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Loop quantization of spherically symmetric midi-superspaces

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We quantize the exterior of spherically symmetric vacuum space-times using a midi-superspace reduction within the Ashtekar new variables. Through a partial gauge fixing we eliminate the diffeomorphism constraint and are left with a Hamiltonian constraint that is first class. We complete the quantization in the loop representation. We also use the model to discuss the issues that will arise in more general contexts in the “uniform discretization” approach to the dynamics.

I. INTRODUCTION

Loop quantum gravity has emerged in recent years as a significant candidate for a theory of quantum gravity. See [1] for recent reviews. The theory has a mathematically rigorous basis for its quantum kinematics [2], which has also been proven to be unique up to certain assumptions [3]. The problem of the dynamics of the full theory has remained unsettled. The origin of the difficulties can be traced back to the kinematical space of states. In this space, infinitesimal diffeomorphisms are not implemented as operators. The issue is further compounded by the fact that one cannot check the consistency of the quantum constraint algebra, since the commutator of two Hamiltonians is proportional to an infinitesimal diffeomorphism. Attempts to circumvent this problem, like representing the Hamiltonian constraint as an operator on states invariant under diffeomorphisms, are faced with other difficulties (see [4] for a more thorough discussion).

These problems have led several researchers to consider alternatives to the usual Dirac approach to the problem of the dynamics. One of the alternatives is the “master constraint” project of Thiemann and collaborators [5] which has similarities to an earlier proposal by Klauder [6]. Others consider the covariant “spin foam” approach as an alternative, since one may bypass the construction of the canonical algebra entirely, at least in some settings. Another approach is the one we have presented in recent papers called “uniform discretizations” [7, 8].

The spherically symmetric case has also specific problems in the traditional approach to loop quantum gravity. It has not been possible up to now to find a particularization of the construction of Thiemann for the Hamiltonian constraint to the spherically symmetric case that has the appropriate algebra of constraints on the diffeomorphism invariant space of states [9].

In spite of these difficulties, the loop approach has been successful in the context of homogeneous cosmologies, giving rise to “loop quantum cosmology” [10]. The reason why the approach works in this context is that there is no issue with the algebra of constraints: the diffeomorphism constraint is gauge fixed by the use of manifestly homogeneous variables and there is only one Hamiltonian constraint. We would like to show that a similar situation emerges in the case of spherically symmetric space-times. We will show that one can fix the diffeomorphism gauge in such a way that one is left with only a set of Abelian Hamiltonian constraints.

Since this is our first approach to this problem, we will not explore the more interesting possibilities of this model, for instance, what happens to the singularity. We will restrict our attention to the exterior of the horizon. We will also fix a gauge to eliminate the diffeomorphism constraint for the sake of simplicity. The resulting Hamiltonian constraint has nevertheless a non-trivial first class algebra (with structure functions) in the continuum, but we will show it can be Abelianized. We will show that the loop quantization can be completed within the traditional Dirac quantization approach. In particular we will construct the kinematical and physical Hilbert spaces and we will show the quantization to be equivalent to the one carried out in terms of traditional variables by Kuchař [11]. We will discuss the implications for the elimination of the singularity, though we will not work out the details in this paper.

We would also like to discuss what lessons one can get from this model for the “uniform discretization” approach to the dynamics of quantum gravity. Since the uniform discretization approach is equivalent to the Dirac quantization procedure if one has Abelian constraints, it does not add anything new to the quantization of this model if one chooses to Abelianize the constraints as we do. We nevertheless would like to discuss what would happen if one had chosen not to Abelianize the constraints. The uniform discretization procedure can be applied, but the treatment of the model becomes more involved, and the degree of complexity increases with how much one chooses to ignore about the particular details of the model.

The organization of this paper is as follows. In section II we set up the classical variables for spherically symmetric space-times in the Ashtekar formulation; in section III we discuss the loop quantization using the traditional Dirac procedure and in section IV we discuss the use of uniform discretizations. We end with a discussion.

II. SPHERICALLY SYMMETRIC SPACE-TIMES

We will use the Ashtekar new variables to describe the spherically symmetric space-times. Previous work on this subject was done by Bengtsson [12], Thiemann and Kastrup [13] which also Abelianize the constraints with the traditional (complex) Ashtekar variables and with the more modern real connection variables by Bojowald and Kastrup [14], and Bojowald and Swiderski [9] so we will not repeat the full construction of spherically symmetric triads and connections (the latter were also discussed in the context of spherically symmetric Yang–Mills theory by Cordero and Teitelboim [15]). Also, in the context of geometrodynamical variables, gauge fixings in spherical symmetry were considered by [16].

One assumes that the topology of the spatial manifold is of the form $\Sigma = R^+ \times S^2$. We will choose a radial coordinate x and study the theory in the range $[0, \infty]$. We will later assume that there is a horizon at $x = 0$, with appropriate boundary conditions as we discuss below.

The invariant connection can be written as,

$$A = A_x(x)\Lambda_3 dx + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2) d\theta + ((A_1(x)\Lambda_2 - A_2(x)\Lambda_1) \sin \theta + \Lambda_3 \cos \theta) d\varphi, \quad (1)$$

where A_x, A_1 and A_2 are real arbitrary functions on R^+ , the Λ_I are generators of $su(2)$, for instance $\Lambda_I = -i\sigma_I/2 \equiv \tau_I$ where σ_I are the Pauli matrices or rigid rotations thereof. The invariant triad takes the form,

$$E = E^x(x)\Lambda_3 \sin \theta \frac{\partial}{\partial x} + (E^1(x)\Lambda_1 + E^2(x)\Lambda_2) \sin \theta \frac{\partial}{\partial \theta} + (E^1(x)\Lambda_2 - E^2(x)\Lambda_1) \frac{\partial}{\partial \varphi}, \quad (2)$$

where again, E^x, E^1 and E^2 are functions on R^+ . One has the following canonical Poisson brackets for the symmetry reduced variables,

$$\{A_x(x), E^x(x')\} = 2\gamma G \delta(x - x'), \quad (3)$$

$$\{A_1(x), E^1(x')\} = \gamma G \delta(x - x'), \quad (4)$$

$$\{A_2(x), E^2(x')\} = \gamma G \delta(x - x'), \quad (5)$$

and all the other brackets vanish; G is Newton's constant and γ is the Immirzi parameter.

Under coordinate transformations of the x coordinate $x'(x)$, A_x transforms like a scalar density $A'_x(x') = (\partial x/\partial x')A_x(x)$ whereas E^x is a scalar. A^1 and A^2 are scalars and E^1 and E^2 are scalar densities.

There are three constraints. The first one is a Gauss law,

$$G(\lambda) = \int dx \lambda ((E^x)' + 2A_1 E^2 - 2A_2 E^1), \quad (6)$$

that generates $U(1)$ gauge transformations on the line. The second one is the remnant of the diffeomorphism constraint,

$$D[N_x] = \int dx N_x (2(A_1)'E^1 + 2(A_2)'E^2 - A_x(E^x)'), \quad (7)$$

and finally the Hamiltonian constraint,

$$H[N] = (2G)^{-1} \int dx N(x) (|E^x| [(E^1)^2 + (E^2)^2])^{-1/2} \times (2E^x (E^1 A'_2 - E^2 A'_1) + 2A_x E^x (A_1 E^1 + A_2 E^2) + (A_1^2 + A_2^2 - 1) ((E^1)^2 + (E^2)^2) - (1 + \gamma^2) (2K_x E^x (K_1 E^1 + K_2 E^2) + ((K^1)^2 + (K^2)^2) ((E^1)^2 + (E^2)^2)), \quad (8)$$

where K_x, K_1 and K_2 are the independent components of the spherically symmetric curvature,

$$K = K_x \Lambda_3 dx + (K_1 \Lambda_1 + K_2 \Lambda_2) d\theta + (K_1 \Lambda_2 - K_2 \Lambda_1) \sin \theta d\varphi, \quad (9)$$

which can be written in terms of the canonical variables A 's and E 's.

It simplifies things if one introduces ‘‘polar’’ type coordinates in the directions 1, 2. To do this we define,

$$A_1 = A_\varphi \cos \beta, \quad (10)$$

$$A_2 = A_\varphi \sin \beta. \quad (11)$$

We introduce a canonical transformation from the variables A_1, A_2, E^1, E^2 to the variables $A_\varphi, \beta, P^\varphi, P^\beta$ through the type III generating function $F = E^1 A_\varphi \cos \beta + E^2 A_\varphi \sin \beta$. This leads to the above relation between $A_{1,2}$ and A_φ and β and defines the conjugate momenta,

$$P^\varphi = 2E^1 \cos \beta + 2E^2 \sin \beta, \quad (12)$$

$$P^\beta = -2E^1 A_\varphi \sin \beta + 2E^2 A_\varphi \cos \beta, \quad (13)$$

and introducing a new angular variable α via $E^1 = E^\varphi \cos(\alpha + \beta)$ and $E^2 = E^\varphi \sin(\alpha + \beta)$, we have that $P^\varphi = 2E^\varphi \cos \alpha$ and $P^\beta = 2E^\varphi A_\varphi \sin \alpha$, and we have rescaled P^φ and P^β by a factor of two so one has the canonical Poisson bracket relations,

$$\{A_\varphi(x), P^\varphi(x')\} = 2\gamma G\delta(x - x'), \quad (14)$$

$$\{\beta(x), P^\beta(x')\} = 2\gamma G\delta(x - x'), \quad (15)$$

with the symplectic structure in A_x, E^x unchanged from before.

In terms of these variables the Gauss law and diffeomorphism constraint simplify,

$$G = P^\beta + (E^x)', \quad (16)$$

$$D = P^\beta \beta' + P^\varphi A_\varphi' - (E^x)' A_x. \quad (17)$$

To write the Hamiltonian constraint it turns out that a further change is desirable towards variables more closely connected with the geometry. Let us start by identifying the metric constructed from the densitized triads we have. To do this we first write the determinant of the metric $\det g = |E^x|(E^\varphi)^2 \sin^2 \theta$. The metric components are,

$$g_{xx} = \frac{(E^\varphi)^2}{|E^x|}, \quad g_{\theta\theta} = |E^x|, \quad g_{\varphi\varphi} = |E^x| \sin^2 \theta. \quad (18)$$

We consider generators of $su(2)$ rotated with respect to the Cartesian basis. Given $\tan(\alpha + \beta) = -E^2/E^1$, we define,

$$\Lambda_E^\varphi = \Lambda_1 \cos(\alpha + \beta) + \Lambda_2 \sin(\alpha + \beta), \quad (19)$$

$$\Lambda_E^\theta = -\Lambda_1 \sin(\alpha + \beta) + \Lambda_2 \cos(\alpha + \beta), \quad (20)$$

so the (undensitized) co-triad can be written as,

$$e = e_x^3 \Lambda_3 dx + e_\theta^\theta \Lambda_E^\theta d\theta + e_\varphi^\varphi \Lambda_E^\varphi d\varphi, \quad (21)$$

where,

$$e_x^3 = \frac{E^\varphi}{\text{sgn}(E^x) \sqrt{|E^x|}}, \quad (22)$$

$$e_\theta^\theta = \sqrt{|E^x|}, \quad (23)$$

$$e_\varphi^\varphi = \sqrt{|E^x|} \sin \theta. \quad (24)$$

From the co-triad we can compute the spin connection,

$$\Gamma_x^3 = -(\alpha + \beta)', \quad \Gamma_\theta^\varphi = \frac{(e_\varphi^\varphi)'}{e_x^3}, \quad \Gamma_\varphi^\theta = -\frac{(e_\varphi^\varphi)'}{e_x^3} \sin \theta, \quad \Gamma_\varphi^3 = \cos \theta, \quad (25)$$

and then from $\gamma K_a^i = A_a^i - \Gamma_a^i$ we can compute the curvature components

$$\gamma K_x = A_x + (\alpha + \beta)', \quad (26)$$

$$\gamma K_1 = A_1 - \Gamma_\varphi \sin(\alpha + \beta), \quad (27)$$

$$\gamma K_2 = A_2 + \Gamma_\varphi \cos(\alpha + \beta), \quad (28)$$

where we define Γ_φ as,

$$\Gamma_\varphi \equiv -\frac{(e_\varphi^\varphi)'}{e_x^3} = -\frac{(E^x)'}{2E^\varphi}, \quad (29)$$

and also $\Gamma_x \equiv \Gamma_x^3$.

As was noted by Bojowald and Swiderski [9], in order to introduce a representation in terms of holonomies one requires certain falloff conditions, in particular $A_\varphi \rightarrow 0$ as $x \rightarrow \infty$. This unfortunately is not true. To see it, note that Γ_φ is a ratio of two densities and therefore a scalar. But asymptotically, $E^x \sim x^2$ and therefore $(E^x)' \sim 2x$ and $E^\varphi \sim x$ where M is the mass of the classical solution considered and therefore $\Gamma_\varphi \sim 1$, which implies that A_φ will not tend to zero asymptotically since $K_\varphi \rightarrow 0$ asymptotically and we have the relation $(A_\varphi)^2 = (\Gamma_\varphi)^2 + (\gamma K_\varphi)^2$ ($K_\varphi = \sqrt{K_1^2 + K_2^2}$). For further discussion of asymptotic properties see Kuchař [11].

To construct a connection with a good asymptotic behavior we consider the following canonical transformation,

$$A_\varphi \rightarrow \bar{A}_\varphi = 2 \cos \alpha A_\varphi, \quad (30)$$

$$\beta \rightarrow \eta = \alpha + \beta, \quad (31)$$

with the following type II generating function

$$F = P^\eta(\alpha + \beta) + 2E^\varphi A_\varphi \cos \alpha, \quad (32)$$

which recovers the above transformation and introduces the canonical momenta P^η and E^φ ,

$$P^\beta = P^\eta, \quad P^\varphi = 2E^\varphi \cos \alpha, \quad (33)$$

which leads us to consider the following canonical variables $\bar{A}_\varphi, E^\varphi, \eta, P^\eta$, in terms of which the Gauss law and diffeomorphism constraint take a simple form,

$$G = P^\eta + (E^x)' \quad (34)$$

$$D = P^\eta \eta' + E^\varphi \bar{A}'_\varphi - (E^x)' A_x. \quad (35)$$

Before continuing it is worthwhile noting that E^x, E^φ are “metric” variables, in the sense that they are invariant under the transformations generated by Gauss’ law. \bar{A}_φ and $A_x + \eta'$ are also invariant and the latter is proportional to K_x . In order to clarify the meaning of \bar{A}_φ notice first that,

$$A_\varphi \Lambda_A^\varphi \equiv A_\varphi \cos \beta \Lambda_2 - A_\varphi \sin \beta \Lambda_1, \quad (36)$$

$$= A_1 \Lambda_2 - A_2 \Lambda_1 = \Gamma_\varphi \Lambda_E^\theta + \gamma K_\varphi \Lambda_E^\varphi, \quad (37)$$

and then note that Λ_E^θ and Λ_E^φ are orthogonal, i.e., $\text{Tr}(\Lambda_E^\theta \Lambda_E^\varphi) = 0$ and also that $\text{Tr}(\Lambda_E^\varphi \Lambda_A^\varphi) = \cos \alpha$. Therefore $A_\varphi \cos \alpha = \gamma K_\varphi$ and it follows that $\bar{A}_\varphi = 2\gamma K_\varphi$. This automatically implies that the connection has the right falloff condition since $K_\varphi \rightarrow 0$. We also have that,

$$\gamma K_x = A_x - \Gamma_x = A_x + (\alpha + \beta)' = A_x + \eta', \quad (38)$$

which confirms what had been noted, that $A_x + \eta'$ is a gauge invariant combination.

It is convenient to relate these variables of the Ashtekar formalism with the canonical variables used for the spherical case by Kuchař. The latter are given by,

$$ds^2 = \Lambda(x)^2 dx^2 + R(x)^2 d\Omega^2, \quad (39)$$

with $\Lambda > 0$ and $R > 0$ and the latter is the curvature of the 2-spheres. We then have,

$$|E^x| = R^2 \quad \frac{(E^\varphi)^2}{|E^x|} = \Lambda^2 \quad (40)$$

$$K_{xx} = -\Lambda N^{-1} \left(\dot{\Lambda} - (N^x \Lambda)' \right) = -\Lambda K_x + 2 \frac{\Lambda^2}{R} K_\varphi \quad (41)$$

$$K_{\theta\theta} = -N^{-1} R (\dot{R} - R' N^x) = -R K_\varphi \quad (42)$$

One therefore has a canonical transformation between the pairs $(\bar{A}_\varphi, E^\varphi, A_x + \eta', E^x)$ and $(P_\Lambda, \Lambda, P_R, R)$ where,

$$\Lambda = \frac{E^\varphi}{\sqrt{|E^x|}}, \quad (43)$$

$$P_\Lambda = \sqrt{|E^x|} \frac{\bar{A}_\varphi}{2\gamma}, \quad (44)$$

$$R = \sqrt{|E^x|}, \quad (45)$$

$$P_R = \sqrt{|E^x|} \frac{A_x + \eta'}{\gamma} + \frac{E^\varphi}{\sqrt{|E^x|}} \frac{\bar{A}_\varphi}{2\gamma}. \quad (46)$$

The Hamiltonian constraint can be obtained from equation (8), and takes the form

$$H = -\frac{E^\varphi}{2\sqrt{|E^x|}} - \frac{A_x \bar{A}_\varphi \sqrt{|E^x|}}{2\gamma^2} - \frac{\bar{A}_\varphi^2 E^\varphi}{8\sqrt{|E^x|} \gamma^2} + \frac{((E^x)')^2}{8\sqrt{|E^x|} E^\varphi} \\ - \frac{\sqrt{|E^x|} (E^x)' (E^\varphi)'}{2(E^\varphi)^2} - \frac{\bar{A}_\varphi \sqrt{|E^x|} \eta'}{2\gamma^2} + \frac{\sqrt{|E^x|} (E^x)''}{2E^\varphi}. \quad (47)$$

We will now fix the spatial diffeomorphism gauge freedom. This will simplify calculations considerably. We would like to have two things: a) that the Gauss law remains as a constraint in order to have usual loop representation techniques for quantization and b) that the expression for the spatial volume retains a simple form since it plays an important role geometrically. We choose a gauge that is suitable for the region exterior to a horizon, $E^x = (x+a)^2$ where a is a positive constant given by $E^x|_{\text{Horizon}} = a^2$, which is equivalent to $R = x+a$ and $x=0$ corresponds to the horizon. This gauge choice commutes with Gauss' law, which therefore remains a first class constraint. Since we do not know the value of E^x at the horizon, a is really a dynamical variable, its conjugate momentum is related to $(A_x + \eta')|_{x=0}$. In that gauge the diffeomorphism constraint takes the form,

$$E^\varphi \bar{A}'_\varphi - (E^x)'(A_x + \eta') = E^\varphi \bar{A}'_\varphi - 2(x+a)(A_x + \eta') = 0, \quad (48)$$

and it can be explicitly solved for $A_x + \eta' = E^\varphi \bar{A}'_\varphi / (2(x+a))$. This also determines strongly the value of the canonical momentum of the variable a .

After solving for that variable, the Hamiltonian constraint takes the form,

$$H = -\frac{E^\varphi}{(x+a)\gamma^2} \left(\frac{\bar{A}_\varphi^2(x+a)}{8} \right)' - \frac{E^\varphi}{2(x+a)} + \frac{3(x+a)}{2E^\varphi} + (x+a)^2 \left(\frac{1}{E^\varphi} \right)' = 0. \quad (49)$$

The constraint only depends on the canonical pair $\bar{A}_\varphi(x), E^\varphi(x)$ and a . The constraints remaining after eliminating the diffeomorphism constraint are first class, but have an algebra with structure functions,

$$\{H(x), H(y)\} = \left(\frac{\bar{A}_\varphi(y)}{2\gamma} H(y) \right)' \delta(x-y) - \frac{\bar{A}_\varphi(y)}{\gamma} H(y) \delta_{,x}(x-y), \quad (50)$$

$$\{G(x), H(y)\} = 0, \quad (51)$$

$$\{G(x), G(y)\} = 0. \quad (52)$$

This reduced system after eliminating the diffeomorphism constraint is the one we will use as starting point for the quantization.

Our gauge choice is suitable for describing the exterior of a black hole. The initial data for the system is given on a spatial surface extending from the Schwarzschild radius to infinity and it determines entirely the spacetime bounded by the horizon and null infinity, since no entering data is given on these two surfaces. The boundary conditions on the horizon correspond to having the $g_{xx} = (E^\varphi)^2 / (x+a)^2$ component be singular, $1/E^\varphi|_{x=0} = 0$ and $\bar{A}_\varphi|_{x=0} = 0$ (as discussed by Bojowald and Swiderski [17]) which corresponds to the isolated horizon boundary condition [9]. For the falloff conditions at infinity we have [11],

$$E^\varphi = x + M(t) + O(x^{-\epsilon}), \quad (53)$$

$$\bar{A}_\varphi = O(x^{-(1+\epsilon)}), \quad (54)$$

where $M(t)$ is a function of t and the above expressions hold in the case that the value of $x \gg M(t)$. One also has that the shift vanishes asymptotically $N^x = O(x^{-\epsilon})$ and the lapse $N = N + O(x^{-\epsilon})$.

In order for the variations of the dynamical variables to preserve the falloff conditions one has to add boundary terms to the action. Given,

$$S(\bar{A}_\varphi, E^\varphi, N, a) = \lim_{x_+ \rightarrow \infty} \int dt \int_0^{x_+} dx \left(E^\varphi \dot{\bar{A}}_\varphi - NH(x) \right). \quad (55)$$

The problematic term is the one stemming from the variation of E^φ in the term $(x+a)^2(1/E^\varphi)'$ in the Hamiltonian. Noting that $\delta E^\varphi_+ = \delta M_+$ and that in problematic term one has,

$$\int dx N(x+a)^2 ((E^\varphi)^{-2} (\delta E^\varphi)'). \quad (56)$$

This term requires that one integrate by parts. This produces an extra term of the form $N(x_+) \delta M_+$ and it would require that the lapse vanish at infinity. This led Kuchař to propose adding a term at infinity of the form $-\int dt M_+ \dot{\tau}_+$ with τ_+ a new dynamical variable such that its variation yields $\dot{M}_+ = 0$ and variation with respect to E^φ_+ is now well defined and yields $\dot{\tau}_+ = N_+$. The resulting action is,

$$S(\bar{A}_\varphi, E^\varphi, N, a, \tau) = \lim_{x_+ \rightarrow \infty} \int dt \int_0^{x_+} dx \left(E^\varphi \dot{\bar{A}}_\varphi - NH(x) \right) - \int dt M_+ \dot{\tau}_+. \quad (57)$$

Through the partial gauge fixing we are now left with a model that is still a midi-superspace but that has only one constraint, the Hamiltonian, with a non-trivial first class algebra.

III. QUANTIZATION

Our objective is to use the spherically symmetric model to explore the properties of the loop quantization approach in a case with field theoretic variables. Although the model in the end has a finite number of degrees of freedom, this is achieved in a non-trivial way through the imposition of the Hamiltonian constraint, all the time treating the model as a field theory.

We will for simplicity concentrate on the exterior of the black hole spacetime. We will therefore not concern ourselves in this first approach on issues like how the use of loop variables may eliminate the singularity, etc. The issues we are interested in probing are: the role of the Hamiltonian constraint in generating the evolution, seeing if the theory is coordinate invariant after quantization and discretization and how the loop treatment compares with the continuum quantization and seeing if one can avoid dealing with anomalies in the quantum constraint algebra.

A. Traditional canonical quantization

In order to compare with the other quantization approaches, we briefly review the traditional canonical quantization of this model, first discussed by Kuchař [11] (see this paper for earlier reference on this subject). In the region $x > 0$ exterior to the horizon, one can fix a gauge globally. For instance $\bar{A}_\varphi = 2\gamma K_\varphi = 0$. The constraints $\bar{A}_\varphi = 0$ and $H(x) = 0$ are second class, since their Poisson bracket is non-vanishing (not even weakly). To quantize one has to impose these constraints strongly. In the action the only remaining contribution after this gauge fixing is an asymptotic one,

$$S = \int dt \dot{\tau} M_+. \quad (58)$$

From here one gets canonically conjugate variables $\tau, P^\tau = -M_+$. τ is the proper time that determines the position of the spatial hypersurfaces of vanishing extrinsic curvature (usual Schwarzschild slicings). For these surfaces one has $\bar{A}_\varphi = 0$ and the quantity E^φ is the solution of,

$$H = 0 = -\frac{(x+a)^2 E'_\varphi}{(E^\varphi)^2} + \frac{3(x+a)}{2E^\varphi} - \frac{E^\varphi}{2(x+a)}, \quad (59)$$

which yields as solution

$$E^\varphi = \frac{(x+a)}{\sqrt{1 - \frac{a}{(x+a)}}} = \frac{R}{\sqrt{1 - \frac{a}{R}}} \quad (60)$$

and one can identify by expanding asymptotically that $M = a/2$ and recalling that $g_{xx} = (E^\varphi)^2/E^x$ one obtains the usual form of the Schwarzschild metric with M the mass of the space-time.

The quantization is straightforward, since the only remaining canonical variables are M and τ . These variables have no dynamics. One has $\hat{A}_\varphi = 0$, $\hat{E}^\varphi = (x + 2\hat{M})/\sqrt{1 - 2\hat{M}/(x + 2\hat{M})}$ and one can for instance introduce an eigenbasis of the mass operator, $\hat{M}\phi(m) = m\phi(m)$.

The operators associated to the other components of the metric can be determined easily. The lapse is determined by the equation fixing the preservation in time of the gauge condition, $\dot{\bar{A}}_\varphi = \{\bar{A}_\varphi, \int N(x)H(x)dx\} = 0$ with solution $N = \sqrt{1 - a/(x+a)}$. The shift is determined by noting that if $\bar{A}_\varphi = 0$ then $K_\varphi = 0$ and therefore equation (42) implies, given that $\hat{R} = 0$, that $N^x = 0$. One then has $g_{00} = N^2$ and this leads to the usual Schwarzschild solution: $\hat{g}_{00} = (1 - 2\hat{M}/R)$.

Different choices of gauge lead to quantum theories in different coordinate systems. In the gauge chosen above $\bar{A}_\varphi = 0$. For an more general gauge with arbitrary \bar{A}_φ one has that

$$E^\varphi = (x + 2M) \left[1 - \frac{2M}{x + 2M} + \frac{\bar{A}_\varphi^2}{4\gamma^2} \right]^{-\frac{1}{2}}. \quad (61)$$

For instance in the gauge $\bar{A}_\varphi = \gamma\kappa x/(x + 2M)^2$, with κ a constant, and which satisfies the boundary condition at the horizon and at infinity, one has that,

$$E^\varphi = \frac{x + 2M}{\sqrt{1 - \frac{2M}{x+2M} + \frac{\kappa x}{4(x+2M)}}} \quad (62)$$

and the lapse and shift are both non-vanishing,

$$N = \sqrt{1 - \frac{2M}{x+2M} + \frac{k^2}{4(x+2M)^2} \left(1 - \frac{2M}{x+2M}\right)^2}, \quad (63)$$

$$N_x = \frac{NA_\varphi}{2\gamma}. \quad (64)$$

In the end all the quantizations are equivalent since in essence we are dealing with a mechanical system and the choices of coordinates are just different choices of quantities that one computes for the mechanical system, and one has only two observables, the mass and the proper time at infinity.

B. Loop quantization

In the loop representation the fundamental operators are associated with holonomies. This requires re-casting the theory of interest in terms of such variables. This usually involves taking limits of holonomies around loops of vanishing area, for instance, to represent the curvature. Moreover, the operators we wish to consider need to be regularized. This is usually achieved by discretizing the theory on a lattice. Therefore the usual loop treatment of theories is a natural framework in which to apply the “uniform discretization” technique.

We will proceed as follows:

- 1) We will identify the space of states of the loop representation for the spherically symmetric case and the operators that are well defined in this space.
- 2) We will particularize the space of states to the gauge we are considering in this paper, where one is left with only the Hamiltonian constraint.
- 3) We will note that one can Abelianize the constraint and, upon discretization, the resulting discrete theory can be treated using the traditional Dirac quantization procedure (in the case of Abelian constraints it is known to be equivalent to the uniform discretization procedure).

C. The space of states in the loop representation for spherical symmetry

The quantization in the loop representation is based on cylindrical functions that depend on the connection through “open holonomies”. The latter are associated with graphs on the spatial manifold (in this case the one dimensional line) composed of a set of edges without intersections $g = \bigcup_i e_i$ where g is the graph and e_i are the edges. The edges’ vertices form a set $V(g)$ composed by all the endpoints of e_i . The \bar{A}_φ correspond to directions transverse to the radial one and are represented in the loop representation via “point holonomies” $\exp(i\mu_v \bar{A}_\varphi(x)) = \exp(2i\gamma\mu_v K_\varphi)$ at each vertex v with $\mu \in \mathbb{R}$ as is usually done for scalar fields [18]. The variable η is an angle and the corresponding point holonomy is given by $\exp(in_v \eta(v))$ with $n_v \in \mathbb{Z}$. Before imposing gauge invariance under the transformations generated by Gauss’ law the spin network states in a basis of spherically symmetric connections are,

$$T_{g, \vec{k}, \vec{n}, \vec{\mu}}(A) = \prod_{e \in g} \exp\left(\frac{i}{2} k_e \int_e A_x dx\right) \prod_{v \in V(g)} \exp(i\mu_v \gamma K_\varphi(v)) \exp(in_v \eta(v)) \quad (65)$$

where $k_e \in \mathbb{Z}$ is the multiplicity of the loop. If we recall that the Gauss law is $P^\eta + (E^x)' = 0$ and that \bar{A}_φ and $A_x + \eta'$ are gauge invariant, one has that the gauge invariant spin networks for the spherically symmetric case are,

$$T_{g, \vec{k}, \vec{\mu}}(A) = \prod_{e \in g} \exp\left(\frac{i}{2} k_e \int_e (A_x + \eta') dx\right) \prod_{v \in V(g)} \exp(2i\mu_v \gamma K_\varphi(v)). \quad (66)$$

In the full three dimensional case the Hamiltonian constraint is written in terms of holonomies along small closed paths. One can do something similar here for the holonomies along the radial direction,

$$\exp\left(i \int_{v_o}^{v_f} A_x dx\right) = 1 + i\epsilon A_x(v) + O(\epsilon^2), \quad (67)$$

where $\epsilon = v_f - v_o$ is the distance between two vertices connected by an edge. For the directions where we have homogeneity we cannot use this argument since we do not have edges that we can make “small”. In that case we can seek an alternative: to expand the “point holonomies” in the limit in which ρ is small,

$$\exp(i\gamma\rho K_\varphi(v)) = 1 + i\gamma\rho K_\varphi + O(\rho^2) \dots \quad (68)$$

It should be noted that quantum mechanically the limits $\epsilon \rightarrow 0$ and $\rho \rightarrow 0$ end up not being well defined and the resulting quantum theories therefore approximate the classical theory well in regions where the relevant variables in the exponent are small in order to make the above expansions accurate.

At this point it is worthwhile mentioning that in the full 3D case, in the loop representation, one represents the Hamiltonian constraint on the space of diffeomorphism invariant states. Here we will not proceed in such a way. To begin with, since we already fixed gauge for the diffeomorphism constraint, there is no sense in representing the Hamiltonian in a space of diffeomorphism invariant states. Moreover, if one were to try to mimic the construction that is done in the 3D case (without fixing the gauge for diffeomorphisms), in the spherical case one would encounter an additional difficulty: since there is no notion of “planar vertices” the constraints will not close the appropriate algebra even if represented on diffeomorphism invariant states, at least with the usual treatment, as discussed in [9]. Therefore even if one started with diffeomorphism invariant states the theory would not be invariant under space-time diffeomorphisms.

D. Loop representation of the partially gauge fixed theory

We will work with a space where the spatial coordinates have been gauge fixed in the way we described in the classical section, and therefore the diffeomorphism constraint is not present. The relevant canonical variables are $\bar{A}_\varphi, E^\varphi$ (we will usually substitute \bar{A}_φ for K_φ since they are simply related) and the asymptotic variables $M = a/2, \tau$.

We will work in a one dimensional lattice L with points $0, \dots, x_N$ in \mathbb{R}^+ with spacing $x_{n+1} - x_n = \epsilon(n)$, and we include the possibility of unequal spacing since we will see it may be useful in certain cases. Strictly speaking, for the example we are considering, there is no need to discretize and one could work directly in the continuum. We will work in a discretized lattice to be able to make better contact later with the “uniform discretization” approach and also for comparison with other situations. We will see that in the end we can correctly remove the lattice regulator and produce a continuum theory.

The Hilbert space in the bulk is given by,

$$\mathcal{H} = L^2(\otimes_N R_{\text{Bohr}}, \otimes_N d\mu_0) \quad (69)$$

where R_{Bohr} is the Bohr compactification of the real line and μ_0 is the measure of integration we discuss below. In this space one can introduce a basis,

$$T_{g, \vec{\mu}}[K_\varphi] = \prod_{v \in V(g)} \exp(i\mu_v \gamma K_\varphi(v)) \quad (70)$$

with $\mu_v \in \mathbb{R}$, g the graph, $V(g)$ vertices of g in L and the notation $\vec{\mu}$ denotes $(\mu_{v_1}, \dots, \mu_{v_p})$ with p the number of vertices in the graph. In the gauge fixed case we are considering a graph is just a collection of vertices. We will now introduce a notation more adapted to the uniform discretization setting by labeling the points in L with an index m, n, \dots . We would then have $K_{\varphi, n} = K_\varphi(x_n)$. We have also rescaled E_n^φ in such a way that $\{K_{\varphi, m}, E_n^\varphi\} = G\delta_{m, n}$ with G Newton’s constant and we choose units where $\hbar = c = 1$ and therefore $G = \ell_{\text{Planck}}^2$. One then has the quantum representation, acting on states $\Psi[K_{\varphi, m}] = \langle K_{\varphi, m} | \Psi \rangle$,

$$\hat{E}_m^\varphi = -i\ell_{\text{Planck}}^2 \frac{\partial}{\partial K_{\varphi, m}}, \quad (71)$$

and therefore,

$$\hat{E}_m^\varphi T_{g, \vec{\mu}} = \sum_{v \in V(g)} \mu_v \gamma \ell_{\text{Planck}}^2 \delta_{m, n(v)} T_{g, \vec{\mu}}, \quad (72)$$

where the δ is a Kronecker delta and $n(v)$ is the position on the lattice L of the vertex v of the spin network.

Given an interval I in L the volume of the corresponding “shell” is given classically by,

$$V(I) = 4\pi \sum_{m \in I} |E_m^\varphi|(x_m + a), \quad (73)$$

and as a quantum operator,

$$\hat{V}(I) T_{g, \vec{\mu}} = \sum_{v \in I} 4\pi |\mu_v|(x_v + \hat{a}) \gamma \ell_{\text{Planck}}^2 T_{g, \vec{\mu}}. \quad (74)$$

We can introduce a basis of loop states $|g, \vec{\mu}\rangle$,

$$\langle K_{\varphi,m} |g, \vec{\mu}\rangle = T_{g,\vec{\mu}}[K], \quad (75)$$

and the Bohr measure guarantees that,

$$\langle g, \vec{\mu} |g', \vec{\mu}'\rangle = \delta_{g,g'} \delta_{\vec{\mu},\vec{\mu}'}. \quad (76)$$

We define the ‘‘holonomy’’ associated with the ‘‘transverse’’ connection \bar{A}_φ at a vertex v . We will use this to construct the Hamiltonian. The definition is,

$$h_\varphi(v, \rho) \equiv \exp\left(\frac{i}{2}\rho\bar{A}_\varphi(v)\right) = \exp(i\rho\gamma K_\varphi(v)), \quad (77)$$

one has that,

$$\hat{h}_\varphi(v_i, \rho)|g, \vec{\mu}\rangle = |g, \mu_{v_1}, \dots, \mu_{v_i} + \rho, \dots\rangle, \quad (78)$$

and in the case of v_i not belonging to the initial spin network, the action adds a vertex with $\mu = \rho$. The co-triad is

$$\hat{E}_m^\varphi |g, \vec{\mu}\rangle = \sum_{v \in V(g)} \mu_v \gamma \ell_{\text{Planck}}^2 \delta_{m,n(v)} |g, \vec{\mu}\rangle. \quad (79)$$

We will now define the inverse operators that arise in the Hamiltonian. It should be noted that similarly with what happens in loop quantum cosmology, the inverse operators are bounded. We will start by computing classically $\text{sgn}(E^\varphi)/\sqrt{|E^\varphi|}$. To do it we note that,

$$D_m \equiv \cos\left(\frac{\rho\gamma K_{\varphi,m}}{2}\right) \left\{ \sqrt{|E_m^\varphi|}, \sin\left(\frac{\rho\gamma K_{\varphi,m}}{2}\right) \right\} - \sin\left(\frac{\rho\gamma K_{\varphi,m}}{2}\right) \left\{ \sqrt{|E_m^\varphi|}, \cos\left(\frac{\rho\gamma K_{\varphi,m}}{2}\right) \right\} \quad (80)$$

$$= -\frac{G \text{sgn}(E_m^\varphi) \gamma \rho}{4\sqrt{|E_m^\varphi|}} \cos^2\left(\frac{\gamma\rho K_{\varphi,m}}{2}\right) - \frac{G \text{sgn}(E_m^\varphi) \gamma \rho}{4\sqrt{|E_m^\varphi|}} \sin^2\left(\frac{\gamma\rho K_{\varphi,m}}{2}\right) \quad (81)$$

$$= -\frac{G \text{sgn}(E_m^\varphi) \gamma \rho}{4\sqrt{|E_m^\varphi|}}, \quad (82)$$

which leads to

$$\frac{\text{sgn}(E_m^\varphi)}{\sqrt{|E_m^\varphi|}} = -\frac{4}{\gamma G \rho} D_m. \quad (83)$$

Quantum mechanically one has

$$\frac{\widehat{\text{sgn}(E_m^\varphi)}}{\sqrt{E_m^\varphi}} |g, \vec{\mu}\rangle = \frac{4i}{\gamma \ell_{\text{Planck}}^2 \rho} \left(\cos\left(\frac{\gamma\rho \hat{K}_{\varphi,m}}{2}\right) \left[\sqrt{|\hat{E}_m^\varphi|}, \sin\left(\frac{\gamma\rho \hat{K}_{\varphi,m}}{2}\right) \right] \right. \quad (84)$$

$$\left. - \sin\left(\frac{\gamma\rho \hat{K}_{\varphi,m}}{2}\right) \left[\sqrt{|\hat{E}_m^\varphi|}, \cos\left(\frac{\gamma\rho \hat{K}_{\varphi,m}}{2}\right) \right] \right) |g, \vec{\mu}\rangle, \quad (85)$$

and noting that,

$$\sqrt{|\hat{E}_m^\varphi|} |g, \vec{\mu}\rangle = \sum_{v \in V(g)} \delta_{m,n(v)} \sqrt{\gamma \ell_{\text{Planck}}} |\mu_v|^{\frac{1}{2}} |g, \vec{\mu}\rangle, \quad (86)$$

one finally has,

$$\frac{\widehat{\text{sgn}(E_m^\varphi)}}{\sqrt{E_m^\varphi}} |g, \vec{\mu}\rangle = \frac{2}{\sqrt{\gamma \ell_{\text{Planck}} \rho}} \sum_{v \in V(g)} \delta_{m,n(v)} \left(|\mu_v + \frac{\rho}{2}|^{\frac{1}{2}} - |\mu_v - \frac{\rho}{2}|^{\frac{1}{2}} \right) |g, \vec{\mu}\rangle. \quad (87)$$

The maximum value of this quantity occurs for $\mu_v = \rho/2$ and is given by $2/(\sqrt{\gamma \ell_{\text{Planck}}})$. For $\mu \gg \rho$ one should recover the classical approximation. In that limit $|g, \vec{\mu}\rangle$ is an eigenstate with eigenvalue $1/(\sqrt{\gamma \ell_{\text{Planck}} \sqrt{|\mu_v|}})$ and satisfies that,

$$|\hat{E}^\varphi| \left(\frac{\widehat{\text{sgn}(E^\varphi)}}{\sqrt{|E^\varphi|}} \right)^2 |g, \vec{\mu}\rangle = |g, \vec{\mu}\rangle, \quad \mu \gg \rho \quad (88)$$

which is the usual relation between the classical variables. In what follows we will either work in the loop representation or the connection representation as needed.

E. The Hamiltonian constraint, Abelianization and traditional Dirac quantization

Here we will show that one can Abelianize the Hamiltonian constraint, discretize it and then quantize the discrete theory using the traditional Dirac quantization procedure. It is known that for Abelian constraints the uniform discretization procedure coincides with the Dirac quantization so we will not work it out explicitly. This quantization will provide a baseline against which to compare quantizations in which we do not Abelianize the constraint. The latter are more realistic since in the general theory it is unlikely that one will be able to Abelianize the constraints.

We start from the classical expression (49) for the Hamiltonian constraint,

$$H = -\frac{E^\varphi}{(x+a)\gamma^2} \left(\frac{\bar{A}_\varphi^2(x+a)}{8} \right)' - \frac{E^\varphi}{2(x+a)} + \frac{3(x+a)}{2E^\varphi} + (x+a)^2 \left(\frac{1}{E^\varphi} \right)' = 0. \quad (89)$$

We then Abelianize the Hamiltonian (49), multiplying by $\frac{2(x+a)}{E^\varphi}$ and grouping terms as

$$H = \left(\frac{(x+a)^3}{(E^\varphi)^2} \right)' - 1 - \frac{1}{4\gamma^2} ((x+a)\bar{A}_\varphi^2)'. \quad (90)$$

We wish to write the discretization in terms of classical quantities that are straightforward to represent in the quantum theory. Here one has to make choices, since there are infinitely many ways of discretizing a classical expression. In particular, we will notice that there exists, for this model, a way of discretizing the constraint in such a way that it remains first class (more precisely, Abelian) upon discretization. This is unusual, and we do not expect such a behavior in more general models.

We now proceed to discretize this expression and to “holonomize” it, that is, to cast it in terms of quantities that are easily representable by holonomies,

$$H_m^\rho = \frac{1}{\epsilon} \left[\left(\frac{(x_m+a)^3 \epsilon^2}{(E_m^\varphi)^2} - \frac{(x_{m-1}+a)^3 \epsilon^2}{(E_{m-1}^\varphi)^2} \right) - \epsilon - \frac{1}{4\gamma^2 \rho^2} \left((x_m+a) \sin^2(\rho \bar{A}_{\varphi,m}) - (x_{m-1}+a) \sin^2(\rho \bar{A}_{\varphi,m-1}) \right) \right], \quad (91)$$

expression that recovers (90) in the limit $\epsilon \rightarrow 0$, $\rho \rightarrow 0$. From now on we will assume the spacing is uniform so we are dropping the m dependence of ϵ . This expression is immediately Abelian since it can be written as the difference of two terms, one dependent on the variables at m and the other at $m-1$. Therefore each term has automatically vanishing Poisson brackets with itself and with the other,

$$H_m^\rho = \frac{1}{\epsilon} \left(\phi^\rho(x_m, E_m^\varphi, \bar{A}_{\varphi,m}) - \phi^\rho(x_{m-1}, E_{m-1}^\varphi, \bar{A}_{\varphi,m-1}) \right), \quad (92)$$

with

$$\phi^\rho(x_m, E_m^\varphi, \bar{A}_{\varphi,m}) = \frac{(x_m+a)^3}{(E_m^\varphi)^2} \epsilon^2 - x_m - \frac{1}{4\gamma^2 \rho^2} (x_m+a) \sin^2(\rho \bar{A}_{\varphi,m}). \quad (93)$$

At the horizon, the boundary condition is $\phi^\rho(0,0,0) = 0$. The condition that the constraints vanish $H_m^\rho = 0$ is equivalent to $\phi^\rho(x_m, E_m^\varphi, \bar{A}_{\varphi,m}) = 0$.

When one has an Abelian set of constraints the uniform discretization quantization procedure is equivalent to the Dirac quantization procedure, as was shown in [8]. So for simplicity we will just follow the Dirac procedure. To implement the constraints as quantum operators as one does in the Dirac procedure, it is convenient to solve the constraints for the E_m^φ ,

$$E_m^\varphi = \pm \frac{(x_m+a)\epsilon}{\sqrt{1 - \frac{a}{x_m+a} + \frac{1}{4\gamma^2 \rho^2} \sin^2(2\rho\gamma K_{\varphi,m})}}, \quad (94)$$

and this relation can be immediately implemented as an operatorial relation and find the states that satisfy it. It should be noted that this relation can be implemented for other gauges as well in a straightforward manner. The states are given by,

$$\Psi[K_{\varphi,m}, \tau, a] = C(\tau, a) \exp \left(\pm \frac{i}{\ell_{\text{Planck}}^2} \sum_m f[K_{\varphi,m}] \right), \quad (95)$$

where $C(\tau, a)$ is a function of the variables at the boundary τ and a , which has to solve the constraint at the boundary (we will discuss this later). The functional f has the form,

$$f[K_{\varphi, m}] = \frac{1}{4\gamma^2\rho^2\left(1 - \frac{a}{x_m+a}\right)}(x_m+a)\epsilon \left[F\left(\sin(2\gamma\rho K_{\varphi, m}), \frac{i}{4\gamma^2\rho^2\left(1 - \frac{a}{x_m+a}\right)}\right) + 2F\left(1, \frac{i}{4\gamma^2\rho^2\left(1 - \frac{a}{x_m+a}\right)}\right) \operatorname{sgn}(\sin(2\gamma\rho K_{\varphi, m})) \right] \quad (96)$$

with $F(\phi, m) \equiv \int_0^\phi (1 - m^2 \sin^2 t)^{-1/2} dt$ the Jacobi Elliptic function of the first kind. Notice that the continuum limit of this expression for the state is immediate, i.e. the sum in m becomes an integral.

We now need to impose the constraints on the boundary, in particular $p^\tau = -a/2$ (in the limit $N \rightarrow \infty$). Quantum mechanically $\hat{p}^\tau = -i\ell_{\text{Planck}}^2 \partial/\partial\tau$ and therefore,

$$C(\tau, a) = C_0(a) \exp\left(-\frac{ia\tau}{2\ell_{\text{Planck}}^2}\right) \quad (97)$$

and $C_0(a)$ is an arbitrary function. This is analogous to the quantization that Kuchař found where one had wavefunctions that only depended on the mass. We have therefore completely solved the theory.

Remarkably, the physical state we found is normalizable in a kinematic space of wavefunctions associated with one superselection sector, that is, the space of K'_φ 's defined in the interval $[0, \pi/(\gamma\rho)]$ (this defines an inner product in the bulk, the picture is easily completed in the boundary by considering functions $C_0(a)$ which are square integrable, as we will do later). In such space one can define operators associated to quantities that are not observables in the theory. This is therefore well suited to treat the problem of time in a relational way, as proposed by Page and Wootters [19]. The main objection to the Page–Wootters construction was that one could not construct conditional probabilities based on states of the kinematical Hilbert space because the physical states were not contained in the kinematical Hilbert space and therefore the conditional probabilities were not well defined on physical states (see Kuchař [20] for a detailed discussion). Here, since the physical states exist in the kinematical space one does not face that problem. Although in the particular example we are considering there is not much point in defining a relational time, given its simple and static nature, it should be noted that the above result exhibits some level of robustness. For instance, it is true every time we are considering models where the variables are handled via the Bohr compactification. This is true, for instance, in cosmological models. It is also true if one couples the present model to a scalar field. It may also hold in more complicated models when one gauge fixes the diffeomorphisms. This is worthwhile further investigation.

The above quantization has taken place in the “connection representation”. We would like to see that one obtains equivalent results in the loop representation. It is more convenient to write (94) operatorially as $\hat{O}_m \Psi = \Psi$ where,

$$\hat{O}_m = \left(\sqrt{1 - \frac{a}{x_m+a} + \frac{1}{4\gamma^2\rho^2} \sin^2(2\gamma\hat{K}_m^\varphi)} \frac{\hat{E}_m^\varphi}{(x_m+a)\epsilon} \right)^2 \quad (98)$$

since this expression is straightforward to represent in the loop representation,

$$\left(1 - \frac{a}{x_m+a} + \frac{1}{8\gamma^2\rho^2}\right) \frac{\mu_m^2 \gamma^2 \ell_{\text{Planck}}^4}{(x_m+a)^2 \epsilon^2} \Psi(\mu_m) - \left[\frac{\mu_m^2 \gamma^2 \ell_{\text{Planck}}^4}{(x_m+a)^2 \epsilon^2} + \frac{2\mu_m \gamma^2 \ell_{\text{Planck}}^4}{(x_m+a)^2 \epsilon^2} \right] \frac{\Psi(\mu_m + 4\rho)}{16\gamma^2\rho^2} - \left[\frac{\mu_m^2 \gamma^2 \ell_{\text{Planck}}^4}{(x_m+a)^2 \epsilon^2} - \frac{2\mu_m \gamma^2 \ell_{\text{Planck}}^4}{(x_m+a)^2 \epsilon^2} \right] \frac{\Psi(\mu_m - 4\rho)}{16\gamma^2\rho^2} = \Psi(\mu_m). \quad (99)$$

The above equation is a recursion relation that implies that in the functions $\Psi(\mu_m)$ the possible values of the μ 's are $\mu = \mu(r) = \pm r + 4n\rho$ with $r \in [0, 2\rho]$ and n an integer. Solutions for different values of r are therefore not connected, and there is therefore a superselection. It is suggestive to compare this expression with the one obtained in loop quantum cosmology, where we see that for each point in the radial direction our wavefunctions have a similar recursion relation to those in loop quantum cosmology for the wavefunction of the universe.

One can also see that the solution to this recursion relation can be obtained via the “loop transform” from the solution in the connection representation we found. Each solution in the connection representation corresponds to a given r-sector of the superselection rule via,

$$\Psi_r(\mu_m) = \frac{\rho\gamma}{\pi} \int_0^{\pi/(\rho\gamma)} dK_\varphi \Psi(K_\varphi) \exp(2\rho K_\varphi \gamma \mu_m(r)). \quad (100)$$

The physical space of states is a Hilbert space with a natural inner product given by the L^2 norm in the variable a , and one must demand that the functions $C_0(a)$ be square integrable. The theory has two independent observables, the mass and its canonically conjugate variable. As mentioned above, the wavefunctions are functions of the mass. One can define observables with more geometric content, for instance the metric in a given gauge, as a function of the mass, using for example equations (62-64), and similarly in Schwarzschild coordinates. The continuum limit is taken trivially, since the variables in a given coordinate system are uniquely defined in terms of the functions of the physical space and one approximate a continuum solution as accurately as one needs by reducing the stepsize. If one keeps the theory discrete, the relation is only approximate. It should be noted that the continuum limit is achieved in the limit $\epsilon \rightarrow 0$ and $\rho \rightarrow 0$. If one adopts the point of view commonly used in loop quantum cosmology, that the quantum of distance should have a minimum value, then one would not expect to take the limits ρ and ϵ going to zero, but to keep the parameters at a minimum value. In such a case one could expect to eliminate the singularity. This is plausible since then the triads would likely not go to zero. However, this would require a more careful analysis using coordinates that actually reach the singularity, which is not the case for the coordinates we have chosen so at the moment we cannot conclusively state what happens at the singularity. Also, it should be noted that in the most recent treatments of loop quantum cosmology the quantity that plays the same role as ρ is not taken to have a constant value, but to depend on the dynamical variables [21]. If one followed a similar approach here, the results we derived would change. The results would remain valid in the asymptotic region far away from the horizon, since there ρ would tend to a constant.

Another interesting aspect is that although we regularized the theory using a lattice, the level of ambiguity of the construction is limited, in part in order to have the constraints be Abelian. Although very limited, this example suggests that the use of a lattice regularization is not necessarily fraught with uncontrollable ambiguities.

An aspect we have not emphasized is that in the calculations we have assumed that $a \geq 0$. Since a is the dynamical variable on which the wavefunctions end up depending, it is problematic to work with an operator with a continuum spectrum and positive eigenvalues only. A better way to handle this is to allow a to take all possible real values and then the gauge fixing condition we introduced should read $E^x = (x + |a|)^2$, and all subsequent equations involving $x + a$ have to be modified accordingly (the conjugate momentum should be modified as well). In this treatment the mass is given by $|a|/2$ and is automatically positive. It is interesting to speculate what would happen if one wished to consider negative masses. What is clear at this stage is that the analysis would differ significantly with what we did in this paper, since in the Cauchy surface one would have to take into account the singularity and one cannot limit oneself to study the “exterior” as in the case of positive mass, where there is a well defined causal boundary at the horizon. The discussion of negative masses is better postponed till one can handle the interior problem and discuss the possibility of eliminating the singularity, since in the negative mass case the singularity has to be faced from the outset.

IV. QUANTIZATION USING UNIFORM DISCRETIZATIONS

Although we have succeeded in quantizing the model using the traditional Dirac quantization, we saw that in order to do this one had to use a property —the Abelianization of the Hamiltonian constraint— that is unlikely to hold in more general models. It is therefore worthwhile asking: what would have happened if we didn’t Abelianize the constraint? In such case the traditional Dirac quantization would not have succeeded and we would have to resort to alternative proposals, like the uniform discretizations. Generically the discrete constraints that are obtained by discretizing a field theory with first class constraints are second class and become first class only in the continuum limit. We would like to develop a quantization strategy for such systems. An immediate answer to the problem is to deal with the second class constraints using the Dirac procedure. This is likely to be very onerous in cases of interest, and it does not take advantage of the fact that in the continuum limit the constraints become first class.

Here is where our *uniform discretization* approach can help. In this approach the discrete theory has no constraints and nevertheless is capable of approximating the continuum theory in a controlled fashion. What we would advocate is to construct the discrete theory using the uniform discretization approach and then proceed to quantize the resulting discrete theory. In some cases one will be able to take the continuum limit in the quantum theory and this completes the quantization of the original continuum theory one started with satisfactorily. In some cases, as it is likely to be the case in the most interesting situations, it might occur that the continuum limit cannot be taken in the quantum theory. In such cases the approach we advocate is the following: what matters in a quantization procedure is to recover in the semiclassical limit the classical theory one started from plus corrections. We know the classical discrete theory we constructed approximates the continuum theory well. We therefore expect the quantum discrete theory to also approximate well the quantum continuum theory, even in cases where we cannot construct the latter exactly via this method, at least for certain states (it will obviously fail, for instance, for states that probe lengths smaller than the lattice spacing). We would therefore end with a quantum theory that approximates semiclassically the classical

theory we started with, plus corrections, i.e. the goal we were trying to achieve.

Since these points of view imply a radical departure from the traditional Dirac method, it is worthwhile investigating how they perform in the face of concrete models where one can carry out the computations explicitly. In this section we would like to apply the uniform discretization approach to the treatment of spherically symmetric midi-superspaces. We will see that the approach is feasible but calculations can become quite involved if one does not choose to Abelianize the constraints.

A. A brief summary of Uniform discretizations

To recap briefly on previous discussions of uniform discretizations we would like to summarize its application to a field theory. We assume one is starting with a theory with variables $q^i(x, t)$. One discretizes the underlying space-time manifold. The action can therefore be approximated by,

$$S = \sum_n L(n, n+1) \quad (101)$$

$$L(n, n+1) \equiv \sum_m \Delta \epsilon L([q_{m,n}^i], [q_{m,n+1}^i]), \quad (102)$$

where $q_{m,n}^i$ represents a discretization of $q^i(x, t)$ with m representing the array points in the spatial discretization and n the ones in the timelike direction. The notation $[q_{m,n}^i]$ means the variable $q_{m,n}^i$ and some neighboring q^i 's determined by the scheme used to discretize spatial derivatives. For instance, if one uses a non-centered first order scheme one would have $[q_{m,n}^i] = (q_{m,n}^i, q_{m+1,n}^i)$. ϵ is the coordinate spatial separation between neighboring points in the lattice and Δ the time-like separation of neighboring points. Though we do not make it explicit, L depends on both ϵ and Δ . We choose a first order approximation for the time derivatives since it simplifies constructing a canonical theory (for treatments with more than two time levels see [22]). One works out the canonical momenta for the discrete action,

$$p_{m,n+1}^k \equiv \frac{\partial L(n, n+1)}{\partial q_{m,n+1}^k}. \quad (103)$$

One can also work out the momenta at n , which via the Lagrange equation,

$$\frac{\partial L(n, n+1)}{\partial q_{m,n}^i} + \frac{\partial L(n-1, n)}{\partial q_{m,n}^i} = 0, \quad (104)$$

yields,

$$p_{m,n}^k \equiv -\frac{\partial L(n, n+1)}{\partial q_{m,n}^k}. \quad (105)$$

These equations imply a canonical transformation between q, p at level n and level $n+1$ via a type I generating function given by $F(q_n, q_{n+1}) = -L(n, n+1)$. It should be noted that we have not made a distinction between variables and Lagrange multipliers. Upon discretization, variables that in the continuum were Lagrange multipliers become evolution variables and they get determined by the evolution equations (this is a generic statement for Lagrange multipliers associated with diffeomorphism invariances, for many details concerning the Dirac treatment of discrete systems and particular examples see [23]). Equations that in the continuum were constraint equations now become evolution equations. The number of degrees of freedom is therefore larger than in the continuum theory. There will be different solutions in the discrete theory that correspond to different approximations or parameterizations of the same solution in the continuum theory. The determination of the Lagrange multipliers has proved problematic in previous examples we have studied. The resulting equations are usually polynomials of high order that can have complex solutions or develop discontinuities and branches. This may translate in that the discrete theory produces solutions that are not close to the constraint surface of the continuum theory throughout the whole evolution, and may depart significantly from it at certain points. This was usually considered a major obstacle to the use of these discrete theories. The uniform discretization approach bypasses this problem.

The approach consists in replacing the evolution equations that we obtained for the dynamical variables of the problem by a set of evolution equations that in a sense we will make precise later on preserves the constraints of the continuum theory at a given level of approximation. The reason we can do this is that there is a large amount of freedom in how one discretizes a theory. In a sense we will be exploiting this freedom to our advantage.

To construct the uniform discretizations we note that since the evolution is given by a canonical transformation, generically one can find an infinitesimal generator for it. We will call this generator \mathbb{H} . In terms of this generator, the discrete evolution of the dynamical variables can be written as,

$$q_{m,n+1}^i = e^{\{\bullet, \mathbb{H}\}} (q_{m,n}^i) \equiv q_{m,n}^i + \{q_{m,n}^i, \mathbb{H}\} + \frac{1}{2} \{\{q_{m,n}^i, \mathbb{H}\}, \mathbb{H}\} + \dots \quad (106)$$

We will assume a discretization has been chosen such that the form of \mathbb{H} is given as $H(q, p) = f(\phi_1(q, p), \dots, \phi_P(q, p))$ where ϕ_1, \dots, ϕ_P are discretizations of the constraints of the continuum theory. To simplify the notation we have dropped the fact that the discretizations of the constraints will generically be functions of $[q_{m,n}^i], [p_{m,n}^i]$. The function f has the following properties: a) $f(x_1, \dots, x_P) = 0 \iff x_a = 0, a = 1 \dots P$ and otherwise $f > 0$; b) $\frac{\partial f(0, \dots, 0)}{\partial x_b} = 0$; c) $\det \frac{\partial^2 f}{\partial x_a \partial x_b} \neq 0, \forall x$ and d) $f(\phi_1(q, p), \dots, \phi_P(q, p))$ is defined for all (q, p) in the complete phase space.

Discretizations of this sort can be constructed by suitable choices when one discretizes the action, as we proceeded. Further discussion on this is in reference [7]. However, we will not use in any way that the discretization was arrived to in this form so we refer the reader to the reference and just take the discretization as a given.

A particularly simple example of such a function, that is useful in many systems is to just consider the sum of the squares of the discretized constraints. It is immediate that the evolution equations we wrote preserve exactly \mathbb{H} . Therefore this means that if one chooses a small initial value for \mathbb{H} one is guaranteeing that the sum square of the constraints is kept small and therefore one is not departing significantly from the constraint surface. This is also the case if one takes a more general f as well.

Let us now show that this evolution does indeed have the correct continuum limit. For concreteness, and since this is general enough to accommodate the example we study in this paper, we consider a system that represents a discretization of a 1+1 dimensional field theory of the continuum with one first class constraint. We assume the spatial direction has a finite extension $[x_0, x_1]$. We have a lattice with N points such that $|x_1 - x_0| = N\epsilon$. The time-like direction starts at t_0 and has spacing Δ . The field has R components $q^i(x), i = 1 \dots R$ and canonical momenta $p^i(x)$. One has that,

$$q_{m,n}^i = q^i(x_0 + m\epsilon, t_0 + n\Delta), \quad (107)$$

$$\frac{p_{m,n}^i}{\epsilon} = p^i(x_0 + m\epsilon, t_0 + n\Delta), \quad (108)$$

$$x = \lim_{\epsilon \rightarrow 0} x_0 + m\epsilon, \quad (109)$$

$$t = \lim_{\Delta \rightarrow 0} x_0 + m\Delta, \quad (110)$$

$$q^i(x, t) = \lim_{\epsilon, \Delta \rightarrow 0} q^i(x_0 + m\epsilon, t_0 + n\Delta), \quad (111)$$

$$p^i(x, t) = \lim_{\epsilon, \Delta \rightarrow 0} p^i(x_0 + m\epsilon, t_0 + n\Delta). \quad (112)$$

We also assume a single (field-theoretic) constraint in the continuum that corresponds to N constraints $\phi_a([p_{n,m}], [q_{n,m}], \epsilon)$ in the discrete theory. The generator is given by $\mathbb{H} = k^2 \sum_{a=1}^N \phi_a^2 / 2$. In units where $\hbar = c = 1$ the constraints have dimensions of length⁻¹ and the constant k has dimensions of length so that the generator is dimensionless. We assume that the constraints have been appropriately rescaled such that,

$$\lim_{\epsilon \rightarrow 0} \mathbb{H} = \frac{k^2}{2} \int dx \phi^2 [q(x), p(x)], \quad (113)$$

where we emphasize that the constraint has a functional dependence on p, q .

Since \mathbb{H} is preserved upon evolution, we take $\mathbb{H} = \delta^2 / 2$ with δ a constant. We define $N_a \equiv k\phi_a([q], [p]) / \delta$ which satisfies $\sum_{a=1}^N N_a^2 = 1$. We note that,

$$\frac{q_{m,n+1}^i - q_{m,n}^i}{k\delta} = \frac{1}{k\delta} \{q_{m,n}^i, \mathbb{H}\} + O(\Delta) = \sum_{a=1}^N \{q_{m,n}^i, \phi_a([q], [p])\} N_a + O(\Delta) \quad (114)$$

and identifying $\Delta = k\delta$ and taking the limit $\epsilon \rightarrow 0$ (and therefore $N \rightarrow \infty$), $\delta \rightarrow 0$ one has that,

$$\dot{q}^i(x, t) = \int dz \{q^i(x, t), \phi[q(z, t), p(z, t)]\} N_z \quad (115)$$

where we have taken the limit of vanishing time and spatial spacings and therefore the number of spatial points and of constraints to infinity, as one would expect in a field theory. In the discrete theory the canonical Poisson brackets are $\{q_{m,n}^i, p_{m,n}^j\} = \delta^{i,j} \delta_{m,n}$ with Kronecker deltas and in the continuum $\{q^i(x, t), p^j(z, t)\} = \delta^{ij} \delta(x - z)$. We have assumed that the model has one spatial dimension, as we consider in this paper, it is straightforward to generalize the construction to more spatial dimensions. Although we did not specify the field theory we were considering, it should be noted that in diffeomorphism invariant theories, to construct the generator one needs to integrate a density of weight one. Therefore one should choose appropriately densitized versions of the constraints.

The above equations are valid if the constraints are first class in the continuum theory and the corresponding quantities in the discrete theory become first class constraints in the continuum limit. Generically, constraints that are first class in the continuum become second class quantities when discretized and will not necessarily become first class in the continuum limit of the discrete theory, which might also fail to exist altogether. To be precise, by “second class” in the discrete theory we mean that the vanishing of the quantities does not imply that their Poisson bracket vanishes. Notice that the equations specify the values of the Lagrange multipliers, they are given, up to a factor, by the values of the constraints. This therefore mandates that the constraints must be first class since only then one is free, in the continuum theory, to pick a particular set of Lagrange multipliers, which is what our method is doing. If the constraints are second class, the Lagrange multipliers are determined by the Dirac procedure and therefore the above procedure can be potentially inconsistent. There are two ways to proceed in this case. One of them is to use Dirac brackets in all the above expressions. Then the method still works as specified. Another possibility is to pretend the constraints are first class and then to study the continuum limit of the theory. If the continuum limit exists and in such limit the constraints become first class, then one may be able to use the discrete theory to construct the continuum limit and use it as a vehicle for quantization. Although it might appear that this latter point of view is less warranted than the first, in practice for cases of interest (as in general relativity in $3 + 1$ dimensions) the use of the Dirac brackets will lead to unsolvable equations and one may opt for the second avenue as a route for quantization. In fact, the latter point of view is likely to become the preferred one for cases of interest. The reason for this is that even in the case in which the continuum limit does not exist, one can use the constructed discrete theory as an approximation of the continuum one since the continuum constraints can be kept small. There might be other reasons why one does not want to consider the continuum limit. For instance, as occurs in loop quantum gravity, the kinematical space is not well suited for taking spatially continuum limits since one expects space to be quantized, as indicated by the quantization of the area, volume, etc. We will see how this operates in the example we discuss in this paper.

B. Spherically symmetric vacuum gravity via uniform discretizations

We would like to treat the model using uniform discretizations. Here one faces two possibilities. Since the discrete constraints are not first class, one can only ensure that the uniform discretizations will have the continuum limit if one treats them using Dirac brackets. However, one does not expect that in more complicated models one will be able to take this path. We will therefore choose to treat the model using a technique more suited for more general models. The technique is to implement the uniform discretizations without using the Dirac brackets. Since one knows that in the continuum limit the constraints become first class, it is possible that in the continuum limit one will recover the desired theory. We will actually use some of the ambiguities in the discretization to make this outcome more likely.

We will proceed to construct the quantity \mathbb{H} which corresponds to the square of the Hamiltonian constraint. Here one faces a significant degree of ambiguity in choosing how to discretize the expression. To guide us in this we would like to require some conditions on the discretization that make it more likely that the continuum quantum limit will be achieved. The first condition is that when we take the continuum limit in the classical theory, the classical continuum constraint algebra should be reproduced. That is, the continuum limit of the Poisson bracket of the constraints of the discrete theory should reproduce the continuum classical constraint algebra. The second condition is that the spectrum of the quantity \mathbb{H} contain the zero eigenvalue in the combined classical and continuum limit. If these conditions are met, then one can be satisfied that one constructed a suitable quantum theory in the continuum limit.

In achieving the above limit one expects two different types of problems. On the one hand, it could be that the constraints have been discretized in such a way that even classically they do not reproduce the constraint algebra in the continuum limit and this may imply that even classically \mathbb{H} does not vanish in the continuum limit. In addition to this problem, one may have quantum anomalies. That is, even if one ensured that the algebra was reproduced classically in the continuum limit, upon quantization one may fail to reproduce the algebra in the continuum limit. The two types of problems can be characterized by the appearance of terms of order ϵ^n with ϵ the spatial separation and n some power in the case of classical effects and the appearance of terms $(\ell_{\text{Planck}}/\epsilon)^m$ with m some power in the case of the quantum anomalies. For things to work out in the limit we should have that $n > 0$ ($m \geq 0$ due to how quantum anomalies arise). This suggests that the classical continuum limit should be taken $\ell_{\text{Planck}} \rightarrow 0$ first and then

$\epsilon \rightarrow 0$.

So what we are arguing is that due to the presence of quantum anomalies one could face the situation where there is no well defined quantum continuum theory if $m > 0$, and nevertheless the discrete quantum theory is able to produce an acceptable continuum classical limit. This has important parallels with proposals by Klauder to extend the more traditional approaches to quantization [24], and proposals on how to handle Thiemann's master constraint in the case in which it does not have zero eigenvalue [25].

We already have a candidate for the constraints that satisfies all the above conditions: the Abelianized version. What we would like to do, however, is to consider other versions of the constraint, where the above problems arise, and show that one can construct a suitable quantization. At this point one has to choose a discretization and a quantization of \mathbb{H} that satisfies the two conditions we outlined above. There is no generic algorithmic procedure to construct such a discretization and quantization. In a generic situation one can imagine proposing parameterized discretizations and checking that the conditions are satisfied by choosing parameters. It is also possible to add terms that vanish in the continuum limit to the constraints to achieve the desired result. In the particular model we are studying we will construct a suitable discretization by starting from the Abelian version of the Hamiltonian constraint (90), and we multiply it times powers of the triad so one gets a non-Abelian constraint. In this case we will choose to multiply it times a fourth power of the triad since this simplifies quantizing it in the loop representation. The classical discrete Hamiltonian constraint we take is,

$$H_m^\rho = \frac{1}{\epsilon^5(x_m + a)^2(x_{m-1} + a)^2} \left[(E_{m-1}^\varphi)^2 ((x_m + a)^3 + \epsilon^3) \epsilon^2 - (E_m^\varphi)^2 ((x_{m-1} + a)^3 + \epsilon^3) \epsilon^2 - (E_m^\varphi)^2 (E_{m-1}^\varphi)^2 \left(\epsilon + \frac{(x_m + a) \sin^2(2\rho\gamma K_{\varphi,m}) - (x_{m-1} + a) \sin^2(2\rho\gamma K_{\varphi,(m-1)})}{4\gamma^2 \rho^2} \right) \right]. \quad (116)$$

The quantization we choose, in order for the operator to be Hermitian is,

$$\hat{H}_m^\rho = \frac{1}{\epsilon^5(x_m + a)^2(x_{m-1} + a)^2} \left[(\hat{E}_{m-1}^\varphi)^2 ((x_m + a)^3 + \epsilon^3) \epsilon^2 - (\hat{E}_m^\varphi)^2 ((x_{m-1} + a)^3 + \epsilon^3) \epsilon^2 - (\hat{E}_m^\varphi)^2 (\hat{E}_{m-1}^\varphi)^2 \epsilon - (\hat{E}_m^\varphi)^2 (\hat{E}_{m-1}^\varphi)^2 \left(\frac{(x_m + a) \sin^2(2\rho\gamma \hat{K}_m^\varphi) - (x_{m-1} + a) \sin^2(2\rho\gamma \hat{K}_{m-1}^\varphi)}{8\gamma^2 \rho^2} \right) - \left(\frac{(x_m + a) \sin^2(2\rho\gamma \hat{K}_m^\varphi) - (x_{m-1} + a) \sin^2(2\rho\gamma \hat{K}_{m-1}^\varphi)}{8\gamma^2 \rho^2} \right) (\hat{E}_m^\varphi)^2 (\hat{E}_{m-1}^\varphi)^2 \right] \quad (117)$$

From here one can construct the operator $\hat{\mathbb{H}} = \sum_m (\hat{H}_m^\rho)^2$, and we do not need to include powers of the determinant of the metric, since one has fixed spatial diffeomorphisms and therefore the spatial integral of the Hamiltonian as a scalar is well defined. The task now is to solve the eigenvalue problem and find the minimum eigenvalue of this positive definite quantum mechanical operator. This is a well defined problem in quantum mechanics, but it is far from trivial. Further studies on how to handle these types of problems are clearly needed.

For the particular model at hand we can however determine an upper bound on the value of the minimum eigenvalue. To do this we use the states (95), which are normalized in the kinematical inner product

$$\prod_m \frac{\gamma\rho}{\pi} \int_0^{\pi/(\gamma\rho)} dK_{\varphi,m} \int_0^\infty da \Psi^*[K_{\varphi,m}, \tau, a] \Psi[K_{\varphi,m}, \tau, a] = 1. \quad (118)$$

We use this inner product to compute the expectation value of $\hat{\mathbb{H}}$ for these states, and we get,

$$\langle \Psi | \hat{\mathbb{H}} | \Psi \rangle = C_1 \frac{\epsilon^3}{a^3} + C_2 \frac{\ell_{\text{Planck}}^2}{a^3} + C_3 \frac{\ell_{\text{Planck}}^4}{a\epsilon^3} + \sum_{n=3}^8 C_{n+1} \frac{\ell_{\text{Planck}}^{2n}}{a\epsilon^{n+1}}, \quad (119)$$

where the constants C_i are of order unity. Since the inner product has an integration over a the above expressions in the powers of a really correspond to expectation values, for instance the first term strictly speaking should be $C_1 \epsilon^3 \langle a^{-3} \rangle$. The expectation value above satisfies the conditions we imposed for the continuum classical limit, i.e. it goes to zero as $\epsilon \rightarrow 0$, $\ell_{\text{Planck}} \rightarrow 0$. Moreover, in the continuum quantum limit ($\epsilon \rightarrow 0$ but ℓ_{Planck} finite) it is divergent. Notice that this problem may arise even for the exact eigenvector corresponding to the minimum eigenvalue of \mathbb{H} . That is, it is possible that one may not be able to use the discrete quantum theory to define a continuum quantum

theory as a limit. The minimum value of bound for the expectation value is achieved (assuming a large value of $\langle a \rangle$) for $\epsilon = \sqrt[3]{\ell_{\text{Planck}}^2 \langle a \rangle}$. For that value $\langle \hat{\mathbb{H}} \rangle \lesssim \ell_{\text{Planck}}^2 / \langle a \rangle^2$. Just for reference, for a Solar sized black hole, this amounts to 10^{-80} .

V. CONCLUSIONS

We have analyzed spherically symmetric quantum gravity in the connection and loop representations, outside the horizon. We were able to find a gauge fixing of the spatial diffeomorphisms that yields a theory with an Abelian Gauss law and a Hamiltonian constraint with a first class algebra with structure functions. A redefinition of the constraints yields an Abelian algebra of constraints. The latter can be quantized using the traditional Dirac procedure and yields a quantum theory in the connection and loop representation that coincides with Kuchař's treatment in terms of the traditional variables. We show one can discretize the theory to regularize the Hamiltonian during quantization in a reasonably unambiguous way and take the continuum limit, which is well defined.

An interesting point is that in the gauge fixed model the physical quantum states of a given superselection sector are part of the kinematical Hilbert space. This allows, for instance, to implement in a well defined way a relational solution to the problem of time. This behavior is also expected in other models and needs to be further explored.

Although we succeeded in implemented a traditional Dirac quantization for the model, this was done at a certain price. On one hand we gauge fixed the diffeomorphisms. On the other we Abelianized the Hamiltonian constraint through a rescaling. Gauge fixing the diffeomorphisms may be viewed as a legitimate approach to gain understanding into models, but ultimately is faced with the usual question of if the quantization obtained is equivalent to a quantization performed in another gauge. The Abelianization of the Hamiltonian is a property that as far as we know, only holds in the gauge fixed model and appears unlikely to hold in other models generically (although an Abelianization without gauge fixing was achieved using the complex form of the Ashtekar variables in [13]).

In view of the mentioned limitations of the traditional approach, we have also made a first exploration of the application of the uniform discretization approach to the system when we do not Abelianize the algebra of constraints. We have not done a complete analysis of such a model, since the calculations needed are significant. We note that it is plausible that one may produce a quantum theory that does not have a continuum limit, but has a classical continuum limit. This behavior may also be expected in more complicated models, where constraints generically cannot be Abelianized. Ambiguities in the discretization can be reduced by choosing things in such a way that the quantum theory approximates the continuum as much as possible. In this example, if one applied this criterium, one would recover the Abelian theory previously mentioned.

Future developments will consist in studying the complete space-time using Kruskal-type coordinates and to see what effect one would have on the singularity due to the use of the loop representations. This should also be compared with mini-superspace treatments of the interior based on the isometry of the interior space-time with Kantowski–Sachs [26], although the latter should be more limited since the isometry is limited to the interior region. We see in our treatment some initial hints of singularity removal since the discrete equations resemble those found in loop quantum cosmology, though further details need to be considered.

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