1-1-2003

Canonical Quantization of General Relativity in Discrete Space-Times

Rodolfo Gambini
Universidad de la Republica

Jorge Pullin
Louisiana State University

Canonical quantization of general relativity in discrete space-times

Rodolfo Gambini[1], and Jorge Pullin[2]
1. *Instituto de Física, Facultad de Ciencias, Universidad de la República, Iguá 4225, CP 11400 Montevideo, Uruguay*
2. *Department of Physics and Astronomy, Louisiana State University, 202 Nicholson Hall, Baton Rouge, LA 70803*
(Dated: October 27th 2002)

It has long been recognized that lattice gauge theory formulations, when applied to general relativity, conflict with the invariance of the theory under diffeomorphisms. Additionally, the traditional lattice field theory approach consists in fixing the gauge in a Euclidean action, which does not appear appropriate for general relativity. We analyze discrete lattice general relativity and develop a canonical formalism that allows to treat constrained theories in Lorentzian signature space-times. The presence of the lattice introduces a "dynamical gauge" fixing that makes the quantization of the theories conceptually clear, albeit computationally involved. Among other issues the problem of a consistent algebra of constraints is automatically solved in our approach. The approach works successfully in other field theories as well, including topological theories like BF theory. We discuss a simple cosmological application that exhibits the quantum elimination of the singularity at the big bang.

arXiv:gr-qc/0206055v2 6 Dec 2002

Lattice approaches have been successful in Yang–Mills theory not only in making the theory finite, but also in making it practically computable. There has been for a long time a hope that similar techniques could help in defining a quantization for general relativity (see [1] for a review). More recently, discretizations on lattices have been used to regularize the Hamiltonian constraint of general relativity by Thiemann [2] and others and also to define the path integral in the "spin foam" approaches (see [3] for a recent review). A problem that has been present since the outset is that the introduction of a lattice breaks the symmetry of general relativity under space-time diffeomorphisms. This has had several practical implications. For instance, discretizations of the constraints of general relativity fail to close an algebra like the one present in the continuum [4]. Moreover, it is well known that if one discretizes the Einstein equations, the evolution equations fail to preserve the constraints. Therefore, strictly speaking, the discrete theory is inconsistent. This has been noted and discussed in the numerical relativity literature [5].

In this letter we would like to point out that what has been lacking is a systematic canonical treatment of the discrete theory, both at a classical and quantum mechanical level. When one treats the discrete theory carefully, one notices that it has an involved canonical structure. Dealing with it is non-trivial, since one needs a generalization of the Dirac procedure to systems in which time is discrete (attempting a purely spatial discretization appears highly unnatural in the case of general relativity, where the constraints mix space and time symmetries). We present such a generalization of the Dirac procedure and show that the resulting theory is consistent, in the sense that constraints do not conflict with evolution equations and can be quantized. In our construction the lattice provides a natural "gauge fixing" for the theory, which then has a "true Hamiltonian" evolution. This gauge fixing may have practical use in classical numerical relativity as well.

Let us illustrate our procedure with a mechanical system to begin with. We take time to be discrete. The action therefore is written as $S = \sum_{n=1}^{N} L(q_n, q_{n+1})$ where we have partitioned the time interval and discretized the derivatives of the Lagrangian. The Lagrange equations of motion are, $\partial S/\partial q_n = \partial L(q_{n-1}, q_n)/\partial q_n + \partial L(q_n, q_{n+1})/\partial q_n = 0$. In order to introduce canonically conjugate momenta one notices that differentiating with respect to the time derivative is equivalent to differentiating with respect to q_{n+1}, or, alternatively, $p_n = \partial L(q_{n-1}, q_n)/\partial q_n$. The evolution of the canonically conjugate pair (q_n, p_n) can be induced through a type 1 canonical transformation generated by, $F_1(q_n, q_{n+1}) \equiv -L(q_n, q_{n+1})$. The resulting equation of motion is $p_{n+1} = \partial L(q_n, q_{n+1})/\partial q_{n+1}$, and together with the definition of the momentum completely determines the evolution of the system and reproduce in the limit the equations of motion of the system [6].

To quantize the resulting system we need to find a unitary operator that implements the finite canonical transformation. Let us particularize to the example of a particle in a potential. In such case the discrete Lagrangian is $L(n, n+1) = m(q_{n+1} - q_n)^2/(2\epsilon) - V(q_n)\epsilon$, where ϵ is the time interval. The canonical transformation yields equations of motion that can be rewritten as determining q_{n+1}, p_{n+1} in terms of q_n, p_n. In order to quantize we need to find a unitary operator that implements quantum mechanically the equations of motion, $\hat{p}_{n+1} = \hat{U}\hat{p}_n\hat{U}^\dagger$, $\hat{q}_{n+1} = \hat{U}\hat{q}_n\hat{U}^\dagger$. This quantization ensures that q_n, p_n are such that $[\hat{q}_n, \hat{p}_n] = i\hbar$. The operator can be readily found. Starting with wave functions that are function of the coordinates $\Psi[q]$, the operator is given by, $\hat{U} = \exp\left(\frac{iV(q)\epsilon}{\hbar}\right)\exp\left(\frac{ip^2\epsilon}{2m\hbar}\right)$.

We therefore see that one can set up a canonical theory for a system with discrete time and quantize the system. What we need to do in order to generalize this to gravity is to consider systems with constraints. The construction proceeds as before, but the resulting discrete equations are radically different from the continuum ones: they are generically inconsistent *unless one chooses particular values of the Lagrange multipliers*. These particular values also make *the constraints automatically satisfied*: the resulting theory is only described through evolution equations. As before, the quantization of the theory is achieved by finding a unitary transformation that implements the finite canonical transformation. This is readily done for the case of the particle, at least for simple potentials. Notice that once we solved the constraints for the multipliers, they do not play any role in the quantization procedure. T. D. Lee [7] was the first to recognize that the discretized mechanics of a particle in a potential could be made consistent through fixing the Lagrange multipliers (in his language, fixing the time interval). Also Friedmann and Jack noticed similar things in the context of Hamiltonian Regge calculus [8].

Solving the discrete constraints by choosing the Lagrange multipliers is a procedure that is only available in the discrete theory. This procedure might be useful in numerical relativity. There it is well known that the discrete evolution equations (for instance, in the ADM formulation) fail to preserve the discrete constraints. Our proposal here consists in the following: take an evolution step with arbitrary, unspecified, lapse and shifts. Then impose the discrete constraints by solving the for the lapse and shift. There are four quantities to be determined and four equations. The construction obviously is only guaranteed to work locally both spatially and in time. Numerical experimentation will be needed to decide if this is really useful in practice.

We have tested the canonical procedure with Yang–Mills theories and BF theories in the lattice and one recovers in the former case the traditional lattice theory. To our knowledge, our approach provides the first lattice treatment for BF theory ever presented. Details can be seen in [6].

Let us now sketch the case of general relativity. We start by considering the Lagrangian written in terms of Ashtekar variables, (see for instance [9] and the book by Ashtekar [10] page 47). To simplify treatment here we will consider the Euclidean case, but there is no obstruction to treating the Lorentzian case, one simply has to replace the action we are considering here with the one introduced by Holst [11] which leads to the real variables of Barbero [12]. The Lagrangian is,

$$L = \int E^{ai} F^i_{a0} + \epsilon_{abc} \left[E^{bi} E^{cj} \epsilon^{ijk} N + N^b E^{ck}\right] F^k_{de} \epsilon^{ade},$$

where E^{ai} and A^j_a are the usual Ashtekar variables [13] and N and N^a are the lapse and shift. This Lagrangian can be discretized as follows,

$$L(n,n+1) = \sum_v \mathrm{Tr}\left[E^a_{n,v} h^a_{n,v} V_{n,v+e_1} (h^a_{n+1,v})^\dagger (V_{n,v})^\dagger \right.$$
$$+K_{1,n,v} h^2_{n,v} h^3_{n,v+e_2} (h^2_{n,v+e_3})^\dagger (h^3_{n,v})^\dagger + \text{cyc.}$$
$$\left. +\, \alpha_{a,n,v} \left(h^a_{n,v} \left(h^a_{n,v}\right)^\dagger - 1\right) + \beta_{n,v} \left(V_{n,v} V^\dagger_{n,v} - 1\right)\right] \quad (1)$$

where $K_{a,n,v} = \epsilon_{abc}\left[E^b_{n,v} E^c_{n,v} N_{n,v} + N^b_{n,v} E^c_{n,v}\right]$ and repeated indices a, b, c are summed from 1 to 3 and "cyc" means cyclic permutations of $1, 2, 3$. In this expression h^a_{n+1} represents an holonomy along the a direction at instant $n+1$, V_n represents the "vertical" (time-like) holonomy. We will assume that the holonomies are matrices of the form $h = h^I T^I$, $V = V^I T^I$ where $T^0 = I/\sqrt{2}$ and $T^a = -i\sigma^a/\sqrt{2}$ where σ_a are the Pauli matrices. The indices n, v represent a label for "time" n and a spatial label for the vertices of the lattice v. The elementary unit vectors along the spatial directions are labeled as e_a, so for instance $n + e_1$ labels the nearest neighbor to n along the e_1 direction. The quantities $E^a_{n,v}$ are elements of the algebra of $su(2)$ and α and β are Lagrange multipliers, the last two terms of the Lagrangian enforcing the condition that the holonomies are elements of $SU(2)$

We need to work out the equations of motion of the action. The variables are $E^a_{n,v}, h^a_{n,v}, V_{n,v}, \alpha^a_{n,v}, \beta^b_{n,v}, N_{n,v}, N^b_{n,v}$. For each variable we compute the canonical momenta at instants n and $n+1$ by $\partial L/\partial q_n$ and $\partial L/\partial q_{n+1}$ as in the examples we discussed. We will not list them explicitly for reasons of space but we will discuss their implications. The equations for α lead to a constraint that implies the spatial holonomy h is a matrix of $SU(2)$. The equations for β lead to a similar conclusion for the holonomy V. The equations for E lead to an evolution equation for the holonomy. The equations for h lead to a constraint relating E with the canonical momentum of the holonomy at instant $n+1$ and can be viewed as an evolution equation for E. The equations for N, N^a give the diffeomorphism and Hamiltonian constraints of general relativity, which are solved to determine the multipliers N, N^a as we discussed above. The equations for V yield Gauss' law. To complete the canonical theory one needs to check that all constraints are preserved in time. This can be done and it leads to a set of second class constraint very similar to the ones we have analyzed in detail in reference [6] for BF theory. One needs to introduce the Dirac brackets, which are identical to those of BF theory and impose the constraints strongly. This concludes the canonical formulation.

Quantization would now proceed by finding a unitary evolution operator that implements the previous evolution equations as operator equations quantum mechanically. Notice that there is no issue of algebra of constraints, since the latter have been solved for the multipliers, the lapse and shift. The procedure is computationally intensive in a generic situation, requiring the simultaneous solution of four non-linear coupled algebraic equations at each lattice point. For particular situations, the procedure might be completed in a straightforward fashion. The study of midi-superspace situations is therefore the next natural step in this program of quantization of gravity. A direct application would be to study the dispersion of waves to attempt to make contact with gamma-ray-burst phenomenology [14].

For concreteness, let us exhibit the construction in a simplified situation, that of a cosmological model. We consider a Friedmann model with flat spatial sections coupled to a scalar field which we take to have a large mass so we can neglect its kinetic term in the action. This is just done in order to yield the simplest possible cosmological model with at least one continuum degree of freedom. Written in terms of Ashtekar's variables, the only non-trivial quantities are one component of the triad field $E(t)$ and one component of the connection $A(t)$ and for the scalar field its value $\phi(t)$ and its canonically conjugate momentum $\Pi(t)$. The continuum Lagrangian is given by $L = E\dot{A} + \Pi\dot{\phi} - NE^2(-A^2 + \Lambda E + m^2\phi^2 E)$ where N is the lapse (with density weight -1, as is usual in the Ashtekar formulation, in terms of the usual lapse α it is given by $N = E^{-3/2}\alpha$) and Λ is a (positive) cosmological constant. The exact solution for this model (in the time slicing in which $\alpha = 1$) has a constant scalar field and the geometry is given by (anti)DeSitter space, $E = \exp(2\sqrt{\Lambda + m^2\phi^2}t)/(\Lambda + m^2\phi^2)$, $A = \exp(\sqrt{\Lambda + \phi^2}t)$. If we discretize the action we get,

$$
\begin{aligned}
L_n &= E_n\left(A_{n+1} - A_n\right) + \pi_n\left(\phi_{n+1} - \phi_n\right) \\
&\quad -N_n E_n^2\left(-A_n^2 + \Lambda E_n + m^2{\phi_n}^2 E_n\right)
\end{aligned} \tag{2}
$$

and the generating function of the canonical transformation that implements the equations of motion is given as usual by $F_1 = -L$. Working out the equations of motion and the definition of the canonically conjugate momenta from the generating function and the Lagrangian, one ends up with a recurrence relation for the variable A_n.

$$
A_{n-1}^2 - A_n^2 + 2\left(A_{n+1} - A_n\right)A_n = 0 \tag{3}
$$

The lapse is completely determined by solving the Hamiltonian constraint. This fixes dynamically the gauge: the evolution step in time is determined by the initial conditions. For instance if we substitute the expression for the lapse in the equations of motion one gets, $E_n = {P^A}_{n+1} = {A_n}^2/(\Lambda + m^2{\phi_n}^2)$, $A_{n+1} = A_n + \left(A_n^2 - \Lambda {P^A}_n - m^2{\phi_n}^2{P^A}_n\right)/2A_n$ where P^A is conjugate momentum to A. We see that for instance the "step" between A_n and A_{n+1} is given by $A_n^2 - \Lambda {P^A}_n - m^2{\phi_n}^2$. Choosing the initial values for A and P^A and the scalar field determines the step.

This simple model exhibits several attractive features. For instance, for large values of the "time label" n, $A \to n^{2/3}$, $E \to n^{4/3}$ and the lapse $N \to n^{-3}$. If we identify $t = \epsilon n$ where ϵ is a constant, we then get $E \to t^{4/3}$ for large times. This approximates the continuum solution if one chooses a lapse in the continuum $\alpha \sim t^{-1}$. We can recover any parameterization we desire in the continuum by redefining t as a function of ϵn. We therefore see in which sense the lattice "fixes the gauge". Indeed, the lattice treatment operates just like a gauge fixed treatment, but one still has available the full reparameterization invariance of the theory when identifying the lattice behavior with the continuum one. Although not a traditional gauge fixing, this procedure may share with the latter some problems. One breaks the gauge symmetries of the theory, but it is yet to be seen, especially in complex situations if one does not face a Gribov problem or that singularities do not appear that prevent from covering portions of the gauge orbits, etc. In particular, by choosing different parameterizations we can choose to start the region that approximates the continuum behavior as early or as late as we desire. One can consider the recursion relation (3) and run it backwards in time. If one does that, one finds that the universe achieves a minimum radius and the recursion continues with the universe re-expanding again. This is a dramatically different behavior than the one in the continuum theory. In the latter, if one gives any initial condition and runs back in time one will always meet an initial singularity (in this simple model, just a coordinate singularity, in other models it is a genuine singularity [20]). In the discrete theory one needs to fine tune the initial data for this to happen, otherwise the singularity is avoided: it is a point that does not fall on the grid. Quantum mechanically, the implications of this fact are more significant. Since the singularity is only achieved in a set of measure zero of the classical theory, quantum mechanically one has zero probability of hitting the singularity. Notice that this mechanism for avoiding the singularity quantum mechanically appears distinct, although it predicts somewhat similar results, to the one recently presented by Bojowald [15] in the context of loop quantization. Singularity avoidance has also been discussed in other contexts [19] with different mechanisms. We would like to stress that the particular way the singularity is avoided is model dependent and in some models the singularity may not be avoided [20]

Quantization proceeds by considering wave functions, for instance, of the momentum ($P = P^A$) and the scalar field, $\hat{P}\Psi(P,\phi) = P\Psi(P,\phi)$, $\hat{A}\Psi(P,\phi) = d\Psi(P,\phi)/dP$. One needs to implement the reality conditions $A^* = -A$, $P^* = P$. These can be implemented via the inner product $< \Psi|\Phi >= \int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} dP \Psi^*(P,\phi)\Phi(P,\phi)$. Since under this inner product P has a continuous spectrum, the probability of having a singularity (which corresponds to $P = 0$) vanishes. This can also be seen from the fact that the expectation value of the volume is always greater than zero. This implements in detail the behavior we had anticipated in the heuristic discussion above.

Going back to the generic discussion of our approach, at this point one might wonder if by taking too seriously the discrete theory we are not neglecting the continuum limit. Two comments can be made. First of all, there appears to be some consensus [16] among researchers in quantum gravity that the ultimate theory may have some fundamental discreteness at the quantum level. This is in part based on the fact that the continuum limit of Yang–Mills theories relies heavily on the renormalizability of the theory, which appears not to be present in the gravitational case. This is therefore motivation to take a look at the discrete theory seriously. Moreover, having a consistent discrete theory, which includes among its solutions some that correspond to solutions of the continuum theory, opens the possibility of defining an averaging procedure that could make results discretization-independent. If one adds up solutions for various discretizations, the solutions that have a continuum counterpart will appear in all terms and will add up. The solutions without a continuum counterpart will therefore be suppressed, since they will be different for different discretizations. Notice that this allows a method of implementing the "sum over all discretizations" that is advocated in the context of spin foam models [17]. Following the procedure outlined above for gravity, one can construct transition amplitudes for the discrete theory. Then, performing the averaging we just discussed for a finite, yet large, number of discretizations will produce a result for the transition amplitude that is approximately discretization-independent (this is only a heuristic argument that could face problems, for instance, discretizations could be unstable and modes that do not correspond to the continuum limit could end up dominating the sum). If the construction were to work one would not need renormalizability in order to define a continuum theory in this way. It should be noted that the general state of the art of these ideas is, at the moment, only speculative, and there is disagreement among various researchers. In fact, the more conventional viewpoint for the continuum limit of lattice gauge theories, applied to quantum gravity does appear to work in sucessfully in $1+1$ dimensions, and progress is being made in higher dimensionalities [18]. Even within this more traditional viewpoint, the "consistent" construction we present in this paper could play a role at the time of selecting the correct measure of integration, since it automatically takes into account the constraints, which is crucial at the time of properly defining the path integral.

Another attractive element of this treatment is that the separation of the lattice points is dynamically determined by the initial conditions. This implies that quantum mechanically, where generically one has a weighted superposition of all possible initial conditions for the system, there is an automatic "averaging" of all possible discretizations per each given quantum state. It should be noted that this averaging feature only occurs for totally constrained systems so it is a case that diffeomorphism invariance actually helps in getting rid of the details of the particular discretization.

Finally, the lattice treatment introduced for gravity, since it operates as a gauge fixing, provides a solution for the "problem of time". However, as it happens with usual gauge fixings one should expect that our construction will only work locally.

Summarizing, we have presented a formulation of general relativity on the lattice that is consistent and well defined classically and can be readily quantized. The formulation can operate in both the Euclidean and Lorentzian signatures and yields naturally a proposal for the continuum limit of the theory. We exhibited its behavior in detail in a particular cosmological situation where one sees the singularity at the big bang disappears due to quantum effects (although this is not a generic feature, see [20]). The immediate future course of action is to explore the formulation in more realistic, midisuperspace models, where detailed predictions of the field theory aspects of quantum gravity could be worked out, some of which may have experimental implications.

We wish to thank Abhay Ashtekar, Martin Bojowald, Doug Arnold, Alejandro Pérez, Richard Price and Oscar Reula for comments. This work was supported by grants nsf-phy0090091, funds from the Horace Hearne Jr. Institute for Theoretical Physics and the Fulbright Commission in Montevideo.

[1] R. Loll, Living Rev. Rel. **1**, 13 (1998).
[2] T. Thiemann, Class. Quan. Grav. **15** 839; 875; 1207; 1249; 1281; 1463 (1998); C. Di Bartolo, R. Gambini, J. Griego, J. Pullin, Phys. Rev. Lett. **84** 2314 (2000);
[3] See J. C. Baez, Lect. Notes Phys. **543**, 25 (2000).
[4] M. Bander, Phys. Rev. D **36**, 2297 (1987); **38**, 1056 (1988).
[5] M. Choptuik, Phys. Rev. D **44**, 3124 (1991).
[6] C. Di Bartolo, R. Gambini, J. Pullin, Class. Quan. Grav. **19**, 5475 (2002).
[7] T. D. Lee, in "How far are we from the gauge forces" Antonino Zichichi, ed. Plenum Press, (1985)
[8] J. L. Friedman and I. Jack, J. Math. Phys. **27**, 2973 (1986).
[9] R. Capovilla, T. Jacobson, J. Dell and L. Mason, Class. Quant. Grav. **8** (1991) 41.
[10] A. Ashtekar, "Lectures on non-perturbative quantum gravity" World Scientific, Singapore (1991).
[11] S. Holst, Phys. Rev. D **53**, 5966 (1996).
[12] J. F. Barbero, Phys. Rev. D **51**, 5507 (1995).
[13] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).
[14] For a review see G. Amelino-Camelia, astro-ph/0201047.
[15] M. Bojowald Phys. Rev. Lett. **87**, 121301 (2001).
[16] S. Kauffman and L. Smolin, Lect. Notes Phys. **541**, 101 (2000).
[17] L. Crane, A. Pérez, C. Rovelli, Phys. Rev. Lett. **87** 181301 (2001).
[18] J. Ambjorn, J. Jurkiewicz and R. Loll, Nucl. Phys. B **610**, 347 (2001) and references therein.
[19] P. Hajicek, gr-qc/0204049 and references therein.
[20] R. Gambini, J. Pullin, "Consistent lattice quantum gravity: applications to cosmology" (in preparation).