - y_n) \dots \delta(x_1 - y_1). \qquad (38)$$

For the readers familiar with the extended loop formalism ([17] chapter 2), the above multitangent corresponds to $X(e)^{a_1 x_1 \cdots a_n x_n}$ in such a notation. In the above expressions, we have adopted the usual "generalized Einstein convention" of extended loops, in which a summation is implied by any repeated index a_i and an integral over the three manifold is implied by any repeated index x_i.

Some general results about the factorization of the geometric factors (see equation (25)) can be derived from the following property of these fields,

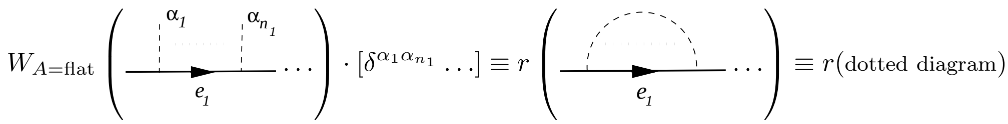
$$\sum_{P(1\dots m, m+1 \dots n)}$$  $$=$$  $$, \qquad (39)$$

where $P(1 \dots m, m+1 \dots n)$ is any permutation of the indices which preserves the ordering of the subsets $(1 \dots m)$ and $(m+1 \dots n)$ between themselves. This equation express the "algebraic constraint" of the multitangent fields, first studied in [39]. Introducing (36) in (32), we get the following diagrammatic expression for the expectation value of the Wilson net,

$$< W_A(s) > = \sum_{n_1=0}^{\infty} \dots \sum_{n_{N(\Gamma)}=0}^{\infty} W_{A=\text{flat}}$$  $$\times$$
$$\times < (A^{\alpha_1}_{a_1 x_1} \dots A^{\alpha_{n_1}}_{a_{n_1} x_{n_1}}) \dots >, \qquad (40)$$

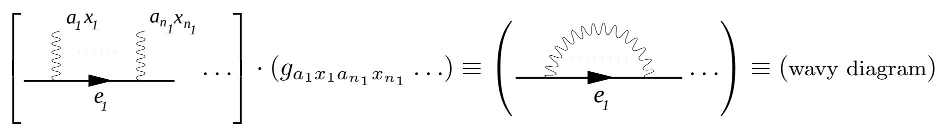
Notice that in the above equation one takes a sum for each edge of the net. In general, $W_{A=\text{flat}}(s)$ defines the "chromatic evaluation" of the spin network s. This quantity is evaluated replacing in the functional integral all the holonomies by identity matrices, $U(e)^{B_e}_{A_e} \to \delta^{B_e}_{A_e}$ for all e. The resulting expression can be computed using recoupling theory. Then, in (40) $W_{A=\text{flat}}($ $\dots)$ has the following meaning: take the chromatic evaluation of the spin network replacing the holonomy $U(e_i)$ by a product of Lie algebra generators $T_{\alpha_1} \dots T_{\alpha_{n_i}}$ in the representation J_i if $n_i \neq 0$, or by the identity matrix if $n_i = 0$.

At this point it is that the convenience of the notation we introduced becomes clear, we see in equation (40) that the expression for the expectation value of the Wilson net becomes really compact, involving a chromatic evaluation of a diagram times another diagram contracted with the propagators. This is quite analogous to what happened in the case of ordinary Wilson loops. To generate from (40) the perturbative series one has to introduce the Feynman rules to express the Green functions $< (A^{\alpha_1}_{a_1 x_1} \ldots A^{\alpha_{n_1}}_{a_{n_1} x_{n_1}}) \ldots >$ in terms of wavy lines (the propagator and the vertex)[8] and dotted lines (the invariant tensors of the group). We immediately see that a mechanism similar to that observed in the case of loops operates here: the contraction of the invariant tensors with $W_{A=\text{flat}}(\ldots)$ generates dotted diagrams (the group factors), whereas the contraction of the two and three point propagators with the multitangent fields generates wavy diagrams (the geometric factors). We schematically represent this procedure in the following way,



(41)

and,



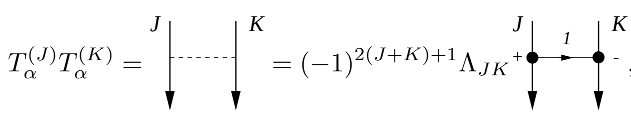
(42)

These relations define the group and geometric factors for spin networks. Notice that in the above examples we have chosen a specific way to connect the indices α_1, α_{n_1} and $a_1 x_1, a_{n_1} x_{n_1}$. In general, the application of the Feynman rules produce, for a given number of indices, a superposition of diagrams with different connectivities of the dotted (or wavy) lines. Therefore, we can schematically write (ignoring the ghosts as we discussed before),

$$< W(s) >= \sum_{m=0}^{\infty} \sum_{\{\text{diagrams}\}} r[(\text{dotted diagrams})^{(m)}](\text{wavy diagrams})^{(m)}\, \kappa^m, \tag{43}$$

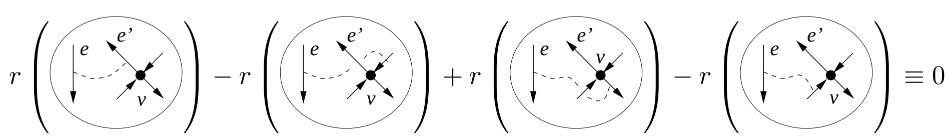
The index m gives the order of the perturbative expansion, and for a fixed m we sum over all the diagrams that contributes to this order. Each term factorizes in the product of a dotted diagram (the group factor) and a wavy diagram (the geometric factor), both having the same connectivity (i.e., in both diagrams the connection of the dotted and wavy lines are identical).

In principle, the group factors can be evaluated using recoupling theory. For $SU(2)$ one uses the following version of the Fierz identity (the standard identity between the Lie algebra generators and the Clebsch–Gordan coefficients in the fundamental representation),



(44)

where $\Lambda_{JK} = \Lambda_J \Lambda_K$ and $\Lambda_J = \sqrt{J(J+1)(2J+1)}$. The Fierz identity allows to rewrite any dotted diagram in terms of the chromatic evaluation of a new spin network where the dotted lines are substituted by edges of spin one. In general, the group factors satisfy some identities which are characteristic of spin networks and which play an important role at the time to identify the topological invariants associated with the perturbative expansion. To introduce these identities, let us consider a generic dotted diagram and let us isolate from it a single dotted line connecting two edges e and e'. We draw this single line enclosed into a circle. We also draw one of the vertices of the selected edges, for example, the origin of e', which we assume to be four-valent. The dotted lines outside the circle remain fixed as we move the generator starting from e' all around the vertex. Then it is easy to prove that,

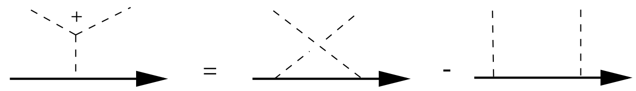


(45)

[8] As in the case of loops, we will ignore the ghosts. It one wishes to include them, one has to incorporate in the analysis the ghost's propagator and the ghost's vertex.

where r is any of the group factors we have considered (we omit indices since the expression is valid at any order). For a given diagram, one can perform these movements with all the dotted lines of the graphs producing a linear system of equations for the group factors (of course, not all the possible movements generate independent equations). This set of identities is valid for any group, and they are rooted in the gauge invariance property of the Wilson net. For this reason, we call these identities the Gauss' law for spin networks.

Another set of identities, called "STU relations" arise from the generator algebra,



(46)

The STU identities, together with Gauss' law, form a set of linear identities that can be solved order by order in the perturbative expansion, allowing to find all group factors in terms of a reduced set of "independent factors".

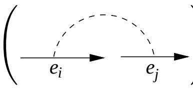
The zeroth order contribution of (43) is given by the chromatic evaluation of the net, which we usually denote in the form,

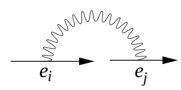
$$< W(s) >^{(0)} = W_{A=\text{flat}}(s) \equiv E(s,0) \tag{47}$$

In the following subsections we are going to study the detailed structure of the group and geometric factors for the first and second order in κ.

B. The first order contribution

Considering equation (43) to first order in κ we have,

$$< W(s) >^{(1)} = \sum_i \sum_{j\leq i} r \left(\right.$$


$$\left. \right)$$


$$\equiv \sum_i \sum_{j\leq i} r_{ij}\varphi_{ij}, \tag{48}$$

where by i and j we denote any pair of edges of the spin network (they can also be the same edge). We usually use the compact notation r_{ij} for the first order group factors, and φ_{ij} for a wavy diagram connecting the edges e_i and e_j.

We notice a difference with the case of loops. In that case there exists only one group factor, r^{11}_G, at first order. Here we have a number of first order group factors equals to $N(\Gamma)(N(\Gamma)+1)/2$. These quantities are not all independent due to the Gauss constraints (45). The Gauss identities generate for a given graph a linear system of equations which allows to express some of the r_{ij} in terms of a small subset, which we call the set of "independent" group factors (IGF). The determination of the IGF allows to construct the topological invariants associated with the spin network. As in the case of loops, the linear combinations of geometric factors that multiply each group factor are (regular) knot invariants.

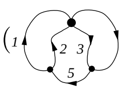
It is clear that the IGF set is not unique (we can choose different free parameters to express the solutions of the Gauss identities).

Let IGF $= \{z_i\}$, $i = 1\ldots d(s)$, be the free parameters with respect to which the linear system of Gauss equations is solved. The number $d(s)$ of parameters depend on the particular spin network being considered. Then, (48) can be rewritten in the following way,

$$< W(s) >^{(1)} = \sum_{i=1}^{D(s)} z_i \mathcal{I}_i(\varphi) \tag{49}$$

where the $\mathcal{I}(\varphi)$'s represent knot invariants formed by the linear combination of geometric factors associated with each independent parameter.

Let us illustrate the procedure with an example. For the net shown in figure 1, $d(s) = 6$ and we choose IGF $= \{r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}\}$. Then one gets

$$< W($$


$$) >^{(1)} = -r_{12}(\varphi_{11}+\varphi_{22}-\varphi_{12}) - r_{34}(\varphi_{33}+\varphi_{44}-\varphi_{34})$$
$$+ \sum_{i=1}^{2}\sum_{j=3}^{4} r_{ij}(\varphi_{ii}+\varphi_{jj}-\varphi_{55}+\varphi_{i5}+\varphi_{j5}+\varphi_{ij}). \tag{50}$$

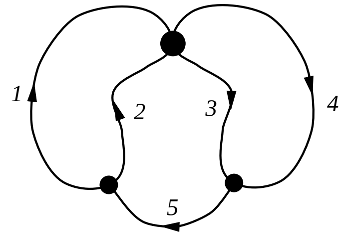
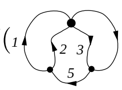


FIG. 1. A spin net with a four-valent and a trivalent vertices.

The invariants have a simple geometric interpretation in terms of linking numbers of loops constructed with the edges of the graph. It is immediate to see that each of the above linear superposition of geometric factors combine to give the self-linking number of a closed path γ with support in the net,

$$< W(\text{[diagram]}) >^{(1)} = -r_{12}\varphi(\gamma_{12}) - r_{34}\varphi(\gamma_{34}) + \sum_{i=1}^{2}\sum_{j=3}^{4} r_{ij}\varphi(\gamma_{ij5}), \tag{51}$$



with[9],

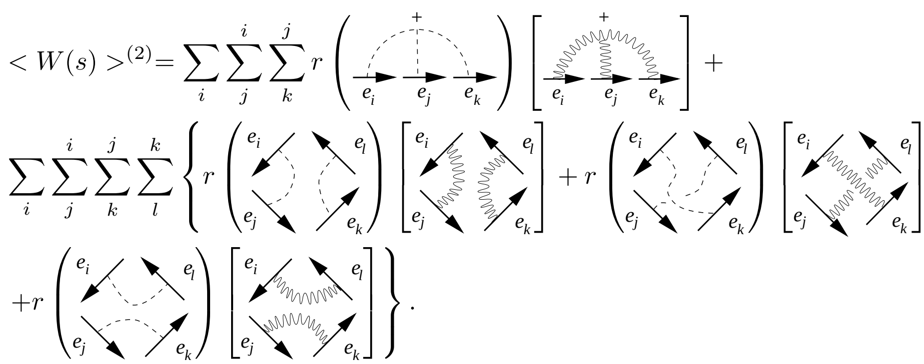
$$\gamma_{ij} \equiv e_i \circ \bar{e}_j \tag{52}$$

$$\gamma_{ij5} \equiv e_i \circ e_j \circ e_5 \tag{53}$$

For any other IGF, the invariants are written as a combination of the above self-linking numbers. Notice that in this example all the first order invariants are framing dependent.

C. The second order contribution

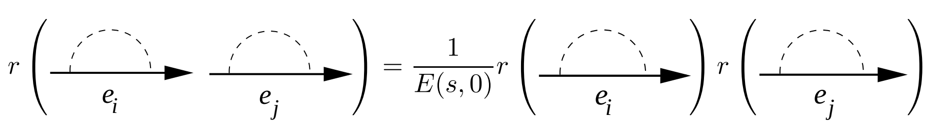
Considering the contribution to second order in κ in (43) we have,



(54)

Guided by the case of ordinary loops, we would like to study if it is possible to extract a pre-factor that is dependent on framing, therefore isolating a framing-independent knot invariant in this expression. In the case of ordinary loops the exponential of the first order contribution was the prefactor that allowed to isolate the Jones polynomial. Here we would like to check if this is possible for spin nets up to second order in the expansion, for four valent intersections. We will see that, as we independently proved in our previous paper, it is possible to extract the prefactor for trivalent vertices, but it is not possible for four and higher valent intersections. This was already anticipated in the work of Labastida, who noted that it does not hold for multiloops, which are a particular case of spin networks.

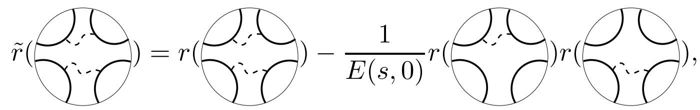
Using the Gauss and STU identities and the algebraic constraint we discussed before, and observing that,



(55)

[9]Overbars denote edges traversed in opposite directions, and the $\circ$ operation is the usual composition law of open paths.

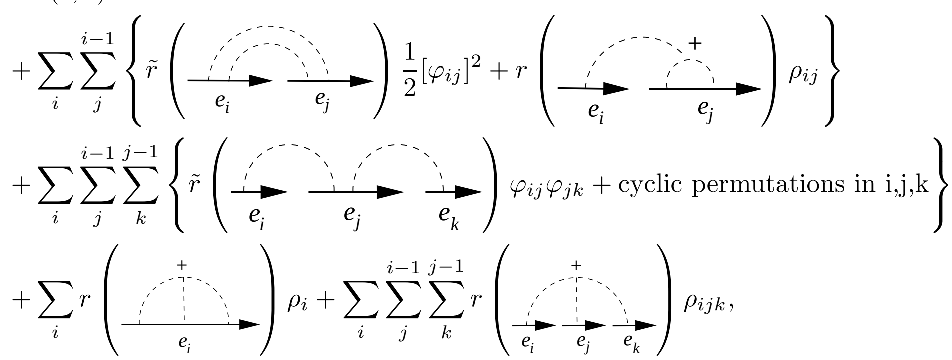
and introducing the "shifted" four-point group factors $\tilde{r}$, defined in the following way,



(56)

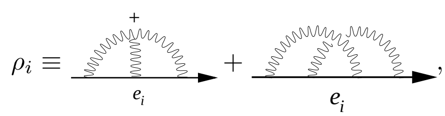
it is found that,

$$< W(s) >^{(2)} = \frac{1}{2E(s,0)}[< W(s) >^{(1)}]^2$$

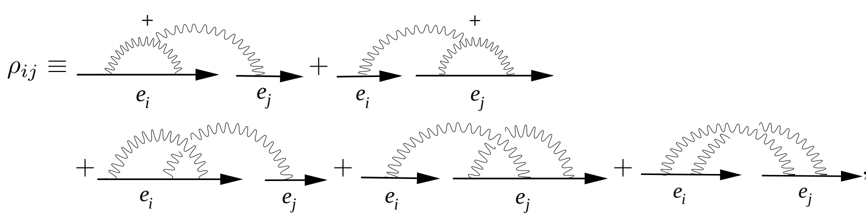


(57)

The quantities φ_{ij} are the first order geometric factors linking the edges e_i and e_j, and,

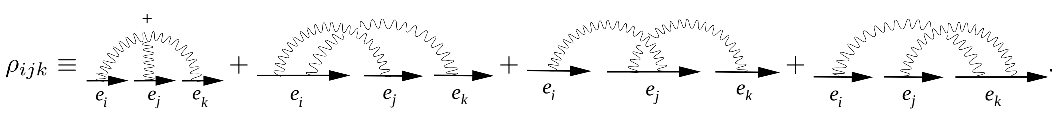


(58)



(59)

and,



(60)

Looking at equation (57) we see that we have not achieved the desired result. If we look at the first term in the right hand side of this equation we identify the expansion to second order of the exponential of the first order correction, exactly like in the case of loops. However, there remain framing dependent contributions in the other terms, therefore one will not be able to extract the framing dependence in an overall prefactor, as happened in the case of loops. The additional framing dependence arises for instance in the terms involving φ_{ij} or ρ_{ijk} when the edges i, j, k share a common vertex.

A remarkable exception happens in the case of trivalent intersections. In this case the group factors r that appear in the above mentioned terms vanish identically. This can be seen straightforwardly applying recoupling at the vertex. In the case of trivalent vertices this yields only a prefactor and the two terms involved in the definition of the $\tilde{r}$, equation (56) vanish. In the case of higher order vertices, the application of recoupling leads to a sum of terms and therefore the two terms in (56) are different. This fact will also appear at the root of our proof that the loop derivative annihilates trivalent framing independent invariants, as we will discuss later on.

D. Topological invariants for spin networks

The analysis of the higher order contributions can be performed using the same techniques we have developed at first and second order. The results can be summarized as follows. To any order in the perturbative expansion, there exist a number of independent group factors for a given group G, $z_a^{(n)}(\vec{J},\vec{I},G)$, based on solving the Gauss and STU relations. In terms of these the perturbative expansion can be written as,

$$< W >^n = \sum_{a=1}^{d^n(s)} z_a^{(n)}(\vec{J},\vec{I},G)\mathcal{I}_a^{(n)}(\Gamma) \tag{61}$$

where the $\mathcal{I}_a^{(n)}$ are (regular or ambient) isotopic invariants of the embedding of the spin network Γ, and $d^n(s)$ is the number of independent group factors at $n-th$ order for a given spin network s. This is a difference with the case of ordinary loops, where the expressions are given once and for all loops. This is related to the fact that the spin nets are a much more general object.

The regular or ambient isotopy properties of the invariants can be analyzed systematically, and, to any order in κ, the invariants are grouped in two categories: framing independent and framing dependent. The framing dependent part cannot be exponentiated like in the case of loops. The general result can be written in the following way,

$$< W(s) > = \sum_{i=0}^{\infty}\sum_{a=1}^{d_i^{\rm fd}(s)} z_{ia}^{\rm fd}(\vec{J},\vec{I},G)\,\mathcal{I}_{ia}^{\rm fd}(\Gamma)\,\kappa^i + \sum_{i=0}^{\infty}\sum_{b=1}^{d_i^{\rm fi}(s)} z_{ib}^{\rm fi}(\vec{J},\vec{I},G)\,\mathcal{I}_{ib}^{\rm fi}(\Gamma)\,\kappa^i. \tag{62}$$

In spite to this dependence with the particular spin network considered, the invariants have the following property: given an embedding Γ, let $\Gamma' \in \Gamma$ be a subgraph obtained by removing some of the edges of Γ. Then, the invariants $\mathcal{I}(\Gamma')$ are obtained from $\mathcal{I}(\Gamma)$ putting zero all the wavy diagrams with legs attached to any of the suppressed edges.

The specific expression of the invariants $\mathcal{I}_{fd}$ and $\mathcal{I}_{fi}$ that appear in expression (62) and their quantity $d_n^{fi}(s)$, depend on the choice of independent invariants picked to solve the constraints among the group factors. We will give a procedure that generates, for a given spin network, a basis of independent invariants that are framing-independent.

A general invariant of $n-th$ order is in general given by a linear combination,

$$\mathcal{I}^{(n)}(\Gamma) = \sum_{a=1}^{d^n(s)} c_a \mathcal{I}_a^{(n)}(\Gamma), \tag{63}$$

where the $\mathcal{I}_a$'s have the following expression,

$$\mathcal{I}_a^{(n)}(\Gamma) = \sum_i D_{ia}^{\rm fi}\alpha_i^{(n){\rm fi}}(\Gamma) + \sum_i D_{ia}^{\rm fd}\alpha_i^{(n){\rm fd}}(\Gamma), \tag{64}$$

where the $D_{ia}^{\rm fd,fi}$ are well defined numbers that depend on the connectivity of the spin net and the α's are products of multitangents contracted with the geometric portion of the Feynman diagrams (propagators and vertices) and can either be framing dependent or independent. We can therefore conclude that an invariant will be framing independent if the choice of coefficient c_a's is such that,

$$\sum_{a=1}^{d^n(s)} c_a D_{ja}^{\rm fd} = 0. \tag{65}$$

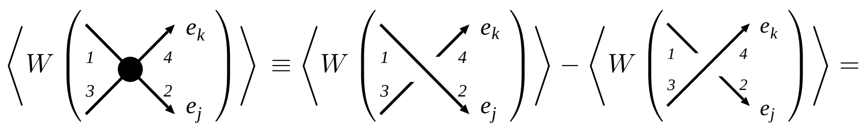
One now solves this system of equations, which will lead to expressing some of the c_a's in terms of a set of independent c_a's. For each of these independent c_a's one has an independent invariant.

Let us consider a concrete example of this procedure. We will carry out the computation for the spin network of the figure (1). The invariants that appear at second order in the calculation have the generic form,

$$\mathcal{I}^{(2)}(\Gamma) = c_1\rho(\gamma_{12}) + c_2\rho(\gamma_{31}) + \sum_{i=1}^{2}\sum_{j=i}^{4} c_{ij}\rho(\gamma_{ij5}), \tag{66}$$

where γ_{ij} are the subloops of the spin network we defined in equations (52,53). The quantities ρ are closely related with the second coefficient of the Alexander-Conway polynomial [28]. These were the expressions one directly got when one performed the calculation in the case of loops. In the case of spin networks one gets combinations of these invariants evaluated in sub-loops that form the spin network. The expression is manifestly framing independent since the coefficients of the Alexander-Conway polynomial are framing-independent. We see that (at least up to this order in the perturbative expansion) the use of spin networks does not introduce new topological invariants, but just combinations of invariants evaluated in the various subloops that form the spin network.

Finally, we show that the invariants we have obtained are the natural generalization to the spin network context of Vassiliev invariants. To see this, consider an arbitrary net, and construct the Vassiliev intersection. Following the same type of argument as we did in the case of ordinary loops one immediately obtains, using variational techniques,



$$= \kappa < \ldots h^{(J_j)}(e_{1j}) T_\alpha^{(J_j)} h^{(J_j)}(e_{2j}) \ldots h^{(J_k)}(e_{3k}) T_\alpha^{(J_k)} h^{(J_k)}(e_{4k}) \ldots > \tag{67}$$

where the indices in parenthesis refer to the representation in which one is considering the holonomies and the Lie algebra basis elements. We therefore reach the same conclusion as in the case of loops: an invariant with n Vassiliev crossings is of order κ^n meaning that the first $n-1$ coefficients in the series expansion vanish for spin networks with n crossings. Therefore, they are Vassiliev invariants of order up to $n-1$.

IV. THE DIFFEOMORPHISM CONSTRAINT

A. Loop differentiability

We now have succeeded in setting a suitable "arena" where we can start discussing the constraints of quantum gravity. The arena will be the space of Vassiliev invariants, and in order to implement the constraints as quantum operators we will take advantage of the fact that these invariants are loop differentiable. This is a result we introduced in a previous paper [23], we only recall the appropriate formula here. The result appears simply by applying the definition of loop derivative [17] to the path integral defining the expectation value of the Wilson loop. One acts with the loop derivative on the Wilson net in the path integral. One obtains as a result a Wilson net with an F_{ab} inserted at the point where the loop derivative acted. One then re-expresses the F_{ab} in terms of a functional derivative of the exponential of the Chern–Simons form with respect to the connection. This functional derivative is integrated by parts to act on the Wilson loop again. The result is the expectation value of a Wilson net with the insertion of two Lie algebra generators,

$$\Delta_{ab}(\pi_o^x) E(\overset{e_j}{\downarrow} o, \kappa) = -2\kappa \sum_k \epsilon_{abc} \int_{e_k} dy^c \delta^3(x-y) \times \tag{68}$$
$$< \cdots U^{(J_j)}(e_{v_j}^o) U^{(J_j)}(\pi_o^x) T_\alpha^{(J_j)} U^{(J_j)}(\pi^{-1}{}_o^x) U^{(J_j)}(e_o^{v'_j}) \cdots U^{(J_k)}(e_{v_k}^y) T_\alpha^{(J_k)} U^{(J_k)}(e_y^{v'_k}) \cdots >,$$

where v_j and v'_j are the start and end points of edge e_j. The loop derivative depends on a path π_o^x going from the basepoint o to a point x, and this operator acts on the space of spin networks with a fixed basepoint $o \in \Gamma$. The links involved in the formula are shown in the figure. Using (44) one can rearrange the above expression in the following

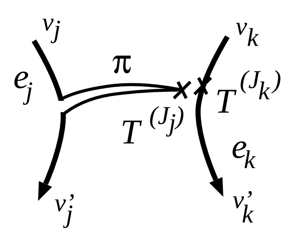
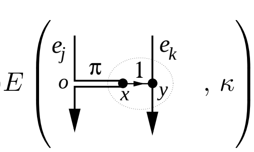


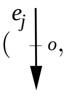
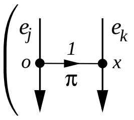
FIG. 2. The basic configuration that defines the loop derivative.

way,

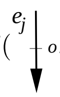
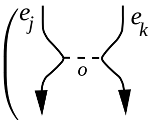
$$\Delta_{ab}(\pi_o^x) E(\overset{e_j}{\downarrow} o, \kappa) = \kappa \sum_k (-1)^{2(J_j+J_k)} \Lambda_{J_j J_k} \epsilon_{abc} \int_{e_k} dy^c \delta^3(x-y) E\left(\cdot , \kappa \right). \tag{69}$$



We now replace the two parallel lines in terms of a sum of irreducible representations. The resulting expression can be cast using recoupling identities[10] as,

$$\Delta_{ab}(\pi_o^x)E(\ $$$$\ ,\kappa) = \kappa \sum_k (-1)^{2(J_j+J_k)} \Lambda_{J_j J_k} \epsilon_{abc} \int_{e_k} dy^c \delta^3(x-y) E \left($$$$,\ \kappa \right). \quad (70)$$

For future reference, we would like to introduce a formula for the loop derivative when the path π added is of infinitesimal length. This is useful in the definition of the Hamiltonian constraint. It could arise when the point x is infinitesimally separated along the same edge as the origin o or it could correspond to two infinitesimally adjacent edges (for instance incident on a vertex). More precisely, in both the diffeomorphism and Hamiltonian constraint one has the loop derivative evaluated along an infinitesimal path multiplied times a regularized Dirac Delta function. In that case one can rearrange the above expression in terms of a chord diagram,

$$\lim_{\epsilon \to 0} \epsilon^2 \int d^3x \chi_\epsilon(x,o) \Delta_{ab}(\pi_o^x) E(\ $$$$\ ,\kappa) = \lim_{\epsilon \to 0} \epsilon^2 \kappa \sum_k \epsilon_{abc} \int_{e_k} dy^c \chi_\epsilon(o,y) E \left($$$$,\ \kappa \right), \quad (71)$$

where χ is a (symmetric, $\chi_\epsilon(x,y) = \chi_\epsilon(y,x)$) regulator such that $\lim_{\epsilon\to 0}\chi_\epsilon(x,y) = \delta^3(x-y)$. We will derive in the appendix the rather remarkable result that the kind of derivative we considered in the previous formula annihilates the framing-independent Vassiliev invariants if the spin networks consider have trivalent or lower intersections. This will allow us, in the companion paper, to find large families of solutions to all the constraints of quantum gravity, albeit with vanishing volume.

Notice that formula (70) implies, if we analyze it order by order in κ, that the loop derivative of a Vassiliev invariant of order n (the coefficient of κ^n in the expansion) is proportional to a Vassiliev invariant of order $n-1$. That is, the loop differentiation operation lowers the order of a Vassiliev invariant by one.

As an instructive example of the consistency of the formalism, we can compare the result predicted for the loop derivative of the first order term in κ by equation (70) and the direct computation of the loop derivative of expression (48),

$$\Delta_{ab}^{(e)}(\pi_o^x) < W(s) >^{(1)} = -\frac{1}{2} \Delta_{ab}^{(e)}(\pi_o^x) \left(\sum_{e_i, e_j \in s} r_{ij} \int_{e_i} dy^c \int_{e_j} dz^d \epsilon_{cdf} \partial^f \frac{1}{4\pi |y-z|} \right) \quad (72)$$

where we denoted that the loop derivative acts along the edge e_i by a superscript. The loop derivative will only act on the loop dependence of the right-hand-side, i.e., on the integrals. To evaluate this action, we recall the well known result (i.e. [17]),

$$\Delta_{ab}^{(e)}(\pi_o^x) \int_e dy^c F_c(y) = \partial_{[a} F_{b]}(x), \quad (73)$$

and therefore get,

$$\Delta_{ab}^{(e_k)}(\pi_o^x) < W(s) >^{(1)} = -\sum_{e_j \in s} r_{kj} \int_{e_j} dz^d \partial_{[a} \epsilon_{b]df} \partial^f \frac{1}{4\pi |x-z|} \quad (74)$$

which we can rearrange as,

$$\Delta_{ab}^{(e_k)}(\pi_o^x) < W(s) >^{(1)} = \sum_{e_j \in s} r_{kj} \left(\int_{e_j} dz^c \epsilon_{abc} \partial_f \partial^f \frac{1}{4\pi |x-z|} - \int_{e_j} dz^f \epsilon_{abc} \partial_f \partial^c \frac{1}{4\pi |x-z|} \right). \quad (75)$$

[10]The recoupling relations reduce the sum over the irreducible representations to a single term with spin one. This is a difference with our previous result -equation (28)- of reference [23], where we used a slightly different version of the Fierz identity. The one we are using here is more useful in calculations.

The last term gives the integral along an edge of a quantity differentiated with respect to the integration variable. This gives contributions only at the ends of the edges, and it gives the same contribution per each incoming edge. Recalling that the summation of all r_{ij}'s for each vertex vanishes, the last term vanishes when added along the whole spin network, vertex per vertex. The remaining term gives the final result,

$$\Delta_{ab}^{(e_k)}(\pi_o^x) < W(s) >^{(1)} = \sum_{e_j \in s} r_{kj} \epsilon_{abc} \int_{e_j} dy^c \delta^3(x-y). \tag{76}$$

This expression coincides with the right hand side of (70) evaluated to first order in κ. The group factors r_{ij} are given in (70) by the chromatic evaluation of the invariant E, which is what one gets at zeroth order in κ.

It is worthwhile emphasizing that although expressions like (70) can be viewed as "formal manipulations" in the path integral that one uses to motivate certain results, yet the perturbative Feynman-diagrammatic calculations yield order by order results that are consistent with the formal manipulations. We will not discuss in detail similar calculations for invariants of higher order, but they can be checked in detail using the formulae for loop derivatives of multitangents, see for instance [17].

An interesting observation is that if we go back to expression (68) we see that the loop derivative introduces in the spin network two Lie algebra generators contracted. This is precisely the same contribution we got in (67) for the Vassiliev derivative. This points out to a very appealing connection between the loop derivative and the Vassiliev derivative. It therefore highlights why Vassiliev invariants play a special role among knot invariants from the point of view of quantum gravity: they are naturally "differentiable" knot invariants, and the derivative in question is nothing else but the loop derivative. This is consistent with the fact that the loop derivative "lowers the order" of a Vassiliev invariant, as we saw before in the example of the invariant of order one being related to the zeroth order invariant. Freidel was the first to notice, and is currently studying the implications of this connection [40].

B. The diffeomorphism constraint and the loop derivative

To define a diffeomorphism constraint we will formally integrate by parts the expression of the constraint in the connection representation acting on a state given by the loop transform. In terms of the new variables, the diffeomorphism constraint is given classically by $C(\vec{N}) = \int d^3x N^a(x) \tilde{E}_i^b F_{ab}^i$. If one wishes to promote this to a quantum operator one needs to choose a factor ordering and a regularization. We will choose the factor ordering in which the triad is at the left. In this factor ordering, the operator formally fails to generate diffeomorhpisms on the wavefunctions in the connection representation. This can be amended if one regularizes the operator by point-splitting the operator E and F and using a symmetric regularization [16]. We will therefore operate with such an operator on a state in the loop representation defined via the loop transform,

$$\begin{aligned} C(\vec{N})\Psi(s) &= \int DA (C(\vec{N})\Psi(A)) W_A(s) \\ &= \int DA \int d^3x \int d^3y \lim_{\epsilon \to 0} \frac{(N^a(x) + N^a(y))}{2} \chi_\epsilon(x,y) \left(\frac{\delta}{\delta A_b^\alpha(x)} F_{ab}^\alpha(y) \Psi(A) \right) W_A(s), \end{aligned} \tag{77}$$

where again χ is a (symmetric, $\chi_\epsilon(x,y) = \chi_\epsilon(y,x)$) regulator such that $\lim_{\epsilon\to 0} \chi_\epsilon(x,y) = \delta^3(x-y)$, and the symmetric regularization refers to taking the symmetric average of the vector $\vec{N}$ at the points x and y. This expression is not gauge invariant due to the point splitting. This can be fixed by introducing small pieces of holonomies along paths π_y^x that connect the points x and y and inserting them to get a gauge invariant expression. If one now formally integrates by parts the functional derivative and represents the F_{ab} using the loop derivative, we get,

$$C(\vec{N})\Psi(s) = \sum_k \int DA \Psi(A) \lim_{\epsilon \to 0} \int d^3x \int_{e_k} dy^b \frac{(N^a(x) + N^a(y))}{2} \chi_\epsilon(x,y) \Delta_{ab}(\pi_y^x) W_A(s). \tag{78}$$

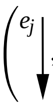
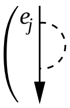
We therefore identify the following operator as the infinitesimal generator of diffeomorphisms in terms of spin nets,

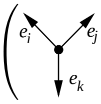
$$C(\vec{N})\Psi(s) = \sum_k \lim_{\epsilon \to 0} \int d^3x \int_{e_k} dy^b \frac{(N^a(x) + N^a(y))}{2} \chi_\epsilon(x,y) \Delta_{ab}(\pi_y^x) \Psi(s). \tag{79}$$

The last expression should be understood as a shorthand notation for (78). This means that the limit should be taken before evaluating the functional integral, otherwise we would fall in the usual pitfall of not having a defined action for a loop derivative on a knot invariant. This action *is defined* by formula (78) for the invariants in question, replacing $\Psi(A)$ by the exponential of the Chern–Simons form. The last expression has been well known for some time to correspond to the infinitesimal generator of diffeomorphisms on Wilson loops evaluated along smooth loops. In the case of spin networks, the expression displaces each edge infinitesimally. The vertices are left fixed, but are reconnected to the shifted edges by retraced paths. Due to the gauge invariance of the intertwiner operators, the functions behave as if the vertex had been shifted along with the edges.

In our previous work in terms of loops [17,6] we have used an (unregulated) diffeomorphism constraint that looked slightly different than the one we consider here, but that coincides in the limit, when acting on functions on which the loop derivative yields a smooth result, like for instance holonomies of smooth connections. The main difference is that in the regulated operator the path π extends out of the original loop and tends to it in the limit, whereas in the unregulated expressions one simply took the path directly along the loop.

The formula (78) involves evaluating the loop derivative for an infinitesimal path. To evaluate this, it is convenient to go back to (68), and notice that the addition of the two Lie algebra generators in the two nearby points corresponds to a chord diagram in which a dashed line starts and ends in a given edge of the spin network. Another possibility is if the action is close to a vertex. In that case one can have the insertion of the two Lie algebra generators in two different links adjacent to the vertex. Concretely, we get,

$C(\vec{N})E\Big($  $,\kappa\Big) = -\frac{\kappa}{2}\lim_{\epsilon\to 0}\int_{e_j} dy^b \int_{e_j} dx^c \big(\frac{N^a(y)+N^a(x)}{2}\big)\epsilon_{abc}\chi_\epsilon(x,y)E\Big($  $,\kappa\Big) = 0$ (80)

$C(\vec{N})E\Big($  $,\kappa\Big) = 0$ (81)

where we see that the contributions vanish in the first expression due to the contraction of the ϵ_{abc} with a symmetric expression in b, c. In the second expression, given the action on an edge i, the contribution arising from connecting that edge with another edge j is equal and opposite to that arising from considering the action on the edge j and considering the connection with the edge i, and they therefore cancel each other as well. This is all only true because we have chosen a symmetric regularization, otherwise the expression would not be symmetric under the interchange of b and c.

We should notice that this is a major departure from what we did in our previous paper [23]. There we did not choose the symmetric regularization, and the diffeomorphisms did not vanish on these invariants. At that point we interpreted the action of the constraint as associated with the change in the value of the framing invariants due to the addition of "writhes" by the diffeomorphism. It should be noted that there is some ambiguity in what one means by diffeomorphism in the case of framing dependent invariants, i.e., if one is talking of diffeomorphisms of "ribbons" or of loops with no width. Choosing the invariants to be diffeomorphism invariant is in line with adopting the point of view that one is considering diffeomorhpisms of "ribbons".

What we find here is that the motivation for the symmetric regularization that came from the connection representation has a natural counterpart in loops: in the connection representation the regularization ensured that the constraint generate diffeomorphisms, and therefore that the Chern–Simons state be annihilated by the constraint. The same is true in the loop representation. With this regularization, the loop transforms of that state are diffeomorphism invariant. In other words, in this regularization, the diffeomorphism considers the framing dependent objects as invariants. All coefficients in the expansion of the expectation value of the Wilson loop, and in particular all the independent components of each coefficient (the Vassiliev invariants) we discussed before, are annihilated by the diffeomorphism constraint. That is, in the regularization chosen, both the framing independent and the framing dependent invariants are annihilated by the diffeomorphism constraint. It is still open to question if the use of framing-dependent objects is warranted in quantum gravity, where the whole formalism has been from the outset set up in terms of loops without any reference to a framing.

V. CONCLUSIONS

To conclude, we summarize on the results found up to now: a) We have a space of wavefunctions of spin networks that are loop differentiable (the Vassiliev invariants and all the coefficients of the expansion of the expectation value of the Wilson loops); b) The diffeomorphism constraint naturally constructed in terms of the loop derivative is well defined and annihilates these states.

We have elucidated several aspects of the generalization of Vassiliev invariants to spin networks. To begin with, we see that in general the expectation value of a Wilson net cannot be factorized as a framing independent invariant times a framing dependent prefactor as was the case for loops. This, however, is possible if the spin networks have trivalent or simpler intersections. It nevertheless possible to identify at each order in the perturbation expansion, and for a given spin network, invariant quantities that are ambient isotopic (framing independent) and we illustrate the procedure to find them up to second order in the perturbative expansion. We have given expressions for the invariants that are given by analytic integrals along the lines that form the spin network. This allows to operate with these wavefunctions in a direct manner with the operators we will discuss in the following paper.

The fact that the operator we chose for the diffeomorphism constraint generates the geometrical action of diffeomorphisms on Wilson nets, suggests that the algebra of constraints will reproduce the expected classical Poisson structure. At the moment however, we have only constructed states in the loop representation where the constraint vanishes and therefore the algebra is reproduced, albeit trivially. In the companion paper we will introduce "habitats" of non-diffeomorphism invariant states related to the ones we introduced here and we will explicitly show that the classical Poisson algebra is reproduced at the quantum level.

Having a well defined action for the loop derivative opens the possibility of introducing quantum versions of the Hamiltonian constraint and to implement in a concrete well defined setting proposals for the Hamiltonian constraint that have been shown to have the correct algebra at a formal quantum level [6]. We will expand on this and other topics in the companion paper.

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APPENDIX A: LOOP DERIVATIVE OF FRAMING INDEPENDENT VASSILIEV INVARIANTS

In this appendix we will prove a property of a limiting behavior of the loop derivative of the framing independent Vassiliev invariants based on *trivalent* intersections. What we will show is that in the limit in which the path on which the loop derivative shrinks to zero, the loop derivative vanishes. The interest of this limit is that it is the one that appears in the loop derivatives that arise in the diffeomorphism and Hamiltonian constraints of quantum gravity in the loop representation that we will discuss in the forthcoming paper.

To prove this statement, we start by defining the invariant $J(s,\kappa)$,

$$E(s,\kappa) \equiv \exp\left[\frac{\kappa}{E(s,0)} < W(s) >^{(1)}\right] J(s,\kappa), \tag{A1}$$

that is, we extract the framing dependence in the prefactor, something that is allowed for trivalent or lower intersections.

Using the recoupling identities for trivalent intersections and equations (71) and (76), one gets,

$$\lim_{\epsilon\to 0}\epsilon^2 \int d^3y \chi_\epsilon(y,o)\Delta_{ab}(\pi_o^y)E(s,\kappa) = \lim_{\epsilon\to 0}\epsilon^2 E(s,\kappa)\int d^3y\chi_\epsilon(y,o)\Delta_{ab}(\pi_o^y)\left[\frac{\kappa}{E(s,0)} < W(s) >^{(1)}\right], \tag{A2}$$

and comparing this expression with the derivative of (A1), and bearing in mind that the loop derivative satisfies Leibnitz' rule, one immediately concludes that

$$\lim_{\epsilon\to 0}\epsilon^2 \int d^3y\chi_\epsilon(y,o)\Delta_{ab}(\pi_o^y)J(s,\kappa) = 0. \tag{A3}$$

Since the result is independent of κ, it implies that all terms in the expansion of $J(s,\kappa)$ in powers of κ is annihilated by the loop derivative as well. Moreover, since we could have carried this construction for *any gauge group* (the only elements we used was that recoupling identities for trivalent intersections are tantamount to extracting a prefactor, which is true for any group), we conclude that the topological invariants are annihilated by this limit of the loop derivative.