10-15-2018

Gravitation in terms of observables

Rodolfo Gambini
Universidad de la Republica Instituto de Fisica

Jorge Pullin
Louisiana State University

Gravitation in terms of observables

Rodolfo Gambini[1] and Jorge Pullin[2]

1. Instituto de Fsica, Facultad de Ciencias,
Igu 4225, esq. Mataojo, 11400 Montevideo, Uruguay.
2. Department of Physics and Astronomy,
Louisiana State University, Baton Rouge, LA 70803-4001.

In the 1960's, Mandelstam proposed a new approach to gauge theories and gravity based on loops. The program was completed for Yang–Mills theories by Gambini and Trias in the 1980's. In this approach, gauge theories could be understood as representations of certain group: the group of loops. The same formalism could not be implemented at that time for the gravitational case. Here we would like to propose an extension to the case of gravity. The resulting theory is described in terms of loops and open paths and can provide the underpinning for a new quantum representation for gravity distinct from the one used in loop quantum gravity or string theory. In it, space-time points are emergent entities that would only have quasi-classical status. The formulation may be given entirely in terms of Dirac observables that form a set of gauge invariant functions that completely define the Riemannian geometry of the spacetime. At the quantum level this formulation will lead to a reduced phase space quantization free of any constraints.

I. INTRODUCTION

There exists a renewed interest in the description in terms of observables of gauge theories and gravity. Recently, Giddings and Donnelly [1] proposed explicit constructions that extend the observables associated to gauge theories to the case of gravitation for weak fields. They note that an important feature of the resulting quantum theory of gravity is the algebra of observables, that becomes non-local. Observable-based techniques are also used in several modern developments attempting to extract information from quantum gauge theories [2]. The most ambitious attempt to describe gravity intrinsically without coordinates was proposed by Mandelstam in the 1960's [3]. The approach did not flourish because the intrinsic description loses completely the notion of space-time point, and becomes difficult to recover it even classically. Paths that end in the same physical point in this description cannot be easily recognized. In the 1980's Gambini and Trias [4] showed that gauge theories arise as representations of the group of loops in certain Lie groups. The complete geometric structure of gauge theories can be recovered from identities obeyed by the infinitesimal generators of the group of loops. The possibility of extending this description to the gravitational case did not appear possible due to the issues we mentioned with Mandelstam's approach. In this paper we will show how to extend the notion of the group of loops and its representations which arise in gauge theories to the gravitational case. This leads to a complete classical description of gravitation without coordinates. The metric is everywhere referred to local frames parallel transported starting from a given point. In such frames it takes the Minkowskian form. The geometrical content of the theory is completely recovered by relations between reference frames obtained by parallel transport along paths that differ by an infinitesimal loop and is given by the Riemann tensor. Although the construction is

based on loops, it differs from the one underlying the usual loop representation of gauge theories and gravity. In the loop representation the objects constructed are gauge invariant whereas in the present construction the objects are both gauge invariant and space-time diffeomorphism invariant. That is, the objects are Dirac observables. This leads to a theory that *does not involve diffeomorphisms* and may allow to bypass at the quantum level the LOST-F [5] theorem that leads to a discrete structure in the Hilbert space of ordinary loop quantum gravity and conflicts with the differentiability of the group of loops. The latter is crucial to recover the kinematics of gauge theories and gravity in this context.

The organization of this paper is as follows: In section II we make a brief review of the group of loops on differential manifolds. In section III we introduce gauge theories as representations of the group of loops. In section IV we recall the Mandelstam approach, in terms of intrinsic paths, to gravity and discuss some of its problems. In section V we extend the loop techniques to intrinsic paths. In section VI we show that an intrinsic description of gravity arises as a representation of the group of loops in the Lorentz group. In section VII we establish the relation between the intrinsic and coordinate descriptions. In section VIII we show that the intrinsic and coordinate representations of gravity are equivalent at the classical level but they are not equivalent at the quantum level. In section IX we present an intrinsic path dependent Lagrangian formalism for arbitrary path dependent fields. In section X we analyze the relation between path dependence and diffeomorphisms. In section XI we show how to extend the Hamiltonian techniques to intrinsic paths. Finally in section XII we present some concluding remarks.

II. THE GROUP OF LOOPS: A BRIEF REVIEW

A. Holonomies and the definition of loops

We will briefly review some notions of the group of loops. For a more extensive treatment see [4, 6].

We start with a set of parametrized curves on a manifold M. We assume they are continuous and piecewise smooth. There is no real need to have the curves parameterized but we do it to fix ideas. A curve p is a map

$$p : [0, s_1] \cup [s_1, s_2] \cdots [s_{n-1}, 1] \ \rightarrow \ M \tag{2.1}$$

that is smooth in each closed interval $[s_i, s_{i+1}]$ and continuous in the whole domain. Given two piecewise smooth curves p_1 and p_2 where the end point of p_1 is the same as the beginning point of p_2, the composition curve $p_1 \circ p_2$ is given by:

$$p_1 \circ p_2(s) = \begin{cases} p_1(2s), & \text{for } s \in [0, 1/2] \\ p_2(2(s-1/2)) & \text{for } s \in [1/2, 1]. \end{cases} \tag{2.2}$$

The curve traversed in the opposite orientation ("opposite curve") is given by

$$p^{-1}(s) := p(1-s). \tag{2.3}$$

We also consider closed curves $l, m, ...$, that is, curves which start and end at the same point o. We call L_o the set of all such closed curves. The set L_o is a semi-group under the composition law $(l, m) \rightarrow l \circ m$. The identity element ("null curve") is defined to be the

constant curve $i(s) = o$ for any s and any parametrization. However, we do not have a group structure, since the opposite curve l^{-1} is not a group inverse in the sense that $l \circ l^{-1} \neq i$.

Holonomies are given by the parallel transport around closed curves. The parallel transport around a closed curve $l \in L_o$ is a map from the fiber over o to itself given by the path ordered exponential,

$$H_A(l) = P \exp \int_l A_a(y) dy^a. \tag{2.4}$$

The holonomy H_A is an element of the group G and the product denotes the right action of G. The main property of H_A is

$$H_A(l \circ m) = H_A(l) H_A(m). \tag{2.5}$$

A change in the choice of the point on the fiber over o from o to o' induces the transformation

$$H'_A(l) = g^{-1} H_A(l) g, \tag{2.6}$$

where g is the holonomy of a path joining o to o'.

In order to transform the set L_o into a group, we need to introduce a further equivalence relation, the idea is to identify all curves yielding the same holonomy. These equivalence classes we will from now on call *loops*. We will denote them with Greek letters, to distinguish them from the individual curves of the equivalence classes. Several definitions of this equivalence relation have been proposed. The simplest one is that the curves yield the same holonomy for any connection. Related to it is that two curves that differ by a retraced path ("tree") are equivalent since retraced paths (paths that go out and back along the same curve) do not contribute to the holonomy. There are other possible definitions but we will not discuss them here (see [6] and [7, 8] for details).

With any of the definitions one can show that the composition between loops is well defined and is again a loop. In other words if $\alpha \equiv [l]$ and $\beta \equiv [m]$ then $\alpha \circ \beta = [l \circ m]$ where by $[]$ we denote the equivalence classes.

With the equivalence relation defined, it makes sense to define an inverse of a loop. Since the composition of a curve with its opposite yields a tree (see figure 1) it is natural, given a loop α, to define its inverse α^{-1} by $\alpha \circ \alpha^{-1} = \iota$ where ι is the set of closed curves equivalent to the null curve (thin loops or trees). α^{-1} is the set of curves opposite to the elements of α. We will also denote inverse loops with an overbar $\alpha^{-1} \equiv \overline{\alpha}$.

We will denote the set of loops base-pointed at o by $\mathcal{L}_o$. Under the composition law given by $\circ$ this set is a non-Abelian group, which is called the group of loops.

We have relations between holonomies of composed loops

$$H(\alpha \circ \beta) = H(\alpha) H(\beta), \tag{2.7}$$

and that inverses are mapped to each other,

$$H(\alpha^{-1}) = (H(\alpha))^{-1}. \tag{2.8}$$

We will define a set of differential operators acting on functions of loops that are related to the infinitesimal generators of the group of loops: the loop and connection derivatives.

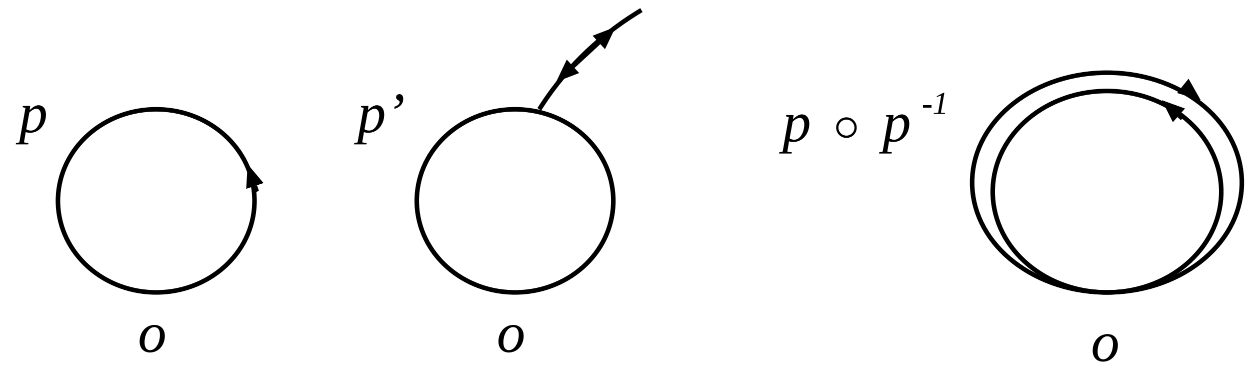

FIG. 1: Curves p and p' differ by a tree. The composition of a curve and its inverse is a tree.

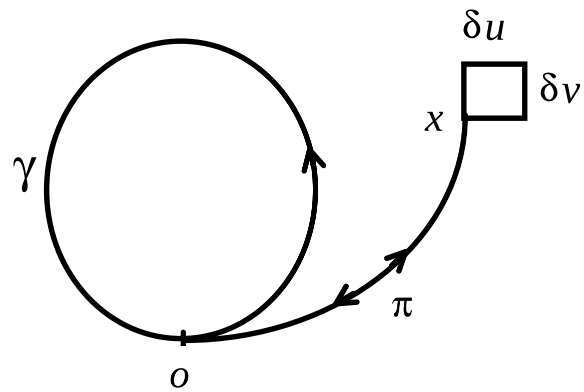

FIG. 2: The infinitesimal loop that defines the loop derivative.

B. The loop derivative

Given $\Psi(\gamma)$ a continuous, complex-valued function of $\mathcal{L}_o$ we want to consider its variation when the loop γ is changed by the addition of an infinitesimal loop $\delta\gamma$ base-pointed at a point x connected by a path π_o^x to the base-point of γ, as shown in figure 2. That is, we want to evaluate the change in the function when changing its argument from γ to $\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma$. In order to do this we will consider a two-parameter family of infinitesimal loops $\delta\gamma$. Notice that no matter what path π one chooses, the added path is infinitesimal due to the invariance of loops under re-tracings —additions of trees— and therefore induces an infinitesimal deformation of γ. Since spacetimes look flat at sufficiently small regions, $\delta\gamma$ may be described in a particular coordinate chart by the curve obtained by traversing the vector u^a from x^a to $x^a + \epsilon_1 u^a$, the vector v^a from $x^a + \epsilon_1 u^a$ to $x^a + \epsilon_1 u^a + \epsilon_2 v^a$, the vector $-u^a$ from $x^a + \epsilon_1 u^a + \epsilon_2 v^a$ to $x^a + \epsilon_2 v^a$ and the vector $-v^a$ from $x^a + \epsilon_2 v^a$ back to x^a as shown in figure 2. We will denote these kinds of curves with the notation $\delta u \delta v \overline{\delta u}\, \overline{\delta v}$.

For a given π and γ a loop differentiable function depends only on the infinitesimal vectors

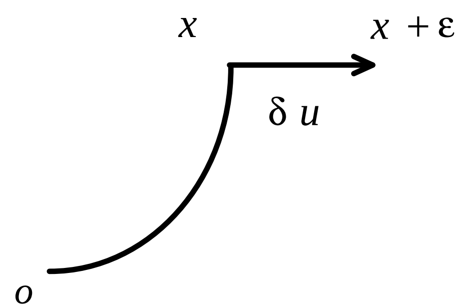

FIG. 3: The extended path defining the Mandelstam derivative, $\pi_E = \pi_o^x \circ \delta u$

$\epsilon_1 u^a$ and $\epsilon_2 v^a$. We will assume it has the following expansion with respect to them,

$$\begin{aligned}\Psi(\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma) = \quad & \Psi(\gamma) + \epsilon_1 u^a Q_a(\pi_o^x)\Psi(\gamma) + \epsilon_2 v^a P_a(\pi_o^x)\Psi(\gamma) \\ & + \tfrac{1}{2}\epsilon_1\epsilon_2(u^a v^b + v^a u^b) S_{ab}(\pi_o^x)\Psi(\gamma) \\ & + \tfrac{1}{2}\epsilon_1\epsilon_2(u^a v^b - v^a u^b)\Delta_{ab}(\pi_o^x)\Psi(\gamma). \end{aligned} \tag{2.9}$$

where Q, P, S, Δ are differential operators on the space of functions $\Psi(\gamma)$. If ϵ_1 or ϵ_2 vanishes or if u is collinear with v then $\delta\gamma$ is a tree and all the terms of the right-hand side except the first one must vanish. This means that $Q = P = S = 0$. Since the antisymmetric combination $(u^a v^b - v^a u^b)$ vanishes in any of these cases, Δ need not be zero. That is, a function is loop differentiable if for any path π_o^x and vectors u, v, the effect of an infinitesimal deformation is completely contained in the path dependent antisymmetric operator $\Delta_{ab}(\pi_o^x)$,

$$\Psi(\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma) = (1 + \tfrac{1}{2}\sigma^{ab}(x)\Delta_{ab}(\pi_o^x))\Psi(\gamma), \tag{2.10}$$

where $\sigma^{ab}(x) = 2\epsilon_1\epsilon_2(u^{[a}v^{b]})$ is the element of area of the infinitesimal loop $\delta\gamma$. We will call this operator the loop derivative.

Loop derivatives do not commute. One can show that,

$$[\Delta_{ab}(\pi_o^x), \Delta_{cd}(\chi_o^y)] = \Delta_{cd}(\chi_o^y)[\Delta_{ab}(\pi_o^x)], \tag{2.11}$$

where we have introduced in the right hand side the loop derivative of functions of open paths from which it is immediate to show that

$$\Delta_{ab}(\pi_o^x)[\Delta_{cd}(\chi_o^y)] = -\Delta_{cd}(\chi_o^y)[\Delta_{ab}(\pi_o^x)]. \tag{2.12}$$

Given a function of an open path $\Psi(\pi_o^x)$, a local coordinate chart at the point x and a vector in that chart u^a, we define the Mandelstam derivative by considering the change in the function when the path is extended from x to $x + \epsilon u$ by the infinitesimal path δu shown in figure 3 as

$$\Psi(\pi_o^x \circ \delta u) = (1 + \epsilon u^a D_a)\Psi(\pi_o^x). \tag{2.13}$$

One can derive a Bianchi identity, based on the fundamental idea that "the boundary of a boundary vanishes" and constructing a tree that run the along the edges of a parallelepiped

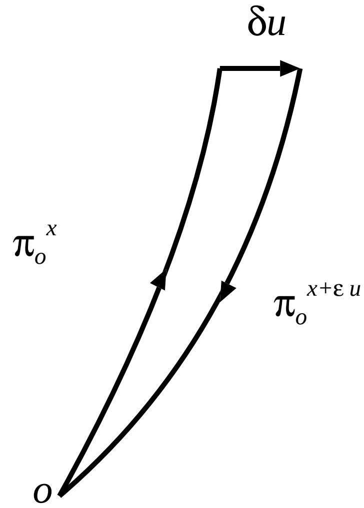

FIG. 4: The path that defines the connection derivative. We assume that the point o is in the same coordinate patch as x.

(see ref. [4]) . The result is,

$$D_a \Delta_{bc}(\pi_o^x) + D_b \Delta_{ca}(\pi_o^x) + D_c \Delta_{ab}(\pi_o^x) = 0. \tag{2.14}$$

There is also a Ricci identity,

$$[D_a, D_b]\Psi(\pi_o^x) = \Delta_{ab}(\pi_o^x)\Psi(\pi_o^x). \tag{2.15}$$

This is the analogue of the usual commutator of covariant derivatives and its relation to the Yang–Mills curvature.

C. The connection derivative

One can introduce a differential operator with properties similar to those of the connection or vector potential of a gauge theory, this allows for a better contact with the usual formulation of gauge theories.

Let us consider a covering of the manifold with overlapping coordinate patches. We attach to each coordinate patch $\mathcal{P}^i$ a path $\pi_o^{y_0^i}$ going from the origin of the loop to a point y_0^i in $\mathcal{P}^i$. We also introduce a continuous function with support on the points of the chart $\mathcal{P}^i$ such that it associates to each point x on the patch a path $\pi_{y_0^i}^x$. Given a vector u at x, the connection derivative of a continuous function of a loop $\Psi(\gamma)$ will be obtained by considering the deformation of the loop given by the path $\pi_o^{y_0^i} \circ \pi_{y_0^i}^x \circ \delta u \circ \pi_{x+\epsilon u}^{y_0^i} \circ \pi_{y_0^i}^o$ shown in figure 4. The path δu goes from x to $x + \epsilon u$. We will say that the connection derivative δ_a exists and is well defined if the loop dependent function of the deformed loop admits an expansion in terms of ϵu^a given by

$$\Psi(\pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o \circ \gamma) = (1 + \epsilon u^a \delta_a(x))\Psi(\gamma), \tag{2.16}$$

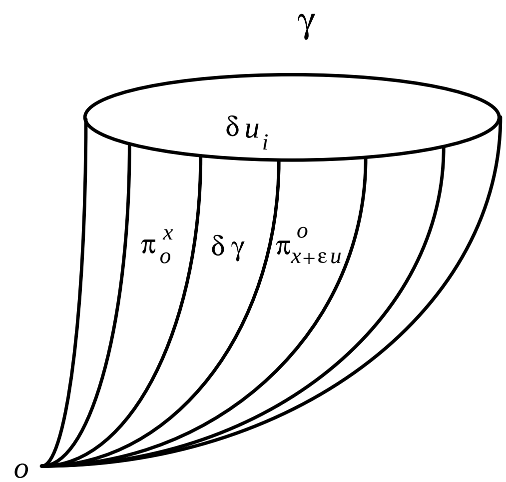

FIG. 5: Generating a finite loop using the infinitesimal generators combining (2.18) and (2.19).

where we have written π_o^x to denote the path $\pi_o^{y_0^i} \circ \pi_{y_0^i}^x$ and similarly for its inverse.

One can show the following relation between the connection and the loop derivatives,

$$\Delta_{ab}(\pi_o^x) = \partial_a \delta_b(x) - \partial_b \delta_a(x) + [\delta_a(x), \delta_b(x)], \tag{2.17}$$

again reminiscent of expressions in ordinary Yang–Mills theory. The loop derivative defined by (2.17) automatically satisfies the Bianchi identities.

The usual relation between connections and holonomies in a local chart in a gauge theory can also be written in this language, it is given by the path ordered exponential,

$$U(\gamma_0) = \mathrm{P}\exp\left(\int_{\gamma_0} dy^a \delta_a(y)\right), \tag{2.18}$$

where $U(\gamma_0)\Psi(\gamma) = \Psi(\gamma_0 \circ \gamma)$. This again is reminiscent of the familiar expression for gauge theories, which yields the holonomy in terms of the path ordered exponential of a connection. Through a second path ordered integral it could be expressed in terms of the loop derivative, embodying the usual non-Abelian Stokes theorem and illustrated in figure 5.

The relation between the connection and the loop derivative can be derived in the following way. Consider a deformation going from π^x to $\pi^{x+\epsilon}$ given by the displacement vector field along the path π defined as follows: Let π^x be given by $x^\alpha(\lambda)$ such that $x^\alpha(\lambda_f) = x^\alpha$ end point of π, and $\pi^{x+\epsilon u}$ be given by $x'^\alpha(\lambda)$ such that $x'^\alpha(\lambda_f) = x^\alpha + \epsilon^\alpha$. Then the displacement field connecting both paths will be given by $x'^\alpha(\lambda) = x^\alpha(\lambda) + \epsilon^\beta w_\beta^\alpha(\lambda)$ for all λ belonging to $[0, \lambda_f]$ and $w_\beta^\alpha(\lambda_f) = \delta_\beta^\alpha$. From this relation and the definition of the derivatives we get

$$\delta_\mu(\pi^x) = \int_0^{\lambda_f} \Delta_{\alpha\beta}(\pi^x(\lambda))\dot{x}^\alpha(\lambda) w_\mu^\beta(\lambda) d\lambda. \tag{2.19}$$

Once one attaches to each point of an open region in the manifold a given path π_o^x, the connection derivative is an ordinary function $\delta_\mu(x) = \delta_\mu(\pi_o^x)$. The substitution of (2.19) for the family of paths π_o^x into (2.18) embodies the general form of the non Abelian Stokes'

theorem allowing to write an arbitrary loop deformation as a "surface" integral of loop derivatives. One may therefore consider the loop derivatives as the infinitesimal generators of the group of loops.

III. KINEMATICS OF YANG-MILLS THEORIES AS REPRESENTATIONS OF THE GROUP OF LOOPS

We would like to recall how the kinematical structure of gauge theories emerges from the group of loops. We consider a map of the group of loops onto some gauge group G,

$$\mathcal{H} : \mathcal{L}_0 \to G, \tag{3.1}$$

i.e.,

$$\gamma \longrightarrow H(\gamma), \tag{3.2}$$

such that $H(\gamma_1)H(\gamma_2) = H(\gamma_1 \circ \gamma_2)$.

Let us consider a specific Lie group, for instance $SU(N)$, with $N^2 - 1$ generators X^i such that $\text{Tr}X^i = 0$ and

$$[X^i, X^j] = C^{ij}_k X^k, \tag{3.3}$$

where C^{ij}_k are the group's structure constants.

Let us compute the action of the connection derivative in this representation. We use the same prescriptions as in the previous section

$$(1 + \epsilon u^a \delta_a(x))H(\gamma) = H(\pi^x_o \circ \delta u \circ \pi^o_{x+\epsilon u} \circ \gamma) = H(\pi^x_o \circ \delta u \circ \pi^o_{x+\epsilon u})H(\gamma). \tag{3.4}$$

Since the loop $\pi^x_o \circ \delta u \circ \pi^o_{x+\epsilon u}$ is close to the identity loop (with the topology of loop space) and since H is a continuous, differentiable representation,

$$H(\pi^x_o \circ \delta u \circ \pi^o_{x+\epsilon u}) = 1 + i\epsilon u^a A_a(x), \tag{3.5}$$

where $A_a(x)$ is an element of the algebra of the group, in our example of $SU(N)$. That is, $A_a(x) = A^i_a(x)X^i$. Therefore, we see that through the action of the connection derivative,

$$\delta_a(x)H(\gamma) = iA_a(x)H(\gamma). \tag{3.6}$$

Following similar steps one obtains the action of the loop derivative,

$$\Delta_{ab}(\pi^x_o)H(\gamma) = iF_{ab}(x)H(\gamma), \tag{3.7}$$

where F_{ab} is an algebra-valued antisymmetric tensor field.

From equation (2.17) we immediately get the usual relation defining the curvature in terms of the potential,

$$F_{ab}(x) = \partial_a A_b(x) - \partial_b A_a(x) + i[A_a, A_b]. \tag{3.8}$$

From (2.18) and (3.5) we also have that,

$$H(\eta) = \mathrm{P}\exp\left(i\oint_{\eta} dy^a A_a(y)\right), \tag{3.9}$$

yielding the usual expression for the holonomy of the connection A_a.

In this framework, matter fields can be included considering open paths. For more details see [6].

Finally, the usual form of the Ricci identity,

$$[D_a, D_a] = iF_{ab}, \tag{3.10}$$

can be obtained directly from the previous expressions, in particular (2.15).

This construction allows to recover any gauge theory with local symmetries associated to a fiber bundle structure. The extension of this construction to gravity is not trivial. In the language of fiber bundles it requires the introduction of a soldering form connecting the fiber to the manifold [9]. This is not the approach we will take in this paper. In the forthcoming sections we will develop a formalism that exploits the properties of the group of loops to construct an intrinsic description of the Riemannian geometry.

IV. BRIEF REVIEW OF MANDELSTAM'S 1962 PROPOSAL FOR QUANTIZING THE GRAVITATIONAL FIELD

Mandelstam starts with a critique of the usual approaches to quantizing the gravitational field, which consider c-number coordinates and q-number metrics and distances. The diffeomorphism invariance of a theory of quantities like the distances that are partially quantized through the metric could be problematic. He is interested in formulating an approach that is coordinate independent and therefore only framed in terms of q-number physical quantities associated to intrinsically defined paths without any ambiguity associated with coordinate conditions, and all distances that appear in the theory will be physical distances. He focuses on paths in space-time (manifold plus metric) constructed by starting from a reference point (for instance infinity in an asymptotically flat situation, notice that it would require a suitable compactification) and constructing an inertial reference frame at the reference point (from now on we call it “the origin”). He then specifies a second point, not by using coordinates, but by considering a path from the origin to the new point. To construct the path he chooses a vector defined in the local reference frame at the origin and parallel transports infinitesimally such reference frame along the integral curve of the vector. At the next point another vector is chosen and so on. For instance, one could move a certain distance along the geodesic the x direction taking the reference frame along this path, then another distance along the y direction defined with respect to the reference frame obtained at the end of the first transport. He wishes to describe the gravitational field in terms of these paths and therefore without referring to a description of the space-time in terms of coordinates defined on an open set of the space-time and their transformations. With the information available about the paths in this intrinsic framework one cannot say if two paths have led to the same point just by the specification of the paths. However, the question can be answered with a knowledge of the Riemann tensor. If all physical measurements (e.g. all gauge invariant functions of all fields) at the ends of the paths are the same or differ by a

Lorentz transformation we can say that they ended in the same point. It is clear that this is not a useful way to distinguish paths in practice. Notice that the construction is such that all along the paths the metric is Minkowskian even though the space-time is not necessarily flat because it results from the parallel transport of the inertial frame F given at o. To have a completely invariant description of the process, the paths are parameterized by the invariant distance traversed (or the proper time in the case of timelike paths).

To flesh out the above ideas, consider two paths π_1 and π_2 such that, after a portion of π_2 common to both paths (that we shall call π_3^z) has been traversed, they differ by a small area $\sigma_{\mu\nu}$. Given a vector a_μ in the frame at the point where the two paths start to differ, the vector at the same point but at the end of the closed path will differ by an amount,

$$da_\lambda = \frac{1}{2}\sigma^{\mu\nu} R_{\mu\nu\lambda}{}^{\sigma}\left(\pi_3^z\right) a_\sigma. \tag{4.1}$$

Mandelstam denotes with x, y, or z the end point of the path π in the intrinsic framework. That is, the components of the end point, given by x^α are the total displacement along each of the unit vectors of the parallel transported reference frame e_α. The above expression is valid in the reference frame parallel transported to z along π_3^z, we denote this by making the components of tensors like the Riemann tensor explicitly path dependent.

The vectors defining the reference frame also get rotated and this difference also contributes to the path dependence of the field variables. So both vectors and the path are rotated. If we think of the paths as curves on space-time, the direction of the portion of the path π_2 following π_3 will be rotated with respect to the original portion of π_1 by an amount proportional to the Riemann tensor at z. In this framework quantities become path dependent for two reasons: the path determines the point where the quantity is observed and in the case of coordinate dependent quantities it also determines the reference frame chosen to describe them. The variation of a vector field $A_\mu(\pi_1^x)$ in a weak gravitational field when one moves along a path like the one described above will be given by,

$$\delta_z A_\mu(\pi_1^x) = \frac{1}{2}\sigma^{\kappa\lambda} R_{\kappa\lambda\mu}{}^{\nu}(\pi_3^x) A_\nu(\pi_1^x) - \frac{1}{2}\sigma^{\kappa\lambda} R_{\kappa\lambda\tau}{}^{\nu}(\pi_3^z)\,(x-z)^\tau\,\frac{\partial A_\mu(\pi_1^x)}{\partial x^\nu} \tag{4.2}$$

The first term is due to the rotation of the reference frame. The second term represents the effects of the change of the path. The above expression is only valid in the linearized case, it ignores higher corrections in the curvature and assumes that points x and z are on the same flat patch in which one can set up coordinates such that quantities like $(x-z)^\tau$ behave as vectors and one can compute a derivative without a non-trivial connection. In the general case of a strong gravitational field there would be terms with higher order powers in the curvature all along the path and one does not have a closed form for the deformation at the end of π_3. In particular it would be very difficult to determine the displacement of the end points under arbitrary deformations. We conclude from this analysis that paths ending at the same physical point cannot be easily recognizable in the intrinsic notation. Teitelboim [10] made some progress on this issue but only for infinitesimally close paths. Moreover, as the previous analysis shows, the end points of two different paths like π_1 and π_2 defined intrinsically could be the same without implying that both paths end at the same physical point. Another related important obstacle for a practical implementation of this intrinsic formalism is that the previous analysis shows that closed loops in space-time will be very difficult to recognize in the intrinsic notation and therefore the groups of loops will not be of any practical use.

V. A NEW INTRINSIC DESCRIPTION: THE GROUP OF LOOPS IN THE GRAVITATIONAL CASE

At the end of the previous section we have sketched some of the obstacles faced by the Mandelstam formulation. Here we will tackle these issues. In first place we will refine the intrinsic description of the paths in such a way that "trees", that is, closed paths from the base point o equivalent to the null path that do not contribute to holonomies, could be easily recognized. Then we will introduce a technique allowing to assign to each physical point, that is to each point of the manifold M, intrinsically described paths that end at that point. These conditions will allow to apply the loop techniques to the intrinsic description of gravitation. In particular they will allow to recognize closed loops in M and to recognize paths ending at the same physical point.

Let us start by a path in a manifold M whose geometry is given. We shall assume that all the paths start at the same point o of M. If the space-time is asymptotically flat we shall choose o at infinity and assume diffeomorphisms and gauge transformations reduce to the identity there. In non asymptotically flat situations, like cosmologies, one could pick a point in the infinite past or future (notice that we are considering spatio-temporal paths). We will describe paths in M intrinsically in terms of a Lorentz reference frame in o. Given a reference frame F in o a path is described as follows: Starting from the origin we parallel transport, for an invariant distance ds, the reference frame with "velocity" $v^\alpha(0)$ to a new point d_1x^α such that the displacement is $d_1x^\alpha = v^\alpha(0)ds$. Starting at this point we proceed to a new point moving further the reference frame with velocity $v^\alpha(ds)$ and displacement $d_2x^\alpha = v^\alpha(ds)ds$. All the displacements are given in terms of invariant distances and the parallel transported reference frame. The intrinsic description of the path π^x may be therefore described by $x^\alpha(s)$ such that $v^\alpha(s) = dx^\alpha/ds$ and $x^\alpha = x^\alpha(s_f)$ is the intrinsic total displacement associated to the end point. We will say that a path is reducible if it contains a portion $x^\alpha(s)$ with $s_0 < s < s_1$ such that for any point s in this interval $v^\alpha(s) = -v^\alpha(2s_1 - s)$. The construction is such that portions of the path followed forward and back along the same curve —following a tree— can be eliminated from the final description of the path. This is because after following a tree one returns to the same initial frame. We will therefore only consider irreducible paths under the equivalence by trees. It will be convenient in certain occasions to use a generic parametrization $x^\alpha(\lambda)$ with λ an arbitrary parameter. The invariant distance may be always recovered by considering $ds = \sqrt{\eta_{\alpha\beta}dx^\alpha dx^\beta}$. At this stage we are not considering null paths, except perhaps as limits.

A. The loop derivative

We have already noticed that in the Mandelstam construction paths ending at the same physical point cannot be easily recognized. They may be identified only indirectly by noticing that all the physical fields defined at the end of two paths ending at the same physical points π^{x_1} and π^{x_2} are related by a Lorentz transformation. Furthermore this difficulty implies that closed loops in physical space will appear as open in intrinsic notation and that there will be hidden relations between path dependent fields ending on different points in intrinsic notation that extend the Eq. (4.2) to the case of strong gravitational fields. Without a satisfactory solution to this problem, the approach proposed by Mandelstam cannot be used in practice.

This difficulty can be solved as follows: given an intrinsically described path π^x that

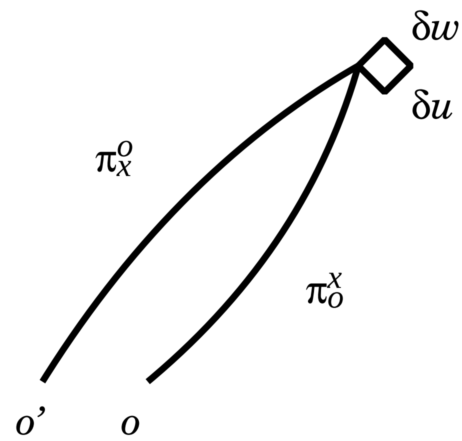

FIG. 6: The path described in the text: the initial and final point will have the same intrinsic coordinate (that is why both paths are labeled by o and x) but would correspond to two different end points of the space-time, o and o'.

arrives to some physical point in M, we are going to show here how to identify other intrinsic paths $\pi^{'x'}$ that arrive at the same physical point[1]. This identification will allow solving the above mentioned problems and applying the loop calculus techniques summarized in the first sections. Let us start by learning how to describe intrinsically closed paths that correspond to the infinitesimal generators of the group of loops, the loop derivatives. The corresponding holonomies associated with these paths determine the Lorentz transformation connecting the reference frame F given initially at o with the frame obtained at the end of the closed path. Recall that the infinitesimal loop added by the loop derivative using the standard notation of section II on a differential manifold M is given by $\pi_o^x \delta\gamma \pi_x^o$ with $\delta\gamma$ obtained traversing the curve $\delta u \delta w \overline{\delta u \delta w}$. However, if we describe this loop using the intrinsic description given above in terms of displacement vectors referred to a local system of reference parallel transported from the origin o to each point of the path after following the closed loop $\delta\gamma$ the reference system will be rotated and, consequently, a vector at x before the rotation will be rotated by an amount $\delta v^\rho = \delta u^\alpha \delta w^\beta R_{\alpha\beta\sigma}{}^\rho v^\sigma$. This rotation as we have discussed implies that if one attempts going back to the origin following π_x^o with the same prescription given to reach x in reverse order one will end up in a different point of the space-time as shown in figure 6.

In order to go back to the origin along the original path in M, we need to take into account the Lorentz rotation suffered by the reference frame after following the closed path, then instead of considering the intrinsic initial displacement $v^\alpha(s)ds$ followed in the opposite direction we consider $(-v^\alpha(s) - \delta u^\rho \delta w^\sigma R_{\rho\sigma}{}^\alpha{}_\beta(\pi_0^x) v^\beta(s))ds$. With this prescription we are now following the physical path pi_o^x in the opposite direction, but now, as the parallel transported reference frame was rotated, the intrinsic displacements needed to keep track

[1] Notice that x and x' will be different in general, the intrinsic total displacements for different paths with the same end point will in general be different in strong gravity. We use this notation only for labeling points along a given path. Also notice that the information about the intrinsic total displacement is redundant because it is contained in the information that defines the path π, as we noted in the introduction of this section.

of this rotation were rotated in the opposite sense. It is important to remark that when one is back at the origin one ends up with a reference frame F' rotated with respect to the original one. Vector components v^β with respect to F will be related to vector components with respect to F' by a Lorentz transformation given by the holonomy,

$$H(\pi_o^x \circ \delta\gamma \circ \Lambda(\delta\gamma)\pi_x^o)^\alpha{}_\beta = \delta^\alpha{}_\beta + \delta u^\rho \delta w^\sigma R_{\rho\sigma}{}^\alpha{}_\beta(\pi_o^x), \tag{5.1}$$

where $\Lambda(\delta\gamma)\pi_x^o$ is the retraced rotated path described above.

Also notice that we have followed a closed path in space-time but the final intrinsic coordinate will be different from the vanishing-initial one. The intrinsic path associated to the infinitesimal generator of the group of loops may be represented in compact notation as $\pi \circ \delta\gamma \circ \Lambda(\delta\gamma)\overline{\pi}$. It will be convenient in order to keep track of the order of infinitesimals to introduce a parameter ϵ with dimensions of length, much smaller than the length associated with the curvature of space-time such that $\delta u = \epsilon u$ and $\delta w = \epsilon w$.

Note that the paths $\epsilon u \epsilon w \epsilon \overline{u} \epsilon \overline{w}$ are only closed for infinitesimal loops, for finite ones they are not closed in a generic curved space-time. In order for it to close —for a small, but fixed, ϵ— one has to consider $\epsilon u \epsilon w \epsilon \overline{u} \epsilon \overline{w^{(1)}}$ where $w^{(1)}$ is given in the appendix. The holonomy induced by both paths coincides at order ϵ^2 but differs by terms $(R\epsilon^2)^2$ with R the typical scale of the curvature of space-time. The proposed description is therefore correct for closed paths with finite ϵ if $R\epsilon^2 \ll 1$, which always holds for classical gravity for sufficiently small ϵ. In the quantum case one expects that ϵ cannot be made smaller than the Planck length ℓ_{Planck} and $R\ell^2_{\text{Planck}}$ could be of order one; for instance, in the region of a black hole corresponding to the classical singularity. This indicates that at those scales the notion of curvature, and consequently the notion of point is completely lost.

In the appendix the path that must be followed to close an intrinsic loop is constructed. The result that is convenient to keep in mind in what follows is,

$$w^{(1)\mu} = w^\mu + \frac{1}{6} R^\mu_{\alpha\rho\gamma} w^\alpha u^\rho w^\gamma \epsilon^2 + \frac{1}{3} R^\mu{}_{\alpha\rho\gamma} u^\alpha u^\rho w^\gamma \epsilon^2. \tag{5.2}$$

It is important to point out that once one has identified closed infinitesimal paths one has everything needed in order to describe generic closed paths —loops— and in terms of them to define a notion of point by associating them with equivalence classes of open paths that differ by closed loops. The notion of closed path that is proposed stops being valid when the notion of point does. This will occur in the deep quantum regime.

Having defined intrinsic descriptions for the infinitesimal generators of the group of loops and the associated holonomies, we can compute the holonomies corresponding to finite deformations by considering the product of infinitesimal generators. Notice that in order to compute the product we need to relate each infinitesimal path to the parallel transported reference frame the path that comes before it. In compact notation, for the product of two infinitesimal generators, we need to consider the closed path,

$$\pi_1 \circ \delta\gamma_1 \circ \Lambda(\delta\gamma_1)\,\overline{\pi_1} \circ \Lambda(\delta\gamma_1)\,\pi_2 \circ \Lambda(\delta\gamma_1)\,\delta\gamma_2 \circ \Lambda(\delta\gamma_2)\,\Lambda(\delta\gamma_1)\,\overline{\pi_2} \tag{5.3}$$

which corresponds to the infinitesimal holonomy $H = H_1 H_2$. Notice that though the group of loops can be defined in an arbitrary differential manifold (as we showed in section 2) without reference to its geometry, the intrinsic loop description depends on the geometry. Taking into account the way we have proceeded to compute the product of infinitesimal

generators, given two loops γ_1 and γ_2 with origin o described in intrinsic notation, one can define a product $\gamma_1 \cdot \gamma_2$ given by following γ_1 and taking into account the rotation of the reference frame at o, then following $\Lambda(\gamma_1)\gamma_2$. This last object represents the loops whose intrinsic displacements are rotated by Λ from the original components. Explicitly, we have that $\gamma_1 \cdot \gamma_2 = \gamma_1 \circ \Lambda(\gamma_1)\gamma_2$, and one can easily convince oneself that intrinsic loops form a group. The generalized Stokes' theorem allows to obtain the holonomy for an arbitrary loop as a product of infinitesimal Lorentz transformations associated to the infinitesimal generators. With this definition of the group of loops one can recognize two paths ending at the same physical point. Two paths π and π' end at the same point if there exists a loop γ such that the open paths $\gamma \cdot \pi = \pi'$.

B. The connection derivative

1. *A particular case*

The fact that the intrinsic description depends on the geometry now implies that the criterion used to recognize that two paths end in the same point does so too. Therefore in an eventual quantum treatment *the notion of point only acquires precise meaning when quantum fluctuations can be neglected.* We do not include in π the information about the intrinsic coordinates of its end point because these coordinates may take arbitrary values for the same physical end point and do not add relevant information. If the manifold is not simply connected besides the infinitesimal generators one needs information about at least one holonomy of a loop γ connecting paths π and π' ending at the same physical point such that γ is a generator of the homotopy group. The equivalence class of paths that end in the same physical point may be represented by any of the paths that end in that point.

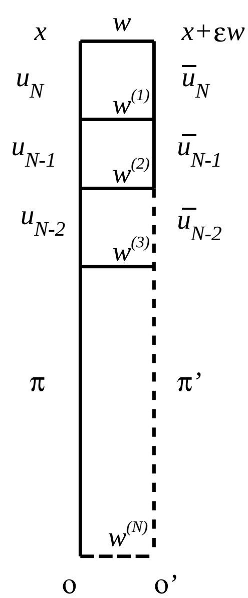

FIG. 7: The holonomy associated with the connection derivative.

We are now going to compute the holonomy associated to a connection derivative, as in (3.5). We will essentially reconstruct figure 4 for a particular path using (2.19). The latter

transforms the path $\pi_o^{o'} \circ \pi'^{\,x+\epsilon w}_{o'}$ to the path $\pi_o^{x+\epsilon w}$ as shown in figure 7, where o is the origin (see below for a more precise discussion of the frames involved). It is computed considering a partition $\epsilon u_1 \cdots \epsilon u_N$ of the path π_o^x and taking the product of loop derivatives,

$$\pi_o^{x-\epsilon u_N} \circ \epsilon u_N \circ \epsilon w \circ \epsilon \overline{u_N} \circ \epsilon \overline{w^{(1)}} \circ \epsilon w^{(1)} \circ \epsilon \overline{u_{N-1}} \circ \epsilon \overline{w^{(2)}} \ldots \tag{5.4}$$

where $\pi_o^{x-\epsilon u_N}$ is the portion of π going from o to $x - \epsilon u_N$. This corresponds to the transformation,

$$\left(\delta_\gamma^\eta + \epsilon^2 u_N^\alpha w^\beta R_{\alpha\beta\gamma}{}^\eta \left(\pi_o^{x-\epsilon u_N}\right)\right) \left(\delta_\eta^\rho - \tilde{u}_{N-1}^\alpha \tilde{w}^{(1)\beta} \epsilon^2 R_{\alpha\beta\eta}{}^\rho \left(\pi_o^{x-\epsilon u_N - \epsilon u_{N-1}}\right)\right) \times \cdots \tag{5.5}$$

and taking into account that the variables with a tilde are Lorentz transformed from the initial ones (e.g. $\tilde{u}^\alpha = \Lambda_1{}^\alpha_\beta u^\beta$ with $\Lambda_1 = \Lambda\left(\epsilon u_N \circ \epsilon w \circ \epsilon \overline{u_N} \circ \epsilon \overline{w^{(1)}}\right)$) we get,

$$\begin{aligned}
&\left(\delta_\gamma^\eta + \epsilon^2 u_N^\alpha w^\beta R_{\alpha\beta\gamma}{}^\eta \left(\pi_o^{x-\epsilon u_N}\right)\right) \left[\delta_\eta^\rho + \epsilon^2 \left(\delta_\sigma^\nu - u_N^\kappa w^\beta \epsilon^2 R_{\kappa\beta\sigma}{}^\nu \left(\pi_o^{x-\epsilon u_N}\right)\right) \times \right.\\
&\times \left. u_{N-1}^\sigma \left(\delta_\mu^\lambda - \epsilon^2 u_N^\chi w^\tau R_{\chi\tau\mu}{}^\lambda \left(\pi_o^{x-\epsilon u_N}\right)\right) w^{(1)\mu} R_{\nu\lambda\eta}{}^\rho \left(\pi_o^{x-\epsilon u_N - \epsilon u_{N-1}}\right)\right] \ldots,
\end{aligned} \tag{5.6}$$

and observing that the corrections introduced by $w^{(1)}, \ldots, w^{(N)}$ grow with the square of the proper distance to the end point x as shown in the appendix, and keeping the result up to order linear in ϵ, we get,

$$H_\gamma{}^\nu \left(\epsilon u_1 \circ \ldots u_N \circ \epsilon w \circ \epsilon \overline{u_N} \circ \ldots \circ \epsilon \overline{w^{(N)}}\right) = \delta_\gamma^\nu + \epsilon w^\rho A_{\rho\gamma}{}^\nu \left(F, \pi_o^x\right), \tag{5.7}$$

with,

$$\begin{aligned}
A_{\rho\gamma}{}^\nu \left(F, \pi_o^x\right) &= \int_0^{s_f} ds \dot{y}^\alpha(s) R_{\alpha\rho\gamma}{}^\nu \left(\pi_o^{y(s)}\right) \\
&+ \frac{1}{6} \int_0^{s_f} ds" \int_{s_f}^{s"} ds' \int_{s_f}^{s'} ds R_{\rho(\beta\alpha)}{}^\mu \left(\pi_o^{y(s)}\right) \dot{y}^\beta(s) \dot{y}^\alpha(s') R_{\mu\delta\gamma}{}^\nu \left(\pi_o^{y(s")}\right) \dot{y}^\delta(s")
\end{aligned} \tag{5.8}$$

where the integral is along π and $A_{\rho\gamma}{}^\nu$ are the Lorentz intrinsic components of the spin connection that depends on the path π referred to the frame F. We add the dependence on F explicitly in the connection since in further usage we will use other frames to which the specification of the paths are referred to. It is important to remark that at order ϵ the quantities $\tilde{u}$ and $\tilde{w}$ are equal to u and w. The loop $\epsilon u_1 \circ \ldots \circ \epsilon u_N \circ \epsilon w \circ \epsilon \overline{u_N} \circ \ldots \circ \epsilon \overline{u^{(1)}} \circ \epsilon \overline{w^{(N)}}$ connects the path $\pi \circ w$ referred to the frame F with the path $\epsilon w^{(N)} \circ \pi_{o'}$ referred to the frame F' that differs from F by the Lorentz transformation (5.7). Both paths end at the same physical point.

2. The general case

The previously defined connection derivative is a particular example of connections relating two "parallel" neighboring paths. But more generally, one can define a connection derivative for each tangent vector in the path manifold. If a path π_o^x is defined by $u^\alpha(\lambda) = dx^\alpha(\lambda)/d\lambda$ in the intrinsic frame parallel transported to the point $x^\alpha(\lambda)$, the tangent at the element π_o^x of the manifold of intrinsic paths may be described by the vector

field $w^\alpha(\lambda)$ as shown in figure 8.

Let us therefore compute the holonomy associated with a generic connection derivative, going from the path $\pi_o^x \circ w$ to the path ${\pi'}_o^{x+\epsilon u}$ as shown in figure (8), where o is the origin. Let us introduce the tangent vector at each point $x^\alpha(\lambda)$ of π_o^x, given by $u^\alpha(\lambda)$. The invariant length s goes from 0 at o to s_f at x and $ds = \sqrt{\eta_{\alpha\beta}u^\alpha u^\beta} d\lambda$. The path π'_o admits a description in terms of displacements $\epsilon w^\alpha(\lambda)$ referred to the frame transported to the point $x^\alpha(\lambda)$ of the path π_o^x. Different displacements $\epsilon w^\alpha(\lambda)$ with the same final value $w^\alpha(\lambda_f) = w^\alpha$ define different connection derivatives.

It is easy to see that [10] the frame transported up to $x^\alpha(\lambda)$ by π_o^x and from there along $w^\alpha(\lambda)$ till P differs from the one transported along π'_o by the infinitesimal Lorentz transformation,

$$\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \Omega^\alpha{}_\beta\left(\lambda\right), \tag{5.9}$$

with,

$$\Omega^\alpha{}_\beta\left(\lambda\right) = \int_o^\lambda \epsilon R_{\gamma\delta}{}^\alpha{}_\beta\left(\lambda'\right) u^\gamma(\lambda') w^\delta(\lambda') d\lambda'. \tag{5.10}$$

We can also compute $u'(\lambda)$ (the tangent to π') in terms of $u(\lambda)$ and $w(\lambda)$ as,

$$u'^\alpha = \Lambda^\alpha{}_\beta u^\beta(\lambda) + \epsilon \frac{dw^\alpha}{d\lambda}, \tag{5.11}$$

which allows to define intrinsically the path π'_o by $u'^\alpha(\lambda) = dx'^\alpha(\lambda)/d\lambda$. The connection derivative of a path dependent vector field $B^\beta(\pi)$ is given by,

$$B^\beta\left(\pi_o^x \epsilon w(\lambda_f) {\pi'}^o{}_{x+\epsilon w}\pi\right) = \left(1 + \epsilon w^\alpha(\lambda_f)\delta_\alpha\left(\pi_o^x\right)\right) B^\beta(\pi) = \Lambda^\beta_\sigma\left(\lambda_f\right) B^\sigma\left(\pi\right), \tag{5.12}$$

and therefore,

$$\delta_\alpha\left(\pi_o^x\right) B^\beta\left(\gamma\right) = A_\alpha{}^\beta{}_\sigma\left(\pi_o^x\right) B^\sigma\left(\gamma\right), \tag{5.13}$$

with $\epsilon w^\alpha A_\alpha{}^\beta{}_\sigma\left(\pi_o^x\right) = \Omega^\beta{}_\sigma\left(\lambda_f\right)$.

As a consequence, choosing displacement vectors $w^\beta(\lambda) = w^\alpha(\lambda_f) E_\alpha^\beta(\lambda)$ with E_α^β a linear transformation such that the evaluation of E_α^β in λ_f is $E_\alpha^\beta = \delta_\alpha^\beta$ one gets,

$$A_\alpha{}^\beta{}_\sigma\left(\pi_o^x\right) = \int_o^{\lambda_f} R_{\gamma\delta}{}^\beta{}_\sigma\left(\lambda'\right) u^\gamma\left(\lambda'\right) E_\alpha^\delta\left(\lambda'\right) d\lambda' = \int_o^{s_f} R_{\gamma\delta}{}^\beta{}_\sigma\left(y\right) E_\alpha^\delta\left(y\right) dx^\gamma, \tag{5.14}$$

with $dx^\alpha = u^\alpha\left(\lambda\right) d\lambda$ and the integral is along π_o^x referred to the frame F. Notice that the connection derivative is not unique and would require to include the information about $E_\alpha^\beta(\lambda)$ for $0 \le \lambda \le \lambda_f$ with the fixed boundary condition $E_\alpha^\beta(\lambda_f) = \delta_\alpha^\beta$. The complete notation would therefore be $A_\alpha{}^\beta{}_\sigma\left(F, \pi_o^x, [E_\alpha]\right)$, where $[E_\alpha]$ defines the tangent vector basis to the path π_o^x.

Notice that in the definition of the connection derivative introduced in section II.C there was an assignment of paths to the points of the manifold. A different assignment corresponds to a gauge change. Here that role is being played by the matrices E_α^β.

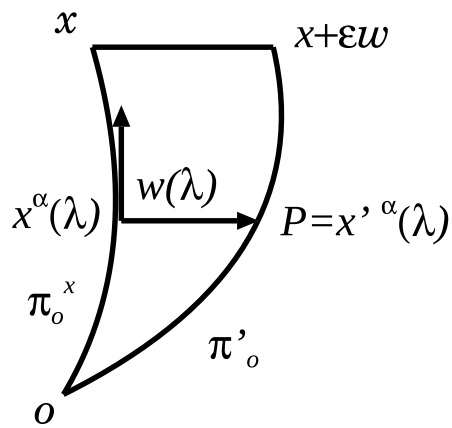

FIG. 8: The path defining the connection derivative.

C. Finite deformations

1. *A finite loop based on "parallel" connections*

We are now in the position to compute the holonomy associated to a closed finite path that extends the path ordered exponentials (2.18) and (3.9) to the gravitational case. We first analyze for simplicity a finite loop generated by "parallel" connections. This relationship allows to obtain $H^{\alpha}{}_{\beta}$ as a path ordered exponential. The construction that follows can be done with the connection (5.7) or the ones stemming from the connection (5.14) associated to figure 8.

To obtain a closed path in intrinsic gravity is non-trivial but crucial for identifying physical points in the manifold. The idea is to construct them by composition of paths associated to connections like those in figures 7 and 8. We wish to define the path of figure 9 in intrinsic notation as a loop referred to the frame F, appropriately parallel transported. Omitting the ϵ's it is given by $\gamma = \pi^x \circ w_1 \circ \ldots \circ w_N \circ \overline{\pi} \circ \overline{\Sigma}^{(n)}_{oo'_N}$. The idea is to obtain it as a product of infinitesimal deformations that we organize in brackets, as shown in figure 9,

$$
\begin{aligned}
\gamma \;=\; & \left(\pi^x \circ w_1 \circ \overline{\pi}^{y_1} \circ \overline{w_1}^{(n)}\right)|_F \left(w_1^{(n)} \circ \pi^{y_1} \circ w_2 \circ \overline{\pi^{y_2}} \circ \overline{w_2}^{(n)} \Lambda_2 \, \overline{w_1}^{(n)}\right)\Big|_{F_1} \\
& \times \left(\Lambda_2 w_1^{(n)} \circ w_2^{(n)} \circ \pi^{y_2} \circ w_3 \circ \overline{\pi^{y_3}} \circ w_3^{(n)} \circ \Lambda_3 \overline{w_2}^{(n)} \circ \Lambda_3 \Lambda_2 \overline{w_1}^{(n)}\right)\Big|_{F_2} \\
& \times \cdots \left(\Sigma_{oo_p} \circ \pi^{y_p} \circ w_{p+1} \circ \overline{\pi^{y_{p+1}}} \circ \overline{w_{p+1}}^{(n)} \circ \overline{\Sigma_{o_{p+1}o}}\right)\Big|_{F_p} \cdots
\end{aligned}
\tag{5.15}
$$

with $\Sigma_{oo_p} = \Lambda_p \Lambda_{p-1} \cdots \Lambda_2 w_1^{(n)} \circ \Lambda_p \ldots \Lambda_3 w_2^{(n)} \circ \cdots \Lambda_p w_{p-1}^{(n)} \circ w_p^{(n)}$ and where the subscript F_p means the frame rotated by $\Lambda_p \cdots \Lambda_1$ of F and Λ_p the infinitesimal Lorentz transformation induced by the closed path $\pi_{p-1} \circ w_p \circ \overline{\pi_p} \circ \overline{w_p}^{(n)}$ (notice the change in notation for Λ's).

The equation for γ leads to an expression very similar to (3.9) for the holonomy,

$$
\begin{aligned}
H^{\alpha}{}_{\beta} \;=\; & \left(\delta^{\alpha}_{\beta_1} + \epsilon w_1^{\rho} A_{\rho}{}^{\alpha}{}_{\beta_1}(F, \pi)\right) \left(\delta^{\beta_1}_{\beta_2} + \epsilon w_2^{\rho} A_{\rho}{}^{\beta_1}{}_{\beta_2}(F_1, \Sigma_{oo_1}\pi_1)\right) \cdots \\
& \times \left(\delta^{\beta_p}_{\beta_{p+1}} + \epsilon w_{p+1}^{\rho} A_{\rho}{}^{\beta_p}{}_{\beta_{p+1}}(F_p, \Sigma_{oo_p}\pi_p)\right) \cdots
\end{aligned}
\tag{5.16}
$$

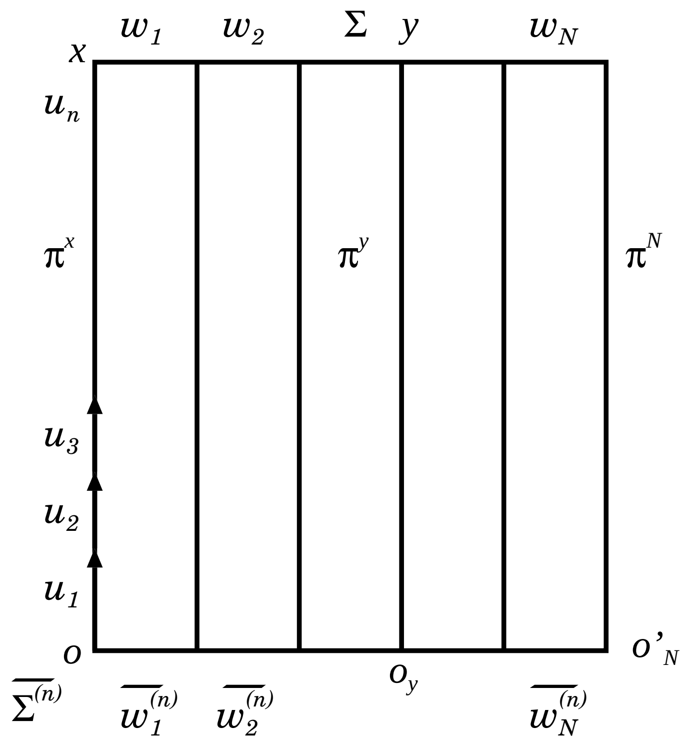

FIG. 9: The path $\gamma = \pi^x \circ \Sigma \circ \overline{\pi^N} \circ \overline{\Sigma^{(n)}}$ used in the construction of the holonomy associated to closed finite path.

that is,

$$H(\gamma) = \mathrm{P}\exp\left(i\int_{\Sigma} dy^{\alpha} A_{\alpha}\left(F_y, \Sigma_{oo_y}\pi^{y}_{o_y}\right)\right). \tag{5.17}$$

We therefore recover the intrinsic version of the non-Abelian Stokes' theorem.

2. A finite loop based on general connections

We now proceed to construct a finite loop based on general connections. The idea is to obtain $\gamma = \pi^x_o \Sigma \overline{\pi}^N$, as shown in figure 10, as a product of infinitesimal deformations that we organize in brackets,

$$\gamma = \left(\pi^x_o \epsilon w_1 \overline{\pi}^{y_1}_1\right)\left(\pi^{y_1}_1 \epsilon w_2 \overline{\pi}^{y_2}_2\right)\ldots\left(\pi^{y_p}_p \epsilon w_p \overline{\pi}^{y_{p+1}}_{p+1}\right)\ldots \tag{5.18}$$

Where $w^{\alpha}_i = w^{\alpha}_i(\lambda_f)$ and $\overline{\pi}^{y_i}_i$ is the path defined by the tangent $u^{\alpha}_i(\lambda)$ referred to the frame parallel transported along $\pi^x_o \circ \epsilon w_i$. We therefore repeat the calculations in equations (5.9) and subsequent ones. We have that,

$$u^{\alpha}_1(\lambda) = \left(\delta^{\alpha}{}_{\beta} + \Omega_1{}^{\alpha}{}_{\beta}(\lambda)\right) u^{\beta}(\lambda) + \epsilon\frac{dw^{\alpha}_1(\lambda)}{d\lambda}, \tag{5.19}$$

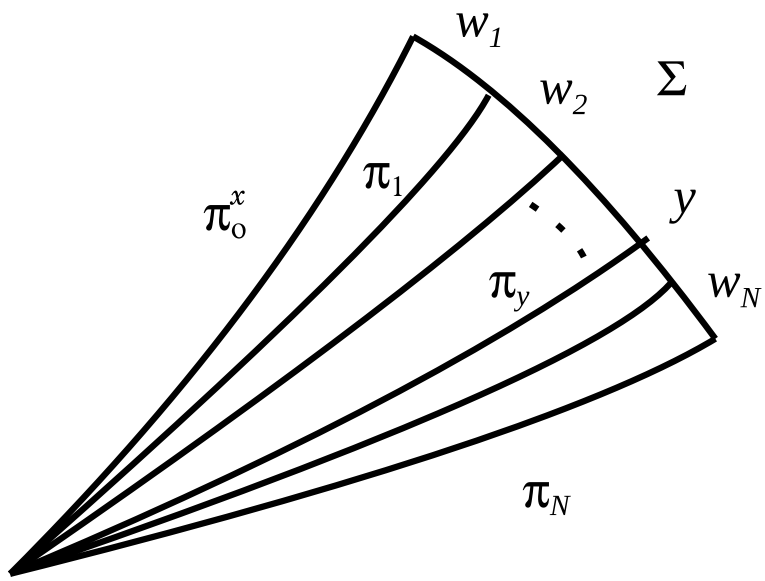

FIG. 10: The path $\gamma = \pi^x \circ \Sigma \circ \overline{\pi^N}$ used in the construction of the holonomy associated to closed finite path.

and

$$\Omega_1{}^{\alpha}{}_{\beta}(\lambda) = \epsilon \int_0^{\lambda} R_{\gamma\delta}{}^{\alpha}{}_{\beta}(\lambda') u^{\gamma}(\lambda') w_1^{\delta}(\lambda') d\lambda', \tag{5.20}$$

where $u^{\gamma}(\lambda)$ is the tangent vector to the path π_o^x. Analogously, $\pi_p^{y_p}$ is the path given by the tangent vector u_p^{α} given by,

$$u_p^{\alpha}(\lambda) = \left(\delta^{\alpha}{}_{\beta} + \Omega_p{}^{\alpha}{}_{\beta}(\lambda)\right) u_{p-1}^{\beta}(\lambda) + \epsilon \frac{dw_p^{\alpha}}{d\lambda}, \tag{5.21}$$

with,

$$\Omega_p{}^{\alpha}{}_{\beta}(\lambda) = \epsilon \int_0^{\lambda} R_{\gamma\delta}{}^{\alpha}{}_{\beta}(\lambda') u_{p-1}^{\gamma}(\lambda') w_p^{\delta}(\lambda') d\lambda'. \tag{5.22}$$

These relations may be written as follows, parametrizing the $u's$ in such a way that $u^{\alpha}(\lambda, \mu = k\epsilon) = u_k^{\alpha}(\lambda)$, and $w^{\alpha}(\lambda, \mu = k\epsilon) = w_k^{\alpha}(\lambda)$ respectively, with $\epsilon = d\mu$,

$$u^{\alpha}(\lambda, \mu) - u^{\alpha}(\lambda, \mu - d\mu) = \Omega^{\alpha}{}_{\beta}(\lambda, \mu) u^{\beta}(\lambda, \mu) d\mu + \frac{dw^{\alpha}(\lambda, \mu)}{d\lambda} d\mu, \tag{5.23}$$

and,

$$\Omega^{\alpha}{}_{\beta}(\lambda, \mu) = \int_0^{\lambda} R_{\gamma\sigma}{}^{\alpha}{}_{\beta}(\lambda', \mu) u^{\gamma}(\lambda', \mu) w^{\sigma}(\lambda', \mu) d\lambda', \tag{5.24}$$

and therefore,

$$\frac{du^{\alpha}(\lambda, \mu)}{d\mu} = \Omega^{\alpha}{}_{\beta}(\lambda, \mu) u^{\beta}(\lambda, \mu) + \frac{dw^{\alpha}(\lambda, \mu)}{d\lambda}. \tag{5.25}$$

If one can solve the above equations one gets an expression for the finite deformation.

The expression for γ leads to,

$$H^{\alpha}{}_{\beta} = (\delta^{\alpha}{}_{\beta_1} + \delta w_1^{\rho} A_{\rho}{}^{\alpha}{}_{\beta_1}(\pi_o^x)) \left(\delta^{\beta_1}{}_{\beta_2} + \delta w_2^{\rho} A_{\rho}{}^{\beta_1}{}_{\beta_2}(\pi_1^{y_1})\right) \ldots \left(\delta^{\beta_p}{}_{\beta} + \delta w_p^{\rho} A_{\rho}{}^{\beta_p}{}_{\beta}(\pi_{p-1}^{y_{p-1}})\right) \ldots \tag{5.26}$$

that is,

$$H(\gamma) = \mathrm{P}\exp\left(i\int_{\Sigma} dy^{\alpha} A_{\alpha}(\pi^{y})\right), \tag{5.27}$$

with the connection given by (5.14). We see that in this case the intrinsic version of the non-Abelian Stokes' theorem takes the standard form. The loop gamma connects the path $\pi_o^x \circ \Sigma$ with π_N, and noticing that w_2 is referred to the frame transported along π_1, etc., we get $\Sigma = \delta w_1 \circ \Lambda_1 \delta w_2 \circ \ldots \circ \Lambda_1 \Lambda_2 \ldots \Lambda_{N-1} \delta w_N$ with $\Lambda_{p\beta}^{\alpha} = \delta_{\beta}^{\alpha} + \Omega_{p\beta}^{\alpha}$ and Ω_p given by equation (5.22) .

To compute explicitly the connection for the path π^y one needs to solve for $u^{\alpha}(\lambda,\mu)$, which requires the solution of (5.25). This might be solved in closed form for particular geometries. One can proceed to solve it iteratively for weak fields. Let us denote by $u_{(0)}, u_{(1)}, u_{(2)}$ the order of iteration computed, we have,

$$u_{(0)}^{\alpha}(\lambda,\mu) \;=\; u^{\alpha}(\lambda,0), \tag{5.28}$$

$$u_{(1)}^{\alpha}(\lambda,\mu) \;=\; u^{\alpha}(\lambda,0) + \int_0^{\mu} \frac{dw^{\alpha}(\lambda,\mu')}{d\lambda} d\mu', \tag{5.29}$$

with,

$$\begin{aligned} \frac{du_{(2)}^{\alpha}}{d\mu} &= \Omega_{(1)}{}^{\alpha}{}_{\beta} u_{(1)}^{\beta} + \frac{dw^{\alpha}(\lambda,\mu)}{d\lambda} \\ &= \int_0^{\lambda} R_{\gamma\delta}{}^{\alpha}{}_{\beta}(\lambda',\mu) u_{(1)}^{\gamma}(\lambda',\mu) w^{\delta}(\lambda',\mu) d\lambda' u_{(1)}^{\beta}(\lambda,\mu) + \frac{dw^{\alpha}(\lambda,\mu)}{d\lambda}, \end{aligned} \tag{5.30}$$

and with $R_{\gamma\delta}{}^{\alpha\beta}(\lambda',\mu) = R_{\gamma\delta}{}^{\alpha\beta}(\pi_{(1)}(\mu))$, with $\pi_{(1)}(\mu)$ defined by $x_{(1)}^{\alpha}(\lambda,\mu)$ such that $\partial_{\lambda} x_{(1)}^{\alpha}(\lambda,\mu) = u_{(1)}^{\alpha}(\lambda,\mu)$ and

$$\begin{aligned} u_{(2)}^{\alpha}(\lambda,\mu) \;=\; & \int_0^{\mu} d\mu' \left\{ \int_0^{\lambda} d\lambda' R_{\gamma\sigma}{}^{\alpha}{}_{\beta}\left(x_{(1)}(\lambda',\mu')\right) u_{(1)}^{\gamma}(\lambda',\mu') w^{\sigma}(\lambda',\mu') u_{(1)}^{\beta}(\lambda,\mu') \right\} \\ & + u_{(1)}^{\alpha}(\lambda,\mu) \end{aligned} \tag{5.31}$$

and by iteration we determine $u^{\alpha}(\lambda,\mu)$ for sufficiently weak fields.

VI. PATH DEPENDENT FIELDS

Let us consider fields with tensor, spinor or internal components. One can start by giving the fields for arbitrary paths at each point $\phi^{(A,I)}(\pi)$ where the index A represents the Lorentz tensor or spinor components and I the internal components. The indices refer to the components in the frame parallel transported along the path. Having recognized the closed loops γ, the fields transform under changes of the reference path by representations of the group of loops. For instance for a vector field with internal group $SU(N)$ in some

representation,
$$A^{\alpha}{}_{I}(\pi') = H(\gamma)^{\alpha}{}_{\beta} H\,(\gamma)_{I}{}^{J} A^{\beta}{}_{J}(\pi), \tag{6.1}$$
if $\pi' = \gamma \circ \Lambda(\gamma)\pi = \gamma \cdot \pi$, which guarantees that π' and π end at the same point on M. $H(\gamma)^{\alpha}{}_{\beta}$ is a holonomy associated with the Lorentz group and $H(\gamma)_{I}{}^{J}$ a holonomy associated with the internal group. The path-dependent fields like $A^{\beta}{}_{J}$ depend on the paths π referred to the frame F chosen as a reference at o. Analogous relations hold for any matter field and should be compared with the corresponding relation in Mandelstam notation (4.2) that cannot even be written explicitly in the case of strong fields.

The notion of covariant derivative of path dependent fields can be introduced using the Mandelstam derivative. Its meaning for gauge theories was analyzed in sections II y III, defined by $(1 + \epsilon u^{\beta} D_{\beta}) A^{\alpha}{}_{I}(\pi^{z}) = A^{\alpha}{}_{I}(\pi_{E}^{z+\epsilon u})$. Where $\pi_{E}^{z+\epsilon u}$ is the path extended in the direction u whose components are given with respect to the frame at the end point z. It compares the field parallel transported from $z + \epsilon u$ to z with the field at z and therefore gives us the component of the space time covariant derivative with respect of the intrinsic basis parallel transported along π. π_E is the extended path shown in figure (3) but now the extension is given in terms of the intrinsic components of u in the frame parallel transported up to z.

A. Symmetries of the path dependent Riemann tensor

As we mentioned in section II one can derive a Bianchi identity by considering a tree that follows the edges of a cube and noticing that "the boundary of a boundary vanishes". If this construction is done at the end point of π one gets

$$\left([D_{\beta}[D_{\gamma}, D_{\delta}]] + [D_{\gamma}[D_{\delta}, D_{\beta}]] + [D_{\delta}[D_{\beta}, D_{\gamma}]]\right) A_{\alpha}(\pi) = D_{[\beta} R_{\gamma\delta]\alpha}{}^{\epsilon}\,(\pi)\, A_{\epsilon}\,(\pi) = 0 \tag{6.2}$$

which implies that the path dependent Riemann tensor satisfies the Bianchi identity. In the intrinsic formalism we are developing, a scalar satisfies $\phi(\pi) = \phi(\pi')$ if $\pi' = \gamma \cdot \pi$ and, applying the same construction with a scalar we get,

$$\left([[D_{\alpha}, D_{\beta}] D_{\gamma}] + [[D_{\beta}, D_{\gamma}], D_{\alpha}] + [[D_{\gamma}, D_{\alpha}], D_{\beta}]\right) \phi(\pi) = R_{[\alpha\beta\gamma]}{}^{\delta} D_{\delta} \phi(\pi) = 0. \tag{6.3}$$

Since by construction the Riemann tensor is antisymmetric in the first two and the last two indices, the above identities imply the remaining algebraic identities of Riemann's tensor are all satisfied.

In what follows, as an application of the techniques developed up to now, we will show that the Riemann tensor has the expected tensorial transformation under changes of path. So we consider a one form along a path with a small closed loop. And then along a path with two small loops. The first term will give rise to a rotation of the form given by the Riemann tensor evaluated in the path π_2 as per (5.1). The second deformation will change the frame of the Riemann tensor, which will therefore be Lorentz transformed by going from the path π_2 to $\delta\gamma_1 \cdot \pi_2$. The paths are shown in figure (11). Let us start by computing,

$$A_{\alpha}\left(\delta\gamma_1 \cdot \delta\gamma_2 \cdot \overline{\delta\gamma_1} \cdot \pi\right) - A_{\alpha}\left(\delta\gamma_2 \cdot \pi\right) \tag{6.4}$$

where A_{α} is a path dependent intrinsic description of a one form, $\delta\gamma_1 = \left(\pi_1 \circ \delta u_1 \circ \delta w_1 \circ \overline{\delta u_1} \circ \overline{\delta w_1} \circ \Lambda\left(\delta\gamma_{1x}\right) \overline{\pi_1}\right)_F$, with $\delta\gamma_2$ similarly defined for π_2. Notice that

for brevity we have slightly changed the notation in that $\delta\gamma_i$ include the path π_i now. We also have that $\overline{\delta\gamma_1} = \left(\pi_1 \circ \delta w_1 \circ \delta u_1 \circ \overline{\delta w_1} \circ \overline{\delta u_1} \circ \Lambda\left(\overline{\delta\gamma_{1x}}\right)\overline{\pi_1}\right)_{F_1}$ where F_1 is the frame rotated with Λ of F. Therefore the variation of the Riemann tensor under a change of path is given by,

$$\sigma_2^{\eta\rho}\delta R_{\eta\rho\alpha}{}^{\beta}(\pi_2)\,A_\beta(\pi) = \left[\left(\delta_\alpha{}^\beta + (\tilde{\sigma})_2^{\eta\rho}\,R_{\eta\rho\alpha}{}^{\beta}(\delta\gamma_1\cdot\pi_2)\right) - \left(\delta_\alpha{}^\beta + \sigma_2^{\eta\rho}R_{\eta\rho\alpha}{}^{\beta}(\pi_2)\right)\right]A_\beta(\pi)\,, \tag{6.5}$$

where

$$\sigma_i^{\eta\rho} = \frac{1}{2}\epsilon^2\left(\delta u_i^\eta \delta w_i^\rho - \delta u_i^\rho \delta w_i^\eta\right) \tag{6.6}$$

and the components of $\tilde{\sigma}_2^{\eta\rho}$ are rotated with $\Lambda(\delta\gamma_1)$. In order to compute $\delta R_{\eta\rho\alpha}{}^{\beta}(\pi_2)$, that

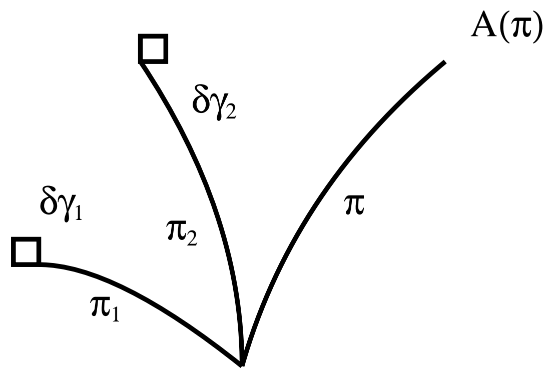

FIG. 11: The path used to show the Lorentz transformation of the Riemann tensor.

represents the variation of R under the deformation $\pi_2 \to \delta\gamma_1\cdot\pi_2$, we note that (6.5) can be rewritten as,

$$\begin{aligned}&\Big[\left(\delta_\alpha{}^\lambda + \sigma_1^{\mu\nu}R_{\mu\nu\alpha}{}^{\lambda}(\pi_1)\right)\left((\tilde{\sigma})_2^{\delta\rho}\,R_{\delta\rho\lambda}{}^{\gamma}(\pi_2)\right)\left(\delta_\gamma{}^\beta - \sigma_1^{\mu'\nu'}R_{\mu'\nu'\gamma}{}^{\beta}(\pi_1)\right)\\ &-\sigma_2^{\delta\rho}R_{\delta\rho\alpha}{}^{\beta}(\pi_2)\Big]\,A_\beta(\pi) = \sigma_2^{\eta\rho}\delta R_{\eta\rho\alpha}{}^{\beta}(\pi_2)\,A_\beta(\pi)\,,\end{aligned} \tag{6.7}$$

and taking into account that

$$(\tilde{\sigma})_2^{\rho\sigma} = \sigma_2^{\rho\sigma} + \sigma_1^{\mu\nu}R_{\mu\nu\epsilon}{}^{\rho}(\pi_1)\,\sigma_2^{\epsilon\sigma} + \sigma_1^{\mu\nu}R_{\mu\nu\epsilon}{}^{\sigma}(\pi_1)\,\sigma_2^{\rho\epsilon}, \tag{6.8}$$

we see that (6.5) can be rewritten as,

$$\begin{aligned}\delta R_{\eta\rho\alpha}{}^{\beta}(\pi_2) \;=\; &\big[\omega_\alpha{}^\lambda(\pi_1)\,R_{\eta\rho\lambda}{}^{\beta}(\pi_2) - \omega_\lambda{}^\beta(\pi_1)\,R_{\eta\rho\alpha}{}^{\lambda}(\pi_2)\\ &+\omega_\eta{}^\gamma(\pi_1)\,R_{\gamma\rho\alpha}{}^{\beta}(\pi_2) + \omega_\rho{}^\gamma(\pi_1)\,R_{\eta\gamma\alpha}{}^{\beta}(\pi_2)\big]\end{aligned} \tag{6.9}$$

with $\omega_\alpha{}^\lambda(\pi_1) = \sigma_1^{\mu\nu}R_{\mu\nu\alpha}{}^{\lambda}(\pi_1)$ and R undergoes a Lorentz transformation under a change of paths.

B. Equations of motion

To illustrate how one would write path dependent equations of motion let us consider a gravitating scalar field,

$$\left(\eta^{\alpha\beta}D_\alpha D_\beta - m^2\right)\phi\left(\pi\right) \;=\; 0 \tag{6.10}$$

$$R_{\alpha\lambda\beta}{}^{\lambda}\left(\pi\right) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\rho}R_{\gamma\lambda\rho}{}^{\lambda}\left(\pi\right) \;=\; \kappa T_{\alpha\beta}\left(\pi\right), \tag{6.11}$$

with

$$T_{\alpha\beta}\left(\pi\right) = D_\alpha\phi\left(\pi\right)D_\beta\phi\left(\pi\right) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}D_\mu\phi\left(\pi\right)D_\nu\phi\left(\pi\right) - m^2\phi^2\left(\pi\right)\eta_{\alpha\beta}, \tag{6.12}$$

and $\kappa = 8\pi G$. Notice that all tensor components are Lorentzian components in the local frame, therefore the metric is the Minkowski one. Recall that in the intrinsic description the physical points are associated with classes of paths that differ by loops. Although scalar fields are only point dependent and $\phi(\pi) = \phi(\gamma\cdot\pi)$, the information about points is given in terms of a path. The intrinsic description of the paths ensures that $\phi(\pi)$ is a diffeomorphism invariant physical observable. The discussion of the next section will allow to reproduce the ordinary equations from the path dependent ones.

VII. RECOVERING THE STANDARD COORDINATE DEPENDENT DESCRIPTION

A. Going from the intrinsic to coordinate description

We have shown that the intrinsic description allows to recognize when open paths lead to the same point. Let us consider an assignment of reference paths that define normal coordinates at each point of a region U sufficiently small around a point P to which we have arrived following a geodesic that starts at o. That is P is intrinsically defined following a geodesic starting at o given by by $z^\alpha(s) = su^\alpha$, where u^α is a vector in the frame F. The point P corresponds to $s = s_P$. A point Q of U is given by $z^\alpha(Q) = s_P u^\alpha + s_Q v^\alpha$ with v^α the vector components relative to the frame parallel transported to P of the tangent at P of the geodesic that joins Q with P. The construction is possible locally since we assume that there exists a unique geodesic at U from P to Q. The quantities $x^a(Q) \equiv z^a(Q) - s_P u^a$ define a chart that maps the points of U to a region of R^4 that are Riemann normal coordinates with origin at P (we denote Riemann coordinates with Latin letters). It is possible to define charts $\overline{x}^a(Q)$ diffeomorphic to x. The intrinsic construction allows to associate to each Q, in addition to its coordinates $x^a(Q)$ the coordinates of the local frame transported from o to that point $e^a_\alpha\left(\pi^Q_R\right)$ with π^Q_R the above mentioned path going from o to P and from there to Q.

The frames transform under changes of path $\pi^Q_R \to \pi'^Q$ (keeping the original coordinates defined by π_R) as,

$$e^a_\beta\left(\pi'^Q\right) = H_\beta{}^\alpha\left(\gamma\right)e^a_\alpha\left(\pi^Q_R\right) = H_\beta{}^\alpha\left(\gamma\right)e^a_\alpha\left(x(Q)\right), \tag{7.1}$$

with H^{α}_{β} the Lorentz transformation associated with the holonomy along the closed loop γ is such that $\pi'^{Q} = \gamma \cdot \pi^{Q}_{R}$. Recall that the index α corresponds to a frame index and the index a is a coordinate index. Under diffeomorphisms $x^a \to \overline{x}^a(x)$, we have that,

$$e^{b}_{\alpha}\left(x'\right) = \frac{\partial x'^{b}}{\partial x^{a}} e^{a}_{\alpha}\left(x\right). \tag{7.2}$$

The metric in this system of coordinates can be specified as usual in terms of tetrads,

$$g^{ab}\left(x\right) = \eta^{\alpha\beta} e^{a}_{\alpha}\left(\pi^{x}_{R}\right) e^{b}_{\beta}\left(\pi^{x}_{R}\right), \tag{7.3}$$

and is independent of the reference path since the holonomies are Lorentz transformations. Since the tetrads are obtained by parallel transport from the origin, and taking into account the definition of the Mandelstam derivative, the intrinsic construction implies immediately that

$$D_{\alpha} e^{b}_{\beta}\left(\pi^{x}_{R}\right) = 0. \tag{7.4}$$

Defining,

$$\nabla_{a} e^{b}_{\beta}\left(x\right) \equiv e^{\alpha}_{a} D_{\alpha} e^{b}_{\beta}\left(\pi^{x}_{R}\right), \tag{7.5}$$

we have that $\nabla_{a} e^{b}_{\beta}\left(x\right) = 0$ and we recover the usual covariant derivative since it compares the tetrad at $x + dx$ with the parallel transported one at that point. As a consequence $\nabla_{a} g^{bc}\left(x\right) = 0$ and the connection is metric compatible.

To show that the torsion is zero we consider a scalar field $\phi\left(x\right) = \phi\left(\pi^{x}_{R}\right) = \phi\left(\pi^{Q}_{R}\right)$. We have that $\phi\left(\pi^{Q}_{R}\right) = \phi\left(\pi'^{Q}\right)$ for any path π arriving at Q, and taking into account the intrinsic version of (2.15), we have that,

$$D_{[a} D_{b]} \phi\left(\pi\right) = \frac{1}{2} \Delta_{\alpha\beta}\left(\pi\right) \phi(\pi) = 0 \tag{7.6}$$

since ϕ is really path independent, and therefore the connection is therefore torsion free.

By construction, since the point P is the origin of the normal coordinates we are using, we have at P that $e^{a}_{\alpha}\left(\pi^{P}_{R}\right) = e^{a}_{\alpha}\left(P\right) = \delta^{a}_{\alpha}$ and for Q, using well known results for normal coordinates we have that,

$$e^{a}_{\alpha}\left(x_{Q}\right) = e^{a}_{\alpha}\left(\pi^{Q}_{R}\right) = \delta^{a}_{\alpha} + \frac{1}{3} R^{a}_{b\alpha c}\left(\pi^{P}_{R}\right) x^{b} x^{c} + O\left(s^{3}_{Q}\right), \tag{7.7}$$

recalling that at second order in Riemann coordinates the Riemann tensor is evaluated at the origin P where intrinsic and Riemann components coincide.

Although the Riemann tensor identities follow from the intrinsic ones given in VIa from the metricity and torsion freedom of the connection, it is immediate to obtain the identities in terms of coordinates from the intrinsic ones taking into account (7.4), and the discussion presented in section VIa, and recalling that at P the tetrad components in Riemann coordinates reduce to the identity.

B. Relating intrinsic and coordinate descriptions of paths and local frames

We would like to relate the paths described in coordinate systems with intrinsic paths and identify the local frames at an arbitrary point of the path in terms of the geometric or intrinsic descriptions of the paths. Let $\gamma^a(\lambda)$ be a curve in an arbitrary coordinate system such that $\gamma^a(0) = x^a_o$, the coordinates of o, and $\gamma(1) = x^a$. We want to determine ${e_\alpha}^a(\lambda = 1) = {e_\alpha}^a(\gamma(\lambda = 1))$ and in general ${e_\alpha}^a(\lambda) = {e_\alpha}^a(\gamma(\lambda))$ and from them the intrinsic components of $\gamma^a(\lambda)$, let us call them $y^a(\lambda)$.

Using that,

$$d\lambda\dot{\gamma}^a \nabla_a {e_\alpha}^b = d\lambda\dot{\gamma}^a \left(\partial_a + {\Gamma_{ad}}^b\right) e^d_\alpha = 0, \tag{7.8}$$

it follows that,

$${e_\alpha}^c(\lambda + d\lambda) = \left(\delta^c_d - d\gamma^a {\Gamma_{ad}}^c\left(\gamma(\lambda)\right)\right) {e_\alpha}^d, \tag{7.9}$$

which can be integrated along the path to give,

$${e_\alpha}^c(\lambda) = \mathrm{P}\left(\exp\left(-\int_0^\lambda d\lambda'\dot{\gamma}^a(\lambda')\Gamma_a\right)\right)^c_d {e_\alpha}^d(0), \tag{7.10}$$

and for ${e_\alpha}^d(0) = \delta^d_\alpha$ one gets the explicit form of the parallel transported local frame along gamma,

$${e_\alpha}^c(\lambda) = \mathrm{P}\left(\exp\left(-\int_0^\lambda d\lambda'\dot{\gamma}^a(\lambda')\Gamma_a\right)\right)^c_\alpha, \tag{7.11}$$

and the intrinsic coordinates are

$$\frac{dy_\alpha}{d\lambda} = \dot{\gamma}_c(\lambda){e_\alpha}^c(\lambda), \tag{7.12}$$

$$y^\alpha(\lambda) = \int_0^\lambda \dot{\gamma}^c(\lambda'){e^\alpha}_c(\lambda')d\lambda'. \tag{7.13}$$

Knowing the geometry, the metric in M and its associated connection allows to determine through (7.11) the intrinsic coordinates associated to any given curve γ.

The inverse correspondence allows to associate to each path π, described intrinsically by $y^\alpha(\lambda)$ and each system of coordinates, the components of the frame parallel transported along π and the curve in coordinates $\gamma^a(\lambda)$ that corresponds to the intrinsic path $y^\alpha(\lambda)$,

$${e_\alpha}^a(\lambda) \equiv {e_\alpha}^a\left(\left[y^\beta\right], \lambda\right), \tag{7.14}$$

$$\dot{\gamma}^a = \dot{y}^\alpha {e_\alpha}^a(\lambda) = \dot{y}^\alpha {e_\alpha}^a\left([y], \lambda\right), \tag{7.15}$$

$$\gamma^a(\lambda) = \int_0^\lambda d\lambda'\dot{y}^\alpha {e_\alpha}^a\left([y], \lambda\right) + x^a_o, \tag{7.16}$$

$$\gamma^a(0) = x^a_o, \tag{7.17}$$

where the brackets denote functional dependence on the $y's$. Notice that at the quantum level the local frames in (7.16) will be promoted to operators. If one describes the path in terms of the intrinsic functions $y^\alpha(\lambda)$, the corresponding path in a given system γ^a will also be given by quantum operators, and therefore the notion of point will only emerge in a semiclassical regime.

The tetrads defined allow to compute the metric,

$$g^{ab}\left([y],\lambda\right)=\eta^{\alpha\beta}e_{\alpha}{}^{a}\left([y],\lambda\right)e_{\beta}{}^{b}\left([y],\lambda\right)=g^{ab}\left(\gamma(\lambda)\right). \tag{7.18}$$

The assignment of frames $e_{\alpha}{}^{a}\left([y],\lambda\right)$ would allow to identify that two different curves have the same endpoints $\gamma_0^a(\lambda_f)=\gamma_1^a(\lambda_f)$. That implies,

$$\int_0^{\lambda_f} d\lambda' \dot{y}_0^{\alpha} e_{\alpha}{}^{a}\left([y_0],\lambda'\right)=\int_0^{\lambda_f} d\lambda' \dot{y}_1^{\alpha} e_{\alpha}{}^{a}\left([y_1],\lambda'\right), \tag{7.19}$$

since the integrals reduce to evaluations at the endpoints of $y's$. This is therefore the condition for two curves whose intrinsic description is known, to have the same endpoints. Since we are arriving at the same point with frames that are parallel transported, they therefore may differ by a Lorentz transformation, $e_{\alpha}{}^{a}\left([y_0],\lambda_f\right)=\Lambda_{\alpha}{}^{\beta}e_{\beta}^{a}\left([y_1],\lambda_f\right)$ with $\Lambda_{\alpha}{}^{\beta}$ the matrix of the Lorentz transformation.

VIII. NON-LOCALITY OF THE OBSERVABLE ALGEBRA

Here we would like to analyze the non-locality of the observable algebra in the linearized case. For that purpose we will define a coordinate system in terms of reference paths for instance using geodesics. In fact it is known that with the resulting Riemann normal coordinates one may cover an arbitrarily large region of spacetime in the linearized case [11]. It is important to remark that here we will not use the second order approximation for Riemann normal coordinates. Let us first start by discussing how the linearized theory emerges from the intrinsic formulation.

A. From intrinsic gravity to linearized gravity

Given such a coordinate system, we may now proceed as we did in section 5 and assign to each point x in U a spin connection, in the non-holonomic description given by the tetrads $e_{\alpha}^{a}(\pi_R^x)$,

$$A_{\mu\alpha\beta}\left(x\right)=A_{\mu\alpha\beta}\left(F,\pi_R^x\right), \tag{8.1}$$

where π_R^x is the reference path defined above.

Analogously,

$$R_{\mu\nu\alpha\beta}\left(x\right)=R_{\mu\nu\alpha\beta}\left(\pi_R^x\right). \tag{8.2}$$

In the linear approximation we can drop the second term in (5.8),

$$A_{\rho\gamma\nu}\left(x\right)=A_{\rho\gamma\nu}\left(\pi_R^x\right)=\int_{s_i}^{s_f} ds\dot{y}^{\alpha}\left(s\right)R_{\alpha\rho\gamma\nu}\left(\pi_R^{y(s)}\right), \tag{8.3}$$

neglecting the correction of A quadratic in R. Taking into account that the R's satisfy,

$$R_{\alpha[\rho\gamma\nu]}\left(\pi\right)=0, \tag{8.4}$$

one gets

$$A_{\rho\gamma\nu}\left(x\right)+A_{\nu\rho\gamma}\left(x\right)+A_{\gamma\nu\rho}\left(x\right)=0, \tag{8.5}$$

and from $\partial_{[\mu}R_{\alpha\beta]\gamma\nu} = 0$, we get that,

$$R_{\alpha\beta\gamma\nu} = \partial_\alpha A_{\beta\gamma\nu} - \partial_\beta A_{\alpha\gamma\nu}. \tag{8.6}$$

Finally, the symmetry $R_{\alpha\beta\gamma\nu} = R_{\gamma\nu\alpha\beta}$ allows to define a superpotential $h_{\rho\alpha}$,

$$A_{\rho\alpha\beta} = h_{\rho\alpha,\beta} - h_{\rho\beta,\alpha}. \tag{8.7}$$

Since the spin connections constructed satisfy (8.5) for any path, it follows that under a change of path

$$A'_{\mu\alpha\beta}(x) = A_{\mu\alpha\beta}(\pi'^x) = A_{\mu\alpha\beta}(x) + \Lambda_{\alpha\beta,\mu}, \tag{8.8}$$

just like in gauge theories, with,

$$\Lambda_{\alpha\beta,\mu} + \Lambda_{\mu\alpha,\beta} + \Lambda_{\beta\mu,\alpha} = 0, \tag{8.9}$$

and therefore define a vector ξ_α,

$$\Lambda_{\alpha\beta} = \xi_{\alpha,\beta} - \xi_{\beta,\alpha}, \tag{8.10}$$

and

$$\delta h_{\mu\alpha} = \xi_{\mu,\alpha} + \xi_{\alpha,\mu}, \tag{8.11}$$

and the components of the Riemann tensor are invariant under these transformations.

Recalling the relationship of spin connections with the tetrads in a linear theory one gets,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{8.12}$$

These relations also hold for any assignment of reference paths which satisfies the above mentioned conditions. That is: i) the end point intrinsic coordinates defined by their total intrinsic displacement along the parallel transported system of coordinates coincide with the coordinates of the local chart in V; and ii) any portion of a reference path is also a reference path.

It is important to emphasize that the intrinsic formulation of linearized gravity also differs from the path dependent description of gauge theories given in sections 2 and 3. Small differences depend on the intrinsic description of the paths and would disappear if one defines paths in a flat manifold and considers the linearized theory as another gauge theory. For instance one would not even have an equation (7.1) in the case of ordinary (non-intrinsic) paths in the flat background manifold.

One should however recall that only the intrinsic theory is given in terms of physical observables. As we shall see a description in terms of observables is always non-local.

B. Non-locality

In the case of gauge theories, like Yang–Mills theory, it is always possible to define local gauge invariant observables, for instance $\mathrm{Tr}\left(F_{\alpha\beta}F^{\alpha\beta}\right)$. However, when gravity is included the observables are always non-local. For example, a scalar field $\phi(x)$ is not observable due to its dependence on diffeomorphisms but $\phi(\pi)$ is since it refers to a specific field at an intrinsically defined point and depends on a non-ambiguous measuring procedure.

If one fixes paths, for instance using geodesics as in the previous section, the gauge is completely fixed and the scalars are observable,

$$\phi\left(x\right)=\phi\left(\pi_R^x\right). \tag{8.13}$$

It is clear that in an eventual quantization, quantum fluctuations in the geometry throughout the path will change the arrival point and therefore the value of the measured field. It is difficult to do an explicit demonstration since the dependence enters through the parallel transport whose expression in intrinsic coordinates is not known explicitly.

However, we can illustrate the dependence on the Riemann tensor by considering the change of a scalar field when one changes the path. For instance, if $x^\nu\left(\pi\right)$ are the Riemann normal coordinates of the end point of a path π that differs from π_R^x by an infinitesimal spatial deformation at y, an intermediate point of π_R^x, we have that the coordinates of the end point (in the linearized case, to keep things simple) change as,

$$x^\nu\left(\pi\right)=x^\nu-\frac{1}{2}\sigma^{\alpha\beta}R_{\alpha\beta\lambda}{}^{\nu}\left(\pi_R^y\right)\left(x_0-z\right)^\lambda. \tag{8.14}$$

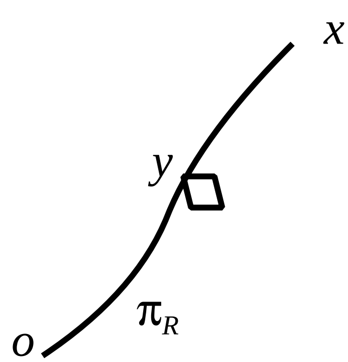

FIG. 12: The path π_R with an infinitesimal deformation at the point y.

Expressions like this suggest that scalar fields evaluated at different points (or dependent on paths that end at different points) will generically have non-vanishing Poisson brackets with the Riemann tensor and therefore among themselves. This is what we mean by non-locality.

As we mentioned before, in this formulation the components of the Riemann tensor can be considered functions of points given by the reference path,

$$R_{\mu\nu\lambda\rho}\left(x\right)=R_{\mu\nu\lambda\rho}\left(\pi_R^x\right), \tag{8.15}$$

and in linearized gravity these quantities are gauge invariant and therefore observables of the theory. We will show explicitly that the non-locality of the theory emerges in the example of the linearized case by noting that non-vanishing Poisson brackets between variables at spatially separated points emerge. The observables will therefore obey a non-local algebra.

The Poisson brackets between Riemann tensors of linearized gravity were computed by

De Witt some time ago.

$$[R_{\mu\nu\sigma\tau}(x), R_{\alpha\beta\gamma\delta}(x')] = \frac{i}{4}(\eta_{\mu\alpha}\eta_{\sigma\gamma} + \eta_{\mu\gamma}\eta_{\sigma\alpha} - \eta_{\mu\sigma}\eta_{\alpha\gamma})\,\Delta(x,x')_{,\nu,\tau,\beta,\delta} + \text{Permutations} \tag{8.16}$$

where $\Delta(x,x')$ is the odd homogeneous propagator of massless fields in flat space-times, and the permutations are the fifteen ones compatible with the symmetries of the Riemann tensor.

The variation of a scalar field under a change of path like we have for a the spatial deformation σ^{ab} (we use Latin indices for spatial components), is given by,

$$\delta\phi(\pi, P) = \phi(\pi) - \phi(\pi_R) = -\frac{1}{2}\sigma^{kl}R_{kl\mu}{}^{\nu}(\pi_R^y)(x-y)^{\mu}\,\partial_\nu\phi(\pi_R). \tag{8.17}$$

It can be easily checked from (8.16) that it does not have vanishing Poisson bracket with the components of the Riemann tensor in y where the deformation in the definition of $\delta\phi$ takes place. What we have shown for the variation $\delta\phi$ also holds for the field itself if one considers the Poisson bracket with the Riemann tensor at a point along the path. Two fields themselves will also have non-vanishing Poisson brackets as long as their paths intersect.

In contrast to what happens in ordinary field theory, gauge invariant observables in the presence of gravity cannot be localized in well defined regions of space-time and therefore one does not have a definition of subsystems stemming from commuting sub-algebras. Donnelly and Giddings have discussed gravitational non-locality for a different set of gravitational observables in references [1].

It should be noted that at a classical level commuting subalgebras are possible by considering observables dependent on non-intersecting paths. At a quantum level however, this is clearly impossible since given the paths π and π' through their intrinsic description it is not possible to know if they have or do not have intersections or common parts when the geometry fluctuates and is not uniquely determined.

IX. THE ACTION IN TERMS OF PATH DEPENDENT FIELDS

Teitelboim [10] was the first to note that the usual action of fields could be expressed as an action of path dependent fields, by gauge fixing using Fadeev–Popov terms. Although his proposal is very suggestive, he does not present a proof of the equivalence with the ordinary action. We will provide a proof for an arbitrary Lagrangian,

$$\mathcal{L}\left(R_{\alpha\beta\gamma}{}^{\rho}(\pi), \phi(\pi), \psi(\pi)\right), \tag{9.1}$$

where ϕ and ψ are fields that could be scalar, vector or spinor.

The quantity $\mathcal{L}$ is a scalar and therefore is independent of π. Let us recall that for scalar quantities π only provides the intrinsic description of the point in the manifold M where it is being evaluated. $\mathcal{L}$, given in terms of path dependent quantities, does not refer to any local chart and therefore there is no explicit reference to its invariance under diffeomorphisms.

If we describe π through the path in the frame parallel-transported from o, $x^\alpha(\lambda)$ with $u^\alpha = dx^\alpha/d\lambda$ the action S is given by $S = \int \mathcal{L}\mathcal{D}x$ with,

$$\mathcal{D}x = \Pi_{\lambda,\alpha}dx^\alpha(\lambda)\delta(\pi - \pi'_R)\Delta_{FP}(\pi_R). \tag{9.2}$$

We are considering a standard path integral integration, the product on λ represents the limit for N going to infinity of the product $\Pi_{i=1}^{N}$ for partitions of the interval $[0, \lambda_f]$ in N portions. In the above expression π_R is a reference path associated to each point of the manifold M, $\delta(\pi-\pi_R)$ fixes a path for each point and Δ_{FP} is the Fadeev–Popov determinant for that choice of path. The Lagrangian $\mathcal{L}$ is a Lorentz scalar and takes the same value for all paths π that reach the end of π_R and is therefore independent of π. The choice of reference paths plays the same role as a gauge fixing in an ordinary gauge theory.

Let us show the equivalence with the usual action in Riemann normal coordinates in a neighborhood U of a point o_1. We consider paths π^U and π_R^U from o_1 to P with P and arbitrary point in U. The path dependence will be restricted to the region in which the normal coordinates are defined so we have that,

$$\pi_R = \pi_{oR}^{o_1} \circ \pi_R^U, \tag{9.3}$$

$$\pi = \pi_{oR}^{o_1} \circ \pi^U, \tag{9.4}$$

and $\pi_{oR}^{o_1}$ a fixed reference path from o to o_1. The paths from o_1 to P, π_R^U are geodesics and take the form $x^\alpha = u^\alpha \lambda$. If we define Riemann normal coordinates associated with the geodesics centered at o_1, z^a, one can identify $z^a = x^\alpha$ and the metric is locally flat at o_1. Let π^U be a path from o_1 to P given by $x_\pi^\alpha(\lambda)$ arbitrary such that $x^\alpha(0) = 0$. If $w^\alpha(\lambda)$ are the infinitesimal displacements referred to the reference path that goes from $x_{\pi_R}^\alpha(\lambda)$ to $x_\pi^\alpha(\lambda)$ and $u_\pi^\alpha = dx_\pi^\alpha/d\lambda$, one has taking into account (5.20) and (5.21), particularized to a geodesic path,

$$u_\pi^\alpha(\lambda) = \Lambda^\alpha{}_\beta(\lambda) u^\beta + \frac{dw^\alpha(\lambda)}{d\lambda}, \tag{9.5}$$

with,

$$\Lambda^\alpha{}_\beta(\lambda) = \delta_\beta^\alpha + \Omega_\beta^\alpha(\lambda) = \delta_\beta^\alpha + \int_0^\lambda d\lambda' R_{\gamma\delta}{}^\alpha{}_\beta(\lambda') w^\delta(\lambda') u^\gamma. \tag{9.6}$$

If we now impose the gauge conditions that say that the reference paths are geodesics going from o_1 to P, $du_\pi^\alpha/d\lambda = d^2 x_\pi^\alpha/d\lambda^2 = 0$,

$$0 = \frac{du_\pi^\alpha(\lambda)}{d\lambda} = R_{\gamma\delta}{}^\alpha{}_\beta(\lambda) u^\gamma w^\delta(\lambda) u^\beta + \frac{d^2 w^\alpha(\lambda)}{d\lambda^2}, \tag{9.7}$$

where u^γ, u^β are the constant vectors that define the geodesic reference path π_R, we get the equation that allows to compute the Fadeev–Popov determinant. In order to do that we note that, integrating (9.5),

$$\delta\left(x_\pi^\alpha(\lambda) - x_R^\alpha(\lambda)\right) = \delta\left(\int_0^\lambda \Omega^\alpha{}_\beta(\lambda') u^\beta d\lambda' + w^\alpha(\lambda)\right) \tag{9.8}$$

where $w^\alpha(\lambda = 0) = w^\alpha(\lambda = \lambda_f) = 0$ since both paths go from o_1 to P. We also recall that,

$$\delta\left(\pi - \pi_R\right) = \Pi_{\lambda,\alpha} \delta\left(x_\pi^\alpha(\lambda) - \lambda u^\alpha\right). \tag{9.9}$$

Let us note that the above expression can be written as,

$$\delta\left(x_\pi^\alpha(\lambda) - x_R^\alpha(\lambda)\right) = \delta\left(M^{\alpha(\lambda)}{}_{\delta(\mu)} w^{\delta(\mu)}\right). \tag{9.10}$$

where α and δ are Lorentz indices and (λ) and (μ) continuous indices that are integrated from 0 to λ when repeated and $w^{\delta(\mu)} \equiv w^{\delta}(\mu)$. The quantity $M^{\alpha(\lambda)}{}_{\delta(\mu)}$ can be computed by first integrating (9.7) for w^α with boundary conditions that vanish for $\lambda = 0$ and $\lambda = \lambda_f$, i.e. $w^\alpha(0) = w^\alpha(\lambda_f) = 0$. With this, in a sufficiently small region U, allowing the second order approximation for Riemann coordinates, we have that,

$$\begin{aligned}
M^{\alpha(\lambda)}{}_{\beta(\mu)} &= \delta^\alpha_\beta \delta(\lambda-\mu) + R_{\gamma\beta}{}^\alpha{}_\delta(0) u^\gamma u^\delta \\
&\quad \times \left[\int_0^\lambda d\lambda' \int_0^{\lambda'} d\lambda" \delta(\lambda" - \mu) - \frac{\lambda}{\lambda_f}\int_0^{\lambda_f} d\lambda' \int_0^{\lambda'} d\lambda" \delta(\lambda" - \mu)\right] \\
&= \delta^\alpha_\beta \delta(\lambda-\mu) + R_{\gamma\beta}{}^\alpha{}_\delta u^\gamma u^\delta \left[\int_0^\lambda \Theta(\lambda'-\mu) d\lambda' - \frac{\lambda}{\lambda_f}\int_0^{\lambda_f} d\lambda' \Theta(\lambda'-\mu)\right] \\
&= \delta^\alpha_\beta \delta(\lambda-\mu) + R_{\gamma\beta}{}^\alpha{}_\delta u^\gamma u^\delta \left[(\lambda-\mu) - \frac{\lambda}{\lambda_f}(\lambda_f - \mu)\right]. \qquad (9.11)
\end{aligned}$$

We now recall that $\delta(Lx) = \delta(x)/\det(L)$ if L is a non-singular matrix and x a vector. Therefore the right hand side of (9.10) can be rewritten involving the determinant of M. This is, by definition, the Fadeev–Popov determinant.

In order to compute the determinant we use that,

$$\det(I+L) = \exp\left(\mathrm{Tr}\ln(I+L)\right), \qquad (9.12)$$

where I is the identity matrix and L is a matrix of norm smaller that one. This holds in this case since $R\lambda_f^2 \ll 1$. Recalling that we are working up to second order in the Riemann coordinates only, we can take $\ln(I+L) \sim L$ and keep only the first order in the exponential. Therefore the determinant is given by $1 + \mathrm{Tr}(L)$. Therefore,

$$\begin{aligned}
\det(I+L) &= 1 - \int_0^{\lambda_f} R_{\gamma\beta}{}^\beta{}_\delta u^\gamma u^\delta (\lambda_f - \lambda)\frac{\lambda}{\lambda_f} d\lambda \\
&= 1 - R_{\gamma\beta}{}^\beta{}_\delta u^\gamma u^\delta \left(\frac{\lambda_f \lambda^2}{2} - \frac{\lambda^3}{3}\right)\Bigg|_0^{\lambda_f} \frac{1}{\lambda_f} \\
&= 1 - R_{\gamma\beta}{}^\beta{}_\delta u^\gamma u^\delta \frac{\lambda_f^2}{6} = 1 - R_{cb}{}^{bd}\left(\pi^{o_1}_{oR}\right)\frac{z^c z^d}{6}, \qquad (9.13)
\end{aligned}$$

with $z^a = u^a \lambda_f$. Recall that in normal coordinates we have that,

$$g_{mn} = \eta_{mn} - \frac{1}{3} R_{manb} z^a z^b, \qquad (9.14)$$

so for the determinant of the metric we have that,

$$\begin{aligned}
\sqrt{-g} &= \sqrt{1 - \frac{1}{3}\eta^{mn} R_{manb} u^a u^b \lambda_f^2} \\
&= 1 - \frac{1}{6}\eta^{mn} R_{manb} z^a z^b. \qquad (9.15)
\end{aligned}$$

And therefore,

$$\mathcal{D}x = \Pi_{\alpha,\lambda} dx^{\alpha}_{\pi}(\lambda)\delta(\pi - \pi_R)\Delta_{FP} = \Pi_a dz^a \sqrt{-g}, \tag{9.16}$$

and we recover the Einstein–Hilbert action in the coordinate system defined by the π_R.

X. SOME COMMENTS ABOUT THE CANONICAL FORMULATION

To get some idea of the issues involved in a canonical quantization of a framework like the one presented, let us consider the particular case of a scalar field in a curved space-time and study its canonical formulation.

A path dependent scalar field $\phi(\pi)$ is such that $\phi(\pi') = \phi(\pi)$ if $\pi' = \gamma \cdot \pi$ with γ a closed loop. Indeed, its description is frame independent and the path only fulfills the role of identifying the point in M where the field is evaluated without introducing coordinate systems.

The equation of motion for a massless scalar field in a Riemannian manifold is,

$$\eta^{\alpha\beta} D_\alpha D_\beta \phi(\pi) = 0, \tag{10.1}$$

and follows from the action,

$$S = \frac{1}{2} \int Du_\pi \Delta_{FP}(\pi)\delta(\pi' - \pi)\eta^{\alpha\beta} D_\alpha \phi(\pi') D_\beta \phi(\pi'), \tag{10.2}$$

following the ideas of previous sections.

Let $\pi = \pi_R$ be an arbitrary assignment of reference paths. By definition of the Mandelstam derivative we have that (see figures 12 and 13),

$$(1 + \epsilon u^\alpha D_\alpha)\, \phi\left(\pi^x_R\right) = \phi\left(\pi^{x+\epsilon u}_{R_E}\right), \tag{10.3}$$

where π_{R_E} represents the extended path.

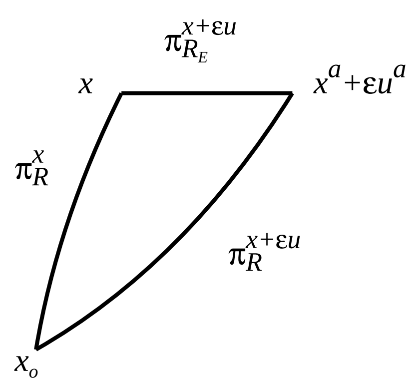

FIG. 13: The path in the Mandesltam derivative

Since the scalar field in $x + \epsilon u$ takes the same value for any path with that final point the covariant derivative becomes an ordinary one and we have that,

$$u^\alpha D_\alpha \phi\left(\pi_R\right) = u^\alpha e_\alpha{}^a\left(\pi_R\right) \left(\partial_a \phi\left(\pi_R\right)\right)\big|_{\pi^x_R} = u^a \partial_a \phi\left(\pi^x_R\right), \tag{10.4}$$

where e_α^a is the frame transported along π_R,

$$D_\alpha \phi\left(\pi_R\right) = e_\alpha{}^a\left(\pi_R\right) \partial_a \phi\left(\pi_R^x\right). \tag{10.5}$$

As a consequence, the action becomes the ordinary one,

$$\begin{aligned} S &= -\frac{1}{2} \int \mathcal{D}u_{\pi_R} \Delta_{FP}\left(\pi_R\right) \delta\left(\pi' - \pi_R\right) \eta^{\alpha\beta} e_\alpha{}^a\left(\pi_R\right) e_\beta{}^b\left(\pi_R\right) \partial_a \phi\left(x\left(\pi_R\right)\right) \partial_b \phi\left(x\left(\pi_R\right)\right), \\ &= -\frac{1}{2} \int dx \sqrt{-g} g^{ab}(x) \partial_a \phi \partial_b \phi. \end{aligned} \tag{10.6}$$

Its equations of motion are,

$$g^{ab} \nabla_a \nabla_b \phi(x) = \eta^{\alpha\beta} e_\alpha{}^a e_\beta{}^b \nabla_a \nabla_b \phi = \eta^{\alpha\beta} D_\alpha D_\beta \phi\left(\pi_R^x\right) = 0, \tag{10.7}$$

where we have used equation (6.5).

Let us proceed to the canonical formulation. In the first place we note that although $\phi(\pi)$ is independent of the path π that arrives at the point x, its canonical conjugate momentum depends of the notion of time used and therefore of the frame transported to x along π. It should be pointed out that the canonical framework is not well suited for the intrinsic formulation since it assumes that the surface can be foliated and that the topology is fixed. These are two hypotheses that are not natural in the intrinsic approach. It will, however, allow us to carry out a first approach towards quantization.

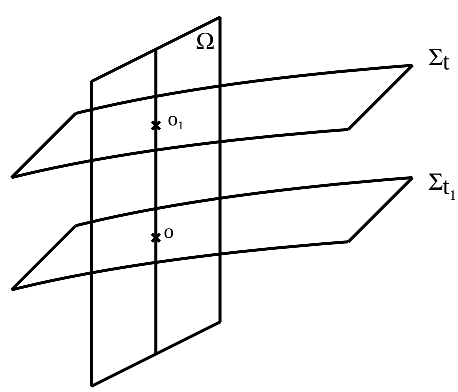

FIG. 14: The foliation of the manifold.

We can start with a manifold $M = \Sigma \times R$ with coordinates adapted and introduce a geometry in M à la ADM, for example. In order to introduce intrinsic reference paths arriving to each point of M adapted to a foliation Σ_t (see figure 14), we introduce a platform through o, a three dimensional hypersurface Ω such that $\Sigma_t \cap \Omega$ is two dimensional. We also introduce a congruence of curves in Σ_t parameterized by t, u, v, w such that $\gamma^a\left(t, u, v, w_0 = 0\right)$ are points on $\Sigma_t \cap \Omega$ and that $\gamma^a\left(t = t_0, 0, 0, 0\right)$ is the origin of the intrinsic description. Given a point t_1, x_1 with $x_1^i = \gamma^i\left(t_1, u_1, v_1, w_1\right)$ in Σ_{t_1} we define reference paths π_R starting in o, $\gamma^a(t, 0, 0, 0)$ with $\gamma^a(t_1, 0, 0, 0) = o_1$. From o_1 we go to $x_1'^{\,i} = \gamma^i(t_1, u_1, v_1, 0)$ through the path $\gamma^a(t_1, \lambda u_1, \lambda v_1, 0)$ with $\lambda \in [0, 1]$ and through $\gamma^a(t_1, u_1, v_1, w)$ to $\gamma^a(t_1, u_1, v_1, w_1)$. Its intrinsic description will depend on the geometry. If we use a $3+1$ ADM notation, we have

that,

$$
\begin{aligned}
g_{ij} &= {}^4g_{ij}, \qquad N = \left(\sqrt{-{}^4g^{00}}\right)^{-1}, \qquad N_i = {}^4g_{0i}, &(10.8)\\
g^{ij}g_{jk} &= \delta^j_k, \qquad {}^4g_{00} = -\left(N^2 - N^iN_i\right), \qquad N^i = g^{ij}N_j, &(10.9)\\
{}^4g^{0i} &= \frac{N^i}{N^2}, \qquad {}^4g^{00} = -\frac{1}{N^2}, \qquad {}^4g^{ij} = g^{ij} - \frac{N^iN^j}{N^2}, &(10.10)\\
\det{}^4g &= -N\det g, \qquad \sqrt{-{}^4g} = N\sqrt{g}, &(10.11)\\
n_\mu &= -N\delta^0_\mu, \qquad {}^4g^{\mu\nu}n_\mu n_\nu = -1. &(10.12)
\end{aligned}
$$

Recalling that the action for $\phi(x) = \phi\left(\pi^x_R\right)$ is,

$$
\begin{aligned}
S &= -\frac{1}{2}\int dx\sqrt{-g}g^{AB}\partial_A\phi\partial_B\phi,\\
&= -\frac{1}{2}\int dxN\sqrt{g}\left[-\frac{1}{N^2}\left(\partial_0\phi\right)^2 + 2\frac{N^i}{N^2}\partial_0\phi\partial_i\phi + \left(g^{ij} - \frac{N^iN^j}{N^2}\right)\partial_i\phi\partial_j\phi\right], \quad (10.13)
\end{aligned}
$$

the canonical momentum is

$$P_\phi = \frac{\sqrt{g}}{N}\partial_0\phi - \frac{N^i}{N}\sqrt{g}\partial_i\phi. \tag{10.14}$$

We can then proceed to do the Legendre transform and obtain the Hamiltonian,

$$
\begin{aligned}
\mathcal{H} &= P_\phi\partial_0\phi - \mathcal{L}\\
&= \frac{NP_\phi^2}{\sqrt{g}} + P_\phi N^i\partial_i\phi - \frac{1}{2}\frac{\sqrt{g}}{N}\left(\frac{NP_\phi}{\sqrt{g}} + N^i\partial_i\phi\right)\\
&\quad + \frac{N^i\sqrt{g}}{N}\partial_i\phi\left(\frac{NP_\phi}{\sqrt{g}} + N^i\partial_i\phi\right) + \frac{1}{2}\left(g^{ij} - \frac{N^iN^j}{N^2}\right)\partial_i\phi\partial_j\phi\sqrt{g}\\
&= \frac{NP_\phi^2}{2\sqrt{g}} + P_\phi N^i\partial_i\phi + \frac{1}{2}\left(g^{ij}\partial_i\phi\partial_j\phi\right)N\sqrt{g}. \qquad (10.15)
\end{aligned}
$$

From it we get the equations of motion

$$
\begin{aligned}
\partial_0\phi &= \frac{NP_\phi}{\sqrt{g}} + N^i\partial_i\phi, &(10.16)\\
\partial_0P_\phi &= -\partial_i\left(P_\phi N^i\right) - \partial_i\left(g^{ij}\partial_j\phi N\sqrt{g}\right). &(10.17)
\end{aligned}
$$

The Poisson brackets are,

$$
\begin{aligned}
\left\{\phi(x), P_\phi(y)\right\}_t &= \left\{\phi(\pi^x), P_\phi(\pi^y)\right\}_t = \delta(x,y), &(10.18)\\
\left\{\phi(x), \phi(y)\right\}_t &= \left\{P_\phi(x), P_\phi(y)\right\}_t = 0. &(10.19)
\end{aligned}
$$

From (10.14,10.16) we get the brackets of the time derivatives,

$$
\begin{aligned}
\left\{\phi(x), \partial_0\phi(y)\right\} &= \frac{N}{\sqrt{g}}\delta(x,y), &(10.20)\\
\left\{\partial_0\phi(x), P_\phi(y)\right\} &= N^i\partial_i\delta(x,y), &(10.21)
\end{aligned}
$$

where ∂_0 is the derivative with respect to the parameter of the foliation Σ_t.

So we see that fixing the reference path has led us to the traditional canonical formulation of scalar field. But we really are interested in the Poisson brackets for arbitrary paths described intrinsically. Let us first consider paths that start from o in Σ_t and have the same end point than that of π^x. In order to do that we will use the technique of going from paths in coordinate systems to intrinsic paths and vice-versa. It will allow us to recognize paths that end in x.

Let π given by $y^\alpha(\lambda)$ that corresponds to $\gamma^a(\lambda)$ with $\gamma^\mu(x) = x^a$, that is,

$$\int_0^{\lambda_f} d\lambda \dot{y}^\alpha(\lambda) e_\alpha{}^a\left([y],\lambda\right) = x^a, \tag{10.22}$$

and π' given by $y'^\alpha(\lambda)$,

$$\int_0^{\lambda_f} d\lambda \dot{y'}^\alpha(\lambda) e_\alpha{}^a\left([y'],\lambda\right) = z^a. \tag{10.23}$$

The Poisson brackets satisfy

$$\{\phi(\pi), P_\phi(\pi')\} = \delta^3\left(\gamma^a(\lambda_f), \gamma'^a(\lambda_f)\right) = \delta^3(x^a, z^a), \tag{10.24}$$

with

$$\gamma^a(\lambda) = \int_0^\lambda d\lambda_1 \dot{y}^\alpha(\lambda_1) e_\alpha{}^a\left([y],\lambda_1\right). \tag{10.25}$$

The advantage of this kind of relation is that it is easily generalizable to the case of quantum gravity where the $e_\alpha{}^a$ are operators.

If we consider π extended to the future region defined along the time component of the local basis $e_\alpha{}^a\left([y],1\right)$, we have that,

$$D_0\phi(\pi) = e_0{}^a \partial_a \phi(\pi), \tag{10.26}$$

$$\{D_0\phi(\pi), \phi(\pi')\} = -\frac{e_0{}^0 N}{\sqrt{g}} \delta\left(\gamma^a(\lambda_f), \gamma'^a(\lambda_f)\right). \tag{10.27}$$

The timelike Mandelstam derivative extends the path π along the zeroth component of the parallel transported frame and is given by (10.26). From it and (10.20) we get (10.27).

Notice that if one were to quantize the gravitational field that equation (10.27) taking into account the relation of the intrinsic and space-time coordinates (10.25) would become an operatorial identity. In particular the arguments of the Dirac delta in (10.27) become operatorial. This will require further study for a complete canonical quantization. As we mentioned, the canonical approach is not the most natural in this context and other approaches that implement directly the algebra of Dirac observables might be preferred.

XI. CONCLUDING REMARKS

We presented an intrinsic framework for the formulation of gravitational theories including general relativity in terms of paths. We solved the problem of defining what is a space-time point, that was problematic in the original proposal on the subject by Mandelstam. The relation of the fields for two paths that arrive at the same point is now under control.

In the intrinsic description of gravity a physical point is given by the equivalence class of paths that differ by loops that may be defined by the repeated action of the loop derivative. In an eventual quantum theory, a fluctuation of the geometry in any region of space-time will change that equivalence class, that is, some of the paths that led to that point will fail to arrive to it. This will induce fluctuations in the points that must be considered as emergent objects of an underlying structure of paths. The fluctuations of the space-time points will be more important in a region where quantum effects are expected to be large, like near where black holes have their classical singularities. Close to a region with big quantum fluctuations the fields will stop being local, in particular scalar fields associated to nearby points will not commute, irrespective of the separation being space-like or time-like. Note that the non-locality is also in time, which makes the causal structure of events become fuzzy. One of the remaining questions is whether is relevant in the context of black hole evaporation.

The intrinsic description naturally operates with space-time paths. However, even if one considers spatial paths one could end up in points that are in the future of where one started. This will require special care at the time of quantization, as was already observed by Mandelstam.

The whole construction is locally Lorentz invariant but there may be a distortion of the invariance, unrelated to the ones due to granular descriptions of space-time, due to the fluctuation of the points. Further studies of the quantization are needed to understand the non-local effects induced by time-like paths. In a forthcoming paper we will discuss the Poisson algebra of path dependent fields including gravity and its quantization.

XII. ACKNOWLEDGEMENTS

We wish to thank Aureliano Skirzewski and Rafael Porto for discussions and especially Miguel Campiglia and Saeed Rastgoo for reading the manuscript and providing useful advice and comments. This work was supported in part by Grant No. NSF-PHY-1603630, funds of the Hearne Institute for Theoretical Physics, CCT-LSU, and Pedeciba and Fondo Clemente Estable FCE_1_2014_1_103803.

Appendix

To understand the effects of curvature on finite closed paths one needs to take into account that for the infinitesimal generator or the loop derivative to close a loop after going along a path $\epsilon u \epsilon w \epsilon \overline{u}_{||}$ with u, v unit vectors, instead of traversing $\overline{w}_{||}$ one needs to go along a different path, which we call $\overline{w}^{(1)}$. We introduce here the notation $u_{||}$ to emphasize that it is at a different point (and referred to a different frame) than u since this will be important in the calculation. To compute it, we consider normal coordinates around an arbitrary point of the manifold o' that can be considered at the end of a path $\pi_o^{o'}$. The geodesics emanating from o' are given in normal coordinates by $x^\mu(s) = v^\mu s$ (the x^μ are the normal coordinates and $v\mu$ are constants). The metric at o' is η and, the Christoffel symbols and metric nearby

are given in normal coordinates in terms of the Riemann tensor computed at o' as,

$$\Gamma^{\mu}_{\alpha\beta} = -\frac{1}{3}\left(R^{\mu}{}_{\alpha\beta\gamma} + R^{\mu}{}_{\beta\alpha\gamma}\right)x^{\gamma}, \tag{12.1}$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}x^{\alpha}x^{\beta}. \tag{12.2}$$

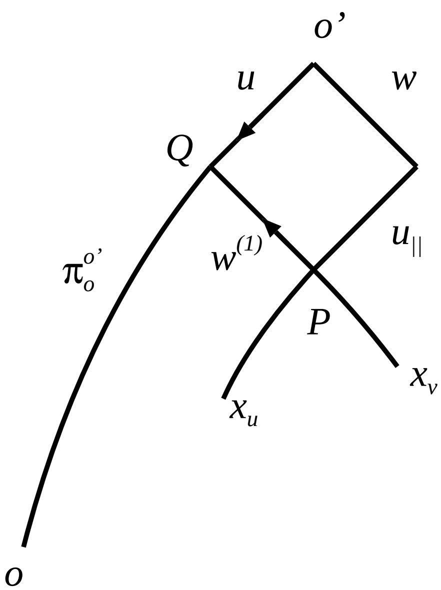

FIG. 15: The geodesics discussed in the appendix.

We compute $u(s)$ by transporting u along ϵw,

$$w^{\alpha}u^{\beta}{}_{;\alpha} = w^{\alpha}\left(\partial_{\alpha}u^{\beta} + \Gamma^{\beta}_{\alpha\rho}u^{\rho}\right) = 0, \tag{12.3}$$

$$w^{\alpha}\partial_{\alpha}u^{\beta} = \frac{du^{\beta}}{ds} = -w^{\alpha}\Gamma^{\beta}_{\alpha\rho}u^{\rho} = \frac{1}{3}R^{\beta}{}_{\alpha\rho\gamma}w^{\alpha}u^{\rho}w^{\gamma}s, \tag{12.4}$$

$$\frac{d^2u^{\beta}}{ds^2} = \frac{1}{3}R^{\beta}{}_{\alpha\rho\gamma}w^{\alpha}u^{\rho}w^{\gamma}. \tag{12.5}$$

Therefore,

$$u^{\beta}(s) = u^{\beta}(0) + \frac{1}{6}R^{\beta}{}_{\alpha\rho\gamma}w^{\alpha}u^{\rho}w^{\gamma}s^2, \tag{12.6}$$

$$u^{\beta}_{||} = u^{\beta}(0) + \frac{1}{6}R^{\beta}{}_{\alpha\rho\gamma}w^{\alpha}u^{\rho}w^{\gamma}\epsilon^2, \tag{12.7}$$

$$\tag{12.8}$$

and transporting w along ϵu we get,

$$w^{\beta}_{||} = w^{\beta}(0) + \frac{1}{6}R^{\beta}{}_{\alpha\rho\gamma}u^{\alpha}w^{\rho}u^{\gamma}\epsilon^2, \tag{12.9}$$

where $u^{\beta}_{||}$ is u parallel transported along ϵw and is shown in figure (15) and $w^{\beta}_{||}$, not shown in

the figure is w parallel transported along ϵu. The vectors $u, w, u_{||}, w_{||}$ do not form a closed loop. To close it we need to compute $w^{(1)}$ which differs by terms of order ϵ^3 from $w_{||}$.

In Riemann coordinates an arbitrary geodesic not necessarily going through o is given by,

$$x^\mu(s) = v_0^\mu + v_1^\mu s + \frac{1}{3} R^\mu{}_{\alpha\beta\rho} v_0^\rho v_1^\alpha v_1^\beta s^2, \tag{12.10}$$

where v_0^μ are the coordinates of a point v_0 and v_1^μ the tangent to the geodesic at the same point.

Let us consider the geodesic x^μ in figure 15 and let us determine the coordinates of the point P, end point of $u_{||}$, and Q, end point of u,

$$x_P^\mu = \epsilon w^\mu - \epsilon u^\mu + \frac{\epsilon^3}{6} R_{\alpha\rho\beta}{}^\mu u^\alpha w^\beta w^\rho - \frac{\epsilon^3}{3} R_{\beta\rho\alpha}{}^\mu w^\rho u^\alpha u^\beta, \tag{12.11}$$

and therefore $x_P - x_Q$ is given by,

$$x_P^\mu - x_Q^\mu = \epsilon w^\mu + \frac{\epsilon^3}{6} R_{\alpha\rho\beta}{}^\mu u^\alpha w^\rho w^\beta + \frac{\epsilon^3}{3} R_{\beta\rho\alpha}{}^\mu w^\rho u^\alpha u^\beta. \tag{12.12}$$

Therefore the components of the vector $w^{(1)\mu}$ at the point Q are given by,

$$\epsilon w^{(1)\mu} = \epsilon w^\mu + \frac{\epsilon^3}{6} R_{\alpha\rho\beta}{}^\mu u^\alpha w^\rho w^\beta + \frac{\epsilon^3}{3} R_{\alpha\rho\beta}{}^\mu u^\beta u^\alpha w^\rho, \tag{12.13}$$

and notice that $w^{(1)}$ is not a unit vector. It is also useful to compute the components of $w^{(1)}$ at the point P. In the intrinsic notation we need to write $w^{(1)}$ in the parallel transported basis to P given by,

$$e_\alpha{}^\mu(P) = \delta_\alpha{}^\mu + \frac{1}{3} R_{\rho\alpha\beta}{}^\mu u^\beta w^\rho \epsilon^2 + \frac{1}{6} R_{\rho\alpha\beta}{}^\mu u^\beta u^\rho \epsilon^2. \tag{12.14}$$

Therefore $\epsilon w^{(1)}$ in intrinsic notation takes the form,

$$\epsilon w^{(1)\alpha} = -\epsilon w^\alpha - \frac{\epsilon^3}{2} R_{\gamma\rho\beta}{}^\alpha u^\beta u^\gamma w^\rho - \frac{\epsilon^3}{2} R_{\gamma\rho\beta}{}^\alpha u^\gamma w^\rho w^\beta. \tag{12.15}$$

---

[1] W. Donnelly and S. B. Giddings, Phys. Rev. D **93**, no. 2, 024030 (2016) Erratum: [Phys. Rev. D **94**, no. 2, 029903 (2016)] doi:10.1103/PhysRevD.94.029903, 10.1103/PhysRevD.93.024030 [arXiv:1507.07921 [hep-th]]; Phys. Rev. D **94**, no. 10, 104038 (2016) doi:10.1103/PhysRevD.94.104038 [arXiv:1607.01025 [hep-th]];arXiv:1706.03104 [hep-th].

[2] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005) doi:10.1103/PhysRevLett.94.181602 [hep-th/0501052]; N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, JHEP **1003**, 110 (2010) doi:10.1007/JHEP03(2010)110 [arXiv:0903.2110 [hep-th]]; N. Arkani-Hamed and J. Trnka, JHEP **1410**, 030 (2014) doi:10.1007/JHEP10(2014)030 [arXiv:1312.2007 [hep-th]]; M. Bullimore and D. Skinner, arXiv:1101.1329 [hep-th]; S. Caron-Huot, JHEP **1107**, 058 (2011)

doi:10.1007/JHEP07(2011)058 [arXiv:1010.1167 [hep-th]].
[3] S. Mandelstam, Ann. Phys. (NY) **19**, 1 (1962); Ann. Phys. (NY) **19**, 25 (1962).
[4] R. Gambini, A. Trias, Phys. Rev. **D23**, 553 (1981).
[5] J. Lewandowski, A. Okolow, H. Sahlmann and T. Thiemann, Commun. Math. Phys. **267**, 703 (2006) doi:10.1007/s00220-006-0100-7 [gr-qc/0504147]; C. Fleischhack, *Phys. Rev. Lett.* 97, 061302 (2006).
[6] R. Gambini, J. Pullin, "Loops, knots, gauge theories and quantum gravity", Cambridge University Press (1996); R. Gambini, A. Trias, Phys. Rev. **D27**, 2935 (1983); R. Gambini "Teorías de Calibre en el espacio de ciclos", unpublished notes, Universidad Simón Bolivar (1986); A. Trias, "El grupo de ciclos en teorías de calibre" unpublished notes.
[7] J. N. Tavares, Int. J. Mod. Phys. A **9**, 4511 (1994) doi:10.1142/S0217751X94001795 [hep-th/9305173].
[8] J. Lewandowski, Class. Quan. Grav. **10**, 879 (1993); M. Reiris, P. Spallanzani, Class. Quan. Grav. 16, 8 (1999).
[9] F. Hehl, J. McCrea, E. Mielke, Y. Ne'eman, Phys. Rep. 258, 1 (1995).
[10] C. Teitelboim, Nucl. Phys. B **396**, 303 (1993).
[11] A. Nesterov, Class. Quan. Grav. 16, 465 (1999).