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Asymptotic Formulae for Restricted Unimodal Sequences

Richard Alexander Frnka
Louisiana State University and Agricultural and Mechanical College, richardfrnka@outlook.com

ASYMPTOTIC FORMULAE FOR RESTRICTED UNIMODAL SEQUENCES

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Richard Alexander Frnka
B.S. in Mathematics and Computer Science, Eckerd College, 2011
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Asymptotic Notation

Big-O Notation: A function $f(x) = O(g(x))$ if and only if there exists a positive real number c such that

$$|f(x)| \leq c|g(x)| \text{ for all } x$$

$\sim$: For $f(n)$ and $\phi(n)$ positive functions of n, as $n \to \infty$, $f/\phi \to 1$,

$\ll$: (Vinogradov Notation)

$$f(x) \ll g(x) \Longleftrightarrow f(x) = O(g(x))$$

q-Series Notation:

$$\begin{aligned}(a;q)_n &= \prod_{k=0}^{n-1}(1-aq^k) \\ &= (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1})\end{aligned}$$

$$(a)_n = (a;q)_n$$

$$(a,b,c,d,\dots;q)_n = (a;q)_n(b;q)_n(c;q)_n(d;q)_n\dots$$

Abstract

Additive enumeration problems, such as counting the number of integer partitions, lie at the intersection of various branches of mathematics including combinatorics, number theory, and analysis. Extending partitions to integer unimodal sequences has also yielded interesting combinatorial results and asymptotic formulae, which form the subject of this thesis.

Much like the important work of Hardy and Ramanujan [11] proving the asymptotic formula for the partition function, Auluck [8] and Wright [26] gave similar formulas for unimodal sequences. Following the circle method of Wright, we provide the asymptotic expansion for unimodal sequences with odd parts. This is then generalized to a two-parameter family of mixed congruence relations, with parts on one side with parts on one side up to the peak satisfying $r \pmod{m}$ and parts on the other side $-r \pmod{m}$, and an asymptotic formula is provided. Techniques used in the proofs include Wright's circle method, modular transformations, and bounding of complex integrals.

Chapter 1
Introduction

A major fact that governs the field of elementary number theory is the Fundamental Theorem of Arithmetic (see [10]), which states that for any number $n \in \mathbb{N}$, we can decompose it in terms of its prime factors. Formally, we can express n as a product of primes:

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \dots p_r^{e_r},$$

where the p_i's are distinct primes with exponents $e_i \in \mathbb{N}$. This provides a unique factorization of natural numbers when writing them multiplicatively.

However, when we consider ways to break down numbers additively, there are various ways to write them in terms of sums of integers. A partition requires that the numbers are written from greatest to smallest (non-increasing order). The prodigious Indian mathematician Srinivasa Ramanujan, along with G.H. Hardy [11], was able to give an asymptotic formula, showing the exponential growth for the number of partitions for large n.

Relaxing the conditions of ordering even more gives way to unimodal sequences, and asymptotic formulas for these were discovered by Auluck [8] and E.M. Wright [26]. A thorough survey of these unimodal sequences is given by Bringmann and Mahlburg in [13].

By restricting the parts of the unimodal sequences, some interesting things happen when trying to count them for large n. Finding asymptotics and exploring different congruence relations on the parts is the main focus of this study, which requires knowledge of complex analysis, basic hypergeometric functions, and modular forms.

1.1 Statement of Main Results

We start with the basic definition of partitions and lay down some of the relevant theorems that are extended to the main results of the paper.

Definition. A partition λ of a natural number $n \in \mathbb{N}$ is an ordered sum of non-increasing integers (parts) $\lambda_1 \ldots \lambda_r$ equaling n. Formally

$$\lambda \mapsto \lambda_1 + \lambda_2 + \cdots + \lambda_r = n$$

with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1.$$

The partition function $p(n)$ counts the number of partitions of size n. The 7 partitions of size 5 are shown below, so $p(5) = 7$:

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1,$$
$$2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1.$$

In 1918, Hardy and Ramanujan published a paper [11] providing an asymptotic expansion of the partition function for large n, showing the exponential growth of $p(n)$:

$$p(n) \sim \frac{1}{2^2 3^{1/2} n} e^{\pi \sqrt{\frac{2n}{3}}}.$$

When adding restrictions to the parts so that each part is $r \pmod m$ and letting $p_{(r,m)}(n)$ count these partitions, an asymptotic formula given by Andrews [5] is

$$p_{(r,m)}(n) \sim \left(\Gamma\left(\frac{r}{m}\right)\pi^{(r/m)-1}2^{-(3/2)-(r/2m)}3^{-(r/2m)}m^{-1/2+(r/2m)}\right)n^{-\frac{1}{2}\left(1+\frac{r}{m}\right)}e^{\pi\sqrt{\frac{2n}{3m}}}.$$

Plugging in $r = 1$ and $m = 2$ gives us the case for odd partitions $p_o(n)$, with asymptotics

$$p_o(n) \sim \frac{1}{4 \cdot 3^{\frac{1}{4}} n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{3}}}.$$

When the parts are allowed to be $\pm r \pmod m$ and letting $p_{(\pm r,m)}(n)$ be the number of partitions,

$$p_{(\pm r,m)}(n) \sim \frac{\csc\frac{\pi r}{m}}{4\pi 3^{\frac{1}{4}} m^{\frac{1}{4}} n^{\frac{3}{4}}} e^{2\pi\sqrt{\frac{n}{3m}}}.$$

Slightly relaxing the restriction on the ordering of the parts by allowing them to increase to a peak and decrease down the other side, we have unimodal sequences.

Definition. A unimodal sequence or "stack" of size n is a sequence of non-zero parts $a_1 \ldots a_r$, c, and $b_s \ldots b_1$ such that

$$n = \sum_{i=1}^{r} a_i + c + \sum_{j=1}^{s} b_j \tag{1.1}$$

and

$$1 \le a_1 \le a_2 \le \cdots \le a_r \le c > b_s \ge \cdots \ge b_2 \ge b_1 \tag{1.2}$$

The stacks of size n will include partitions of n while adding extra sequences. The stacks of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1,$$
$$1+3, \quad 1+2+1, \quad 1+1+1+1$$

Letting $s(n)$ be the stack counting function, then $s(4) = 7$.

Questions about the asymptotics for the stack function were first introduced by the physicist Temperley [24] in 1952, who was studying particle configurations on specific lattices. The entropy of the system depends on the exponential bound of $s(n)$, which had not been previously discovered. This was provided by Auluck in 1952 [8], when he showed

$$s(n) \sim \frac{1}{2^3 3^{3/4} n^{5/4}} e^{\pi\sqrt{2n}}.$$

This was proved using a different method by Wright in 1971 [26] and he also gave a way to calculate lower order terms, providing the asymptotic expansion in powers of $n^{-\frac{1}{2}}$.

We specifically work with odd stacks, where each of the parts must be odd along with the typical restrictions (2.5) and (2.6). Andrews studied combinatorial properties of a related function ($xo(q)$ in [6]), which counts odd stacks with specified summits. The first new result of this thesis provides an asymptotic formula for $s_o(n)$, the number of odd stacks of size n.

Theorem 1. For large n,

$$s_o(n) \sim \frac{1}{2^{\frac{13}{4}} 3^{\frac{1}{4}} n^{\frac{3}{4}}} e^{\pi\sqrt{2n/3}}.$$

Andrews also defined a "mixed parity stack" in [6] where the parts on the left side of the stack up to the peak are odd, and the parts on the right side are even (or vice-versa). We extend this idea to mixed congruence stacks, where the parts on the left up to the peak are $r \pmod m$ and on the right $-r \pmod m$. More formally, it is a sequence $a_1, \ldots, a_k, c, b_s, \ldots, b_1$ where

$$n = a_1 + a_2 + \cdots + a_k + c + b_s + \cdots + b_2 + b_1,$$

and

$$a_i \equiv r(\text{mod } m), \quad c \equiv r(\text{mod } m), \quad b_j \equiv -r(\text{mod } m) \quad \text{for all } i \text{ and } j,$$

with the usual conditions

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_r \leq c > b_s \geq \cdots \geq b_2 \geq b_1.$$

We require that $\gcd(r, m) = 1$, since if r and m share a common factor, we can reduce down to a smaller case. It is also assumed that $1 \leq r < \frac{m}{2}$.

The second major result in this work is an asymptotic formula for the number of mixed congruence stacks of size n, denoted $s_{(r,m)}$.

THEOREM 2. For $0 < r < m/2$,

$$s_{(r,m)}(n) \sim \frac{\csc\left(\frac{\pi r}{m}\right)}{2^3 3^{1/4} m^{1/4} n^{3/4}} \exp\left(2\pi\sqrt{\frac{n}{3m}}\right).$$

1.2 Outline of Thesis

The thesis will provide complete proofs of the main theorems and some computational results to demonstrate the accuracy of the asymptotic formulae that are established. This introductory chapter just outlines the major theorems that lead into the main results, but more detail will be given for each statement along the way.

In Chapter 2, we provide preliminary definitions and results for additive enumeration problems. Generating functions for partitions and unimodal sequences are established and the q-series notation that is necessary in the main proofs is explained. Some historical context of the relevant theorems and results are also discussed in full detail.

In Chapter 3, the proof of Theorem 1 is given following the circle method of Wright used in [26]. Lemmas provided by Wright are expounded upon and adapted for our formulation, and full details of the integral bounds are demonstrated. After the proof of the main theorem, we provide some tables comparing actual values to our exponential approximation and discuss error bounds.

In Chapter 4, the proof of Theorem 2 is shown. Again, the circle method of Wright is used, although there are many significant differences from the previous case. Establishing the modular inversion formula for our infinite products requires special attention, and we must carefully treat the false theta function that arises from the generating function. Some more computational results are shown for smaller cases to test out the asymptotic expansion.

A full list of references is given. Some larger tables are provided in the appendices for actual values of $s(n), s_o(n)$, and $s_{(r,m)}(n)$. Also provided is the Maple code used to generate values and do computations.

Chapter 2
Additive Arithmetic

When considering the ways to write a natural number n in terms of additive parts, there are a lot of assumptions that can be made that completely change the nature of the problem. One typical assumption is that the parts are all natural numbers themselves, so that each part must be greater than 0. Ordering of the parts also plays a major role.

As in many cases of number theory and combinatorics, the questions that govern these additive problems are simple. How can we write one number as a sum of other numbers? Serious study of these problems goes back to Euler. It was not until the early 1900's when renowned combinatorist Major Percival MacMahon [21] made great strides in calculating and computing large sizes of the partition function that the breadth of the subject of additive enumeration was fully understood. In 1918, when Srinivasa Ramanujan and G. Hardy published their paper [11] for an asymptotic formula for the partition function, it was a ground-breaking treatise that opened the door for a new era of mathematics combining complex analysis, combinatorics, and number theory.

The physicist Temperley [24] posed questions for practical applications of these enumerative functions when considering alignment of particles in crystal configurations. If the particles stack evenly when allowed to fall in to place, the ways they can align are a visual representation of these additive arithmetic ideas (see Figure 2.3). One can then calculate the probabilities that the crystals will form certain configurations, which feeds into the statistical mechanics problems that Temperley was studying.

The first case to consider is the number of compositions of a number n.

DEFINITION. A composition is a way to break down an integer $n \in \mathbb{N}$ into parts where order matters:

$$n = a_1 + a_2 + \cdots + a_r.$$

Order mattering means that we can have the same parts, but in different order in separate, unique compositions. The compositions of 5 therefore are

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+3,$$
$$2+2+1, \quad 2+1+2, \quad 2+1+1+1, \quad 1+4, \quad 1+3+1,$$
$$1+2+2, \quad 1+2+1+1, \quad 1+1+3, \quad 1+1+2+1, \quad 1+1+1+2,$$
$$1+1+1+1+1.$$

The fundamental question is to ask how many there are for each n. The answer is actually quite simple, which is why it is our beginning case:

THEOREM 3. The number of compositions $C(n)$ for any $n \in \mathbb{N}$ is 2^{n-1}.

Proof. For the given n, write a 1 n times with spaces between them.

$$1 \,\_\, 1 \,\_\, 1 \,\_\, 1 \,\_\, \ldots \,\_\, 1.$$

Then either put a comma or a + sign in each of the spaces to form the different compositions. There are two choices for each space and $n-1$ spaces, so there are 2^{n-1} possibilities. $\square$

This case completes one side of the additive arithmetic problem under consideration. The next case starts on the other side of the spectrum, and offers much more of a challenge to solve.

2.1 Partitions

Partitions are an excellent example of the beauty inherent in number theory and mathematics, where a simple question can have far-reaching and unsuspected consequences. There are connections to modular forms, combinatorics, complex analysis, and representation theory buried within the deep reaches of partition theory. We start with the basic definition of partitions and lay down some of the relevant theorems and results that are extended to the main results of the paper (See [5] as a good reference for partitions).

DEFINITION. A partition λ of a natural number $n \in \mathbb{N}$ is an ordered sum of non-increasing integers (parts) $\lambda_1 \dots \lambda_r$ equaling n. Formally

$$\lambda \mapsto \lambda_1 + \lambda_2 + \cdots + \lambda_r = n$$

with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1.$$

We can represent partitions visually which adds accessibility to them as a mathematical object. Usually, dots are used to represent partitions in what are called "Ferrers graphs", and they are typically drawn horizontally with the largest part of the partitions at the top as seen in Figure 2.1.

For this paper, we focus on the "Young Tableaux" visual representation. Usually, they are drawn like the Ferrers graphs, but we will rotate them so that the partitions look like staircases that either stay on the same level or decrease as we move

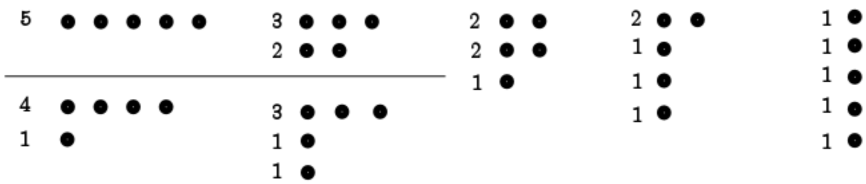

FIGURE 2.1: Ferrers Graph for the partitions of size 5

from left to right as in Figure 2.2. This different approach will be made obvious in the next section when we extend the idea of partitions to unimodal sequences and beyond.

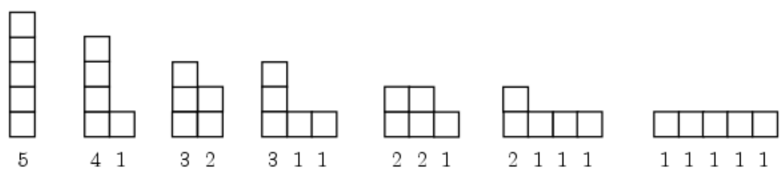

FIGURE 2.2: Young Tableaux for the partitions of size 5

We define the partition function $p(n)$ as the number of partitions of size n. Hence $p(5) = 7$. As the size of n grows, the partition function increases exponentially. This can be seen in Table 2.1. In fact, finding the number of partitions for large n is a rather difficult problem, and baffled many of the best mathematical minds. Even approximating $p(n)$ was difficult, until the problem was approached by the mathematician Ramanujan, who was fascinated with the theory of partitions.

TABLE 2.1: Some small values of the partition function

n	p(n)	n	p(n)	n	p(n)	n	p(n)	n	p(n)
1	1	6	11	11	56	16	231	21	792
2	2	7	15	12	77	17	297	22	1002
3	3	8	22	13	101	18	385	23	1255
4	5	9	30	14	135	19	490	24	1575
5	7	10	42	15	176	20	627	25	1958

The generating function for partitions allows us to computationally calculate out the number that exist for large n, rather than listing out all of the possibilities by hand and counting them. Without loss of generality, we assume that there is 1 partition of size 0, specifically the "empty partition". Thus,

$$\begin{aligned}
P(q) := 1 + \sum_{n\geq 1} p(n)q^n &= \frac{1}{\prod_{k\geq 1}(1-q^k)}. \\
&= \frac{1}{(1-q)(1-q^2)(1-q^3)\dots} \\
&= (1+q+q^2+q^3+\dots)(1+q^2+q^4+q^6+\dots) \\
&\quad * (1+q^3+q^6+q^9+\dots)(1+q^4+q^8+q^{12}+\dots)\dots
\end{aligned}$$

Each $\frac{1}{1-q}$ is a geometric series that expands into an infinite sum. When multiplying these out, exponents are added together and contribute to the coefficient of the powers in the final expansion. The partition

$$18 = 4+4+3+2+2+2+1$$

makes a contribution for the coefficient of q^{18} in the product expansion

$$\begin{aligned}
P(q) &= \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)\dots} \\
&= (1+\boxed{q^{1*1}}+q^{1*2}+q^{1*3}+\dots)(1+q^{2*1}+q^{2*2}+\boxed{q^{2*3}}+\dots) \\
&\quad * (1+\boxed{q^{3*1}}+q^{3*2}+q^{3*3}+\dots)(1+q^{4*1}+\boxed{q^{4*2}}+q^{4*3}+\dots)\dots,
\end{aligned}$$

where for all of the other geometric series, we only multiply by 1.

Following the q-series notation (established in the Nomenclature section)

$$(a;q)_n = \prod_{k=1}^{n-1}(1-aq^k) = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}),$$

we can write

$$P(q) = \frac{1}{\prod_{k\geq 1}(1-q^k)} = (q;q)_\infty^{-1}.$$

We turn our attention to some relevant results in the theory of partitions that will help lead in to the original research done for this work. If we let $p_o(n)$ denote the number of partitions with parts that are specifically odd, then the generating function is

$$P_o(q) := 1 + \sum_{n\geq 1} p_o(n)q^n = (q;q^2)_\infty^{-1}. \tag{2.1}$$

Letting $p_d(n)$ be the number of partitions whose parts are distinct, where there is only one of each size part showing up in a certain partition, then

$$P_d(q) := 1 + \sum_{n\geq 1} p_d(n)q^n = (-q;q)_\infty. \tag{2.2}$$

An important early result due to Euler (see [10]) is that the number of odd partitions of size n are the same, hence

$$p_d(n) = p_o(n).$$

These kinds of identities between different kinds of partitions with various combinatorial definitions is part of what makes partition theory an interesting topic, and the simplicity of the definitions make it accessible to the layman and experienced mathematicians alike. There are countless relations such as these, and each incorporates a different style when considering the proofs. Some famous relations with higher meaning in a deeper context are the Rogers-Ramanujan identities and the Göllnitz-Gordon identities (see [5] for a good reference and many examples).

A common problem when considering combinatorial objects is to guess how many there are as we take higher and higher n. We saw in Table 2.1 that the partition function grows quite rapidly. But to what degree does it grow? We now turn our focus to understanding the asymptotics of this function, which require the use of different methods in complex analysis and partition theory.

It was the celebrated theorem of Hardy and Ramanujan that first established the precise asymptotic behavior of the partition function:

THEOREM 4. As $n \to \infty$,

$$p(n) \sim \frac{1}{2^2 3^{1/2} n} e^{\pi \sqrt{\frac{2n}{3}}}. \tag{2.3}$$

Letting $x(n) = \frac{1}{2^2 3^{1/2} n} e^{\pi \sqrt{\frac{2n}{3}}}$, we see in Table 2.2 that this approximation is very close to the actual value for the partition function for significantly large n.

TABLE 2.2: Large partitions and asymptotics

n	$p(n)$	$x(n)$
100	190569292	199280893
1000	$2.41 * 10^{31}$	$2.44 * 10^{31}$
10000	$3.62 * 10^{106}$	$3.63 * 10^{106}$

Thus we can closely approximate how the partition grows with larger and larger n. It is these style of results we seek in the main theorems of this work.

Let $p_{(r,,m)}(n)$ be the number of partitions where the parts are congruent to $r \pmod{m}$. The following theorem has been shown several ways, and may be found in [5]:

THEOREM 5. As $n \to \infty$,

$$p_{(r,m)}(n) \sim \left(\Gamma\left(\frac{r}{m}\right) \pi^{(r/m)-1} 2^{-(3/2)-(r/2m)} 3^{-(r/2m)} m^{-1/2+(r/2m)} \right) n^{-\frac{1}{2}\left(1+\frac{r}{m}\right)} e^{\pi \sqrt{\frac{2n}{3m}}}. \tag{2.4}$$

Plugging in $r = 1$ and $m = 2$ gives us the case for odd partitions $p_o(n)$, with asymptotics

$$p_o(n) \sim \frac{1}{4 \cdot 3^{\frac{1}{4}} n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{3}}}.$$

Andrews also gives the case when the parts are allowed to be $\pm r \pmod{m}$. Letting $p_{(\pm r,m)}(n)$ be the number of of these partitions,

$$p_{(\pm r,m)}(n) \sim \frac{\csc \frac{\pi r}{m}}{4\pi 3^{\frac{1}{4}} m^{\frac{1}{4}} n^{\frac{3}{4}}} e^{2\pi\sqrt{\frac{n}{3m}}}.$$

As one of the applications of partitions, when thinking about lattice structure of crystals [24], there may be certain properties in the crystal that make use of (2.4), especially the exponent, thus it is a relevant topic to investigate. It also invites speculation into future possibilities in this area. In fact, we can take multiple congruence relations and combine them in an extension of these kinds of ideas.

2.2 Unimodal Sequences

After partitions, the next step is to slightly relax the restrictions on the ordering. A unimodal sequence of size n is a sequence of positive integers such that the parts are listed in increasing order until a " peak" is reached, and then the numbers decrease on the other side of the peak. The term unimodal describes the single "mode" or peak of the staircase.

Unimodality occurs naturally in a number of combinatorial functions including certain polynomials whose coefficients have unimodal properties. One can look at Richard Stanley's book Enumerative Combinatorics [23] for some typical phenomena and occurences of unimodality in enumeration problems.

One may even continue the idea of changing the restriction on the ordering of the parts of a sequence of positive integers summing to n by adding more peaks. Adding an increasing staircase on either side of a one-mode sequence would make a "one-and-a-half mode" sequence creating another interesting combinatorial object. These extend the idea of n-tuple partitions, which serve as a superset of "multi-modal sequences". Very little work has been done after the unimodal case, and further studies in the future will attack these kinds of problems.

As in the summary on partitions, we will define and initialize the generating functions for unimodal sequences and point to some important results in the literature for these interesting structures.

Definition. A unimodal sequence or "stack" of size n is a sequence of non-zero parts $a_1 \dots a_r$, c, and $b_s \dots b_1$ such that

$$n = \sum_{i=1}^{r} a_i + c + \sum_{j=1}^{s} b_j, \tag{2.5}$$

and

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_r \leq c > b_s \geq \cdots \geq b_2 \geq b_1. \tag{2.6}$$

Like partitions, the visual representation of these stacks brings more meaning to the definitions. When drawing the parts in a Young Tableaux, one gets a staircase that goes up on one side and comes down on the other. In Figure 2.3, we see the stacks of size 4.

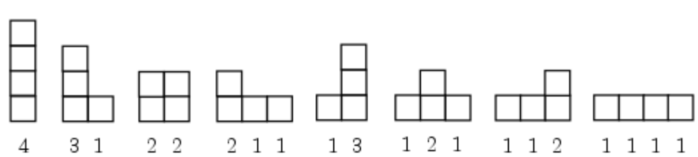

FIGURE 2.3: The stacks of size 4

The notation previously established for the q-factorial is still very relevant and alleviates some tedium in the following analysis. If we let $s(n)$ be the number of stacks of size n, the generating function is

$$
\begin{aligned}
S(q) :&= \sum_{n\geq 1} s(n)q^n = \sum_{m\geq 1} \frac{q^m}{(q;q)_{m-1}^2(1-q^m)} \\
&= \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)(1-q)} + \frac{q^3}{(1-q)(1-q^2)(1-q^3)(1-q)(1-q^2)} \dots \\
&= q + 2q^2 + 4q^3 + 8q^4 + 15q^5 + 27q^6 + 47q^7 + \dots
\end{aligned}
$$

Here, the index m increments over the peak of the stacks, and the geometric series go up to the exponent of the peak on the left, and go up to one less than the peak on the right. Note that here we have dropped the convention of including the "empty" stack of size 0.

The stack function $s(n)$ grows quickly as n increases. From Table 2.3, we can see that it is probably infeasible to even list out all the stacks of size 10.

TABLE 2.3: The number of stacks for $n = 1 \dots 25$

n	$s(n)$	n	$s(n)$	n	$s(n)$	n	$s(n)$	n	$s(n)$
1	1	6	27	11	330	16	2604	21	15880
2	2	7	47	12	512	17	3804	22	22277
3	4	8	79	13	784	18	5504	23	31048
4	8	9	130	14	1183	19	7898	24	43003
5	15	10	209	15	1765	20	11240	25	59220

The infinite sum for the generating function of stacks cannot be analyzed directly, although it is useful for computing smaller values of the stack function. Using it in its current form becomes computationally expensive around $n = 1000$, so it must be adjusted before any asympotic formulas can be found. Fortunately, it can be rewritten as an infinite q-series multiplied by a false theta function, specifically

$$S(q) = \frac{1}{(q;q)_\infty^2} \sum_{n\geq 1} (-1)^{n-1} q^{\frac{n(n-1)}{2}}. \tag{2.7}$$

Analytic proofs for this formula can be found in [8] and [26], and a combinatorial proof is also given by [23]. For a full expose of stacks and different variations of unimodal sequences, see [13], [8], and [25]−[27]. Auluck first showed using the constant term method that

$$s(n) \sim \frac{1}{2^3 \cdot 3^{\frac{3}{4}} n^{\frac{5}{4}}} e^{2\pi\sqrt{\frac{1}{3}n}}. \tag{2.8}$$

This was proven using a different method by Wright in 1971 [26] and he also gave a way to calculate lower order terms, providing the asymptotic expansion in powers of $n^{-\frac{1}{2}}$.

In relation to partitions, the constant in the exponent outstrips the asymptotics for $p(n)$. On the far end of the spectrum exists the exact number of compositions at 2^{n-1} for any given n, and what happens in between is a mystery that must eventually be solved to truly understand the nature of multi-modal sequences and multi-partitions.

Chapter 3
Odd Stacks

We shift our attention to stacks whose parts have specific congruence relations. Comparable to the situation with partitions, these restrictions have a major effect on the asymptotic relations involved, and make for interesting analysis.Unlike with the partition case, we will see that there are certain congruence relations that we cannot consider, but we still attack the ones that are within reach.

A future investigation may be made into stacks that have parts that are not congruent to a certain relation. This kind of cancellation will still allow the use of eta-products and their transformations for infinite q-series over a certain base.

The first case to consider is when stacks have exclusively even parts. If we let $s_e(n)$ be the number of stacks of n with even parts, one can quickly tell that $s_e(n) = 0$ for n odd. The generating function is

$$\begin{aligned} S_e(q) := \sum_{n\geq 0} s_e(n)q^n &= 1 + \sum_{m\geq 1} \frac{q^{2m}}{(q^2;q^2)_{m-1}(q^2;q^2)_m} \\ &= 1 + 1q^2 + 2q^4 + 4q^6 + 8q^8 + 15q^{10} + \dots \end{aligned}$$

Even stacks are not particularly interesting as they can be obtained directly from normal stacks by the fact that if $\{a_1, a_2, \dots a_k, c, b_l, \dots, b_2, b-1\}$ is a normal stack of size n, then $\{2a_1, 2a_2, \dots, 2a_k, 2c, 2b_l, \dots, 2b_2, 2b_1\}$ is an even stack of size $2n$. Thus we have the relation

$$s_e(n) = \begin{cases} s(\frac{n}{2}) \text{ for } n \text{ even} \\ 0 \text{ for } n \text{ odd} \end{cases}. \tag{3.1}$$

The generating function can be rewritten as

$$S_e(q) = 1 + \sum_{m\geq 1} \frac{q^{2m}}{(q^2;q^2)_{m-1}(q^2;q^2)_m} = \frac{1}{(q^2;q^2)_\infty^2} \sum_{n\geq 1} (-1)^{n-1} q^{n(n+1)}. \qquad (3.2)$$

Using (3.1), and plugging in to equation (2.8), we see

Corollary.

$$s_e(n) \sim \frac{1}{2^{\frac{7}{4}} 3^{\frac{3}{4}} n^{\frac{5}{4}}} e^{\pi\sqrt{\frac{2n}{3}}}. \qquad (3.3)$$

3.1 Generating Functions

We consider the case for stacks whose parts must be odd. Denoting the number of these stacks as $s_o(n)$, the first few terms are easy to calculate by hand by listing out the possible combinations. The odd stacks for $n = 5$ are shown in Figure 3.1.

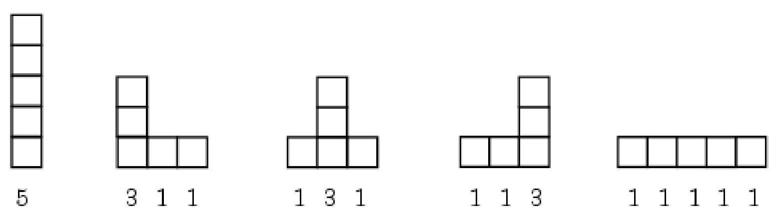

FIGURE 3.1: The odd stacks for $n = 5$.

The generating function is given as

$$\begin{aligned} S_o(q) = \sum_{n=0}^{\infty} s_o(n) q^n &= 1 + \sum_{m=1}^{\infty} \frac{q^{2m-1}}{(q;q^2)_m (q;q^2)_{m-1}} \\ &= \frac{q}{(1-q)} + \frac{q^3}{(1-q)^2(1-q^3)} + \frac{q^5}{(1-q)^2(1-q^3)^2(1-q^5)} + \dots \\ &= 1 + q + q^2 + 2q^3 + 3q^4 + 5q^5 + 8q^6 + 12q^7 + 18q^8 + 26q^9 + 37q^{10} + \dots \end{aligned}$$

Again, the sum increments over the odd peak of size $2m-1$ and the geometric series representing the left side of the stack go up to $1 - q^{2m-1}$ while the series from the right can only go up to $1 - q^{2m-3}$. In the case of odd summitted stacks, there is an extra $1-q^{2m-1}$ term in the denominator as we can have the maximum size to the left and right of the peak.

The behavior of this infinite summand is difficult to analyze and approximate in this form. It can be used to pick off the lower-valued coefficients, but this gets computationally expensive for larger numbers. It was used to list the first 50 values of the odd stack function in Table 3.1.

TABLE 3.1: Odd stacks for $n = 1 \ldots 50$.

n	$s_o(n)$	n	$s_o(n)$	n	$s_o(n)$	n	$s_o(n)$	n	$s_o(n)$
1	1	11	52	21	881	31	8490	41	59764
2	1	12	72	22	1126	32	10444	42	71726
3	2	13	98	23	1434	33	12807	43	85912
4	3	14	133	24	1815	34	15660	44	102711
5	5	15	178	25	2288	35	19102	45	122562
6	8	16	236	26	2874	36	23236	46	145986
7	12	17	312	27	3594	37	28196	47	173592
8	18	18	408	28	4478	38	34138	48	206062
9	26	19	530	29	5562	39	41232	49	244204
10	37	20	686	30	6883	40	49692	50	288954

As with equation (2.7), we wish to find another form of the generating function with some infinite q-series multiplied by some kind of sum.

THEOREM 6.

$$\sum_{n\geq 1} \frac{q^{2n-1}}{(q;q^2)_{n-1}} = \frac{1}{(q;q^2)_\infty^2} \sum_{n\geq 1}(-1)^{n-1}q^{n(n+1)} + \sum_{k\geq 1}(-1)^k \left(q^{3k^2+2k} - q^{3k^2-2k}\right). \quad (3.4)$$

Proof. One would expect the main term by looking at the summation form for the standard stacks in (2.7). Because of some slight hiccups when dealing with odd parts, the combinatorial mechanics used by Stanley in [23] cannot be replicated, with the source of the problem being the other sum, which serves as a remainder term.

We need the following formula from Ramanujan's "lost notebook", a proof of which is given by Andrews [4] using Heine's transformation formula.

$$1+\sum_{n\geq 1}\frac{q^n}{(1+aq)(1+aq^2)\dots(1+aq^n)(1+bq)\dots(1+bq^n)}$$
$$=(1+a^{-1})\left(1+\sum_{n\geq 1}\frac{(-1)^n q^{n(n+1)/2}b^n a^{-n}}{(1+bq)\dots(1+bq^n)}\right)-\frac{a^{-1}\sum_{n\geq 0}(-1)^n q^{n(n+1)/2}b^n a^{-n}}{\prod_{j\geq 1}(1+aq^j)(1+bq^j)}.$$

Sending $q\mapsto q^2$, $a\mapsto -q$, and $b\mapsto -q^{-1}$ gives

$$1+\sum_{n\geq 1}\frac{q^{2n}}{(1-q^3)(1-q^5)\dots(1-q^{2n+1})(1-q))(1-q^3)\dots(1-q^{2n-1})}$$
$$=(1-q^{-1})\left(1+\sum_{n\geq 1}\frac{(-1)^n q^{n(n-1)}}{(1-q)(1-q^3)\dots(1-q^{2n-1})}\right)$$
$$+(q^{-1})\frac{\sum_{n\geq 0}(-1)^n q^{n(n-1)}}{(1-q)(1-q^3)^2(1-q^5)^2\dots}.$$

Multiplying everything by $\frac{q}{1-q}$, we can now use q-series notation nicely and have

$$\frac{q}{(1-q)}+\sum_{n\geq 1}\frac{q^{2n+1}}{(q;q^2)_{n+1}(q;q^2)_n}=\sum_{n\geq 1}\frac{(-1)^{n-1}q^{n(n-1)}}{(q;q^2)_n}+\frac{1}{(q;q^2)_\infty^2}\sum_{n\geq 2}(-1)^n q^{n(n-1)}.$$

Reindexing the summands leaves us with

$$\sum_{n\geq 1}\frac{q^{2n-1}}{(q;q^2)_n(q;q^2)_{n-1}}=\frac{1}{(q;q^2)_\infty^2}\sum_{n\geq 1}(-1)^{n-1}q^{n(n+1)}+\sum_{k\geq 1}(-1)^{k-1}\frac{q^{n^2-n}}{(q;q^2)_n}. \quad (3.5)$$

The first term on the right-hand-side makes up our main term. The second summation is a remainder term that we can manipulate to demonstrate the sparsity of the terms. Again, we need a q-series identity, this time courtesy of L.J. Rogers

[16]:

$$1+\sum_{n\geq 1}\frac{(-1)^n y^{2n} q^{n(n+1)/2}}{(yq;q)_n}=\sum_{n\geq 0}(-1)^n y^{3n} q^{n(3n+1)/2}(1-y^2q^{2n+1}). \tag{3.6}$$

Sending $q\mapsto q^2$ and $y\mapsto q^{-1}$, (3.6) becomes

$$1+\sum_{n\geq 1}\frac{(-1)^n q^{n^2-n}}{(q;q^2)_n}=\sum_{n\geq 0}(-1)^n q^{3n^2-2n}(1-q^{4n}).$$

Multiplying by -1, the left side of the equation is exactly the right-most summation in (3.5). Now expanding out the right hand side, and noting that the terms cancel when $n=0$, we have finished the proof. $\square$

In [6], Andrews showed a similar formula for odd stacks with summits (Theorem 2 for $xo(q)$), where the peak is explicitly specified in each stack.

3.2 Asymptotic Analysis

Now that we have rewritten the generating function, we can concisely write

$$S_o(q)=\frac{1}{(q;q^2)_\infty^2}L(q)+R(q), \tag{3.7}$$

where

$$L(q):=\sum_{n\geq 1}(-1)^{n-1}q^{n(n-1)}, \tag{3.8}$$

and

$$R(q):=\sum_{n\geq 1}(-1)^k q^{3k^2+2k}+(-1)^{k-1}q^{3k^2-2k}. \tag{3.9}$$

The remainder term $R(q)$ expanded out is

$$\sum_{r\geq 1}(-1)^k\left(q^{3k^2+2k}-q^{3k^2-2k}\right)=q-q^5-q^8+q^{16}+q^{21}-q^{33}-q^{44}+q^{56}+\ldots.$$

The coefficients are quadratically distributed and insignificant when compared with those of the main term as they only contribute ± 1, even for large values of q.

This remainder term was originally surprising and found computationally by subtracting from the actual amount of odd stacks the main term that was expected from following Stanley's proof. These terms are the "generalized octagonal numbers", but the distribution of the plus and minus signs are quite strange. There must be some combinatorial reasoning behind these remainder terms, but so far, the explanation is beyond reach. As these terms contribute very little to the asymptotic analysis, we shift our focus to the main term

$$V(q) := \sum_{n\geq 1} v(n)q^n = \frac{1}{(q;q^2)_\infty^2} L(q) = \frac{1}{(q;q^2)_\infty^2} \sum_{n\geq 1} (-1)^{n-1} q^{n(n-1)}. \tag{3.10}$$

By (3.7), $s_o(n) \sim v(n)$, so this will effectively determine the asymptotic result for odd stacks, which is our main theorem of the chapter.

Theorem 7. As $n \to \infty$,

$$v(n) \sim \frac{1}{2^{\frac{9}{4}} 3^{\frac{1}{4}} (n + \frac{1}{12})^{\frac{3}{4}}} \exp\left(\pi \sqrt{\frac{2n}{3} + \frac{1}{18}}\right).$$

The function $L(q)$ is a "false theta function". It is false in the regards that it does not observe the same functional equations that a normal theta function would follow, although it has a similar shape. Without these nice transformations, it becomes tougher to analyze than a typical theta function.

We must know what kind of contribution the false theta function contributes to the main term. Plugging $2z$ in for z in Lemma 1 of [26] (p. 110), we can rewrite $L(q)$ in powers of z, with a way to express coefficients up to a desired power.

Lemma 8. If $|\arg(z)| < \frac{\pi}{2} - c$ for some constant c, then as $|z| \to 0$,

$$L(e^{-z}) = \sum_{s=0}^{k-1} \alpha_s(z)^s + O(z^k), \tag{3.11}$$

where

$$\alpha_s = \frac{(s)!}{2(s+1)!} + \frac{(-1)^s}{(s)!} \sum_{h\geq 0}^{\lfloor \frac{1}{2}(s-1) \rfloor} \frac{2^{2s-2h-1}}{(s-h)} \binom{s}{2h+1} B_{2s-2h}.$$

The circle method, when used by Hardy and Ramanujan to prove their asymptotic formula for the partition function, represented a major shift in thinking and ushered in a new era of using complex analysis to solve difficult problems and find coefficients of generating functions. The method can be summarized in a few steps which will be followed thusly:

1. Use Cauchy's formula to transform coefficients of generating functions into integrals,

2. Integrate over an appropriate circle that will allow easier calculations,

3. Bound the integrals over the major and minor arcs,

4. Manipulate the integrals to avoid or shift around simple poles.

Before using the circle method, we need another way to represent the infinite q-series in the main term (3.10). The modular inversion formula of the Dedekind Eta function is well known (see Chapter 20 of Iwaniec-Kowalski [12] for a good reference) and gives

$$(q;q)_\infty = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i \tau}{12} - \frac{\pi i}{12\tau}} \left(1 + O(e^{-\frac{2\pi i}{\tau}})\right). \tag{3.12}$$

Using this and the fact that

$$\frac{1}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}, \tag{3.13}$$

we obtain

$$\frac{1}{(q;q^2)_\infty} = \frac{1}{\sqrt{2}} e^{-\frac{\pi i \tau}{12} + \frac{\pi i}{24\tau}} \left(1 + O(e^{-\frac{\pi i}{\tau}})\right). \tag{3.14}$$

What we have in (3.13) is referred to as an "eta-quotient". The Dedekind-Eta function is

$$\eta(\tau) := q^{1/24}(q;q)_\infty$$

and forms one of the initial examples of modular forms (it is a holomorphic modular form of weight $\frac{1}{2}$). These eta-quotients are weakly holomorphic modular forms for certain congruence subgroups. Because of the nice transformation formulas, we are able to come up with a good exponential bound for these eta-quotients.

To better follow the circle method used by Wright in [26], we similarly adopt the convention of using $x = e^{-z}$ instead of $q = e^{2\pi i \tau}$. With this notation, we make the transformation $\tau \to \frac{iz}{2\pi}$ in the above equation to obtain

$$\frac{1}{(x;x^2)_\infty} = \frac{1}{\sqrt{2}} e^{\frac{z}{24} + \frac{\pi^2}{12z}} \left(1 + O(e^{-\frac{2\pi^2}{z}})\right). \tag{3.15}$$

Since we are working with $(q;q^2)_\infty^{-2}$, we square the main term and define the function

$$w(z) := \frac{1}{2} e^{\frac{z}{12} + \frac{\pi^2}{6z}}.$$

An important term in our asymptotic analysis is

$$w(z)x^{-n} = \frac{1}{2} e^{(n+\frac{1}{12})z + \frac{\pi}{6z}}.$$

The Arithmetic-Geometric Mean inequality states that for x and $y \geq 0$,

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Hence, we have that

$$\frac{1}{2}\left(\left(n+\frac{1}{12}\right)z+\frac{\pi^2}{6z}\right) \geq \sqrt{\left(n+\frac{1}{12}\right)z \cdot \frac{\pi^2}{6z}}. \tag{3.16}$$

Multiplying by z and bringing everything to one side, we have the quadratic equation

$$\frac{1}{2}\left(n+\frac{1}{12}\right)z^2 - \left(\pi\sqrt{\frac{(n+\frac{1}{12})}{6}}\right)z+\frac{\pi^2}{6} \geq 0.$$

Completing the square and solving for z, we see that equality occurs when $z = \frac{\pi}{\sqrt{6(n+\frac{1}{12})}}$. Thus we define the constant κ

$$\kappa := \frac{\pi}{\sqrt{6(n+\frac{1}{12})}}.$$

In the $x-$plane, we now define the circle $\mathscr{C}$ to be the circle where $|x| = e^{-\kappa} = e^{-\frac{\pi}{\sqrt{6(n+\frac{1}{12})}}}$. Breaking this circle into two parts, let $\mathscr{C}_1$ be the major arc where $|\arg(x)| \leq 6\kappa$ (this choice for the size of the arc is explained in Remark 10), and let $\mathscr{C}_2 = \mathscr{C} - \mathscr{C}_1$ be the minor arc (see Figure 3.2).

The major arc establishes the main asymptotic bound at the major pole of $x = 1$. The minor arc accounts for the other part of the circle away from the pole at $x = 1$. Then using Cauchy's formula, the coefficients in our Fourier expansion of $V(x)$ are

$$v(n) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{L(x)(x;x^2)_\infty^{-2}}{x^{n+1}}dx. \tag{3.17}$$

We cannot directly calculate this integral. What we would like to do is approximate

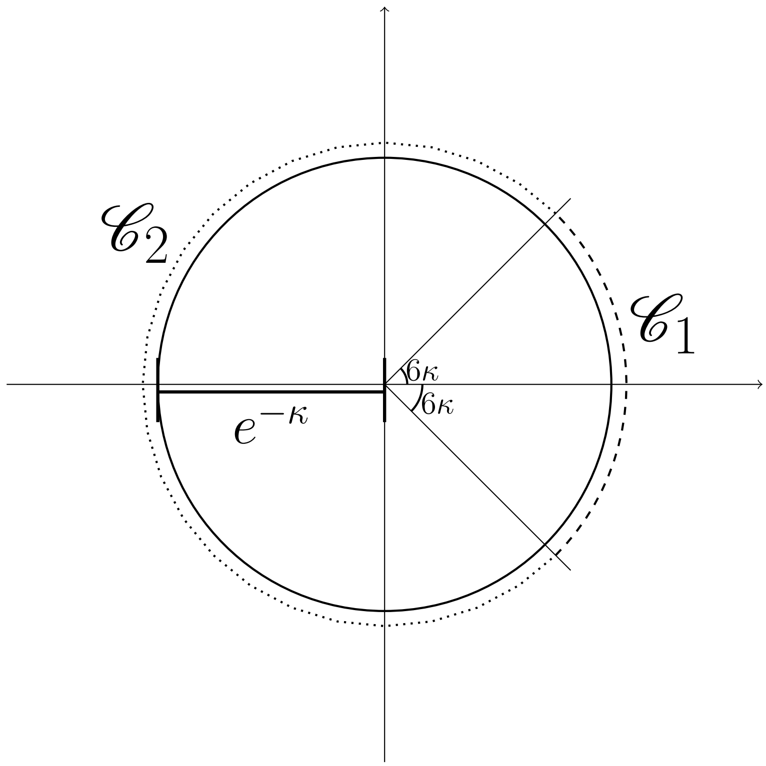

FIGURE 3.2: The circle $\mathscr{C}$ with major and minor arcs

the integral using

$$v_s(n) = \frac{1}{2\pi i} \int_{\mathscr{C}_1} \frac{z^s w(z)}{x^{n+1}} dx, \tag{3.18}$$

which incorporates the asymptotic bound we found in Lemma 8. We will see in Proposition 11 that $v_s(n) \sim e^{2N}$ where

$$N := \frac{\pi^2}{6\eta} = \pi \sqrt{\frac{\left(n + \frac{1}{12}\right)}{6}},$$

so we need to show that $v(n)$ is approximated by $\sum_s \alpha_s v_s(n)$ with a small error term compared to e^{2N}.

Proposition 9.

$$v(n) = \sum_{s=0}^{k-1} \alpha_s v_s + O(N^{-k} e^{2N}).$$

Proof. We take the difference between $v(n)$ and $\sum_{s=0}^{k-1} \alpha_s v_s(n)$, break the problem into 3 parts, and handle them individually:

$$2\pi i \left(v(n) - \sum_{s=0}^{k-1} \alpha_s v_s(n) \right) = \int_{\mathscr{C}_1} \left(L(x) - \sum_{s=0}^{k-1} \alpha_s z^s \right) w(z) x^{-n-1} dx \qquad (E_1)$$
$$+ \int_{\mathscr{C}_1} L(x)((x;x^2)^{-2} - w(z)) x^{-n-1} dx \qquad (E_2)$$
$$+ \int_{\mathscr{C}_2} L(x)(x;x^2)^{-2} x^{-n-1} dx. \qquad (E_3)$$

To bound E_1, consider the function

$$\begin{aligned} \left| w(z) x^{-n} \right| &= \left| \frac{1}{2} e^{(n+\frac{1}{12})z + \frac{\pi^2}{6z}} \right| \\ &\ll \frac{1}{2} \left| e^{(n+\frac{1}{12})z} \right| \left| e^{\frac{\pi^2}{6z}} \right|. \end{aligned}$$

On the circle $\mathscr{C}$, we can let $z = \kappa + i\nu$ so $\mathrm{Re}(\frac{1}{z}) = \frac{\kappa}{\kappa^2+\nu^2} \leq \frac{1}{\kappa}$. Using our arithmetic-geometric mean inequality (3.16),

$$\begin{aligned} \frac{1}{2} \left| e^{(n+\frac{1}{12})z} \right| \left| e^{\frac{\pi^2}{6z}} \right| &= \frac{1}{2} e^{(n+\frac{1}{12})\kappa} e^{\frac{\pi^2}{6}\left(\frac{\kappa}{\kappa^2+\nu^2}\right)} \\ &\ll \frac{1}{2} e^{\left(n+\frac{1}{12}\right)\kappa} e^{\frac{\pi^2}{6\kappa}} && (3.19) \\ &\ll \frac{1}{2} e^{2\left(\frac{\pi}{\sqrt{6}} \sqrt{n+\frac{1}{12}}\right)} \\ &= O\left(e^{2N}\right). && (3.20) \end{aligned}$$

For the integral E_1, we are on $\mathscr{C}_1$ so $|z| \leq |\kappa + 6\kappa i| = O(\kappa)$. Along with Lemma 8, this implies

$$\left| L - \sum_{s=0}^{k-1} \alpha_s z^s \right| \ll |z|^k \ll \kappa^k. \qquad (3.21)$$

Combining (3.20) and (3.21),

$$\begin{aligned}
|E_1| &= \left| \int_{\mathscr{C}_1} \left(L(x) - \sum_{s=0}^{k-1} \alpha_s z^s \right) w(z) x^{-n-1} dx \right| \\
&\leq |e^z| \left| w(z) x^{-n} \right| \left| L - \sum_{s=0}^{k-1} \alpha_s z^s \right| \\
&= O(\kappa^k e^{2N}) \\
&= O(N^{-k} e^{2N}).
\end{aligned}$$

To bound E_2, we know from our modular inversion formula (3.14) that

$$\left|(x;x^2)_\infty^2 - w(z)\right| \ll \left| w(z) e^{-\frac{2\pi^2}{z}} \right|. \tag{3.22}$$

Also, from Lemma 8 and (3.21),

$$|L(x)| \leq \left| \sum_{s=0}^{k-1} \alpha_s z^s + cz^k \right| = O(1). \tag{3.23}$$

On $\mathscr{C}_1$, $\mathrm{Re}\frac{1}{z} \leq \frac{\kappa}{\kappa^2\sqrt{1+36}} \leq \frac{1}{\sqrt{37}\kappa}$. Thus $\left|e^{-\frac{4\pi^2}{z}}\right| \leq e^{-\frac{4\pi^2}{\sqrt{37}\kappa}} \sim e^{-cN}$ for some $c > 0$. Using this along with (3.20), (3.22), and (3.23), we can bound E_2:

$$\begin{aligned}
|E_2| &= \left| \int_{\mathscr{C}_1} L(x) \Big((x;x^2)^{-2} - w(z) \Big) x^{-n-1} dx \right| \\
&\ll |L(x)| \left| w(x)^2 x^{-n} \right| \left| e^{-\frac{4\pi^2}{z}} \right| \\
&\ll e^{2N - \frac{4\pi^2}{\sqrt{37}\kappa}} \\
&= e^{2N-cN} \\
&\ll N^{-m} e^{2N} \qquad \text{for any } m.
\end{aligned}$$

When working with E_3, we are far away from the simple pole at $x = 1$, so we can consider the logarithm in order to bound the integral on the minor arc. Taking the log of the infinite product turns it in to a sum of infinite logs;

$$\begin{aligned}\log((x;x^2)_\infty^{-1}) &= \log\left(\frac{(x^2;x^2)_\infty}{(x;x)_\infty}\right)\\ &= \sum_{h\geq 1} -\log(1-x^h)+\log(1-x^{2h}).\end{aligned}$$

Using the Taylor Series

$$\log(1-x) = -\sum_{m\geq 1}\frac{x^m}{m},$$

the log becomes

$$\begin{aligned}\log((x;x^2)_\infty^{-1}) &= -\sum_{m\geq 1}\frac{1}{m}\left(\sum_{h\geq 1}(-x^{hm}+x^{2hm})\right)\\ &= -\sum_{m\geq 1}\frac{1}{m(1-x^{2m})}-\frac{1}{m(1-x^m)}\\ &= -\sum_{m\geq 1}\frac{1}{m(1-x^{2m})}-\left(\frac{1}{2m(1-x^{2m})}+\frac{1}{(2m-1)(1-x^{2m-1})}\right)\\ &= -\sum_{m\geq 1}\frac{1}{2m(1-x^{2m})}-\frac{1}{(2m-1)(1-x^{2m-1})}\\ &= \sum_{m\geq 1}\frac{(-1)^{m-1}}{m(1-x^m)}.\end{aligned}$$

In this form, we can now bound the log

$$\begin{aligned}\left|\log(x;x^2)_\infty^{-1}\right| &= \left|\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m(1-x^m)}\right|\\ &\leq \sum_{m=1}^{\infty}\frac{1}{m|1-x^m|}.\end{aligned}$$

Shifting the sum, we can add in and subtract out a term and since $\frac{1}{|1-x|} \leq \frac{1}{1-|x|}$,

$$\begin{aligned}
\sum_{m\geq 1} \frac{1}{m|1-x^m|} &= \sum_{m\geq 2} \frac{1}{m|1-x^m|} + \frac{1}{|1-x|} + \left(\frac{1}{1-|x|} - \frac{1}{1-|x|}\right) \\
&\leq \frac{1}{1-|x|} + \sum_{m\geq 2} \frac{1}{m(1-|x|^m)} + \left(\frac{1}{|1-x|} - \frac{1}{1-|x|}\right) \\
&= \sum_{m\geq 1} \frac{1}{m(1-|x|^m)} + \left(\frac{1}{|1-x|} - \frac{1}{1-|x|}\right) \\
&= (|x|;|x|)_\infty^{-1} - \left(\frac{1}{1-|x|} - \frac{1}{|1-x|}\right). \qquad (3.24)
\end{aligned}$$

On all of $\mathscr{C}$, $|x| = e^{-\kappa}$ so plugging in $\tau = \frac{i\kappa}{2\pi}$ in the modular inversion formula (3.14),

$$(|x|;|x|)_\infty^{-1} \sim \sqrt{\frac{\kappa}{2\pi}} e^{\frac{\pi^2}{6\kappa}}. \qquad (3.25)$$

On $\mathscr{C}_2$, we have that $|\arg(x)| > 6\kappa$, so

$$\begin{aligned}
1-x &= 1-e^{-\kappa-\mathrm{Im}(z)} \\
&\gg 1-(1-\kappa)\left(\cos(6\kappa) - i\sin(6\kappa)\right) \\
&\sim 1-(1-\kappa)(1-i\sin(6\kappa)) \\
&\sim \kappa - i6\kappa
\end{aligned}$$

Thus $|1-x| \gg \sqrt{37}\kappa$ and

$$\frac{1}{|1-x|} \ll \frac{1}{\sqrt{37}\kappa}. \qquad (3.26)$$

Taking the Taylor expansion of $1-|x| = 1-e^{-\kappa} \sim \kappa$,

$$\frac{1}{1-|x|} \sim \frac{1}{\kappa}. \qquad (3.27)$$

Plugging (3.25), (3.26), and (3.27) into (3.24), we see

$$\left|\log(x;x^2)_\infty^{-1}\right| \ll \frac{\pi^2}{6\kappa} - \frac{1}{\kappa} + \frac{1}{\sqrt{37}\kappa}. \tag{3.28}$$

On $\mathscr{C}_2$, we have that $|L(x)| \ll \kappa^{-1}$. Also $|x^{-n}| = e^{\kappa n}$ and $n\kappa \sim N$, so we can finally bound E$_3$:

$$\begin{aligned}
|E_3| &= \left|\int_{\mathscr{C}_2} L(x)(x;x^2)^{-2}x^{-n-1}dx\right| \\
&\ll |L(x)x^{-n}| \left|\exp\left(\log\left((x;x^2)^{-2}\right)\right)\right| \\
&\ll \kappa^{-1}e^N \exp\left(2\left(\frac{\pi^2}{6\kappa} - \frac{1}{\kappa} + \frac{1}{\sqrt{37}\kappa}\right)\right) \\
&= \kappa^{-1}\exp\left(N + 2\left(N - \frac{6}{\pi^2}N\left(1 - \frac{1}{\sqrt{37}}\right)\right)\right) \\
&= O(N^{-k}e^{2N}).
\end{aligned}$$

Remark 10. Notice here that one could have chosen any $c > 5.54327$ when defining the arc $|\arg(x)| > c\kappa$ on $\mathscr{C}_2$ to ensure that there is an exponential savings over $2N$. Any c would work that ensures

$$1 - \frac{6}{\pi^2}\left(1 - \frac{1}{\sqrt{1+c^2}}\right) < \frac{1}{2}.$$

Comparing the terms for each of the integrals, it is easy to see that all E_i satisfy the claimed bound. □

Now that we have shown that the error bound comparing $v(n)$ to the approximating sum is small enough, we can now bound each $v_s(n)$.

Proposition 11.

$$v_s(n) = \frac{1}{2}\kappa^{s+1} I_{-s-1}(2N) + O\left(e^{\frac{38N}{37}}\right).$$

where I_{-s-1} is the modified Bessel function of the first kind [1].

Proof. Let $\mathscr{D}$ be the rectangle whose endpoints are $\pm\kappa \pm 6\kappa i$ (traversed counter-clockwise) as seen on the right side of Figure 3.3. Now we want to integrate over

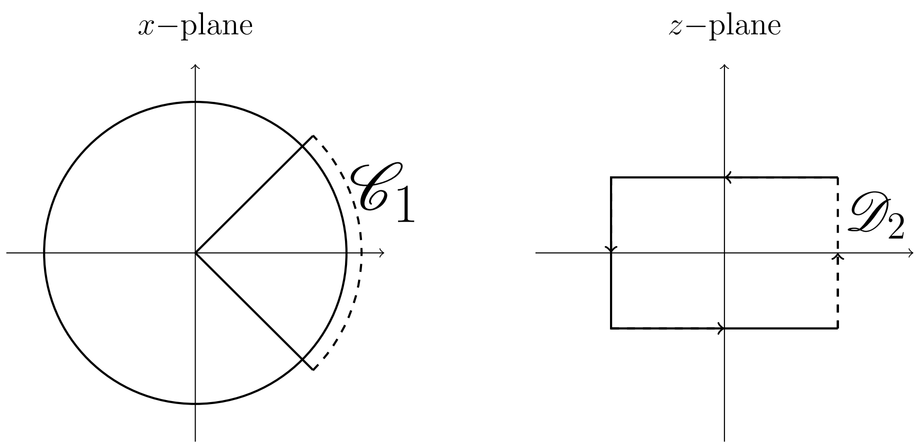

FIGURE 3.3: The transformation of the circle onto the z-plane

$\mathscr{D}$, so let

$$\begin{aligned} W_s &= \frac{1}{2\pi i}\int_{\mathscr{D}} z^s w(z) e^{nz} dz \\ &= \int_{-\kappa-6\kappa i}^{\kappa-6\kappa i} + \int_{\kappa-6\kappa i}^{\kappa+6\kappa i} + \int_{\kappa+6\kappa i}^{-\kappa+6\kappa i} + \int_{-\kappa+6\kappa i}^{-\kappa-6\kappa i} \ldots dz \\ &= \int_{\mathscr{D}_1} + \int_{\mathscr{D}_2} + \int_{\mathscr{D}_3} + \int_{\mathscr{D}_4}. \end{aligned}$$

Then the integral over $\mathscr{D}_2$ is exactly the image of $\mathscr{C}_1$ in the $z-$plane, thus

$$v_s(n) = \frac{1}{2\pi i}\int_{\mathscr{D}_2} z^s w(z) e^{nz} dz.$$

Sending $z \mapsto t\kappa$, we get

$$\begin{aligned} W_s &= \frac{1}{2\pi i}\int_{\mathscr{D}} z^s w(z) e^{nz} dz \\ &= \frac{1}{4\pi i}\kappa^{s+1}\int_{D'} t^s e^{N(t+\frac{1}{t})} dt \\ &= \frac{1}{2}\kappa^{s+1} I_{-s-1}(2N), \end{aligned}$$

as the term over $\mathscr{D}'$ is exactly the integral form of the modified Bessel function [1]. In the $t-$plane on the contour $\mathscr{D}'$, the endpoints of the rectangle are now given by $\pm 1 \pm 6i$. Consider the integral on $\mathscr{D}'_1$:

$$\left|\int_{-1-6i}^{1-6i} t^s e^{N(t+\frac{1}{t})} dt\right| \leq \int_{-1-6i}^{1-6i} |t^s| \left|e^{N(t+\frac{1}{t})}\right| dt.$$

If we let $t = u - 6i$, then $-1 \leq u \leq 1$, so $\mathrm{Re}\left(t + \frac{1}{t}\right) = u + \frac{u}{36+u^2}$. This function is increasing on the interval, so it achieves a maximum value when $u = 1$. Hence

$$\left|\int_{-1-6i}^{1-6i} t^s e^{N(t+\frac{1}{t})} dt\right| \ll e^{\frac{38N}{37}}.$$

Similarly on $\mathscr{D}_3$, we get

$$\left|\int_{-1+6i}^{1+6i} t^s e^{N(t+\frac{1}{t})} dt\right| \ll e^{\frac{38N}{37}}.$$

On $\mathscr{D}_4$, we can let $t = -1 + vi$ where $-6 \leq v \leq 6$. Then $\mathrm{Re}(t + \frac{1}{t}) = -1 - \frac{1}{1+v^2} < -1$ for every value $v \in \mathbb{R}$. Therefore

$$\left|\int_{-1+6i}^{1+6i} t^s e^{N(t+\frac{1}{t})} dt\right| \ll e^{-N}.$$

Now $W_s = \int_{\mathscr{D}_1} + v_s(n) + \int_{\mathscr{D}_3} + \int_{\mathscr{D}_4}$ and we have bounded the other terms, thus

$$v_s(n) = W_s + O(e^{38N/37}).$$

$\square$

Now we must put it all together to finish off Theorem 1. From Propositions 9 and 11, we have

$$v(n) = \sum_{s=0}^{k} \alpha_s v_s + O\left(N^{(-k-1)} e^{2N}\right). \tag{3.29}$$

We use Hankel's approximation for the Bessel Function in Proposition 11 (given in [2]) and pull off the term when $s = 0$:

$$\begin{aligned} v_0(n) &= \frac{1}{2}\kappa I_{-1}(2N) + O(e^{38N/37}) \\ &= \frac{\pi^{3/2} e^{2N}}{24N^{3/2}} \left[1 - \frac{4s^2+8s+3}{2^4 N} + \frac{(4s^2+8s+3)(4s^2+8s-5)}{2!(2^4N)^2} \cdots \right] + O(e^{38N/37}). \end{aligned} \tag{3.30}$$

Putting (3.30) into (3.29) and pulling off the main term,

$$\begin{aligned} v(n) &= \sum_{s=0}^{k} \alpha_s v_s + O\left(N^{(-k-1)} e^{2N}\right) \\ &= \frac{e^{2N}\pi^{3/2}}{24N^{3/2}} \left(\sum_{s\geq 0} \sum_{r \geq 0} \alpha_s \beta_{r,s} + O(N^{-k}) \right). \end{aligned} \tag{3.31}$$

where the $\beta_{r,s}$ terms could be computed for further asymptotic expansions. Now $N = \pi 6^{-1/2} \sqrt{n + \frac{1}{12}}$, and when $s = 0$, $\alpha_s \beta_{r,s} = \frac{1}{2}$, so

$$v(n) = \frac{1}{2^{\frac{13}{4}} 3^{\frac{1}{4}} (n + \frac{1}{12})^{\frac{3}{4}}} \exp\left(\pi\sqrt{\frac{2n}{3} + \frac{1}{18}}\right) \left(1 + O(N^{-1})\right).$$

3.3 Some Computational Results

We are interested in the accuracy of our asymptotic formula. From (3.31) we have the ability to calculate extra terms in asymptotic expansion in powers of $n^{-\frac{1}{2}}$.

We let

$$x(n) = \frac{1}{2^{\frac{13}{4}} 3^{\frac{1}{4}} n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Then in Table 3.2, we have a good visual comparison for $s_o(n)$ versus $x(n)$ for every 200 values of n. The error term is calculated by

$$\frac{x(n) - s_o(n)}{s_o(n)}$$

After $n > 1200$, the error becomes less than 2% and gets smaller for larger n.

TABLE 3.2: Odd Stacks and the Asymptotic Comparison

n	$s_o(n)$	$x(n)$	Error
200	$8.11805419 \times 10^{12}$	$8.53219581 \times 10^{12}$	0.05101488730
400	$1.643858100 \times 10^{19}$	$1.702157640 \times 10^{19}$	0.03546506850
600	$1.241596810 \times 10^{24}$	$1.277279220 \times 10^{24}$	0.02873913400
800	$1.672428800 \times 10^{28}$	$1.713866250 \times 10^{28}$	0.02477680720
1000	$7.428956630 \times 10^{31}$	$7.593083590 \times 10^{31}$	0.02209286910
1200	$1.495288490 \times 10^{35}$	$1.525376810 \times 10^{35}$	0.02012208360
1400	$1.648770140 \times 10^{38}$	$1.679431500 \times 10^{38}$	0.01859650070
1600	$1.127582010 \times 10^{41}$	$1.147168800 \times 10^{41}$	0.01737061950
1800	$5.215002110 \times 10^{43}$	$5.300308540 \times 10^{43}$	0.01635788900
2000	$1.737396460 \times 10^{46}$	$1.764331160 \times 10^{46}$	0.01550290890
2200	$4.373452080 \times 10^{48}$	$4.438042710 \times 10^{48}$	0.01476879700
2400	$8.633254960 \times 10^{50}$	$8.755237820 \times 10^{50}$	0.01412941720
2600	$1.376599020 \times 10^{53}$	$1.395274150 \times 10^{53}$	0.01356613860
2800	$1.816193780 \times 10^{55}$	$1.839922140 \times 10^{55}$	0.01306488290
3000	$2.022331330 \times 10^{57}$	$2.047843550 \times 10^{57}$	0.01261525130
3200	$1.932342260 \times 10^{59}$	$1.955933820 \times 10^{59}$	0.01220879060
3400	$1.606788770 \times 10^{61}$	$1.625811680 \times 10^{61}$	0.01183908700
3600	$1.176783970 \times 10^{63}$	$1.190318110 \times 10^{63}$	0.01150095460
3800	$7.670180640 \times 10^{64}$	$7.756010420 \times 10^{64}$	0.01119005980
4000	$4.489621080 \times 10^{66}$	$4.538571540 \times 10^{66}$	0.01090302700
4200	$2.378760090 \times 10^{68}$	$2.404062740 \times 10^{68}$	0.01063690830
4400	$1.148858730 \times 10^{70}$	$1.160794540 \times 10^{70}$	0.01038927740
4600	$5.089272120 \times 10^{71}$	$5.140969610 \times 10^{71}$	0.01015813080
4800	$2.079345890 \times 10^{73}$	$2.100017980 \times 10^{73}$	0.009941629760
4999	$7.734432000 \times 10^{74}$	$7.809760760 \times 10^{74}$	0.009739404650

Chapter 4
Mixed Congruence Relations

The next goal was to explore other congruence relations, but the techniques that currently exist cannot find non-trivial asymptotics for any stacks with parts specifically congruent to $a \pmod b$. What we can do is deal with mixed congruence stacks, as we are able to use various forms of Heine's transformation formula for hypergeometric series to manipulate the generating functions. The infinite q-series that emerge satisfy nice modular properties that allow us to bound them and derive asymptotic formulas, as was done in the previous chapter.

Definition. A <u>mixed congruence stack</u> is a unimodal sequence where on one side the parts are all congruent to $r \pmod m$ and on the other side, the parts are $-r \pmod m$ for $1 \le r \le m/2$ and $\gcd(r, m) = 1$. More formally, it is a sequence $a_1, \ldots, a_k, c, b_s, \ldots, b_1$ where

$$n = a_1 + a_2 + \cdots + a_k + c + b_s + \cdots + b_2 + b_1$$

and

$$a_i \equiv r(\text{mod } m), \quad c \equiv r(\text{mod } m), \quad b_j \equiv -r(\text{mod } m) \quad \text{for all } i \text{ and } j,$$

with the usual conditions

$$1 \le a_1 \le a_2 \le \cdots \le a_r \le c > b_s \ge \cdots \ge b_2 \ge b_1.$$

Letting $s_{(r,m)}$ be the number of "mixed stacks" with parts satisfying these congruences, the generating function becomes

$$S_{(r,m)}(q) := \sum_{n\geq 0} s_{(r,m)}(n)q^n = \sum_{k\geq 0} \frac{q^{km+r}}{(q^r;q^m)_{k+1}(q^{m-r};q^m)_k}. \tag{4.1}$$

Like the case for regular and odd stacks, the k index in the generating function refers to the height of the peak. Then we have parts up to $km+r$ on the left and parts up to $km-r$ on the right of the stack. Shown in Figure 4.1, we see the mixed stacks for the relation $1 \pmod 4$ of size 12.

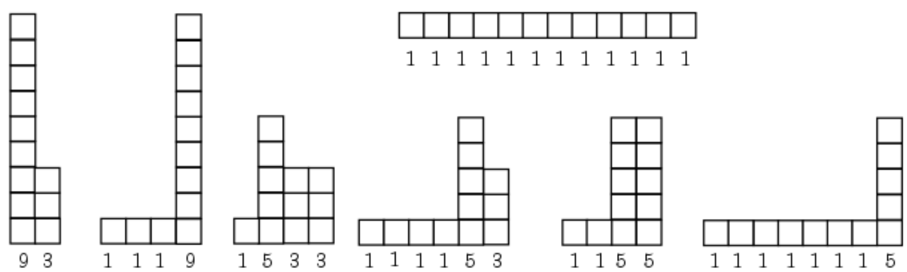

FIGURE 4.1: Mixed congruence stacks for $s_{(1,4)}(12)$

The main theorem of the chapter provides an asymptotic formula for mixed stacks dependent on r and m.

Theorem 12. For large n, and $r < m$

$$h(n) = \frac{\csc\frac{\pi r}{m}}{2^3 3m\left(\frac{r(m-r)}{6m^2} - \frac{1}{36} + \frac{n}{3m}\right)^{3/4}} e^{\left(2\pi\sqrt{\frac{r(m-r)}{6m^2} - \frac{1}{36} + \frac{n}{3m}}\right)} \left(1 + O(N^{-1})\right). \tag{4.2}$$

4.1 Proof of Theorem 2

Mixed stacks as a more general case offer a broader class of stacks that even includes the odd case when taking $r = 1$ and $m = 2$. Much of the proof of the main result follows the same methodology of the previous chapter, demonstrating the power of the circle method as refined by Wright. As the parameters r and m add another level of complexity to the problem, we must be careful when establishing the modular inversions of our infinite product that comes from rewriting the generating function. Once we have accomplished this goal, we can slightly manipulate the techniques from the case of odd stacks to garner the main result.

One of the beautiful equations from Ramanunjan's Lost Notebook is given as Equation (1.1) and explored in depth in [4]:

$$\begin{aligned} 1+\sum_{n\geq 1}\frac{q^n}{\prod_{j=1}^{n}(1+aq^j)(1+q^j/a)} &= (1+a)\sum_{n\geq 0} a^{3n}q^{\frac{n}{2}(3n+1)}\left(1-a^2q^{2n+1}\right) \\ &\quad - a\sum_{n\geq 0}\frac{(-1)^n a^{2n}q^{\frac{n}{2}(n+1)}}{\prod_{j\geq 1}(1+aq^j)(1+q^j/a)}. \end{aligned} \tag{4.3}$$

Sending $q \mapsto q^m$ and $a \mapsto -q^{-r}$, multiplying by $\frac{q^r}{(1-q^r)}$ and reorganizing, (4.1) becomes

$$\begin{aligned} S_{(r,m)}(q) &= \frac{1}{(q^r,q^{m-r};q^m)_\infty}\sum_{n\geq 0}(-1)^n q^{\frac{mn(n+1)}{2}-2rn} \\ &\quad + \sum_{n\geq 0}(-1)^{n-1}q^{\frac{mn(3n+1)}{2}-3rn}\left(1-q^{(2n+1)m-2r}\right). \end{aligned} \tag{4.4}$$

Much like equation (3.4) for the odd case, we have a sparse remainder term plus a more significant term that involves an infinite product multiplied by a false theta function. Letting

$$R_{(m,n)}(q) = \sum_{n\geq 0}(-1)^{n-1}q^{\frac{mn(3n+1)}{2}-3rn}\left(1-q^{(2n+1)m-2r}\right)$$

and

$$H(q) := \sum_{n\geq 1} h(n)q^n = \frac{1}{(q^r,q^{m-r};q^m)_\infty}L_{(r,m)}(q), \tag{4.5}$$

where $L_{(r,m)}(q)$ is the false theta function

$$L_{(r,m)}(q) = \sum_{n\geq 0}(-1)^n q^{\frac{mn(n+1)}{2}-2rn}, \tag{4.6}$$

we have

$$S_{(r,m)}(q) = H(q) + R(q).$$

As before, the $R(q)$ offers contributions to the size of the mixed stacks rather sparsely and $s_{(r,m)}(n) \sim h(n)$. Thus we turn our attention to the main term $H(q)$.

Aside from the odd case, the first non-trivial case is where $m = 3$ and $r = 1$, we have

$$\frac{1}{(q, q^2; q^3)_\infty} = \frac{(q^3; q^3)_\infty}{(q; q)_\infty} \tag{4.7}$$

and can use the modular inversion formula (3.14) to obtain

$$\frac{1}{(q, q^2; q^3)_\infty} = \frac{1}{\sqrt{3}} e^{\frac{-\pi i \tau}{6}+\frac{\pi i}{18\tau}} \left(1 + O\left(e^{-\frac{2\pi i}{3\tau}}\right)\right). \tag{4.8}$$

In the case for large m, we must use some more advanced techniques to establish a modular inversion formula for the general case.

Proposition 13. For $q = e^{2\pi i \tau}$, and for $0 < r < m$,

$$\frac{1}{(q^r, q^{m-r}; q^m)_\infty} = \frac{1}{2\sin(\frac{\pi(m-r)}{m})} e^{\left(\frac{-\pi i r(m-r)}{m}+\frac{\pi i m}{6}\right)\tau+\frac{\pi i}{6m\tau}} \left(1 + O\left(e^{\frac{-2\pi i}{m\tau}}\right)\right). \tag{4.9}$$

We offer two ways to prove this proposition, although the second is important in establishing the modular transformations used in the first. However, as Klein forms offer an interesting look into the connection between partitions, stacks, and modular forms on congruence subgroups, it is significant to keep the ideas used in both proofs.

Proof. The generalized Klein form is given in [3] and [20] as

$$t_{(r,s)}^{(N)}(\tau) := -\frac{\zeta_{2N^2}^{s(r-N)}}{2\pi i} q^{\frac{r(r-N)}{2N^2}} \left(1-\zeta_N^s q^{\frac{r}{N}}\right) \prod_{n\geq 1} \frac{\left(1-\zeta_N^s q^{n+\frac{r}{N}}\right)\left(1-\zeta_N^{-s} q^{n-\frac{r}{N}}\right)}{(1-q^n)^2}, \quad (4.10)$$

where the roots of unity $\zeta_n := e^{2\pi i/n}$ are used. These Klein forms observe a transformation law for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ (explicitly given by Bajpai et. al in [3]):

$$t_{(r,s)}^{(N)}(\gamma\tau) = (c\tau+d)^{-1} t_{(ra+sc,rb+sd)}^{(N)}(\tau), \quad (4.11)$$

where $ra+sc$ and $rb+sd$ are given modulo m. In particular, we let $s=0$, $N=m$, and let $\tau \mapsto m\tau$ to obtain

$$t_{(r,0)}^{(m)}(m\tau) = -\frac{1}{2\pi i} q^{\frac{r(r-m)}{2m}} \frac{(q^r;q^m)_\infty (q^{m-r};q^m)_\infty}{(q^m;q^m)_\infty^2}. \quad (4.12)$$

Rearranging, we have that

$$F_{(r,m)}(\tau) := \frac{1}{(q^r,q^{m-r};q^m)_\infty} = \frac{-1}{2\pi i} \frac{q^{\frac{r(r-m)}{m}}}{(q^m;q^m)_\infty^2 t_{(r,0)}^{(m)}(m\tau)}. \quad (4.13)$$

Now $t_{(r,0)}(m\tau)$ is a weight 1 modular form on the congruence subgroup $\Gamma_1(m)$ and applying the transformation law for $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$$\begin{aligned} t_{(r,0)}(m\tau) &= m\tau t_{(0,m-r)}(Sm\tau) \\ &= m\tau t_{(0,m-r)}\left(\frac{-1}{m\tau}\right). \end{aligned}$$

Combining (4.10) and (4.13), we have an exact formula for $F(\tau)$:

$$\begin{aligned} F_{(r,m)}(\tau) &= \frac{1}{(q^r;q^m)_\infty (q^{m-r};q^m)_\infty} \\ &= \frac{q^{\frac{r(r-m)}{m}}}{2m\tau(q^m;q^m)^2 \sin\left(\frac{\pi(m-r)}{m}\right)} \prod_{n\geq 1} \frac{\left(1-e^{\frac{-2\pi i n}{m\tau}}\right)^2}{\left(1-e^{-\frac{2\pi i}{m\tau}(r\tau+n)}\right)\left(1-e^{\frac{2\pi i}{m\tau}(r\tau-n)}\right)}. \end{aligned} \tag{4.14}$$

Using (3.14), we see that asymptotically,

$$\frac{1}{(q^r,q^{m-r};q^m)_\infty} = \frac{1}{2\sin\left(\frac{\pi r}{m}\right)} e^{\left(\frac{-\pi i r(m-r)}{m}+\frac{\pi i m}{6}\right)\tau+\frac{\pi i}{6m\tau}} \left(1+O\left(e^{\frac{-2\pi i}{m\tau}}\right)\right). \tag{4.15}$$

$\square$

The second proof directly uses theta functions.

Proof. One of the reasons that we can study this particular congruence is that it can be expressed in terms of Jacobi's theta function (see [9] for a good reference), which is

$$\Theta(w) = \Theta(w;\tau) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{\pi i n^2\tau+2\pi i n(w+\frac{1}{2})}, \tag{4.16}$$

and has an equivalent product form by the Triple Product Identity,

$$\Theta(w;\tau) = -iq^{\frac{1}{8}} e^{-\pi i w}(q;q)_\infty (e^{2\pi i w};q)_\infty (e^{-2\pi i w}q;q)_\infty. \tag{4.17}$$

The theta function satisfies the following transformation formulas:

$$\Theta(-w;\tau) = -\Theta(w;\tau), \tag{4.18}$$

$$\Theta\left(\frac{w}{\tau};-\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\, e^{\frac{\pi i w^2}{\tau}}\Theta(w;\tau). \tag{4.19}$$

The Dedekind eta function satisfies the inversion formula (similar to that of $(q;q)_\infty$ from (3.12))

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau). \tag{4.20}$$

We can express our function in terms of the theta functions under a certain transformation. Indeed, the proof of the transformation formula for the Klein forms is derived from the transformation properties of the theta function and eta function.

We have for $m \geq 2$ and $1 \leq r < d$, that

$$F_{r,m}(q) = \left(q^r, q^{m-r}; q^m\right)_\infty^{-1} = -iq^{\frac{m}{12}-\frac{r}{2}}\frac{\eta(m\tau)}{\Theta(r\tau; m\tau)}. \tag{4.21}$$

Letting $w \to r\tau$ and $\tau \to m\tau$ in (4.19),

$$\Theta(r\tau; m\tau) = \frac{i}{\sqrt{-im\tau}} e^{-\frac{\pi i r^2 \tau}{m}}\Theta\left(\frac{r}{m}; -\frac{1}{m\tau}\right), \tag{4.22}$$

where by plugging in $\frac{r}{m}$ for w and $\frac{-1}{m\tau}$ for τ in (4.17), we have

$$\Theta\left(\frac{r}{m}; \frac{-1}{m\tau}\right) = -ie^{-\frac{\pi i}{4m\tau}-\frac{\pi i r}{m}}\left(e^{-\frac{2\pi i}{m\tau}}; e^{-\frac{2\pi i}{m\tau}}\right)_\infty \left(e^{\frac{2\pi i r}{m}}; e^{-\frac{2\pi i}{m\tau}}\right)_\infty \left(e^{-\frac{2\pi i}{m}\left(r+\frac{1}{\tau}\right)}; e^{-\frac{2\pi i}{m\tau}}\right)_\infty. \tag{4.23}$$

Using the inversion formula (4.20), we also have that

$$\eta(m\tau) = \frac{1}{\sqrt{-im\tau}}\eta\left(\frac{-1}{m\tau}\right), \tag{4.24}$$

so

$$\eta(m\tau) = \frac{1}{\sqrt{-im\tau}} e^{\frac{-\pi i}{12m\tau}}\left(e^{\frac{-2\pi i}{m\tau}}; e^{\frac{-2\pi i}{m\tau}}\right)_\infty. \tag{4.25}$$

Finally, plugging (4.23) and (4.25) into (4.21), we have

$$\begin{aligned} F_{r,m}(q) &= \frac{-q^{\frac{m}{12}-\frac{r}{2}} e^{-\frac{\pi i}{12 m\tau}} \left(e^{\frac{-2\pi i}{m\tau}}; e^{\frac{-2\pi i}{m\tau}}\right)_\infty}{e^{-\frac{\pi i}{m}\left(r^2\tau+\frac{1}{4\tau}+r\right)} \left(e^{-\frac{2\pi i}{m\tau}}; e^{-\frac{2\pi i}{m\tau}}\right)_\infty \left(e^{\frac{2\pi i r}{m}}; e^{-\frac{2\pi i}{m\tau}}\right)_\infty \left(e^{-\frac{2\pi i}{m}\left(r-\frac{1}{\tau}\right)}; e^{-\frac{2\pi i}{m\tau}}\right)_\infty} \\ &= \frac{e^{\pi i\tau\left(\frac{m}{6}-r+\frac{r^2}{m}\right)+\frac{\pi i}{6m\tau}}}{ie^{\frac{-\pi i r}{m}}\left(1-e^{\frac{2\pi i r}{m}}\right)\left(e^{\frac{2\pi i r}{m}-\frac{2\pi i r}{m\tau}}, e^{\frac{-2\pi i r}{m}-\frac{2\pi i}{m\tau}}; e^{\frac{-2\pi i}{m\tau}}\right)_\infty}. \end{aligned} \tag{4.26}$$

Letting $\tau \to \frac{iz}{2\pi}$,

$$F_{r,m}(x) = \frac{1}{2\sin\left(\frac{\pi r}{m}\right)} e^{-\left(\frac{m}{12}+\frac{r^2}{2m}-\frac{r}{2}\right)z+\frac{\pi^2}{3mz}} \left(1+O\left(e^{\frac{-4\pi^2}{mz}}\right)\right). \tag{4.27}$$

□

The generalized false theta function is a little more difficult to manipulate using Euler-Maclaurin summation. Thus we use a result found in [19], where they explored the function

$$f_{a,b}(\tau) = \sum_{n\geq 1} (-1)^n q^{(an^2+bn)/2} \tag{4.28}$$

and showed the following:

Lemma 14 (Kim, Kim, Seo (2015)). As $y \to 0^+$, with $|x| \leq y \leq \frac{\sqrt{3}}{8}$,

$$\left| f_{a,b}(\tau) - \left(-\frac{1}{2} + \frac{b}{8}(-2\pi i\tau) + \frac{ab}{32}(-2\pi i\tau)^2 + \frac{b(6a^2-b^2)}{384}(-2\pi i\tau)^3\right)\right| < cy^4. \tag{4.29}$$

where $c = 105\pi^4 a^4 b^1 4 e^{\frac{\pi\sqrt{3}b^2}{32a}}$.

If we let $a \mapsto m, b \mapsto (-4r+m)$ in (4.28), then

$$L_{r,m}(q) = \sum_{n\geq 0} (-1)^{n-1} q^{-2rn+\frac{mn}{2}(n+1)} = q^{2r} f_{m,-m-4r}(q). \tag{4.30}$$

Now letting $z = -2\pi i\tau$, we can find constants for the false theta function in powers of z.

LEMMA 15. For $|arg(z)| < \frac{\pi}{2} - c$ where c is some arbitrary constant, taking $|z| \to 0$ gives

$$L_{r,m}(x) = \frac{1}{2} - \sum_{s=1}^{k-1} a_s z^s + O(z^k), \tag{4.31}$$

where

$$a_s \cong \frac{(4r+m)(m^{s-1})}{2^{2s+1}}.$$

REMARK 16. We effectively reset the notation used from the previous chapter to pertain to mixed stacks. All terminology is exclusively restricted to this chapter, although it mirrors the methodology used before.

With $x = e^{-z}$,

$$F_{(r,m)}(z) = \frac{1}{2\sin\left(\frac{\pi r}{m}\right)} e^{\left(\frac{r(m-r)}{2m} - \frac{m}{12}\right)z + \frac{\pi^2}{3mz}} \left(1 + O\left(e^{\frac{-4\pi^2}{mz}}\right)\right), \tag{4.32}$$

hence we let

$$w(z) := \frac{1}{2\sin\left(\frac{\pi r}{m}\right)} e^{\left(\frac{r(m-r)}{2m} - \frac{m}{12}\right)z + \frac{\pi^2}{3mz}},$$

since

$$w(z)x^{-n} = \frac{1}{2\sin\left(\frac{\pi r}{m}\right)} e^{\left(\frac{r(m-r)}{2m} - \frac{m}{12} + n\right)z + \frac{\pi^2}{3mz}}.$$

Using the Arithmetic-Geometric Mean Inequality on the exponent above gives

$$\left(\frac{r(m-r)}{2m} - \frac{m}{12} + n\right)z + \frac{\pi^2}{3mz} \geq 2\sqrt{\frac{\pi^2}{3m}\left(\frac{r(m-r)}{2m} - \frac{m}{12} + n\right)},$$

where equality occurs when $z = \kappa$ with

$$\kappa := \frac{\pi}{\sqrt{\frac{3r(m-r)}{2} - \frac{m^2}{4} + 3mn}}.$$

Note that this κ aligns with that of the previous chapter when $r = 1$ and $m = 2$.

Hence we now have the circle $\mathscr{C} = e^{-\kappa}$ that we wish to integrate over. The major arc $\mathscr{C}_1$ is where $arg(x) < \rho\kappa$ for fixed $\rho > 0$ (this choice of ρ will be explained later). The minor arc is the rest of the circle $\mathscr{C}_2 = \mathscr{C} - \mathscr{C}_1$ away from the pole at $x = 1$. We can now use the circle method over $\mathscr{C}$, establishing the integrals

$$h(n) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{L(x) x^{-(n+1)}}{(x^r; x^m)_\infty (x^{m-r}; x^m)_\infty} dx. \tag{4.33}$$

and

$$h_s(n) = \frac{1}{2\pi i} \int_{\mathscr{C}_1} \frac{z^s w(z)}{x^{n+1}} dx. \tag{4.34}$$

Again, the main term of each $h_s(n)$ will be shown to be e^{2N} where now

$$N := \frac{\pi^2}{3m\kappa} = \pi \sqrt{\frac{r(m-r)}{6m^2} - \frac{1}{36} + \frac{n}{3m}}.$$

The goal will be to show that the difference between $h(n) - \sum_{s=0}^{k-1} \alpha_s h_s(n)$ is asymptotically less than e^{2N}.

Proposition 17. For $m \geq 3$, $0 \leq r < m$, and as $n \to \infty$,

$$h(n) = \sum_{s=0}^{k-1} \alpha_s h_s(n) + O\left(N^{-k} e^{2N}\right).$$

Proof. Breaking up the difference of the integrals into 3 pieces, we have

$$2\pi i \left(h - \sum_{s=0}^{k-1} \alpha_s h_s \right) = \int_{\mathscr{C}_1} \left(L(x) - \sum_{s=0}^{k-1} \alpha_s z^s \right) w(z) x^{-n-1} dx \qquad (E_1)$$

$$+ \int_{\mathscr{C}_1} L(x) \left((x^r, x^{m-r}; x^m)_\infty^{-1} - w(z) \right) x^{-n-1} dx \qquad (E_2)$$

$$+ \int_{\mathscr{C}_2} L(x) (x^r, x^{m-r}; x^m)_\infty^{-1} x^{-n-1} dx. \qquad (E_3)$$

Note that

$$\left| w(z) x^{-n} \right| \leq \left| e^{\left(\frac{r(m-r)}{2m} - \frac{m}{12} + n \right) z} \right| \left| e^{\frac{\pi^2}{3mz}} \right|,$$

and letting $z = \kappa + i\nu$ on the circle $\mathscr{C}$, the real part of $\left(\frac{1}{z}\right)$ is $\frac{\kappa}{\kappa^2 + \nu^2}$, so

$$\frac{1}{2 \sin\left(\frac{\pi r}{m}\right)} \left| e^{\left(\frac{r(m-r)}{2m} - \frac{m}{12} + n \right) z} \right| \left| e^{\frac{\pi^2}{3mz}} \right| \ll e^{\left(\frac{r(m-r)}{2m} - \frac{m}{12} + n \right) \kappa + \frac{\pi^2}{3m} \left(\frac{\kappa}{\kappa^2 + \nu^2} \right)} \tag{4.35}$$

$$\ll e^{2\pi \sqrt{\frac{r(m-r)}{6m^2} - \frac{1}{36} + \frac{n}{3m}}}$$

$$= O\left(e^{2N}\right). \tag{4.36}$$

Over $\mathscr{C}_1$, we see that $|z| \leq |\kappa + \rho\kappa i| = O(\kappa)$. Along with Lemma 15,

$$\left| L_{r,m}(x) - \sum_{s=0}^{k-1} a_s z^s \right| \ll |z|^k \ll \kappa^k. \tag{4.37}$$

Using (4.36) and (4.37),

$$|E_1| = \left| \int_{\mathscr{C}_1} \left(L_{r,m}(x) - \sum_{s=0}^{k-1} \alpha_s z^s \right) w(z) x^{-n-1} dx \right|$$

$$\leq |e^z| \left| w(z) x^{-n} \right| \left| L_{r,m}(x) - \sum_{s=0}^{k-1} \alpha_s z^s \right|$$

$$= O(\kappa^k e^{2N}) = O(N^{-k} e^{2N}). \tag{4.38}$$

The modular inversion formula in (4.32) shows that

$$\left| \left(\left(x^r, x^{m-r}; x^m\right)_\infty - w(z) \right) \right| \ll \left| w(z) e^{\frac{-2\pi^2}{mz}} \right|. \tag{4.39}$$

Asymptotically, (4.37) really tells us that

$$|L_{(r,m)}(x)| \leq \left| \sum_{s=0}^{k-1} a_s z^s + c z^k \right| = O(1). \tag{4.40}$$

Hence we can bound E_2 by using (4.36), (4.39), and (4.40):

$$\begin{aligned} |E_2| &= \left| \int_{\mathscr{C}_1} L_{r,m}(x) \left((x^r, x^{m-r}; x^m)_\infty^{-1} - w(z) \right) x^{-n-1} dx \right| \\ &\ll |L_{r,m}(x)| \left| w(x) x^{-n} \right| \left| e^{\frac{-2\pi^2}{mz}} \right| \\ &\leq e^{2N - \frac{4\pi^2}{\kappa\sqrt{1+\rho^2}}} \\ &= O(N^{-k} e^{2N}). \end{aligned} \tag{4.41}$$

As we move away from the pole at $x = 1$ along the minor arc, there are still some problematic spots, specifically at mth roots of unity. We must separately handle the cases where we are near and far away from these roots.

Firstly, suppose x on $\mathscr{C}_2$ is not close to a root of unity. Explicitly, we have that $x = e^{-\kappa + iy}$ and want to insure that y is sufficiently far enough from a root of unity, thus we require for $1 \leq a \leq m-1$ that

$$\left| y - \frac{2\pi i a}{m} \right| > \rho\kappa.$$

Now we can use the logarithm for a good asymptotic expansion:

$$\begin{aligned}\log(F(q)) &= \log\left(\left(x^r, x^{m-r}; x^m\right)_\infty^{-1}\right) \\ &= -\log\left(\prod_{n\geq 1}\left(1-x^{mn-r}\right)\left(1-x^{(n-1)m+r}\right)\right) \\ &= -\sum_{n\geq 1}\log\left(\left(1-x^{mn-r}\right)+\left(1-x^{(n-1)m+r}\right)\right).\end{aligned}$$

The Taylor Series Expansion for $\log(1-x^k)$ is

$$-\log(1-x^k) = \sum_{n\geq 1}\frac{x^{nk}}{n}$$

so the above becomes

$$\begin{aligned}-\sum_{n\geq 1}\log\left(\left(1-x^{mn-r}\right)+\left(1-x^{(n-1)m+r}\right)\right) &= \sum_{n\geq 1}\sum_{j\geq 1}\frac{x^{((n-1)m+r)j}+x^{(mn-r)j}}{j} \\ &= \sum_{j\geq 1}\frac{x^{rj}}{j}\sum_{n\geq 0}\left(x^{mj}\right)^n + \sum_{j\geq 1}\frac{x^{(m-r)j}}{j}\sum_{n\geq 0}\left(x^{mj}\right)^n \\ &= \sum_{j\geq 1}\frac{x^{rj}+x^{(m-r)j}}{j}\sum_{n\geq 0}\left(x^{mj}\right)^n.\end{aligned}$$

The sum on the right is a geometric series, so

$$\log\left(F(q)\right) = \sum_{j\geq 1}\frac{x^{rj}+x^{(m-r)j}}{j\left(1-x^{mj}\right)}. \tag{4.42}$$

Now taking the absolute value, we can bound the log term:

$$
\begin{aligned}
|\log(F(q))| &= \left|\log\left(\left(x^r, x^{m-r}; x^m\right)_\infty^{-1}\right)\right| = \left|\sum_{j\geq 1} \frac{x^{rj} + x^{(m-r)j}}{j\left(1-x^{mj}\right)}\right| \\
&\leq \sum_{j\geq 2} \frac{\left|x^{rj}\right| + \left|x^{(m-r)j}\right|}{j\left|1-x^{mj}\right|} + \frac{|x^r| + |x^{m-r}|}{|1-x^m|} + \left(|x^r| + |x|^{m-r}\right)\left(\frac{1}{1-|x|^m} - \frac{1}{1-|x|^m}\right) \\
&\leq \sum_{j\geq 1} \frac{|x^{rj}| + |x^{(m-r)j}|}{j(1-|x^{mj}|)} + \left(|x^r| + |x^{m-r}|\right)\left(\frac{1}{|1-x^m|} - \frac{1}{1-|x|^m}\right) \\
&= \log\left((|x^r|, |x^{m-r}|; |x^m|)_\infty\right)^{-1} + \left(|x^r| + |x^{m-r}|\right)\left(\frac{1}{1-|x|^m} - \frac{1}{|1-x^m|}\right). \quad (4.43)
\end{aligned}
$$

Now $|x| = e^{-\kappa}$, so by our modular inversion formula

$$\log(|x^r|, |x^{m-r}|; |x^m|) \sim \log(w(\kappa)) \sim \frac{\pi^2}{3m\kappa}. \tag{4.44}$$

On $\mathscr{C}_2$, $|\arg(x)| > \rho\kappa$, so

$$\frac{1}{|1-x^m|} \ll \frac{1}{m\sqrt{1+\rho^2}\kappa}, \tag{4.45}$$

and by using the Taylor expansion of $1-|x^m| \leq 1-e^{-m\kappa} \sim m\kappa$, we can approximate

$$\frac{1}{1-|x|^m} \sim \frac{1}{m\kappa}. \tag{4.46}$$

Plugging (4.44), (4.45), and (4.46) into (4.43), we have that

$$|\log(F(x))| \sim \frac{\pi^2}{3m\kappa} - \frac{1}{m\kappa} - \frac{1}{\sqrt{1+\rho^2}m\kappa}. \tag{4.47}$$

Finally, we can bound E_3:

$$
\begin{aligned}
|E_3| &= \left| \int_{\mathscr{C}_2} L(x)F(x)x^{-n-1}dx \right| \\
&\ll |L(x)x^{-n}||\exp(\log(F(x)))| \\
&\ll e^N \exp\left(\frac{1}{m\kappa} \left(\frac{\pi^2}{3} - 1 + \frac{1}{\sqrt{1+\rho^2}} \right) \right) \\
&= \exp\left(N + \frac{1}{m\kappa} \left(\frac{\pi^2}{3} - 1 + \frac{1}{\sqrt{1+\rho^2}} \right) \right) \\
&= \exp\left(N + N - \frac{3N}{\pi^2} + \frac{3N}{\pi^2\sqrt{1+\rho^2}} \right) \\
&= O(N^{-1}e^{2N}). && (4.48)
\end{aligned}
$$

The goal was to beat the bound e^{2N}, hence we motivate the choice for the constant $\rho > 0$, and have successfully bounded E_3 under the assumption that x was away from a root of unity.

Suppose now that x is close to a mth root of unity and $\gcd(r, m) = 1$ (otherwise we can reduce the case down). Following the techniques used by Bringmann and Mahlburg in [15], we want to shift away from the root of unit by a small amount that still allows us to use the periodicity of the theta functions.

The mth roots of unity for $x = e^{-z}$ exist at $z = \kappa - \frac{2\pi i a}{m}$ for $1 \leq a \leq m-1$. We shift by $\frac{2\pi k}{m}$ for some k so that $\left|\frac{2\pi}{m}(a-k)\right| < \rho N$. With this shift in mind, we can rewrite

$$
\begin{aligned}
x = e^{-z} &= e^{-\kappa + \frac{2\pi i k}{m} + \frac{2\pi i(a-k)}{m}} \\
&= \zeta_m^k e^{-\kappa + \frac{2\pi i(a-k)}{m}} \\
&= \zeta_m^k e^{-z'} && (4.49)
\end{aligned}
$$

where $\zeta_m = e^{\frac{2\pi i}{m}}$ and $z' = \kappa - \frac{2\pi i(a-k)}{m}$. Then we can write $F_{(r,m)}(x)$ in terms of z':

$$\begin{aligned}
F_{(r,m)}(x) &= (x^r, x^{m-r}; x^m)_\infty^{-1} && (4.50)\\
&= \left(\zeta_m^{kr} e^{-rz'}, \zeta_m^{(m-r)k} e^{-(m-r)z'}; e^{-mz'}\right)_\infty^{-1} \\
&= \left(\zeta_m^{(kr \pmod m)} e^{-rz'}, \zeta_m^{(-kr \pmod m)} e^{-(m-r)z'}; e^{-mz'}\right)_\infty^{-1} \\
&= \left(e^{2\pi i\phi + 2\pi i \frac{irz'}{2\pi}}, e^{-2\pi i \phi + 2\pi i \frac{i(m-r)z'}{2\pi}}; e^{2\pi i \frac{imz'}{2\pi}}\right)_\infty^{-1} && (4.51)
\end{aligned}$$

where $\phi := \left\{\frac{kr}{m}\right\} = 1 - \lfloor \frac{kr}{m} \rfloor$ is the fractional part. Then we can express this in terms of theta and eta functions:

$$\begin{aligned}
F_{(r,m)}(x) &= \frac{-ie^{\frac{\pi i}{4} - \frac{imz'}{2\pi}} e^{-\pi i(\phi + \frac{irz'}{2\pi})} \left(e^{\frac{imz'}{2\pi}}; e^{\frac{imz'}{2\pi}}\right)_\infty}{\Theta\left(\phi + \frac{irz'}{2\pi}, \frac{imz'}{2\pi}\right)} \\
&= \frac{-ie^{\frac{\pi i}{6}\frac{imz'}{2\pi} - \pi i(\phi + \frac{irz'}{2\pi})} \eta\left(\frac{imz'}{2\pi}\right)}{\Theta\left(\phi + \frac{irz'}{2\pi}, \frac{imz'}{2\pi}\right)}. && (4.52)
\end{aligned}$$

Now using the transformations of the eta and theta functions in (4.52), we have

$$F_{(r,m)}(x) = \frac{-e^{\pi i(\frac{iz'}{\pi}(\frac{m}{12} - \frac{r}{2}) - \phi) + \frac{2\pi^2}{mz'}(\phi^2 + \phi irz' - \frac{r^2 z'^2}{4\pi^2})} \eta\left(\frac{-2\pi}{imz'}\right)}{\Theta\left(\frac{-2\pi i\phi}{mz'} + \frac{r}{m}; \frac{2\pi i}{mz'}\right)}. \quad (4.53)$$

For the eta function, the approximation is

$$\eta\left(\frac{-2\pi}{imz'}\right) \sim e^{\frac{\pi i}{12}\cdot\frac{-2\pi}{imz'}} = e^{\frac{-\pi^2}{6mz'}} \quad (4.54)$$

and for the theta function,

$$\begin{aligned}
\Theta\left(\frac{-2\pi i\phi}{mz'} + \frac{r}{m}; \frac{2\pi i}{mz'}\right) &\sim -ie^{\frac{\pi i}{4}\frac{2\pi i}{mz'} - \pi i(\frac{-2\pi i\phi}{mz'} + \frac{r}{m})}(1 - e^{2\pi i(\frac{-2\pi i\phi}{mz'} + \frac{r}{m})}) \\
&= -ie^{\frac{-\pi^2}{2mz'} - \frac{2\pi^2\phi}{mz'}} \zeta_{2m}^{-r}\left(1 - \zeta_m^r e^{\frac{4\pi^2\phi}{mz'}}\right). && (4.55)
\end{aligned}$$

Putting (4.54) and (4.55) into (4.53), the main exponential term is

$$e^{\frac{2\pi^2\phi^2}{mz'}-\frac{\pi^2}{6mz'}+\frac{\pi^2}{2mz'}-\frac{2\pi^2\phi}{mz'}} = e^{\frac{\pi^2}{mz'}\left(\frac{1}{3}-2\phi(1-\phi)\right)}.$$

Bounding this term and noticing that $2\phi(1-\phi) > 0$ for $\gcd(r,m) = 1$,

$$\begin{aligned}
\left|e^{\frac{\pi^2}{mz'}\left(\frac{1}{3}-2\phi(1-\phi)\right)}\right| &\leq e^{\mathrm{Re}\left(\frac{1}{z'}\right)\frac{\pi^2}{m}\left(\frac{1}{3}-2\phi(1-\phi)\right)} \\
&\leq e^{\frac{\pi^2}{\kappa m}(\frac{1}{3}-2\phi(1-\phi))} \\
&= e^{\frac{\pi^2}{3\kappa m}-\frac{\pi^2}{\kappa m}2\phi(1-\phi)} \\
&= e^{N(2-c)} \qquad \text{for some } c > 0,
\end{aligned}$$

hence we still receive our exponential savings over e^{2N}. Thus we have successfully handled the case when x is close to an mth root of unity and have shown that $h_s(n)$ is still a good approximation. We have essentially broken up the minor arcs into pieces between the roots of unity, and deal with each individual arc. Thus we can see that

$$h(n) - \sum_{s=0}^{k-1} \alpha_s h_s(n) = O\left(N^{-k}e^{2N}\right).$$

□

4.2 Asymptotic Expansion

Lastly, we need to bound the approximating integral h_s, and calculate out the final asymptotic result. The technique is similar to the previous chapter, where we make a transformation from the x-plane to the z-plane.

PROPOSITION 18.

$$h_s(n) = \frac{\csc\frac{\pi r}{m}}{4}\kappa^{s+1}I_{-s-1}(2N) + O\left(e^{\frac{(\rho^2+2)N}{\rho^2+1}}\right).$$

Proof. The proof follows exactly the same method as in the odd case, except the contour $\mathscr{D}$ is the rectangle whose endpoints are $\pm\kappa \pm \rho\kappa i$ (traversed counter-clockwise). Then

$$\begin{aligned} W_s &= \frac{\csc\frac{\pi r}{m}}{4\pi i} \int_{\mathscr{D}} z^s w(z) e^{nz} dz \\ &= \int_{-\kappa-\rho\kappa i}^{\kappa-\rho\kappa i} + \int_{\kappa-\rho\kappa i}^{\kappa+\rho\kappa i} + \int_{\kappa+\rho\kappa i}^{-\kappa+\rho\kappa i} + \int_{-\kappa+\rho\kappa i}^{-\kappa-\rho\kappa i} \ldots dz \\ &= \int_{\mathscr{D}_1} + \int_{\mathscr{D}_2} + \int_{\mathscr{D}_3} + \int_{\mathscr{D}_4}, \end{aligned}$$

with the integral over $\mathscr{D}_2$ as exactly the image of $\mathscr{C}_1$ in the $z-$plane. Hence

$$h_s(n) = \frac{\csc\frac{\pi r}{m}}{4\pi i} \int_{\mathscr{D}_2} z^s w(z) e^{nz} dz.$$

Sending $z \mapsto t\kappa$, we get

$$\begin{aligned} W_s &= \frac{\csc\frac{\pi r}{m}}{4\pi i} \int_{\mathscr{D}} z^s w(z) e^{nz} dz \\ &= \frac{\csc\frac{\pi r}{m}}{8\pi i} \kappa^{s+1} \int_{\mathscr{D}'} t^s e^{N(t+\frac{1}{t})} dt \\ &= \frac{\csc\frac{\pi r}{m}}{4} \kappa^{s+1} I_{-s-1}(2N). \end{aligned}$$

In the $t-$plane on the contour $\mathscr{D}'$, the endpoints of the rectangle are now given by $\pm 1 \pm \rho i$. Consider the integral on $\mathscr{D}_1'$:

$$\left| \int_{-1-\rho i}^{1-\rho i} t^s e^{N(t+\frac{1}{t})} dt \right| \leq \int_{-1-\rho i}^{1-\rho i} |t^s| \left| e^{N(t+\frac{1}{t})} \right| dt.$$

If we let $t = u - \rho i$, then $-1 \leq u \leq 1$, so $Re\left(t + \frac{1}{t}\right) = u + \frac{u}{\rho^2+u^2}$. This function is increasing on the interval, so it achieves a maximum value when $u = 1$. Hence

$$\left| \int_{-1-\rho i}^{1-\rho i} t^s e^{N(t+\frac{1}{t})} dt \right| \ll e^{\frac{(\rho^2+2)N}{\rho^2+1}}.$$

Similarly on $\mathscr{D}_3$, we get

$$\left| \int_{-1+\rho i}^{1+\rho i} t^s e^{N(t+\frac{1}{t})} dt \right| \ll e^{\frac{(\rho^2+2)N}{\rho^2+1}}.$$

On $\mathscr{D}_4$, we can let $t = -1+vi$ where $-\rho \leq v \leq \rho$. Then $Re(t+\frac{1}{t}) = -1-\frac{1}{1+v^2} < -1$ for every value $v \in \mathbb{R}$. Therefore

$$\left| \int_{-1+\rho i}^{1+\rho i} t^s e^{N(t+\frac{1}{t})} dt \right| \ll e^{-N}.$$

Now $W_s = \int_{\mathscr{D}_1} + V_s + \int_{\mathscr{D}_3} + \int_{\mathscr{D}_4}$ and we have bounded the other terms, thus

$$\begin{aligned} h_s(n) &= W_s + O\left(e^{\frac{(\rho^2+2)N}{\rho^2+1}}\right) \\ &= \frac{\csc\frac{\pi r}{m}}{4} \kappa^{s+1} I_{-s-1}(2N) + O\left(e^{\frac{(\rho^2+2)N}{\rho^2+1}}\right). \end{aligned} \tag{4.56}$$

□

We have $\kappa = \frac{3mN}{\pi^2}$. If we use Hankel's approximation for the Bessel function in (4.56), we can write

$$\begin{aligned} h_0(n) = & \frac{\csc\frac{\pi r}{m}}{4} \kappa I_{-1}(2N) + O\left(e^{\frac{(\rho^2+2)N}{\rho^2+1}}\right) \\ = & \frac{\csc\frac{\pi r}{m} \pi^{3/2} e^{2N}}{24 m N^{3/2}} \left[1 - \frac{4s^2+8s+3}{2^4 N} + \frac{(4s^2+8s+3)(4s^2+8s-5)}{2!(2^4 N)^2} \dots \right] \\ & + O\left(e^{\frac{(\rho^2+2)N}{\rho^2+1}}\right). \end{aligned} \tag{4.57}$$

From Proposition 17,

$$h(n) = \sum_{s=0}^{k} \alpha_s h_s(n) + O\left(N^{-k-1} e^{2N}\right), \tag{4.58}$$

and plugging (4.57) into (4.58) and pulling on the first term,

$$h(n) = \frac{\csc\frac{\pi r}{m} e^{2N}\pi^{3/2}}{24mN^{3/2}} \left(\sum_{s\geq 1} \sum_{r\geq 0} \alpha_s \beta_{r,s} + O(N^{-k}) \right). \tag{4.59}$$

for a constant $\beta_{r,s}$. Finally, $N = \pi\sqrt{\frac{r(m-r)}{6m^2} - \frac{1}{36} + \frac{n}{3m}}$, and when $s = 0$, the coefficients $\alpha_s\beta_{r,s} = 1$, hence

$$h(n) = \frac{\csc\frac{\pi r}{m}}{2^3 3m\left(\frac{r(m-r)}{6m^2} - \frac{1}{36} + \frac{n}{3m}\right)^{3/4}} e^{\left(2\pi\sqrt{\frac{r(m-r)}{6m^2} - \frac{1}{36} + \frac{n}{3m}}\right)} \left(1 + O(N^{-1})\right). \tag{4.60}$$

4.3 Computational Results

We would like to see the precision of the asymptotic formula, and check that the exponent is correct. Let

$$x_{(r,m)}(n) = \frac{\csc\frac{\pi r}{m}}{2^3 3^{1/4} m^{1/4} n^{\frac{3}{4}}} e^{2\pi\sqrt{\frac{n}{3m}}}.$$

It is easy to see that $x_{(1,3)}(n)$ provides a good approximation of $s_{(1,3)}(n)$ in Table 4.1. The constants are now very close, and the order of growth is clearly similar.

TABLE 4.1: Large values of $s_{(1,3)}(n)$ versus the asymptotic formula

n	$s_{(1,3)}(n)$	$x_{(1,3)}(n)$
50	1.1369×10^4	1.1976×10^4
100	3.1671×10^6	3.286×10^6
150	2.6053×10^8	2.684×10^8
200	1.1151×10^{10}	1.144×10^{10}
250	3.1201×10^{11}	3.193×10^{11}
300	6.4320×10^{12}	6.568×10^{12}
350	1.0499×10^{14}	1.070×10^{14}
400	1.4227×10^{15}	1.449×10^{15}
450	1.6547×10^{16}	1.683×10^{16}
500	1.6924×10^{17}	1.72×10^{17}

The error, as expected, falls within the range of $n^{-1/2}$, as this would be the next term in the asymptotic expansion.

In the case with modulo 5 and larger bases, the $r = 1$ stack relation will always be larger than the other congruences, yet they are of the same order and differ by a ratio that can be calculated using the constants in terms of r. In this case, the difference is a ratio of $\frac{\csc(\pi/5)}{\csc(2\pi/5)}$. Again, Table 4.2 shows the quality of $x_{(1,5)}(n)$ and $x_{(2,5)}$ as approximations of the number of mixed stacks with base 5. The relative error is of order $n^{-1/2}$ due to the truncated terms of the asymptotic expansion.

TABLE 4.2: Values for $s_{(1,5)}, s_{(2,5)}$ and the asymptotic formulas

n	$s_{(1,5)}(n)$	$x_{(1,5)}(n)$	$s_{(2,5)}(n)$	$x_{(2,5)}(n)$
50	549	551	309	340
100	37778	37955	21816	23458
150	1069054	1.07×10^6	625183	6.6326×10^6
200	1.8640514×10^7	1.870×10^7	1.0984153×10^7	1.1559×10^7
250	2.36608672×10^8	2.373×10^8	1.40148552×10^8	1.46660×10^8
300	2.3879880×10^9	2.394×10^9	1.419853827×10^9	1.4798×10^9
350	$2.0215725788 \times 10^{10}$	2.0265×10^{10}	$1.2055380004 \times 10^{10}$	1.2524×10^{10}
400	$1.487166661 \times 10^{11}$	1.4905×10^{11}	$8.88954532 \times 10^{10}$	9.2120×10^{11}
450	$9.746112753 \times 10^{12}$	9.7668×10^{12}	$5.83714717 \times 10^{12}$	6.0362×10^{12}
500	$5.794700666 \times 10^{12}$	5.8063×10^{12}	$3.4762972081 \times 10^{12}$	3.5885×10^{12}

Chapter 5
Conclusion

We have effectively used the circle method of Wright [26] along with some important refinements found in [15]. A new class of unimodal sequences has been found and analyzed.

5.1 Final Results

The final results of this work are the asymptotic formula for odd stacks and an asymptotic formula for mixed congruence stacks.

THEOREM 1. For large n,

$$s_o(n) \sim \frac{1}{2^{\frac{13}{4}} 3^{\frac{1}{4}} n^{\frac{3}{4}}} e^{\pi\sqrt{2n/3}}.$$

THEOREM 2. For $0 < r < m/2$,

$$s_{(r,m)}(n) \sim \frac{\csc\left(\frac{\pi r}{m}\right)}{2^3 3^{1/4} m^{1/4} n^{3/4}} \exp\left(2\pi\sqrt{\frac{n}{3m}}\right).$$

In Table 3.2 we computationally demonstrated the accuracy of the asymptotic formula for odd stacks. There are some closely related questions to these for slightly augmented stacks that can be explored in the future.

5.2 Future Work

This area of restricted unimodal sequences is full of possible questions that can be explored in future work. Different congruence relations and restrictions on the parts of stacks will be interesting ways to approach new problems. We still need to solve the problem of stacks with a specific congruence on all of the parts. What has yielded better results is a stack that has parts not congruent to a certain modulo class. These allow the use of eta-quotients and the modular transformations that they bring.

Adding even more restrictions on shapes of the unimodal sequence adds a whole new layer to the stacks that are already in place. In [13], Bringmann and Mahlburg explore asymptotics of different variations of stacks that have intriguing practical applications.

If we specify the summit for a stack, this fixes the peak for the sequence and we are left with a partition up to the size of the peak on the left and right. These "stacks with summits" add an interesting wrinkle to each kind of stacks.

Essentially, one can think of these summitted stacks as "double partitions", as can be seen from the generating function. These serve as a super-set of stacks, although their asymptotic formulas are similar. In [6], Andrews presents these as "convex compositions" and even analyzes them in the odd case.

Other types of stacks include receding, strict, left-strict with the option of specifying the summit in each of them. Exploring congruence relations on the different kinds of stacks will yield asymptotic results similar to those of this paper.

References

[1] G. Arfken. *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, pp. 633-634, 1985.

[2] M. Abramowitz and I.A. Stegun,"Modified Spherical Bessel Functions." 10.2 in *Handbook of Mathematical Functions*, 9th printing. New York: Dover, pp. 443-445, 1972.

[3] Bajpai, J., Kimport, S., Liang, J., Ma, D., Ricci, J., "Bilateral Series and Ramanujan's Radial Limits," *Proc. Amer. Math. Soc.* **143** (2015), no. 2, 479-492.

[4] G. Andrews, "An Introduction to Ramanujan's 'Lost' Notebook", *Ramanujan: Essays and Surveys (AMS)* **22**, Providence, RI (2001) 165-184

[5] G. Andrews, *The Theory of Partitions*, Cambridge University Press, Cambridge, 1998.

[6] G. Andrews, "Concave and Convex Compositions," *Ramanujan J.* **31** (2013), no. 1-2, 67-82.

[7] G. Andrews, "Ramanujan's 'lost' notebook I: Partial θ-functions." *Adv. in Math.* 41 (1981), no. 2, 137-172.

[8] F. Auluck, "On Some New Types of Partitions Associated with Generalized Ferrers Graphs," *Math. Proc. Cambridge Philos. Soc.*, **47** (1951) 679-686.

[9] H. Davenport, *Multiplicative Number Theory*, 2nd ed. New York: Springer-Verlage. 1980.

[10] G.H. Hardy and E.M. Wright, *Introduction to the Theory of Numbers*, 6th ed, Oxford University Press, New York, 2008.

[11] G.H. Hardy and S. Ramanujan, "Asymptotic Formulae in Combinatory Analysis," *Proc. Lond. Math. Soc.* (2) **17**, 75-115 (1919).

[12] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Colloquium Publications (AMS); Volume 53, pp. 449-467.

[13] K. Bringmann and K. Mahlburg, "Asymptotic Formulas for Stacks and Unimodal Sequences," *J. Combin. Theory Ser. A* **126** (2014) 194-215

[14] K. Bringmann and K. Mahlburg, "Asymptotic Inequalities for Positive Crank and Rank Moments," *Trans. Amer. Math. Soc.* **366** (2014), no. 2, 1073-1094.

[15] K. Bringmann and K. Mahlburg, "Schur's Second Partition Theorem and Mixed Mock Modular Forms", arXiv:1307.1800 [math.NT], 2013

[16] L.J. Rogers, "On two theorems of combinatory analysis and some allied identities", *Proc. London Math. Soc.* (2), 16 (1916) 315-336.

[17] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Second Edition. Encycl. of Math. and its Appl., **96**. Cambridge University Press, Cambridge, 2004.

[18] A. Ingham, "A Tauberian theorem for partitions", *Ann. of Math.* **42** (1941), 1075-1090.

[19] Byungchan Kim, Eunmi Kim, and Jeehyeon Seo, "Asymptotics for q-expansions involving partial theta functions." *Discrete Math.* 338 (2015), no.2, 180-189.

[20] Daniel S. Kubert and Serge Lang, *Modular Units*, Grundlehren der Mathmatischen Wissenschaften, vol. 244, Springer-Verlag, New Yourk, 1982.

[21] P.A. MacMahon, *Combinatory Analysis* Cambridge University Press, Cambridge, 1916.

[22] H. Rademacher," On the expansion of the partition function in a series", *Ann. of Math.* (2) **44**, (1943), 416-422.

[23] R. Stanley, *Enumerative Combinatorics*, Vol. 1 and 2, second edition, Cambridge University Press, 2011.

[24] H.V. Temperley, "Statistical Mechanics and the partition of numbers II. The form of crystal surfaces", *Proc. Camb. Phil. Soc.* **48** (1952), 683-97.

[25] E.M. Wright, "Stacks", *Quart. J. Math. Ser.* (2) **19** (1968) 313-320.

[26] E.M. Wright, "Stacks (II)", *Quart. J. Math.* (Oxford) **22** (1971) 107-16.

[27] E.M. Wright, "Stacks (III)", *Qrt. J. Math.* (Oxford) **23** (1972), 153-158.

[28] E.M. Wright, "Asymptotic Partition Formulae II: Weighted Partitions," *Proc. London Math. Soc.* (92) **36** (1933), 117-141.

Appendix A: Tables of Values

TABLE 5.1: Values of $s(n)$ and $s_o(n)$ for $n = 1 \dots 200$.

n	$s(n)$	$s_o(n)$
1	1	1
2	2	1
3	4	2
4	8	3
5	15	5
6	27	8
7	47	12
8	79	18
9	130	26
10	209	37
11	330	52
12	512	72
13	784	98
14	1183	133
15	1765	178
16	2604	236
17	3804	312
18	5504	408
19	7898	530
20	11240	686
21	15880	881
22	22277	1126
23	31048	1434
24	43003	1815
25	59220	2288
26	81098	2874
27	110484	3594
28	149769	4478
29	202070	5562
30	271404	6883
31	362974	8490
32	483439	10444
33	641368	12807
34	847681	15660
35	1116325	19102

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Table 5.1 – continued from previous page

n	$s(n)$	$s_o(n)$
36	1464999	23236
37	1916184	28196
38	2498258	34138
39	3247088	41232
40	4207764	49692
41	5436972	59764
42	7005688	71726
43	9002752	85912
44	11538936	102711
45	14752316	122562
46	18814423	145986
47	23938188	173592
48	30387207	206062
49	38487496	244204
50	48641220	288954
51	61344055	341366
52	77205488	402681
53	96974176	474322
54	121567834	557904
55	152110204	655306
56	189974638	768682
57	236837795	900480
58	294742961	1053522
59	366177506	1231042
60	454164484	1436708
61	562373990	1674734
62	695254782	1949936
63	858193804	2267766
64	1057704607	2634468
65	1301654610	3057155
66	1599533747	3543874
67	1962777146	4103812
68	2405146194	4747400
69	2943185034	5486430
70	3596759104	6334321
71	4389698140	7306274
72	5350554461	8419458
73	6513505598	9693342
74	7919417515	11149942

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Table 5.1 – continued from previous page

n	$s(n)$	$s_o(n)$
75	9617107420	12814088
76	11664829463	14713860
77	14132034817	16880962
78	17101440701	19351078
79	20671475778	22164484
80	24959151174	25366559
81	30103447306	29008304
82	36269285199	33147174
83	43652201764	37847762
84	52483826126	43182578
85	63038314304	49233133
86	75639876987	56090912
87	90671609978	63858472
88	108585811849	72650868
89	129916066296	82597042
90	155291343494	93841318
91	185452486464	106545338
92	221271428869	120889976
93	263773630036	137077382
94	314164195520	155333608
95	373858325220	175911212
96	444516723375	199092082
97	528086816326	225190956
98	626850631289	254558956
99	743480454072	287587472
100	881103407093	324712860
101	1043376420550	366421246
102	1234573123869	413253823
103	1459684593136	465813098
104	1724535987821	524769422
105	2035921623554	590868144
106	2401761177785	664937952
107	2831280370652	747899688
108	3335219695019	840775944
109	3926075572884	944702224
110	4618378657772	1060938748
111	5429015015732	1190883340
112	6377596398201	1336086292
113	7486887098530	1498266100

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Table 5.1 – continued from previous page

n	$s(n)$	$s_o(n)$
114	8783295562218	1679326759
115	10297440523198	1881377446
116	12064802379100	2106753518
117	14126472544591	2358039580
118	16530014795238	2638095488
119	19330455177456	2950084322
120	22591418786903	3297502988
121	26386434945538	3684216590
122	30800434628041	4114495504
123	35931468076552	4593055860
124	41892673618556	5125104914
125	48814533881504	5716389976
126	56847459676882	6373251904
127	66164748365378	7102684778
128	76965968893204	7912400316
129	89480833975356	8810898382
130	103973626934016	9807545244
131	120748261197752	10912658254
132	140154059657040	12137598444
133	162592354340738	13494872955
134	188524018840611	14998246078
135	218478062716152	16662860510
136	253061432596753	18505371196
137	292970185973612	20544090520
138	339002223699823	22799146702
139	392071794103140	25292658330
140	453226007438402	28048923711
141	523663633415592	31094627354
142	604756487706053	34459067046
143	698073756323220	38174400180
144	805409649325165	42275912366
145	928814829558157	46802312178
146	1070632116625895	51796050976
147	1233537034794704	57303671512
148	1420583843057030	63376189670
149	1635257772037876	70069508914
150	1881534281038860	77444871590
151	2163946257467024	85569352498
152	2487660193634399	94516394732

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Table 5.1 – continued from previous page

n	$s(n)$	$s_o(n)$
153	2858562513183166	104366392668
154	3283357362503305	115207328965
155	3769677355315651	127135465786
156	4326208940043592	140256096538
157	4962834276884824	154684366240
158	5690791741179818	170546161426
159	6522857442769004	187979077600
160	7473550441231688	207133473642
161	8559364679844754	228173615435
162	9799031027332393	251278918370
163	11213813246666762	276645300076
164	12827842171702833	304486647340
165	14668492911718570	335036408768
166	16766810484626299	368549327220
167	19157989955081386	405303317812
168	21881917883528162	445601505650
169	24983782737365798	489774440646
170	28514762831464663	538182497370
171	32532801421941586	591218477658
172	37103479725331160	649310437096
173	42300999955275604	712924746292
174	48209291906563723	782569409178
175	54923258263624338	858797663812
176	62550175608633750	942211880770
177	71211270158493856	1033467786600
178	81043489505906047	1133279043334
179	92201494198884326	1242422204656
180	104859895795623841	1361742082262
181	119215771216923840	1492157560708
182	135491486711685327	1634667887886
183	153937868712429950	1790359482328
184	174837763207256225	1960413304903
185	198510030177023832	2146112829762
186	225314025056681859	2348852665880
187	255654625290496145	2570147887584
188	289987866773533810	2811644118904
189	328827262551774598	3075128435890
190	372750884498114729	3362541158046
191	422409298075852612	3675988587006

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Table 5.1 – continued from previous page

n	$s(n)$	$s_o(n)$
192	478534450650200826	4017756771616
193	541949625441016096	4390326386804
194	613580586031980978	4796388801016
195	694468050743969106	5238863429446
196	785781652051190348	5720916481202
197	888835554010898114	6245981195096
198	1005105920304447487	6817779684170
199	1136250447463711006	7440346522586
200	1284130202107009978	8118054194025

TABLE 5.2: Table of Mixed Congruence Stack Values

n	$s_{(1,3)}(n)$	$s_{(1,5)}(n)$	$s_{(2,5)}(n)$	$s_{(1,7)}(n)$	$s_{(2,7)}(n)$	$s_{(3,7)}(n)$
1	1	1	0	1	0	0
2	1	1	1	1	1	0
3	1	1	0	1	0	1
4	2	1	1	1	1	0
5	2	1	0	1	0	0
6	3	2	1	1	1	1
7	4	2	1	1	0	0
8	6	2	1	2	1	0
9	7	2	1	2	1	1
10	10	3	2	2	1	1
11	12	4	1	2	1	0
12	17	5	3	2	1	1
13	20	5	2	2	1	1
14	27	6	4	3	2	1
15	32	7	3	4	1	1
16	42	9	5	5	3	1
17	50	10	5	5	1	2
18	64	12	6	5	4	2
19	76	13	7	5	2	1
20	96	16	9	6	4	3
21	114	18	9	7	3	3
22	141	22	12	9	4	2
23	167	24	12	10	5	3
24	205	28	16	11	5	5
25	242	31	16	11	6	3
26	294	37	20	12	6	4

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Table 5.2 – continued from previous page

n	$s_{(1,3)}(n)$	$s_{(1,5)}(n)$	$s_{(2,5)}(n)$	$s_{(1,7)}(n)$	$s_{(2,7)}(n)$	$s_{(3,7)}(n)$
27	346	41	22	13	7	6
28	417	47	26	16	9	6
29	489	52	28	18	8	5
30	585	61	34	21	11	8
31	684	68	36	22	9	8
32	813	78	43	24	14	8
33	948	86	46	25	12	9
34	1120	98	55	28	16	12
35	1303	109	59	31	15	11
36	1531	124	69	36	18	12
37	1776	137	75	39	20	14
38	2077	155	86	43	21	17
39	2403	171	94	45	24	16
40	2799	193	108	49	25	18
41	3229	214	117	53	28	21
42	3747	241	135	60	32	24
43	4312	265	146	66	32	22
44	4986	297	167	74	39	27
45	5724	327	181	79	37	31
46	6597	366	205	85	47	31
47	7556	403	224	90	45	32
48	8682	449	252	99	54	41
49	9922	493	275	108	55	41
50	11369	549	309	120	61	43
51	12964	603	337	130	67	49
52	14816	669	377	141	71	55
53	16860	734	410	149	79	55
54	19221	812	459	161	84	62
55	21830	890	499	173	91	68
56	24829	984	556	191	101	74
57	28146	1077	606	208	104	76
58	31943	1187	672	227	120	86
59	36144	1299	732	242	120	94
60	40936	1430	811	260	140	100
61	46238	1563	881	276	142	104
62	52266	1718	975	300	160	120
63	58936	1875	1060	324	168	126
64	66496	2057	1169	354	183	132
65	74861	2244	1270	380	198	145

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Table 5.2 – continued from previous page

n	$s_{(1,3)}(n)$	$s_{(1,5)}(n)$	$s_{(2,5)}(n)$	$s_{(1,7)}(n)$	$s_{(2,7)}(n)$	$s_{(3,7)}(n)$
66	84314	2459	1399	409	210	162
67	94772	2679	1519	434	229	165
68	106560	2931	1669	466	244	180
69	119597	3190	1812	499	263	198
70	134258	3486	1988	542	286	213
71	150464	3793	2156	583	299	220
72	168649	4139	2363	630	332	245
73	188742	4497	2560	671	343	263
74	211242	4902	2801	717	383	278
75	236091	5323	3035	763	396	295
76	263861	5795	3314	820	436	327
77	294515	6288	3589	879	459	344
78	328711	6838	3914	949	495	366
79	366436	7412	4236	1014	532	394
80	408450	8053	4614	1085	564	430
81	454770	8723	4990	1152	609	449
82	506275	9467	5429	1230	647	482
83	563024	10247	5868	1312	692	521
84	626030	11110	6378	1410	745	559
85	695408	12015	6887	1508	784	583
86	772324	13017	7477	1616	854	634
87	856965	14067	8073	1718	890	681
88	950676	15224	8752	1830	974	719
89	1053731	16442	9443	1942	1017	761
90	1167685	17780	10231	2075	1102	831
91	1292916	19188	11030	2213	1163	875
92	1431223	20733	11938	2369	1244	927
93	1583118	22359	12864	2522	1328	995
94	1750678	24138	13910	2686	1408	1070
95	1934579	26016	14980	2846	1506	1121
96	2137218	28065	16185	3028	1598	1200
97	2359473	30227	17419	3217	1701	1284
98	2604116	32584	18803	3434	1818	1368
99	2872263	35071	20228	3656	1916	1438
100	3167122	37778	21816	3895	2063	1545
101	3490096	40639	23454	4129	2161	1646
102	3844898	43745	25279	4383	2331	1742
103	4233279	47026	27161	4642	2445	1840
104	4659533	50587	29251	4936	2622	1981

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Table 5.2 – continued from previous page

n	$s_{(1,3)}(n)$	$s_{(1,5)}(n)$	$s_{(2,5)}(n)$	$s_{(1,7)}(n)$	$s_{(2,7)}(n)$	$s_{(3,7)}(n)$
105	5125830	54350	31414	5244	2768	2093
106	5637133	58426	33804	5582	2945	2211
107	6196114	62737	36286	5921	3129	2354
108	6808514	67398	39021	6282	3311	2519
109	7477598	72328	41860	6644	3524	2644
110	8210009	77656	44986	7043	3728	2811
111	9009712	83291	48237	7461	3954	2996
112	9884396	89370	51802	7926	4204	3177
113	10838859	95804	55516	8404	4430	3340
114	11881995	102736	59585	8916	4731	3567
115	13019587	110072	63824	9428	4966	3782
116	14261917	117972	68457	9977	5309	3993
117	15615936	126330	73295	10544	5577	4218
118	17093531	135318	78566	11170	5940	4501
119	18703020	144834	84078	11824	6263	4746
120	20458148	155056	90075	12535	6636	5013
121	22368835	165875	96344	13255	7028	5312
122	24450974	177490	103159	14019	7418	5647
123	26716364	189781	110291	14797	7862	5937
124	29183372	202962	118024	15639	8301	6284
125	31866002	216917	126125	16521	8775	6663
126	34785467	231867	134900	17485	9293	7050
127	37958350	247690	144093	18481	9783	7408
128	41409151	264633	154035	19540	10389	7863
129	45157455	282562	164463	20616	10908	8318
130	49231553	301744	175720	21759	11591	8764
131	53654521	322047	187534	22944	12173	9236
132	58459020	343748	200275	24232	12903	9808
133	63672195	366711	213644	25579	13587	10326
134	69331727	391245	228049	27023	14349	10880
135	75469493	417202	243176	28505	15149	11498
136	82128945	444912	259444	30066	15961	12167
137	89347455	474232	276539	31669	16858	12781
138	97175117	505508	294909	33388	17761	13495
139	105655662	538594	314211	35186	18727	14255
140	114846822	573872	334932	37124	19767	15035
141	124799687	611185	356714	39136	20783	15796
142	135580765	650941	380070	41262	21978	16696
143	147249661	692990	404629	43440	23064	17606

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Table 5.2 – continued from previous page

n	$s_{(1,3)}(n)$	$s_{(1,5)}(n)$	$s_{(2,5)}(n)$	$s_{(1,7)}(n)$	$s_{(2,7)}(n)$	$s_{(3,7)}(n)$
144	159882949	737764	430941	45745	24398	18527
145	173550078	785110	458606	48139	25608	19487
146	188339166	835504	488223	50709	27040	20600
147	204331024	888781	519378	53398	28427	21663
148	221627014	945451	552686	56258	29945	22780
149	240321041	1005362	587733	59205	31534	24002
150	260529649	1069054	625183	62306	33159	25324
151	282361817	1136372	664579	65504	34928	26573
152	305951485	1207909	706646	68905	36725	27978
153	331424987	1283502	750911	72464	38637	29480
154	358936169	1363789	798133	76261	40675	31016
155	388631320	1448624	847829	80208	42704	32545
156	420687057	1538680	900811	84366	45010	34300
157	455272600	1633813	956562	88648	47195	36079
158	492590610	1734764	1015962	93159	49749	37895
159	532836615	1841384	1078480	97854	52165	39803
160	576243049	1954470	1145031	102850	54921	41934
161	623035604	2073894	1215078	108072	57647	44021
162	673480729	2200500	1289610	113597	60587	46223
163	727838686	2334170	1368044	119309	63665	48585
164	786414746	2475831	1451450	125301	66827	51114
165	849508762	2625365	1539232	131505	70230	53587
166	917470295	2783768	1632512	138062	73712	56292
167	990644498	2950951	1730691	144915	77391	59164
168	1069431449	3127975	1834964	152182	81293	62132
169	1154228081	3314770	1944696	159740	85230	65110
170	1245492059	3512497	2061185	167676	89592	68438
171	1343679459	3721092	2183775	175879	93839	71853
172	1449313554	3941805	2313829	184492	98648	75343
173	1562917953	4174615	2450692	193457	103319	79001
174	1685090614	4420856	2595828	202945	108514	83025
175	1816432193	4680536	2748538	212850	113729	87020
176	1957625631	4955113	2910400	223282	119295	91218
177	2109359429	5244612	3080711	234094	125116	95702
178	2272413195	5550604	3261128	245410	131121	100449
179	2447575495	5873177	3450953	257142	137520	105173
180	2635735833	6214009	3651961	269496	144103	110287
181	2837796700	6573230	3863412	282390	151032	115671
182	3054772465	6952672	4087234	295994	158337	121245

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Table 5.2 – continued from previous page

n	$s_{(1,3)}(n)$	$s_{(1,5)}(n)$	$s_{(2,5)}(n)$	$s_{(1,7)}(n)$	$s_{(2,7)}(n)$	$s_{(3,7)}(n)$
183	3287695254	7352501	4322673	310150	165774	126915
184	3537721221	7774695	4571758	324975	173880	133108
185	3806029909	8219501	4833753	340326	181912	139489
186	4093938392	8689035	5110833	356408	190801	146061
187	4402793744	9183609	5402228	373146	199600	152916
188	4734095603	9705532	5710277	390776	209207	160346
189	5089381305	10255176	6034220	409169	218950	167837
190	5470357495	10835034	6376531	428474	229273	175665
191	5878777456	11445587	6736466	448499	240054	183966
192	6316582444	12089505	7116679	469427	251198	192738
193	6785770725	12767366	7516398	491142	263006	201551
194	7288549295	13482083	7938495	513927	275178	210985
195	7827194175	14234325	8382208	537694	287953	220911
196	8404213218	15027235	8850567	562666	301375	231177
197	9022197188	15861629	9342860	588645	315113	241680
198	9683994408	16740884	9862336	615804	329901	253026
199	10392552068	17665952	10408267	643953	344745	264723
200	11151103605	18640514	10984153	673380	360898	276798

Appendix B: Sample Maple Code

Maple 17 was used to generate this code. All rights reserved to MapleSoft.

Maple was chosen as the language of choice due to the "series" functionality. When expanding in q powers as we do so often, it is quick and easy to implement. Most series expansions are written in terms of the simplest generating functions, and no effort was made to optimize the algorithms. Calculating the false theta function and infinite q-series and then multiplying out to get the series expansion would provide a faster means of computation.

A constant that limits the length of our series. (K<=300 is suggested for reasonable computational time.)

$K := 500 :$

Odd Summitted Stacks

$V := series\left(sum\left(q^{(2\,k-1)} \cdot product\left(\left(1-q^{(2\,m-1)}\right)^{(-2)}, m=1..k\right), k=1..K\right), q, K\right) :$

Double Odd Partitions

$D1 := series\left(product\left(\left(1-q^{(2\,k-1)}\right)^{(-2)}, k=1..K\right), q, K\right) :$

Odd Stacks

$V1 := series\left(sum\left(\left(1-q^{(2\,k-1)}\right) \cdot q^{(2\,k-1)} \cdot product\left(\left(1-q^{(2\,m-1)}\right)^{(-2)}, m=1..k\right), k=1..K\right), q, K\right); V10 := series\left(sum\left(\left(1-q^{k}\right) \cdot q^{k} \cdot product\left(\left(1-q^{m}\right)^{(-2)}, m=1..k\right), k=1..K\right), q, K\right) :$

A quick check to make sure that V = D1-V1.

$series(V-D1+V1, q, K)$

$$-1+\mathrm{O}\left(q^{1000}\right) \tag{1}$$

Even Stacks

$V2 := series\left(sum\left(\left(1-q^{(2\,k)}\right) \cdot q^{(2\,k)} \cdot product\left(\left(1-q^{(2\,m)}\right)^{(-2)}, m=1..k\right), k=1..K\right), q, K\right)$

This is the Target Series that Odd Stacks should be close to from following Stanley.

$MD := series\left(product\left(\left(1-q^{(2\,k-1)}\right)^{(-2)}, k=1..K\right) \cdot \left(sum\left((-1)^{m-1} q^{(m\cdot(m+1))}, m=1..K\right)\right), q, K\right) :$

A check to see that they are close, plus/minus a few terms. This is the series of remainder terms.

$series(V1-MD, q, K)$

$$q-q^{5}-q^{8}+q^{16}+q^{21}-q^{33}-q^{40}+q^{56}+q^{65}-q^{85}-q^{96}+q^{120}+q^{133}-q^{161}-q^{176}+q^{208}+q^{225}-q^{261}-q^{280}+q^{320}+q^{341}-q^{385}-q^{408}+q^{456}+q^{481}-q^{533}-q^{560}+q^{616}+q^{645}-q^{705}-q^{736}+q^{800}+q^{833}-q^{901}-q^{936}+\mathrm{O}\left(q^{1000}\right) \tag{2}$$

My previous answer for a formula for the remainder terms.

$series\left(sum\left((-1)^{k} q^{3\,k^2+2\,k}+(-1)^{k-1} q^{3\,k^2-2\,k}, k=1..K\right), q, K\right)$

$$q-q^{5}-q^{8}+q^{16}+q^{21}-q^{33}-q^{40}+q^{56}+q^{65}-q^{85}-q^{96}+q^{120}+q^{133}-q^{161}-q^{176}+q^{208}+q^{225}-q^{261}-q^{280}+q^{320}+q^{341}-q^{385}-q^{408}+q^{456}+q^{481}-q^{533}-q^{560}+q^{616}+q^{645}-q^{705}-q^{736}+q^{800}+q^{833}-q^{901}-q^{936}+\mathrm{O}\left(q^{1008}\right) \tag{3}$$

The remainder series shows up in a much nicer form when using "Ramanujan's Formula".

$$R := series\left(\frac{(1-q)}{q}\cdot sum\left((-1)^{n-1} q^{n\cdot(n+1)}\cdot product\left((1-q^{2m-1})^{-1}, m=1..n\right), n=1..K\right), q, K\right)$$

$$q - q^{5} - q^{8} + q^{16} + q^{21} - q^{33} - q^{40} + q^{56} + q^{65} - q^{85} - q^{96} + q^{120} + q^{133} - q^{161} - q^{176} + q^{208} + q^{225} - q^{261} - q^{280} + q^{320} + q^{341} - q^{385} - q^{408} + q^{456} + q^{481} - q^{533} - q^{560} + q^{616} + q^{645} - q^{705} - q^{736} + q^{800} + q^{833} - q^{901} - q^{936} + \mathrm{O}(q^{999}) \quad \textbf{(4)}$$

Thus we have found an alternate formula for Odd Stacks, which we can show by using "Ramanujan's Formula".

$series(V1 - R - MD, q, K)$

$$-1 + \mathrm{O}(q^{1000})$$

The Asymptotic Formula for Odd Stacks;

$$f := x \rightarrow \frac{1}{2^{\left(\frac{13}{4}\right)}\cdot 3^{\left(\frac{1}{4}\right)}\cdot x^{\left(\frac{3}{4}\right)}}\cdot \exp\left(\mathrm{Pi}\cdot\left(\frac{2x}{3}\right)^{\left(\frac{1}{2}\right)}\right):$$

Example of Creating Mixed Congruence Stacks for fixed r and m

$r := 1 : m := 5 :$

$$V15 := series\left(\frac{q^{r}}{(1-q^{r})} + sum\left(\left(q^{(m\cdot k+r)}\cdot product\left((1-q^{(m\cdot m1+r)})^{(-1)}, m1=0..k\right)\cdot product\left((1-q^{(m\cdot m2-r)})^{(-1)}, m2=1..k\right)\right), k=1..\mathrm{K}\right), q, \mathrm{K}\right):$$

Example for Building Arrays that can be easily exported to LaTeX

$C := array(1..200, 1..3):$

for i **from** 1 **to** 200 **do** $C[i,1] := i : C[i,2] := coeff(V1, q, i) : C[i,3] := \mathrm{evalf}(\mathrm{f}(\mathrm{i}))$: **end do**:

Vita

Richard Alexander Frnka was born in Arlington, Texas. He was raised most of his life in Granbury, Texas where he attended high school, played basketball, soccer, and tennis, ran track and cross-country, and represented his school in the academic decathlon as well as other academic contests. He attended Eckerd College in St. Petersburg, Florida where he majored in Mathematics and Computer Science with a minor in Physics, receiving his Bachelors of Science degree in May 2011. Starting his graduate education at Louisiana State University in August of 2011, he first received a Masters of Science degree in mathematics in 2013, and anticipates receiving a Doctor of Philosophy in Mathematics in May 2017.