

7-23-2020

Gravitation in terms of observables 2: The algebra of fundamental observables

Rodolfo Gambini
Facultad de Ciencias

Saeed Rastgoo
York University

Jorge Pullin
Louisiana State University

Follow this and additional works at: https://digitalcommons.lsu.edu/physics_astronomy_pubs

Recommended Citation

Gambini, R., Rastgoo, S., & Pullin, J. (2020). Gravitation in terms of observables 2: The algebra of fundamental observables. *Classical and Quantum Gravity*, 37 (14) <https://doi.org/10.1088/1361-6382/ab8eb8>

This Article is brought to you for free and open access by the Department of Physics & Astronomy at LSU Digital Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Digital Commons. For more information, please contact ir@lsu.edu.

Gravitation in terms of observables 2: the algebra of fundamental observables

Rodolfo Gambini¹, Saeed Rastgoo², Jorge Pullin³

*1. Instituto de Física, Facultad de Ciencias,
Iguá 4225, esq. Mataojo, 11400 Montevideo, Uruguay.*

*2. Department of Physics and Astronomy, York University,
4700 Keele Street, Toronto, Ontario M3J 1P3, Canada.*

*3. Department of Physics and Astronomy,
Louisiana State University, Baton Rouge, LA 70803-4001, USA.*

In a previous paper, we showed how to use the techniques of the group of loops to formulate the loop approach to gravity proposed by Mandelstam in the 1960's. Those techniques allow to overcome some of the difficulties that had been encountered in the earlier treatment. In this approach, gravity is formulated entirely in terms of Dirac observables without constraints, opening attractive new possibilities for quantization. In this paper we discuss the Poisson algebra of the resulting Dirac observables, associated with the intrinsic components of the Riemann tensor. This provides an explicit realization of the non-local algebra of observables for gravity that several authors have conjectured.

Contents

I. Introduction	2
II. Previous results	3
III. Procedure for computing the Poisson brackets for intrinsic components of the Riemann tensor	5
IV. Explicit computation	8
A. General Poisson bracket to be computed	8
B. Relation between partial and Mandelstam derivatives	10
C. Terms $\Theta_1 - \Theta_3$	11
D. Terms $\Theta_4 - \Theta_7$	13
V. Full Poisson brackets	15
VI. Conclusions	20
VII. Acknowledgments	20
References	20

I. INTRODUCTION

In 1962 Mandelstam [1] published two articles introducing path dependent techniques for the description of gauge theories and gravity at the classical and quantum level. The case of gravity resulted too complicated, but this motivated many physicists to study gauge theories using related loop techniques. The hope was that holonomies could allow a better understanding of the confinement phase. Makeenko and Migdal [2], and Polyakov [3] proposed different non-perturbative loop techniques in 1979, but the hopes raised by them were soon abandoned because the very elegant equations for Wilson loops were difficult to regularize and renormalize in a non-perturbative way. In the early 80's, Gambini and Trias [4] introduced the techniques of the group of loops and the Hamiltonian treatment of Yang Mills in the space of loops. This technique was subsequently used in Loop Quantum Gravity with an approach closer to the Hamiltonian formulation of Yang Mills theory. However, the original idea of Mandelstam was much more powerful. In fact, the most ambitious attempt to describe gravity intrinsically without coordinates and purely in terms of observables was proposed by Mandelstam in his second paper of 1962. It could serve as the basis for a coordinate independent approach to the quantization of gravitation. This paradigm did not flourish because the intrinsic description loses completely the notion of space-time point, and it becomes difficult to recover this notion even classically. That is because in this description, the paths that end in the same physical point cannot be easily recognized.

Recently, there has been a renewed interest in the description in terms of observables of gauge theories and gravity. Donnelly and Giddings [5] have proposed explicit constructions that extend the observables associated to gauge theories to the case of gravitation in the weak field limit. They note that an important feature of the resulting quantum theory of gravity is the algebra of observables, which becomes non-local. Observable-based techniques are also used in several modern developments attempting to extract information from quantum gauge theories [6].

In a previous paper [7] we have shown how to extend the notion of the group of loops and its representations arising in gauge theories to the gravitational case proposed by Mandelstam. This leads to a complete classical description of gravitation without coordinates. The metric is everywhere referred to local frames parallel transported starting from a given point. In such frames it takes the Minkowskian form. The geometrical content of the theory is completely recovered by relations between reference frames obtained by parallel transport along paths that differ by an infinitesimal loop and is given by the Riemann tensor. Although the construction that we presented there was based on loops, it differs from the one underlying the usual loop representation of gauge theories and gravity. In the loop representation the objects constructed are gauge invariant whereas in the present construction the objects are both gauge invariant and space-time diffeomorphism invariant. That is, the objects are Dirac observables. This leads to a theory that does not involve diffeomorphisms and may allow to bypass at the quantum level the LOST-F [8] theorem that leads to a discrete structure in the Hilbert space of ordinary loop quantum gravity and conflicts with the differentiability of the group of loops. The latter is crucial to recover the kinematics of gauge theories and gravity in this context.

In this paper we will show how to determine Poisson brackets among path dependent Riemann observables that are consistent with Einstein's equations. In his original papers Mandelstam had computed an algebra of Poisson brackets but it was unclear whether it was compatible with the Einstein equations. It turns out that the resulting algebra among

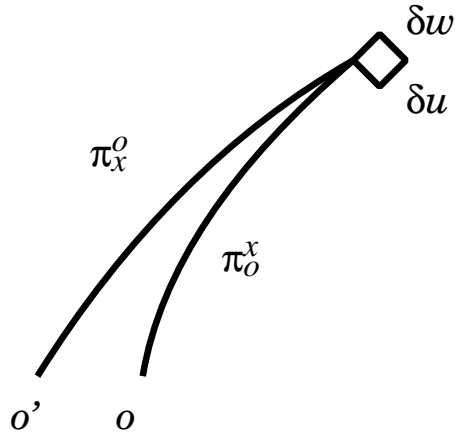


FIG. 1: The path has the same intrinsic initial and final coordinates (that is why both paths are labeled by o and x) but would correspond to two different end points of the space-time, o and o' .

Riemann observables evaluated on arbitrary paths is non-local. Observables do not organize themselves into local commuting sub-algebras (as occurs in usual field theories), and therefore the principle of locality [9] must apparently be reformulated or abandoned, as Donnelly and Giddings [5] conjectured. In section II we review the Mandelstam intrinsic formulation and the techniques allowing to determine physical points. In section III we determine the Poisson algebra of path dependent Riemann tensors in vacuum, finally in section IV we concluded with a discussion of the non-locality of the algebra of the gravitational theory and some final remarks.

II. PREVIOUS RESULTS

Mandelstam's construction starts by the *intrinsic* specification of paths in a manifold. By that he means the following: starting from a chosen initial point (in asymptotically flat manifolds it could be infinity) one parallel transports a frame along a curve a certain invariant distance and then follows another distance along a different direction and so on. The important point is that the direction is defined by the parallel transported frame. Therefore one characterizes curves by a series of instructions of how to proceed with respect to a local frame. This has similarities with how a GPS provides instructions to a driver to follow a path. Diffeomorphisms in space-time affect the curves but not the set of instructions that is given intrinsically. The trouble with these sets of instructions is that it is difficult to determine if two paths end at the same point. This hampered the development of Mandelstam's framework in the 1960's.

Suppose one considers two paths that intrinsically are sets of instructions opposite of each other. Clearly, if one were to follow them, one would return to the same point. Suppose, however, that one adds an infinitesimal loop between them. The loop would alter the frame with respect to which the initial instruction of the return path is specified. As a consequence, one would end up with a path that does not return to the same point, as shown in figure 1.

If one wished the path to start and end at the same point, one would have to correct the set of instructions of the return path to undo the rotation of the frame that took place due

to the addition of the infinitesimal loop. This way, if one considers the frame one started with and evaluates the parallel transport of it along the corrected path, one gets back to the same point. The initial and final frame of such a closed path would be related by a Lorentz transformation (holonomy) given by,

$$H(\pi_o^x \circ \delta\gamma \circ \Lambda(\delta\gamma)\pi_x^o)^\alpha_\beta = \delta^\alpha_\beta + \delta u^\rho \delta w^\sigma R_{\rho\sigma}{}^\alpha{}_\beta(\pi_o^x), \quad (2.1)$$

where $\Lambda(\delta\gamma)\pi_x^o$ is the retraced rotated path described above and R is the Riemann tensor.

The addition of an infinitesimal loop is associated with the generator of the group of loops. The composition (product) of such infinitesimal generators can be used to construct finite loops. This allows to reverse the construction: two intrinsically defined paths will end at the same point if they differ by a loop. This was the missing piece in Mandelstam's 1960's construction that we added.

In this framework, matter fields become path dependent, and they are given by the fields evaluated at the endpoints of paths. Under a change of path, they transform with appropriate holonomies. For instance, for a vector field with internal group $SU(N)$ in some representation,

$$A^\alpha_I(\pi') = H(\gamma)^\alpha_\beta H(\gamma)_I^J A^\beta_J(\pi), \quad (2.2)$$

if $\pi' = \gamma \circ \Lambda(\gamma)\pi = \gamma \cdot \pi$, which guarantees that π' and π end at the same point on M . Here, we introduced the dot as a shorthand for the composition of loops in the intrinsic formulation, incorporating the Lorentz rotation of the previous loop. Also, $H(\gamma)^\alpha_\beta$ is a holonomy associated with the Lorentz group and $H(\gamma)_I^J$ a holonomy associated with the internal group.

In our previous paper we saw that it is possible to identify when two paths described in such a way end at the same point. Indeed, for a given geometry, two open paths π, π' whose local bases transported to their ends differ by a Lorentz transformation and $\pi' = \gamma \cdot \pi$ with γ , then both paths end at the same point in the manifold M . The notion of closed loops also depends on the geometry. This implies that at the quantum level, when the geometry fluctuates, so do the points, and they become fuzzy objects.

The intrinsic quantization is therefore *nonequivalent* to the usual one. Let us be more explicit using the technique developed in section VII of our previous paper [7]. Given a path $\gamma^a(\lambda)$ in a differential manifold in a given coordinate system, with $\gamma^a(0) = x_o^a$ the coordinates of o and $\gamma^a(1) = x^a$ a local point, the frame transported along γ^a is,

$$e_\alpha{}^c(\lambda) = P \left(\exp \left(- \int_0^\lambda d\lambda' \dot{\gamma}^a(\lambda') \Gamma_a \right) \right)_d^c e_\alpha{}^d(0), \quad (2.3)$$

where Γ_a is the connection in the given coordinate system, and the intrinsic coordinates are given by

$$y^\alpha(\lambda) = \int_0^\lambda \dot{\gamma}^c(\lambda') e_\alpha{}^c(\Gamma, \lambda') d\lambda', \quad (2.4)$$

which implies that upon quantization of the geometry (and therefore of Γ), to a curve $\gamma^a(\lambda)$ in M corresponds an operator $\hat{y}^\alpha(\lambda)$. On the contrary, if one considers intrinsic coordinates as the primary description of the path and one uses,

$$\gamma^a(\lambda) = \int_0^\lambda d\lambda' \dot{y}^\alpha e_\alpha{}^a([y], \lambda') + x_o^a, \quad (2.5)$$

where $e_\alpha^a([y], \lambda')$ is the tetrad transported from the origin with the prescription given by the function y up to the point with parameter λ' , one would get for the intrinsic trajectory $\gamma(\lambda)$, upon quantization, an operator,

$$\hat{\gamma}^a(\lambda) = \int_0^\lambda d\lambda' \hat{e}_\alpha^a([y], \lambda') \dot{y}^\alpha + x_o^a, \quad (2.6)$$

and therefore the traditional notion of curve only is recovered in the semi-classical approximation. In the usual quantization scheme one is given a curve that remains classical and quantizes the geometric operators, like the metric. Intrinsically defined curves become quantum operators, as they depend on the metric. Conversely, if one were to take the intrinsic description of the curve as a starting point for a quantization, the latter would be classical whereas the curve itself becomes a quantum operator as shown in (2.6).

III. PROCEDURE FOR COMPUTING THE POISSON BRACKETS FOR INTRINSIC COMPONENTS OF THE RIEMANN TENSOR

If one takes as reference paths in the action presented in section IX of the companion paper the ones used in Riemann or Fermi normal coordinates, one recovers the standard Einstein–Hilbert action in those coordinates. It is well known [10] that in order to have geodesics that do not cross each other, one must have $s \ll |R_o|^{-1/2}$ where R_o is the typical size of the curvature and s is the length of the geodesic. In this region of validity we can use the Palatini first order action,

$$S = \int d^4x \sqrt{-g} g^{ab} R_{ab}(\Gamma), \quad (3.1)$$

and recalling that in normal coordinates $g_{ab} = \eta_{ab} + h_{ab}$ with h of second order (in s) and Γ_{ab}^c is an independent first order quantity and the R_{ab} 's are zeroth order quantities,

$$R_{ab} = \Gamma_{ab,c}^c - \Gamma_{ac,b}^c + \Gamma_{ab}^c \Gamma_{cd}^d - \Gamma_{ac}^d \Gamma_{bd}^c. \quad (3.2)$$

Recall that either in Riemann or Fermi normal coordinates the gauge is partially fixed. This will not be relevant because we are going to compute the relation for diffeomorphism-invariant quantities at the end, and working in a specific coordinate system simplifies the canonical analysis.

Let us restrict the action considering its expansion up to second order in s . The analysis is valid for arbitrary Riemann tensors in a sufficiently small region. The action will then read,

$$S_2 = \int d^4x \sqrt{1 + h_d^d} (\eta^{ab} - h^{ab}) R_{ab}(\Gamma), \quad (3.3)$$

where h and Γ are considered independent variables. The variation with respect to h yields $R_{ab} - \eta_{ab}R/2$ with $R = \eta^{cd}R_{cd}$. Variation with respect to Γ leads to,

$$(\sqrt{-g}g^{ab})_{;c} = \left(\sqrt{1 + h_d^d} (\eta^{ab} - h^{ab}) \right)_{;c} + \eta^{ad} \Gamma_{dc}^b + \eta^{bd} \Gamma_{dc}^a - \eta^{ab} \Gamma_{dc}^d, \quad (3.4)$$

which implies that the first order connection takes the form,

$${}^{(1)}\Gamma_{ab}^c = \frac{1}{2}\eta^{cd}(h_{ad,b} + h_{bd,a} - h_{ab,d}). \quad (3.5)$$

Introducing adapted three dimensional quantities,

$$\begin{aligned} g_{ij} &= \eta_{ij} + h_{ij}, & N &= (-\eta^{00} + h^{00})^{-1/2} = 1 - \frac{h^{00}}{2}, \\ N_i &= h_{0i}, & \sqrt{g} &= \sqrt{1 + h_i^i}, & \sqrt{-4g} &= N\sqrt{g} \\ \pi^{ij} &= (\Gamma_{pq}^0 - \eta_{pq}\eta^{rs}\Gamma_{rs}^0)\eta^{pi}\eta^{qj} = \Gamma_{pq}^0\eta^{ip}\eta^{jq} - \Gamma_{rs}^0\eta^{rs}\eta^{ij}, \end{aligned} \quad (3.6)$$

$$\Gamma_{pq}^0 = \frac{1}{2}(h_{pq,0} - N_{p,q} - N_{q,p}), \quad (3.7)$$

and defining $\pi = \pi^{ij}\eta_{ij}$ one gets,

$$2\left(\pi_{ij} - \frac{1}{2}\pi\eta_{ij}\right) = h_{ij,0} - N_{i,j} - N_{j,i}. \quad (3.8)$$

In terms of these the quantities the second order Lagrangian takes the form,

$$\mathcal{L} = \pi^{ij}h_{ij,0} - \left(1 - \frac{h_{00}}{2}\right)R^0 - N_i R^i, \quad (3.9)$$

where,

$$R^0 = -\sqrt{1 + h_a^a}{}^3R + \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right), \quad (3.10)$$

$$R^i = -2\pi_{,j}^{ij}. \quad (3.11)$$

As the action is partially gauge fixed, the total Hamiltonian includes a true Hamiltonian plus a linear combination of constraints, leading to a Lagrangian,

$$\mathcal{L}_2 = \pi^{ij}\partial_0 h_{ij} - \left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \frac{h^{00}}{2}{}^3R + 2N_i\pi_{,j}^{ij}, \quad (3.12)$$

which allows to determine the Hamiltonian and the constraints and to define Poisson brackets that lead to canonical equations for h_{ij} and π^{ij} ,

$$\{h_{ij}(x), \pi^{kl}(y)\} = \delta_{ij}{}^{kl}\delta^3(x-y), \quad (3.13)$$

with $\delta_{ij}{}^{kl} = \delta_i^k\delta_j^l + \delta_i^l\delta_j^k$. Therefore,

$$\begin{aligned} \{h_{ij}(x), h_{,0}^{kl}(y)\} &= 2\kappa \left\{ h_{ij}(x), \pi^{kl}(y) - \frac{1}{2}\pi(y)\eta^{kl} \right\} \\ &= \kappa (2\delta_{ij}{}^{kl} - \delta_{ij}{}^{mn}\eta_{mn}\eta^{kl}) \delta^3(x-y) \\ &= \kappa (2\delta_i^k\delta_j^l + 2\delta_i^l\delta_j^k - 2\eta_{ij}\eta^{kl}) \delta^3(x-y), \end{aligned} \quad (3.14)$$

with x, y Riemann or Fermi normal coordinates.

To compute the Poisson brackets between intrinsic Riemann tensors, we use Riemann normal coordinates x^a around a point p_0 . Given an intrinsic path $y^\alpha(s)$ the corresponding curve in Riemann normal coordinates is $\gamma^a(y, R)$. We are interested in computing the Poisson bracket between the components of Riemann tensors in p_0 and p .

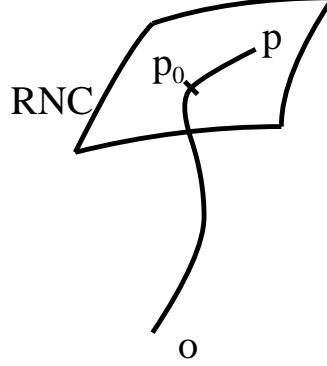


FIG. 2: The Riemann normal coordinates used in the computations of the Poisson brackets.

So we have an intrinsic path and we know the Riemann tensor at p_0 . The metric there is flat and the connection vanishes. In a neighborhood of that point we work in Riemann normal coordinates, and we wish to relate the Riemann tensor in such coordinates at p with the intrinsic Riemann tensor there.

We have the curves $y^\alpha = su^\alpha$ with u^α a unitary vector at p_0 and s the geodesic distance. In Riemann normal coordinates they have the same expression, $x^a = su^a$ (we are using Latin letters for the Riemann coordinates and Greek letters for the intrinsic ones). The metric at p_0 is flat and the tetrads are $e_\alpha^a = \delta_\alpha^a$, and $e_a^\alpha = \delta_a^\alpha$. The tetrads at p are given by the parallel transport (see [10]),

$$\begin{aligned} e_a^{(\alpha)}(p) &= \delta_a^\alpha + \frac{1}{s} \int_0^{s_p} d\sigma \int_0^\sigma d\sigma' \sigma' e_b^{(\alpha)} R^b{}_{cda} u^c u^d \\ &= \delta_a^{(\alpha)} + \frac{1}{s} \int_0^{s_p} d\sigma \int_0^\sigma d\sigma' \sigma' u^c u^d + O(R^2) = O(s_p^4), \end{aligned} \quad (3.15)$$

and

$$R_{\alpha\beta\gamma\delta}(\pi_o^{p_0} \circ \pi_{p_0}^p) = e_\alpha^a e_\beta^b e_\gamma^c e_\delta^d R_{abcd}(x = us_p), \quad (3.16)$$

where the tetrads are evaluated at $(\pi_o^{p_0} \circ \pi_{p_0}^p)$. Defining the linearized tetrad at that point, $e_\alpha^a = \delta_\alpha^a + \delta e_\alpha^a$, we wish to compute the Poisson bracket between the Riemann tensor at p and p_0 in intrinsic coordinates,

$$\{R_{\mu\nu\lambda\rho}(\pi_o^{p_0}), R_{\alpha\beta\gamma\delta}(\pi_o^{p_0} \circ \pi_{p_0}^p)\}. \quad (3.17)$$

Notice that we are restricting ourselves to paths going to p and p_0 that are continuations of each other. One could have reached p and p_0 by different paths. Such a calculation can be inferred from the present one by adding additional loops to the paths. However, all the relevant information for the general computation is present in the one shown in equation (3.17) so we will concentrate on it.

Given that at p_0 the intrinsic and Riemann normal coordinates coincide, we have,

$$R_{\mu\nu\lambda\rho}(\pi_o^{p_0}) = \delta_\mu^m \delta_\nu^n \delta_\lambda^l \delta_\rho^r R_{mnlr}(p_0). \quad (3.18)$$

Also, given that and the expansion of the tetrads, we get,

$$R_{\alpha\beta\gamma\delta}(\pi_o^{p_0} \circ \pi_{p_0}^p) = \delta_\alpha^a \delta_\beta^b \delta_\gamma^c \delta_\delta^d R_{abcd}(p) + e_\alpha^a \delta_\beta^b \delta_\gamma^c \delta_\delta^d R_{abcd}(p) + \dots \quad (3.19)$$

where the dots mean the repetition of the same construction for the other indices

With this we can reduce the computation of the Poisson bracket of the Riemann tensor in intrinsic coordinates to that in Riemann normal coordinates using the Poisson brackets we already presented. We will discuss this in the following section.

IV. EXPLICIT COMPUTATION

A. General Poisson bracket to be computed

We would like to compute two fundamental non-trivial Poisson brackets. The first one, which we call P_1 , involves the Riemann tensor with one zeroth index and the Riemann tensor with spatial indices. The second one, which we denote by P_2 , involves two Riemann tensors with one zeroth components. The other Poisson brackets can be readily derived from these ones using the equations of motion as they involve second time derivatives. As is usual in Poisson bracket computations, it is convenient to smear the functions, at least for one of the terms, we do so with a test function $\phi(y)$,

$$\begin{aligned} P_1 = & \left\{ R_{0IJK}(x), \int d^3y \phi(y) R_{ABCD}(y) \right. \\ & + \int d^3y \phi(y) \delta e_A^m(y) R_{mBCD}(y) + \int d^3y \phi(y) \delta e_B^m(y) R_{AmCD}(y) \\ & \left. + \int d^3y \phi(y) \delta e_C^m(y) R_{ABmD}(y) + \int d^3y \phi(y) \delta e_D^m(y) R_{ABCm}(y) \right\}, \quad (4.1) \end{aligned}$$

and

$$\begin{aligned} P_2 = & \left\{ R_{0IJK}(x), \int d^3y \phi(y) R_{0BCD}(y) \right. \\ & + \int d^3y \phi(y) \delta e_0^m(y) R_{mBCD}(y) + \int d^3y \phi(y) \delta e_B^m(y) R_{0mCD}(y) \\ & \left. + \int d^3y \phi(y) \delta e_C^m(y) R_{0BmD}(y) + \int d^3y \phi(y) \delta e_D^m(y) R_{0BCm}(y) \right\}, \quad (4.2) \end{aligned}$$

where A, B, C, \dots are spatial indices while a, b, c, \dots are spacetime indices, both in Riemann normal coordinates. These two brackets can be considered as special cases of

$$\begin{aligned}
P = & \left\{ R_{0IJK}(x), \int d^3y \phi(y) R_{aBCD}(y) \right. \\
& + \int d^3y \phi(y) \delta e_a^m(y) R_{mBCD}(y) + \int d^3y \phi(y) \delta e_B^m(y) R_{amCD}(y) \\
& \left. + \int d^3y \phi(y) \delta e_C^m(y) R_{aBmD}(y) + \int d^3y \phi(y) \delta e_D^m(y) R_{aBCm}(y) \right\}. \quad (4.3)
\end{aligned}$$

Using the notations introduced by Mandelstam,

$${}_{i \leftrightarrow j}^A f_{ij} = f_{ij} - f_{ji}, \quad (4.4)$$

$${}_{a \leftrightarrow b}^S F_{ab} = F_{ab} + F_{ba} - \eta_{ab} F_r{}^r, \quad (4.5)$$

it can be written as

$$\begin{aligned}
P = & \int d^3y \phi(y) \{R_{0IJK}(x), R_{aBCD}(y)\} \\
& - {}_{a \leftrightarrow B}^A \int d^3y \phi(y) \{R_{0IJK}(x), \delta e_B^m(y) R_{maCD}(y)\} \\
& - {}_{C \leftrightarrow D}^A \int d^3y \phi(y) \{R_{0IJK}(x), \delta e_D^m(y) R_{mCaB}(y)\}, \quad (4.6)
\end{aligned}$$

or, expanding the products,

$$\begin{aligned}
P = & \int d^3y \phi(y) \{R_{0IJK}(x), R_{aBCD}(y)\} \\
& - {}_{a \leftrightarrow B}^A \int d^3y \phi(y) \{R_{0IJK}(x), \delta e_B^m(y)\} R_{maCD}(y) \\
& - {}_{a \leftrightarrow B}^A \int d^3y \phi(y) \delta e_B^m(y) \{R_{0IJK}(x), R_{maCD}(y)\} \\
& - {}_{C \leftrightarrow D}^A \int d^3y \phi(y) \{R_{0IJK}(x), \delta e_D^m(y)\} R_{mCaB}(y) \\
& - {}_{C \leftrightarrow D}^A \int d^3y \phi(y) \delta e_D^m(y) \{R_{0IJK}(x), R_{mCaB}(y)\}. \quad (4.7)
\end{aligned}$$

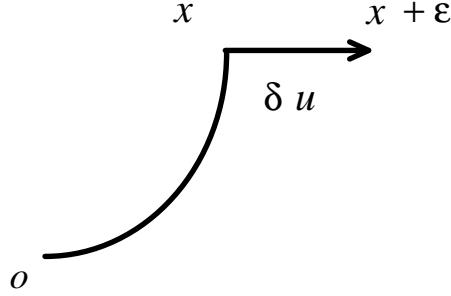


FIG. 3: The path that defines the Mandelstam derivative, $\pi_E = \pi_o^x \circ \delta u$

Hence, in principle we need to compute the following Poisson brackets,

$$\Theta_1 = \{R_{0IJK}(x), R_{ABCD}(y)\}, \quad (4.8)$$

$$\Theta_2 = \{R_{0IJK}(x), R_{0BCD}(y)\}, \quad (4.9)$$

$$\Theta_3 = \{R_{0IJK}(x), R_{0B0D}(y)\}, \quad (4.10)$$

$$\Theta_4 = \{R_{0IJK}(x), \delta e_B^M(y)\}, \quad (4.11)$$

$$\Theta_5 = \{R_{0IJK}(x), \delta e_B^0(y)\}, \quad (4.12)$$

$$\Theta_6 = \{R_{0IJK}(x), \delta e_0^0(y)\}, \quad (4.13)$$

$$\Theta_7 = \{R_{0IJK}(x), \delta e_0^M(y)\} \quad (4.14)$$

to get to the final results. Now we compute the terms (4.8) to (4.14) and then substitute them into (4.7) to find the Poisson brackets.

B. Relation between partial and Mandelstam derivatives

The expressions we have involve partial derivatives, which correspond to Riemann normal coordinates in the calculations we are interested in. We need to translate those into path dependent derivatives like the Mandelstam derivatives we discussed in our previous paper. Here we briefly recall the definition of such derivatives (for more details see for instance [11]). Given a path dependent function $\Psi(\pi_o^x)$ and a vector u^α , the Mandelstam derivative is obtained by considering its change when the path is extended from x to $x + \epsilon u$ through an infinitesimal path δu shown in figure (3),

$$\Psi(\pi_o^x \circ \delta u) = (1 + \epsilon u^\alpha D_\alpha) \Psi(\pi_o^x). \quad (4.15)$$

In our case we need to adapt this definition to the situation we wish to consider, illustrated in figure (4). Notice that although it appears similar to (3) we need to take into account that w^α is rotated since the Mandelstam derivative must be referred to the frame parallel transported along a geodesic from x to y with $u^\alpha = y^\alpha - x^\alpha$.

This yields,

$$\frac{\partial}{\partial y^a} = D_a^y - \frac{1}{6} R_{bac}{}^n (y^b - x^b) (y^c - x^c) D_n^y, \quad (4.16)$$

where by D_a^y we denote the Mandelstam derivative acting at point y .

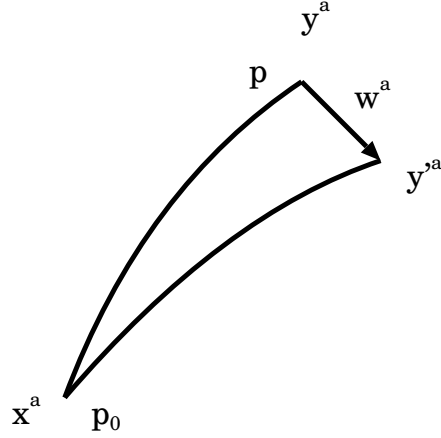


FIG. 4: The quantities involved in the Mandelstam derivative applied to our case.

Hence for the time derivatives we have

$$\frac{\partial}{\partial y^0} = D_0^y - \frac{1}{6} R_{b0c}{}^n (y^b - x^b) (y^c - x^c) D_n^y, \quad (4.17)$$

$$\frac{\partial}{\partial x^0} = D_0^x - \frac{1}{6} R_{b0c}{}^n (x^b - x^b) (x^c - x^c) D_n^x = D_0^x. \quad (4.18)$$

and we also get a relationship we will need in future computations:

$$\frac{\partial}{\partial x^J} \frac{\partial}{\partial y^c} \frac{\partial}{\partial y^a} = D_J^x D_c^y D_a^y + \frac{1}{6} R_{Jac}{}^n D_n^y. \quad (4.19)$$

C. Terms $\Theta_1 - \Theta_3$

The terms (4.8)-(4.10) become

$$\Theta_1 = \overset{A}{J \leftrightarrow KC} \overset{A}{C \leftrightarrow D} \left\{ \frac{\partial}{\partial x^J} \frac{\partial}{\partial x^I} h_{0|K}(x), \frac{\partial}{\partial y^C} \frac{\partial}{\partial y^{[B]} h_{A]D}(y) \right\}, \quad (4.20)$$

$$\Theta_2 = \overset{A}{J \leftrightarrow KC} \overset{A}{C \leftrightarrow D} \left\{ \frac{\partial}{\partial x^J} \frac{\partial}{\partial x^I} h_{0|K}(x), \frac{\partial}{\partial y^C} \frac{\partial}{\partial y^{[B]} h_{0]D}(y) \right\}, \quad (4.21)$$

$$\Theta_3 = \overset{A}{J \leftrightarrow K0' \leftrightarrow D} \left\{ \frac{\partial}{\partial x^J} \frac{\partial}{\partial x^I} h_{0|K}(x), \frac{\partial}{\partial y^{0'}} \frac{\partial}{\partial y^{[B]} h_{0]D}(y) \right\}, \quad (4.22)$$

where in the last term we distinguish the $0'$ index that is being interchanged with D from other 0 indices.

The Poisson bracket in term (4.20) can be written as

$$\Theta_1 = \left\{ \frac{\partial}{\partial x^J} \frac{\partial}{\partial x^I} h_{0|K}(x), \frac{\partial}{\partial y^C} \frac{\partial}{\partial y^{[B]} h_{A]D}(y) \right\}. \quad (4.23)$$

Using the Poisson bracket in Riemann normal coordinates (3.14) one can easily obtain,

$$\begin{aligned} \left\{ \frac{\partial}{\partial x^0} h_{IK}(x), h_{bd}(y) \right\} &= -2\kappa (\eta_{bI}\eta_{dK} + \eta_{bK}\eta_{dI} - \eta_{bd}\eta_{IK}) \delta^{(3)}(y-x) \\ &= -2\kappa \underset{I \leftrightarrow K}{S} \eta_{bK}\eta_{dI} \delta^{(3)}(y-x), \end{aligned} \quad (4.24)$$

noticing that the lapse and shift commute with the canonical variables.

With these expressions we have,

$$\Theta_1 = -\frac{\kappa}{2} \underset{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K}{A} \underset{A}{A} \underset{A}{A} \underset{S}{S} \frac{\partial}{\partial x^J} \frac{\partial}{\partial y^C} \frac{\partial}{\partial y^A} [\eta_{BI}\eta_{DK} \delta^{(3)}(y-x)]. \quad (4.25)$$

However, in terms of Mandelstam derivatives, using (4.19), we get

$$\frac{\partial}{\partial x^J} \frac{\partial}{\partial y^C} \frac{\partial}{\partial y^A} \delta^{(3)}(y-x) = D_J^x D_c^y D_a^y \delta^{(3)}(y-x) + \frac{1}{6} R_{Jac}{}^n D_n^y \delta^{(3)}(y-x). \quad (4.26)$$

Using this in (4.25) yields

$$\begin{aligned} \Theta_1 &= -\frac{\kappa}{2} \underset{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K}{A} \underset{A}{A} \underset{A}{A} \underset{S}{S} D_J^x D_c^y D_a^y [\eta_{BI}\eta_{DK} \delta^{(3)}(y-x)] \\ &\quad - \frac{\kappa}{12} \underset{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K}{A} \underset{A}{A} \underset{A}{A} \underset{S}{S} R_{JAC}{}^n D_n^y [\eta_{BI}\eta_{DK} \delta^{(3)}(y-x)]. \end{aligned} \quad (4.27)$$

It should be noted that in this expression the arguments of the Dirac delta are really the paths going from o to x and y so that the Mandelstam derivative can act on them. We keep the usual notation $\delta^{(3)}(x-y)$ as a shorthand in this and future similar expressions.

The Poisson bracket in term (4.21) can be written as

$$\Theta_2 = \left\{ \frac{\partial}{\partial x^J} \frac{\partial}{\partial x^I} h_{0|K}(x), \frac{\partial}{\partial y^C} \frac{\partial}{\partial y^B} h_{0|D}(y) \right\} \quad (4.28)$$

and one can check that it vanishes.

For the Θ_3 term, (4.22), we have

$$\Theta_3 = \underset{J \leftrightarrow K \leftrightarrow D}{A} \underset{A}{A} \left\{ \frac{\partial}{\partial x^J} \frac{\partial}{\partial x^I} h_{0|K}(x), \frac{\partial}{\partial y^{0'}} \frac{\partial}{\partial y^B} h_{0|D}(y) \right\}. \quad (4.29)$$

So we get

$$\begin{aligned} \Theta_3 &= \frac{\kappa}{2} \underset{J \leftrightarrow KI \leftrightarrow K}{A} \underset{S}{S} \eta^{CF} \frac{\partial}{\partial x^J} \frac{\partial}{\partial y^F} \frac{\partial}{\partial y^B} \eta_{CI}\eta_{DK} \delta^{(3)}(y-x) \\ &\quad - \frac{\kappa}{2} \underset{J \leftrightarrow KI \leftrightarrow K}{A} \underset{S}{S} \eta^{CF} \frac{\partial}{\partial x^J} \frac{\partial}{\partial y^F} \frac{\partial}{\partial y^C} \eta_{BI}\eta_{DK} \delta^{(3)}(y-x) \\ &\quad + \frac{\kappa}{2} \underset{J \leftrightarrow KI \leftrightarrow K}{A} \underset{S}{S} \eta^{CF} \frac{\partial}{\partial x^J} \frac{\partial}{\partial y^F} \frac{\partial}{\partial y^D} \eta_{BI}\eta_{CK} \delta^{(3)}(y-x) \\ &\quad - \frac{\kappa}{2} \underset{J \leftrightarrow KI \leftrightarrow K}{A} \underset{S}{S} \eta^{cd} \frac{\partial}{\partial x^J} \frac{\partial}{\partial y^D} \frac{\partial}{\partial y^B} \eta_{cI}\eta_{dK} \delta^{(3)}(y-x). \end{aligned} \quad (4.30)$$

Using (4.19) in above, we get, in terms of paths,

$$\begin{aligned}
\Theta_3 = & \frac{\kappa}{2} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} D_J^x D_F^y D_B^y [\eta_{CI} \eta_{DK} \delta^{(3)}(y-x)] \\
& - \frac{\kappa}{2} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} D_J^x D_F^y D_C^y [\eta_{BI} \eta_{DK} \delta^{(3)}(y-x)] \\
& + \frac{\kappa}{2} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} D_J^x D_F^y D_D^y [\eta_{BI} \eta_{CK} \delta^{(3)}(y-x)] \\
& - \frac{\kappa}{2} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{cd} D_J^x D_D^y D_B^y [\eta_{cI} \eta_{dK} \delta^{(3)}(y-x)] \\
& + \frac{\kappa}{12} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} R_{JBF}{}^n D_n^y [\eta_{CI} \eta_{DK} \delta^{(3)}(y-x)] \\
& - \frac{\kappa}{12} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} R_{JCF}{}^n D_n^y [\eta_{BI} \eta_{DK} \delta^{(3)}(y-x)] \\
& + \frac{\kappa}{12} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} R_{JDF}{}^n D_n^y [\eta_{BI} \eta_{CK} \delta^{(3)}(y-x)] \\
& - \frac{\kappa}{12} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{cd} R_{JBD}{}^n D_n^y [\eta_{cI} \eta_{dK} \delta^{(3)}(y-x)]. \tag{4.31}
\end{aligned}$$

This contains second derivatives of the metric and therefore requires the equations of motion, which we assume are in vacuum. Hence, we get,

$$\begin{aligned}
\Theta_3 = & \frac{\kappa}{2} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} D_J^x D_F^y D_B^y [\eta_{CI} \eta_{DK} \delta^{(3)}(y-x)] \\
& - \frac{\kappa}{2} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} D_J^x D_F^y D_C^y [\eta_{BI} \eta_{DK} \delta^{(3)}(y-x)] \\
& + \frac{\kappa}{2} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} D_J^x D_F^y D_D^y [\eta_{BI} \eta_{CK} \delta^{(3)}(y-x)] \\
& - \frac{\kappa}{2} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{cd} D_J^x D_D^y D_B^y [\eta_{cI} \eta_{dK} \delta^{(3)}(y-x)] \\
& + \frac{\kappa}{12} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} R_{JBF}{}^n D_n^y [\eta_{CI} \eta_{DK} \delta^{(3)}(y-x)] \\
& - \frac{\kappa}{12} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} R_{JCF}{}^n D_n^y [\eta_{BI} \eta_{DK} \delta^{(3)}(y-x)] \\
& + \frac{\kappa}{12} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{CF} R_{JDF}{}^n D_n^y [\eta_{BI} \eta_{CK} \delta^{(3)}(y-x)] \\
& - \frac{\kappa}{12} A_{J \leftrightarrow KI \leftrightarrow K} S \eta^{cd} R_{JBD}{}^n D_n^y [\eta_{cI} \eta_{dK} \delta^{(3)}(y-x)]. \tag{4.32}
\end{aligned}$$

D. Terms $\Theta_4 - \Theta_7$

In this case, we note that

$$\delta e_a{}^m = \frac{1}{s} \int_0^s d\sigma \int_0^\sigma d\sigma' \sigma' R^m{}_{cda} u^c u^d(y'), \tag{4.33}$$

where $y'^a = y^a + (\sigma' - \sigma) u^a$ and $u^a = (y^a - x^a) / \sigma$.

If we now expand $R^m{}_{cda}$ around y , with, $y'^l = y^l + (\sigma' - \sigma) u^l = y^l + \epsilon^l$, we get,

$$R^m{}_{cda}(y') = R^m{}_{cda}(y) + \partial_h R^m{}_{cda}(y) \epsilon^h + \mathcal{O}(\epsilon^2), \tag{4.34}$$

which yields δe_a^m as

$$\begin{aligned}
\delta e_a^m &\approx \frac{1}{s} \int_0^s d\sigma \int_0^\sigma d\sigma' \sigma' \left(R^m{}_{cda}(y) + \partial_h R^m{}_{cda}|_y \epsilon^h + \partial_j \partial_h R^m{}_{cda}|_y \epsilon^h \epsilon^j \right) u^c u^d \\
&= R^m{}_{cda}(y) u^c u^d \left(\frac{1}{s} \int_0^s d\sigma \int_0^\sigma d\sigma' \sigma' \right) \\
&= \frac{1}{6} R^m{}_{cda}(y) (y^c - x^c) (y^d - x^d)
\end{aligned} \tag{4.35}$$

where we neglect terms of higher order in epsilon and we have also used

$$\frac{1}{s^3} \int_0^s d\sigma \int_0^\sigma d\sigma' \sigma' = \frac{1}{s^3} \frac{1}{6} s^3 = \frac{1}{6}. \tag{4.36}$$

Hence, we can write

$$\begin{aligned}
\delta e_a^m &\approx \frac{1}{6} R^m{}_{cda}(y) (y^c - x^c) (y^d - x^d) \\
&\approx -\frac{1}{6} \eta^{mf} R_{DaCf}(y) (y^C - x^C) (y^D - x^D),
\end{aligned} \tag{4.37}$$

where we have assumed that the path going from x to y is spatial, so spatial indices need only be considered. Thus the Poisson brackets in (4.11) - (4.13) can be written as

$$\{R_{0IJK}(x), \delta e_a^m(y)\} = -\frac{1}{6} \eta^{mf} \{R_{0IJK}(x), R_{DaCf}(y)\} (y^C - x^C) (y^D - x^D). \tag{4.38}$$

Now we compute Θ_4 to Θ_6 using this expression.

The Θ_4 term corresponds to (4.38) with $m \rightarrow M$, $a \rightarrow B$ and $f \rightarrow F$, where the last one is a consequence of $\eta^{0I} = 0$ for the outside η^{mf} . Hence we have

$$\begin{aligned}
\Theta_4 &= \{R_{0IJK}(x), \delta e_B^M(y)\} \\
&= -\frac{1}{6} \eta^{MF} \{R_{0IJK}(x), R_{DBCF}(y)\} (y^C - x^C) (y^D - x^D) \\
&= -\frac{1}{6} \eta^{MF} \Theta_1 \Big|_{D \rightarrow F, A \rightarrow D} (y^C - x^C) (y^D - x^D) \\
&= \eta^{MF} \frac{\kappa}{12} \left(\begin{matrix} A & A & A & S & D_J^x D_C^y D_D^y \\ D \leftrightarrow BC \leftrightarrow FJ \leftrightarrow KI \leftrightarrow K \end{matrix} [\eta_{BI} \eta_{FK} \delta^{(3)}(y-x)] \right) (y^C - x^C) (y^D - x^D) \\
&\quad + \eta^{MF} \frac{\kappa}{12 \cdot 6} \left(\begin{matrix} A & A & A & S & R_{JDC}^n D_n^y \\ D \leftrightarrow BC \leftrightarrow FJ \leftrightarrow KI \leftrightarrow K \end{matrix} [\eta_{BI} \eta_{FK} \delta^{(3)}(y-x)] \right) (y^C - x^C) (y^D - x^D),
\end{aligned} \tag{4.39}$$

where we have used the expression (4.8). The notation in Θ_1 means substituting D for F and A for D in it. We will use a similar notation in the following terms.

The Θ_5 term corresponds to (4.38) with $m \rightarrow 0$, $a \rightarrow B$, and as a consequence $f \rightarrow 0$.

Hence

$$\begin{aligned}
\Theta_5 &= \{R_{0IJK}(x), \delta e_B^0(y)\} \\
&= -\frac{1}{6}\eta^{00} \{R_{0IJK}(x), R_{D0C0}(y)\} (y^C - x^C) (y^D - x^D) \\
&= \frac{1}{6} \{R_{0IJK}(x), R_{0CBD}(y)\} (y^C - x^C) (y^D - x^D) \\
&= \frac{1}{6} \Theta_2 \Big|_{B \leftrightarrow C} (y^C - x^C) (y^D - x^D) \\
&= 0
\end{aligned} \tag{4.40}$$

The Θ_6 term corresponds to (4.38) with $m \rightarrow 0$, $a \rightarrow 0$, and as a consequence $f \rightarrow 0$. Hence using 4.37 we get

$$\begin{aligned}
\Theta_6 &= \{R_{0IJK}(x), \delta e_0^0(y)\} \\
&= \left\{ R_{0IJK}(x), \frac{1}{6} R_{D0C0}(y) (y^C - x^C) (y^D - x^D) \right\} \\
&= \frac{1}{6} (y^C - x^C) (y^D - x^D) \Theta_3 \Big|_{B \rightarrow C}.
\end{aligned} \tag{4.41}$$

The Θ_7 term corresponds to (4.38) with $m \rightarrow M$, $a \rightarrow 0$, and as a consequence and $f \rightarrow F$. Hence using 4.37 we get

$$\begin{aligned}
\Theta_7 &= \{R_{0IJK}(x), \delta e_0^M(y)\} \\
&= -\frac{1}{6}\eta^{MF} (y^C - x^C) (y^D - x^D) \{R_{0IJK}(x), R_{D0CF}(x)\} \\
&= \frac{1}{6}\eta^{MF} (y^C - x^C) (y^D - x^D) \Theta_2 \Big|_{D \rightarrow F, B \rightarrow D} \\
&= 0
\end{aligned}$$

V. FULL POISSON BRACKETS

Now that we have all the necessary terms, we put them together to get the first Poisson bracket (4.1). Using (4.7), and some of symmetries of A and S operators and the symmetries

of the Riemann tensor, we get for (4.1),

$$\begin{aligned}
P_1 = P|_{a=A} &= \int d^3y \phi(y) \Theta_1 \\
&\quad - \int_{A \leftrightarrow B} d^3y \phi(y) \Theta_4 R_{MACD}(y) \\
&\quad + \int_{A \leftrightarrow B} d^3y \phi(y) \delta e_B^M(y) \Theta_1 \Big|_{B \rightarrow M} \\
&\quad - \int_{C \leftrightarrow D} d^3y \phi(y) \Theta_4 \Big|_{B \rightarrow D} R_{MCAB}(y) \\
&\quad + \int_{C \leftrightarrow D} d^3y \phi(y) \delta e_D^M(y) \Theta_1 \Big|_{D \rightarrow M}. \tag{5.1}
\end{aligned}$$

Recalling (3.19), we obtain the first Poisson bracket

$$\begin{aligned}
P_1 &= \{R_{0IJK}(\pi_o^x), R_{ABCD}(\pi_o^x \circ \pi_x^y)\} \\
&= -\frac{\kappa}{2} \int_{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A A S D_J^x D_C^x D_A^x [\delta^3(x-y)] \eta_{BI} \eta_{DK} \\
&\quad + \kappa \int_{A \leftrightarrow BJ \leftrightarrow KI \leftrightarrow K} A A S D_J^x [\delta^3(x-y) R^F_{ADC}(\pi_o^x)] \eta_{BI} \eta_{FK} \\
&\quad + \frac{\kappa}{4} \int_{A \leftrightarrow BJ \leftrightarrow KI \leftrightarrow K} A A S D_J^x [\delta^3(x-y) R^F_{CDA}(\pi_o^x)] \eta_{BI} \eta_{FK} \\
&\quad - \frac{\kappa}{4} \int_{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A S D_J^x [\delta^3(x-y) R^F_{CBA}(\pi_o^x)] \eta_{FI} \eta_{DK} \\
&\quad + \kappa \int_{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A S D_J^x [\delta^3(x-y) R^F_{CBA}(\pi_o^x)] \eta_{FI} \eta_{DK} \\
&\quad - \frac{\kappa}{4} \int_{A \leftrightarrow BJ \leftrightarrow KI \leftrightarrow K} A A S D_F^x [\delta^3(x-y) R^F_{ADC}(\pi_o^x)] \eta_{BI} \eta_{JK} \\
&\quad - \frac{\kappa}{4} \int_{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A S D_F^x [\delta^3(x-y) R^F_{CBA}(\pi_o^x)] \eta_{DI} \eta_{JK} \\
&\quad + \frac{\kappa}{12} \int_{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A A S D_F^x [\delta^3(x-y) R^F_{CAJ}(\pi_o^x)] \eta_{BI} \eta_{DK} \\
&\quad + \frac{\kappa}{12} \int_{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A A S D_F^x [\delta^3(x-y) (R^F_{JBC}(\pi_o^x) + R^F_{CBJ}(\pi_o^x))] \eta_{AI} \eta_{DK} \\
&\quad - \frac{\kappa}{12} \int_{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A A S D_F^x [\delta^3(x-y) (R^F_{JDA}(\pi_o^x) + R^F_{ADJ}(\pi_o^x))] \eta_{BI} \eta_{CK} \\
&\quad - \frac{\kappa}{12} \int_{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A A S D_A^x [\delta^3(x-y) (R^F_{JBC}(\pi_o^x) + R^F_{CBJ}(\pi_o^x))] \eta_{FI} \eta_{DK} \\
&\quad + \frac{\kappa}{4} \int_{A \leftrightarrow BJ \leftrightarrow KI \leftrightarrow K} A A S D_A^x [\delta^3(x-y) R^F_{JDC}(\pi_o^x)] \eta_{BI} \eta_{FK} \\
&\quad - \frac{\kappa}{4} \int_{A \leftrightarrow BJ \leftrightarrow KI \leftrightarrow K} A A S D_B^x [\delta^3(x-y) R^F_{ADC}(\pi_o^x)] \eta_{JI} \eta_{FK} \\
&\quad + \frac{\kappa}{12} \int_{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A A S D_C^x [\delta^3(x-y) (R^F_{JDA}(\pi_o^x) + R^F_{ADJ}(\pi_o^x))] \eta_{BI} \eta_{FK} \\
&\quad - \frac{\kappa}{4} \int_{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A S D_C^x [\delta^3(x-y) R^F_{JBA}(\pi_o^x)] \eta_{FI} \eta_{DK} \\
&\quad - \frac{\kappa}{4} \int_{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A A S D_D^x [\delta^3(x-y) R_{ABC}^F(\pi_o^x)] \eta_{JI} \eta_{FK}, \tag{5.2}
\end{aligned}$$

where we did not include contributions due to the Ricci tensor as we are assuming we are in vacuum. One can check that the above expression for the Poisson bracket is compatible with the vacuum equations by contracting it with η^{IJ} and seeing that it vanishes. This is most easily seen considering each of the terms in the expansion of (4.3) and showing that their contraction with η^{IJ} vanishes. Therefore,

$$\{R_{0K}(\pi_o^x), R_{ABCD}(\pi_o^x \circ \pi_x^y)\} = 0. \quad (5.3)$$

Introducing the symmetrizer notation for pairs of indices,

$$T_{X \leftrightarrow Y} f_{XY} = f_{XY} + f_{YX}, \quad (5.4)$$

the expression can be made more compact,

$$\begin{aligned} P_1 = & -\frac{\kappa}{2} A_{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A_{S} D_J^x D_C^x D_A^x [\delta^3(x-y)] \eta_{BI} \eta_{DK} \\ & + \frac{\kappa}{12} A_{A \leftrightarrow BC \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A_{S} D_F^x [\delta^3(x-y) R_{JAC}^F(\pi_o^x)] \eta_{BI} \eta_{DK} \\ & + T_{A \leftrightarrow C; B \leftrightarrow D} \left\{ A_{A \leftrightarrow BJ \leftrightarrow K} A_{I \leftrightarrow K} \left[\kappa S D_J^x [\delta^3(x-y) R_{ADC}^F(\pi_o^x)] \right] \eta_{BI} \eta_{FK} \right. \\ & - \frac{\kappa}{4} S_{I \leftrightarrow KF \leftrightarrow B} T D_F^x [\delta^3(x-y) R_{ADC}^F(\pi_o^x)] \eta_{BI} \eta_{JK} \\ & + \frac{\kappa}{12} A_{C \leftrightarrow DI \leftrightarrow KJ \leftrightarrow CA \leftrightarrow C} S T D_J^x [\delta^3(x-y) R_{ABC}^F(\pi_o^x)] \eta_{DI} \eta_{FK} \\ & \left. + \frac{\kappa}{12} A_{C \leftrightarrow DI \leftrightarrow KJ \leftrightarrow CA \leftrightarrow F} S T A D_A^x [\delta^3(x-y) R_{JBC}^F(\pi_o^x)] \eta_{DI} \eta_{FK} \right\}. \quad (5.5) \end{aligned}$$

We can now proceed to compute the second Poisson bracket. We are interested in computing,

$$P_2 = \{R_{0IJK}(\pi_o^x), R_{0BCD}(\pi_o^x \circ \pi_x^y)\}. \quad (5.6)$$

we have,

$$\begin{aligned} P_2 = & P|_{a=0} \\ = & \int d^3y \phi(y) R_{0BCD}(y) \Theta_6 \\ & - \int d^3y \phi(y) R_{M0CD}(y) \Theta_4 \\ & + \int d^3y \phi(y) \delta e_0^M(y) \Theta_1 \Big|_{A \rightarrow M} \\ & - \frac{A}{C \leftrightarrow D} \int d^3y \phi(y) R_{MC0B}(y) \Theta_4 \Big|_{B \rightarrow D} \\ & - \frac{A}{C \leftrightarrow D} \int d^3y \phi(y) \delta e_D^0(y) \Theta_3 \Big|_{D \rightarrow C}. \quad (5.7) \end{aligned}$$

Carrying out all the above computations we get

$$\begin{aligned}
P_2 = & -\kappa \frac{A}{J \leftrightarrow KI \leftrightarrow K} S D_J^x [\delta^3(x-y) R^F{}_{0CD}(\pi_o^x)] \eta_{BI} \eta_{FK} \\
& + \frac{\kappa}{4} \frac{A}{J \leftrightarrow KI \leftrightarrow K} S D_F^x [\delta^3(x-y) R^F{}_{0CD}(\pi_o^x)] \eta_{BI} \eta_{JK} \\
& + \frac{\kappa}{4} \frac{A}{J \leftrightarrow KI \leftrightarrow K} S D_B^x [\delta^3(x-y) R^F{}_{0CD}(\pi_o^x)] \eta_{JI} \eta_{FK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_M^x [\delta^3(x-y) (R^M{}_{JC0}(\pi_o^x) + R^M{}_{CJ0}(\pi_o^x))] \eta_{BI} \eta_{DK} \\
& + \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_J^x [\delta^3(x-y) (R^M{}_{CB0}(\pi_o^x) + R^M{}_{BC0}(\pi_o^x))] \eta_{MI} \eta_{DK} \\
& + \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_B^x [\delta^3(x-y) (R^M{}_{JC0}(\pi_o^x) + R^M{}_{CJ0}(\pi_o^x))] \eta_{MI} \eta_{DK} \\
& + \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_C^x [\delta^3(x-y) (R^M{}_{BJ0}(\pi_o^x) + R^M{}_{JB0}(\pi_o^x))] \eta_{MI} \eta_{DK} \\
& + \kappa \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_J^x [\delta^3(x-y) R^F{}_{CB0}(\pi_o^x)] \eta_{DI} \eta_{FK} \\
& - \frac{\kappa}{4} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_F^x [\delta^3(x-y) R^F{}_{CB0}(\pi_o^x)] \eta_{DI} \eta_{JK} \\
& - \frac{\kappa}{4} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_D^x [\delta^3(x-y) R^F{}_{CB0}(\pi_o^x)] \eta_{JI} \eta_{FK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_J^x [\delta^3(x-y) (R^M{}_{0BD}(\pi_o^x) + R^M{}_{DB0}(\pi_o^x))] \eta_{MI} \eta_{CK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_J^x [\delta^3(x-y) (R^M{}_{0CD}(\pi_o^x) + R^M{}_{DC0}(\pi_o^x))] \eta_{BI} \eta_{MK} \\
& + \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_J^x [\delta^3(x-y) (R_{BDC0}(\pi_o^x) + R_{CDB0}(\pi_o^x))] \eta_{IK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_F^x [\delta^3(x-y) (R_{JDB0}(\pi_o^x) + R_{BDJ0}(\pi_o^x))] \eta^{MF} \eta_{MI} \eta_{CK} \\
& + \frac{\kappa}{6} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_F^x [\delta^3(x-y) (R^F{}_{DJ0}(\pi_o^x) + R^F{}_{0JD}(\pi_o^x))] \eta_{BI} \eta_{CK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_F^x [\delta^3(x-y) (R_{JDC0}(\pi_o^x) + R_{CDJ0}(\pi_o^x))] \eta^{MF} \eta_{BI} \eta_{MK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_B^x [\delta^3(x-y) (R^M{}_{DJ0}(\pi_o^x) + R^M{}_{0JD}(\pi_o^x))] \eta_{MI} \eta_{CK} \\
& + \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_B^x [\delta^3(x-y) (R_{CDJ0}(\pi_o^x) + R_{JDC0}(\pi_o^x))] \eta_{IK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_C^x [\delta^3(x-y) (R^M{}_{DJ0}(\pi_o^x) + R^M{}_{0JD}(\pi_o^x))] \eta_{BI} \eta_{MK} \\
& + \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} A S D_C^x [\delta^3(x-y) (R_{JDB0}(\pi_o^x) + R_{BDJ0}(\pi_o^x))] \eta_{IK}. \tag{5.8}
\end{aligned}$$

and as before there is a consistency relation with the equations of motion,

$$\eta^{AB} \{R_{0ABC}(\pi_o^x), R_{0IJK}(\pi_o^x \circ \pi_x^y)\} = 0. \tag{5.9}$$

And the expression can also be made more compact through the symmetrizer, as before,

$$\begin{aligned}
P_2 = & -\kappa \frac{A}{J \leftrightarrow KI \leftrightarrow K} \frac{S}{D} D_J^x [\phi(x) R^F_{0CD}(x)] \eta_{BI} \eta_{FK} \\
& -\kappa \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} \frac{A}{S} D_J^x [\phi(x) R^F_{CB0}(x)] \eta_{DI} \eta_{FK} \\
& + \frac{\kappa}{4} \frac{A}{J \leftrightarrow KI \leftrightarrow KF \leftrightarrow B} \frac{S}{T} D_F^x [\phi(x) R^F_{0CD}(x)] \eta_{BI} \eta_{JK} \\
& + \frac{\kappa}{4} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow KF \leftrightarrow D} \frac{A}{S} D_F^x [\phi(x) R^F_{CB0}(x)] \eta_{IJ} \eta_{DK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow KJ \leftrightarrow CF \leftrightarrow B} \frac{A}{S} D_F^x [\phi(x) R^F_{JC0}(x)] \eta_{BI} \eta_{DK} \\
& + \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow KB \leftrightarrow CJ \leftrightarrow C} \frac{A}{S} D_J^x [\phi(x) R^F_{BC0}(x)] \eta_{FI} \eta_{DK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow KI \leftrightarrow CI \leftrightarrow BJ \leftrightarrow B} \frac{A}{S} D_I^x [\phi(x) R_{0BDJ}(x)] \eta_{CK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow KI \leftrightarrow CI \leftrightarrow B} \frac{A}{S} D_J^x [\phi(x) R_{0IDB}(x)] \eta_{CK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow KK \leftrightarrow JC \leftrightarrow J} \frac{A}{S} D_K^x [\phi(x) R_{0CDJ}(x)] \eta_{IB} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow KK \leftrightarrow J} \frac{A}{S} D_C^x [\phi(x) R_{0KDJ}(x)] \eta_{BI} \\
& + \frac{\kappa}{6} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K0 \leftrightarrow D} \frac{A}{S} D_F^x [\phi(x) R^F_{0JD}(x)] \eta_{BI} \eta_{CK} \\
& - \frac{\kappa}{12} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow KJ \leftrightarrow C} \frac{A}{S} D_J^x [\phi(x) R_{C0}(x)] \eta_{BI} \eta_{DK} \\
& + \frac{\kappa}{6} \frac{A}{C \leftrightarrow DJ \leftrightarrow KI \leftrightarrow K} \frac{A}{S} D_J^x [\phi(x) R_{D0}(x)] \eta_{BI} \eta_{CK}. \tag{5.10}
\end{aligned}$$

The remaining Poisson brackets are pretty straightforward to compute. Some of them vanish,

$$\{R_{0A0C}(\pi_o^x), R_{0I0K}(\pi_o^x \circ \pi_x^y)\} = 0, \tag{5.11}$$

$$\{R_{0A0C}(\pi_o^x), R_{IJKL}(\pi_o^x \circ \pi_x^y)\} = 0, \tag{5.12}$$

and the others can be written in terms of $P_{1,2}$ using the equations of motion and are non-vanishing,

$$\{R_{0A0C}(\pi_o^x), R_{0IJK}(\pi_o^x \circ \pi_x^y)\} = \{R_{BABC}(\pi_o^x), R_{0IJK}(\pi_o^x \circ \pi_x^y)\}. \tag{5.13}$$

It is possible to extend the algebra to arbitrary paths using the deformation techniques here developed. The analysis must be extended to paths in any spatial or time-like direction,

$$\{R_{abcd}(\pi_o^x), R_{ijkl}(\eta_o^y)\} = \Delta_{abcd,ijkl}(\pi_o^x, \eta_o^y, [R]), \tag{5.14}$$

Where Δ is a path-dependent distribution that takes non vanishing values when π and η end on the same point for the geometry given by the intrinsic Riemann tensor R . The problem of dynamics, that up to now was unsolvable in loop quantum gravity arises in this context as the computation of the algebra for arbitrary paths and its operatorial implementation.

VI. CONCLUSIONS

As Donnelly and Giddings [5] put it, and results from (5.14), “The physical observables in a gravitational theory therefore do not organize themselves into local commuting sub-algebras” [as occurs in usual field theories]: “the principle of locality must apparently be reformulated or abandoned, and in fact we lack a clear definition of the coarser and more basic notion of a quantum subsystem of the Universe.”

Locality expresses the idea that quantum processes can be localized in space and time [and, at the level of observable quantities, that causally separated processes are exempt from any uncertainty relations restricting their co-measurability.

The quantum implementation of the Poisson algebra of intrinsic Riemann tensors presented in this paper could provide an approximate notion of quantum subsystem and allow to determine uncertainty relations restricting the co-measurability of physical observables.

The quantization of the general framework laid out in this paper is clearly a tall order. However, it can be the starting point for the analysis of simplified situations, like minisuperspaces. We plan on pursuing this in the future in order to identify the fundamental elements of a quantum version of this framework.

VII. ACKNOWLEDGMENTS

This work was supported in part by Grant No. NSF-PHY-1603630, NSF-PHY-1903799, funds of the Hearne Institute for Theoretical Physics, CCT-LSU, and Pedeciba and Fondo Clemente Estable FCE_1_2019_1_155865.

-
- [1] S. Mandelstam, *Ann. Phys. (NY)* **19**, 1 (1962); *Ann. Phys. (NY)* **19**, 25 (1962).
 - [2] Y. M. Makeenko and A. A. Migdal, *Phys. Lett.* **88B**, 135 (1979) Erratum: [*Phys. Lett.* **89B**, 437 (1980)].
 - [3] A. Polyakov, *Nucl.Phys B*164, 171 (1979).
 - [4] R. Gambini, A. Trias, *Phys. Rev.* **D23**, 553 (1981).
 - [5] W. Donnelly and S. B. Giddings, *Phys. Rev. D* **93**, no. 2, 024030 (2016) Erratum: [*Phys. Rev. D* **94**, no. 2, 029903 (2016)] doi:10.1103/PhysRevD.94.029903, 10.1103/PhysRevD.93.024030 [arXiv:1507.07921 [hep-th]]; *Phys. Rev. D* **94**, no. 10, 104038 (2016) doi:10.1103/PhysRevD.94.104038 [arXiv:1607.01025 [hep-th]]; arXiv:1706.03104 [hep-th].
 - [6] R. Britto, F. Cachazo, B. Feng and E. Witten, *Phys. Rev. Lett.* **94**, 181602 (2005) doi:10.1103/PhysRevLett.94.181602 [hep-th/0501052]; N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *JHEP* **1003**, 110 (2010) doi:10.1007/JHEP03(2010)110 [arXiv:0903.2110 [hep-th]]; N. Arkani-Hamed and J. Trnka, *JHEP* **1410**, 030 (2014) doi:10.1007/JHEP10(2014)030 [arXiv:1312.2007 [hep-th]]; M. Bullimore and D. Skinner, arXiv:1101.1329 [hep-th]; S. Caron-Huot, *JHEP* **1107**, 058 (2011) doi:10.1007/JHEP07(2011)058 [arXiv:1010.1167 [hep-th]].
 - [7] R. Gambini and J. Pullin, *Class. Quant. Grav.* **35**, no. 21, 215008 (2018) doi:10.1088/1361-6382/aae449 [arXiv:1802.02661 [gr-qc]].

- [8] J. Lewandowski, A. Okolow, H. Sahlmann and T. Thiemann, *Commun. Math. Phys.* **267**, 703 (2006) doi:10.1007/s00220-006-0100-7 [gr-qc/0504147]; C. Fleischhack, *Phys. Rev. Lett.* **97**, 061302 (2006).
- [9] R. Haag, *Local quantum physics: Fields, particles, algebras.* (Texts and monographs in physics). Springer, Berlin, Germany, (1992).
- [10] A. Nesterov, *Class. Quan. Grav.* **16**, 465 (1999).
- [11] R. Gambini, J. Pullin, “Loops, knots, gauge theories and quantum gravity”, Cambridge University Press (1996); R. Gambini, A. Trias, *Phys. Rev.* **D27**, 2935 (1983); R. Gambini “Teorías de Calibre en el espacio de ciclos”, unpublished notes, Universidad Simón Bolívar (1986); A. Trias, “El grupo de ciclos en teorías de calibre” unpublished notes.