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Moduli Spaces of Flat GSp-Bundles

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MODULI SPACES OF FLAT GSP-BUNDLES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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Abstract

A classical problem in the theory of differential equations is the classification of first-order singular differential operators up to gauge equivalence. A related algebro-geometric problem involves the construction of moduli spaces of meromorphic connections. In 2001, P. Boalch constructed well-behaved moduli spaces in the case that each of the singularities are diagonalizable. In a recent series of papers, C. Bremer and D. Sage developed a new approach to the study of the local behavior of meromorphic connections using a geometric variant of fundamental strata, a tool originally introduced by C. Bushnell for the study of p-adic representation theory. Not only does this approach allow for the generalization of diagonalizable singularities, but it is adaptable to the study of flat G-bundles for G a reductive group.

In this dissertation, the objects of study are irregular singular flat GSp_{2n}-bundles. The main results of this dissertation are two-fold. First, the local theory of fundamental strata for GSp_{2n}-bundles is made explicit; in particular, the fundamental strata necessary for the construction of well-behaved moduli spaces are shown to be associated to uniform symplectic lattice chain filtrations. Second, a construction of moduli spaces of flat GSp_{2n}-bundles is given which has many of the geometric features that have been important in the work of P. Boalch and others.

Chapter 1

Introduction

A fundamental problem in the theory of differential equations is the classification of first-order singular differential operators up to gauge equivalence. A modern variant of this problem, rephrased in the language of algebraic geometry, involves the construction of moduli spaces of meromorphic connections on a complex algebraic curve C.

Meromorphic connections are well-understood in the following situation. Let $V \cong \mathscr{O}_C^n$ be a trivializable rank n vector bundle over the Riemann sphere $\mathbb{P}^1$, and let ∇ be a meromorphic connection on V with an irregular singularity at the point y. Fix a local parameter z at y. Then, once one picks a trivialization, ∇ can be expressed locally as:

$$\nabla = d + \left(M_{-r}z^{-r} + M_{-r+1}z^{-r+1} + \dots\right)\frac{dz}{z}$$

with $M_i \in \mathfrak{gl}_n(\mathbb{C})$ and $r \geq 0$. Important information about the connection can be extracted from an analysis of the leading term M_{-r}. For example, if the leading term is not nilpotent, then r is an invariant of the connection known as the *slope.* Roughly, the slope is a measure of the “irregularity” of the connection. For example, if the slope is 0, then the connection is said to be *regular singular.* If the slope is positive, then the connection is *irregular singular.*

More can be said when the leading term is regular semisimple, for $r > 0$. In this case, a classical result due to Wasow [19] guarantees that all terms of ∇ can be simultaneously diagonalized; i.e., there exists a trivialization with the property that

$$\nabla = d + \left(D_{-r}z^{-r} + D_{-r+1}z^{-r+1} + \dots\right)\frac{dz}{z}$$

where D_i is a diagonal matrix. The diagonal one-form appearing in this expression is known as a *formal type.* Meromorphic connections with formal types at each irregular singularity have been studied extensively in recent years. In particular, P. Boalch [2] constructed moduli spaces of meromorphic connections on $\mathbb{P}^1$ with specified formal types at each irregular singularity.

However, there are many interesting meromorphic connections that do not have regular semisimple leading terms. For example, E. Frenkel and B. Gross [9] considered a connection of the form

$$\begin{aligned} \nabla &= d + \begin{bmatrix} 0 & z^{-1} \\ 1 & 0 \end{bmatrix} \tfrac{dz}{z} \\ &= d + \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \tfrac{dz}{z}. \end{aligned} \tag{1.1}$$

In this case, the leading term is not regular semisimple. Moreover, it is nilpotent. Hence, the classical analysis of leading terms fails to produce a canonical form for the connection matrix, and fails to compute the slope.

In a recent series of papers [5, 6, 7], C. Bremer and D. Sage generalized the notion of leading term using a geometric theory of *fundamental strata.* Fundamental strata were originally developed by C. Bushnell [8] to study p-adic representation theory. The key observation of Bremer and Sage was that fundamental strata play an analogous role in the p-adic setting to that of a nonnilpotent leading term in the geometric setting. Some of the primary benefits of the approach of studying connections via fundamental strata are as follows:

- While it is not always the case that a connection has a nonnilpotent leading term, it is always the case that a connection contains a fundamental stratum.

- C. Bremer and D. Sage defined a generalization of regular semisimple leading term called a *regular stratum.* Many of the interesting connections which do not have a regular semisimple leading term, such as the Frenkel–Gross connection in Equation (1.1), contain a regular stratum. Connections containing a regular stratum can be "diagonalized" to a generalized formal type ([5, Theorem 4.13]). A construction of moduli spaces of connections on $\mathbb{P}^1$ with these generalized formal types is described in [5, Theorem 5.6].

- The approach of studying connections via strata is purely Lie theoretic. In particular, this approach can be adapted to the study flat G-bundles, for G a reductive group.

The focus of this dissertation is on flat GSp_{2n}-bundles, where GSp_{2n} is the general symplectic group. The two primary accomplishments are as follows.

1. *The local theory:* The theory of regular strata and formal types for flat GSp_{2n}-bundles is made explicit.

2. *The global theory:* Constructions of moduli spaces of flat GSp_{2n}-bundles analogous to the constructions of Boalch, Bremer, and Sage are described.

More specifically, it is shown in Chapter 4 that regular strata can be described in terms of *uniform symplectic lattice chain filtrations.* This mirrors the situation for flat GL_n-bundles [5, 7], where (strongly uniform) regular strata can be described in terms of uniform lattice chain filtrations. The construction of moduli spaces of flat GSp_{2n}-bundles in Chapter 5 is very similar to the construction for GL_n. This may suggest that constructions of moduli spaces of other flat G-bundles can be carried out similarly.

There are a few important things to note about the work in this dissertation. First, it is not known whether general fundamental strata for GSp_{2n}-bundles can be described in terms of lattice chains. L. Morris [14] shows that, in the case that G is a classical group, the fundamental strata in the p-adic setting can be described in terms of a generalization of the class of lattice chain filtrations known as *C-filtrations.* A potential next-step project involves adapting these results to the geometric setting. Second, the definition of stratum containment for GSp_{2n}-

bundles is simplified in Proposition 3.7. This result depends on the category of representations for GSp_{2n} satisfying certain properties (Lemma 2.2), which may not be true for the category of representations of general reductive groups. Another next-step project involves the investigation of analogous results to Proposition 3.7 for other groups. Finally, many of the results necessary for the construction of moduli spaces in this dissertation do not depend on regular strata being defined in terms of lattice chain filtrations. This may suggest that many of aspects of the constructions of moduli spaces in this dissertation are generalizable. A final next-step project involves an investigation into the construction of moduli spaces for general flat G-bundles.

Chapter 2

The Symplectic Vector Space, Group, and Algebra

This chapter recalls some well-known definitions and facts related to the representation theory of the symplectic and general symplectic group. For those inclined to skim or skip this chapter, please note the following highlights:

- The standard ordered symplectic basis is described in (2.1).
- Lemma 2.2 is a critical component of the proof of Theorem 4.13.
- Equation 2.2 is useful for explicit computations of the moduli spaces described in Chapter 5.

2.1 Symplectic Vector Spaces

Let $k \supset \mathbb{R}$ be an algebraically closed field of characteristic 0. A symplectic k-vector space (V, ω) is a k-vector space V equipped with a nondegenerate, skew-symmetric bilinear form ω. It is straight-forward to show that any finite-dimensional symplectic vector space has a symplectic basis

$$\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$$

with the property that $\omega(e_i, f_j) = -\omega(f_j, e_i) = \delta_{i,j}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$. With respect to the basis order

$$(e_1, e_2, \ldots, e_n, f_n, f_{n-1}, \ldots, f_1), \tag{2.1}$$

the bilinear form ω is represented by the matrix

$$J = \begin{bmatrix} 0 & K \\ -K & 0 \end{bmatrix}$$

where K is the $n \times n$ matrix with 1's on the anti-diagonal:

$$K = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

This particular basis order ensures that the Borel subgroup for the symplectic group defined in Section 2.3 is upper triangular. In this paper, this ordered symplectic basis will be called the *standard symplectic basis*; unless otherwise stated, elements, endomorphisms, and bilinear forms of symplectic vector spaces will be expressed with respect to this ordered symplectic basis (when expressed in coordinates). Note that much of the literature (e.g., [16]) uses the basis order $(e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n)$. When it is convenient, the standard symplectic basis will be written as $(e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n})$ (where $e_{n+i} = f_{n+1-i}$ for f_i as defined as in (2.1)).

Recall that the transpose A^t of a square matrix A is defined by $[A^t]_{i,j} = [A]_{j,i}$. Define the *weird transpose* $A^{\clubsuit}$ of a $k \times k$ square matrix A by $[A^{\clubsuit}]_{i,j} = A_{k+1-j,k+1-i}$.

For example, the weird transpose of a 3×3 matrix is illustrated below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{\clubsuit} = \begin{bmatrix} a_{33} & a_{23} & a_{13} \\ a_{32} & a_{22} & a_{12} \\ a_{31} & a_{21} & a_{11} \end{bmatrix}.$$

2.2 Symplectic and General Symplectic Groups

Let (V, ω) be a symplectic vector space. The *general symplectic group* $\mathrm{GSp}(V)$ (or the group of *symplectic similitudes*) is the group of automorphisms g of V preserving ω up to an invertible scalar; i.e., the group of $g \in \mathrm{GL}(V)$ such that, for all $v, w \in V$, $\omega(gv, gw) = \lambda(g)\omega(v, w)$ for some invertible scalar $\lambda(g)$. The *symplectic group* $\mathrm{Sp}(V)$ is the subgroup of $\mathrm{GSp}(V)$ consisting of elements preserving the symplectic form; i.e., $\mathrm{Sp}(V)$ consists of all $g \in \mathrm{GSp}(V)$ for which $\lambda(g) = 1$. In coordinates,

$$\mathrm{GSp}_{2n} := \mathrm{GSp}(k^{2n}) = \left\{ g \in \mathrm{GL}_{2n} \,|\, g^t J g = \lambda(g) J \text{ for some } \lambda(g) \in \mathrm{G}_m \right\}.$$

The general symplectic group is a central extension of the symplectic group

$$1 \to \mathrm{Sp}_{2n} \to \mathrm{GSp}_{2n} \to \mathrm{G}_m \to 1.$$

Note that $\mathrm{Sp}_2 \cong \mathrm{SL}_2$ and $\mathrm{GSp}_2 \cong \mathrm{GL}_2$.

The *(matrix) symplectic Lie algebra* $\mathfrak{sp}_{2n}$ is the set of endomorphisms X of k^{2n} satisfying $\omega(Xv, w) + \omega(v, Xw) = 0$. In coordinates,

$$\mathfrak{sp}_{2n} = \left\{ X \in \mathfrak{gl}_{2n} : X^t J + JX = 0 \right\}.$$

Consequently, elements X in $\mathfrak{sp}_{2n}$ have block form

$$X = \left[\begin{array}{c|c} A & B \\ \hline C & -A^{\clubsuit} \end{array}\right]$$

where $B = B^{\clubsuit}$ and $C = C^{\clubsuit}$. The *(matrix) general symplectic Lie algebra* $\mathfrak{gsp}_{2n}$ is the direct sum

$$\mathfrak{gsp}_{2n} = \mathfrak{sp}_{2n} \oplus k\text{-sp}\{I\}$$

where I is the $2n \times 2n$ matrix with 1's on the diagonal and 0's elsewhere.

Unlike the general linear Lie algebras, the symplectic Lie algebras are not necessarily associative. This fact is a consequence of the following useful lemma.

Lemma 2.1. *Suppose $X \in \mathfrak{sp}_{2n}$. Then X^m is symplectic if and only if m is odd or $X^m = 0$.*

Proof. This follows immediately from the fact that $X \in \mathfrak{sp}_{2n}$ if and only if $X^m J = X^{m-1} J X = \ldots = (-1)^m X^m$. □

Note that the above lemma does not extend to all elements of $\mathfrak{gsp}_{2n}$; in particular, $(\gamma I)^m \in \mathfrak{gsp}_{2n}$ for all field elements γ and positive integers m.

2.3 The Root System of Type C and the Category of Representations

The symplectic group Sp_{2n} is connected, simply connected, and semisimple, and the general symplectic group GSp_{2n} is connected and reductive. Fix a maximal torus $T_{\mathrm{Sp}_{2n}} \subset \mathrm{Sp}_{2n}$ of the form

$$T_{\mathrm{Sp}_{2n}} = \left\{\mathrm{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}) | a_i \in k^*\right\} \cong \mathrm{G}_m^n,$$

and a maximal torus $T \subset \mathrm{GSp}_{2n}$ of the form

$$T = \left\{ \mathrm{diag}(a_1, \ldots, a_n, a_{n+1}, \frac{a_n a_{n+1}}{a_{n-1}}, \ldots, \frac{a_n a_{n+1}}{a_1}) | a_i \in k^* \right\} \cong \mathrm{G}_m^{n+1} .$$

The Lie algebra $\mathfrak{t}_{\mathrm{Sp}_{2n}}$ of $T_{\mathrm{Sp}_{2n}}$ has the form

$$\{(a_1, \ldots, a_n, -a_n, \ldots, -a_1) | a_i \in k\} \cong k^n.$$

We will regularly specify a point in $\mathfrak{t}_{\mathrm{Sp}_{2n}}$ by referring to its first n coordinates $(a_1, \ldots, a_n)$.

Denote the characters and cocharacters of T by X^* and X_* respectively. The group of characters X^* are generated by $\{\chi_i\}_{i=1}^n$, where

$$\chi_i(\mathrm{diag}(a_1, \ldots, a_n, a_{n+1}, \ldots, \frac{a_n a_{n+1}}{a_{n-1}})) = a_i.$$

The set of roots with respect to T is $\Phi = \{\pm\chi_i \pm \chi_j\}_{1 \leq i,j \leq n} - \{0\}$, and corresponds to the Dynkin diagram C_n for $n \geq 2$. Fix simple roots

$$\Delta = \{\chi_1 - \chi_2, \chi_2 - \chi_3, \ldots, \chi_{n-1} - \chi_n, 2\chi_n\} .$$

The Borel subalgebra $\mathfrak{b}$ and subgroup B corresponding to this set of simple roots are upper triangular with respect to the standard symplectic basis defined in Section 2.1.

Let $\mathrm{Rep}(\mathrm{GSp}_{2n})$ denote the category of finite-dimensional representations of GSp_{2n}. The following result will be useful in Section 3.7.

Lemma 2.2. $\mathrm{Rep}(\mathrm{GSp}_{2n})$ *is generated from the standard representation V and the operations of taking subrepresentations, duals, direct sums, and tensor products.*

Proof. The representations of the Lie algebra $\mathfrak{sp}_{2n}$ can be generated as such [10]. Since Sp_{2n} is simply connected, it follows from [11, Theorem 5.6] that the representations of $\mathfrak{sp}_{2n}$ and $\mathrm{Rep}(\mathrm{Sp}_{2n})$ correspond functorially. Finally, since $\mathrm{Rep}(\mathrm{GSp}_{2n})$ is generated from $\mathrm{Rep}(\mathrm{Sp}_{2n})$ and the determinant representation $\bigwedge^{2n} V$ via direct sum, the claim follows. □

2.4 The Weyl Group $W(C_n)$ and Regular Elements

The Weyl group $W = W(C_n)$ of type C_n (for $n \geq 2$) is isomorphic to the hyperoctahedral group of order $2^n n!$. In [4, Proposition 24], Carter gives a clear and concise description of the elements and conjugacy classes of W, which is largely repeated verbatim here. The elements of W act on the weights $\{\pm\chi_i\}_{i=1}^n$ for the standard representation by permuting elements of $\{\chi_i\}_{i=1}^n$ and changing the signs on a subset of them. To elaborate, each element $w \in W$ determines a permutation of $\{1, 2, \ldots, n\}$, and this permutation can be expressed as a product of cycles. Let $(k_1 k_2 \ldots k_r)$ be such a cycle. Then w acts via

$$\chi_{k_1} \mapsto \pm\chi_{k_2} \mapsto \pm\chi_{k_3} \mapsto \ldots \mapsto \pm\chi_{k_n} \mapsto \pm\chi_{k_1}.$$

The cycle is said to be *positive* if $w^r(\chi_1) = \chi_1$ and *negative* if $w^r(\chi_1) = -\chi_1$. The lengths of cycles together with their signs give a set of positive and negative integers called the *signed cycle-type of* w. For example, in $W(C_2)$, the identity element has signed cycle-type $(1\ 1)$, the involution given by $\chi_i \mapsto -\chi_i \mapsto \chi_i$ for $i \in \{1, 2\}$ has signed cycle-type $(-1 - 1)$, and the Coxeter element given by $\chi_1 \mapsto \chi_2 \mapsto -\chi_1 \mapsto -\chi_2 \mapsto \chi_1$ has signed cycle-type (-2). Conjugacy classes in W

(which will henceforth be called Weyl classes for brevity) are determined by their signed cycle-type.

In Chapter 4, Weyl classes consisting of *regular elements* are of particular interest. More information on regular elements for Coxeter groups can be found in [18]. The regular Weyl classes in $W(C_n)$ are characterized by the following signed cycle-types:

$$\begin{aligned} \left(d^{\frac{n}{d}}\right) &= (d\,d\ldots d) && \text{for } d|n \text{ odd, and} \\ \left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right) &= \left(-\frac{d}{2}\ -\frac{d}{2}\cdots -\frac{d}{2}\right) && \text{for } d|2n \text{ even.} \end{aligned}$$

Note that there is exactly one regular Weyl class in $W(C_n)$ consisting of order d elements for each divisor d of $2n$. Also note that, for all n, the conjugacy class (1^n) of the identity element and the class $(-n)$ of Coxeter elements are regular. For quick reference, the regular Weyl classes for $2 \leq n \leq 9$ are given below.

n	Regular classes in $W(C_n)$
2	$(1^2), (-1^2), (-2)$
3	$(1^3), (-1^3), (3), (-3)$
4	$(1^4), (-1^4), (-2^2), (-4)$
5	$(1^5), (-1^5), (5), (-5)$
6	$(1^6), (-1^6), (3^2), (-2^3), (-3^2), (-6)$

2.5 Symplectic Flags

The *symplectic complement* W^ω of a linear subspace $W \subset V$ is defined to be $W^\omega = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}$. Notice that a linear subspace W of a symplectic vector space V is not necessarily symplectic; in fact, this is only the case precisely when $W \cap W^\omega = \emptyset$. Recall that a flag $\mathscr{F}$ in a k-vector space V is a sequence $\{F^i\}_{i=0}^m$ of k-subspaces $F^i \subset V$ that is decreasing; i.e., $F^i \supsetneq F^{i+1}$ for all $0 \leq i < m$. A *symplectic flag* in a symplectic k-vector space V is a sequence $\{F^i\}_{i=0}^m$ of k-subspaces $F^i \subset V$ that is decreasing and closed under symplectic complements. In other words, $\mathscr{F}$ satisfies the following properties:

- $F^i \supsetneq F^{i+1}$ for all $0 \leq i \leq m-1$, and

- for every i, there exists some j such that $F^j = (F^i)^\omega$.

Note that in order for a k-linear subspace to belong to a symplectic flag, it must be comparable to its symplectic complement via inclusion; i.e., it must be isotropic, coisotropic, or Langrangian. For example, $F = k\text{-span}\,\{e_1, f_1\}$ is not comparable to its symplectic complement $F^\omega = k\text{-span}\,\{e_2, f_2\}$.

The *signature* of a symplectic flag $\{F^i\}_{i=0}^m$ is the sequence $(\dim(F^i))_{i=0}^m$. A *complete symplectic flag* is a symplectic flag with signature $(\dim(V) - i)_{i=0}^{\dim(V)}$. There is a simply transitive action of $\mathrm{GSp}(V)$ on the space of all symplectic flags in V with orbits parametrized by signature. Parabolic subgroups (resp. Borel subgroups) can be defined alternatively as the stabilizers of symplectic flags (resp. complete symplectic flags).

A *standard symplectic flag* in k^{2n} is a symplectic flag where each subspace F^i is defined as the k-span of some subset of the standard symplectic basis. A symplectic flag (and its parabolic subgroup and subalgebra) is *uniform* if $d_i - d_{i+1}$ is a fixed constant for all i.

The set of all complete symplectic flags in a symplectic vector space is a principal homogeneous space called the *symplectic flag variety,* and is commonly identified with $B\backslash \mathrm{GSp}_{2n}$. The expression below provides a distinct set of representatives for this quotient group:

$$\bigsqcup_{w \in W} Bw\left(w^{-1}U^{\mathrm{op}}w \cap U\right) \tag{2.2}$$

where U is the unipotent radical of B. This expression is useful for explicit constructions of the moduli spaces described in Chapter 5.

Chapter 3

Flat GSp-Bundles and Strata

The focus of this chapter is the local theory of flat GSp_{2n}-bundles. In particular, formal flat GSp-bundles are discussed in Section 3.3, and strata are discussed in Section 3.5. A key component of a stratum is a specification of a well-behaved filtration on the loop algebra $\widehat{\mathfrak{gsp}_{2n}}$. Specifically, the role of a well-behaved filtration on the loop algebra is played by the Moy–Prasad filtrations [15], which is a class of filtrations indexed by points in the Bruhat–Tits building for $\widehat{\mathrm{GSp}_{2n}}$. These are discussed in Section 3.4. In Chapter 4, it is shown that the Moy–Prasad filtrations corresponding to symplectic lattice chains are sufficient for the study of a special type of strata, called *regular strata*. Symplectic lattice chains are discussed in Section 3.1, and regular strata are discussed in Section 4.4. Note that much of this chapter consists of restatements of the theory introduced or discussed in [5, 6, 7] for the specific case that $\mathrm{G} = \mathrm{GSp}_{2n}$.

The following conventions are fixed henceforth. Let $F := k(\!(z)\!)$ be the field of formal Laurent series over k with ring of integers $\mathfrak{o} = k[\![z]\!]$. Write $\Delta^\times := \mathrm{Spec}(F)$ for the formal punctured disk, and $\Omega^1 := \Omega^1_{F/k}$ for the space of differential one-forms on $\Delta^\times$. Denote the Euler vector field $z\frac{d}{dz}$ by τ. Let ι_τ be the inner derivation (or contraction) by τ, so that the order -1 one-form $\nu := \frac{dz}{z}$ is the unique one-form satisfying $\iota_\tau(\nu) = 1$. Any one-form $\omega \in \Omega^1$ can be written $\omega = \iota_\tau(\omega)\nu$ where ι_τ

is the inner derivation of a differential form by τ. For a given reductive group G, denote the loop group $\mathrm{G}(F)$ by $\widehat{\mathrm{G}}$ and the loop algebra $\mathfrak{g} \otimes F$ by $\widehat{\mathfrak{g}}$. If $V \in \mathrm{Rep}(\mathrm{G})$, denote the representation $V \otimes F$ of $\widehat{\mathrm{G}}$ by $\widehat{V}$.

3.1 Symplectic Lattice Chains

Let V be a $2n$-dimensional vector space over F. An $\mathfrak{o}$-lattice $L \subset V$ is a finitely generated $\mathfrak{o}$-module with $L \otimes_{\mathfrak{o}} F \cong V$. If L is a lattice, then the set $L^* = \{v \in V : \omega(v, L) \subseteq \mathfrak{o}\}$ is a lattice, called the *dual lattice* of L. Two lattices L and L' are defined to be *homothetic* if $L = fL'$ for some $f \in F^*$. The space of lattices homothetic to L is precisely $\{z^i L\}_{i \in \mathbb{Z}}$. A *symplectic lattice* is a lattice L with the property that the set of lattices generated from L by homothety and duality (or, equivalently, the set of lattices homothetic to L or L^*) is totally ordered by inclusion (i.e., a chain). For example, the dual of the lattice $L = \mathfrak{o}\text{-span}\,\{e_1, e_2, f_2, zf_1\}$ is $L^* = \mathfrak{o}\text{-span}\,\{z^{-1}e_1, e_2, f_2, f_1\}$. The set of lattices homothetic to L or L^* can be totally ordered by inclusion

$$\ldots \supsetneq z^{-1}L^* \supsetneq z^{-1}L \supsetneq L^* \supsetneq L \supsetneq zL^* \supsetneq zL \supsetneq \ldots .$$

Hence L is a symplectic lattice. On the other hand, the lattice $L' = \mathfrak{o}\text{-span}\{ze_1, z^{-1}e_2, f_2, f_1\}$ is not symplectic, since the dual lattice $(L')^* = \mathfrak{o}\text{-span}\{e_1, e_2, zf_2, z^{-1}f_1\}$ is not comparable to L' via inclusion. This discussion naturally motivates the next definition.

Definition 3.1. *A* symplectic lattice chain $\mathscr{L}$ *is a collection of lattices* $\{L^i\}_{i\in\mathbb{Z}}$ *that is decreasing and closed under homothety and duality. In other words,* $\mathscr{L}$ *satisfies the following properties:*

- $L^i \supsetneq L^{i+1}$,
- $L^{i+e} = zL^i$ *for some fixed* $e = e_{\mathscr{L}}$ *(called the* period of $\mathscr{L}$*), and*
- *if* $L \in \mathscr{L}$*, then* $L^* \in \mathscr{L}$*.*

Note that any lattice appearing in a symplectic lattice chain is automatically a symplectic lattice. A *complete symplectic lattice* is a symplectic lattice chain with period $e = 2n$.

There is a simply transitive action of $\mathrm{GSp}(V)$ on the variety of symplectic lattice chains. A *parahoric subgroup* $P \subset \mathrm{GSp}(V)$ is the stabilizer of a symplectic lattice chain $\mathscr{L}$; i.e.,

$$P = \left\{g \in \mathrm{GSp}(V) : gL^i = L^i \text{ for all } i\right\}.$$

A *parahoric subalgebra* $\mathfrak{p} \subset \mathfrak{gsp}(V)$ is the Lie algebra of a parahoric subgroup P. Equivalently,

$$\mathfrak{p} = \left\{X \in \mathfrak{gsp}(V) : XL^i \subset L^i \text{ for all } i\right\}.$$

An *Iwahori subgroup* I is the stabilizer of a complete symplectic lattice chain, and an *Iwahori subalgebra* $\mathfrak{i}$ is the Lie algebra of an Iwahori subgroup I.

Suppose that $V = V_k \otimes_k F$ for a given symplectic k-vector space V_k. There is a distinguished lattice $V_{\mathfrak{o}} = V_k \otimes_k \mathfrak{o}$, and a map $\rho : V_{\mathfrak{o}} \to V_k$ induced by the k-linear "evaluation 0" map $\mathfrak{o} \to k$ (determined by $z \mapsto 0$). Any subspace $W \subset V_k$ that is comparable to its symplectic complement (i.e., any W that is isotropic, coisotropic, or Langrangian) determines a symplectic lattice $\rho^{-1}(W) \subset V$. Moreover, any symplectic flag $\mathscr{F} = \{V_k = F^0 \supsetneq F^1 \supsetneq \ldots \supsetneq F^e = 0\}$ determines a period e symplectic lattice chain $\mathscr{L}_{\mathscr{F}} = \{L^i\}_{i \in \mathbb{Z}}$ by setting $L^{qe+r} = z^q \rho^{-1}(F^r)$ for all integers q and $0 \leq r \leq e-1$:

$$\mathscr{L}_{\mathscr{F}} = \left\{ \ldots \supsetneq z^{-1}\rho^{-1}(F^{e-1}) \supsetneq V_{\mathfrak{o}} \supsetneq \rho^{-1}(F^1) \supsetneq \ldots \supsetneq \rho^{-1}(F^{e-1}) \supsetneq zV_{\mathfrak{o}} \supsetneq \ldots \right\}.$$

The corresponding parahoric subgroup may be referred to as $\rho^{-1}(Q)$, where Q is the stabilizer of the $\mathscr{F}$.

A symplectic lattice chain (and its associated parahoric subgroup and algebra) is called *standard* if it corresponds to a standard symplectic flag, and *uniform* if it corresponds to a uniform symplectic flag. Alternatively, $\mathscr{L}$ is uniform if $\dim_k L^i/L^{i+1} = 2n/e_{\mathscr{L}}$ for all i. Note that the period of a uniform symplectic lattice chain must divide $2n$.

Example 3.2. *An example of a period* 1 *standard uniform symplectic lattice chain is given by* $\mathscr{L} = \{L^i\}_{i \in \mathbb{Z}}$*, where* $L^i = z^i \mathfrak{o}^4$*. The corresponding parahoric subgroup is* $P = \mathrm{GSp}_4(\mathfrak{o})$*, with parahoric subalgebra* $\mathfrak{p} = \mathfrak{gsp}_4(\mathfrak{o})$*.*

Example 3.3. *A period 2 standard uniform symplectic lattice chain $\mathscr{L} = \{L^i\}_{i\in\mathbb{Z}}$ is generated from the following two lattices by homothety:*

$$L^0 = \begin{bmatrix} \mathfrak{o} \\ \mathfrak{o} \\ \mathfrak{o} \\ \mathfrak{o} \end{bmatrix} \quad \supsetneq \quad L^1 = \begin{bmatrix} \mathfrak{o} \\ \mathfrak{o} \\ z\mathfrak{o} \\ z\mathfrak{o} \end{bmatrix}$$

Equivalently, set $L^{2i} = z^i L^0$ and $L^{2i+1} = z^i L^1$ for all integers i. Its corresponding parahoric subgroup and subalgebra, respectively, are

$$P = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \cap \mathrm{GSp}_4(F), \quad \mathfrak{p} = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \cap \mathfrak{gsp}_4(F)$$

Example 3.4. *A period 4 standard uniform symplectic lattice chain is generated by the following four lattices by homothety:*

$$L^0 = \begin{bmatrix} \mathfrak{o} \\ \mathfrak{o} \\ \mathfrak{o} \\ \mathfrak{o} \end{bmatrix} \quad \supsetneq \quad L^1 = \begin{bmatrix} \mathfrak{o} \\ \mathfrak{o} \\ \mathfrak{o} \\ z\mathfrak{o} \end{bmatrix} \quad \supsetneq \quad L^2 = \begin{bmatrix} \mathfrak{o} \\ \mathfrak{o} \\ z\mathfrak{o} \\ z\mathfrak{o} \end{bmatrix} \quad \supsetneq \quad L^3 = \begin{bmatrix} \mathfrak{o} \\ z\mathfrak{o} \\ z\mathfrak{o} \\ z\mathfrak{o} \end{bmatrix}$$

This special parahoric subgroup (resp. subalgebra) is called a standard Iwahori subgroup (resp. subalgebra), and is denoted by I and $\mathfrak{i}$.

$$I = \begin{bmatrix} \mathfrak{o}^* & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & \mathfrak{o}^* & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o}^* & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o}^* \end{bmatrix} \cap \mathrm{GSp}_4(F), \quad \mathfrak{i} = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ z\mathfrak{o} & z\mathfrak{o} & z\mathfrak{o} & \mathfrak{o} \end{bmatrix} \cap \mathfrak{gsp}_4(F)$$

Any symplectic lattice chain in V induces a filtration on $\mathfrak{gsp}(V)$ as follows. Let $\mathscr{L}$ be a symplectic lattice chain in V with parahoric subalgebra $\mathfrak{p} \subset \mathfrak{gsp}(V)$. Define $\{\mathfrak{p}^s\}_{s\in\mathbb{Z}}$ by setting

$$\mathfrak{p}^s = \left\{X \in \mathfrak{gsp}(V) : XL^i \subset L^{i+s} \text{ for all } i\right\}.$$

This is a filtration on $\mathfrak{gsp}(V)$ with $\mathfrak{p}^0 = \mathfrak{p}$ and $\mathfrak{p}^{i+e_{\mathscr{L}}} = z\mathfrak{p}^i$ for all i. Since any symplectic lattice chain $\mathscr{L}$ is also a lattice chain, it follows that $\mathfrak{p}^i_{\mathfrak{gsp}(V)} = \mathfrak{p}^i_{\mathfrak{gl}(V)} \cap \mathfrak{gsp}(V)$.

There is a corresponding filtration on $\mathrm{GSp}(V)$ by congruence subgroups P^s for $s \geq 0$. In contrast to the analogous filtration for $\mathrm{GL}(V)$ (as in, e.g., [5, Section 2.1]), it is not the case that $P^s = I + \mathfrak{p}^s$ for all $s > 0$.

3.2 Formal Flat Vector Bundles

A *formal flat vector bundle* (U, ∇) on $\Delta^\times$ is a finite-dimensional F-vector space U equipped with a connection ∇. A *connection* ∇ is a k-derivation $\nabla : U \to \Omega^1(U)$, which means that:

- ∇ is k-linear, and

- ∇ satisfies a "Leibniz rule"; i.e., for $f \in F, u \in U$, ∇ satifies $\nabla(fu) = \frac{df}{dz}u + f\nabla(u)$.

Define the *rank* of a connection to be the dimension of U. It is often desirable to express a connection "in coordinates." With this goal in mind, fix a trivialization of U; i.e., fix a vector space isomorphism $\phi : U \to F^n$. Define $\nabla^\phi := \phi\nabla\phi^{-1}$, and set $[\nabla]_\phi$ equal to the matrix in $M_n(\Omega^1) = M_n(F) \otimes_F \Omega^1$ with columns given by $\nabla^\phi(e_i)$ with respect to the standard basis for F^n. Then

$$\nabla^\phi = d + [\nabla]_\phi,$$

where d is the usual exterior derivative on F^n. To see this, let $\sum_{i=1}^n f_i e_i$ be an arbitrary element of F^n. Then

$$\begin{aligned}
\nabla^\phi \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} &= \nabla^\phi \left(\textstyle\sum_{i=1}^n f_i e_i \right) \\
&= \textstyle\sum_{i=1}^n \nabla^\phi(f_i e_i) && \text{by } k\text{-linearity of } \phi \text{ and } \nabla \\
&= \textstyle\sum_{i=1}^n \phi\nabla(f_i\phi^{-1}(e_i)) && \text{by } F\text{-linearity of } \phi \\
&= \textstyle\sum_{i=1}^n \phi \left(\frac{df_i}{dz}\phi^{-1}(e_i) + f_i\nabla(\phi^{-1}(e_i)) \right) && \text{by the Leibniz rule for } \nabla \\
&= \textstyle\sum_{i=1}^n \frac{df_i}{dz} e_i + f_i\nabla^\phi(e_i) && \text{by } F\text{-linearity of } \phi
\end{aligned}$$

$$= d \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} + [\nabla]_\phi \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

We call $[\nabla]_\phi \in M_n(\Omega^1)$ the *connection matrix* of ∇ with respect to ϕ, and we call the expression $d + [\nabla]_\phi$ the *matrix-form* for ∇ with respect to ϕ. Note that it is common in the literature to suppress the naming of a trivialization ϕ in the notation and simply write $\nabla = d + [\nabla]$.

A connection corresponds to a first-order system of differential equations in a natural way. To see this, first define $\nabla_\tau := \iota_\tau(\nabla)$ (where ι_τ is the usual inner-derivation, or contraction, of a differential form by a vector field). Then ∇_τ can also be expressed in a matrix-form with respect to a triviliazation. Let ϕ be a trivialization. Then

$$\nabla_\tau^\phi = \tau + [\nabla_\tau]_\phi,$$

where $[\nabla_\tau]_\phi \in M_n(F)$. The horizontal section of ∇_τ^ϕ — i.e., the solutions to $\tau + [\nabla_\tau]_\phi$ — are exactly the solutions $v \in F^n$ to the first-order system of algebraic differential equations $\tau(v) = -[\nabla_\tau]_\phi(v)$. We will call the matrix $[\nabla_\tau]_\phi = \iota_\tau([\nabla]_\phi)$ the (contracted) connection matrix for ∇_τ with respect to ϕ, and $\tau + [\nabla_\tau]_\phi$ is the (contracted) matrix-form for ∇_τ with respect to ϕ. Note that $[\nabla]_\phi = [\nabla_\tau]_\phi \frac{dz}{z}$; contraction can be viewed as a way to express a matrix-form for ∇ in terms of Laurent series instead of one-forms.

Some low rank examples of formal vector bundles can be described as follows. The trivial connection ∇_{triv} on F^n is defined by $\nabla_{\text{triv}}(e_i) = 0$. Hence $\nabla^{\phi}_{\text{triv}} = d$, where $\phi : F^n \to F^n$ is the identity. An example of a connection on F^2 with a diagonal connection matrix is given by ∇_1, where $\nabla_1(e_1) = ae_1$ and $\nabla_1(e_2) = be_2$. Then

$$\nabla_1^{\phi} = d + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_{\phi} \frac{dz}{z},$$

where ϕ is the identity isomorphism on F^2. The Frenkel–Gross connection ∇_{FG} [9] on F^2 is defined by $\nabla_{\text{FG}}(e_1) = z^{-1}e_2$ and $\nabla_{\text{FG}}(e_2) = z^{-1}e_1$. Then

$$\nabla_{\text{FG}}^{\phi} = d + \begin{bmatrix} 0 & 1 \\ z^{-1} & 0 \end{bmatrix}_{\phi} \frac{dz}{z},$$

where ϕ again is the identity on F^2.

There is a left simply-transitive action of $\text{GL}_n(F)$ on the space of trivializations. To elaborate, if $\phi : V \to F^n$ is a trivialization of V and $g : F^n \to F^n$ is an automorphism, then $g \cdot \phi := g \circ \phi : V \to F^n$ is a trivialization of V. One of the primary reasons we use trivializations $\phi : V \to F^n$ instead of frames $\psi : F^n \to V$ is that the natural action of $\text{GL}_n(F)$ on frames is a right action instead of a preferable left action. The action of $\text{GL}_n(F)$ naturally corresponds to an action of $\text{GL}_n(F)$ on the connection matrix, given by

$$g \cdot [\nabla]_{\phi} := [\nabla]_{g \cdot \phi} = g[\nabla]_{\phi} g^{-1} - (dg)g^{-1}.$$

This action is known as the *gauge-change action.* An action of $\mathrm{GL}_n(F)$ on contracted connection matrices is defined by

$$g \cdot [\nabla_\tau]_\phi := [\nabla_\tau]_{g\cdot\phi} = g[\nabla_\tau]_\phi g^{-1} - \tau(g)g^{-1}.$$

Similar actions can be defined on $\widehat{\mathfrak{gl}_n} \otimes \Omega^1$ and $\widehat{\mathfrak{gl}_n}$. Each of these actions are also known as gauge change actions.

Note that if $X \in \mathrm{End}(U \otimes_F \Omega^1)$, the formula $d + X$ is not a priori well-defined. However, if U is endowed with some fixed k-structure, say $U = U_k \otimes F$, then $d+X$ does make sense: $(d+X)(fv) = \frac{df}{dz}v + fT(v)$ for $f \in F$ and $v \in U_k$. In subsequent sections, we will frequently consider flat vector bundles on $\widehat{V} = V \otimes F$, where V is a k-vector space. In this case, expressions of the form $d + X$ are defined in terms of the natural k-structure.

A flat vector bundle (U, ∇) is called *regular singular* if there exists an $\mathfrak{o}$-lattice $L \subset U$ with the property that $\nabla_\tau(L) \subset L$. This implies that there exists a trivialization for which there is a trivialization ϕ for which $[\nabla]$ has at worst a simple pole. Otherwise, ∇ is called *irregular singular.* The deviation of an irregular singular connection from the regular singular case is measured by an invariant known as the slope. One of the primary results of the work of C. Bremer and D. Sage in [5, 7] is a novel formulation of the slope in terms of fundamental strata.

3.3 Formal Flat GSp_{2n}-Bundles

We now turn our attention from flat vector bundles to flat GSp_{2n}-bundles. The necessary definitions and mechanics of flat G-bundles for arbitrary connected reductive groups G are compiled in [7, Section 2.4]. We recall that exposition here (mostly verbatim) for the specific case that $\mathrm{G} = \mathrm{GSp}_{2n}$. A *formal principal* GSp_{2n}*-bundle* $\mathcal{G}$ is a principal GSp_{2n}-bundle over $\Delta^\times$. A GSp_{2n}-bundle $\mathcal{G}$ induces a tensor functor from $\mathrm{Rep}(\mathrm{GSp}_{2n})$ to the category of formal vector bundles via $(V, \rho) \mapsto V_{\mathcal{G}} := \mathcal{G} \times_{\mathrm{GSp}_{2n}} V$. By Tannakian formalism, this tensor functor uniquely determines $\mathcal{G}$. Formal principal GSp_{2n}-bundles are trivializable; hence it is always possible to choose a trivialization $\phi : \mathcal{G} \to \widehat{\mathrm{GSp}_{2n}}$. A trivialization ϕ induces a compatible collection of maps $\phi_V : V_{\mathcal{G}} \to \widehat{V}$. As was the case for formal vector bundles, there is a left action of $\widehat{\mathrm{GSp}_{2n}}$ on the set of trivializations.

A *flat structure* ∇ on a principal GSp_{2n}-bundle $\mathcal{G}$ is a formal derivation that determines a compatible family of flat connections ∇_V (which we usually write as simply ∇) on $V_{\mathcal{G}}$ for all $(V, \rho) \in \mathrm{Rep}(\mathrm{GSp}_{2n})$. In practice, once a trivialization has been fixed, ∇ may be expressed in terms of a one-form with coefficients in $\widehat{\mathfrak{gsp}_{2n}}$. This means that there exists an element $[\nabla]_\phi \in \Omega^1(\widehat{\mathfrak{gsp}_{2n}})$, called the *connection matrix* of ∇ with respect to the trivialization ϕ, for which the induced connection ∇ on $\widehat{V}$ is given by $d + (d\rho)([\nabla]_\phi)$. We will formally write

$$\nabla_\phi = d + [\nabla]_\phi$$

for the flat structure on $\widehat{\mathrm{GSp}_{2n}}$ induced by ϕ. The left action of $\widehat{\mathrm{GSp}_{2n}}$ on trivializations corresponds to a *gauge change action* of $\widehat{\mathrm{GSp}_{2n}}$ on the connection matrix.

3.4 The Bruhat–Tits Building $\mathscr{B}(\widehat{\mathrm{GSp}_{2n}})$ and Moy–Prasad Filtrations

Much of the relevant definitions and general theory involved with Bruhat–Tits buildings and Moy–Prasad filtrations are summarized in [7]. The notation used here is also consistent. Denote the reduced Bruhat–Tits building and the enlarged building of $\widehat{\mathrm{GSp}_{2n}}$ by $\overline{\mathscr{B}}(\widehat{\mathrm{GSp}_{2n}})$ and $\mathscr{B}(\widehat{\mathrm{GSp}_{2n}})$ respectively. We will refer to both as the Bruhat–Tits building, or simply the building, of $\widehat{\mathrm{GSp}_{2n}}$. The standard apartments $\overline{\mathscr{A}_0}$ (resp. $\mathscr{A}_0$) are affine spaces isomorphic to $X_*(T \cap \widehat{\mathrm{Sp}_{2n}}) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$). This isomorphism induces a canonical isomorphism $\overline{\mathscr{A}_0} \cong (\mathfrak{t}_{\mathrm{Sp}_{2n}})_{\mathbb{R}}$. The image of $x \in \overline{\mathscr{A}_0}$ via this isomorphism is denoted $\widetilde{x}$. As discussed in Section 2.3,

$$\widetilde{x} = (x_1, \ldots, x_n, -x_n, \ldots, -x_1)$$

can be specified via its first n coordinates; for the sake of being concise, we will often commit a mild abuse of notation and write

$$\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$$

or

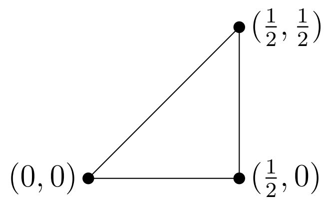
$$x = (x_1, \ldots, x_n).$$

The point corresponding to $(0, \ldots, 0) \in (\mathfrak{t}_{\mathrm{Sp}_{2n}})_{\mathbb{R}}$ is denoted by o.

The standard apartments are endowed with a cell structure induced by the affine roots. With respect to the roots Φ and simple roots Δ defined in Section 2.3, the affine roots are given by

$$\Phi + \mathbb{Z} = \{\chi + r : \chi \in \Phi, r \in \mathbb{Z}\}$$

and the simple affine roots are given by $\Delta \cup \{1 - 2\chi_1\}$ (note that $2\chi_1$ is the highest root with respect to Δ). The fundamental alcove is the intersection of the positive half-spaces determined by the simple affine roots. For example, the fundamental alcove in $\overline{\mathscr{B}}(\mathrm{GSp}_4)$ is illustrated below with coordinates for the 0-simplices labelled.



Given a point in the Bruhat–Tits building for $\widehat{\mathrm{GSp}_{2n}}$, Moy and Prasad [15] define compatible, "periodic" $\mathbb{R}$-filtrations $\left\{\widehat{V}_{x,r}\right\}_{r \in \mathbb{R}}$ on each representation $\widehat{V}$ for $V \in \mathrm{Rep}(\mathrm{GSp}_{2n})$. When considering filtrations corresponding to the reduced building, it is enough to consider $\mathrm{Rep}([\mathrm{GSp}_{2n}, \mathrm{GSp}_{2n}]) = \mathrm{Rep}(\mathrm{Sp}_{2n})$. Note that the weights for a representation $V \in \mathrm{Rep}(\mathrm{Sp}_{2n})$ are closed under inversion. This implies that $\mathrm{Crit}_x(V)$ is symmetric about 0 for all $V \in \mathrm{Rep}(\mathrm{Sp}_{2n})$.

There are corresponding $\mathbb{R}_{\geq 0}$-filtrations $\left\{(\widehat{\mathrm{GSp}_{2n}})_{x,r}\right\}_{r \in \mathbb{R}_{\geq 0}}$ on the parahoric subgroups $(\widehat{\mathrm{GSp}_{2n}})_x := (\widehat{\mathrm{GSp}_{2n}})_{x,0}$. As is the case in general [7, Section 2.6], $(\widehat{\mathrm{GSp}_{2n}})_{x+} := (\widehat{\mathrm{GSp}_{2n}})_{x,0+}$ is the pro-unipotent radical of $(\widehat{\mathrm{GSp}_{2n}})_x$; i.e., there is a split short exact sequence

$$1 \to (\widehat{\mathrm{GSp}_{2n}})_{o+} \to (\widehat{\mathrm{GSp}_{2n}})_{x+} \to U_x \to 1,$$

where U_x is the unipotent subgroup of the parabolic corresponding to $(\widehat{\mathrm{GSp}_{2n}})_x$. Likewise, there is a split short exact sequence

$$1 \to (\widehat{\mathrm{GSp}_{2n}})_{o+} \to (\widehat{\mathrm{GSp}_{2n}})_x \to Q_x \to 1,$$

where Q_x is the parabolic corresponding to $(\widehat{\mathrm{GSp}_{2n}})_x$.

Moy–Prasad filtrations can also be defined on the smooth k-dual space $\widehat{\mathfrak{gsp}_{2n}}^\vee$ with respect to a fixed nondegenerate invariant symmetric bilinear form [7, Section 2.6]. We define one such bilinear form below. The trace function $\mathrm{Tr} : \mathfrak{gsp}_{2n} \times \mathfrak{gsp}_{2n} \to k$ is a symmetric, invariant k-bilinear form. It is easily checked (by considering k-basis elements for $\mathfrak{gsp}_{2n}$) that Tr is also nondegenerate. Note that the Killing form is also a nondegenerate invariant symmetric bilinear form, but Tr has the advantage of being simpler computationally. Define $\langle \cdot, \cdot \rangle_\nu : \widehat{\mathfrak{gsp}_{2n}} \times \widehat{\mathfrak{gsp}_{2n}} \to k$ by

$$\langle A, B \rangle_\nu = \mathrm{Res}[\mathrm{Tr}(AB)\nu].$$

This nondegenerate invariant symmetric k-bilinear form induces an isomorphism $\widehat{\mathfrak{gsp}_{2n}} \to \widehat{\mathfrak{gsp}_{2n}}^\vee$ given by $X \mapsto \langle X, \cdot \rangle_\nu$. As in [7, Section 2.6], set $(\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,r}$ equal to the image of $(\widehat{\mathfrak{gsp}_{2n}})_{x,r}$ under this isomorphism. Given $\alpha \in (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,r}$, we will denote by α_ν an element in $(\widehat{\mathfrak{gsp}_{2n}})_{x,r}$ for which $\alpha(\cdot) = \langle \alpha_\nu, \cdot \rangle_\nu$. We will call such an element corresponding to α a *functional representative.* The set $\alpha_\nu + (\widehat{\mathfrak{gsp}_{2n}})_{x,-r+} = \alpha_\nu + (\widehat{\mathfrak{gsp}_{2n}})^\perp_{x,r}$ is precisely the set of functional representatives for α.

3.5 Strata and Stratum Containment for Flat GSp_{2n}-Bundles

In [7], Bremer and Sage develop a geometric theory of strata for flat G-bundles. The necessary definitions are recalled in this section; in particular, definitions of strata, fundamental strata, and stratum containment for flat GSp_{2n}-bundles are given. In addition, a simplified version of stratum containment for flat GSp_{2n}-bundles is described in Proposition 3.7.

Definition 3.5. *A* $\widehat{\mathrm{GSp}_{2n}}$*-stratum is a triple* (x, r, β) *with*

- $x \in \mathscr{B}(\widehat{\mathrm{GSp}_{2n}})$,
- $r \geq 0$ *a real number (known as the* depth *of the stratum), and*
- $\beta \in ((\widehat{\mathfrak{gsp}_{2n}})_{x,r}/(\widehat{\mathfrak{gsp}_{2n}})_{x,r+})^\vee$.

By [7, Proposition 3.6], the quotient $((\widehat{\mathfrak{gsp}_{2n}})_{x,r}/(\widehat{\mathfrak{gsp}_{2n}})_{x,r+})^\vee$ may be identified with $(\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r}/(\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r+}$. We call $\widetilde{\beta} \in (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r}$ a representative for β if β corresponds to $\widetilde{\beta} + (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r+} \in (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r}/(\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r+}$. Recall [7, Section 2.6] that $(\widehat{\mathfrak{gsp}_{2n}})^\vee \cong \Omega^1(\widehat{\mathfrak{gsp}_{2n}})$ (via $\langle X, \cdot \rangle_\nu \mapsto X\frac{dz}{z}$). Denote by $X_{\widetilde{\beta}}$ the one-form corresponding to the functional $\widetilde{\beta}$. A stratum is associated to a flat GSp_{2n}-bundle as follows.

Definition 3.6. *A flat* GSp_{2n}*-bundle* $(\mathcal{G}, \nabla)$ contains *the* $\widehat{\mathrm{GSp}_{2n}}$*-stratum* (x, r, β) *with respect to the trivialization* ϕ *if, for any (or equivalently, for some) represen-*

tative $\widetilde{\beta} \in (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r}$ *of* β,

$$\left(\nabla_\phi - i\frac{dz}{z} - X_{\widetilde{\beta}}\right)(\widehat{V}_{x,i}) \subset \Omega^1(\widehat{V})_{x,(i-r)+} \tag{3.1}$$

for all $i \in \mathbb{R}$ *and* $V \in \mathrm{Rep}(\mathrm{GSp}_{2n})$.

Stratum containment is well-behaved both with respect to conjugation. By [7, Lemma 4.2], if $(\mathcal{G}, \nabla)$ contains a stratum (x, r, β) with respect to ϕ, then it contains $(gx, r, g\beta)$ with respect to $g \cdot \phi$.

Stratum containment for flat GSp_{2n}-bundles is significantly simplified by the result below. Note that an analogous result holds for $\mathrm{G} = \mathrm{GL}_n$ by [7, Corollary A.2].

Proposition 3.7. *The* $\widehat{\mathrm{GSp}_{2n}}$*-stratum* (x, r, β) *is contained in the flat* GSp_{2n}*-bundle* $(\mathcal{G}, \nabla)$ *with respect to* ϕ *if and only if (3.1) holds for the standard representation (with respect to* ϕ*).*

Proof. The forward direction is obvious. Suppose (3.1) holds for the standard representation. By [7, Proposition A.1], the subset of $\mathrm{Rep}(\mathrm{GSp}_{2n})$ satisfying (3.1) is closed under taking subrepresentations, duals, direct sums, and tensor products. Then (3.1) holds for all representations in $\mathrm{Rep}(\mathrm{GSp}_{2n})$ by Lemma 2.2. Therefore, $(\mathcal{G}, \nabla)$ contains (x, r, β) (with respect to ϕ). $\square$

There is a useful equivalent definition of stratum containment in the case that (x, r, β) has $x \in \mathscr{A}_0$ [7, Proposition 4.3]. Given $x \in \mathscr{A}_0$, we say that the flat GSp_{2n}-bundle contains (x, r, β) with respect to a trivialization ϕ if

- $[\nabla]_\phi - \widetilde{x}\frac{dz}{z} \in \widehat{\mathfrak{gsp}_{2n}}^\perp$, and

- $([\nabla]_\phi - \widetilde{x}\frac{dz}{z}) + (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r+}$ determines the functional $\beta \in ((\widehat{\mathfrak{gsp}_{2n}})_{x,r}/(\widehat{\mathfrak{gsp}_{2n}})_{x,r+})^\vee$.

If $r > 0$, then β is simply the functional induced by $[\nabla]_\phi$.

When studying strata contained in flat G-bundles, it is often desirable that the strata be fundamental. A theory of fundamental strata is developed in [7]. A few important results from that paper are recounted below, expressed for the particular case that $\mathrm{G} = \mathrm{GSp}_{2n}$. A stratum (x, r, β) is *fundamental* if the corresponding coset $\widetilde{\beta} + (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,-r+}$ does not contain a nilpotent element. There is a useful equivalent definition of fundamental strata in the case that $x \in \mathscr{A}_0$. Let $\widetilde{\beta}_0$ denote the unique homogeneous representative in $(\widehat{\mathfrak{gsp}_{2n}})^\vee_x(r)$ for $\widetilde{\beta}$. A stratum (x, r, β) (with $(\widehat{\mathfrak{gsp}_{2n}})_o \in \mathscr{A}_0$) is fundamental if and only if $\widetilde{\beta}_0$ is not nilpotent. The depth of a fundamental stratum contained in a flat GSp_{2n}-bundle is an invariant of the connection known as the *slope.* Every flat GSp_{2n}-bundle contains a fundamental stratum.

The strata that are of interest for the construction of moduli spaces in Chapter 5 are called *regular strata.* These are defined in [6, Section 4]. Regular strata for flat GSp_{2n}-bundles are particularly nice in that the filtrations of interest correspond to symplectic lattice chains as discussed in Section 3.1; this is the main result of Chapter 4.

Chapter 4

Tori of Regular Type and Compatible Filtrations for GSp_{2n}

In order to study regular $\widehat{\mathrm{G}}$-strata (for G a reductive group), it is necessary to compare Moy–Prasad filtrations with the natural filtrations on the maximal tori in $\widehat{\mathrm{G}}$. In [6, Section 3], Bremer and Sage define compatibility between points in $\mathscr{B}(\widehat{\mathrm{G}})$ and maximal tori in $\widehat{\mathrm{G}}$. A study of regular strata is simplified if attention is restricted to only those Moy–Prasad filtrations compatible with a maximal torus corresponding to a regular strata. Furthermore, since strata and tori behave well with respect to conjugation, it is possible to choose a set of "canonical" tori from each relevant conjugacy class of tori. The first step then is to identify these relevant conjugacy classes.

It is well-known (see, e.g., [13, Lemma 2]) that there is a correspondence between the set of conjugacy classes of tori in $\widehat{\mathrm{G}}$ (resp. CSA's in $\widehat{\mathfrak{g}}$) and the set of Weyl classes for G. A maximal torus $S \subset \widehat{\mathrm{G}}$ is said to be **of type** γ if γ is the Weyl class corresponding to S. By [6, Corollary 4.10], the maximal tori S for which there exist S-regular strata are precisely the maximal tori S of type γ a regular Weyl class. One of the primary results in this chapter is a concrete definition of a canonical CSA in $\widehat{\mathfrak{gsp}_{2n}}$ of type γ for each regular class γ (see Section 2.4 for the classification of regular Weyl classes in $W(C_n)$). These CSA's are each compatible with a unique filtration satisfying certain properties, described below.

Theorem 4.1. *The CSA defined in Section 4.1.3 (resp. Section 4.1.5) is of type* $\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)$ *(resp.* $\left(d^{\frac{n}{d}}\right)$*). Each of these CSA's are compatible with a unique point* $x \in \overline{\mathscr{A}_0}$ *(defined in Section 4.2.1 and 4.2.2, respectively) satisfying the following properties:*

- $\mathrm{Crit}_x(\mathfrak{gsp}_{2n}) = \frac{1}{d}\mathbb{Z}$*, where* d *is the order of an element in* γ*, and*

- $\mathfrak{o}^{2n} \in \left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$.

These points have the property that $\left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$ *corresponds to a uniform symplectic lattice chain of period the corresponding integer* d*.*

Definitions of CSA's for each regular type are given in Section 4.1. The corresponding compatible points are given in Section 4.2. Finally, an explicit tame corestriction is defined for each of these specified CSA's of regular type in Section 4.3. Tame corestriction is an important tool in the theory of regular strata and formal types.

4.1 Tori of Regular Type

First we recall the situation for $\mathrm{G} = \mathrm{GL}_n$ and $\gamma = (n)$ the Coxeter class for $W(A_{n-1})$ outlined in [6, Remark 3.14]. Note that a torus of arbitrary type $\gamma \in W(A_{n-1})$ can be defined by taking a block-diagonal embedding of appropriately sized tori of Coxeter type.

4.1.1 A Torus in $\widehat{\mathrm{GL}_n}$ of Type $(n) \in W(A_n)$

Let ξ be a primitive n^{th} root of unity and define u to be an n^{th} root of z. Define $\omega_{\mathrm{GL}_n,(n)} := \sum_{i=1}^{n-1} e_{i,i+1} + ze_{n,1}$. This element of $\widehat{\mathfrak{gl}_n}$ is regular semisimple over $E := k(\!(u)\!)$. In particular, $\mathrm{Ad}(g_{\mathrm{GL}_n,(n)}) \sum_{i=1}^{n} \xi^i u e_{i,i} = \omega_{\mathrm{GL}_n,(n)}$, where $g_{\mathrm{GL}_n,(n)}$ is the Vandermonde matrix $\sum_{i,j=1}^{n} \xi^{(i-1)j} u^{i-1} e_{i,j}$ (with inverse $g_{\mathrm{GL}_n,(n)}^{-1} = \frac{1}{n} \sum_{i=1}^{n} \xi^{(n-i)(j-1)} u^{1-i} e_{i,j}$). Set $\Gamma := \mathrm{Gal}(E/F)$. The corresponding CSA is

$$\begin{aligned}
\big(\mathrm{Ad}(g_{\mathrm{GL}_n,(n)})(\mathfrak{t}(E))\big)^{\Gamma} &= \Big(\mathrm{Ad}(g_{\mathrm{GL}_n,(n)})(\mathfrak{z}_{\widehat{\mathfrak{gl}_n}}(\textstyle\sum_{i=1}^{n} \xi^i u e_{i,i}))\Big)^{\Gamma} \\
&= \Big(\mathfrak{z}_{\widehat{\mathfrak{gl}_n}}(\mathrm{Ad}(g_{\mathrm{GL}_n,(n)})(\textstyle\sum_{i=1}^{n} \xi^i u e_{i,i}))\Big)^{\Gamma} \\
&= \Big(\mathfrak{z}_{\widehat{\mathfrak{gl}_n}}(\omega_{\mathrm{GL}_n,(n)})\Big)^{\Gamma} \\
&= \mathfrak{z}_{\widehat{\mathfrak{gl}_n}}(\omega_{\mathrm{GL}_n,(n)}) \\
&= \textstyle\bigoplus_{i=0}^{n-1} F \cdot \omega_{\mathrm{GL}_n,(n)}^{i}.
\end{aligned}$$

The corresponding torus is $F[\omega_{\mathrm{GL}_n,(n)}]^{\times}$.

It remains to show that this torus is indeed a Coxeter torus (i.e., a torus of Coxeter type). Let σ be the generating element of Γ determined by $u \mapsto \xi u$. Then

$$\sigma(g_{\mathrm{GL}_n,(n)}) = \sum_{i,j=1}^{n} \xi^{(i-1)(j+1)} u^{i-1} e_{i,j} = g_{\mathrm{GL}_n,(n)} \left(e_{1,n} + \sum_{i=2}^{n} e_{i,i-1} \right).$$

Since $g^{-1}\sigma(g) = (e_{1,n} + \sum_{i=2}^{n} e_{i,i-1})$ is a representative for the n-cycle

$$w : \chi_1 \mapsto \chi_2 \mapsto \ldots \mapsto \chi_n \mapsto \chi_1$$

in $W(A_n)$, it follows that this torus has Coxeter type and $g_{\mathrm{GL}_n,(n)}$ is a w-diagonalizer.

Example 4.2. *Let $n = 4$. Then*

$$\omega_{\mathrm{GL}_4,(4)} = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ z & & & \end{bmatrix}.$$

The corresponding Cartan subalgebra of type (4) *in* $\widehat{\mathfrak{gl}_4}$ *is given by*

$$\left\{ \begin{bmatrix} a & b & c & d \\ dz & a & b & c \\ cz & dz & a & b \\ bz & cz & dz & a \end{bmatrix} : a, b, c, d \in F \right\}.$$

Now let $\mathrm{G} = \mathrm{GSp}_{2n}$. An explicit definition of a torus for each regular type in $W(C_n)$ is given in the following sections.

4.1.2 Type $(-n)$

There are two slightly different cases depending on whether n is even or odd. In either case, let ξ be a primitive $2n^{\text{th}}$ root of unity and let u be a $2n^{\text{th}}$ root of z. First suppose that n is even. Define

$$\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}} := \sum_{i=1}^{n-1} e_{i,i+1} - \sum_{i=n}^{2n-1} e_{i,i+1} + z e_{2n,1}.$$

Then $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}} \in \widehat{\mathfrak{gsp}_{2n}}$ is a regular semisimple element over $E := k((u))$. For convenience, the diagonalizers for the tori in $\widehat{\mathrm{GSp}_{2n}}$ are defined in terms of the diagonalizer $g_{\mathrm{GL}_{2n},(2n)}$ for $\widehat{\mathrm{GL}_{2n}}$, using the natural embedding of $\widehat{\mathrm{GL}_{2n}}$ into $\widehat{\mathrm{GSp}_{2n}}$.

Set

$$\alpha := \left(\sum_{i=1}^{n} e_{i,i} + \sum_{i=n+1}^{2n} (-1)^i e_{i,i} \right), \ \beta := \left(\sum_{i=1}^{n} e_{i,n+1-i} + \sum_{i=n+1}^{2n} e_{i,i} \right), \text{ and}$$

$$\gamma := \left(\sum_{i=1}^{n} \xi^{n-i} e_{i,i} + \sum_{i=n+1}^{2n} e_{i,i} \right).$$

Define $g_{\mathrm{GSp}_{2n},(-n),\mathrm{even}} := \alpha g_{\mathrm{GL}_{2n},(2n)} \beta\gamma$. Then

$$\begin{aligned}
\mathrm{Ad}(g_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}) \left(u \textstyle\sum_{i=1}^{n} \xi^{n+1-i} e_{i,i} + u \sum_{i=n+1}^{2n} \xi^i e_{i.i} \right) &= \\
\mathrm{Ad}(\alpha g_{\mathrm{GL}_{2n},(2n)} \beta) \left(u \textstyle\sum_{i=1}^{n} \xi^{n+1-i} e_{i,i} + u \sum_{i=n+1}^{2n} \xi^i e_{i.i} \right) &= \\
\mathrm{Ad}(\alpha g_{\mathrm{GL}_{2n},(2n)}) \left(u \textstyle\sum_{i=1}^{2n} \xi^i e_{i,i} \right) &= \\
\mathrm{Ad}(\alpha)(\omega_{\mathrm{GL}_{2n},(2n)}) &= \\
\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}.
\end{aligned}$$

The corresponding CSA is

$$\mathfrak{z}_{\widehat{\mathfrak{gsp}_{2n}}}(\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}) = F \cdot I \oplus \bigoplus_{i=1}^{n} \omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}^{2i-1}.$$

This can easily be proven by noticing that I and any power of $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$ commutes with $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$, but exactly I and the odd powers of $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$ are similitudes by Lemma 2.1. Since $\widehat{\mathfrak{gsp}_{2n}}$ has rank $n+1$, these linearly independent elements span the CSA, proving the claim. The corresponding torus is $F[\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}]^\times \cap \widehat{\mathrm{GSp}_{2n}}$. We remark that it may not be immediately clear why the factor of γ is present in the definition of $g_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$; this factor makes $g_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$ a similitude. The proof of this fact is tedious and is therefore omitted.

Suppose that n is odd. Define

$$\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{odd}} := \sum_{i=1}^{n} e_{i,i+1} - \sum_{i=n+1}^{2n-1} e_{i,i+1} + z e_{2n,1}.$$

Again, $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}} \in \widehat{\mathfrak{gsp}_{2n}}$ is a regular semisimple element over E. The diagonalizer $g_{\mathrm{GSp}_{2n},(-n),\mathrm{odd}}$ is defined similarly: $g_{\mathrm{GSp}_{2n},(-n),\mathrm{odd}} := \alpha g_{\mathrm{GL}_{2n},(2n)} \beta\gamma$. Then

$$\mathrm{Ad}(g_{\mathrm{GSp}_{2n},(-n),\mathrm{odd}}) \left(u \sum_{i=1}^{n} \xi^{n+1-i} e_{i,i} + u \sum_{i=n+1}^{2n} \xi^{i} e_{i.i} \right) = \omega_{\mathrm{GSp}_{2n},(-n),\mathrm{odd}}.$$

It remains to show that the associated tori $S_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$ and $S_{\mathrm{GSp}_{2n},(-n),\mathrm{odd}}$ corresponds to the Coxeter class $(-n)$ in $W(C_n)$. Note that the coset representative in $W(C_n)$ corresponding to $g^{-1}_{\mathrm{GSp}_{2n},(-n),\mathrm{even}} \sigma(g_{\mathrm{GSp}_{2n},(-n),\mathrm{even}})$ is the same coset representative corresponding to $(g_{\mathrm{GL}_{2n},(2n)}\beta)^{-1}\sigma(g_{\mathrm{GL}_{2n},(2n)}\beta)$; the column and row scaling matrices α and γ do not change the representative. Then the coset representative is given by

$$\begin{aligned} y^{-1} g^{-1}_{\mathrm{GL}_{2n},(2n)} \sigma(g_{\mathrm{GL}_{2n},(2n)})\beta &= \beta^{-1} \left(e_{1,2n} + \textstyle\sum_{i=2}^{2n} e_{i,i-1} \right) \beta \\ &= \textstyle\sum_{i=1}^{n-1} e_{i,i+1} + e_{n,2n} + e_{n+1,1} + \sum_{i=n+2}^{2n} e_{i,i-1}, \end{aligned}$$

which corresponds to the $(-n)$-cycle

$$w : \pm\chi_n \mapsto \pm\chi_{n-1} \mapsto \ldots \mapsto \pm\chi_2 \mapsto \pm\chi_1 \mapsto \mp\chi_n.$$

Hence the torus $S_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$ is indeed of type $(-n)$ and $g_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$ is a w-diagonalizer. The analogous statements all follow similarly for the case that n is odd.

We can more generally define a regular semisimple element

$$\omega_{\mathrm{GSp}_{2n},(-n)} := \sum_{i=1}^{n-1} e_{i,i+1} + (-1)^{n+1} e_{n,n+1} - \sum_{i=n+1}^{2n-1} e_{i,i+1} + z e_{2n,1},$$

which equals $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$ when n is even and $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{odd}}$ when n is odd. Below are two examples of $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{even}}$ and its centralizer in the case that $n = 2$ and $n = 3$.

Example 4.3. *Let $n = 2$. Note that $\omega_{\mathrm{GL}_4,(4)}$ is not an element of $\widehat{\mathfrak{gsp}_4}$ via the obvious embedding of $\widehat{\mathfrak{gsp}_4}$ into $\widehat{\mathfrak{gl}_4}$. Instead we take*

$$\omega_{\mathrm{GSp}_4,(-2)} = \left[\begin{array}{cc|cc} & 1 & & \\ & & -1 & \\ \hline & & & -1 \\ z & & & \end{array}\right].$$

The corresponding Cartan subalgebra of type (-2) in $\widehat{\mathfrak{gsp}_4}$ is given by

$$\left\{ \left[\begin{array}{cc|cc} a & b & & c \\ cz & a & -b & \\ \hline & -cz & a & -b \\ bz & & -cz & a \end{array}\right] : a, b, c \in F \right\}.$$

Example 4.4. *Let $n = 3$. Then*

$$\omega_{\mathrm{GSp}_6,(-3)} = \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & -1 & \\ z & & & & & \end{array}\right].$$

The corresponding Cartan subalgebra of type (-3) in $\widehat{\mathfrak{gsp}_6}$ is given by

$$\left\{ \left[\begin{array}{ccc|ccc} a & b & & c & & d \\ dz & a & b & & -c & \\ & dz & a & b & & c \\ \hline cz & & dz & a & -b & \\ & -cz & & -dz & a & -b \\ bz & & cz & & -dz & a \end{array}\right] : a, b, c, d \in F \right\}.$$

4.1.3 Type $\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)$ (for $d|2n$ Even)

The main result of this section is the definition of an explicit CSA of type $\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)$ as a block-diagonal embedding of $\frac{2n}{d}$ copies of CSA's in $\widehat{\mathfrak{gsp}_d}$ of type $\left(-\frac{d}{2}\right)$. We first define the $\frac{2n}{d}$ block-embeddings of copies of $\omega_{\mathrm{GSp}_{2d},(-d)}$ into $\widehat{\mathfrak{gsp}_{2n}}$. The j^{th} block-embedding $\omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j}$ of $\omega_{\mathrm{GSp}_{2d},(-d)}$ is defined explicitly by

$$\begin{aligned}\omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j} := & \textstyle\sum_{i=1}^{\frac{d}{2}} e_{\left(\frac{d}{2}\right)(j-1)+i,\left(\frac{d}{2}\right)(j-1)+i+1} + (-1)^{\frac{d}{2}+1} e_{\left(\frac{d}{2}\right)(j-1),2n-\left(\frac{d}{2}\right)j+1} - \\ & - \textstyle\sum_{i=1}^{\frac{d}{2}} e_{2n-\left(\frac{d}{2}\right)j+i,2n-\left(\frac{d}{2}\right)j+i+1} + z e_{2n-\left(\frac{d}{2}\right)(j-1),\left(\frac{d}{2}\right)(j-1)+1}.\end{aligned}$$

Let ξ be a primitive d^{th}-root of unity and let u be a d^{th}-root of z. Define

$$\omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)} := \textstyle\sum_{j=1}^{\frac{2n}{d}} a_j \omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j},$$

for some nonzero a_j with distinct modulus (e.g., $a_j = j$). Then $\omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)}$ is regular semisimple over $E := k(\!(u)\!)$, with corresponding CSA given by

$$\mathfrak{z}_{\widehat{\mathfrak{gsp}_{2n}}}\left(\omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)}\right) = F \cdot I \oplus \bigoplus_{j=1}^{\frac{2n}{d}} \left(\bigoplus_{i=1}^{\frac{d}{2}} F \cdot \omega^{2i-1}_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j}\right).$$

Define g' to be an appropriate block-diagonal embedding of copies of $g_{\mathrm{GSp}_d,(-d),\mathrm{even}}$ or $g_{\mathrm{GSp}_d,(-d),\mathrm{odd}}$, depending on whether d is even or odd respectively. It is straightforward to see from arguments in Section 4.1.2 (using the block-diagonal form of g') that

$$\begin{aligned}\mathrm{Ad}(g')\left(u\sum_{i=1}^{\frac{d}{2}} \xi^{\frac{d}{2}+1-i} e_{(j-1)(\frac{2n}{d})+i,(j-1)(\frac{2n}{d})+i} + u\sum_{i=\frac{d}{2}+1}^{d} \xi^{i} e_{2n-j\frac{2n}{d}+i,2n-j\frac{2n}{d}+i}\right) \\ = \omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j}.\end{aligned}$$

Hence g' is a w-diagonalizer, for w the $\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)$-cycle defined by

$$w : \pm\chi_{\frac{d}{2}j} \mapsto \pm\chi_{\frac{d}{2}j-1} \mapsto \ldots \mapsto \pm\chi_{\frac{d}{2}(j-1)+1} \mapsto \mp\chi_{\frac{d}{2}j},$$

for all $1 \leq j \leq \frac{2n}{d}$.

Example 4.5. *Let $n = 2$. Then*

$$\omega_{\mathrm{GSp}_4,((-1)^2),1} = \left[\begin{array}{cc|cc} & & & 1 \\ & & & \\ \hline & & & \\ z & & & \end{array}\right] \text{ and } \omega_{\mathrm{GSp}_4,((-1)^2),2} = \left[\begin{array}{cc|cc} & & & \\ & & 1 & \\ \hline & z & & \\ & & & \end{array}\right].$$

The corresponding CSA $\mathfrak{z}\left(\omega_{\mathrm{GSp}_4,((-1)^2),1} + 2\omega_{\mathrm{GSp}_4,((-1)^2),2}\right)$ of type $((-1)^2)$ is given by

$$\left\{ \left[\begin{array}{cc|cc} a & & & b \\ & a & c & \\ \hline & cz & a & \\ bz & & & a \end{array}\right] : a, b, c \in F \right\}.$$

4.1.4 Type (n) (for n Odd)

Let ξ be a primitive $2n^{\text{th}}$ root of unity and let u be an n^{th} root of z. Define

$$\omega_{\mathrm{GSp}_{2n},(n)} := \sum_{i=1}^{n-1} e_{i,i+1} + ze_{n,1} - \left(\sum_{i=n+1}^{2n} e_{i,i+1} + ze_{2n,n+1} \right).$$

Then $\omega_{\mathrm{GSp}_{2n},(n)} \in \widehat{\mathfrak{gsp}_{2n}}$ is regular semisimple over $E := k(\!(u)\!)$. This regular semisimple element can be viewed as a block-diagonal embedding of copies of $\omega_{\mathrm{GL}_n,(n)}$ and $-\omega_{\mathrm{GL}_n,(n)}$:

$$\omega_{\mathrm{GSp}_{2n},(n)} = \left[\begin{array}{c|c} \omega_{\mathrm{GL}_n,(n)} & \\ \hline & -\omega_{\mathrm{GL}_n,(n)} \end{array}\right].$$

To define the diagonalizer $g_{\mathrm{GSp}_{2n},(n)}$, first set automorphisms g', β, and γ as follows:

$$g' := \sum_{i,j=1}^{n} (\xi^2)^{(i-1)j} u^{i-1} e_{i,j} + \sum_{i,j=1}^{n} (\xi^2)^{(i-1)j} u^{i-1} e_{i+n,j+n},$$

$$\beta := \left(\sum_{i=1}^{n} e_{i,n+1-i} + \sum_{i=n+1}^{2n} e_{i,i} \right), \ \gamma := \left(\sum_{i=1}^{n} \xi^{n-i} e_{i,i} + \sum_{i=n+1}^{2n} e_{i,i} \right).$$

The automorphism g' can be viewed as a block-diagonal embedding of two copies of the diagonalizer $g_{\mathrm{GL}_n,(n)} = \sum_{i,j=1}^{n} (\xi^2)^{(i-1)j} u^{i-1} e_{i,j}$ for $\omega_{\mathrm{GL}_n,(n)}$. Note that the expression for $g_{\mathrm{GL}_n,(n)}$ here appears slightly from that in Section 4.1.1; this is because ξ is now assumed to be a $2n^{\text{th}}$-root of unity. Define $g_{\mathrm{GSp}_{2n},(n)} := g'\beta\gamma$. To show that $g_{\mathrm{GSp}_{2n},(n)}$ is a w-diagonalizer, first note that

$$\begin{aligned}
\mathrm{Ad}(g_{\mathrm{GSp}_{2n},(n)}) \left(u \textstyle\sum_{i=1}^{n} (\xi^2)^{n+1-i} e_{i,i} - u \sum_{i=1}^{n} (\xi^2)^{i} e_{i+n,i+n} \right) &= \\
\mathrm{Ad}(g'\beta) \left(u \textstyle\sum_{i=1}^{n} (\xi^2)^{n+1-i} e_{i,i} - u \sum_{i=1}^{n} (\xi^2)^{i} e_{i+n,i+n} \right) &= \\
\mathrm{Ad}(g') \left(u \textstyle\sum_{i=1}^{n} (\xi^2)^{i} e_{i,i} - u \sum_{i=1}^{n} (\xi^2)^{i} e_{i+n,i+n} \right) &= \\
\omega_{\mathrm{GSp}_{2n},(n)}.
\end{aligned}$$

The corresponding CSA is

$$\mathfrak{z}_{\widehat{\mathfrak{gsp}_{2n}}}(\omega_{\mathrm{GSp}_{2n},(n)}) = F \cdot I \oplus \bigoplus_{i=1}^{n} \omega_{\mathrm{GSp}_{2n},(n)}^{2i-1}.$$

This can be proven using the same arguments in Section 4.1.2. The purpose of the factor of γ in the definition of $g_{\mathrm{GSp}_{2n},(n)}$ is that it makes $g_{\mathrm{GSp}_{2n},(n)}$ a similitude.

Let σ be the generator of $\mathrm{Gal}(E/F)$ determined by $u \mapsto \xi^2 u$. It remains to show that $g_{\mathrm{GSp}_{2n},(n)}^{-1} \sigma(g_{\mathrm{GSp}_{2n},(n)})$ is a representative for $w \in (n)$. The factor z in $g_{\mathrm{GSp}_{2n},(n)}$ can be ignored in this computation; hence we consider the automorphism

$\beta^{-1}(g')^{-1}\sigma(g')\beta$. The upper-left block is given by

$$\left(\sum_{i=1}^{n} e_{i,n+1-i}\right)^{-1} \left(g_{\mathrm{GL}_n,(n)}^{-1}\sigma(g_{\mathrm{GL}_n,(n)})\right) \left(\sum_{i=1}^{n} e_{i,n+1-i}\right),$$

which represents the permutation $\chi_n \mapsto \chi_{n-1} \mapsto \ldots \mapsto \chi_2 \mapsto \chi_1 \mapsto \chi_n$. Meanwhile, the lower-right block is given by and the lower-right block is given by $g_{\mathrm{GL}_n,(n)}^{-1}\sigma(g_{\mathrm{GL}_n,(n)})$, which represents the permutation $-\chi_n \mapsto -\chi_{n-1} \mapsto \ldots \mapsto -\chi_2 \mapsto -\chi_1 \mapsto -\chi_n$. Hence the above product represents a positive n-cycle defined by

$$\pm\chi_n \mapsto \pm\chi_{n-1} \mapsto \ldots \mapsto \pm\chi_2 \mapsto \pm\chi_1 \mapsto \mp\chi_n.$$

An example of $\omega_{\mathrm{GSp}_{2n},(-n),\mathrm{odd}}$ and its centralizer is given below.

Example 4.6. *Let $n = 3$. Then*

$$\omega_{\mathrm{GSp}_6,(3)} = \left[\begin{array}{ccc|ccc} & 1 & & & & \\ & & 1 & & & \\ z & & & & & \\ \hline & & & & -1 & \\ & & & & & -1 \\ & & & -z & & \end{array}\right].$$

The corresponding Cartan subalgebra of type (3) *in* $\widehat{\mathfrak{gsp}_6}$ *is given by*

$$\left\{ \left[\begin{array}{ccc|ccc} a+d & b & c & & & \\ cz & a+d & b & & & \\ bz & cz & a+d & & & \\ \hline & & & a-d & -b & -c \\ & & & -cz & a-d & -b \\ & & & -bz & -cz & a-d \end{array} \right] : a,b,c,d \in F \right\}.$$

4.1.5 Type $\left(d^{\frac{n}{d}}\right)$ (for d Odd)

The main result of this section is the definition of an explicit CSA of type $\left(d^{\frac{n}{d}}\right)$ as a block-diagonal embedding of $\frac{n}{d}$ copies of CSA's in $\widehat{\mathfrak{gsp}_{2d}}$ of type (d). First we define the $\frac{n}{d}$ block-embeddings of copies of $\omega_{\mathrm{GSp}_{2d},(d)}$ into $\widehat{\mathfrak{gsp}_{2n}}$. For $d > 1$, define

$$\begin{aligned} \omega_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right),j} := & \sum_{i=1}^{d-1} e_{d(j-1)+i,d(j-1)+i+1} + ze_{dj,d(j-1)+1} - \\ & - \sum_{i=1}^{d-1} e_{2n-dj+i,2n-dj+i+1} - ze_{2n-d(j-1),2n-dj+1} \end{aligned}$$

for all $1 \leq j \leq \frac{n}{d}$. Let ξ be a primitive $2d^{\text{th}}$-root of unity and let u be a d^{th}-root of z. Define

$$\omega_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right)} := \sum_{j=1}^{\frac{n}{d}} a_j \omega_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right),j},$$

for some nonzero a_j with distinct modulus (e.g., $a_j = j$). Then $\omega_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right)}$ is regular semisimple over $E := k(\!(u)\!)$, with corresponding CSA given by

$$\mathfrak{z}_{\widehat{\mathfrak{gsp}_{2n}}}\left(\omega_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right)}\right) = F \cdot I \oplus \bigoplus_{j=1}^{\frac{n}{d}} \left(\bigoplus_{i=1}^{d} F \cdot \omega^{2i-1}_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right),j} \right).$$

Define g' to be a block-diagonal embedding of copies of $g_{\mathrm{GSp}_{2n},(n)}$. It is straightforward to see from arguments in Section 4.1.4 (and the block-diagonal form of g') that

$$\mathrm{Ad}(g')\left(u\sum_{i=1}^{d}\xi^{2(d+1-i)}e_{(j-1)d+i,(j-1)d+i}-u\sum_{i=1}^{d}\xi^{2i}e_{2n-jd+i,2n-jd+i}\right)=\omega_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right),j}.$$

Hence g' is a w-diagonalizer, for w the $\left(d^{\frac{n}{d}}\right)$-cycle defined by

$$\pm\chi_{dj}\mapsto\pm\chi_{dj-1}\mapsto\ldots\mapsto\pm\chi_{d(j-1)+1}\mapsto\pm\chi_{d(j-1)}\mapsto\mp\chi_{dj},$$

for all $1\leq j\leq\frac{2n}{d}$.

In the specific case that $d=1$, define

$$\omega_{\mathrm{GSp}_{2n},(1^n),j}:=ze_{j,j}-ze_{2n+1-j,2n+1-j}$$

for each $1\leq j\leq n$. The corresponding CSA

$$\mathfrak{z}_{\widehat{\mathfrak{gsp}_{2n}}}(\sum_{i=1}^{n}j\omega_{\mathrm{GSp}_{2n},(1^n),j}=F\cdot I\oplus\bigoplus_{i=1}^{n}F\cdot\omega_{\mathrm{GSp}_{2n},(1^n),j}$$

is the usual split CSA in $\widehat{\mathfrak{gsp}_{2n}}$. It is easy to see that this CSA is of type (1^n).

4.2 Compatible Filtrations

Let S be a maximal torus in $\widehat{\mathrm{G}}$ with CSA $\mathfrak{s}$. Compatibility and graded compatibility between a Cartan subalgebra $\mathfrak{s}$ and a point x in $\mathscr{B}$ is originally defined in [6, Definition 3.2]. We recall the definition here.

Definition 4.7. *A point $x\in\mathscr{B}$ is* compatible *with $\mathfrak{s}$ if the filtration induced by x on $\mathfrak{s}$ is the (rescaled) Moy-Prasad filtration on $\mathfrak{s}$, i.e., $\mathfrak{s}_r=\widehat{\mathfrak{g}}_{x,r}\cap\mathfrak{s}$ for all*

r. *If* $x \in \mathscr{A}_0$, x *is* graded compatible *with* $\mathfrak{s}$ *if* $\mathfrak{s}(r) = \widehat{\mathfrak{g}}_x(r) \cap \mathfrak{s}$ *for all* r *and* $\mathfrak{s}(0) \subset \widehat{\mathfrak{t}}(0) = \mathfrak{t}$.

For each regular class γ in $W(C_n)$, we describe a point $x \in \overline{\mathscr{A}_0}$ compatible with the corresponding CSA of type γ (as defined in Sections 4.1.3 and 4.1.5) that is unique with respect to the following properties:

- $\mathrm{Crit}_x(\mathfrak{gsp}_{2n}) = \frac{1}{d}\mathbb{Z}$, where d is the order of an element in γ, and
- $\mathfrak{o}^{2n} \in \left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$.

These points are particularly well-behaved in that $\left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$ corresponds to a uniform symplectic lattice chain of period the corresponding integer d.

4.2.1 Type $\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)$ (for $d|2n$ Even)

Suppose that $\mathfrak{s}$ is a CSA of type $(-n)$ as defined in Section 4.1.2. Assume that $x \in \overline{\mathscr{A}_0}$ is graded compatible with $\mathfrak{s}$. Then

$$\omega_{\mathrm{GSp}_{2n},(-n)} \in \mathfrak{s}\left(\frac{1}{2n}\right) = \left(\widehat{\mathfrak{gsp}_{2n}}\right)_x\left(\frac{1}{2n}\right) \cap \mathfrak{s}$$

implies that $1-2\chi_1(x) = 2\chi_n(x) = (\chi_i - \chi_{i+1})(x) = \frac{1}{2n}$ for all $1 \le i \le n-1$. These equations uniquely determine x to be the barycenter of the fundamental alcove, with coordinates

$$x_i = \frac{2n-2i+1}{4n}.$$

Then

$$\widehat{V}_{x,\frac{2n-2i+1}{4n}+k} = \begin{cases} z^k \cdot \mathfrak{o}\text{-sp}\left\{e_1,\ldots,e_i,ze_{i+1},ze_{i+2},\ldots,ze_{2n}\right\}, & 1 \le i < 2n \\ z^k \cdot \mathfrak{o}^{2n}, & i = 2n \end{cases}$$

has $\mathrm{Crit}_x(V) = \frac{1}{4n} + \frac{1}{2n}\mathbb{Z}$. This filtration is uniform and corresponds to an obvious complete symplectic lattice chain with $\widehat{V}_{x,\frac{-2n+1}{4n}} = \mathfrak{o}^{2n}$.

Now consider the more general case. Suppose $\mathfrak{s}$ is the CSA of type $\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)$ defined in Section 4.1.3. Suppose $x \in \overline{\mathscr{A}_0}$ is graded compatible with $\mathfrak{s}$. Then

$$\omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)} \in \mathfrak{s}\left(\frac{1}{d}\right) = \left(\widehat{\mathfrak{gsp}_{2n}}\right)_x \left(\frac{1}{d}\right) \cap \mathfrak{s}$$

implies that

$$1 - 2\chi_{\frac{d}{2}(j-1)+1}(x) = 2\chi_{\frac{d}{2}j}(x) = \left(\chi_{\frac{d}{2}(j-1)+i} - \chi_{\frac{d}{2}(j-1)+i+1}\right)(x) = \frac{1}{d}$$

for all $1 \leq i \leq \frac{d}{2} - 1$ and $1 \leq j \leq \frac{2n}{d}$. These equations uniquely determine x to be the barycenter of the facet of the fundamental alcove determined by the vanishing of the roots $\chi_{\frac{d}{2}j} - \chi_{\frac{d}{2}j+1}$ for each $1 \leq j \leq \frac{2n}{d}$. The coordinates for x are given by

$$x_i = \frac{d - 2(i - \lfloor \frac{2i-2}{d} \rfloor \frac{d}{2}) + 1}{2d}.$$

The filtration $\left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$ is uniform with $\mathrm{Crit}_x(V) = \frac{1}{2d} + \frac{1}{d}\mathbb{Z}$ and $\widehat{V}_{x,\frac{-d+1}{2d}} = \mathfrak{o}^{2n}$. In particular, this point corresponds to a uniform symplectic lattice chain filtration of period d.

Example 4.8. *Let $\mathfrak{s}$ be the CSA in $\widehat{\mathfrak{gsp}_4}$ of type (-2) defined in Example 4.3. This CSA is graded compatible with $x = \left(\frac{3}{8}, \frac{1}{8}\right)$. A period of the Moy–Prasad filtration at this point is given by*

$$\widehat{V}_{x,-\frac{3}{8}} = \mathfrak{o}^4 \supsetneq \widehat{V}_{x,-\frac{1}{8}} = \mathfrak{o}\text{-}sp\left\{e_1, e_2, e_3, ze_4\right\} \supsetneq$$

$$\widehat{V}_{x,\frac{1}{8}} = \mathfrak{o}\text{-}sp\left\{e_1, e_2, ze_3, ze_4\right\} \supsetneq \widehat{V}_{x,\frac{3}{8}} = \mathfrak{o}\text{-}sp\left\{e_1, ze_2, e_3, ze_4\right\}.$$

This filtration corresponds in a natural way (via translation and scaling of the index) to the complete symplectic lattice chain defined in Example 3.4.

Example 4.9. *Let $\mathfrak{s}$ be the CSA in $\widehat{\mathfrak{gsp}_4}$ of type $((-1)^2)$ defined in Example 4.5. This CSA is graded compatible with $x = \left(\frac{1}{4}, \frac{1}{4}\right)$. A period of the Moy–Prasad filtration at this point is given by*

$$\widehat{V}_{x,-\frac{1}{4}} = \mathfrak{o}^4 \supsetneq \widehat{V}_{x,\frac{1}{4}} = \mathfrak{o}\text{-}sp\left\{e_1, e_2, ze_3, ze_4\right\}.$$

This filtration corresponds to the period 2 uniform symplectic lattice chain defined in Example 3.3.

Example 4.10. *Let $\mathfrak{s}$ be the CSA in $\widehat{\mathfrak{gsp}_4}$ of type (-2) as in Example 4.3. This CSA is graded compatible with $x = \left(\frac{3}{8}, \frac{1}{8}\right)$. A period of the Moy–Prasad filtration at this point is given by*

$$\widehat{V}_{x,-\frac{3}{8}} = \mathfrak{o}^4 \supsetneq \widehat{V}_{x,-\frac{1}{8}} = \mathfrak{o}\text{-}sp\left\{e_1, e_2, e_3, ze_4\right\} \supsetneq$$

$$\widehat{V}_{x,\frac{1}{8}} = \mathfrak{o}\text{-}sp\left\{e_1, e_2, ze_3, ze_4\right\} \supsetneq \widehat{V}_{x,\frac{3}{8}} = \mathfrak{o}\text{-}sp\left\{e_1, ze_2, e_3, ze_4\right\}.$$

This filtration corresponds in a natural way (via translation and scaling of the index) to the complete symplectic lattice chain defined in Example 3.4.

4.2.2 Type $\left(d^{\frac{n}{d}}\right)$ (for d Odd)

Suppose that $\mathfrak{s}$ is a CSA of type (n) (where n is odd) as defined in Section 4.1.4. Assume that $x \in \overline{\mathscr{A}_0}$ is graded compatible with $\mathfrak{s}$. Then

$$\omega_{\mathrm{GSp}_{2n},(n)} \in \mathfrak{s}\left(\frac{1}{n}\right) = \left(\widehat{\mathfrak{gsp}_{2n}}\right)_x \left(\frac{1}{n}\right) \cap \mathfrak{s}$$

implies that $\chi_i - \chi_{i+1} = \frac{1}{n}$ for all $1 \leq i \leq n-1$. An arbitrary point in the locus defined by these equations has coordinates $x_i = \frac{n-i}{n} + c$ for some constant c. In order for a point in this locus to satisfy $\mathrm{Crit}_x(\mathfrak{gsp}_{2n}) = \frac{1}{n}\mathbb{Z}$, it must be the case that $c \in \frac{1}{n}\mathbb{Z}$ (otherwise the critical numbers have $2n$ steps). A point in this discrete locus with the property that $\left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$ contains $\mathfrak{o}^{2n}$ has the property that $\chi_i = -\chi_i$ for all $1 \leq i \leq n$; this defines the unique point x with coordinates

$$x_i = \frac{n-2i+1}{2n}.$$

The corresponding filtration has $\mathrm{Crit}_x V = \frac{1}{n}\mathbb{Z}$ and $\mathfrak{o}^{2n} \in \widehat{V}_{x,\frac{-n+1}{2n}}$. To describe the filtration explicitly, first note that any element of $\frac{1}{n}\mathbb{Z}$ can be written uniquely as $\frac{n-2i+1}{2n} + k$ for some $1 \leq i < n$ and $k \in \mathbb{Z}$. The filtration $\left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$ is defined by

$$\widehat{V}_{x,\frac{n-2i+1}{2n}+k} = \begin{cases} z^k \cdot \bigoplus_{l=0}^{1} \mathfrak{o}\text{-sp}\left\{e_{nl+1}, \ldots, e_{nl+i}, ze_{nl+i+1}, ze_{nl+i+2}, \ldots, ze_{nl+n}\right\}, & 1 \leq i < n \\ z^k \cdot \mathfrak{o}^{2n}, & i = n \end{cases}$$

Now consider the more general case. Suppose $\mathfrak{s}$ is the CSA of type $\left(d^{\frac{n}{d}}\right)$ defined in Section 4.1.5. Suppose $x \in \overline{\mathscr{A}_0}$ is graded compatible with $\mathfrak{s}$. Either $d = 1$ or $d > 1$. First suppose that $d = 1$. The conditions $\mathrm{Crit}_x(\mathfrak{gsp}_{2n}) = \frac{1}{d}\mathbb{Z}$ and $\mathfrak{o}^{2n} \in \left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$

uniquely determine the origin, which corresponds to the period 1 symplectic lattice chain filtration $\left\{z^k\mathfrak{o}^{2n}\right\}_{k\in\mathbb{Z}}$.

Now suppose $d>1$. Then

$$\omega_{\mathrm{GSp}_{2d},(d)} \in \mathfrak{s}\left(\frac{1}{d}\right) = \left(\widehat{\mathfrak{gsp}_{2n}}\right)_x\left(\frac{1}{d}\right)\cap\mathfrak{s}$$

implies that $\chi_{d(j-1)+i}-\chi_{d(j-1)+i+1}=\frac{1}{d}$ for all $1\le i\le d-1$ and $1\le j\le\frac{n}{d}$. There is a unique point x in this locus satisfying these equations with the property that $\mathfrak{o}^{2n}\in\left\{\widehat{V}_{x,r}\right\}_{r\in\mathbb{R}}$; its coordinates are given by

$$x_i=\frac{d-2\left(i-\lfloor\frac{i-1}{d}\rfloor\right)+1}{2d}.$$

The corresponding filtration has $\mathrm{Crit}_x(V)=\frac{1}{d}\mathbb{Z}$ and $\mathfrak{o}^{2n}\in\widehat{V}_{x,\frac{-d+1}{2d}}$. In particular, this filtration corresponds to a uniform symplectic lattice chain filtration with period d.

Example 4.11. *Let $\mathfrak{s}$ be the CSA in $\widehat{\mathfrak{gsp}_6}$ of type (3) defined in Example 4.6. This CSA is graded compatible with $x=\left(\frac{1}{3},0,-\frac{1}{3}\right)$. A period of the Moy–Prasad filtration at this point is given by*

$$\widehat{V}_{x,-\frac{1}{3}}=\mathfrak{o}^6 \supsetneq \widehat{V}_{x,0}=\mathfrak{o}\text{-}sp\left\{e_1,e_2,ze_3,e_4,e_5,ze_6\right\}$$

$$\supsetneq \widehat{V}_{x,\frac{1}{3}}=\mathfrak{o}\text{-}sp\left\{e_1,ze_2,ze_3,e_4,ze_5,ze_6\right\}.$$

4.3 Tame Corestriction

In [6, Lemma 4.2], Bremer and Sage show that there exists a tame corestriction map for general connected, reductive groups G over algebraically closed fields k with characteristic 0. In Proposition 4.12 below, we give a reformulation of [6, Lemma 4.2] with $\mathrm{G} = \mathrm{GSp}_{2n}$. The proof includes explicit definitions of tame corestrictions.

Given a torus S in $\widehat{\mathrm{GSp}_{2n}}$, write $\rho_{\mathfrak{s}} : \widehat{\mathfrak{gsp}_{2n}}^{\vee} \to \mathfrak{s}^{\vee}$ for the restriction map.

Proposition 4.12. *Take $x \in \mathscr{A}_0$, and let (x, r, β) be a graded regular stratum with connected centralizer S. There is a morphism of $\mathfrak{s}$-bimodules $\pi_{\mathfrak{s}} : \widehat{\mathfrak{gsp}_{2n}} \to \mathfrak{s}$ satisfying the following properties:*

1. *$\pi_{\mathfrak{s}}$ restricts to the identity on $\mathfrak{s}$,*

2. *$\pi_{\mathfrak{s}}\left(\left(\widehat{\mathfrak{gsp}_{2n}}\right)_{x,l}\right) = \mathfrak{s}_l$ and $\pi_{\mathfrak{s}}^*(s_l^{\vee}) \subset \left(\widehat{\mathfrak{gsp}_{2n}}\right)_{x,l}^{\vee}$,*

3. *the kernel of the restriction map*

$$\overline{\rho}_{s,l} : (\pi_{\mathfrak{s}}^*(\mathfrak{s}^{\vee}) + (\widehat{\mathfrak{gsp}_{2n}})_{x,l-r}^{\vee})/(\widehat{\mathfrak{gsp}_{2n}})_{x,(l-r)+} \to \mathfrak{s}^{\vee}/\mathfrak{s}_{(l-r)+}^{\vee}$$

   is given by the image of $\mathrm{ad}^*((\widehat{\mathfrak{gsp}_{2n}})_{x,-r}^{\vee})(\beta)$, where $\widetilde{\beta} \in (\widehat{\mathfrak{gsp}_{2n}})_{x,-r}^{\vee}$ is any representative of β,

4. *if $Z \in \mathfrak{s}$ and $X \in \widehat{\mathfrak{gsp}_{2n}}$, then $\langle Z, X\rangle_{\nu} = \langle Z, \pi_{\mathfrak{s}}(X)\rangle_{\nu}$,*

5. *$\pi_{\mathfrak{s}}$ (resp. $\pi_{\mathfrak{s}}^*$) commutes with the adjoint action of $N(S)$, and*

6. *the image $\pi_{\mathfrak{s}}^*(\mathfrak{s}_l^{\vee})$ consists of those elements in $\left(\widehat{\mathfrak{gsp}_{2n}}\right)_{x,l}^{\vee}$ stabilized by S.*

Proof. We proceed by defining a map $\pi_{\mathfrak{s}}$ for each S of regular type γ as defined in Section 4.1, and then show that each of these maps $\pi_{\mathfrak{s}}$ satisfies the conditions listed above. For $\gamma = \left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)$, define $\psi_s^j(X) := \frac{1}{d}\langle \omega^{-s}_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j}, X\rangle_\nu$ and

$$\pi_{\mathfrak{s}}(X) := \sum_{s=-\infty}^{\infty} \sum_{j=1}^{\frac{2n}{d}} \psi_s^j(X)\omega^s_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j}. \tag{4.1}$$

For $\gamma = \left(d^{\frac{n}{d}}\right)$, define $\psi_s^j(X) := \frac{1}{2d}\langle \omega^{-s}_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right),j}, X\rangle_\nu$ and

$$\pi_{\mathfrak{s}}(X) := \sum_{s=-\infty}^{\infty} \sum_{j=1}^{\frac{n}{d}} \psi_s^j(X)\omega^s_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right),j}. \tag{4.2}$$

It is easily checked that for small enough s, $\psi_s^j(X) = 0$, and that $\pi_{\mathfrak{s}}$ is a $\mathfrak{s}$-map. A direct (but tedious) computation shows that for $n \in N(T)$, $\pi_{\mathfrak{s}}(\mathrm{Ad}(n)(X)) = \mathrm{Ad}(n)(\pi_{\mathfrak{s}}(X))$, proving (5). Since ψ_s^j vanishes on all entries not on the j^{th} "block" corresponding to $\omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j}$ (resp. $\omega_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right),j}$), it follows that $\pi_{\mathfrak{s}}$ vanishes off the collection of $\frac{2n}{d}$ (resp. $\frac{n}{d}$) blocks. Since

$$\psi_s^i(\omega_{\mathrm{GSp}_{2n},\left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right),j}) = \psi_s^i(\omega_{\mathrm{GSp}_{2n},\left(d^{\frac{n}{d}}\right),j}) = \delta_{j,s}\delta_{i,j},$$

it follows that $\pi_{\mathfrak{s}}$ is the identity on $\mathfrak{s}$, proving (1).

To prove the first part of (2) — i.e., to show $\pi_{\mathfrak{s}}\left(\left(\widehat{\mathfrak{gsp}_{2n}}\right)_{x,l}\right) = \mathfrak{s}_l$ — first note that the compatibility of x with $\mathfrak{s}$ implies that $\mathfrak{s}_l \subset (\widehat{\mathfrak{gsp}_{2n}})_{x,l}$. Then, by (1), $\mathfrak{s}_l = \pi_{\mathfrak{s}}(\mathfrak{s}_l) \subset \pi_{\mathfrak{s}}((\widehat{\mathfrak{gsp}_{2n}})_{x,l})$. To show the reverse inclusion, let $X \in (\widehat{\mathfrak{gsp}_{2n}})_{x,l}$. It suffices to show that $\langle X, Z\rangle_{\frac{dz}{z}} = 0$ for any $Z \in \mathfrak{s}_{-l+} = \mathfrak{s}_l^\perp$. This follows immediately, since

$$\mathfrak{s}_{-l+} = \mathfrak{s} \cap (\widehat{\mathfrak{gsp}_{2n}})_{x,-l+} \subset (\widehat{\mathfrak{gsp}_{2n}})^\perp_{x,l}.$$

To show the second part of (2), let $\alpha \in \mathfrak{s}_l^\vee$. It suffices to show that $\pi_\mathfrak{s}^*(\alpha)((\widehat{\mathfrak{gsp}_{2n}})_{x,l}^\perp) = 0$. But this is true because

$$\begin{aligned}
\pi_\mathfrak{s}^*(\alpha)((\widehat{\mathfrak{gsp}_{2n}})_{x,l}^\perp) &= \pi_\mathfrak{s}^*(\alpha)((\widehat{\mathfrak{gsp}_{2n}})_{x,-l+}) && \text{by [6, Proposition 2.2]} \\
&= \alpha(\pi_\mathfrak{s}((\widehat{\mathfrak{gsp}_{2n}})_{x,-l+})) \\
&= \alpha(\mathfrak{s}_{-l+}) && \text{as shown above} \\
&= 0,
\end{aligned}$$

concluding the proof of (2).

We now consider statement (6). Begin by taking $\alpha \in \mathfrak{s}_l^\vee$ and $s \in S$. If $X \in \widehat{\mathfrak{gsp}_{2n}}$, then

$$\begin{aligned}
\mathrm{Ad}^*(s)\pi_\mathfrak{s}^*(\alpha)(X) &= \pi_\mathfrak{s}^*(\alpha)\left(\mathrm{Ad}(s^{-1})(X)\right) \\
&= \alpha(\pi_\mathfrak{s}(\mathrm{Ad}(s^{-1})(X))) \\
&= \alpha(\mathrm{Ad}(s^{-1})\pi_\mathfrak{s}(X)) && \text{by (5)} \\
&= \alpha(\pi_\mathfrak{s}(X)) && \text{since } \pi_\mathfrak{s}(X) \subset \mathfrak{s} \\
&= \pi_\mathfrak{s}^*(\alpha)(X).
\end{aligned}$$

Then S stabilizes $\pi_\mathfrak{s}^*(\mathfrak{s}^\vee)$. Recall that (2) implies that $\pi_\mathfrak{s}^*(\mathfrak{s}_l^\vee) \subset (\widehat{\mathfrak{gsp}_{2n}})_{x,l}^\vee$. Conversely, suppose that $\alpha \in \widehat{\mathfrak{gsp}_{2n}}^\vee$ is stabilized by S. Choose a functional representative Y for α so that $\alpha(\cdot) = \langle Y, \cdot \rangle_\nu$. Then $\mathrm{Ad}^*(s)(\alpha) = \alpha$ for all $s \in S$ if and only if $\mathrm{Ad}(s)Y = Y$ for all $s \in S$. This implies that $Y \in \mathfrak{s}$. By (4), it follows that $\alpha(X) = \langle Y, X \rangle_\nu = \langle Y, \pi_\mathfrak{s}(X) \rangle_\nu = \alpha(\pi_\mathfrak{s}(X))$ for any $X \in \widehat{\mathfrak{gsp}_{2n}}$. Then $\alpha = \pi_\mathfrak{s}^*(\alpha)$, where $\alpha \in \mathfrak{s}^\vee \subset \widehat{\mathfrak{gsp}_{2n}}^\vee$, and so $\alpha \in \pi_\mathfrak{s}^*(\mathfrak{s}^\vee)$. This implies that the image $\pi_\mathfrak{s}^*(\mathfrak{s}^\vee)$ consists of those elements in $\widehat{\mathfrak{gsp}_{2n}}^\vee$ stabilized by S. Then (6) follows by as a corollary of (2).

The proof of (3) is a corollary of the results proven above. Its proof can be found in the proof of [6, Lemma 4.2]. □

4.4 Regular Strata

In [6, Section 4], Bremer and Sage define regular and graded regular $\widehat{\mathrm{G}}$-strata.

Theorem 4.13. *Any regular $\widehat{\mathrm{GSp}_{2n}}$-stratum can be based at a point x corresponding to a uniform symplectic lattice chain.*

Proof. Let (x, r, β) be a regular stratum with corresponding torus $Z^0(\widetilde{\beta})$ of type γ. By [6, Proposition 4.9], (x, r, β) is conjugate to a graded regular stratum (and regular) $(g\cdot x, r, \mathrm{Ad}^*(g)(\beta))$ with $g\cdot x \in \mathscr{A}_0$. By [6, Corollary 4.10], the set of $x \in \mathscr{A}_0$ which support an S-regular stratum for some S of type γ is precisely Π_γ. In the case that $\gamma = \left(\left(-\frac{d}{2}\right)^{\frac{2n}{d}}\right)$ (for $d|2n$ even), the locus Π_γ is precisely given by the standard uniform symplectic lattice chain filtrations period d. Hence $g^{-1}\cdot x$ corresponds to a uniform symplectic lattice chain filtration. In the case that $\gamma = \left(d^{\frac{n}{d}}\right)$ (for d odd), the discrete subset determined by the intersection of Π_γ is precisely given by standard uniform symplectic lattice chain filtrations of period d. In this case, $g\cdot x$ can without loss of generality be chosen to be one of these discrete points, and thus x corresponds to a uniform symplectic lattice chain filtration. In either case, (x, r, β) is based at a point x corresponding to a uniform symplectic lattice chain filtration. □

By the above theorem, the defining data (x, r, β) for a regular $\widehat{\mathrm{GSp}_{2n}}$-stratum can be replaced by the equivalent data $(P_{\mathscr{L}_x}, e_{\mathscr{L}_x} r, \beta)$, where $\mathscr{L}_x$ is the uniform symplectic lattice chain corresponding to x and

$$\beta \in (\mathfrak{p}_{\mathscr{L}_x}^{e_{\mathscr{L}_x} r} / \mathfrak{p}_{\mathscr{L}}^{e_{\mathscr{L}} r+1})^\vee = ((\widehat{\mathfrak{gsp}_{2n}})_{x,r} / (\widehat{\mathfrak{gsp}_{2n}})_{x,r+})^\vee,$$

mirroring [5, Definition 2.13]. Furthermore, by Proposition 3.7, the definition of stratum containment of regular $\widehat{\mathrm{GSp}_{2n}}$-strata (x, r, β) in flat GSp_{2n}-bundles can be reformulated into a lattice chain theoretic definition analogous to [5, Definition 4.1]. Thus many of the results in [5] for $\widehat{\mathrm{GL}_n}$-strata defined in terms of lattice chains translate into results for regular $\widehat{\mathrm{GSp}_{2n}}$-strata defined in terms of symplectic lattice chains. It is important to note that it is not known whether the same is true for fundamental $\widehat{\mathrm{GSp}_{2n}}$-strata in general.

4.5 Useful Lemmas

In this section, some of the lattice-theoretic results in [5] are translated into results in the setting of GSp_{2n}-bundles. Throughout, assume that x is a point in $\mathscr{B}(\widehat{\mathrm{GSp}_{2n}})$ corresponding to a uniform symplectic lattice chain $\mathscr{L}_x$. We will frequently make use of the fact that there is an embedding of $\widehat{\mathrm{GSp}_{2n}}$ into $\widehat{\mathrm{GL}_{2n}}$ with the property that $(\widehat{\mathrm{GSp}_{2n}})_{x,r} = P_{\mathscr{L}_x}^{e_{\mathscr{L}_x} r} \cap \widehat{\mathrm{GSp}_{2n}}$ for all nonnegative $r \in \mathrm{Crit}_x(\mathfrak{gsp}_{2n})$, where $P_{\mathscr{L}_x}$ is the parahoric subgroup in $\widehat{\mathrm{GL}_{2n}}$ stabilizing the lattice $\mathscr{L}_x$ in F^{2n}. Similarly, $(\widehat{\mathfrak{gsp}_{2n}})_{x,r} = \mathfrak{p}_{\mathscr{L}_x}^{e_{\mathscr{L}_x} r} \cap \widehat{\mathfrak{gsp}_{2n}}$ for all $r \in \mathrm{Crit}_x(\mathfrak{gsp}_{2n})$, where $\mathfrak{p}_{\mathscr{L}_x} \subset \widehat{\mathfrak{gl}_{2n}}$ is the parahoric corresponding to $\mathscr{L}_x$.

Lemma 4.14. *If $p \in (\widehat{\mathrm{GSp}_{2n}})_x$, then $\tau(p)p^{-1} \in (\widehat{\mathfrak{gsp}_{2n}})_{x+}$.*

Proof. Since $p \in (\widehat{\mathrm{GSp}_{2n}})_x \subset P_{\mathscr{L}_x}$, [5, Lemma 4.4] implies that $\tau(p)p^{-1} \in \mathfrak{p}^1_{\mathscr{L}_x}$. But since the gauge action of $(\widehat{\mathrm{GSp}_{2n}})_x$ on $(\widehat{\mathfrak{gsp}_{2n}})_x$ is an action, it follows that $\tau(p)p^{-1} \in (\widehat{\mathfrak{gsp}_{2n}})_x$. Hence

$$\tau(p)p^{-1} \in \mathfrak{p}^1_{\mathscr{L}_x} \cap (\widehat{\mathfrak{gsp}_{2n}})_x = (\widehat{\mathfrak{gsp}_{2n}})_{x+}.$$

□

Lemma 4.15. *Suppose that (x, r, β) is an S-regular stratum. Let β^1_ν, $\beta^2_\nu \in \mathfrak{s}$ be functional representatives for β. If $\mathrm{Ad}(g)(\beta_n u^1) = \beta^2_\nu$ for some $g \in \widehat{\mathrm{GSp}_{2n}}$, then $\beta^1_\nu = \beta^2_\nu$.*

Proof. Let (x, r, β), β^1_ν, β^2_ν, and $g \in \widehat{\mathrm{GSp}_{2n}}$ satisfy the hypotheses. Let $(P_{\mathscr{L}_x}, e_{\mathscr{L}_x} r, \beta')$ be the corresponding regular $\widehat{\mathrm{GL}_{2n}}$-stratum, with corresponding torus S'. Let $(\beta^1_\nu)'$ and $(\beta^2_\nu)'$ represent the embeddings of β^1_ν and β^2_ν respectively in $\mathfrak{s}'$. Since $\mathrm{Ad}(g)((\beta^1_\nu)') = (\beta^2_\nu)'$, [5, Lemma 3.20] implies that $(\beta^1_\nu)' = (\beta^2_\nu)'$, and therefore $\beta^1_\nu = \beta^2_\nu$. □

Lemma 4.16. *Suppose that (x, r, β) is an S-regular $\widehat{\mathrm{GSp}_{2n}}$-stratum and that $\beta_\nu \in \mathfrak{s}_{-r}$ is a functional representative for β. Let $A \in (\widehat{\mathfrak{gsp}_{2n}})^\vee$ be the functional determined by β_ν. Then A determines an element $A_i \in (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,i}$ by restriction. The stabilizer of A_i under the coadjoint action of $(\widehat{\mathrm{GSp}_{2n}})_{x,i}$ is given by $S_i(\widehat{\mathrm{GSp}_{2n}})_{x,(r-i)+}$ whenever $r \geq 2i$, and $(\widehat{\mathrm{GSp}_{2n}})_{x,i}$ whenever $r < 2i$.*

Proof. Let (x, r, β), β_ν, and A satisfy the hypotheses. Let $(P_{\mathscr{L}_x}, e_{\mathscr{L}_x} r, \beta')$ be the corresponding regular $\widehat{\mathrm{GL}_{2n}}$-stratum, S' the corresponding torus in $\widehat{\mathrm{GL}_{2n}}$, and β'_ν the embedding of β_ν into $\widehat{\mathfrak{gl}_{2n}}$. Then the embedding A' of A into $\widehat{\mathfrak{gl}_{2n}}^\vee$ is induced by β'_ν. By [5, Lemma 3.21], A' induces an element $A'_{e_{\mathscr{L}_x} i} \in (\mathfrak{p}_{\mathscr{L}_x}^{e_{\mathscr{L}_x} i})^\vee$ by restriction, for i a nonnegative multiple of $\frac{1}{e_{\mathscr{L}_x}}$, and the stabilizer of $A'_{e_{\mathscr{L}_x} i}$ under the coadjoint action of $P_{\mathscr{L}_x}^{e_{\mathscr{L}_x} i}$ is given by $(S')^{e_{\mathscr{L}_x} i} P^{e_{\mathscr{L}_x} r - e_{\mathscr{L}_x} i + 1}$ whenever $r e_{\mathscr{L}_x} \geq 2 i e_{\mathscr{L}_x}$, and $P_{\mathscr{L}_x}^{i e_{\mathscr{L}_x}}$ whenever $r e_{\mathscr{L}_x} < 2 i e_{\mathscr{L}_x}$. Set A_i equal to the restriction of $A'_{e_{\mathscr{L}_x} i}$ to $\widehat{\mathfrak{gsp}_{2n}}^\vee$. Then A_i is exactly the restriction of A to $(\widehat{\mathfrak{gsp}_{2n}})^\vee_{x,i}$. The stabilizer of A_i under the coadjoint action of $(\widehat{\mathrm{GSp}_{2n}})_{x,i} = P_{\mathscr{L}_x}^{e_{\mathscr{L}_x} i} \cap \widehat{\mathrm{GSp}_{2n}}$ is given by the symplectic similitudes stabilizer of $A'_{e_{\mathscr{L}_x} i}$; i.e.,

$$(S')^{e_{\mathscr{L}_x} i} P^{e_{\mathscr{L}_x} r - e_{\mathscr{L}_x} i + 1} \cap \widehat{\mathrm{GSp}_{2n}} = S_i (\widehat{\mathrm{GSp}_{2n}})_{x,(r-i)+}$$

whenever $r \geq 2i$ and $(\widehat{\mathrm{GSp}_{2n}})_{x,i}$ whenever $r < 2i$. $\square$

Chapter 5

Moduli Spaces of Flat GSp_{2n}-Bundles on $\mathbb{P}^1$

In this chapter, we describe the construction of the moduli space

$$\mathscr{M}^*\left(A^1,\dots,A^m\right)$$

of "framable" flat GSp_{2n}-bundles on $\mathbb{P}^1(\mathbb{C})$ with singular points $\{y_1,\dots,y_m\}$ and formal type A^i at y_i. We also construct a second moduli space

$$\widetilde{\mathscr{M}}^*\left(A^1,\dots,A^m\right)$$

of "framed" flat GSp_{2n}-bundles on $\mathbb{P}^1$ with formal types A^i at y_i. Finally, we show that the prior moduli space is the Hamiltonian reduction of the latter via a torus action. The main result of the chapter is Theorem 5.5.

5.1 Flat GSp_{2n}-Bundles on $\mathbb{P}^1$

Set $k=\mathbb{C}$. Denote by $F_y\cong F$ the field of Laurent series at $y\in\mathbb{P}^1$, and let $\mathfrak{o}_y\subset F_y$ be the ring of power series. Let $\mathcal{G}$ be a trivializable principal GSp_{2n}-bundle on $\mathbb{P}^1$. The space of global trivializations is a $\mathrm{GSp}_{2n}(\mathbb{C})$-torsor. If a basepoint is fixed, each global trivialization ϕ can be identified with an element $g\in\mathrm{GSp}_{2n}(\mathbb{C})$. Given a flat structure ∇ for $\mathcal{G}$, we will write $[\nabla]$ for the connection matrix of ∇ with respect to the basepoint, and $g\cdot[\nabla]$ for $[\nabla]_\phi$.

Define $(\widehat{\mathrm{GSp}_{2n}})_y := \mathrm{GSp}_{2n}\otimes_{\mathcal{O}_{\mathbb{P}^1}} F_y$. The inclusion

$$\mathrm{GSp}_{2n}(\mathbb{C})=\Gamma(\mathbb{P}^1;V)\subset(\widehat{\mathrm{GSp}_{2n}})_y$$

induces a natural $\mathbb{C}$-structure in $(\widehat{\mathrm{GSp}_{2n}})_y$. Furthermore, there is a designated maximal parahoric subgroup $\mathrm{G}_y := \mathrm{GSp}_{2n} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathfrak{o}_y \cong \mathrm{GSp}_{2n}(\mathfrak{o})$.

Let S_y be a torus of regular type γ in $\mathrm{GSp}_{2n}(F_y)$ with the property that $S_y(\mathfrak{o}_y) \subset \mathrm{GSp}_{2n}(\mathfrak{o}_y)$. By Theorem 4.1, there is a unique point x_y that is compatible with S_y and satisfies $\mathfrak{o}_y^{2n} \in \{V_{x,r}\}_{r\in\mathbb{R}}$, where V is the standard representation of $\mathrm{GSp}_{2n}(\mathbb{C})$. In the following, $(\mathcal{G}, \nabla)$ is a flat GSp_{2n}-bundle on $\mathbb{P}^1$, and $A_y \in \mathscr{A}(S_y, r)$ is a formal type associated to $(\mathcal{G}, \nabla)$ at y; i.e., the formal completion $(\mathcal{G}_y, \nabla_y)$ at y has formal type A_y. Denote the corresponding $\mathrm{GSp}_{2n}(F_y)$-stratum by (x_y, r_y, β_y). It can be assumed without loss of generality that S_y has the block-form described in Chapter 4. Let U_y be the unipotent group U_x corresponding to x as defined in Section 3.4.

Definition 5.1. *A* compatible framing *for* $(\mathcal{G}, \nabla)$ *at* y *is an element* $g \in \mathrm{GSp}_{2n}(\mathbb{C})$ *such that the formal completion* $(\mathcal{G}_y, \nabla_y)$ *contains the stratum* (x_y, r_y, β_y). *The flat* GSp_{2n}*-bundle is called* framable *at* y *if there exists such a* g.

Let $\mathbf{A} = \{A_1, \ldots, A_m\}$ be a collection of formal types A_i at points $y_i \in \mathbb{P}^1$.

Definition 5.2. *The category* $\mathscr{C}^*(\mathbf{A})$ *of framable flat* GSp_{2n}*-bundles with formal types* $\mathbf{A}$ *consists of flat* GSp_{2n}*-bundles* $(\mathcal{G}, \nabla)$, *where*

- $\mathcal{G}$ *is a trivializable principal* GSp_{2n}*-bundle on* $\mathbb{P}^1$,
- ∇ *is meromorphic with singular points* $\{y_i\}$,
- $(\mathcal{G}, \nabla)$ *is framable and has formal type* A^i *at* y_i,

and with morphisms given by principal GSp_{2n}*-bundle morphisms compatible with the flat structures. The moduli space of this category is denoted by* $\mathscr{M}^*(\mathbf{A})$*.*

Following [5] and [2], we define an extended moduli space $\widetilde{\mathscr{M}}^*(\mathbf{A})$ for which $\mathscr{M}^*(\mathbf{A})$ can be resolved via symplectic reduction.

Definition 5.3. *The category* $\widetilde{\mathscr{C}}^*(\mathbf{A})$ *of* framed *flat* GSp_{2n}*-bundles with formal types* $\mathbf{A}$ *consists of triples* $(\mathcal{G}, \nabla, \mathbf{g})$*, where*

- $(\mathcal{G}, \nabla)$ *satisfies the first two conditions of Definition 5.2,*
- $\mathbf{g} = (U_{y_1} g_1, \ldots, U_{y_m} g_m)$*, where* g_i *is a compatible framing for* ∇ *at* y_i*,*
- *the formal type* $(A')^i$ *for* $(\mathcal{G}, \nabla)$ *at* y_i *satisfies* $(A')^i|_{\mathfrak{s}_+} = A^i|_{\mathfrak{s}_+}$*.*

The moduli space of this category is denoted by $\widetilde{\mathscr{M}}^*(\mathbf{A})$*.*

Symplectic reduction is an important tool in our constructions of moduli spaces. A list of well-known general properties of moment maps is given below for reference.

Lemma 5.4. *Let* X, X_i *be symplectic varieties, and let* K *and* H *be Lie groups.*

1. *Let* $\rho : H \to K$ *be a Lie group homomorphism, and let* $\rho^* : \mathfrak{h}^\vee \to \mathfrak{g}^\vee$ *be the pullback. Suppose that* K *acts on a variety* X *in a Hamiltonian fashion, with moment map* μ_G*. Then the induced action of* H *on* X *is also Hamiltonian, with moment map* $\mu_H = \rho^* \circ \mu_K$*.*
2. *In particular, this applies if* H *is a Lie subgroup of* K *and* ρ *is the inclusion map.*

3. *Suppose X_1 and X_2 are equipped with Hamiltonian actions by K, with corresponding moment maps $\mu_1 : X_1 \to \mathfrak{g}^\vee$ and $\mu_2 : X_2 \to \mathfrak{g}^\vee$ respectively. Then the induced diagonal action of K on the product $X_1 \times X_2$ is also Hamiltonian with moment map $\mu_1 + \mu_2$.*

In Section 5.2, the formal types A_i at y_i are used to define the *extended orbits* $\mathscr{M}_i$ and $\widetilde{\mathscr{M}_i}$. The proof of the following theorem is given in Section 5.3.

Theorem 5.5. *Let $\mathscr{M}^*(\mathbf{A})$ and $\widetilde{\mathscr{M}}^*(\mathbf{A})$ be the moduli spaces defined above.*

1. *The moduli space $\mathscr{M}^*(\mathbf{A})$ is a symplectic reduction of the product $\prod_i \mathscr{M}_i$:*

$$\mathscr{M}^*(\mathbf{A}) \cong \left(\prod_i \mathscr{M}_i \right) /\!\!/_0 \, \mathrm{GSp}_{2n}(\mathbb{C}).$$

2. *Similarly,*

$$\widetilde{\mathscr{M}}^*(\mathbf{A}) \cong \left(\prod_i \widetilde{\mathscr{M}}_i \right) /\!\!/_0 \, \mathrm{GSp}_{2n}(\mathbb{C}).$$

   Moreover, $\widetilde{\mathscr{M}}^*(\mathbf{A})$ is a symplectic manifold.

3. *Let $S_i = S_{y_i}$. There is a Hamiltonian action of $S_i^\flat$ on $\widetilde{\mathscr{M}}^*(\mathbf{A})$, and $\mathscr{M}^*(\mathbf{A})$ is a symplectic reduction of $\widetilde{\mathscr{M}}^*(\mathbf{A})$ by the group $\prod_i T_i^\flat$.*

5.2 Extended Orbits

In this section, we construct the symplectic manifolds that make up the "local pieces" of the moduli spaces $\widetilde{\mathscr{M}}(\mathbf{A})$ and $\widetilde{\mathscr{M}}^*(\mathbf{A})$. Following the convention of [5], we call these symplectic manifolds *extended orbits.* Without loss of generality, let $y = 0$ be our singular point. We suppress the subscript y from F_y, $\mathfrak{o}_y$, A_y, U_y, etc. In this section, $\widehat{\mathrm{GSp}_{2n}}$ is equivalent to $\mathrm{GSp}_{2n}(F_y)$, and $\widehat{\mathfrak{gsp}_{2n}}$ is equivalent to $\mathfrak{gsp}_{2n} \otimes_{\mathbb{C}} F_y$.

The study of extended orbits requires a familiarity with the relationship between actions on funtionals and their functional representatives. For example, the following lemma (the proof of which follows immediately from definitions) relates the coadjoint action with the adjoint action.

Lemma 5.6. *The map $\widehat{\mathfrak{gsp}_{2n}} \to (\widehat{\mathfrak{gsp}_{2n}})_x^\vee$ determined by ν (i.e., the map defined by $X \mapsto \langle X, \cdot \rangle_\nu$) intertwines the adjoint action of $(\widehat{\mathrm{GSp}_{2n}})_x$ on $\widehat{\mathfrak{gsp}_{2n}}$ with the coadjoint action of $(\widehat{\mathrm{GSp}_{2n}})_x$ on $(\widehat{\mathfrak{gsp}_{2n}})_x^\vee$.*

Proof. Let $Y \in (\widehat{\mathfrak{gsp}_{2n}})_x$ and $p \in (\widehat{\mathrm{GSp}_{2n}})_x$. Then

$$\begin{aligned}
\mathrm{Ad}^*(p)(\langle X, \cdot \rangle_\nu)(Y) &= \mathrm{Ad}(p^{-1})^*(\langle X, \cdot \rangle_\nu)(Y) \\
&= \langle X, \mathrm{Ad}(p^{-1})(Y) \rangle_\nu \\
&= \mathrm{Res}\,[\mathrm{Tr}(X\,\mathrm{Ad}(p^{-1})(Y))\nu] \\
&= \mathrm{Res}\,[\mathrm{Tr}(\mathrm{Ad}(p)(X)Y)\nu] \\
&= \langle \mathrm{Ad}(p)(X), Y \rangle_\nu \\
&= \langle \mathrm{Ad}(p)(X), \cdot \rangle_\nu(Y)
\end{aligned}$$

□

The following proposition relates the coadjoint action with the gauge change action.

Proposition 5.7. *The map $\widehat{\mathfrak{gsp}_{2n}} \to (\widehat{\mathfrak{gsp}_{2n}})_x^\vee$ determined by ν intertwines the gauge action of $(\widehat{\mathrm{GSp}_{2n}})_x$ on $\widehat{\mathfrak{gsp}_{2n}}$ with the coadjoint action of $(\widehat{\mathrm{GSp}_{2n}})_x$ on $(\widehat{\mathfrak{gsp}_{2n}})_x^\vee$.*

Proof. Let $p \in (\widehat{\mathrm{GSp}_{2n}})_x$ and let $\langle X, \cdot \rangle_\nu|_{(\widehat{\mathfrak{gsp}_{2n}})_x} \in (\widehat{\mathfrak{gsp}_{2n}})_x^\vee$ be an arbitrary functional, so that $X \in \widehat{\mathfrak{gsp}_{2n}}$. Note that Lemma 4.14 and [7, Proposition 3.6(1)] implies that $\tau(p)p^{-1} \in (\widehat{\mathfrak{gsp}_{2n}})_{x+} = (\widehat{\mathfrak{gsp}_{2n}})_x^\perp$. Then

$$
\begin{aligned}
\langle p \cdot X, \cdot \rangle|_{(\widehat{\mathfrak{gsp}_{2n}})_x} &= \langle \mathrm{Ad}(p)(X) - \tau(p)p^{-1}, \cdot \rangle|_{(\widehat{\mathfrak{gsp}_{2n}})_x} \\
&= \langle \mathrm{Ad}(p)(X), \cdot \rangle|_{(\widehat{\mathfrak{gsp}_{2n}})_x} - \langle \tau(p)p^{-1}, \cdot \rangle|_{(\widehat{\mathfrak{gsp}_{2n}})_x} && \text{by bilinearity of } \langle \cdot, \cdot \rangle \\
&= \langle \mathrm{Ad}(p)(X), \cdot \rangle|_{(\widehat{\mathfrak{gsp}_{2n}})_x} && \text{since } \tau(p)p^{-1} \in (\widehat{\mathfrak{gsp}_{2n}})_x^\perp \\
&= \mathrm{Ad}^*(p)(\langle X, \cdot \rangle|_{(\widehat{\mathfrak{gsp}_{2n}})_x}) && \text{by Lemma 5.6.}
\end{aligned}
$$

$\square$

Suppose that A is a formal type at 0 stabilized by a torus S with $S(\mathfrak{o}) \subset (\widehat{\mathrm{GSp}_{2n}})_x$. Further, suppose that the corresponding regular stratum (x, r, β) has $r > 0$ (so ∇ is irregular singular at 0). Recall from Section 3.4 that Q_x is the parabolic subgroup corresponding to $(\widehat{\mathrm{GSp}_{2n}})_x$ and that U_x is its unipotent radical. Note that $(\widehat{\mathrm{GSp}_{2n}})_{o+} \subset (\widehat{\mathrm{GSp}_{2n}})_x \subset (\widehat{\mathrm{GSp}_{2n}})_o$ and $(\widehat{\mathrm{GSp}_{2n}})_x / (\widehat{\mathrm{GSp}_{2n}})_{o+} \cong Q_x$. For any subgroup $H \subset (\widehat{\mathrm{GSp}_{2n}})_o$, there is a natural projection of functionals $p_{\mathfrak{h}} : (\widehat{\mathfrak{gsp}_{2n}})_o^\vee \to \mathfrak{h}^\vee$ defined by $p_{\mathfrak{h}} := i_{\mathfrak{h}}^*$, where $i_{\mathfrak{h}} : \mathfrak{h} \to (\widehat{\mathfrak{gsp}_{2n}})_o$ is the natural inclusion. Denote the projection of A onto $(\widehat{\mathfrak{gsp}_{2n}})_{x+}^\vee$ by A^+; i.e., $A^+ = p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(A)$. Denote the $(\widehat{\mathrm{GSp}_{2n}})_x$-coadjoint orbit of A by $\mathscr{O}$ and the $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-coadjoint orbit of A^+ by $\mathscr{O}^+$.

Definition 5.8. *Let A be a formal type with positive depth $r > 0$. Let Q_x be the parabolic subgroup corresponding to $(\widehat{\mathrm{GSp}_{2n}})_x$, and let U_x be the unipotent radical of Q_x. Define the extended orbits $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$ by*

- *$\mathscr{M}(A) \subset (Q_x \backslash \mathrm{GSp}_{2n}(\mathbb{C})) \times (\widehat{\mathfrak{gsp}_{2n}})_x^\vee$ is the subvariety defined by*

$$\mathscr{M}(A) = \left\{(Q_x, \alpha) : p_{(\widehat{\mathfrak{gsp}_{2n}})_x}(\mathrm{Ad}^*(g)(\alpha)) \in \mathscr{O}\right\}, \text{ and}$$

- *$\widetilde{\mathscr{M}}(A) \subset (U_x \backslash \mathrm{GSp}_{2n}(\mathbb{C})) \times (\widehat{\mathfrak{gsp}_{2n}})_x^\vee$ is defined by*

$$\widetilde{\mathscr{M}}(A) = \left\{(U_x, \alpha) : p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(g)(\alpha)) \in \mathscr{O}^+\right\}.$$

Proposition 5.9. *The extended orbits $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$ are isomorphic to symplectic reductions of $T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}$ and $T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}^+$ respectively:*

$$\begin{aligned} \mathscr{M}(A) &\cong T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O} /\!\!/_0 (\widehat{\mathrm{GSp}_{2n}})_x \\ \widetilde{\mathscr{M}}(A) &\cong T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}^+ /\!\!/_0 (\widehat{\mathrm{GSp}_{2n}})_{x+}. \end{aligned}$$

The natural symplectic form on the product $T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}$ descends to a symplectic form on $\mathscr{M}(A)$ (resp. $\widetilde{\mathscr{M}}(A)$).

Note that $T^*((\widehat{\mathrm{GSp}_{2n}})_o)$ is not finite-dimensional. However, [5, Remark 5.12] notes that it is still possible to treat extended orbits as if they were finite-dimensional.

Proof. We proceed to prove the second isomorphism. The proof of the first is similar. The group $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ acts on $T^*((\widehat{\mathrm{GSp}_{2n}})_o)$ by the usual left action $p \cdot (g, \alpha) = (pg, \alpha)$, and on $\mathscr{O}^+$ by the coadjoint action $p \cdot \beta = \mathrm{Ad}^*(p)(\beta)$. On each factor, the action of $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ is Hamiltonian with respect to the standard symplectic forms on cotangent bundles and coadjoint orbits. By Lemma 5.4(3), the moment map for the diagonal action of $(\widehat{\mathrm{GSp}_{2n}})_{x+}$

$$p \cdot (g, \alpha, \beta) := (pg, \alpha, \mathrm{Ad}^*(p)(\beta)) \tag{5.1}$$

is given by the sum of the moment maps on the factors; i.e.,

$$\mu_{(\widehat{\mathrm{GSp}_{2n}})_{x+}} : T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}^+ \to (\widehat{\mathfrak{gsp}_{2n}})^\vee_{x+}$$

is defined by

$$\mu_{(\widehat{\mathrm{GSp}_{2n}})_{x+}}(g, \alpha, \beta) = p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(-\mathrm{Ad}^*(g)(\alpha)) + \beta. \tag{5.2}$$

Hence

$$\mu^{-1}_{(\widehat{\mathrm{GSp}_{2n}})_{x+}}(0) = \left\{(g, \alpha, \beta) : p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(-\mathrm{Ad}^*(g)(\alpha)) = \beta\right\}.$$

Since $U_x \cong (\widehat{\mathrm{GSp}_{2n}})_{x+}/(\widehat{\mathrm{GSp}_{2n}})_{o+}$, it follows that

$$U_x \backslash \mathrm{GSp}_{2n}(\mathbb{C}) \cong (\widehat{\mathrm{GSp}_{2n}})_{x+} \backslash (\widehat{\mathrm{GSp}_{2n}})_o.$$

Thus it is possible to define a map $\mu^{-1}_{(\widehat{\mathrm{GSp}_{2n}})_{o+}}(0) \to \widetilde{\mathscr{M}}(A)$ via

$$(g, \alpha, \beta) \mapsto ((\widehat{\mathrm{GSp}_{2n}})_{x+} g, \alpha).$$

Since the fibers are $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-orbits, this map identifies

$$(\widehat{\mathrm{GSp}_{2n}})_{x+} \backslash \left(\mu^{-1}_{(\widehat{\mathrm{GSp}_{2n}})_{x+}}(0)\right) \cong \widetilde{\mathscr{M}}(A).$$

$\square$

We want to describe an action of $\mathrm{GSp}_{2n}(\mathbb{C})$ on $\mathscr{M}(A)$ (resp. $\widetilde{\mathscr{M}}(A)$). Let $\mathrm{res}: (\widehat{\mathfrak{gsp}_{2n}})_o^\vee \to \mathfrak{gsp}_{2n}(\mathbb{C})^\vee$ be the restriction map dual to the inclusion $\mathfrak{gsp}_{2n}(\mathbb{C}) \to (\widehat{\mathfrak{gsp}_{2n}})_o$. Explicitly, $\mathrm{res}(\alpha)$ is the map corresponding the usual residue of $\alpha_\nu \nu$.

Proposition 5.10. *There is a Hamiltonian left action of* $\mathrm{GSp}_{2n}(\mathbb{C})$ *on* $\mathscr{M}(A)$ *defined by*

$$h \cdot (Q_x g, \alpha) := (Q_x g h^{-1}, \mathrm{Ad}^*(h)(\alpha))$$

with moment map $\mu_{\mathrm{GSp}_{2n}(\mathbb{C})} : \mathscr{M}(A) \to \mathfrak{gsp}_{2n}(\mathbb{C})$ *given by*

$$\mu_{\mathrm{GSp}_{2n}(\mathbb{C})}(Q_x g, \alpha) = \mathrm{res}(\alpha).$$

Likewise, there is a Hamiltonian left action of $\mathrm{GSp}_{2n}(\mathbb{C})$ *on* $\widetilde{\mathscr{M}}(A)$ *defined by*

$$h \cdot (U_x g, \alpha) := (U_x g h^{-1}, \mathrm{Ad}^*(h)(\alpha))$$

with moment map $\widetilde{\mu}_{\mathrm{GSp}_{2n}(\mathbb{C})} : \widetilde{\mathscr{M}}(A) \to \mathfrak{gsp}_{2n}(\mathbb{C})$ *given by*

$$\widetilde{\mu}_{\mathrm{GSp}_{2n}(\mathbb{C})}(U_x g, \alpha) = \mathrm{res}(\alpha).$$

Proof. We prove the second part of the statement. The proof of the first statement is similar. The usual Hamiltonian left action of $(\widehat{\mathrm{GSp}_{2n}})_o$ on its cotangent bundle $T^*((\widehat{\mathrm{GSp}_{2n}})_o)$ restricts to a Hamiltonian left action of $\mathrm{GSp}_{2n}(\mathbb{C})$ on $T^*((\widehat{\mathrm{GSp}_{2n}})_o)$, given by

$$\rho(h)(g, \alpha) = (g h^{-1}, \mathrm{Ad}^*(h)\alpha).$$

Extend this action to $T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}^+$:

$$\rho(h)(g, \alpha, \beta) = (g h^{-1}, \mathrm{Ad}^*(h)\alpha, \beta).$$

The corresponding moment map is given by $\mu_\rho(g,\alpha) = \operatorname{res}(\alpha)$. It is clear that this action commutes with the action of $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ described in (5.1). Furthermore, μ_ρ is $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-invariant, since

$$\mu_\rho(p \cdot (g,\alpha,\beta)) = \mu_\rho(pg,\alpha,\mathrm{Ad}^*(p)(\beta)) = \operatorname{res}(\alpha) = \mu_\rho(g,\alpha,\beta).$$

By [5, Lemma 5.13], ρ descends to a natural action on $\widetilde{\mathscr{M}}(A)$, which is the one described above. $\square$

Lemma 5.11. $\mathrm{GSp}_{2n}(\mathbb{C})$ *acts freely on* $\widetilde{\mathscr{M}}(A)$.

Proof. We want to show that if $h \in \mathrm{GSp}_{2n}(\mathbb{C})$ fixes some arbitrary point $(U_x g, \alpha)$ in the framed extended orbit, then $h = 1$. We begin by showing that it is sufficient to assume that $g = 1$. To see this, first note the following general fact about group actions: h fixes x if and only if ghg^{-1} fixes $g \cdot x$. Now suppose that the claim is true for $g = 1$; i.e., suppose that $h \cdot (U_x, \alpha) = (U_x, \alpha)$ implies that $h = 1$. Then $ghg^{-1} \cdot (U_x, g\alpha) = (U_x, g\alpha)$ implies that $ghg^{-1} = 1$. But $ghg^{-1} = 1$ if and only if $h = 1$, and, by our general statement about group actions, ghg^{-1} fixes $(U_x, g\alpha)$ if and only if h fixes $(U_x g, \alpha)$. Hence h fixes $(U_x g, \alpha)$ implies that $h = 1$, proving the claim. Thus we assume that $h \in \mathrm{GSp}_{2n}(\mathbb{C})$ fixes some $(U_x, \alpha) \in \widetilde{\mathscr{M}}(A)$.

By [7, Proposition 3.6], there exists a functional representative $\alpha_\nu = \sum_{i=-r}^{0} t^i M_i$ (for $M_i \in \mathfrak{gsp}_{2n}(\mathbb{C})$) for α with terms only in nonpositive degrees. Since $\mathrm{Ad}^*(h)(\alpha) =$

α, it follows that $\mathrm{Ad}(h)(\alpha_\nu) = \alpha_\nu + X$ for some $X \in (\widehat{\mathfrak{gsp}_{2n}})_{x+}$. Then

$$\begin{aligned} \alpha_\nu + X &= \mathrm{Ad}(h)(\alpha_\nu) \\ &= \mathrm{Ad}(h)(\textstyle\sum_{i=-r}^{0} t^i M_i) \\ &= \textstyle\sum_{i=-r}^{0} t^i \,\mathrm{Ad}(h)(M_i) \end{aligned}$$

The fact that $\mathrm{Ad}(h)(M_i) \in \mathfrak{gsp}_{2n}(\mathbb{C})$ implies that $\alpha_\nu + X$ has no terms of positive degree; i.e., $X = 0$ and $\mathrm{Ad}(h)(\alpha_\nu) = \alpha_\nu$.

We next proceed to show that h is $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-conjugate to an element of $S(\mathfrak{o})$; the reason for proving this will become apparent in the next paragraph. Since $(U, \alpha) \in \widetilde{\mathscr{M}}(A)$, it is possible to choose $p \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ such that

$$\mathrm{Ad}^*(p)(p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\alpha)) = A^+.$$

Then $\mathrm{Ad}^*(p)(\alpha) \in \pi_{\mathfrak{s}}^*(\mathfrak{s}_{-r}^\vee) + (\widehat{\mathfrak{gsp}_{2n}})_{x,r-r}^\vee$. By [6, Proposition 4.9(2)], there exists a $p' \in P^r \subset P^1$ and a representative $\widetilde{A^+} \in \pi_{\mathfrak{s}}^*(\mathfrak{s}_{-r}^\vee)$ for A^1 with the property that $\mathrm{Ad}^*(p')(\mathrm{Ad}^*(p)(\alpha)) = \widetilde{A^+}$. Setting $q = p'p \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$, we find that

$$\begin{aligned} \mathrm{Ad}^*(qhq^{-1})(\widetilde{A^+}) &= \mathrm{Ad}^*(q)\,\mathrm{Ad}^*(h)\,\mathrm{Ad}^*(q^{-1})(\widetilde{A^+}) \\ &= \mathrm{Ad}^*(q)\,\mathrm{Ad}^*(h)(\alpha) \\ &= \mathrm{Ad}^*(q)(\alpha) && \text{as shown above} \\ &= \widetilde{A^+}. \end{aligned}$$

By Proposition 4.12(6), it follows that $qhq^{-1} \in S$. Moreover, since $qhq^{-1} \in (\widehat{\mathrm{GSp}_{2n}})_o$, it follows that $qhq^{-1} \in S(\mathfrak{o})$.

We have shown that h is $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-conjugate to an element of $S(\mathfrak{o})$. Recall from Section 3.4 that $(\widehat{\mathrm{GSp}_{2n}})_{x+} \cong U_x \ltimes (\widehat{\mathrm{GSp}_{2n}})_{o+}$. Then there exists some $g \in$

$(\widehat{\mathrm{GSp}_{2n}})_{o+}$ and $u \in U_x$ such that $guhu^{-1}g^{-1} \in S(\mathfrak{o})$. This implies that $uhu^{-1} \in g^{-1}S(\mathfrak{o})g \cap U_x = \{1\}$. Therefore $h = 1$, completing the proof of the proposition. $\square$

Define $S^\flat \subset T$ to be subgroup generated by the rational cocharacters of S. Then $S^\flat = S(\mathfrak{o}) \cap \mathrm{GSp}_{2n}(\mathbb{C})$ and $S(\mathfrak{o}) = S_{0+} \rtimes S^\flat$.

Lemma 5.12. *If $(Q_x g_1, \alpha)$ and $(Q_x g_2, \alpha)$ both lie in $\mathscr{M}(A)$, then $g_2 = pg_1$ for some $p \in Q_x$. Moreover, if $(U_x g_1, \alpha)$ and $(U_x g_2, \alpha)$ both lie in $\widetilde{\mathscr{M}}(A)$, then $g_2 = usg_1$ for some $u \in U_x$ and $s \in S^\flat$.*

Proof. To prove the first statement, we first show that it is sufficient to prove the following specialized statement: if (Q_x, α) and $(Q_x g, \alpha)$ both lie in $\mathscr{M}(A)$, then $g \in Q_x$. To see this, suppose $(Q_x g_1, \alpha)$ and $(Q_x g_2, \alpha)$ both lie in $\mathscr{M}(A)$. Acting on these two elements by g_1^{-1}, we find that $(Q_x, \mathrm{Ad}^*(g_1)(\alpha))$ and $(Q_x g_2 g_1^{-1}, \mathrm{Ad}^*(g_1)(\alpha))$ satisfy the conditions of our specialized statement. Hence $g_2 g_1^{-1} \in Q_x$; equivalently, there exists $p \in Q_x$ such that $g_2 = pg_1$. The reformulation of the second statement is as follows: if (U_x, α) and $(U_x g, \alpha)$ both lie in $\widetilde{\mathscr{M}}(A)$, then $g = us$ for some $u \in U_x$ and $s \in S^\flat$.

We proceed to prove our specialized statement. Let $(Q_x, \alpha), (Q_x g, \alpha) \in \mathscr{M}(A)$. Then $p_{(\widehat{\mathrm{GSp}_{2n}})_{x+}}(\alpha) \in \mathscr{O}$. Let α_ν be a functional representative for α. Then it is possible to choose $p_1 \in (\widehat{\mathrm{GSp}_{2n}})_x$ such that $\mathrm{Ad}(p_1)(\alpha_\nu) \in \mathfrak{s} + (\widehat{\mathfrak{gsp}_{2n}})_{x+}$ is a representative for A. By [6, Proposition 4.9(2)], there exists some $q_1 \in (\widehat{\mathrm{GSp}_{2n}})_{x,r+}$ such that $A_\nu := \mathrm{Ad}(q_1 p_1)(\alpha_\nu) \in \mathfrak{s}$ is a "diagonal" functional representative for

A. Likewise, we can choose $p_2 \in (\widehat{\mathrm{GSp}_{2n}})_x$ and $q_2 \in (\widehat{\mathrm{GSp}_{2n}})_{x,r+}$ such that $A'_\nu := \mathrm{Ad}(q_2 p_2)(\mathrm{Ad}(g)(\alpha_\nu)) \in \mathfrak{s}$ is another "diagonal" functional representative for A. Since A_ν and A'_ν are representatives in $\mathfrak{s}$ for A with $\mathrm{Ad}(q_2 p_2 g p_1^{-1} q_1^{-1})(A_\nu) = A'_\nu$, it follows by Lemma 4.15 that $A_\nu = A'_\nu$. The fact that $q_2 p_2 g p_1^{-1} q_1^{-1}$ centralizes the regular semisimple element $A_\nu \in \mathfrak{s}$ implies that $q_2 p_2 g p_1^{-1} q_1^{-1} \in S \cap (\widehat{\mathrm{GSp}_{2n}})_o = S(\mathfrak{o})$. Note that $p_1, p_2, q_1, q_2 \in (\widehat{\mathrm{GSp}_{2n}})_x$ and $S(\mathfrak{o}) \subset (\widehat{\mathrm{GSp}_{2n}})_x$. Hence $g \in (\widehat{\mathrm{GSp}_{2n}})_x \cap \mathrm{GSp}_{2n}(\mathbb{C}) = Q_x$.

To prove the second statement, we let $(U, \alpha), (Ug, \alpha) \in \widetilde{\mathscr{M}}(A)$. By similar arguments to those used above, it is possible to choose $s' \in S(\mathfrak{o})$ and $p_i \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ with the property that $g = p_1^{-1} s' p_2$. Since $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ is normal in $(\widehat{\mathrm{GSp}_{2n}})_x$, it follows that there exists $q \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ with the property that $s' p_2 = q s'$, giving $g = p_1^{-1} q s'$. The fact that $S(\mathfrak{o}) = S_{0+} \rtimes S^\flat$ implies that $s' = s'' s$ for some $s'' \in S_{0+}$ and $s \in S^\flat$. Note that $u := p_1^{-1} q s'' = g s^{-1} \in \mathrm{GSp}_{2n}(\mathbb{C})$. Hence there is a factorization $g = us$ with $s \in S^\flat$ and $u \in \mathrm{GSp}_{2n}(\mathbb{C}) \cap (\widehat{\mathrm{GSp}_{2n}})_{x+} = U_x$, as desired. $\square$

Lemma 5.13. *Let $\alpha \in (\widehat{\mathfrak{gsp}_{2n}})_o^\vee$ be a functional satisfying $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\alpha) = A^+$. Then $s \in S(\mathfrak{o})$ implies that $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(s)\alpha) = A^+$.*

Proof. Let α_ν and $(A^+)_\nu$ be functional representatives for α and A^+, respectively, which satisfy the hypothesis. Then

$$\alpha_\nu + (\widehat{\mathfrak{gsp}_{2n}})_{x+} = (A^+)_\nu + (\widehat{\mathfrak{gsp}_{2n}})_{x+} \in \mathfrak{s} + (\widehat{\mathfrak{gsp}_{2n}})_{x+}.$$

The claim now follows, since $S(\mathfrak{o})$ stabilizes elements of $\mathfrak{s}$ and preserves $(\widehat{\mathfrak{gsp}_{2n}})_{x+}$.

$\square$

Proposition 5.16 below states that $\mathscr{M}(A)$ is a symplectic reduction of $\widetilde{\mathscr{M}}(A)$ by an action of $S^\flat$. The action of $S^\flat$ on $\widetilde{\mathscr{M}}(A)$ is defined in the following lemma.

Lemma 5.14. *There is an action of $S^\flat$ on $\widetilde{\mathscr{M}}(A)$ given by*

$$s \cdot (U_x g, \alpha) = (U_x s g, \alpha).$$

Proof. Let $(U_x g, \alpha) \in \widetilde{\mathscr{M}}(A)$. We show that $s \cdot (U_x g, \alpha) \in \widetilde{\mathscr{M}}(A)$. To start, recall that $S^\flat \subset \mathrm{GSp}_{2n}(\mathbb{C})$, so $U_x s g$ makes sense. Since $(U_x g, \alpha) \in \widetilde{\mathscr{M}}(A)$, it follows from definitions that $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(g)(\alpha)) \in \mathscr{O}^+$, and thus there exists $u \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ such that

$$\mathrm{Ad}^*(u)(p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(g)(\alpha))) = A^+.$$

It suffices to show that there exists a $u' \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ such that

$$\mathrm{Ad}^*(u')(p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(sg)(\alpha)) = A^+.$$

The element $sus^{-1} \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ has the property that

$$\begin{aligned}
&\mathrm{Ad}^*(sus^{-1})(p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(sg)(\alpha))) = \\
&= p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(sus^{-1})\,\mathrm{Ad}^*(sg)(\alpha)) && \text{since } sus^{-1} \in (\widehat{\mathrm{GSp}_{2n}})_o \\
&= \mathrm{Ad}^*(s)A^+ && \text{as shown above} \\
&= A^+ && \text{by Lemma 5.13,}
\end{aligned}$$

completing the proof. $\square$

Let $(U_x g, \alpha) \in \widetilde{\mathscr{M}}(A)$. Then there exists some $u \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ such that

$$p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(ug)(\alpha)) = A^+. \tag{5.3}$$

Define a map $\mu_{S^\flat} : \widetilde{\mathscr{M}}(A) \to (\mathfrak{s}^\flat)^\vee$ by

$$\mu_{S^\flat}(U_x g, \alpha) = -(\mathrm{Ad}^*(ug)(\alpha))|_{\mathfrak{s}^\flat}.$$

Lemma 5.15. *The map $\mu_{S^\flat}$ is well-defined; i.e., if $u, u' \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ satisfy (5.3), then*

$$-(\mathrm{Ad}^*(ug)(\alpha))|_{\mathfrak{s}^\flat} = -(\mathrm{Ad}^*(u'g)(\alpha))|_{\mathfrak{s}^\flat}.$$

Proof. Let $\widetilde{A} = \mathrm{Ad}^*(ug)(\alpha)$. Then

$$\mathrm{Ad}^*(u'g)(\alpha) = \mathrm{Ad}^*(u'u^{-1})\,\mathrm{Ad}^*(ug)(\alpha) = \mathrm{Ad}^*(u'u^{-1})(\widetilde{A}).$$

The claim is equivalent to the following: if $u' \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ satisfies (5.3), then $\widetilde{A}|_{\mathfrak{s}^\flat} = (\mathrm{Ad}^*(u'u^{-1})\widetilde{A})|_{\mathfrak{s}^\flat}$. By Lemma 4.16, $u'u^{-1} \in S(\mathfrak{o})(\widehat{\mathrm{GSp}_{2n}})_{x,r}$. Hence, the following statement is stronger than the above claim: if $s \in S(\mathfrak{o})$ and $p \in (\widehat{\mathrm{GSp}_{2n}})_{x,r}$, then $\widetilde{A}|_{\mathfrak{s}^\flat} = (\mathrm{Ad}^*(sp)\widetilde{A})|_{\mathfrak{s}^\flat}$. The following statement is even stronger, and is proved in the following paragraph: if $s \in S(\mathfrak{o})$ and $p \in (\widehat{\mathrm{GSp}_{2n}})_{x,r}$, then

$$p_{\mathfrak{s}\cap(\widehat{\mathfrak{gsp}_{2n}})_x}(\widetilde{A}) = p_{\mathfrak{s}\cap(\widehat{\mathfrak{gsp}_{2n}})_x}(\mathrm{Ad}^*(sp)(\widetilde{A})).$$

Let $\widetilde{A}_\nu$ be a functional representative for $\widetilde{A}$, and let $X \in \mathfrak{s} \cap (\widehat{\mathfrak{gsp}_{2n}})_x$. Then

$$\begin{aligned}
p_{\mathfrak{s}\cap(\widehat{\mathfrak{gsp}_{2n}})_x}(\mathrm{Ad}^*(sp)(\widetilde{A}))(X) &= \langle \mathrm{Ad}(sp)\widetilde{A}_\nu, X\rangle_\nu \\
&= \langle \pi_{\mathfrak{s}}(\mathrm{Ad}(sp)\widetilde{A}_\nu), X\rangle_\nu && \text{by Proposition 4.12(4)} \\
&= \langle \mathrm{Ad}(s)\pi_{\mathfrak{s}}(\mathrm{Ad}(p)\widetilde{A}_\nu), X\rangle_\nu && \text{by Proposition 4.12(5)} \\
&= \langle \pi_{\mathfrak{s}}(\mathrm{Ad}(p)\widetilde{A}_\nu), X\rangle_\nu && \text{since } S \text{ stabilizes } \mathfrak{s}.
\end{aligned}$$

Hence $\pi_{\mathfrak{s}}(\mathrm{Ad}(p)\widetilde{A}_\nu)$ is a functional representative for $p_{\mathfrak{s}\cap(\widehat{\mathfrak{gsp}_{2n}})_x}(\mathrm{Ad}^*(ug)(\widetilde{A}))$. However, Proposition 4.12(5) implies that

$$\pi_{\mathfrak{s}}(\mathrm{Ad}(p)\widetilde{A}_\nu) - \pi_{\mathfrak{s}}(\widetilde{A}_\nu) \in (\widehat{\mathfrak{gsp}_{2n}})_{x+} = ((\widehat{\mathfrak{gsp}_{2n}})_x)^\perp.$$

Thus $\pi_{\mathfrak{s}}(\widetilde{A}_\nu)$ is a functional representative both for $p_{\mathfrak{s}\cap(\widehat{\mathfrak{gsp}_{2n}})_x}(\mathrm{Ad}^*(sp)(\widetilde{A}))$ and for $p_{\mathfrak{s}\cap(\widehat{\mathfrak{gsp}_{2n}})_x}(\widetilde{A})$ by similar arguments to those used above. The claim follows. □

Proposition 5.16. *Set $\Lambda := A|_{\mathfrak{s}^\flat}$. The action of $S^\flat$ on $\widetilde{\mathscr{M}}(A)$ defined in Lemma 5.14 is Hamiltonian with moment map $\mu_S^\flat$. Moreover,*

$$\mathscr{M}(A) \cong \widetilde{\mathscr{M}}(A) /\!\!/_{-\Lambda}\, S^\flat.$$

Proof. We proceed by first defining a Hamiltonian action of $S^\flat$ on $T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}^+$. Given $\beta \in \mathscr{O}^+$, $\alpha \in (\widehat{\mathfrak{gsp}_{2n}})_o^\vee$ can be chosen such that $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\alpha) = \beta$. Define an action of $S^\flat$ on $\mathscr{O}^+$ by $s \cdot \beta = p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(s)(\alpha))$. This can be shown to be a well-defined action by similar arguments as those used in the proof of Lemma 5.14.

A moment map for this action of $S^\flat$ can be described as follows. Consider the semidirect product $S^\flat \ltimes (\widehat{\mathrm{GSp}_{2n}})_{x+} \subset (\widehat{\mathrm{GSp}_{2n}})_x$ with Lie algebra $\mathfrak{s}^\flat \times (\widehat{\mathfrak{gsp}_{2n}})_{x+}$.

Then $A^+ \in \mathscr{O}^+$ can be lifted to $\widetilde{A} \in (\mathfrak{s}^\flat \times (\widehat{\mathfrak{gsp}_{2n}})_{x+})^\vee$. Let $\widetilde{\mathscr{O}} \subset (\mathfrak{s}^\flat \times (\widehat{\mathfrak{gsp}_{2n}})_{x+})^\vee$ be the $S^\flat \rtimes (\widehat{\mathrm{GSp}_{2n}})_{x+}$-coadjoint orbit of $\widetilde{A}$. Since $S^\flat$ stabilizes both A^+ (by Lemma 5.13) and $\mathfrak{s}^\flat$, it follows that $S^\flat$ stabilizes $\widetilde{A}$. Thus the action action of $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ on $\widetilde{\mathscr{O}}$ is transitive. In Lemma 5.17 below, it is shown that the natural map $\widetilde{\pi} : \widetilde{\mathscr{O}} \to \mathscr{O}^+$ defined by $\widetilde{A} \mapsto A^+$ is an $S^\flat$-equivariant symplectic isomorphism. Therefore, the moment map $\widetilde{\mu} : \mathscr{O}^+ \to (\mathfrak{s}^\flat)^\vee$ is given by

$$\widetilde{\mu}(\beta) = \pi_{S^\flat}(\widetilde{\pi}^{-1}(\beta)),$$

where $\pi_{S^\flat}$ is the projection $(\mathfrak{s}^\flat \times (\widehat{\mathfrak{gsp}_{2n}})_{x+})^\vee \to (\mathfrak{s}^\flat)^\vee$.

Note that there is amguity in the choice of lift for A^+; an arbitrary lift of A^+ has the form $\widetilde{A}+\gamma$ for $\gamma \in ((\widehat{\mathfrak{gsp}_{2n}})_{x+})^\perp \cong (\mathfrak{s}^\flat)^\vee$. To see the change in the moment map, let $\beta \in \mathscr{O}^+$ be arbitrary; i.e., let $\beta = \mathrm{Ad}^*(u)A^+$ for some $u \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$. Then

$$\widetilde{\pi}^{-1}(\beta) = \widetilde{\pi}^{-1}(\mathrm{Ad}^*(u)A^+) = \mathrm{Ad}^*(u)\left(A^+ + \gamma\right).$$

Let $z \in \mathfrak{s}^\flat$. Since

$$\mathrm{Ad}^*(u)\left(A^+ + \gamma\right)(z) = \mathrm{Ad}^*(u)\left(A^+\right)(z) + \gamma(z) = \beta(z) + \gamma(z),$$

it follows that the corresponding moment map differs from $\widetilde{\mu}$ by γ.

The action of $S^\flat$ on $\widetilde{\mathscr{M}}(A)$ defined in Lemma 5.14 descends from a Hamiltonian action of $S^\flat$ on $T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}^+$. To see this, let $(g, \alpha, \beta) \in T^*((\widehat{\mathrm{GSp}_{2n}})_o) \times \mathscr{O}^+$. Then

$$s \cdot (g, \alpha, \beta) = (sg, \alpha, s \cdot \beta),$$

with $s\cdot\beta$ defined as above, defines a Hamiltonian action of $S^\flat$ on $T^*((\widehat{\mathrm{GSp}_{2n}})_o)\times\mathscr{O}^+$ with moment map μ' given by the sum of the natural moment map on the cotangent bundle $T^*((\widehat{\mathrm{GSp}_{2n}})_o)$ and $\widetilde{\mu}$. Note that the action of $S^\flat$ preserves $\mu^{-1}_{(\widehat{\mathrm{GSp}_{2n}})_{x+}}(0)$, since $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(sg)(\alpha))+\mathrm{Ad}^*(s)(\beta)=0$ if and only if $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(g)(\alpha))+\beta=0$. Moreover, the map $\Phi:\mu^{-1}_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(0)\to\widetilde{\mathscr{M}}(A)$ is $S^\flat$-equivariant, since

$$s\cdot\Phi(g,\alpha,\beta)=s\cdot(U_xg,\alpha)=(U_xsg,\alpha)=\Phi(sg,\alpha,s\cdot\beta)=\Phi(s\cdot(g,\alpha,\beta)).$$

In order to invoke [5, Lemma 5.13], it is necessary to show that the restriction of μ' to $\mu^{-1}_{(\widehat{\mathrm{GSp}_{2n}})_{x+}}(0)$ is $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-invariant. With this goal in mind, let $(g,\alpha,\beta)\in\widetilde{\mathscr{M}}(A)$. Define $\psi(g,\alpha)$ to be the projection of $\mathrm{Ad}^*(g)(\alpha)$ onto $(\mathfrak{s}^\flat\times(\widehat{\mathfrak{gsp}_{2n}})_{x+})^\vee$. Let $u\in(\widehat{\mathrm{GSp}_{2n}})_{x+}$. Then

$$\begin{aligned}\mu'(u\cdot(g,\alpha,\beta) &= \mu'(ug,\alpha,\mathrm{Ad}^*(u)\beta)\\ &= \pi_{S^\flat}\left(-\mathrm{Ad}^*(u)\psi(g,\alpha)\right)+\pi_{S^\flat}\widetilde{\pi}^{-1}(\mathrm{Ad}^*(u)(\beta))\\ &= \pi_{S^\flat}\left(-\mathrm{Ad}^*(u)\psi(g,\alpha)+\mathrm{Ad}^*(u)\widetilde{\pi}^{-1}(\beta)\right).\end{aligned}$$

Since $(g,\alpha,\beta)\in\mu^{-1}_{(\widehat{\mathrm{GSp}_{2n}})_{x+}}(0)$, it follows that $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(g)(\alpha))\in\mathscr{O}^+$. Hence $\psi(g,\alpha)$ (the projection of $\mathrm{Ad}^*(g)(\alpha)$ onto $\mathfrak{s}^\flat\times(\widehat{\mathfrak{gsp}_{2n}})_{x+}$) must lie in a $S^\flat\times(\widehat{\mathrm{GSp}_{2n}})_{x+}$-coadjoint orbit containing $\widetilde{A}-\gamma$ (where $\widetilde{A}$ is the specified lift of A^+ to $\mathfrak{s}^\flat\times(\widehat{\mathfrak{gsp}_{2n}})_{x+}$) for some $\gamma\in(\mathfrak{s}^\flat)^\vee$. Thus

$$\pi_{S^\flat}(-\psi(g,\alpha)+\widetilde{\pi}^{-1}(\beta))=\pi_{S^\flat}(-\mathrm{Ad}^*(u)\psi(g,\alpha)+\mathrm{Ad}^*(u)\widetilde{\pi}^{-1}(\beta))=\gamma.$$

Hence μ' is $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-invariant. By [5, Lemma 5.13], the action of $S^\flat$ on $\widetilde{\mathscr{M}}(A)$ is Hamiltonian, and the corresponding moment map descends from μ'.

It remains to give an explicit formula for this moment map; in particular, $\gamma = \mu_{S^\flat}(U_x g, \alpha)$. By $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-invariance, it is possible to assume without loss of generality that $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}^*(g)(\alpha)) = A^+$. By construction, $\widetilde{\mu}(A^+) = 0$, so $\gamma = -\mathrm{Ad}^*(g)(\alpha)|_{\mathfrak{s}^\flat} = \mu_{S^\flat}(U_x g, \alpha)$.

Finally, it remains to prove that $\mathscr{M}(A) \cong \widetilde{\mathscr{M}}(A) /\!\!/_{-\Lambda} S^\flat$. Suppose that $(U_x g, \alpha) \in \widetilde{\mathscr{M}}(A)$ is in the fiber $\mu_{S^\flat}^{-1}(-\Lambda)$ of $-\Lambda$. We show that $(Q_x g, \alpha) \in \mathscr{M}(A)$. Since $(U_x g, \alpha) \in \widetilde{\mathscr{M}}(A)$, fix $u \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ such that $p_{(\widehat{\mathfrak{gsp}_{2n}})_{x+}}(\mathrm{Ad}(ug)(\alpha) = A^+$. Let α_ν be a functional representative for α. Then $\mathrm{Ad}(ug)(\alpha_\nu) \in A_\nu + (\widehat{\mathfrak{gsp}_{2n}})_x$ for any functional representative A_ν for A. Then $\pi_{\mathfrak{s}}(\mathrm{Ad}(ug)(\alpha_\nu)) = A_\nu + z$ for some $z \in \mathfrak{s}(\mathfrak{o})$. However, note that the assumption that $(U_x g, \alpha) \in \mu_{S^\flat}^{-1}(-\Lambda)$ implies that

$$-A|_{\mathfrak{s}^\flat} = -\Lambda = \mu_{S^\flat}(U_x g, \alpha) = -\left(\mathrm{Ad}^*(ug)(\alpha)\right)|_{\mathfrak{s}^\flat};$$

i.e., the restrictions of $\mathrm{Ad}^*(ug)(\alpha)$ and A to $(\mathfrak{s}^\flat)^\vee$ agree. Hence $z \in \mathfrak{s} \cap (\widehat{\mathfrak{gsp}_{2n}})_{x+}$, and so $\mathrm{Ad}(ug)\alpha - A$ lies in the kernel of $\rho_{\mathfrak{s},r}$ as defined in Proposition 4.12(3). Hence there exists some $p \in (\widehat{\mathrm{GSp}_{2n}})_{x,r}$ such that $\mathrm{Ad}(p)\,\mathrm{Ad}(ug)\alpha_n u \in A_\nu + (\widehat{\mathfrak{gsp}_{2n}})_{x+}$, so $p_{(\widehat{\mathfrak{gsp}_{2n}})_x}(\mathrm{Ad}^*(pug)\alpha) = A$. Then $\mathrm{Ad}^*(g)(\alpha) \in \mathscr{O}$. Thus there is a map $\mu_{S^\flat}^{-1}(-\Lambda) \to \mathscr{M}(A)$ given by $(U_x g, \alpha) \mapsto (Q_x g, \alpha)$. To show that the fibers of this map are $S^\flat$-orbits, suppose that $(U_x g_1, \beta_1)$ and $(U_x g_2, \beta_2)$ both map to $(Q_x g, \alpha)$. Then $\beta_1 = \beta_2 = \alpha$. By Lemma 5.12, it follows that there exists some $s \in S^\flat$ such that

$$(U_x g_2, \alpha) = (U_x s g_1, \alpha) = s \cdot (U_x, g_1);$$

i.e., $(U_x g_1, \alpha)$ and $(U_x g_2, \alpha)$ lie in the same $S^\flat$-orbit. The desired isomorphism follows. $\square$

Lemma 5.17. *The map $\widetilde{\pi} : \widetilde{\mathscr{O}} \to \mathscr{O}^+$ is an $S^\flat$-equivariant symplectic isomorphism.*

Proof. Let $\widetilde{\pi} : \widetilde{\mathscr{O}} \to \mathscr{O}^+$ be the natural map (described in the proof above) defined by $\widetilde{A} \mapsto A^+$. Then $\widetilde{\pi}$ is a $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-map; i.e., $\mathrm{Ad}^*(u)\widetilde{A} \mapsto \mathrm{Ad}^*(u)A^+$ for all $u \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$. We first show that $\widetilde{\pi}$ is $S^\flat$-equivariant. Since $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ acts transitively on $\widetilde{\mathscr{O}}$, it follows that an arbitrary element of $\widetilde{\mathscr{O}}$ has the form $\mathrm{Ad}^*(u)\widetilde{A}$ for some $u \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$. Let $u \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$ and $s \in S^\flat$. Then

$$
\begin{aligned}
\widetilde{\pi}(s \cdot \mathrm{Ad}^*(u)\widetilde{A}) &= \widetilde{\pi}(\mathrm{Ad}^*(s)\,\mathrm{Ad}^*(u)\widetilde{A}) \\
&= \widetilde{\pi}(\mathrm{Ad}^*(s)\,\mathrm{Ad}^*(u)\,\mathrm{Ad}^*(s^{-1})\widetilde{A}) \quad \text{since } S^\flat \text{ stabilizes } \widetilde{A} \\
&= \widetilde{\pi}(\mathrm{Ad}^*(sus^{-1})\widetilde{A}) \\
&= \mathrm{Ad}^*(sus^{-1})A^+ \quad \text{since } sus^{-1} \in (\widehat{\mathrm{GSp}_{2n}})_{x+} \\
&= \mathrm{Ad}^*(s)\,\mathrm{Ad}^*(u)A^+ \\
&= s \cdot \widetilde{\pi}(\mathrm{Ad}^*(u)\widetilde{A})
\end{aligned}
$$

Next we show that $\widetilde{\pi}$ is injective. Suppose that $\mathrm{Ad}^*(u)\widetilde{A}, \mathrm{Ad}^*(u')\widetilde{A}$ have the same image via $\widetilde{\pi}$ for some $u, u' \in (\widehat{\mathrm{GSp}_{2n}})_{x+}$. Since $\widetilde{\pi}$ is a $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-map, it can be assumed without loss of generality that $u' = 1$. But $\mathrm{Ad}^*(u)\widetilde{A}, \widetilde{A}$ have the same image if and only if $\mathrm{Ad}^*(u)A^+ = A^+$, which is true if and only if u is in the stabilizer of A^+. Thus, to prove that $\widetilde{\pi}$ is injective, it suffices to prove that the stabilizer of of $\widetilde{A}$ in $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ is equal to the stabilizer of A^+ in $(\widehat{\mathrm{GSp}_{2n}})_{x+}$.

With this goal in mind, let $\widetilde{A}_\nu \in (\widehat{\mathfrak{gsp}_{2n}})_{x,-r}$ be a functional representative for $\widetilde{A}$. Then $\widetilde{A}_\nu \in (\mathfrak{s} + (\widehat{\mathfrak{gsp}_{2n}})_x) \cap (\mathfrak{s}^\flat)^\perp$. Lemma 4.16 implies that the stabilizer of A^+ is $(S(\mathfrak{o}) \cap (\widehat{\mathrm{GSp}_{2n}})_{x+})(\widehat{\mathrm{GSp}_{2n}})_{x,r}$ if $r > 0+$ and $(\widehat{\mathrm{GSp}_{2n}})_{x+}$ if $r = 0+$. Since the stabilizer of $\widetilde{A}$ is a subgroup of the stabilizer of A^+, it suffices to show that this group stabilizes $\widetilde{A}$. Since $\widetilde{A}_\nu \in \mathfrak{s} + (\widehat{\mathfrak{gsp}_{2n}})_x$, it follows that $S \cap (\widehat{\mathrm{GSp}_{2n}})_{x+}$ stabilizes $\widetilde{A}$. To show that $(\widehat{\mathrm{GSp}_{2n}})_{x,r}$ stabilizes $\widetilde{A}$, let $u \in (\widehat{\mathrm{GSp}_{2n}})_{x,r}$, $z \in \mathfrak{s}^\flat$, and $x \in (\widehat{\mathfrak{gsp}_{2n}})_{x+}$. Then

$$\mathrm{Ad}^*(u)(\widetilde{A})(z + X) = \widetilde{A}(\mathrm{Ad}(u^{-1}z + \mathrm{Ad}(u^{-1})X) = \widetilde{A}(\mathrm{Ad}(u^{-1}z) + \widetilde{A}(X).$$

Thus it is sufficient to check that $\mathrm{Ad}^*(u)\widetilde{A}(z) = \widetilde{A}(z)$. By Proposition 4.12(4),

$$\mathrm{Ad}^*(u)\widetilde{A}(z) = \langle \mathrm{Ad}(u)\widetilde{A}_\nu, z\rangle_\nu = \langle \pi_{\mathfrak{s}}(\mathrm{Ad}(u)\widetilde{A}_\nu), z\rangle_\nu.$$

Then Proposition 4.12(3) implies that $\pi_{\mathfrak{s}}(\mathrm{Ad}(u)\widetilde{A}_\nu) + (\widehat{\mathfrak{gsp}_{2n}})_{x+} = \pi_{\mathfrak{s}}(\widetilde{A}_\nu) + (\widehat{\mathfrak{gsp}_{2n}})_{x+}$, so

$$\mathrm{Ad}^*(u)\widetilde{A}(z) = \langle \pi_{\mathfrak{s}}(\mathrm{Ad}(u)\widetilde{A}_\nu), z\rangle_\nu = \langle \pi_{\mathfrak{s}}(\widetilde{A}_\nu), z\rangle_\nu = \widetilde{A}(z),$$

proving the claim.

Finally, we show that $\widetilde{\pi}$ preserves the natural symplectic form on each coadjoint orbit. Since $\widetilde{\mathscr{O}}$ and $\mathscr{O}^+$ are $(\widehat{\mathrm{GSp}_{2n}})_{x+}$-orbits, it suffices by transitivity to show that the symplectic forms are the same at $\widetilde{A}$ and A^+. This is equivalent to $\widetilde{A}([X_1 + z_1, X_2 + z_2]) = A^+([X_1, X_2])$ for $z_1, z_2 \in \mathfrak{s}^\flat$ and $X_1, X_2 \in (\widehat{\mathfrak{gsp}_{2n}})_{x+}$. Note that the diagonal entries of $[z_i, X_j]$ are all 0, so $\mathrm{Tr}(D[z_i, X_j]) = 0$ for any diagonal D. Hence

$$\begin{aligned}\widetilde{A}([X_1 + z_1, X_2 + z_2]) &= \widetilde{A}([X_1, X_2] + [z_1, X_2] + [X_1, z_2] + [z_1, z_2]) \\ &= \widetilde{A}([X_1, X_2]) + \widetilde{A}([z_1, X_2]) + \widetilde{A}([X_1, z_2]) \\ &= \widetilde{A}([X_1, X_2]) \\ &= A^+([X_1, X_2]).\end{aligned}$$

The last step follows since $[X_1, X_2] \in (\widehat{\mathfrak{gsp}_{2n}})_{x+}$, and the restriction of $\widetilde{A}$ to $(\widehat{\mathfrak{gsp}_{2n}})_{x+}$ is A^+. This completes the proof. □

5.3 Construction of Moduli Spaces

This section contains a proof of Theorem 5.5. Let $\mathcal{G}$ be a trivializable principal GSp_{2n}-bundle on $\mathbb{P}^1$, and let ∇ be a flat structure with singularities at $\{y_1, \ldots, y_m\}$. Assume that ∇ has compatible framings $\{U_1 g_1, \ldots, U_m g_m\}$ for each singularity, and that ∇ has formal type $A_i \in (\mathfrak{gsp}_{2n}(F_{y_i}))_{x_{y_i}}$ at y_i. Define $\mathscr{O}_i \subset (\mathfrak{gsp}_{2n}(F_{y_i}))_{x_{y_i}}$ to be the coadjoint orbit of A_i under $(\mathrm{GSp}_{2n}(F_{y_i}))_{x_{y_i}}$, and define $\mathscr{O}_i^+ \subset (\mathfrak{gsp}_{2n}(F_{y_i}))_{x_{y_i}+}$ to be the coadjoint orbit of A_i^+ under $(\mathrm{GSp}_{2n}(F_{y_i}))_{x_{y_i}+}$. For each y_i, set $\mathscr{M}_i = \mathscr{M}(A^i)$ and $\widetilde{\mathscr{M}_i} = \widetilde{\mathscr{M}}(A^i)$. Fix a global trivialization as a basepoint.

The following definition mirrors [5, Definition 5.21].

Definition 5.18. *The* principal part $[\nabla_{y_i}]^{pp}$ *of* ∇ *at* y_i *is the image of* $[\nabla_{y_i}]$ *in* $(\mathfrak{gsp}_{2n}(F_{y_i})_o^\vee$ *by the residue-trace pairing.*

Let $[\nabla_i]^{\mathrm{pp}}$ be the principal part of ∇ at y_i. It is a consequence of the Duality Theorem ([17, Theorem II.2]) that ∇ is uniquely determined by the collection

$\{[\nabla_i]^{\mathrm{pp}}\}_{i=1}^m$. Moreover, the Residue Theorem ([17, Proposition II.6]) implies that $\sum_{i=1}^m \mathrm{res}([\nabla_i]^{\mathrm{pp}}) = 0$.

A proof of Theorem 5.5 is given below.

Proof. Consider the map $\mathscr{M}_i \to (\mathfrak{gsp}_{2n}(F_{y_i}))^\vee$ defined by $(Q_{x_i}g, \alpha_i) \mapsto \alpha_i$. This map is injective. To see this, let $(Q_{x_i}g, \alpha_i)$ and $(Q_{x_i}h, \alpha_i)$ be elements in $\mathscr{M}_i$. Lemma 5.12 implies that $g = ph$ for some $p \in Q_{x_i}$. Then $(Q_{x_i}g, \alpha_i) = (Q_{x_i}ph, \alpha_i) = (Q_{x_i}h, \alpha_i)$, proving injectivity. This map identifies elements of the extended orbit $\mathscr{M}_i$ with the principal parts of a framable flat structure ∇ at y_i with formal type A_i. Hence any element of $\prod_i \mathscr{M}_i$ with $\sum_i \mathrm{res}(\alpha_i) = 0$ corresponds to a connection ∇ that is framable with formal type A_i at each y_i.

By Proposition 5.10, there is a Hamiltonian action of the global gauge group $\mathrm{GSp}_{2n}(\mathbb{C})$ on each extended orbit $\mathscr{M}_i$ given by $h \cdot (Q_{x_i}g, \alpha_i) = (Q_{x_i}gh^{-1}, Ad^*(h)(\alpha_i))$. The corresponding moment maps $\mathscr{M}_i \to (\mathfrak{gsp}_{2n}(\mathbb{C}))^\vee$ is given by $(Q_{x_i}g, \alpha) \mapsto \mathrm{res}(\alpha)$. Then the diagonal action of $\mathrm{GSp}_{2n}(\mathbb{C})$ on the product $\prod_i \mathscr{M}_i$ is given by

$$h \cdot \prod_i (Q_{x_i}g_i, \alpha_i) = \prod_i (Q_{x_i}g_i h^{-1}, Ad^*(h)(\alpha_i)).$$

By Lemma 5.4(3), the corresponding moment map $\mu_{\mathrm{GSp}_{2n}(\mathbb{C})} : \prod_i \mathscr{M}_i \to (\mathfrak{gsp}_{2n}(\mathbb{C}))^\vee$ is given by

$$\mu_{\mathrm{GSp}_{2n}(\mathbb{C})}\left(\prod_i (Q_i g_i, \alpha_i)\right) = \sum_i \mathrm{res}(\alpha_i).$$

A moment map $\widetilde{\mu} : \prod_i \widetilde{\mathscr{M}}(A)_i \to \mathfrak{gsp}_{2n}(\mathbb{C})^\vee$ can be defined similarly. These actions of $\mathrm{GSp}_{2n}(\mathbb{C})$ correspond to the actions of the global gauge group $\mathrm{GSp}_{2n}(\mathbb{C})$ on global trivializations. The Residue Theorem translates into an m-tuple lying in $\mu^{-1}(0)$, and two flat structures are the same when they lie in the same global gauge group orbit. Hence,

$$\mathscr{M}^*(\mathbf{A}) \cong \left(\prod_i \mathscr{M}_i\right) /\!\!/_0 \operatorname{GSp}_{2n}(\mathbb{C}).$$

Likewise,

$$\widetilde{\mathscr{M}}^*(\mathbf{A}) \cong \left(\prod_i \widetilde{\mathscr{M}}_i\right) /\!\!/_0 \operatorname{GSp}_{2n}(\mathbb{C}).$$

This completes the proof of (1) and (2).

It remains to show that $\mathscr{M}^*(\mathbf{A})$ is a symplectic reduction of $\widetilde{\mathscr{M}}^*(\mathbf{A})$. By Lemma 5.11, $\operatorname{GSp}_{2n}(\mathbb{C})$ acts freely on $\widetilde{\mathscr{M}}_i$, so the diagonal action of $\operatorname{GSp}_{2n}(\mathbb{C})$ on $\prod_i \widetilde{\mathscr{M}}_i$ is free. Let $\Lambda_i = A_i|_{\mathfrak{s}^\flat_{y_i}}$. The action of $\prod_i S^\flat_{y_i}$ on $\prod_i \widetilde{\mathscr{M}}(A)_i$ commutes with the action of $\operatorname{GSp}_{2n}(\mathbb{C})$. By [5, Lemma 5.13], there is a Hamiltonian action of $\prod_i S^\flat_{y_i}$ on $\widetilde{\mathscr{M}}^*(\mathbf{A})$. Similarly, there is a Hamiltonian action of $\operatorname{GSp}_{2n}(\mathbb{C})$ on

$$\prod_i \left(\widetilde{\mathscr{M}}_i /\!\!/_{-\Lambda_i} S^\flat_i\right) \cong \left(\prod_i \widetilde{\mathscr{M}}_i\right) /\!\!/_{\prod_i(-\Lambda_i)} \prod_i S^\flat_i.$$

Taking an iterated symplectic reduction of $\prod_i \widetilde{\mathscr{M}}_i$ by the actions of $\operatorname{GSp}_{2n}(\mathbb{C})$ and $\prod_i S^\flat_i$ is independent of order; i.e.,

$$\left(\left(\prod_i \widetilde{\mathscr{M}}_i\right) /\!\!/_0 \operatorname{GSp}_{2n}(\mathbb{C})\right) /\!\!/_{\prod_i(-\Lambda_i)} \prod_i S^\flat_i \cong \prod_i \left(\widetilde{\mathscr{M}}_i /\!\!/_{-\Lambda_i} S^\flat_i\right) /\!\!/_0 \operatorname{GSp}_{2n}(\mathbb{C}).$$

Therefore,

$$\begin{aligned}
\mathscr{M}^*(\mathbf{A}) &\cong \left(\prod_i \mathscr{M}_i\right) /\!\!/_0 \, \mathrm{GSp}_{2n}(\mathbb{C}) && \text{by (1)} \\
&\cong \left(\prod_i \left(\widetilde{\mathscr{M}_i} /\!\!/_{-\Lambda_i} S_i^\flat\right)\right) /\!\!/_0 \, \mathrm{GSp}_{2n}(\mathbb{C}) && \text{by Proposition 5.16} \\
&\cong \left(\left(\prod_i \widetilde{\mathscr{M}_i}\right) /\!\!/_0 \, \mathrm{GSp}_{2n}(\mathbb{C})\right) /\!\!/_{\prod_i(-\Lambda_i)} \prod_i S_i^\flat && \text{as shown above} \\
&\cong \widetilde{\mathscr{M}^*}(\mathbf{A}) /\!\!/_{\prod_i(-\Lambda_i)} \prod_i S_i^\flat && \text{by (2),}
\end{aligned}$$

completing the proof of (3). □

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Vita

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