

7-21-2003

Perturbative evolution of particle orbits around Kerr black holes: Time-domain calculation

Ramón López-Alemán
Universidad de Puerto Rico, Recinto de Rio Piedras

Gaurav Khanna
Long Island University

Jorge Pullin
Louisiana State University

Follow this and additional works at: https://digitalcommons.lsu.edu/physics_astronomy_pubs

Recommended Citation

López-Alemán, R., Khanna, G., & Pullin, J. (2003). Perturbative evolution of particle orbits around Kerr black holes: Time-domain calculation. *Classical and Quantum Gravity*, 20 (14), 3259-3268. <https://doi.org/10.1088/0264-9381/20/14/320>

This Article is brought to you for free and open access by the Department of Physics & Astronomy at LSU Digital Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Digital Commons. For more information, please contact ir@lsu.edu.

Perturbative evolution of particle orbits around Kerr black holes: time domain calculation

Ramón López-Alemán¹, Gaurav Khanna², and Jorge Pullin³

¹*Physical Sciences Department, University of Puerto Rico - Río Piedras, San Juan, PR, 00931*

²*Natural Science Division, Long Island University, Southampton, NY 11968 and*

³*Department of Physics and Astronomy, Louisiana State University,
202 Nicholson Hall, Baton Rouge, LA 70803-4001*

We consider the problem of the gravitational waves produced by a particle of negligible mass orbiting a Kerr black hole. We treat the Teukolsky perturbation equation in the time domain numerically as a 2+1 partial differential equation. We model the particle by smearing the singularities in the source term by the use of narrow Gaussian distributions. We have been able to reproduce earlier results for equatorial circular orbits that were computed using the frequency domain formalism. The time domain approach is however geared for a more general evolution, for instance of nearly geodesic orbits under the effects of radiation reaction.

PACS numbers: 04.25.-g, 04.25.Nx, 04.70.-s

I. INTRODUCTION

It will not be long before the current set of interferometric gravitational wave detectors such as LIGO, VIRGO, GEO and TAMA achieve sensitivities at which detections are expected to be possible with a reasonable event rate. The ground-based interferometric detectors are now starting full-blown “science runs” and the plans for more sensitive second and even third generation detectors are in the drawing boards. Even LISA, the space-based observatory, seems to be a possibility in a reasonably near future.

This work tries to help in setting the stage for realistic calculations of what now seems to be a very likely occurrence in active galactic nuclei: the capture of compact stellar-sized objects by the supermassive black holes lurking in the center of most galaxies. This type of events will generate gravitational waves within LISA’s planned sensitivity band.

Since the captured objects are small compared to the capturing black hole, they can be treated in a first approximation by black hole perturbation theory. But one thorny problem still remains that has stalled progress in this area. We can model the waves coming from many kinds of stationary particle orbits around the black hole, but real orbits will not be stationary. The emission of gravitational waves will back-react on the object making the orbit decay in time. One of the challenges in this area is to find workable prescriptions for these radiation reaction forces that can be used in numerical simulations that can produce accurate waveforms and energy fluxes for observers at large distances from the hole to be used as templates for the interferometric detectors. The prescriptions that have been worked out so far involve calculating quite complex integrals along the past light cone of the particle [1, 2]. Since almost all perturbative treatments of particles around rotating holes are done in the frequency domain, applying these radiation reaction schemes has proved challenging. Frequency domain computations become computationally expensive when one is dealing with realistic non-circular orbits.

We try here to work out a simple case of equatorial circular particle orbits around a rapidly rotating black hole to show how to treat this problem perturbatively in the time domain. We hope this might help to open the door to the use of the proposed fully relativistic radiation reaction schemes in the calculation of the more complex but interesting problem of modeling the orbital decay and gravitational radiation emission of particles in more general elliptic and/or inclined orbits. Up to now, almost all treatments have been based on the separability of the equation in the frequency domain and calculate the energy and waveforms for the first few ℓ multipoles of the spheroidal harmonics expansion once the radial part has been dealt with.

Using the frequency domain approach, quite a few detailed simulations of the gravitational waves emitted by a particle in a bound orbit around a black hole have been carried out in recent years. The first results for radiation emitted by a particle in orbit around a Schwarzschild black hole were carried out by Detweiler, who pioneered these techniques [3]. In a series of six papers, Poisson and various collaborators studied in detail the gravitational wave emission to infinity and into the horizon of a particle in circular orbit around a non-rotating hole [4, 5, 6, 7, 8, 9].

Other groups have been doing extensive work on Post-Newtonian expansions for various types of particle orbits around Kerr holes (see [10] and references therein). Work has been done on eccentric orbits around Schwarzschild [11], circular orbits slightly inclined away from the equatorial plane of Kerr holes [12], and particles with spin orbiting Kerr holes on equatorial circular orbits [13].

For single static orbits (ignoring the decay by radiation reaction) to get good accuracies in the frequency domain approach involves summing for values of ℓ up to 12 for quite a few of the above-harmonic frequencies, requiring

Authors	Type of result
Detweiler (1978) Poisson (1993,1995)	equatorial circular orbits around Kerr BH
Kojima & Nakamura (1984)	infall trajectories with ang mom $\neq 0$
Sasaki & Nakamura (1982)	infall along symmetry axis of Kerr BH
Tanaka <i>et al</i> (1996)	spinning particles in circ equatorial orbits
Hughes (2000)	slightly eccentric and non-equatorial orbits

TABLE I: A summary of some of the more relevant frequency domain results for a binary black hole system in the particle limit that have appeared in the recent literature.

close to 3000 iterations using a numerical algorithm that involve complex Green's function calculations [14]. Further discussion of the number of ℓ modes needed, and also a discussion of the self-force in the case of a scalar field can be seen in the paper by Burko [15].

II. TIME DOMAIN EVOLUTION OF THE TEUKOLSKY EQUATION

We will perform the numerical evolution with the latest version of a code originally developed by Laguna and collaborators [16] to evolve the Teukolsky equation in the time domain from a generic initial data slice. The method analyzes the radiation at “infinity” by dealing with the $s = -2$ version of the equation. Then the equation is rewritten as a first order matrix equation and numerically integrated. The code has been tested treating scalar fields, scattering of gravitational waves and analysis of quasi-normal ringing and power law tails of the outgoing radiation.

We start with the original Teukolsky equation [17] before separation is performed,

$$\begin{aligned}
& \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \frac{\partial^2 \psi}{\partial \phi^2} - \Delta^{-s} \frac{\partial}{\partial r} \left(\Delta^{s+1} \frac{\partial \psi}{\partial r} \right) \\
& - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) - 2s \left[\frac{a(r-M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right] \frac{\partial \psi}{\partial \phi} - 2s \left[\frac{M(a^2 - r^2)}{\Delta} - r - ia \cos \theta \right] \frac{\partial \psi}{\partial t} \\
& + [s^2 \cot^2 \theta - s] \psi = 4\pi(r^2 + a^2 \cos^2 \theta)T
\end{aligned} \tag{1}$$

where $\Delta = r^2 - 2Mr + a^2$ postulating an ansatz for the ψ function of the form,

$$\psi = \Psi(r, t, \theta) e^{im\phi}. \tag{2}$$

To match this ϕ dependence, we write the matter source term as a similar Fourier decomposition

$$T = T_m(r, t, \theta) e^{im\phi}. \tag{3}$$

With a modicum of algebra one can rewrite the Teukolsky equation as,

$$-\frac{\partial^2 \Phi}{\partial t^2} - A \frac{\partial \Phi}{\partial t} + b^2 \frac{\partial^2 \Phi}{\partial r^{*2}} + \frac{c}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) - V \Phi = T_f. \tag{4}$$

This is possible with the following definitions

$$\Phi = \Psi \sqrt{(r^2 + a^2) \Delta^s}, \tag{5}$$

$$A = \frac{1}{\Sigma^2} (2s [r\Delta - M(r^2 + a^2)] + i [4Marm + 2sa\Delta \cos \theta]), \tag{6}$$

$$b^2 = \frac{(r^2 + a^2)^2}{\Sigma^2} \qquad c = \frac{\Delta}{\Sigma^2}, \tag{7}$$

$$V = \frac{1}{\Sigma^2} [\Delta(s^2 \cot^2 \theta - s) + m^2(\Delta \sin^{-2} \theta - a^2) + 2sm\Delta \cot \theta \sin^{-1} \theta - 2ismar(r - M)] + f \frac{(r^2 + a^2)}{\Sigma^2 \Delta^s} \frac{\partial^2 f}{\partial r^{*2}} \quad (8)$$

$$f^2 = (r^2 + a^2) \Delta^s, \quad (9)$$

$$T_f = -\frac{4\pi(r^2 + a^2 \cos^2 \theta) T f \Delta}{(r^2 + a^2)^2 - a^2 \Delta \sin \theta} \quad (10)$$

where $\Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta$.

One now introduces an auxiliary field Π that converts the Teukolsky equation to a set of 2 coupled PDEs which are first order in space and time

$$\Pi = \frac{\partial \Phi}{\partial t} + b \frac{\partial \Phi}{\partial r^*}, \quad (11)$$

$$\frac{\partial \Pi}{\partial t} + b \frac{\partial \Pi}{\partial r^*} = b(A - \frac{\partial b}{\partial r^*}) \frac{\partial \Phi}{\partial r^*} + \frac{c}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) - A\Pi - V\Phi - T_f. \quad (12)$$

The code we are using produces stable evolutions of these equations using a two-step Lax-Wendroff method.

As in other applications of this Teukolsky code, we impose boundary conditions at the edges of the polar computational domain, i.e. at the black hole horizon, at the rotation axis and at the far end of the radial grid. The condition near the horizon is $\Phi = \Pi = 0$, due to the known asymptotic behavior of the fields [16, 17]. At the outer boundary we impose usual outgoing boundary conditions. Since the wave equation has a potential, errors due to reflection from this boundary bounce back into the computational domain. We deal with this by making the computational domain in the radial direction large enough that any numerical reflections will not make it back to the point where we compute the waveforms and energy flux in the time allotted for the simulation. At the axis of rotation of the black hole one imposes either the condition $\Phi = 0$ or $\partial_\theta \Phi = 0$ depending on the parity of the field specified by the azimuthal integer m .

The initial data we take for the fields is zero. Since we are considering the Teukolsky equation with a source we are therefore violating the constraints of general relativity on the initial data. This would correspond to the particle “appearing suddenly” and therefore generates an artificial burst of radiation. To get more realistic initial data slices, one could try constructing initial data sets that correctly satisfy the Hamiltonian and momentum constraints with prescriptions like that of Bowen and York [18] in the limit in which one black hole is very small, for instance, or modifications of particle limit data sets such as those of Lousto and Price [19]. These prescriptions specify the (g_{ij}, \tilde{K}_{ij}) sets for use in standard ADM evolutions. To convert these quantities to the set $(\Phi, \delta_t \Phi)$ needed for the Teukolsky evolution, one can use the formulae developed in reference [20]. We plan to explore the extent of the benefits that this more accurate initial data specification may bring to enhance the orbital simulations in the particle limit case future work.

The next step is to construct the source term for a general geodesic trajectory of the particle moving in the background Kerr geometry. To include the effect of radiation reaction one treats it as a slight deviation of the background geodesic orbital motion.

In the time domain Teukolsky equation the main source term is given by

$$T = 2\rho^{-4} T_4, \quad (13)$$

where

$$\begin{aligned} T_4 = & (\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)[(\delta^* - 2\tau^* + 2\alpha)T_{nm^*} \\ & - (\Delta + 2\gamma - 2\gamma^* + \mu^*)T_{m^*m^*}] + (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi) \\ & \times [(\Delta + 2\gamma + 2\mu^*)T_{nm^*} - (\delta^* - \tau^* + 2\beta^* + 2\alpha)T_{nn}]. \end{aligned} \quad (14)$$

All the differential operators and Newman–Penrose coefficients, as well as the choice of Kinnersley tetrad vectors follow the notation of [17]. By choosing their values in Boyer-Lindquist coordinates and expanding everything we can end up with an explicit but quite complicated form of the source term that is amenable for numerical calculations[21].

We obtain the $r(t)$, $\theta(t)$, and $\phi(t)$ by discretizing and integrating the geodesic equations in the Kerr background,

$$\Sigma^2 \left(\frac{dr}{d\tau} \right)^2 = [E(r^2 + a^2) - aL_z]^2 - \Delta [r^2 + (L_z - aE)^2 + Q] , \quad (15)$$

$$\Sigma^2 \left(\frac{d\theta}{d\tau} \right)^2 = Q - \cot^2 \theta L_z^2 - a^2 \cos^2 \theta (1 - E^2) , \quad (16)$$

$$\Sigma \left(\frac{d\phi}{d\tau} \right) = \csc^2 \theta L_z + aE \left(\frac{r^2 + a^2}{\Delta} - 1 \right) - \frac{a^2 L_z}{\Delta} , \quad (17)$$

$$\Sigma \left(\frac{dt}{d\tau} \right) = E \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] + aL_z \left(1 - \frac{r^2 + a^2}{\Delta} \right) . \quad (18)$$

The source term is singular in those points where the particle is located at each timestep. We treat this [28] by using the approximation in which we substitute the delta function by a very narrow Gaussian function (whose width is only of a few gridpoints) :

$$\delta(x - x(t)) \approx \frac{1}{\sqrt{2\pi}\sigma} \exp \left(\frac{-(x - x(t))^2}{2\sigma^2} \right) \quad \text{for } \sigma \text{ small.} \quad (19)$$

One must show that this substitution is adequate by showing convergence of the obtained waveform as $\sigma \rightarrow 0$, and one must take special care in normalizing the mass correctly as the particle moves in the background Kerr spacetime so that integration over the whole Gaussian density profile gives a constant mass throughout the whole trajectory and to satisfy $\nabla_\mu T^{\mu\nu} = 0$. This translates in that the Gaussian must approximately integrate to one on the spatial manifold. To achieve this with the correct three-metric of the background black hole spacetime, we normalize the mass-density of the particle by the factor,

$$N = \frac{\sqrt{\gamma^{(3)}}}{\sqrt{g^{(3)}}} , \quad (20)$$

where $\gamma^{(3)}$ is the 3-metric of flat space and $g^{(3)}$ is the 3-metric of a slice of constant Boyer-Lindquist time of the Kerr spacetime. We then normalize the particle mass as it moves thru the space by multiplying the quantity μ in the source term expression by this factor N . We handle the Dirac deltas in the radial and angular direction through the Gaussian approximation, whereas the $\delta(\phi - \phi(t))$ function we handled analytically. The normalizations presented are common in the SPH literature [22].

III. NUMERICAL RESULTS

We now want to calculate the energy flux emitted in the form of gravitational waves from the binary that an orbiting observatory such as LISA would be able to detect. In this paper we focus on calculating fluxes from circular equatorial orbits without the effects of radiation reaction mainly to test the viability of this approach and to show stability, convergence, and agreement with previous results of high accuracy frequency domain calculations[23]. In the next section we will explain the advantages of working in the time domain for future calculations of eccentric and inclined particle orbits.

The code monitors the value of ψ at a set point in the grid far from the horizon and computes the energy and angular momentum fluxes in gravitational radiation according to the formulas [24, 25] :

$$\frac{dE}{dt} = \lim_{r \rightarrow \infty} \left\{ \frac{1}{4\pi r^6} \int_{\Omega} d\Omega \left| \int_{-\infty}^t d\tilde{t} \psi(\tilde{t}, r, \theta, \varphi) \right|^2 \right\} \quad (21)$$

$$\begin{aligned} \frac{dL_z}{dt} = & - \lim_{r \rightarrow \infty} \left\{ \frac{1}{4\pi r^6} \mathbf{Re} \left[\int_{\Omega} d\Omega \left(\partial_\varphi \int_{-\infty}^t d\tilde{t} \psi(\tilde{t}, r, \theta, \varphi) \right) \right. \right. \\ & \left. \left. \times \left(\int_{-\infty}^t dt' \int_{-\infty}^{t'} d\tilde{t} \bar{\psi}(\tilde{t}, r, \theta, \varphi) \right) \right] \right\} \quad d\Omega = \sin \theta d\theta d\varphi. \quad (22) \end{aligned}$$

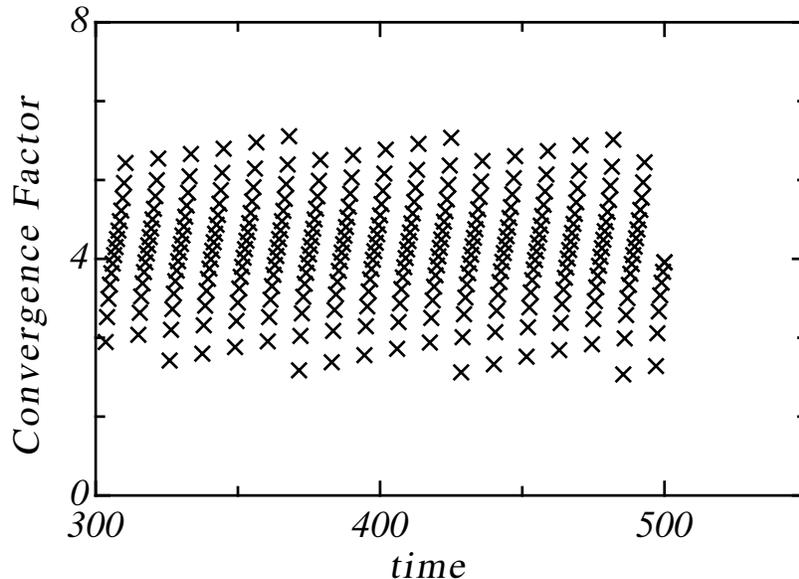


FIG. 1: Analysis of convergence factor \mathcal{C} calculated from evolved data at a point at a radius of $100M$ in the equatorial plane of the black hole. Results show that the code has second order convergence (factor=4). Oscillations in the convergence factor for the fields measured at a point is common in systems where the solution itself oscillates. One can get a smoother plot by showing convergence of quantities in L2 norm, for example.

The code produces output files listing (a) the real and imaginary parts of the ψ field at representative locations of the 2-D grid, and (b) the computed energy and angular momentum fluxes for different values of m . And all this is done periodically at preselected times along the length of the simulation for a circular orbital geodesic and from here the fluxes are postprocessed.

We made standard convergence tests by assuming that the numerical values included errors that decrease as a power of the grid spacing h . Specifically we assume that the value of the Teukolsky function Φ at any point in the computational domain goes like

$$\Phi_{(h)} = \Phi_{exact} + Ch^a + \mathbf{O}(h^{a+1}). \quad (23)$$

Grids typically used in this work are $-40M \leq r_i^* \leq 460M$ and $0 \leq \theta_j \leq \pi$ and $i \simeq 6400$ and $j \simeq 60$. Therefore, if we run at various resolutions (L: 2000X20, M: 4000X40, H: 8000X80) we can get a quantity \mathcal{C} that measures the order of convergence a of the code. This is defined as,

$$\mathcal{C} = \frac{\Phi_L - \Phi_M}{\Phi_M - \Phi_H}, \quad (24)$$

and results for it are depicted in figure 1 where we see second order convergence of the evolution code. We also verified that the results are independent of the Gaussian width σ for both the radial and angular Gaussian distributions used to approximate the location of the particle in the discrete spatial grid (see figure 5).

Since equatorial orbits have been treated with great precision in the frequency domain method, it is important to show that our method gives comparable results if one looks at quantities like the average gravitational energy flux at large distances from the horizon. Finn and Thorne [23] published a comprehensive work that tabulates and includes all previous results for high-precision gravitational wave flux and wave amplitudes coming from many kinds of representative compact objects in equatorial circular orbits around massive central black holes. In Table II we present representative comparisons with the results of formulas (3.8) and (3.10) of that paper using the precise relativistic corrections included in Tables III - VI that they also provide.

IV. DISCUSSION

We have shown that it is possible to integrate the Teukolsky equation in the time domain for the problem of a particle orbiting around a black hole with the particle was represented as a Gaussian and obtain results reasonably close to

m mode	Time domain energy flux	Finn-Thorne flux
1	2.5×10^{-10}	2.2×10^{-10}
2	1.0×10^{-07}	1.4×10^{-07}
3	3.9×10^{-08}	3.1×10^{-08}
4	1.0×10^{-08}	8.0×10^{-09}

TABLE II: Comparisons of gravitational wave energy fluxes detected at infinity using the frequency domain solution of the Teukolsky-Sasaki-Nakamura equation $\dot{E}_{\infty m}$ as calculated in Ref.[23] with the results measured numerically at $r = 100M$ while evolving the Teukolsky equation in the time domain under the "particle-as-a-gaussian-distribution" approximation discussed here. The particle here is in a circular, equatorial orbit of radius $2.0 r_{isco}$ about a black hole with Kerr parameter $a/M = 0.9$ and $\mu/M = 0.01$.

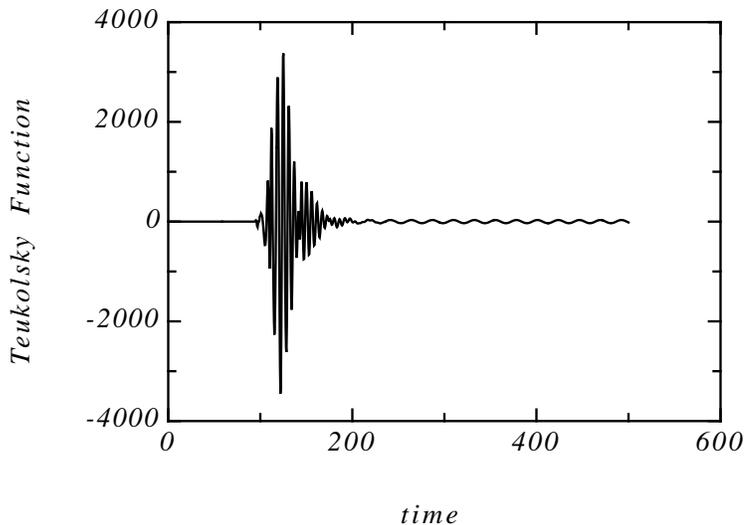


FIG. 2: Representative data showing the evolution of the real part of the Teukolsky function Φ for the $m = 3$ mode of a particle orbiting a rapidly rotating Kerr hole of $a = 0.9M$ at a radius of $2 r_{isco}$ ($r_{isco} = 2.32M$) at an observation point $r/M = 100$, $\theta = \pi/2$.

those of the frequency-domain calculations. The time domain approach is expected to become more useful for the case of more complicated orbits. We plan in future papers to discuss how to use better approximations to the Gaussian like those of reference [26], and how to incorporate radiation-reaction effects. Also, the small discrepancies with the Thorne–Finn results need to be further investigated to define well a “window of reliability” for the results of the code. Another element to be incorporated is to take into account the radiation flux falling into the black hole, since this will also influence trajectories corrected by radiation-reaction. The first approach will be to correct the orbit trajectory using energy-balance arguments, as for instance in [14, 27]. Further possibilities include incorporating proposed radiation reaction forces directly into the integration of the orbit.

Acknowledgments

We are grateful to Pablo Laguna who continually provided help with the use of his Teukolsky code during this work and for comments on the manuscript. We also thank Eric Poisson for providing results for comparison and comments. We are indebted to Lior Burko, Scott Hughes, Carlos Lousto and Karl Martel for comments on the manuscript. RLA is grateful for support in form of a FIPI (Institutional Fund for Starting Research) grant of the Office of the Dean of Research and Graduate Studies of the Rio Piedras campus at the University of Puerto Rico. GK acknowledges

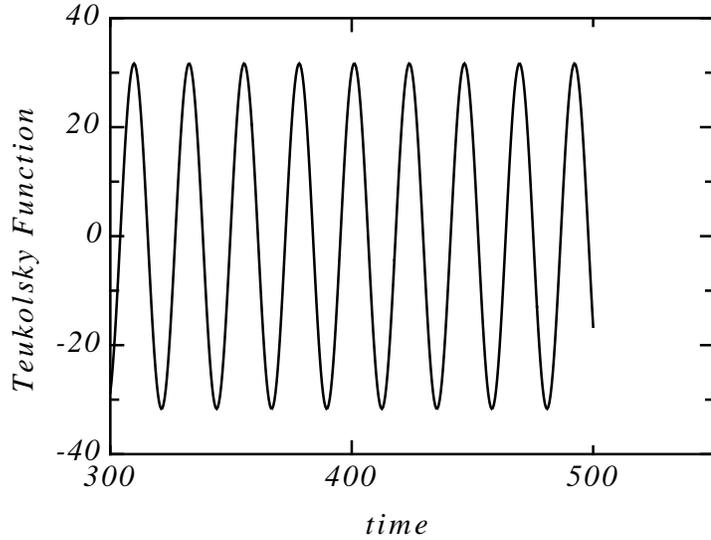


FIG. 3: More detail of the evolution of the Teukolsky function for the same simulation showing the stability of the evolution after the system has settled during a few static circular orbits.

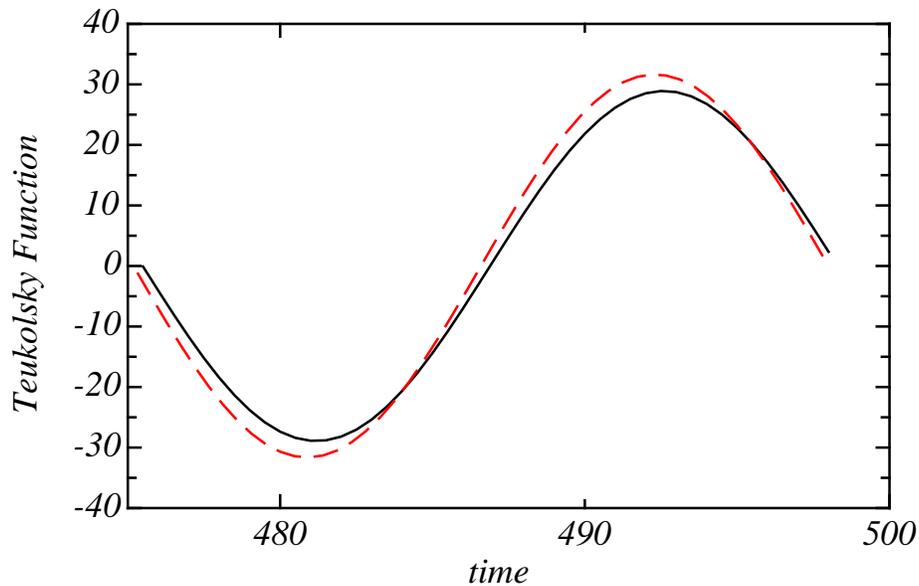


FIG. 4: Teukolsky function insensitivity to the radial and angular width of the gaussian distribution that models the particle in orbit. The parameters for this evolution were the same as used for the other figures. The solid curve corresponds to radial and angular widths of $0.75M$ and 0.47 rad respectively, while dotted curve corresponds to widths of $0.25M$ and 0.16 rad.

research support from Long Island University. This work was supported by NSF grants PHY-0140236, PHY-9800973 and by funds of the Horace Hearne Institute for Theoretical Physics at LSU.

-
- [1] Y. Mino and M. Sasaki and T. Tanaka, Phys. Rev. D, **55**, 3457 (1997)
 [2] T.C. Quinn and R.W. Wald, Phys. Rev. D, **56**, 3381 (1997); Phys. Rev. D, **60**, 064009 (1999)
 [3] S.L. Detweiler, Astrophys. J. **225**, 687 (1978)

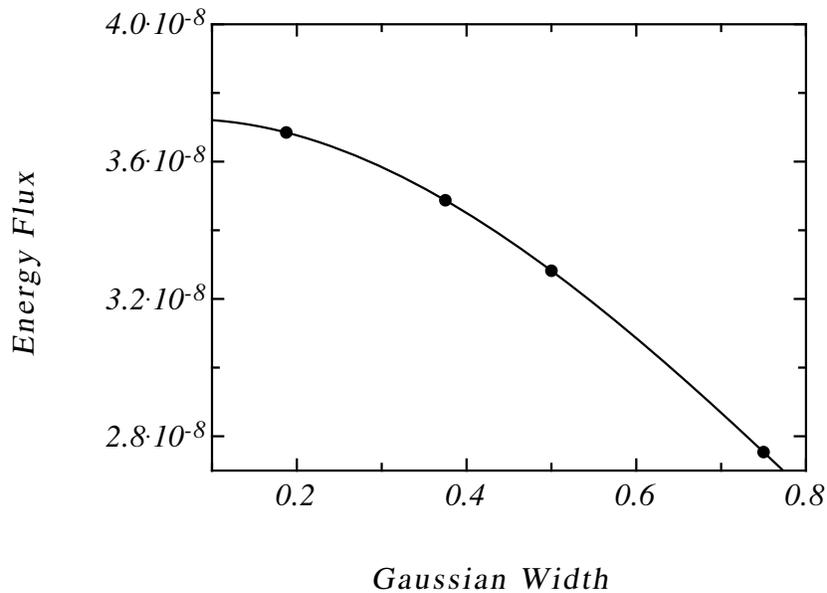


FIG. 5: Convergence of the code when one makes the Gaussian narrower. Of course one cannot expect convergence in the traditional sense when one makes the Gaussian narrower, since at some point it will become too narrow for the finite resolution of the code to model it. What can be expected, and is shown in the figure is that the results tend to a constant value for a while when the Gaussian is made narrower, and shortly before it becomes too narrow for the code to resolve.

- [4] E. Poisson, Phys. Rev. D **47**, 1497 (1993)
- [5] E. Poisson, Phys. Rev. D, **52**, 5719 (1995)
- [6] E. Poisson, Phys. Rev. D, **48**, 1860 (1993)
- [7] T. Apostolatos, D. Kennefick, A. Ori and E. Poisson, Phys. Rev. D, **47**, 5376 (1993)
- [8] E. Poisson and M. Sasaki, Phys. Rev. D, **51**, 5753 (1995)
- [9] C. Cutler, L.S. Finn, E. Poisson and G.J. Sussman, Phys. Rev. D, **47**, 1511 (1993)
- [10] Y. Mino, M. Sasaki, M. Shibata, H. Tagoshi and T. Tanaka, Prog. Theor. Phys. Suppl., **128**, 121 (1997)
- [11] T. Tanaka, M. Shibata, M. Sasaki and T. Nakamura, Prog. Theor. Phys., **90**, 65 (1993)
- [12] M. Shibata, M. Sasaki, H. Tagoshi and T. Tanaka, Phys. Rev. D, **51**, 1646 (1995)
- [13] S.Suzuki and K. Maeda, Phys. Rev. D, **58**, 023005 (1998)
- [14] S.A. Hughes, Phys. Rev. D, **61**, 084004 (2000)
- [15] L. Burko, Phys. Rev. Lett. **84**, 4529 (2000)
- [16] W. Krivan, P. Laguna, P. Papadopoulos and N. Andersson, Phys. Rev. D **56**, 3395 (1997)
- [17] S.A. Teukolsky, Astrophys. J., **185**, 635 (1973)
- [18] J. Bowen and J. York, Phys. Rev. D, **21**, 2047 (1980)
- [19] C.O. Lousto and R.H. Price, Phys. Rev.D, **57**, 1073 (1998)
- [20] M. Campanelli, C.O. Lousto, J.Baker, G. Khanna and J. Pullin, Phys. Rev. D, **58** 084019 (1998)
- [21] R. Lopez-Aleman, “*Evolving Particle Trajectories Perturbatively Around Rotating Black Holes In The Time Domain*”, Ph.D. thesis Penn State Univ. (2001)
- [22] P. Laguna, W. Miller, W. Zurek, Astroph. J. **404**, 678 (1993).
- [23] L.S. Finn and K.S. Thorne, Phys. Rev. D, **62**, 124021 (2000)
- [24] G. Khanna, R. Gleiser, R. Price and J. Pullin, New Journal of Phys., **2**, 3.1 (2000) (<http://www.njp.org>)
- [25] M. Campanelli and C.O. Lousto, Phys. Rev. D, **59**, 124022 (1999)
- [26] R. Price, C. Lousto, Phys. Rev. D **56**, 6439 (1997).
- [27] L. Burko, gr-qc/0208034.
- [28] An alternate way to treat the polar angle dependence of the source term is to expand it using spin-weighted spherical harmonics and keep the as many terms as needed. One must note that this approach has limited applicability and is hard to extend for more generic orbits (inclined orbits, for example). However, it is very useful for testing and verificational purposes.