

12-1-2004

Fundamental decoherence from relational time in discrete quantum gravity: Galilean covariance

Rodolfo Gambini
Universidad de la Republica Instituto de Fisica

Rafael A. Porto
Carnegie Mellon University

Jorge Pullin
Louisiana State University

Follow this and additional works at: https://digitalcommons.lsu.edu/physics_astronomy_pubs

Recommended Citation

Gambini, R., Porto, R., & Pullin, J. (2004). Fundamental decoherence from relational time in discrete quantum gravity: Galilean covariance. *Physical Review D - Particles, Fields, Gravitation and Cosmology*, 70 (12) <https://doi.org/10.1103/PhysRevD.70.124001>

This Article is brought to you for free and open access by the Department of Physics & Astronomy at LSU Digital Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Digital Commons. For more information, please contact ir@lsu.edu.

Fundamental decoherence from relational time in discrete quantum gravity: Galilean covariance

Rodolfo Gambini¹, Rafael A. Porto² and Jorge Pullin³

1. *Instituto de Física, Facultad de Ciencias, Iguá 4225, esq. Mataojo, Montevideo, Uruguay.*

2. *Department of Physics, Carnegie Mellon University, Pittsburgh PA 15213*

3. *Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001*

(Dated: August 14th 2004)

We have recently argued that if one introduces a relational time in quantum mechanics and quantum gravity, the resulting quantum theory is such that pure states evolve into mixed states. The rate at which states decohere depends on the energy of the states. There is therefore the question of how this can be reconciled with Galilean invariance. More generally, since the relational description is based on objects that are not Dirac observables, the issue of covariance is of importance in the formalism as a whole. In this note we work out an explicit example of a totally constrained, generally covariant system of non-relativistic particles that shows that the formula for the relational conditional probability is a Galilean scalar and therefore the decoherence rate is invariant.

I. INTRODUCTION

We have recently introduced a new technique for discretizing physical theories [1]. When applied to general relativity it yields a discrete theory that is constraint-free yet it approximates well the continuum theory under certain circumstances [2, 3]. The lack of constraints allows to tackle some of the fundamental open problems of canonical quantum gravity. For instance one can introduce a relational time [4, 5, 6] a la Page–Wootters [7]. That is, one promotes all quantities in the theory to quantum operators and chooses one that is called “clock” and then one computes conditional probabilities for the other variables to take given values when the “clock” variable shows a certain “time”. The resulting quantum theory approximates ordinary quantum mechanics well when the clock variable chosen behaves in a semi-classical fashion with small quantum fluctuations. If one chooses as clock a variable that is in a quantum regime, the resulting theory is still valid but it will not resemble ordinary quantum mechanics.

We have also argued that, due to the fact that one cannot have a perfectly classical clock in nature, the resulting theory will have small but non-vanishing departures from ordinary quantum mechanics. In particular a pure state does not remain pure forever but evolves into a mixed state.

Since one is approximating a constrained continuum theory with a discrete theory that is unconstrained, the resulting relational discrete theory is formulated in terms of variables that are observables for the discrete theory. But they are not necessarily the discrete counterpart of Dirac observables of the continuum theory. Therefore the issue of how to reconcile the predictions of the discrete relational theory with the covariance of the continuum theory is of importance. In particular, the conditional probabilities must remain invariant when one changes coordinates and both the clock variable and the observed variable change values. To tackle the covariance problem in complete generality is beyond the scope of this paper. What we intend to do here is to analyze a simple model where calculations can be worked out concretely and in particular to probe the following issue. Since the prediction for the time of decoherence of pure states results in a formula that involves the energy of the states, it is may not be immediately apparent in what sense it is Galilean invariant. We would like to discuss in a simple model how to interpret the formula in a way that the invariance is manifest.

The organization of this paper is as follows. In section II we present the model we will study and in section III we will show the emergence of Galilean invariance. We end with a discussion.

II. THE MODEL

We consider the following model. It consists of two non-interacting particles moving in separate potentials in $1 + 1$ dimensions. One of the particles we will assume is much more massive than the other and it will determine the variable we choose as a clock. The other particle we will call the “system” particle. The potential affecting the clock particle will be a constant force field. We will assume the particle is far away from the turning point, since in this regime we know our discrete approach approximates well the continuum [2]. For the system particle we will assume it behaves quantum mechanically and is in a potential that gives rise to bound states. As is well known [8], the best way to understand Galilean transformations in quantum mechanics is to study them as a limit of Lorentz transformations.

We will therefore choose Lorentz invariant and reparameterization invariant action for the particles,

$$S = \int d\tau \left[- \left(Mc + \frac{U(q^0, q)}{c} \right) \sqrt{(\dot{q}^0)^2 - \dot{q}^2} - \left(mc + \frac{V(\phi^0, \phi)}{c} \right) \sqrt{(\dot{\phi}^0)^2 - \dot{\phi}^2} \right] \quad (1)$$

where q^0, q are the space-time coordinates of the ‘‘clock’’ particle and ϕ^0, ϕ are the space-time coordinates of the ‘‘system’’ particle. We have kept the speed of light explicit in order to consider later the non-relativistic limit. We start by assuming that the time-like coordinates of both particles can be synchronized (since we will work in the Newtonian limit this poses no conceptual problem) $q^0 = \phi^0$, and the reference system has been chosen such that $\dot{q}^0 \gg \dot{q}$ and $\dot{\phi}^0 \gg \dot{\phi}$, and also $Mc^2 \gg U(q^0, q)$ and $mc^2 \gg V(\phi^0, \phi)$ (non-relativistic limit). For concreteness we assume that in the reference frame given, $U(q^0, q) = \alpha q$ and $V(\phi^0, \phi) = V(\phi)$ and the latter has bound states. With these assumptions the action becomes,

$$S = \int d\tau \left[-Mc\dot{q}^0 - \frac{\alpha q}{c}\dot{q}^0 + Mc\frac{\dot{q}^2}{2\dot{q}^0} - mc\dot{\phi}^0 - \frac{V(\phi)}{c}\dot{\phi}^0 + Mc\frac{\dot{\phi}^2}{2\dot{\phi}^0} + \lambda(q^0 - \phi^0) \right] \quad (2)$$

where λ is a Lagrange multiplier associated with the constraint that imposes the synchronization. It is immediate to see that if one chooses $q^0 = \phi^0 = ct$ one will obtain the ordinary action for two non-relativistic particles with t the ordinary non-relativistic time. We will not do this here, since we are interested in handling a totally constrained system, since it is in such systems where the introduction of a relational time is meaningful since they have no preferred notion of time.

To understand better the constraint structure of the theory, we will rewrite the action in first-order form. We define the canonical momenta,

$$p_0 = \frac{\partial L}{\partial \dot{q}^0} = -Mc - \frac{\alpha q}{c} - \frac{Mc}{2} \frac{\dot{q}^2}{(\dot{q}^0)^2}, \quad (3)$$

$$p = \frac{\partial L}{\partial \dot{q}} = Mc \frac{\dot{q}}{\dot{q}^0}, \quad (4)$$

$$\pi_0 = \frac{\partial L}{\partial \dot{\phi}^0} = -mc - \frac{V(\phi)}{c} - \frac{Mc}{2} \frac{\dot{\phi}^2}{(\dot{\phi}^0)^2}, \quad (5)$$

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = m \frac{\dot{\phi}}{\dot{\phi}^0}. \quad (6)$$

From where we can get two constraints, in addition to the one we had before $\phi^0 - q^0 = 0$,

$$p_0 = -Mc - \frac{\alpha q}{c} - \frac{p^2}{2Mc} = -\frac{1}{c} H_1(p, q), \quad (7)$$

$$\pi_0 = -mc - \frac{V(\phi)}{c} - \frac{\pi^2}{2mc} = -\frac{1}{c} H_2(\phi, \pi). \quad (8)$$

If we rearrange the latter two constraints into their sum and difference,

$$\pi_0 + p_0 + \frac{H_1}{c} + \frac{H_2}{c} = 0, \quad (9)$$

$$\pi_0 - p_0 + \frac{H_1}{c} - \frac{H_2}{c} = 0, \quad (10)$$

one readily sees that the last constraint together with $q^0 - \phi^0 = 0$ are second class, whereas they both commute with (9). One imposes the second class constraints strongly and is left with a theory with one constraint, whose action is,

$$S = \int d\tau \left((p_0 + \pi_0) \dot{q}^0 + p\dot{q} + \pi\dot{\phi} + N \left[p_0 + \pi_0 + \frac{H_1}{c} + \frac{H_2}{c} \right] \right) \quad (11)$$

$$= \int d\tau \left(\tilde{p}_0 \dot{q}^0 + p\dot{q} + \pi\dot{\phi} + N \left[\tilde{p}_0 + \frac{H_1}{c} + \frac{H_2}{c} \right] \right) \quad (12)$$

where we introduced the shorthand $\tilde{p}_0 \equiv p_0 + \pi_0$. This action is very natural for the system under study (in fact we could have started the calculation simply by considering this action from the outset).

III. GALILEAN INVARIANCE

To probe the invariance of the decoherence effect of interest, we would like to study two different situations. One in which the system particle is a potential $V(\phi)$ and another in which the potential is of the form $V(\phi - \beta q^0)$, this will represent a system that is bound by a potential around some minimum that is fixed or that is moving, respectively. This corresponds to adopting the active point of view of the Galilean transformation. We will do this with our consistent discretization techniques [1, 2, 3]. We refer the reader to our previous papers for details on the technique. We start by discretizing the action in the first of the two cases of interest. The integral in the action becomes replaced by a discrete sum $S = \sum_0^N L(n, n+1)$ and we absorb the time interval $\epsilon = \tau_{n+1} - \tau_n$ and where,

$$L(n, n+1) = \tilde{p}_{0n} (q_{n+1}^0 - q_n^0) + p_n (q_{n+1} - q_n) + \pi_n (\phi_{n+1} - \phi_n) - N_n \left[\tilde{p}_{0n} + \frac{H_1(p_n, q_n)}{c} + \frac{H_2(\pi_n, \phi_n)}{c} \right]. \quad (13)$$

We now implement the canonical transformation that materializes the time evolution between instant n and $n+1$ with the Lagrangian $-L(n, n+1)$ playing the role of generating function of a type I canonical transformation,

$$P_{n+1}^{\tilde{p}_0} = \frac{\partial L(n, n+1)}{\partial \tilde{p}_{0n+1}} = 0, \quad (14)$$

$$P_n^{\tilde{p}_0} = -\frac{\partial L(n, n+1)}{\partial \tilde{p}_{0n}} = -(q_{n+1}^0 - q_n^0) + N_n, \quad (15)$$

$$P_{n+1}^q = \frac{\partial L(n, n+1)}{\partial q_{n+1}^0} = \tilde{p}_{0n}, \quad (16)$$

$$P_n^q = -\frac{\partial L(n, n+1)}{\partial q_n^0} = \tilde{p}_{0n}, \quad (17)$$

$$P_{n+1}^p = \frac{\partial L(n, n+1)}{\partial p_{n+1}} = 0, \quad (18)$$

$$P_n^p = -\frac{\partial L(n, n+1)}{\partial p_n} = -(q_{n+1} - q_n) + N_n \frac{p_n}{M}, \quad (19)$$

$$P_{n+1}^q = \frac{\partial L(n, n+1)}{\partial q_{n+1}} = p_n, \quad (20)$$

$$P_n^q = -\frac{\partial L(n, n+1)}{\partial q_n} = p_n + \alpha N_n, \quad (21)$$

$$P_{n+1}^\pi = \frac{\partial L(n, n+1)}{\partial \pi_{n+1}} = 0, \quad (22)$$

$$P_n^\pi = -\frac{\partial L(n, n+1)}{\partial \pi_n} = -(\phi_{n+1} - \phi_n) - N_n \frac{\pi_n}{m}, \quad (23)$$

$$P_{n+1}^\phi = \frac{\partial L(n, n+1)}{\partial \phi_{n+1}} = \pi_n, \quad (24)$$

$$P_n^\phi = -\frac{\partial L(n, n+1)}{\partial \phi_n} = \pi_n + N_n \frac{\partial V(\phi_n)}{\partial \phi_n}, \quad (25)$$

$$P_{n+1}^N = \frac{\partial L(n, n+1)}{\partial N_{n+1}} = 0, \quad (26)$$

$$P_n^N = -\frac{\partial L(n, n+1)}{\partial N_n} = \tilde{p}_{0n} + Mc + \alpha \frac{q_n}{c} + \frac{(p_n)^2}{2Mc} + mc + \frac{V(\phi_n)}{c} + \frac{(\pi_n)^2}{2mc}. \quad (27)$$

The system has constraints and we will use them to eliminate some of the variables and yield a system of evolution equations in a more explicit form. The resulting system is the following,

$$q_{n+1}^0 = q_n^0 + N_n, \quad (28)$$

$$P_{n+1}^q = P_n^q, \quad (29)$$

$$q_{n+1} = q_n + N_n \frac{P_{n+1}^q}{M}, \quad (30)$$

$$P_{n+1}^q = P_n^q - \alpha N_n, \quad (31)$$

$$\phi_{n+1} = \phi_n + N_n \frac{P_{n+1}^\phi}{m}, \quad (32)$$

$$P_{n+1}^\phi = P_n^\phi - N_n \frac{\partial V(\phi_n)}{\partial \phi_n}, \quad (33)$$

$$0 = P_{n+1}^{q_0} + Mc + \alpha \frac{q_n}{c} + \frac{(P_{n+1}^q)^2}{2Mc} + mc + \frac{V(\phi_n)}{c} + \frac{(P_{n+1}^\phi)^2}{2mc}. \quad (34)$$

The last equation determines the Lagrange multiplier N_n . To see this, we first rewrite it entirely in terms of variables at $n+1$,

$$P_{n+1}^{q_0} + Mc + \alpha \frac{q_{n+1} - N_n P_{n+1}^q}{Mc} + \frac{(P_{n+1}^q)^2}{2Mc} + mc + \frac{V(\phi_{n+1} - N_n P_{n+1}^\phi)}{mc} + \frac{(P_{n+1}^\phi)^2}{2mc} = 0. \quad (35)$$

Since we are ultimately interested in studying the system in a regime close to the continuum limit, we make the assumption that the lapse N_n is small and expand the term involving the potential to first order in N_n ,

$$P_{n+1}^{q_0} + Mc + \alpha \frac{q_{n+1} - N_n P_{n+1}^q}{Mc} + \frac{(P_{n+1}^q)^2}{2Mc} + mc + \frac{V(\phi_{n+1})}{mc} - \frac{N_n P_{n+1}^\phi}{mc} \frac{\partial V(\phi_{n+1})}{\partial \phi_{n+1}} + \frac{(P_{n+1}^\phi)^2}{2mc} = 0. \quad (36)$$

We can now solve explicitly for the Lagrange multiplier,

$$N_n = \left(\alpha \frac{P_{n+1}^q}{Mc} + \frac{P_{n+1}^\phi}{mc} V'(\phi_n) \right)^{-1} C_{n+1} \quad (37)$$

Where C_{n+1} is the constraint of the continuum theory discretized as if all variables were at $n+1$,

$$C_{n+1} = P_{n+1}^{q_0} + Mc + \alpha \frac{q_{n+1}}{c} + \frac{(P_{n+1}^q)^2}{2Mc} + mc + \frac{V(\phi_{n+1})}{c} + \frac{(P_{n+1}^\phi)^2}{2mc}. \quad (38)$$

We now assume that $\alpha \gg V'(\phi)$. This is due to the fact that we are assuming the clock to be classical and large and α is therefore associated with a macroscopic force whereas $V(\phi)$ is the potential in which the system is bound, and the latter is microscopic in nature. With this assumption we make sure there are no singularities in the computation of the Lagrange multiplier. Recall that the discrete description departs from the continuum one close to the turning point of the orbit.

One can now substitute the expression of the Lagrange multiplier in the evolution equations. The resulting system of equations can be viewed as a canonical transformation between instant n and instant $n+1$ for the remaining variables of the problem. The next step consists in quantizing the system by representing the discrete evolution through a unitary operator, i.e. $\hat{z}_n^i = \hat{U}^\dagger \hat{z}_{n+1}^i \hat{U}$ where the z^i 's are all the phase space variables of the problem. All these calculations can be worked out explicitly for a simple system like the one we are considering, we will not show all the details here for reasons of space, the reader can see similar treatments in [4, 9].

Since we are interested in the continuum limit, a shortcut can be taken by considering the Hamiltonian associated with the unitary transformation $\hat{U} = e^{i\hat{H}}$ [9]. The Hamiltonian is obtained by taking the logarithm of the unitary operator as a power series. This power series is convergent at all points in phase space except for a small region around the turning point of the orbit of the clock system. The Hamiltonian is obviously conserved upon evolution (except at the turning point). The first term in the expansion of the Hamiltonian is,

$$H_n \sim \frac{Mc(C_n)^2}{\alpha p_n} \left[1 + O\left(\frac{McC_n}{(p_n)^2}\right) \right], \quad (39)$$

and to simplify notation, from now on we call $P_n^q = p_n$ and $P_n^{q_0} = p_{0n}$.

For the quantization we consider wavefunctions $\psi_n(q^0, q, \phi)$ forming a Hilbert space at the ‘‘instant’’ n . Isomorphic Hilbert spaces exist at all other discrete instants. With the Hamiltonian we will study the evolution operator $\hat{U}(n, n_0) = e^{i\hat{H}(n-n_0)}$ and its action on the states, $\psi_n(q^0, q, \phi) = \hat{U}(n, n_0)\psi_{n_0}(q^0, q, \phi)$, and we are working in the Schrödinger representation. The explicit form of the quantum Hamiltonian is,

$$\hat{H} = \frac{Mc}{\alpha \hat{p}} \left(\hat{p}_0 + \hat{H}_1 + \hat{H}_2 \right)^2. \quad (40)$$

It is to be noted that the expression in parenthesis is the constraint that one has in the continuum theory. In the consistent discretization approach the constraint of the continuum theory is not enforced exactly (what is enforced is equation (34) which corresponds to the constraint of the continuum theory but with the momenta evaluated one instant after the configuration variables). In the continuum limit, it nevertheless is enforced quite approximately and therefore the norm of \hat{H} is going to be small.

We consider a quantum state in which the clock has a semiclassical behavior, so it is described by a coherent state peaked at $\langle \hat{H}_1 \rangle_{n_0} = \bar{E}$, $\langle \hat{q}_0 \rangle_{n_0} = 0$, $\langle \hat{q} \rangle_{n_0} = \bar{q}$, $\langle \hat{p} \rangle_{n_0} = \bar{p}$ and $\langle \hat{p}_0 \rangle_{n_0} = \bar{p}_0$. We then have for the wavefunction,

$$\Psi_{n_0} = \psi_{n_0}(q, q^0) \varphi_{n_0}(\phi) \quad (41)$$

with

$$\psi_{n_0}(q, q^0) = (2\pi\sigma_1^2)^{-1/4} \exp\left[-\frac{(q - \bar{q})^2}{4\sigma_1^2} + i\bar{p}q\right] (2\pi\sigma_0^2)^{-1/4} \exp\left[-\frac{(q^0)^2}{4\sigma_0^2} + i\bar{p}_0q^0\right] \quad (42)$$

where σ_1 is the dispersion in the variable q and σ_0 is the dispersion in the variable q^0 . We have also assumed that $\bar{E} \gg |\bar{p}_0 + \bar{E}| \gg \langle \hat{H}_2 \rangle_{n_0}$. The first inequality is in order to be in the continuum limit. The second inequality is in order to simplify calculations, and implies that we are accepting as ‘‘continuum limit’’ a regime where the constraint of the continuum theory is well enforced with respect to the scale of energies of relevance for the ‘‘clock’’ system, but the error in enforcement is large with respect to the energies of the system under study. It would be desirable to extend the results of this paper to regimes that approximate even further the continuum theory, but the calculations would be more involved.

The fundamental equation to be studied is the conditional probability,

$$P(\phi \in \Delta\phi | q^0 \in \Delta t) = \frac{\sum_n \text{Tr} \left(\hat{U}^\dagger(n) \hat{P}_{\phi, q^0} \hat{U}(n) \rho_{q^0} \times \rho_q \times \rho_\phi \right)}{\sum_n \text{Tr} \left(\hat{U}^\dagger(n) \hat{P}_{q^0} \hat{U}(n) \rho_{q^0} \times \rho_q \times \rho_\phi \right)} \quad (43)$$

with \hat{P}_{ϕ, q^0} is the projector onto the eigenstate labeled by the values ϕ, q^0 and $\rho_{q^0}, \rho_q, \rho_\phi$ the density matrices associated with the state Ψ_{n_0} . From now on we will use natural units where $c = \hbar = 1$.

Let us analyze the denominator of this expression. Taking the trace on the ϕ, q spaces by integrating, we get,

$$\begin{aligned} \text{Den} &= \sum_n \text{Tr} \left(\hat{U}^\dagger(n) \hat{P}_{q^0} \hat{U}(n) \rho_{q^0} \times \rho_q \times \rho_\phi \right) \quad (44) \\ &= \sum_n \text{Tr} \left[\exp \left(-i \frac{(\hat{p}_0 + \bar{E})^2}{\frac{\alpha\bar{p}}{M}} (n - n_0) - 2i \frac{\hat{p}_0 + \bar{E}}{\frac{\alpha\bar{p}}{M}} \langle \hat{H}_2 \rangle (n - n_0) \right) \times \right. \\ &\quad \left. \times \hat{P}_{q^0} \exp \left(i \frac{(\hat{p}_0 + \bar{E})^2}{\frac{\alpha\bar{p}}{M}} (n - n_0) + 2i \frac{\hat{p}_0 + \bar{E}}{\frac{\alpha\bar{p}}{M}} \langle \hat{H}_2 \rangle (n - n_0) \right) \rho_{q^0} \right], \end{aligned}$$

and the term involving \hat{H}_2^2 from U cancels with that of \hat{U}^\dagger since \hat{H}_2^2 commutes with \hat{P}_{q^0} . We have also replaced \hat{H}_1 by \bar{E} and \hat{p} by \bar{p} since the trace implies taking the expectation value of quantities depending on q and p . Since ρ_{q^0} represents a state very peaked at $\langle \hat{p}_0 \rangle = \bar{p}_0$ and $\langle \hat{q}^0 \rangle = 0$ and since $|\bar{p}_0 + \bar{E}| \gg \langle \hat{H}_2 \rangle$, we have that,

$$\text{Den} = \sum_n \text{Tr} \left[\hat{P}_{q^0} \exp \left(i \frac{(\hat{p}_0 + \bar{E})^2}{\frac{\alpha\bar{p}}{M}} (n - n_0)^2 \right) \rho_{q^0} \exp \left(-i \frac{(\hat{p}_0 + \bar{E})^2}{\frac{\alpha\bar{p}}{M}} (n - n_0)^2 \right) \right] \equiv \sum_n \text{Tr} (\rho_n(q^0)) \quad (45)$$

where $\rho_n(q^0) \equiv \hat{P}_{q^0} \rho_{n, q^0} \equiv \hat{P}_{q^0} \hat{U}(n) \rho_{q^0} \hat{U}^\dagger(n)$ represents the wavepacket of a ‘‘free particle’’ which evolves with the effective Hamiltonian

$$\hat{H}_{\text{eff}} = \frac{(\hat{p}_0 + \bar{E})^2}{\frac{\alpha\bar{p}}{M}} \quad (46)$$

It is instructive to realize that one can write,

$$\text{Tr} [\rho_n(q^0)] = (2\pi\sigma_0^2(n))^{-1/4} \exp \left[-\frac{(q^0 - \bar{q}_0(n))^2}{4\sigma_0^2(n)} \right], \quad (47)$$

where

$$\bar{q}^0(n) = 2 \frac{(\bar{p}_0 + \bar{E})(n - n_0)}{\frac{\alpha \bar{p}}{M}} \equiv t_{\max}(n), \quad (48)$$

which shows that the clock “displays a time” in the neighborhood of \bar{q}^0 when we are at the level n of the discrete theory. We have defined $t_{\max}(n)$, the most likely value of the clock “time” for a given n level in the discrete theory, and we have chosen the clock in such a way that t_{\max} grows linearly with n .

The width of the packet grows with n as,

$$\sigma_0^2(n) = \sigma_0^2 \left(1 + \frac{1}{4\sigma_0^4} \left(\frac{M}{\alpha \bar{p}} \right)^2 (n - n_0)^2 \right). \quad (49)$$

We now should introduce some relevant scales. We will assume the characteristic mass of the clock system is about a kilogram. The potential of the clock system is characterized by the macroscopic constant α , which we will assume is of the order of 10 Newton, which in natural units corresponds to $\alpha \sim 10^{22} m^{-2}$, which implies that if we have $\sigma_0 \sim 10^{-10} s \sim 1m$, then,

$$\frac{1}{4\sigma_0^2} \left(\frac{M}{\alpha \bar{p}} \right)^2 \sim 10^{-34} m^2, \quad (50)$$

and we have assumed $\bar{p}/M \sim 10^{-5}$ so we are in a non-relativistic regime.

As we discussed in [4], the sums that appear in the numerator and denominator for the conditional probability should be large enough to involve the complete evolution of interest for the system, but they should not be infinite, otherwise one gets an indeterminate quotient of two diverging quantities for the conditional probability. Given the value computed above for the quantity multiplying $(n - n_0)^2$, it is natural to bound the value of $n - n_0 \ll 10^{17}$, that is we assume that the sums go from n_0 to a maximum value $N \ll 10^{17}$, let's say $N \sim 10^{14}$, otherwise the packet representing the clock will spread too much and we would be out of the semiclassical regime. Notice that we also have that $\bar{E} \geq 10^{26} m^{-1}$, and recalling that $|\bar{p}_0 + \bar{E}|$ has to be smaller than \bar{E} , and choosing it to be $10^{17} m^{-1}$ yields $\bar{q}^0 \sim 10^4 s \sim 3$ hours, which is a reasonable number. Summarizing, by bounding the number of steps we find that the denominator is a quantity of order unity. Its precise value is not of great interest, since we can choose it by fixing the normalization of the probability.

Let us analyze the numerator,

$$\text{Numer} = \sum_n \text{Tr} \left[\hat{P}_{\phi, q^0} \exp \left(i \frac{(\hat{p}_0 + \bar{E})^2 + 2(\hat{p}_0 + \bar{E})\hat{H}_2 + \hat{H}_2^2}{\frac{\alpha \bar{p}}{M}} \right) \rho_{q^0} \rho_q \rho_\phi \exp \left(-i \frac{(\hat{p}_0 + \bar{E})^2 + 2(\hat{p}_0 + \bar{E})\hat{H}_2 + \hat{H}_2^2}{\frac{\alpha \bar{p}}{M}} \right) \right], \quad (51)$$

and observing that the projector is independent of q , and one can therefore substitute H_1 by its expectation value \bar{E} . Using now that the clock is semiclassical and $\bar{p}_0 + \bar{E} \gg \langle \hat{H}_2 \rangle$ to neglect terms quadratic in \hat{H}_2 we can write,

$$\begin{aligned} P(\phi \in \Delta\phi | q^0 \in \Delta t) &= \sum_n \text{Tr} \left[\hat{P}_{\phi, q^0} \exp \left(i \frac{[(\hat{p}_0 + \bar{E})^2 + 2(\bar{p}_0 + \bar{E})\hat{H}_2] (n - n_0)}{\frac{\alpha \bar{p}}{M}} \right) \rho_\phi(n_0) \rho_{q^0}(n_0) \right. \\ &\quad \left. \times \exp \left(-i \frac{[(\hat{p}_0 + \bar{E})^2 + 2(\bar{p}_0 + \bar{E})\hat{H}_2] (n - n_0)}{\frac{\alpha \bar{p}}{M}} \right) \right] \text{Den}^{-1} \end{aligned} \quad (52)$$

$$\begin{aligned} &= \sum_n \text{Tr} \left[\hat{P}_\phi \exp \left(i \hat{H}_2 t_{\max}(n) \right) \rho_\phi \exp \left(-i \hat{H}_2 t_{\max}(n) \right) \right] \times \\ &\quad \times \text{Tr} \left[\hat{P}_{q^0} \exp \left(i \frac{(\hat{p} + \bar{E})^2}{\frac{\alpha \bar{p}}{M}} (n - n_0) \right) \rho_{q^0} \exp \left(-i \frac{(\hat{p} + \bar{E})^2}{\frac{\alpha \bar{p}}{M}} (n - n_0) \right) \right] \text{Den}^{-1}, \end{aligned} \quad (53)$$

where we have replaced \hat{p}_0 with \bar{p}_0 since the clock is approximately classical and its energy dominates in \bar{p}_0 , and we have separated the expression into two pieces, one dependent on the ϕ variable and one dependent on the q_0 variable to make more explicit the separation between clock and system.

The last trace divided by Den can be written as $\mathcal{P}_n(q^0)$ and satisfies that $\sum_n \mathcal{P}_n(q^0) = 1$.

Following the discussion in [6], in order to make contact with ordinary quantum mechanics we assume the spacing in n is small compared with the values of n and introduce a continuous variable $v = n\epsilon$. We choose ϵ such that

$$\epsilon = 2\frac{\bar{p} + \bar{E}}{\frac{\alpha\bar{p}}{E}}, \quad (54)$$

so we have that $\epsilon \leq 1m$ with the choice of scales we made for the problem. We choose $n_0 = 0$ and we can then write a good continuum limit approximation for $\mathcal{P}_n(q^0)$, as in [6],

$$\mathcal{P}_v(q^0) = \delta(v - q^0) + \sigma_0^2(q^0)\delta''(v - q^0), \quad (55)$$

with $\sigma_0^2(q^0)$ given by (48,49),

$$\sigma_0^2(q^0) = \sigma_0^2 \left(1 + \frac{1}{4\sigma_0^4} \frac{(q^0)^2}{4(\bar{p}_0 + \bar{E})^2} \right) \quad (56)$$

and with $t_{\max}(n) = \epsilon n = v$, so we can write,

$$P(\phi \in \Delta\phi | t \in \Delta t) = \int dv \text{Tr} \left[\hat{P}_\phi \exp(i\hat{H}_2 v) \rho_\phi \exp(-i\hat{H}_2 v) \right] (\delta(v - q^0) + \sigma_0^2(q^0)\delta''(v - q^0)) = \text{Tr} \left[\tilde{\rho}_2(q^0) \hat{P}_\phi \right] \quad (57)$$

where

$$\tilde{\rho}_2(q^0) = \int dv \mathcal{P}_v(q^0) \hat{U}_v \rho_\phi(v=0) \hat{U}_v^\dagger, \quad (58)$$

and this density matrix satisfies a Schrödinger equation modified due to the fact that we are considering a quantum clock as shown in detail in [6],

$$\frac{\partial \tilde{\rho}_2}{\partial q^0} = i[\hat{H}_2, \tilde{\rho}_2] - \sigma(q^0)[\hat{H}_2, [\hat{H}_2, \tilde{\rho}_2]], \quad (59)$$

with $\sigma(q^0) = d\sigma_0^2(q^0)/dq^0$. This expression is just the first two terms in a power series in terms of the dispersion of the quantum clock, which for realistic systems is a very small quantity.

To try to get a handle on a rough value for this quantity in the case of a realistic system, we note that the macroscopic clock particle is subject to decoherence due to interaction with the environment. If we characterize such decoherence by a time t_D , we have,

$$\sigma \sim \frac{1}{4\sigma_0^2} \frac{q^0}{2(\bar{p}_0 + \bar{E})^2} |_{q^0=t_D}. \quad (60)$$

If $t_D \sim 1s \sim 10^{10}m$, which is a rather large decoherence time for a macroscopic system, then $\sigma \sim 10^{-24}m$. In reference [10] we have estimated theoretical limits as to how small a dispersion is attainable with optimal realistic clocks.

In order to study the Galilean covariance of the conditional probability, the procedure is simple. We have to repeat the calculation assuming a boost with velocity $-\beta$ has been performed on the system 2 respect to the system 1, in such a way that the potential it now sees is of the form $V(\phi - \beta q^0)$. For instance, the system 2 can be an electron in a central potential given by a nucleus. The relational analysis goes along exactly as before, with two differences. The Hamiltonian for the second system becomes,

$$H'_2 = V(\phi - \beta q^0) + \frac{\pi^2}{2m}, \quad (61)$$

and the initial state of the system is given by $\rho'_{q^0} \times \rho'_\phi = \hat{U}_G (\rho_{q^0} \times \rho_\phi) \hat{U}_G^\dagger$ with

$$\hat{U}_G = \exp \left[i\beta\hat{\pi}\hat{q}^0 - im\beta\hat{\phi} \right]. \quad (62)$$

In other words, the initial state is the one corresponding to the Galilean boost \hat{U}_G to the original state [8]. Notice however that in traditional treatments of Galilean invariance in quantum mechanics the variable that we here take as \hat{q}^0 is a classical parameter t . Our treatment can be considered a relational generalization of the usual Galilean

transformations of quantum mechanics. In ordinary quantum mechanics Schrödinger's equation has a time derivative that acts on the parameter in \hat{U}_G . In the relational treatment the equation has a term involving \hat{p}_0 instead of the time derivative. Notice that \hat{p}_0 is minus the total energy instead of just the "system energy". Therefore the presence of the operator \hat{q}^0 in \hat{U}_G has the same effect in the relational treatment as the derivative with respect to the parameter has in ordinary quantum mechanics: they both induce a change in the energy of the system due to the boost, $\hat{p}_0 \rightarrow \hat{p}_0 + \beta\hat{\pi}$.

To study the changes in the conditional probability we go back in the derivation to equation (51),

$$P'(\phi \in \Delta\phi | q^0 \in \Delta t) = \sum_n \text{Tr} \left[\hat{P}_{\phi, q^0} \exp \left(i \frac{[(\hat{p}_0 + \bar{E})^2 + 2(\bar{p}_0 + \bar{E}) \hat{H}'_2]}{\frac{\alpha \bar{P}}{M}} (n - n_0) \right) \right. \\ \left. \times \hat{U}_G \rho_\phi(n_0) \rho_{q^0}(n_0) \hat{U}_G^\dagger \exp \left(-i \frac{[(\hat{p}_0 + \bar{E})^2 + 2(\bar{p}_0 + \bar{E}) \hat{H}'_2]}{\frac{\alpha \bar{P}}{M}} (n - n_0) \right) \right] \text{Den}^{-1}. \quad (63)$$

The value of the denominator actually does not change due to the boost, although its form changes. We will address this point later on.

To understand the covariance it is convenient to commute \hat{U}_G with the exponential; let us therefore analyze the product,

$$B = \exp \left(i \frac{[(\hat{p}_0 + \bar{E})^2 + 2(\bar{p}_0 + \bar{E}) \hat{H}'_2]}{\frac{\alpha \bar{P}}{M}} t_{\max}(n) \right) \exp \left(i \frac{m\beta^2}{2} \hat{q}^0 \right) \exp(i\beta\hat{\pi}\hat{q}^0) \exp(-im\beta\hat{\phi}), \quad (64)$$

where we have used the fact that,

$$\hat{U}_G = \exp \left(i \frac{m\beta^2}{2} \hat{q}^0 \right) \exp(i\beta\hat{\pi}\hat{q}^0) \exp(-im\beta\hat{\phi}), \quad (65)$$

which can be shown using the Baker–Campbell–Hausdorff formula.

We wish to commute \hat{U}_G to the left. We start by noting that in the subspace of ϕ, π , the variable q^0 behaves as an external parameter, as if it were a classical time t . Following the calculations of [8] for an ordinary quantum system one has,

$$\exp(i\hat{H}'(t - t_0)) \hat{U}_{t_0} \psi(t_0) = \hat{U}_t \exp(i\hat{H}(t - t_0)) \psi(t_0), \quad (66)$$

which just states that the evolution of the Galilean transformed state should coincide with the Galilean transform of the original evolved state. That is,

$$\exp(i\hat{H}'(t - t_0)) \hat{U}_{t_0} \psi(t_0) = \exp \left(im \frac{\beta^2}{2} (t - t_0) \right) \exp(i\beta\hat{\pi}(t - t_0)) \hat{U}_{t_0} \exp(i\hat{H}(t - t_0)) \psi(t_0). \quad (67)$$

We can therefore write for our system,

$$P'(\phi \in \Delta\phi | q^0 \in \Delta t) = \sum_n \text{Tr} \left[\hat{P}_{\phi, q^0} \exp \left(i \left[\frac{m\beta^2}{2} t_{\max}(n) + i\beta\hat{\pi} t_{\max}(n) \right] \right) \exp \left(i \frac{(\hat{p}_0 + \hat{E})^2}{\frac{\alpha \bar{P}}{M}} (n - n_0) \right) \hat{U}_G \right. \\ \times \exp(i\hat{H}'_2 t_{\max}(n)) \rho_\phi(n_0) \rho_{q^0}(n_0) \exp(-i\hat{H}'_2 t_{\max}(n)) \hat{U}_G^\dagger \exp \left(-i \frac{(\hat{p}_0 + \hat{E})^2}{\frac{\alpha \bar{P}}{M}} (n - n_0) \right) \\ \left. \times \exp \left(-i \left[\frac{m\beta^2}{2} t_{\max}(n) + i\beta\hat{\pi} t_{\max}(n) \right] \right) \right] \text{Den}^{-1}. \quad (68)$$

We still need to commute $\exp \left(i \frac{(\hat{p}_0 + \hat{E})^2}{\frac{\alpha \bar{P}}{M}} (n - n_0) \right)$ with \hat{U}_G . Noting that the expression for \hat{U}_G (65) can be written as,

$$U_G = \exp \left(i \left[\frac{\beta^2 m}{2} + \beta\hat{\pi} \right] \hat{q}_0 \right) \exp(-im\beta\hat{\phi}) \quad (69)$$

we see that only the first term in the exponential has a non-trivial commutator.

To proceed we note that if one has two operators \hat{A}, \hat{B} such that $[[\hat{A}, \hat{B}], \hat{A}] = 0$, one has that,

$$e^{\hat{A}} e^{\hat{B}} = e^{\frac{1}{2}[\hat{A}, \hat{B}]} e^{\hat{A} + \hat{B}}. \quad (70)$$

If we now take $A = a(\hat{p}_0 + \bar{E})^2$ and $B = b\hat{q}_0$ we have the following identities,

$$\exp\left(ia(\hat{p}_0 + \bar{E})^2\right) \exp(ib\hat{q}_0) = \exp(iab(\hat{p}_0 + \bar{E})) \exp\left(ia(\hat{p}_0 + \bar{E})^2 + ib\hat{q}_0\right), \quad (71)$$

and,

$$\exp(ib\hat{q}_0) \exp\left(ia(\hat{p}_0 + \bar{E})^2\right) = \exp(-iab(\hat{p}_0 + \bar{E})) \exp\left(-\frac{i}{6}ab^2\right) \exp\left(ia(\bar{p}_0 + \bar{E})^2 + ib\hat{q}_0\right), \quad (72)$$

therefore,

$$\exp\left(ia(\bar{p}_0 + \bar{E})^2\right) \exp(ib\hat{q}_0) = \exp(2iab(\hat{p}_0 + \bar{E})) \exp\left(\frac{i}{6}ab^2\right) \exp(ib\hat{q}_0) \exp\left(ia(\hat{p}_0 + \bar{E})^2\right). \quad (73)$$

Taking $a = \frac{i(n-n_0)}{\frac{\alpha\bar{p}}{M}}$ and $b = \frac{m}{2}\beta^2 + \beta\hat{\pi}$ and substituting $\hat{p}_0 + \bar{E}$ by $\bar{p}_0 + \bar{E}$ in (68) we finally have,

$$P'(\phi \in \Delta\phi | q^0 \in \Delta t) = \sum_n \text{Tr} \left[\hat{P}_{\phi, q^0} \hat{U}_G \exp\left(i \frac{[(\hat{p}_0 + \bar{E})^2 + 2(\bar{p}_0 + \bar{E})\hat{H}_2]}{\frac{\alpha\bar{p}}{M}}(n - n_0)\right) \rho_\phi \rho_{q_0} \right] \quad (74)$$

$$\times \exp\left(-i \frac{[(\hat{p}_0 + \bar{E})^2 + 2(\bar{p}_0 + \bar{E})\hat{H}_2]}{\frac{\alpha\bar{p}}{M}}(n - n_0)\right) \hat{U}_G^\dagger \text{Den}^{-1}. \quad (75)$$

It is remarkable that all the terms involving t_{\max} in (68) have cancelled with the terms stemming from the commutation we just did. We now can address the point we postponed before, namely the change in the denominator of the expression. Basically, a similar calculation to the one we just did starting from (45) and performing the commutations shows that the denominator is actually invariant, using the cyclicity of the trace and the fact that unlike the numerator, it does not involve the projector on the ϕ space.

Now since

$$\hat{U}_G^\dagger \hat{P}_{\phi, q^0} \hat{U}_G = \hat{P}_{\phi - \beta q^0, q^0}, \quad (76)$$

we therefore have,

$$P'(\phi \in \Delta\phi | q^0 \in \Delta t) = P(\phi - \beta q^0 \in \Delta\phi | q^0 \in \Delta t) \quad (77)$$

Which shows that the conditional probability is Galilean invariant.

IV. CONCLUSIONS

We have shown in a simple model that considering a quantum clock in quantum mechanics leads to a modification of Schrödinger equation, and that the resulting probabilities are Galilean invariant. Since the probabilities are invariant, then physical predictions from this framework will also be invariant. In particular the rate of decoherence predicted in [5, 6, 10] should be invariant. This is an interesting point since the rate of decoherence predicted is proportional to the difference of energies of states of the system in an basis of energy eigenstates. One could ask the question, how can this formula be Galilean invariant since the energy is not? The answer has to do with the fact that in order to have an energy basis as the one assumed in the calculation (with discrete spectrum) one has to consider systems analogous to a particle in a potential. In such systems, at least if they are isolated, the difference between energy levels is a Galilean invariant and therefore the decoherence rate is a Galilean invariant.

It remains to be studied how the decoherence presented would transform under Lorentz transformations. Milburn [11] recently studied decoherence in a Lorentz invariant setting and his treatment could provide a framework to analyze our proposal in some detail. This is more problematic, since we are considering corrections to quantum mechanics, and if one goes to the relativistic domain one first has to contend with the usual difficulties of defining a relativistic quantum mechanics. Although it appears that the use of a relational time could yield a well defined theory, the details remain to be studied.

V. ACKNOWLEDGMENTS

This work was inspired by questions by Ted Jacobson and Daniel Sudarsky at the Ladek Zdroj meeting on theoretical physics, JP thanks the organizers, Jerzy Kowalski-Glikman and Giovanni Amelino-Camelia for hospitality. This work was supported by grant nsf-phy0244335 and funds from the Horace Hearne Jr. Laboratory for Theoretical Physics and the Abdus Salam International Center for Theoretical Physics.

-
- [1] C. Di Bartolo, R. Gambini, J. Pullin, *Class. Quan. Grav.* **19**, 5475 (2002) [arXiv:gr-qc/0205123]; C. Di Bartolo, R. Gambini, R. Porto, J. Pullin, “Dirac-like approach for consistent discretizations of classical constrained theories,” arXiv:gr-qc/0405131.
 - [2] R. Gambini, J. Pullin, *Class. Quan. Grav.* **20**, 3341 (2003) [arXiv:gr-qc/0212033].
 - [3] R. Gambini, J. Pullin, *Phys. Rev. Lett.* **90**, 021301, (2003) [arXiv:gr-qc/0206055].
 - [4] R. Gambini, R.A. Porto and J. Pullin, In “Recent developments in gravity,” K. Kokkotas, N. Stergioulas, editors, World Scientific, Singapore (2003) [arXiv:gr-qc/0302064].
 - [5] R. Gambini, R. Porto, J. Pullin, *Class. Quant. Grav.* **21**, L51 (2004) [arXiv:gr-qc/0305098].
 - [6] R. Gambini, R. Porto, J. Pullin, *New J. Phys.* **6**, 45 (2004) [arXiv:gr-qc/0402118].
 - [7] D. N. Page and W. K. Wootters, *Phys. Rev. D* **27**, 2885 (1983); W. Wootters, *Int. J. Theor. Phys.* **23**, 701 (1984); D. N. Page, “Clock time and entropy” in “Physical origins of time asymmetry,” J. Halliwell, J. Perez-Mercader, W. Zurek (editors), Cambridge University Press, Cambridge UK, (1992).
 - [8] H. Brown, P. Hollands, *Am. J. Phys.* **67**, 204 (1999).
 - [9] R. Gambini and J. Pullin, *Int. J. Mod. Phys. D* **12**, 1775 (2003) [arXiv:gr-qc/0306095].
 - [10] R. Gambini, R. A. Porto and J. Pullin, “Realistic clocks, universal decoherence and the black hole information paradox,” arXiv:hep-th/0406260.
 - [11] G. J. Milburn, “Lorentz invariant intrinsic decoherence,” arXiv:gr-qc/0308021.