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Late time tails in the Kerr spacetime

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Abstract

Outside a black hole, perturbation fields die off in time as $1/t^n$. For spherical holes $n = 2\ell + 3$ where ℓ is the multipole index. In the nonspherical Kerr spacetime there is no coordinate-independent meaning of "multipole," and a common sense viewpoint is to set ℓ to the lowest radiatiable index, although theoretical studies have led to very different claims. Numerical results, to date, have been controversial. Here we show that expansion for small Kerr spin parameter a leads to very definite numerical results confirming previous theoretical predictions.

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Perturbation fields outside a spherically symmetric black hole die off in time in a way that has been well understood for more than 35 years[1]. This understanding is closely tied to the fact that the spherical background allows the fields to be decomposed into multipoles, each of which evolves from initial data independently and can be studied with a relatively simple 1+1 computer code. The evolved radiation field starts with oscillations characteristic of the details of the initial data, then undergoes an epoch of quasinormal ringing, and lastly falls off in time t in the form of a late-time "tail" $t^{-2\ell-3}$, where ℓ is the multipole index[2].

For the nonspherical Kerr black hole the situation has been anything but clear since Kerr perturbations cannot be separated into independently evolving multipoles[3]. Certain symmetries do apply, however. Perturbations can be separated into nonmixing azimuthal Fourier modes $e^{im\phi}$ and into modes even or odd with respect to reflection through the equatorial plane. This has given rise to what we shall call a "common sense" viewpoint in which approximate spherical symmetry applies to the distant radiation field, and the late-time behavior of the evolution of initial data is dominated by the lowest multipole index (i.e., the most slowly dying tail) compatible with the azimuthal and equatorial symmetries of the initial data. Thus, for example, a scalar perturbation field whose initial data has $m = 0$ and that is symmetric with respect to the equator will, at late time, be predominately a monopole and will have the t^{-3} tail of a monopole.

Supporting this viewpoint is the fact that, without spherical symmetry in the background, there are no preferred angular coordinates. The radial r and polar θ coordinates are mixed differently in different systems of coordinates used to describe the Kerr spacetime, such as Boyer-Lindquist (BL) coordinates[4] or Kerr coordinates[5]. A "multipole" is specific to the coordinate choice, and therefore cannot determine a physical effect, like the rate of decrease of the field.

Theoretical work has argued against the common sense viewpoint, and claims have appeared of numerical results to support both sides of the argument. The argument for something other than "common sense" was first given by Hod[6–10], (see also Barack and Ori[11, 12]). Hod considered initial data that has only a single multipole $Y_{\ell m}$ in BL coordinates. By looking at the zero frequency limit of a Fourier transform, Hod argued that the tails of a massless scalar function would have the following dependence on time and on multipolarity of the initial data:

$$\Psi \propto \begin{cases} Y_{\ell m}/t^{2\ell+3} & \ell = m \ \text{ or } \ell = m+1 \\ Y_{mm}/t^{\ell+m+1} & \ell - m \geq 2 \ \text{ (even)} \\ Y_{m+1\ m}/t^{\ell+m+2} & \ell - m \geq 2 \ \text{ (odd)}\,. \end{cases} \quad (1)$$

More recently Poisson[13] has come to the same result, though with an approximate weak-field analysis.

The common sense results seemed so compelling, that numerical work was immediately sought that would settle the issue, but numerical tests required rather delicate 2+1 codes. Krivan[14] was the first to attempt this work, using a scalar field with an initial outgoing pulse with BL multipole indices $\ell, m = 4, 0$. The Eq. (1) prediction for this case is a t^{-5} monopole tail while the common sense prediction is a t^{-3} monopole. (This $\ell, m = 4, 0$ case is the simplest scalar case for which there are controversial predications, and we will consider it here as the primary test case.) Krivan's results weakly suggested a $t^{-5.5}$ law, but Krivan pointed out serious numerical problems caused by angular differencing. Subsequent computations of the $\ell, m = 4, 0$ case [15, 16] found the common sense t^{-3} result, but those computations did not start with a pure BL multipole on an initial hypersurface of constant BL time, so that neither of those studies represented the same problem as that to which Eq. (1) and Krivan's results apply.

We present here a new approach for computationally probing the late time evolution of tails in the Kerr spacetime and, in principle, in other nonspherical spacetimes. The advantages of this approach are: (i) it gives a clear meaning to "multipoles" since it uses a spherical operator for evolution; (ii) there is no angular differencing, and hence it avoids the errors pointed out by Krivan; (iii) the method gives convergent, clear numerical answers to the controversies of tails in the Kerr spacetime. The new approach expands fields and the equations that govern them in powers of the spin parameter a, of the Kerr metric.

We show here the approach as applied to a scalar field ψ, both for simplicity, and because previous numerical work has all been for a scalar field. We make a further, minor simplification by choosing the scalar field to have no ϕ (azimuthal) dependence. More details and the case for more general fields will be published elsewhere[17].

In Boyer-Lindquist coordinates, the Teukolsky equation for this case (equivalent to $\psi_{,\alpha}^{\ ;\alpha} = 0$) takes the explicit form

$$L[\psi] \equiv \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} - \Delta\frac{\partial^2\psi}{\partial r^2} - 2(r-M)\frac{\partial\psi}{\partial r} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = 0 \tag{2}$$

where Δ has the usual meaning $r^2 - 2Mr + a^2$, with M the mass and $a = J/M$ the angular momentum parameter.

For any initial data, the field ψ that evolves will depend on the spin parameter a and we expand both ψ and the operator of Eq. (2) in powers of a/M:

$$\psi = \psi^{(0)} + a/M\,\psi^{(1)} + (a/M)^2\,\psi^{(2)} + \cdots \qquad L = L^{(0)} + (a/M)^2\,L^{(2)} + \cdots \tag{3}$$

Here the $\psi^{(n)}$ are functions of t, r and θ. The equations that result for the even powers of a/M are

$$L^{(0)}[\psi^{(0)}] = 0 \tag{4}$$
$$L^{(0)}[\psi^{(2)}] = -L^{(2)}[\psi^{(0)}] \tag{5}$$
$$L^{(0)}[\psi^{(4)}] = -L^{(4)}[\psi^{(0)}] - L^{(2)}[\psi^{(2)}] \tag{6}$$
$$L^{(0)}[\psi^{(6)}] = -L^{(6)}[\psi^{(0)}] - L^{(4)}[\psi^{(2)}] - L^{(2)}[\psi^{(4)}] \tag{7}$$

and so forth. The evolved fields can be solved order by order. Since the right hand sides in the above sequence are treated as known driving terms, only the operator $L^{(0)}$ need be inverted, but $L^{(0)}$ is just the spherically symmetric Schwarzchild operator, so the multipoles contained in the solutions will only be those that appear in the driving terms.

We focus now on the primary test case $\ell = 4$, and on the question of the exponent for the late time tail. Since the initial data has $\ell = 4$ there will be an evolved field that is zero order in a/M, i.e., the purely $\ell = 4$ field that is evolved by the Schwarzschild operator $L^{(0)}$ in Eq. (4). The $\ell = 4$ zeroth order field $\psi^{(0)}$ will provide a source term on the right hand side of Eq. (5). But $L^{(2)}$ (the only term in the full set of $L^{(n)}$ that is not spherically symmetric) contains a term $-M^2\sin^2\theta\partial_t^2$, and the $\sin^2\theta$ multiplied by $P_\ell(\cos\theta)$ gives multipoles of order $\ell-2,\ \ell$, and $\ell+2$. Thus $\psi^{(2)}$ in Eq. (5) will be driven by source terms with $\ell =$ 2,4, and 6. This changing of the multipole order by $L^{(2)}$ will happen again in Eqs. (6) and Eq. (7), since they also contain $L^{(2)}$. As a result of this multipole mixing, initial zero-order $\ell = 4$ data results in a monopole field with terms of order 4,6,8,.... The sequence of multipole couplings from the zeroth order $\ell = 4$ to the final fourth order monopole can happen in only one way:

$$\psi^{(0)}_{\ell=4} \to \psi^{(2)}_{\ell=2} \to \psi^{(4)}_{\ell=0}\,. \tag{8}$$

The sixth order monopole, however, has contributions from the zeroth order $\ell = 4$ field along 3 different paths of coupling:

$$\psi^{(0)}_{\ell=4} \to \psi^{(2)}_{\ell=4} \to \psi^{(4)}_{\ell=2} \to \psi^{(6)}_{\ell=0} \qquad \psi^{(0)}_{\ell=4} \to \psi^{(2)}_{\ell=2} \to \psi^{(4)}_{\ell=2} \to \psi^{(6)}_{\ell=0} \qquad \psi^{(0)}_{\ell=4} \to \psi^{(2)}_{\ell=2} \to \psi^{(4)}_{\ell=0} \to \psi^{(6)}_{\ell=0} \tag{9}$$

The four wave equations (4)–(7) can now be decomposed into multipoles, resulting in six 1+1 wave equations involving radius and time. There is, however, a technical complication: singularities appear in the driving terms at

$r = 2M$. This coordinate effect, due to the fact that the r coordinate location of Kerr horizon depends on a, can be removed by introducing a new radial coordinate ρ

$$r = M + \sqrt{\rho^2 - 2\rho M + M^2 - a^2}\,, \tag{10}$$

so that the Kerr horizon is at $\rho = 2M$ independent of a. Along with this new radius, we use its associated "tortoise" version $\rho^* = \rho + 2M\log(\rho - 2M)$.

The 1+1 wave equations are most conveniently written if we introduce the notation $\psi^{(\rm order)}(t,\rho,\theta) = \rho^{-1}\sum_\ell f_\ell^{(\rm order)}(t,\rho)P_\ell(\cos\theta)$, for multipole decompositions. The six equations we need then are

$$\partial_t^2 f_4^{(0)} - \partial_{\rho*}^2 f_4^{(0)} + \frac{1-2M/\rho}{\rho^2}\left(20 + \frac{2M}{\rho}\right) f_4^{(0)} \;=\; 0 \tag{11}$$

$$\partial_t^2 f_2^{(2)} - \partial_{\rho*}^2 f_2^{(2)} + \frac{1-2M/\rho}{\rho^2}\left(6 + \frac{2M}{\rho}\right) f_2^{(2)} \;=\; -\frac{4}{21}\frac{M^2}{\rho^2}\left(1 - \frac{2M}{\rho}\right)\partial_t^2 f_4^{(0)} \tag{12}$$

$$\begin{aligned}\partial_t^2 f_4^{(2)} - \partial_{\rho*}^2 f_4^{(2)} + \frac{1-2M/\rho}{\rho^2}\left(20 + \frac{2M}{\rho}\right) f_4^{(2)} \;&=\; \frac{2}{77}\frac{M^2(19 + 20M/\rho + 38M^2/\rho^2)}{\rho^2(1-M/\rho)}\partial_t^2 f_4^{(0)} \\ -\frac{M^2}{\rho^2(1-M/\rho)^2}\partial_{\rho*}^2 f_4^{(0)} + \frac{M^2(1-2M/\rho}{\rho^3(1-M/\rho)^3}\partial_{\rho*} f_4^{(0)} \;&-\; \frac{M^2(1-2M/\rho)(1-4M/\rho+2M^2/\rho^2)}{\rho^4(1-M/\rho)^3} f_4^{(0)}\end{aligned} \tag{13}$$

$$\partial_t^2 f_0^{(4)} - \partial_{\rho*}^2 f_0^{(4)} + \frac{1-2M/\rho}{\rho^2}\left(\frac{2M}{\rho}\right) f_0^{(4)} \;=\; -\frac{2}{15}\frac{M^2}{\rho^2}\left(1 - \frac{2M}{\rho}\right)\partial_t^2 f_2^{(2)}\,. \tag{14}$$

$$\begin{aligned}\partial_t^2 f_2^{(4)} - \partial_{\rho*}^2 f_2^{(4)} + \frac{1-2M/\rho}{\rho^2}\left(6 + \frac{2M}{\rho}\right) f_2^{(4)} \;&=\; \frac{2}{21}\frac{M^2(5 + 6M/\rho + 10M^2/\rho^2)}{\rho^2(1-M/\rho)}\partial_t^2 f_2^{(2)} - \frac{M^2}{\rho^2(1-M/\rho)^2}\partial_{\rho*}^2 f_2^{(2)} \\ -\frac{M^2(1-2M/\rho)(1-4M/\rho+2M^2/\rho^2)}{\rho^4(1-M/\rho)^3} f_2^{(2)} \;&+\; \frac{M^2(1-2M/\rho)}{\rho^3(1-M/\rho)^3}\partial_{\rho*} f_2^{(2)} - \frac{4}{21}\frac{M^2(1-2M/\rho)}{\rho^2}\partial_t^2 f_4^{(2)}\,.\end{aligned} \tag{15}$$

$$\begin{aligned}\partial_t^2 f_0^{(6)} - \partial_{\rho*}^2 f_0^{(6)} + \frac{1-2M/\rho}{\rho^2}\left(\frac{2M}{\rho}\right) f_0^{(6)} \;&=\; \frac{2}{3}\frac{M^2(1 + 2M^2/\rho^2)}{\rho^2(1-M/\rho)}\partial_t^2 f_0^{(4)} - \frac{M^2}{\rho^2(1-M/\rho)^2}\partial_{\rho*}^2 f_0^{(4)} \\ +\frac{M^2(1-2M/\rho)}{\rho^3(1-M/\rho)^3}\partial_{\rho*} f_0^{(4)} \;&-\; \frac{2}{15}\frac{M^2(1-2M/\rho)}{\rho^2}\partial_t^2 f_2^{(4)}\,.\end{aligned} \tag{16}$$

Numerical computations with these equations were carried out on a $t, \rho*$ characteristic grid, with no boundary conditions. (Computations were carried out only in the domain of dependence of the initial spatial grid.) All six fields $f_4^{(0)}, f_4^{(2)}, f_2^{(2)}, f_2^{(4)}, f_0^{(4)}, f_0^{(6)}$, were evolved simultaneously. Initial data for $f_4^{(0)}$, at $t = 0$, was chosen to be a Gaussian pulse and was made (approximately) outgoing by taking the initial value at grid point ρ^* to be replicated after a time step Δt, at the spatial grid point $\rho^* + \Delta t$. Initial data were taken to be zero for all higher order fields, so that the final monopole was only the result of the initial pure $\ell = 4$ data. Except for the final monopole, the resulting late-time power-laws, at every order, followed the common-sense $2\ell + 3$ rule. The zeroth-order $\ell = 4$ field, for example, falls off as $1/t^{11}$; the second- and fourth-order quadrupole fields fall off as $1/t^7$.

Only the monopole gives rule-breaking results. Results are shown in Fig. 1 for the effective power-law index $n_{\rm effective} = d\log f/d\log t$ of both $f_0^{(4)}$ and $f_0^{(6)}$, the fourth- and sixth-order monopoles. The cascade of equations (11)–(16) links the highest order and lowest order fields, in effect, by high-order derivatives, so the results shown in Fig. 1 required smoothing of the computational output as will be described in a longer paper[17]. The results in Fig. 1 show clearly that the late time tail is characterized by a $1/t^5$ fall off, not the more "sensible" $1/t^3$ fall off. The default grid size used was $\Delta t = \Delta\rho* = 0.09M$, but the results for $n_{\rm effective}$ were unchanged for a reasonable variation in the grid size.

From a numerical point of view the $1/t^5$ result is truly remarkable. The $\ell > 0$ fields die off very quickly, so it should be valid to consider the late-time monopole evolving without source. The monopole *should* have the exponent $n = 3$. Indeed, small modifications in the computation do change the exponent from $n = 5$ to $n = 3$. We see this change if we put in nonzero initial data for $f_0^{(4)}$, or for $f_2^{(2)}$. We see this change also if we arbitrarily turn off the evolution of the $f_4^{(0)}$ field at some intermediate time and let the higher order fields continue to evolve. The overwhelming tendency for the exponent to be 3 rather than 5 convinces us that there is no error we have overlooked in our program; any error would almost surely lead to $n = 3$.

The delicacy of the the $n = 5$ result underscores the numerical advantages of the 1+1 computations in Eqs. (11) – (16) over 2+1 codes, even though the set of equations contains second derivatives applied several times on the right hand source terms. The method turns out to be accurate enough that we have been able to go one step further, and treat the case of initial $\ell, m = 6, 0$ data (as yet only to fourth order) for which the common sense monopole exponent is $n = 3$, while Eqs. (1) predicts $n = 7$. We have found, with an accuracy equivalent to that shown in Fig. 1, that the late-time monopole has index $n = 7$.

In a forthcoming paper[17] we shall present further details of the method used and shall also provide a wider set of numerical examples, including those for nonaxisymmetric initial data and for gravitational perturbations, and tests of Poisson's approximation[13].

During a revision of this paper we learned from Gaurav Khanna[18] that he had been able to evolve the scalar Teukolsky equation with a 2+1 code from pure BL initial data. A comparison of results, for small and moderate a/M showed excellent agreement with our results.

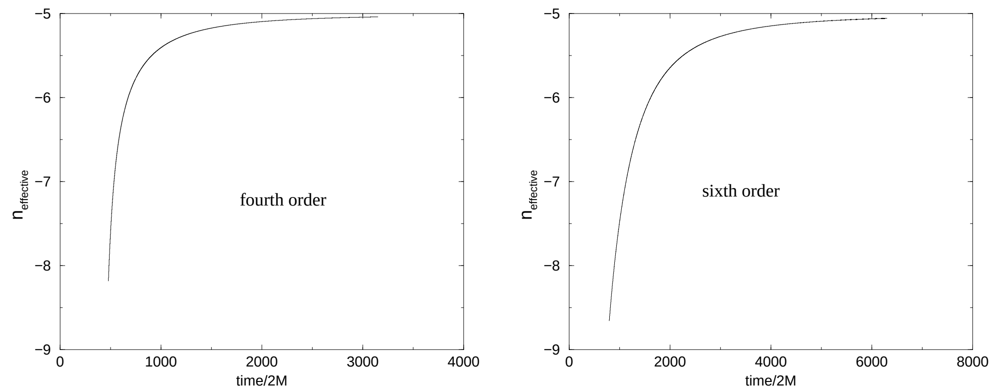

FIG. 1: The effective power-law index $d \log f/dt$ for both the fourth order and sixth order terms in the monopole moment. Initial data in both cases was an outgoing pulse with initial profile $\exp[-(\rho * +100M)^2/(40M)^2]$

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