Estimation of Nonlinear Frontier Functions and Efficiency Measurements for Airframe Production Programs.

Banani Dhar

Louisiana State University and Agricultural & Mechanical College

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ESTIMATION OF NONLINEAR FRONTIER FUNCTIONS AND EFFICIENCY MEASUREMENTS FOR AIRFRAME PRODUCTION PROGRAMS

The Louisiana State University and Agricultural and Mechanical Col. PH.D. 1986

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Submitted to the Graduate Faculty of the
Louisiana State University and Agricultural
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of the Requirements for the Degree
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in
Interdepartmental Program in Business Administration
(Quantitative Business Analysis)

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ABSTRACT

This research examines measures of economic efficiency in aircraft production. In particular, a type of nonlinear frontier estimation is contrasted with more traditional methods for estimating a dynamic cost function. This cost function is grounded in economic theory, and it is consistent with our knowledge of the aircraft production process. The model includes the effects of both learning and production rate on total program costs.

Our model differs from more traditional cost specifications in that we make no attempt to measure production rate. Our model links direct labor requirements to fixed delivery schedules, under the assumption that the firm attempts to optimize production rate over time. The implication is that the optimal direct labor time path is purely a function of time. This makes our model particularly convenient for application since production rate on aircraft programs is extremely difficult to measure.

The usefulness of our model is demonstrated by analyzing the central hypothesis of the thesis. It is shown through various sensitivity analyses that an alternative procurement policy could have resulted in a
lower total program cost to the government for an airframe production program. It is also explained how this model fits within a comprehensive decision support system that is being designed for the Department of Defense.
CHAPTER I
INTRODUCTION

The impact of production rate on cost has been studied by economists for many years. The relationship between cost and output rate has been discussed extensively in the traditional neoclassical economic theory literature. On the other hand, the engineering literature contains many studies of progress functions. Economic theory considers output rate an important factor in determining program costs, but most engineering studies consider cumulative output to be the most important factor. Early cost studies in economic theory and the engineering literature were somewhat contradictory in nature. The idea of combining learning effects and production rate started developing in the early 1950's. Alchian (1959) provided a theoretical development to the problem of linking the economic and engineering approaches. Preston and Keachie (1964), and Oi (1967) also considered the interaction of learning and production rate. All of these were heuristic approaches to the problem, and the results obtained were very general in nature. In addition, they were data free. Rosen (1972) made an attempt to solve the problem directly, but the functional form of his model was not specified for empirical estimation. Washburn (1972) and Womer (1979) obtained cost relations in which both the learning and rate effects were considered. Both models are consistent with
existing theory, and Womer's model is suitable for empirical estimation. Brueckner and Raymon (1983) provided a model for a firm where learning by doing was included as an important factor. Womer and Gulledge (1983) considered a modified model which includes learning, production rate and delivery schedules. This model permits a production program to be modelled as a series of tasks connected by experience. This research is an extension of the Womer and Gulledge model, where constrained optimization of the cost function is considered.

1. Research Problem

Due to cost overruns and a continuing need for better planning estimates, it is necessary to develop new techniques and to modify old techniques to obtain better cost estimates. Along with these techniques, a better understanding of the factors and forces that determine costs is required. The sensitivity of program costs to alternative policy decisions must be accurately estimated, to meet the challenge of providing a better policy.

The problem of interest is to reprice aircraft under hypothetical changes in procurement quantities. This is a Department of Defense (DoD) planning problem that has received much study in the Office of the Assistant Secretary of Defense [OASD(PA&E)] and by many private researchers. The problem is described in the following section.
2. The Aircraft Repricing Problem

Each year DoD planners must decide how many units of a particular weapon system to procure. This is a very complicated and highly politicized decision, and therefore many diverse attributes are considered when making this decision. OASD(PA&E) has the responsibility of providing an estimate of the unit price of each system that is being considered for procurement. The responsibility covers many weapon systems, but this dissertation is only concerned with a single weapon system, aircraft.

Presently the pricing of aircraft (and other items) is being accomplished with a manual system. The work is tedious and time consuming since it is often necessary to provide the unit prices associated with many hypothetical procurement quantities. A level of accuracy is required in these projections, but the level of detail is not very refined. The projections are highly aggregated, and they are never intended to be used for aggregate production planning at the contractor level.

The Institute for Defense Analyses is presently constructing a decision support system to aid DoD planners in repricing aircraft. The description of a prototype version is presented in a paper by Balut, et al. (1986). The model "mimics" the manual approach that is presently being used by DoD planners. In short, the variable cost of
the procured quantities is modelled with a learning curve, and the annual fixed cost is spread according to the number of units produced during the year. The sum of the unit variable and unit fixed costs may be used to obtain an estimate of price.

Strictly speaking, the prototype model is not theoretically correct. It has been demonstrated repeatedly [Gulledge and Womer (1986)] that the learning curve is not the best way to model variable costs. The learning curve does not take into consideration that production rate is changing over time. The constructors of the prototype decision support system were aware of this problem when the model was constructed. However, under the pressure to produce implementable results, this problem was considered to be a refinement that could be included at a later date. This refinement is nontrivial, and it is relevant to the objectives of this research.

The solution to the aircraft repricing problem has eluded researchers for years [Chapter 7 in Gulledge and Womer (1986)]. The solution of the problem is complicated by many factors. For example:

1. limited access to contractor accounting records;
2. aircraft that are procured in a given year are not produced in the same year;
3. production rate is very difficult to measure; and
4. the different components of price do not follow the same learning curve; e.g., indirect costs do not follow the same learning curve as direct costs.

One method that is currently being used [Gardner (1985)] to reprice aircraft requires estimating equations of the form:

$$P_t = \beta_0 x_1^{\beta_1} x_2^{\beta_2}$$  (1.1)

where

- $P_t$ = unit flyaway cost (price),
- $x_1$ = a proxy for production rate,
- $x_2$ = cumulative proxy,
- $\beta_0$, $\beta_1$, $\beta_2$ = parameters.

This model considers only the first complicating factor mentioned above; all others are ignored. Models of this type provide erratic predictions.

The approach advocated by Balut, Gulledge and Womer (1986) is an alternative to the above approach. The approach is contrasted with the traditional cost accounting approach in Fig. 1. If contractor cost accounting data were available, detailed estimates of direct costs would be obtained, and then the indirect overhead would be allocated in proportion to the direct cost. Since these records are not available, the Balut; et al., approach separates the fixed and variable cost statistically, models the variable cost with a mathematical model, then distributes the fixed overhead in proportion to the variable cost.
ESTIMATING APPROACHES

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Figure 1. Alternative Estimating Approaches. Source: S.J. Balut.
This dissertation is concerned with only one segment of this modelling procedure: the mathematical modelling of variable cost. Much research is needed in establishing variable cost profiles. The current aircraft repricing model uses the learning curve. Since this is theoretically incorrect [Gulledge and Womer (1986)], it seems necessary to investigate more appropriate models. While the previous cost models describe the allocation of resources over the time horizon of the program, they have considered the estimation of "average" cost functions. This research uses results from the frontier estimation literature to properly define the cost function. Nonlinear programming is used to derive a cost function for a particular airframe program. This cost function expresses the minimum cost attainable from the given input combination by controlling the disturbance term to be of one sign only. The frontier function is discussed in the next section.

3. The Frontier Function

In neoclassical microeconomic theory a transformation function provides a description of production technology. It describes the maximum amount of output that can be produced for a given level of input usage. The inverse of this transformation function, the resource requirement function, describes minimum amount of input required for the production of a given output with given amounts of all
other inputs. The cost function describes the minimum cost of producing output with a given production technology and input prices. The characteristic that is common to all of the above definitions is the concept of optimality. Each function specifies a maximum or minimum value that can be achieved under the constraints imposed by technology and prices. Each function describes a boundary; that is a frontier.

In many studies, cost functions, production functions, or resource requirement functions are estimated from historical data. Quite often ordinary least squares is used to estimate the parameters in these functions. Strictly speaking, this approach to parameter estimation is not appropriate for estimating the above functions. That is, least squares fits an "average" function with some historical points falling above the estimated surface and some falling below. This is a violation of the definition of a frontier function. For example, the proper estimation of a cost function requires that all of the data points fall above or on the cost function.

Since all observed points fall above or on the least cost frontier, the distance a production unit operates above the frontier is a measure of inefficiency. There are many methods and approaches for estimating this distance. In general, the interpretation of this distance depends on the particular application. For example, the distances may
be used to estimate the inefficiency of firms in an industry if the investigator is estimating an industry function. In short, the definition of frontier is equally applicable at both the micro and macro level.

In this research a cost frontier for a single production process is estimated. The estimation is unique in several respects. The function is a priori specified, and the resulting cost function is highly nonlinear. This leads to a nonlinear frontier estimation problem. The data for the problem is by production lot by quarter, therefore, the estimation requires pooling time series and cross-sectional data.

After estimation, the frontier cost function is used to provide a measure of firm efficiency. The function is used to test whether or not greater efficiency could have been achieved if the firm could have benefited from a more even production delivery schedule. This research is not concerned with comparing efficiency across firms. All efficiency measurements are for the single firm, and they are relative to the historical realized delivery schedule.

The point of departure for this research is a paper by Womer and Gulledge (1983). In this paper a cost function is derived for "batches" of production units. The parameters in this function are estimated by ordinary least squares. This research revises the model so that it applies to individual production units, and the parameters
are estimated in such a way that the estimated relationships conform to the proper definition of a cost function.¹

After estimation and diagnostic checking, the model is used to explore the previously mentioned efficiency analysis. Since the data for this study are taken from a military program, this analysis is of interest to Department of Defense analysts. A current issue in acquisitions research is the appropriateness of multiyear procurement contracts. The hypothesis is that multiyear contracts should lead to reduced procurement costs since defense contractors can construct more realistic production plans. Multiyear procurement should lead to more balanced delivery schedules, and hence greater production efficiency. This hypothesis is explored with the sensitivity analysis in this research.

4. **Scope and Methodology**

This research like many previous cost models deals with a production function where both learning and production rate are modelled to influence program cost. The major purpose is to develop an approach for estimating a minimum cost frontier.

¹The author would like to thank John F. Muth for suggesting this approach.
The work of this dissertation is divided into the following tasks:

1. presentation of a theoretical model of aircraft production and its solution to obtain a cost function;
2. estimation of the frontier cost function;
3. measurement of efficiency of the production program relative to the realized delivery schedule; and
4. sensitivity analysis, where the efficiency of alternative delivery schedules is considered.

The remainder of this dissertation is organized in the following fashion. The relevant literature is reviewed in Chapter II. In Chapter III, the theoretical model is solved to obtain the optimal time path of resource use and then to obtain an estimable cost function. This nonlinear cost function is first estimated from the C-141 program data, without controlling the error terms. The estimated parameters are then compared with estimates obtained by Womer and Gulledge (1983) to see how much the estimates vary when the "average" cost function is estimated for the same aircraft program using a different model. A description of the C-141 data is included in Chapter III. After this presentation, the cost function is estimated by forcing the observed values to lie on or above the minimum cost frontier.

The efficiency of this production program is measured in Chapter IV. This efficiency is then compared with
alternative delivery schedule efficiencies. Sensitivity analysis includes the effect of increasing or reducing delivery schedules at any point during the program. Of particular interest are those schedules that could be associated with a more even multiyear procurement program. In particular the following question is explored. Will the cost to the government be reduced if a more even procurement policy is pursued?
CHAPTER II
LITERATURE REVIEW

The literature review is divided into two sections. The first section contains articles where both learning and production rate are considered as cost determinants. The second section contains articles on frontier estimation. These articles are presented in detail since they lay the foundation for this research.

1. Alchian, A. A. (1959)

In this paper, the author suggests some propositions which are empirically valid and are designed to eliminate the ambiguities and errors in the relationship between costs and outputs.

Alchian defines cost as the change in equity, which is measured as reduction in wealth. The characteristics of output are defined by the expression

$$V = \sum_{T}^{T+m} X(t) \, dt,$$  \hspace{1cm} (2.1)

where $V$ is the planned volume of output, $X(t)$ is the output rate at time $t$, $T$ is the time of completion of the first unit output and $m$ is the length of interval over which the output is made available. The author considers changes in only one of the variables $V$, $X(t)$ or $T$ at a time, assuming the other two constant. The propositions relate to changes
in cost due to changes in above variables.

The first proposition states that the faster the rate of production, the higher is the cost of production. This is reasonable because for higher production rates, either overtime or more labor force is necessary, which implies an increase in cost.

Proposition two states that marginal cost increases with increase in output rate. Costs are higher because more resources are used.

The third proposition states that cost increases with the total planned volume of output. If production proceeds at a constant rate, the production time horizon is longer, and therefore cost increases.

Proposition four describes one form of the learning effect. According to this proposition marginal cost decreases at a uniform rate with increase in planned total output. Techniques used to produce a larger planned volume may be different from those used to produce a smaller volume; this will lower cost.

Since marginal cost decreases with larger planned volume, the average cost per unit of total volume also decreases. This is expressed in proposition five. The author uses graphical and numerical illustrations to provide an intuitive and heuristic interpretation for the above propositions.
Propositions two and four express two opposite ideas. According to proposition two, the cost per unit is larger for higher output rates. Proposition four, on the other hand, shows that cost per unit is lower with increases in planned volume. These propositions are not useful for examining simultaneous changes in planned volume and output rate. If the effect of an increase in total planned output dominates the effect of a higher output rate, then it may be possible to obtain higher production rates at a lower unit cost.

Proposition seven states that cost decreases if $T$, the time between decision to produce and the delivery of output is increased. This makes sense because a longer time horizon for the same planned volume means a lower output rate, which decreases cost.

Proposition eight deals with short and long-run effects on cost. The short-run is usually defined in terms of some fixed and some varied inputs, whereas in the long-run, all inputs are variable, with varying costs. The choice of input variation depends only on the economic cost; technological changes are assumed constant. The distinction between the short and long-run is important in explaining the paths of price or output over time as demand changes.

Proposition nine states that, with increases in cumulative output, future cost declines. Knowledge
increases with increases in production, so future cost is lowered due to increases in efficiency with knowledge. The lower cost is not only for a larger volume, but also for any future volume. This is another form of the learning effect. This proposition is the most relevant for the research presented in this dissertation.

The last part of the paper deals with cost curves where capital value concepts are used instead of the time rate of change definition of cost. Alchian considers two families of curves, one for different values of V, and the other for different time profiles of X(t). However, the assumptions underlying these cost curves remain unexplained. With the introduction of planned volume as a variable, simple price and rate of output relationships are no longer sufficient. In order to determine cost per unit, it now becomes necessary to consider both the planned volume and output rate effects.

In this paper, Alchian made an attempt to explain the theoretical integration of two apparently incongruent conceptions, the increase in unit cost with higher output rate and the learning effect that shows a decline in unit cost as accumulated total output increases. He showed that cost depends on both production rate and planned production volume. The learning effect is explained in both propositions four and nine. In proposition four, the learning effect is due to the production of a larger volume
and the use of various techniques. In proposition nine, the learning effect is due to the experience gained as output is produced and knowledge is acquired. Both of these effects are usually incorporated in the learning curve discussion.

The simultaneous effect of a change in output rate and a change in planned volume was discussed by Alchian, but the cost impact of such a change was not ascertained.


This paper presents an algebraic and graphic integration of cost functions and learning curves. The first part of the paper deals with a model that describes cost-output relations, and the second part is an application for testing these relations.

The cost function in economic theory is a static curve showing different levels of cost and output. The learning curve shows the relationship between the level of unit production costs and the accumulated level of output over time.

The integration of the static cost function and the dynamic progress function is attempted by considering three variables, total (C) or average (Cₜ) production costs; the level of output per production period, qₜ; and the accumulated level of total output, V. The authors show graphically that the cost surface slopes downward as V increases, when the learning effect is present.
The cost-output relation described here is suitable for a series of short-run periods. The production process proceeds through one short-run period after another and thereby achieves an accumulated level of output. The firm acquires experience and knowledge which in turn reduces unit production costs.

The authors illustrate the hypothesized cost-output relation using an example from the production and assembly of five different pieces of radar equipment. The results of the regression analysis indicated that unit cost declines with accumulation of output over time due to the learning phenomenon.

This paper stresses the importance of the learning effect in determining production costs, but the paper does not say anything about how the costs will be affected when changes in output rate and cumulative output are considered simultaneously. Still, this is included in the literature review since it represents one of the first attempts to empirically integrate the learning curve with the economic cost function.


According to Oi, the progress function is a dynamic concept which is a consequence of long-run production planning. Dynamic production theory explains the neoclassical concept of a progress function. The
productivity gain with a larger volume of output over time is usually considered to be a result of learning and technical progress, but Oi offers an alternative hypothesis. The lowering of cost is due to the presence of an intertemporal production plan and not the improvement of technology or learning and experience with accumulated production. His hypothesis is that a dynamic theory of production could generate propositions similar to those provided by Alchian (1959).

The stability of the progress function depends on the continuity of production. Changes in product design or any form of disruption may actually have an opposite effect. For different input factors, there can be different progress functions, and fluctuations in factor prices may cause instability in the progress function. The volume effect is hypothesized to be different from output effects of classical economic theory where average cost declines as a result of an increased flow of output per unit time.

The model considers the minimization of the capital value cost of a specified output program. Two important results are expressed as theorems. Theorem one states that, the cost of producing any given flow of output can be reduced by considering later delivery. The production function has three variables: output flow, capital and labor. If the firm is characterized by intertemporal factor substitution, then the cost of the output flow with
a later delivery date is less than the cost for an earlier date. However, this result does not usually hold if the firm does not practice intertemporal factor substitution. With a later delivery date, a firm can adopt a method of production which enables it to lower cost. The availability of extra time may help the firm remove restrictions imposed on the production process, which in turn helps in cost reduction.

The second theorem states that, when output is produced in several consecutive periods, the cost is lower than the combined cost of an unrelated output program with the same output flow. This is the learning effect when the output flows are technically related. Oi then deduces the propositions of Alchian's cost model using these two theorems. The inter-temporal production function and the learning curve exhibit similar volume effects. Both are specific to the firm and the product line within the firm.

The limitation of this model is that it attempts to link the traditional neoclassical approach of economics and progress functions in a heuristic fashion, but does not provide any direct method for cost estimation that involves production rate and learning. The results obtained here are very general in nature, and hence it is not possible to apply them to real data for estimation of cost functions.

A model of a firm is considered where learning affects the production process. Knowledge is considered as an input variable, and learning and marketable goods are considered as output variables. Knowledge is gained through actual production.

A model is developed for the optimal rate of learning when the present value of the firm is maximized and the production effects on learning are considered. Rosen provides two alternative formulations. The first connects the rate of learning to total output, and the second connects the rate of learning to market inputs. The functional form of the model for the first formulation provides an estimable relation,

\[
X_t = F(L_T, Z_0 + \beta \sum_{j=0}^{t-1} X_j) \tag{2.2}
\]

where \( X_j \) = the output in time period \( j \),

\( j = 0, 1, 2, \ldots, t \)

\( L_T \) = market input,

\( Z_0 \) = initial knowledge, which is exogeneous to the problem, and

\( \beta \) = a constant.

This relation shows that knowledge increases with accumulated output.
The functional equation for maximizing present value is given by

\[ V_N(Z_0) = \max_{L_0} \{ (pX_0 - wL_0) + V_{N-1}(Z_1)/(1+r) \} \quad (2.3) \]

where \( V_N \) and \( V_{N-1} \) are present values for profit over horizons of lengths \( N \) and \( (N-1) \) respectively; \( p \) and \( w \) are market prices associated with output, \( X \), and input, \( L \), respectively; and \( r \) is the discount rate. This is a dynamic programming problem where the optimal decisions in future periods depend on the optimal decisions of the past.

A solution for the one-period horizon problem is initially established, then this solution is used to find a solution for the two-period horizon. The same procedure is then extended for the \( N \)-period horizon, where \( N \) is finite.

In a second formulation of the same problem, learning is considered proportional to experience related to inputs and not outputs. The functional form of the model in this case is

\[ X_t = F(\gamma T, Z_0 + \frac{1}{\gamma} \sum_{j=0}^{t-1} L_j) \]

(2.4)

where \( \gamma \) is a constant. Rosen showed that the marginal cost of learning in any period is equal to the future marginal product of knowledge. The solution to this problem in terms of flow is

\[ \gamma q(t) - \gamma q(t+1) = p F_Z(t+1) - \gamma rq(t) \]

(2.5)
where $\gamma q(t)$ is the marginal cost of learning in any period $t$.

A supply and demand diagram relating $q$ and $Z$ shows that a stable choice of initial $q(0)$ is necessary for the above equilibrium condition [equation (2.5)] to hold. For values greater than $q(0)$, the system will not converge, and for values less than $q(0)$, the system converges rapidly with the final accumulated value of knowledge less than optimal.

Finally, Rosen generalizes the model for an infinite time horizon. Modifications of $\beta$ or $\gamma$ may impose an upper bound on accumulated knowledge, and thus a stationary solution can be obtained. In this case, the progress function can not be the ordinary sum of previous outputs or inputs, the weighted sum must be used to define the progress function.

Rosen shows that the accumulation of knowledge with production over time is an important factor in determining the cost of production and should not be ignored. The cost reduction is not due to technological change or increasing returns to scale in purchased inputs, but due to learning. Every firm goes through an early phase of development when profit is actually negative. This is because more resources are devoted to learning at this stage. In the later stages of the firm's life, learning through experience advances productivity.
This model is firm specific and does not explain what happens if the learning achieved through experience is transferred to competitive firms. The model is applicable when external effects are unimportant. If productivity gains depend partially on exogeneous factors and not solely on learning by doing, then this model cannot be applied.

This paper attempts to provide a solution using a direct problem solving approach. The market structure is specified and a straightforward recursive relation is obtained. However, the functional form of this model does not facilitate empirical estimation. The paper is basically a theoretical exercise.

5. Washburn, A. R. (1972)

Washburn discusses a model and characteristics of optimal production schedules when production involves learning. He also presents an application for a production program for large jet airplanes.

The concept of a production line is used as a frame of reference. The model considers the maximization of profits, which is formulated as a calculus of variations problem. The model is stated as follows:

\[
CV = \max_{0}^{T} \int F[N(t), \dot{N}(t)] e^{-\alpha t} \, dt \quad (2.6)
\]
subject to: \( N(0) = 0, \)
\( N(t) = \) a fixed quantity,
\( \dot{N}(t) > 0, \)
\( \alpha = \) discount rate

where \( N(t) \) represents the total production up to time \( t \).

The function \( F \) is modelled in such a way that it reflects the fact that production efficiency ultimately declines with production rate. Otherwise, an unacceptable conclusion, such as efficiency is independent of the rate of production, may result.

Within this model two conflicting phenomena exist simultaneously. One is the learning phenomenon, a unit produced earlier in the program costs more than a later unit. The other phenomenon states that a given unit costs more to produce when production rate is higher than the same unit when production rate is low.

For increasing production rate, the author considers two alternatives: use of overtime and hiring additional manpower. The first alternative means an increase of the time utilization of the plant for increasing production rate. For the second alternative, Washburn defines a standard man hour (SMH), which is the amount of work accomplished by one man in one hour when he is a member of a standard crew. Two results are obtained -- (i) efficiency remains constant as long as the crew size is less than the standard, but decreases with increases in
crew size, and (ii) efficiency is the same when the ratios of crew to standard crew are equal — the proportionality rule. Usually, the alternative for which the increase in the rate of spending on labor is smallest is used. With the proportionality rule, the cheapest cost per SMH depends only on the ratio of the rate of production of SMH's to the standard crew size. Then,

$$\text{Cost}/\text{SMH} = W h(\frac{\dot{X}}{C}), \quad (2.7)$$

where \( h(0) = 1 \), \( W \) = basic wage, \( C \) = standard crew, and \( \dot{X} \) = rate of production of a SMH. Washburn uses the functional form \( h(Y) = 1 + ay^b \), where the parameters \( a \) and \( b \) are greater than zero. This form is consistent with the model assumption that each unit is made on an assembly line. Washburn defines the argument of \( h(\cdot) \), taking the assembly line as a whole, as muscle factor \( \dot{Y} \).

If \( p \) = the sale price minus material cost, and \( \lambda(t) \) = overhead costs that are independent of production, then the total rate of each inflow at time \( t \) is

$$F(N, \dot{N}) - \lambda(t) \quad (2.8)$$

where

$$F(N, \dot{N}) = \dot{N} \{ p - WHg(N) h[\frac{\dot{N} Hg(N)}{C}] \} \quad (2.9)$$

Washburn uses the standard logarithmic form of improvement curve

$$g(N) = (N_o + N)^{-B} \quad (2.10)$$
where $B$ is the learning constant and $N_o$ is the effective experience at the start of the manufacturing program.

The Euler equation is

$$\frac{d}{dt} \left[ \frac{\partial}{\partial N} F(N, \dot{N}) e^{-at} \right] = \frac{\partial}{\partial N} F(N, \dot{N}) e^{-at} . \quad (2.11)$$

Note that the Euler equation does not contain $\lambda(t)$. The solution to Euler's equation shows optimal $\dot{N}(t)$ to be a continuous function, a result that is important for the model to be useful. There are no internal gaps in production, and Euler's equation is solved to obtain the muscle factor, $\dot{Y}$. If profits are not discounted, then the optimal strategy is to produce in such a way that SMH's are produced at constant cost.

Washburn describes three market situations where this model can be applied. The first is the simplest market situation, where $n$ units are to be constructed in a fixed time $T$, so there are two additional constraints, $N(0) = 0$ and $N(T) = n$. The solution to this boundary value problem gives the optimal production rate. The second constraint is related to a time limited market. In this case the learning effect can overpower production efficiency, and an infinite production rate will lead to infinite profits. Since infinite profits are impossible, the optimal production rate $\dot{N}$ is either zero or the associated muscle factor $\dot{Y}$ increases to a local maximum. The solution
requires a one-dimensional search for finding the actual optimal production schedule.

Finally, Washburn describes an application of this model to a jet airplane production program. Initial profits are assumed to be zero, which makes learning essential for making profit. Washburn shows that, because of the importance of rapid production, the optimal strategy has an initial muscle factor of 1.4 which involves a large labor penalty. When all costs of designing and tooling, etc., are considered, this program with high production rate appears to be reasonable.

This model describes some properties of optimal production schedules. It shows that the optimal production schedule is a continuous function with no internal gaps. This model is also useful in determining optimal production rate. The model has only a few input parameters and hence it is not possible to infer anything about the details of an optimal operation from the results obtained. It is a "gross" planning model.

The improvement curve $g(N)$ does not depend explicitly on time; learning is achieved through actual production. The cost relation obtained here is quite suitable for estimation. This model can be extended to include other important factors such as labor wage levels and crew size, etc., which are associated with the production process.

Womer describes a model of the firm engaged in production to order. The production function and the theory of learning by doing are combined to obtain a cost function. The author shows that it is possible to obtain a specific order size with a continuous production time path while minimizing total discounted production cost.

Production rate varies throughout the time period considered for the program. A class of inputs whose use rate varies throughout is related to output, and the relative change in prices within the class of inputs is assumed constant. This class is thus represented as a single composite resource. Any other resource, which does not vary during the program, does not belong to this class. The former class accounts for variable cost, while later for fixed costs.

The output rate, \( q(t) \), is related to the variable use rate, \( x(t) \), and cumulative production, \( Q(t) \), by the relation

\[
q(t) = A Q^\delta(t) x^{1/\gamma(t)} \tag{2.12}
\]

where \( A \) is a constant, \( \gamma \) a scale parameter and \( \delta \) is the learning parameter. This functional form considers the joint effects of learning from resources and experience and the effect of changing production rate. The assumption is \( \gamma > 1 \), which implies decreasing returns to the variable
factor and that production rate has no absolute maximum. If the above relation is solved for \( x(t) \), when \( q(t) \) is assumed constant, the resulting relation reduces to a learning curve.

Discounted program cost is given by

\[
C = \int_{0}^{T} x(t) e^{-\rho t} dt \tag{2.13}
\]

where \( \rho \) is the discount rate. The objective is to minimize the cost of producing \( V \) units of output by time \( T \).

This problem is solved by considering the transformation \( Z(t) = q(t)/Q^\delta(t) \). \( Z(t) \) is the new state variable, and \( z(t) = dZ/dt \) is the new control variable, the time rate of change of the state variable. This is a calculus of variations problem, so Euler's equation is solved to obtain the optimal solution. Total discounted production cost is

\[
C(V,T) = \frac{A^{-\gamma} V^{(1-\delta)\gamma}}{(1-\delta)\gamma} \left(\frac{\rho}{\gamma-1}\right)^{1-\gamma} \left[\exp\left(\frac{\rho T}{\gamma-1}\right) - 1\right]^{1-\gamma} \tag{2.14}
\]

which is only a function of \( V \) and \( T \).

The effect of volume on cost is due to both the parameters \( \gamma \) and \( \delta \). Since cost rises at an increasing, constant or decreasing rate with volume when \( \gamma \) is greater than, equal to or less than \( (1 - \delta)^{-1} \), the model does not require that unit cost decline with volume. If \( \gamma > 1 \), the
model requires that the discounted cost must decline with the time horizon. The above relation describes a planning situation. If the time path is chosen optimally, then this gives the production cost for any program.

Womer presents an example where he examines three production programs with different time horizons. In all the three cases, discounted cost increases at an increasing rate throughout the program, but the rate of increase is less, as the time horizon increases.

If the cost function is integrated up to \( t \), an equation is obtained that is applicable to the production period. The cost function for the production situation is stated as

\[
C(t/V, t) = \left( \frac{\rho}{\gamma - 1} \right)^{\gamma - 1} \frac{A^{-\gamma}}{(1-\delta)^{\gamma}} V^{(1-\delta)(\gamma-1)}
\]

\[
[Q(t)]^{1-\delta} \left[ \frac{\rho T}{e^{\gamma - 1}} - 1 \right]^{1-\gamma}.
\]

(2.15)

This relation shows that the impact of volume, \( V \), on total discounted cost depends on both the parameters \( \delta \) and \( \gamma \). Learning and returns to scale both influence volume, whereas cumulative production is influenced only by learning through the parameter \( \delta \). In practice, the relative change in resource prices may be negligible, in which case the relation between cumulative discounted cost and cumulative output remains valid.

Usually learning curves are estimated as a function of cumulative output and are used to calculate cost on new
programs without taking into consideration the factor returns parameter, \( y \), and hence cost estimates obtained from the learning curve can be misleading. The learning curve implicitly assumes a constant production rate, therefore scale effects are ignored.

In the example considered by Womer, the curve which corresponds to the planning situation is different for different values of \( y \). By changing \( y \), discounted cumulative average cost also changes. Thus, the returns to scale parameter does affect production cost, and it should be introduced in learning curve estimation. The learning curve model does help in knowing the effect of learning at any particular time if the contractor holds his production rate constant.


This is an extension of work done earlier in the area of military airframe program cost estimation. The model considers the effects of learning, production rate and facility size on total program cost. This model can be used to explain the production and cost behavior of a particular airframe project. The relationship between total program cost and both endogeneous and exogeneous changes of production rate during the production period is described. Like Washburn (1972), the production line is considered as a frame of reference.
The production function includes a learning component, and the discounted cost of production is then minimized to obtain the optimal path of resource use. The variables used in the analysis are:

- \( i \) = an index for a batch of airframes in the same lot \((j)\) all of which are to be delivered at time \( t_{ij} \),
- \( n_j \) = the number of batches in lot \( j \),
- \( m \) = the total number of lots in the production program,
- \( D_{ij} \) = the number of airframes in batch \( i \) of lot \( j \),
- \( E_{ij} \) = a measure of experience prior to the midpoint of batch \( i \),
- \( V \) = the number of airframes in the production process in the facility at time \( t \),
- \( t_j \) = date work begins for all the batches of lot \( j \),
- \( t_{ij} \) = date work ends for batch \( i \) of lot \( j \),
- \( q_{ij}(t) \) = production rate at time \( t \) on batch \( i \) of lot \( j \),
- \( Q_{ij}(t) \) = cumulative production on batch \( i \) of lot \( j \) at time \( t \),
- \( x_{ij}(t) \) = rate of resource use at time \( t \) on batch \( i \) of lot \( j \),
- \( \delta \) = a parameter describing learning prior to batch \( i \),
- \( \epsilon \) = a parameter describing learning on batch \( i \),
- \( \gamma \) = a parameter describing returns to the variable resources,
- \( \alpha \) = a parameter associated with decreases in labor productivity as a batch of airplanes nears completion,
- \( \nu \) = a parameter describing returns to the length of the production line, and
- \( \eta \) = a parameter describing returns to the size of the batch.
The production function has the form,

$$q_{ij}(t) = A V^\nu D_{ij}^\eta Q_{ij}^\varepsilon(t) (t_{ij} - t)^\alpha x_{ij}^{1/\gamma(t)} E_{ij}^\delta$$  \hspace{1cm} (2.16)

where $E_{ij}^\delta$ describes learning by doing. The terms $Q_{ij}^\varepsilon(t)$ and $(t_{ij} - t)^\alpha$ show that the nature of work changes along the production line. The effect of speed of the production line is expressed by the terms $D_{ij}^\eta$ and $x_{ij}^{1/\gamma(t)}$. The variable $i$ is an index for a batch of airframes in the same lot $j$, all of which are to be delivered at time $t_{ij}$. Note that $\gamma > 1$ ensures diminishing returns to the variable resources. The term $V^\nu$ describes the effect due to alternative numbers of airframes being in the facility over time. Total cost is minimized

$$\text{Min } C = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \int_{t_j}^{t_{ij}} x_{ij}(t) e^{-\rho t} dt.$$  \hspace{1cm} (2.17)

In this function cost is measured in units of the variable resource and $\rho$ is the discount rate. The subproblems are additive and cost is monotone nondecreasing, so the solution is obtained by minimizing each subproblem. The solution is obtained by solving Euler's equation of the calculus of variations to obtain an optimum time path of resource use over an interval of time for an individual batch of airframes. Resource use rises in the beginning at an increasing rate up to a point of inflection, then at a decreasing rate to a maximum, and then declines. The
decline is due to decreases in the marginal productivity of labor as the delivery date approaches. This crowding effect requires that the rate of labor use fall to provide an optimal production rate. The time path clearly shows that the labor requirements on a program vary over time. The time dimension of the production program is thus explained by this model.

Next, the authors show that the model is sensitive in capturing the effect of changes in the delivery schedule and changes in lot release dates. The net effect of late delivery is a slight rise in program costs. Compression of the delivery schedule at the end of the program also results in a slight increase in cost.

This model provides a functional form where parameters are estimable. The model is sensitive to exogeneous delivery schedule effects. This model does not include hiring and firing costs, so the work force predicted by the model may not be appropriate. Changes in the delivery schedule may occur when these costs are included in the cost function.

The number of airframes (V) in the plant in a given time period is assumed to remain the same, which may not be true. Revision of the contract may require changing V, so this model can be extended to include V as a decision variable.
Potentially, this model can be used to obtain a cost profile for a new program by adjusting the scaling factor. Application of this model to other products can also be considered.

8. Aigner, D. J. and Chu, S. F. (1968)

This paper presents an estimation technique for a deterministic production process. An industry production function is a frontier of potential attainment for given input combinations.

In micro-economic theory, a firm's production function expresses the maximum product obtainable by the firm from a given combination of input factors during the period of time required to produce output. The maximum output applies to all other firms in the same industry, thus the function so defined is called industry production function. Due to pure random shocks in the production process or due to differences in technical or economic efficiency, many constituent firm outputs lie below the frontier. So, the firms actual output may either be greater or smaller than what the industry production function permits. Aigner and Chu conclude that frontier estimation is more appropriate than the 'average' concept for ascertaining the maximum production capacity of an Industry.
The second section describes how programming methodology is used to obtain the required production surface. Aigner and Chu tried to obtain an estimated function,

\[ A \hat{x}_1^\alpha \hat{x}_2^\beta = \hat{x}_o \]  

(2.18)

such that

\[ A \hat{x}_1^\alpha \hat{x}_2^\beta > x_o . \]  

(2.19)

where \( x_o \) is output rate, \( x_1 \) and \( x_2 \) are inputs, and \( A, \alpha, \) and \( \beta \) are parameters. The estimated function was obtained by minimizing sum of the squared residuals subject to the given constraint. In some cases, it is possible to estimate the parameters within the framework of linear programming.

This paper points out the distinction between average functions and frontier functions as predictors of capacity. For the average function, approximately fifty percent of firm outputs for a selected combination of inputs lie above the predicted output. The frontier function, under a fixed technology gives the output, which only a few firms at the most can produce for any given combination of inputs. Thus the frontier function is a relatively better predictor.

An approach to the estimation of frontier production functions is discussed. The error term is considered to be made up of two components, one normal and the other from a one-sided distribution.

The model is given by,

\[ Y_i = f(x_i, \beta) + \epsilon_i, \quad i = 1, 2, \ldots, N \quad (2.20) \]

where \( \epsilon_i < 0 \) and \( \epsilon_i \) has the structure

\[ \epsilon_i = u_i + v_i, \quad i = 1, 2, \ldots, N \quad (2.21) \]

In this model, \( Y_i \) is the output obtainable from input \( x_i \) and \( \beta \) is a vector of parameters. The error component \( v_i \) represents symmetric disturbances. The \( \{v_i\} \) are assumed independently and identically distributed as \( N(0, \sigma_v^2) \).

The error component \( u_i \) is assumed to be distributed independently of \( v_i \) and satisfies \( u_i < 0 \). The \( \{u_i\} \) are from a \( N(0, \sigma_u^2) \) distribution, truncated above zero. With this error structure the frontier \( Y_i < f(x_i, \beta) + v_i \) itself becomes stochastic. The non-positive disturbance, \( u_i \), ensures that each firm's output must lie on or below its frontier.

The distribution function of the sum of a symmetric normal random variable and a truncated normal random
variable is considered for estimation. The density function of $\varepsilon$ is

$$f(\varepsilon) = \frac{2}{\sigma} f^*(\frac{\varepsilon}{\sigma}) \left[1 - F^*(\varepsilon, \lambda, \sigma^{-1})\right], \quad -\infty < \varepsilon < \infty \tag{2.22}$$

where $\sigma^2 = \sigma_u^2 + \sigma_v^2$ and $\lambda = \sigma_u / \sigma_v$. The functions $f^*(\cdot)$ and $F^*(\cdot)$ are the standard normal density and distribution functions respectively. '\(\lambda\)' is interpreted as an indicator of the relative variability of the two sources of random error that distinguishes firms from one another. Maximum likelihood estimation is applied to the relevant log-linear function for obtaining estimates of the parameters and the asymptotic standard error of the maximum likelihood estimates.

This paper describes a linear model with an error specification which seems appropriate for the estimation of an industry production function using cross-section data. The method is not applicable to nonlinear production functions and nonlinear frontier estimation constraints. Additional research is required to obtain estimated values of parameters in nonlinear estimation and measurement of the performance of production program.
CHAPTER III
THE NONLINEAR FRONTIER COST FUNCTION

In this chapter, a frontier cost function is estimated using the C-141 program. This estimation procedure will eventually be integrated into the decision support system for repricing aircraft (Balut, et al., 1986). This chapter is organized as follows. First the theoretical model is presented. A discussion of the data is presented, and the empirical estimation procedure is described.

1. The Model

Earlier, Womer and Gulledge (1983) developed a model which was used to analyze the C-141 airframe program. The sensitivity analyses on the model suggested some model revisions to accommodate the realities of data availability and to meet the requirements of program management. Taking these considerations into account, the Womer and Gulledge model is modified to focus on the production of a single airframe, instead of modelling the production of a batch of airframes. This model, like the previous one, augments a homogeneous production function with a learning hypothesis. The discounted cost of production is minimized subject to the production function constraint. The solution to the dynamic optimization model yields the optimal time path of direct labor requirements.
The production function in this revised model is of the following form:

\[ q_i(t) = A(i - 1/2)^\delta Q_i^\epsilon(t) (t_{di} - t)^\alpha x^{1/\gamma(t)} V^\nu \]  

(3.1)

where \( A \) is a constant. The input \( x \) is assumed to be a composite of many inputs whose use rate is variable throughout the production period. Since factor prices are assumed to be constant over the relevant time period, cost is measured in the units of variable composite resource.

The variables in the above function are:

- \( i = \) the sequence number of airframe \((i = 1, 2, \ldots, n)\),
- \( V = \) the average number of airframes in process in the facility,
- \( t_{si} = \) the date work begins on airframe \( i \); work on all airframes in the same lot is assumed to start on the lot release date,
- \( t_{di} = \) the delivery date for airframe \( i \),
- \( q_i(t) = \) the production rate at time \( t \) on airframe \( i \),
- \( Q_i(t) = \) the cumulative work performed on airframe \( i \) at time \( t \), i.e.,
  \[
  Q_i(t) = \int_{t_{si}}^{t} q(\tau) \, d\tau ,
  \]
- \( x_i(t) = \) the rate of resource use at time \( t \) on airframe \( i \),
- \( \delta = \) a parameter describing learning prior to airframe \( i \),
- \( \epsilon = \) a parameter describing learning on airframe \( i \),
- \( \gamma = \) a parameter describing returns to the variable composite resource,
\[ a = \] a parameter associated with decreases in labor productivity as an airframe nears completion, and \\
\[ v = \] a parameter describing returns to the length of the production line.

This production function conforms to economic production theory, and it also accommodates the fact that the nature of work along the production line changes from position to position.

To motivate an understanding of the production function specification presented in equation (3.1), the concept of production cost drivers is introduced. In the cost analysis literature, cost drivers are those factors that are the major determinants of cost. This analysis integrates this concept with the neoclassical production function.

The first production cost driver is the concept of learning by doing. The basic idea is that as the cumulative number of airframes produced increases, the unit costs (or at least labor hours) decreases. Following the lead of Washburn (1972), the concept of a production line is adopted as a frame of reference. Learning by doing affects cost by affecting efficiency at each position on the production line. That is, as the number of airframes passing each position on the line increases, yielding more experience, the efficiency at the position increases, thus lowering the labor cost. The term \((i - 1/2)\delta\) describes
this learning by doing.

The second production cost driver is a different learning effect. Over time, learning how to produce more efficiently may take place due to events other than experience at a position on the production line. For example, early in a production program labor hours may be spent to learn how to produce more efficiently. Later in the program this may result in increased efficiency independent of experience at a point on a line. If this is the case, then positions late in the production line may benefit from the experience of earlier positions. Thus work at the later positions proceeds more efficiently than work at early positions on the same airframe. The terms $Q_i(t)$ and $(t_{di} - t)^\alpha$ describe learning that occurs over time during the process of producing airframe $i$.

Furthermore, it is assumed that as the delivery date is approached, it is more difficult to substitute labor for time in the production process.

A third production cost driver is the speed of the production line. If the speed of the line is increased, more labor will be required at each position on the line. Furthermore, due to diminishing returns, the additional labor required is expected to be more than in proportion to the increase in speed. The term $x_i^{1/\gamma(t)}$ captures the effect of the speed of the production line.
The fourth cost driver is the length of the production line. One way to increase delivery rate is to increase the number of positions on the production line, reducing the amount of work to be done at each position, and increasing the total amount of work accomplished per unit of time. An increase in the length of the line may result in crowded facilities and overuse of tools and other fixed resources. This adversely affects the efficiency of production and may result in increased unit costs. This last effect involves an interaction among the airframes that are in the facility at the same point in time. The term $V^v$ is intended to capture the effect of working on alternative numbers of airframes in the same facility.

The learning parameters, $\delta$ and $\varepsilon$, are both expected to be between 0 and 1. However, the effect of learning while producing an airframe and the effect of the learning prior to production cannot be separated with our data, so the $\varepsilon$ cannot be estimated. Due to diminishing returns to the variable composite resource, $\gamma$ is expected to be greater than 1. Since the efficiency of production decreases with increase in number of airframes in the same facility, $v$ is expected to be negative.

The objective of most firms is to minimize discounted cost. The problem is

$$\text{Minimize } C = \sum_{i=1}^{n} \int_{t_{di}}^{t_{si}} x_i(t) e^{-\rho t} dt$$

(3.2)
subject to:

\[ q_i(t) = A(i - 1/2)^6 Q_i^e(t) (t_{di} - t)^\alpha x_i^{1/\gamma(t)} V^v, \]

\[ Q_i(t_{di}) = 1, Q_i(t_{si}) = 0 \]

where \( \rho \) = the discount rate, and
\( C = \) discounted variable program cost measured in labor hour units.

Since total cost is monotonic nondecreasing and the subproblems are additive, the solution can be obtained by minimizing each of the subproblems. The representative problem for the \( i^{th} \) airframe may then be stated as

Minimize \( C_i = \int_{t_{si}}^{t_{di}} x_i(t) e^{-\rho t} dt \) \hspace{1cm} (3.3)

subject to:

\[ q_i(t) = A(i - 1/2)^6 Q_i^e(t) (t_{di} - t)^\alpha x_i^{1/\gamma(t)} V^v, \]

\[ Q_i(t_{di}) = 1, Q_i(t_{si}) = 0 \]

\(^1\)The objective of the firm is in fact dependent upon the wording of the contract between the contracting parties. It is assumed in this research that the contract is written in such a way as to induce the desired behavior. Possible contract forms are fixed price or award fee. This issue and others related to contractor motivation are discussed by Boger, et al. (1982).
where \( C_i \) = discounted variable cost of airframe \( i \).

This is an optimal control problem which may be solved directly by minimizing the Hamiltonian function. However, the problem can be transformed into the problem of Lagrange, which can be solved using classical variational techniques. At this point the subscript \( i \) is dropped. The solution to the model is as follows. Since this problem is nonlinear, a direct analytical solution to the problem may be difficult to obtain. The strategy of this solution is to absorb the constraint into the objective functional and work with the unconstrained problem. An analytical solution to this problem is found by transforming the problem so that the Euler equation is a function of only the time derivative of the state variable. This permits an easy solution to the differential equation.

To begin the solution, equation (3.1) is solved for the variable composite resource. The resulting resource requirement function is

\[
x(t) = q^\gamma(t) A^{-\gamma}(i - 1/2)^{-\gamma \delta} Q^{-\gamma \epsilon}(t) (t_{di} - t)^{-\gamma \alpha} V^{-\gamma \nu}.
\]

(3.4)

A transformation is desired that yields one state variable and one control variable, the control variable being the time rate of change of the state variable. Let

\[
Z(t) = V^{-\nu} A^{-1}(i - 1/2)^{-\delta} Q^{1-\epsilon}(t)/(1 - \epsilon).
\]

(3.5)
This implies that

\[ z(t) = \frac{dZ}{dt} = V^{-\nu} A^{-1} (i - 1/2)^{-\delta} Q^{-\epsilon}(t) q(t) . \]  

(3.6)

For the transformed problem, \( Z(t) \) is the new state variable; and its time derivative, \( z(t) \), is the control variable. Formation of the new objective functional requires absorbing the constraint. From equation (3.6),

\[ z^Y(t) = A^{-\nu} (i - 1/2)^{-\nu \delta} Q^{-\nu \epsilon}(t) q^Y(t) V^{-\nu} . \]  

(3.7)

After substituting equation (3.7) into equation (3.4) the following expression for \( x(t) \) in terms of the new control variable is obtained:

\[ x(t) = z^{-\nu}(t) (t_{di} - t)^{-\alpha \nu} . \]  

(3.8)

The new boundary conditions are \( Z(t_{si}) = 0, Z(t_{di}) = A^{-1} V^{-\nu} (i - 1/2)^{-\nu \delta}/(1-\epsilon) \). After substituting into the objective functional, the following transformed problem is obtained:

\[
\text{Minimize } C' = \int_{t_{si}}^{t_{di}} z^Y(t) (t_{di} - t)^{-\alpha \nu} e^{-\rho t} dt
\]

\[
= \int_{t_{si}}^{t_{di}} I(z,t) dt 
\]  

(3.9)
subject to:

\[ Z(t_{s_i}) = 0, \ Z(t_{d_i}) = A^{-1} V^{-\nu} (i - 1/2)^{-\delta/(1-\epsilon)} \].

Since the intermediate function, \( I \), does not depend explicitly on the state variable, \( Z(t) \), the Lagrange-Euler equation is

\[ \frac{\partial I}{\partial Z} = \gamma Z^{-1}(t) (t_{d_i} - t)^{-\gamma \alpha} e^{-\rho t} = k_0 \], \hspace{1cm} (3.10)

or,

\[ Z^{-1}(t) = \frac{k_0}{\gamma} (t_{d_i} - t)^{\gamma \alpha} e^{\rho t} \].

Therefore, the optimal expression for \( z(t) \) is,

\[ z(t) = k_1 (t_{d_i} - t)^{\gamma/\gamma'} e^{\rho t/(\gamma-1)} \], \hspace{1cm} (3.11)

where

\[ k_1 = [k_0/\gamma]^{1/(\gamma-1)} \].

This also provides a solution for the optimal time path of resource usage; i.e.,

\[ x(t) = k_1' (t_{d_i} - t)^{\gamma/\gamma'} e^{\rho \gamma t/(\gamma-1)} \]. \hspace{1cm} (3.12)

The optimal solution is of transient significance since the value of \( k_1 \) is unknown. We want an optimal expression for \( x(t) \) that is in terms of the variables and parameters of
the original problem. To obtain the integration constant, we proceed as follows:

\[ Z(t) = \int z(\tau) \, d\tau + k_2 , \]

\[ = \int k_1 (t_{di} - \tau)^{\alpha\gamma/(\gamma-1)} e^{\rho\tau/(\gamma-1)} \, d\tau + k_2 . \]

(3.13)

Let \( \omega = \rho(t_{di} - \tau)/(\gamma-1) \), then

\[ Z(u) = \int k_1 [(\gamma-1)/\rho]^{\alpha\gamma/(\gamma-1)} \omega^{\alpha\gamma/(\gamma-1)} \]

\[ e^{-\omega + \rho t_{di}/(\gamma-1)} \quad Jd\omega \]

(3.14)

where \( J \) is the Jacobian of the transformation, and \( u = \rho(t_{di} - t)/(\gamma-1) \). The integral may now be written as

\[ Z(u) = \int k_3 \omega^{\alpha\gamma/(\gamma-1)} e^{-\omega} \, d\omega + k_4 , \]

(3.15)

where

\[ k_3 = k_1 [\gamma/(\gamma-1)]^{\alpha\gamma/(\gamma-1)} e^{\rho t_{di}/(\gamma-1)} \cdot J . \]

The above integral may be written as the difference between two incomplete gamma functions:

\[ Z(u) = k_3 \int_0^{\rho(t_{di} - t)/(\gamma-1)} \omega^{\alpha\gamma/(\gamma-1)} e^{-\omega} \, d\omega \]

\[ - k_3 \int_0^{\rho(t_{di} - t_{si})/(\gamma-1)} \omega^{\alpha\gamma/(\gamma-1)} e^{-\omega} \, d\omega + k_4 . \]
This result may be stated as

\[ Z(u) = - k_3 \left[ \Gamma \left( \frac{\rho(t_{di} - t_{si})}{(\gamma-1)}, \frac{\alpha\gamma}{(\gamma-1)} + 1 \right) \right] - \Gamma(u, \frac{\alpha\gamma}{(\gamma-1)} + 1) \right] + k_4 \]  

(3.16)

where \( \Gamma \) is the incomplete gamma function. Now \( Z(t_{si}) = 0 \) implies \( Z(u(t_{si})) = 0 \), and therefore \( k_4 = 0 \). The other boundary condition gives \( u(t_{di}) = 0 \) and \( Z(t_{di}) = A^{-1} V^{-\gamma} (i - 1/2)^{-\delta}/(1-\varepsilon) \); Therefore,

\[ Z(u(t_{di})) = - k_3 \left[ \Gamma \left( \frac{\rho(t_{di} - t_{si})}{(\gamma-1)}, \frac{\alpha\gamma}{(\gamma-1)} + 1 \right) \right] = A^{-1} V^{-\gamma} (i - 1/2)^{-\delta}/(1-\varepsilon) \]  

(3.17)

Equation (3.17) is now solved for \( k_3 \). The resulting expression is

\[ - k_3 = A^{-1} V^{-\gamma}(i - 1/2)^{-\delta} (1-\varepsilon)^{-1} \]

\[ r^{-1} \left[ \frac{\rho(t_{di} - t_{si})}{(\gamma-1)}, \frac{\alpha\gamma}{(\gamma-1)} + 1 \right] \]  

(3.18)

Also,
After substituting for $k_3$, the following expression is obtained:

\[
 z(t) = A^{-1} \nu^{-\nu}(i - 1/2)^{-\delta} (1-\epsilon)^{-1} \\
 \Gamma^{-1} \left[ \frac{\rho(t_{di} - t_{si})}{\gamma - 1}, \frac{\alpha \gamma}{\gamma - 1} + 1 \right] \left( \frac{\rho}{\gamma - 1} \right)^{-1} \\
 \left[ \rho(t_{di} - t)/(\gamma - 1) \right]^{\alpha \gamma}/(\gamma - 1) \\
 e^{-\rho(t_{di} - t)/(\gamma - 1)}. \tag{3.20}
\]

This formulation for optimum $z(t)$, along with equation (3.11), provides a direct solution for $k_1$. Equating equations (3.11) and (3.20), we obtain

\[
k_1 = A^{-1} \nu^{-\nu}(i - 1/2)^{-\delta} (1-\epsilon)^{-1} \\
 \Gamma^{-1} \left[ \frac{\rho(t_{di} - t_{si})}{\gamma - 1}, \frac{\alpha \gamma}{\gamma - 1} + 1 \right] \\
 \left( \frac{\rho}{\gamma - 1} \right)^{\alpha \gamma}/(\gamma - 1) + 1 e^{-\rho t_{di} / (\gamma - 1)}. 
\]

Substitution of $k_1$ in equation (3.12) yields the following optimum time path of resource use:
\[ x_i(t) = B(i - 1/2)^{-\gamma} \Gamma^{-\gamma}[\rho(t_{di} - t_{si})/(\gamma-1), \beta_1] V^{-\gamma} \]

\[ (t_{di} - t)^{\alpha\gamma}/(\gamma-1) e^{-\gamma\rho(t_{di} - t)/(\gamma-1)} , \]  \hspace{1cm} (3.21)

where

\[ B = A^{-\gamma} (1 - \epsilon)^{-\gamma} \Gamma_{\rho/(\gamma-1)}^{\gamma^2/(\gamma-1) + \gamma} , \]

\[ \beta_1 = \alpha\gamma/(\gamma-1) + 1 , \] and

\[ \Gamma( , , ) \] is the incomplete gamma function. If \( T_1 \) and \( T_2 \) represent beginning and ending dates for a period, then the appropriate expression for the per period cost of airframe \( i \) is

\[ X_i(T_2) - X_i(T_1) = \int_{T_1}^{T_2} x_i(t) \, dt \]  \hspace{1cm} (3.22)

where \( x(t) = \frac{dX}{dt} \). Using equation (3.21), the above integral is

\[ X_i(T_2) - X_i(T_1) = \]

\[ \beta_0(i - 1/2)^{-\gamma} \Gamma^{-\gamma}[\rho(t_{di} - t_{si})/(\gamma-1), \beta_1] \]

\[ V^{-\gamma} \Gamma_{(T_1, T_2)} \{ \Gamma[\gamma\rho(t_{di} - T_1)/(\gamma-1), \beta_1] \]

\[ - \Gamma[\gamma\rho(t_{di} - T_2)/(\gamma-1), \beta_1] \} \]

where
\[ \beta_0 = B[(\gamma-1)/\gamma \rho]^{\gamma/(\gamma-1)} + 1. \]

Because of the nature of the data, the left hand side of equation (3.22) is unobservable. The observable quantity is direct manhours in the lot, i.e.,
\[ \sum_{i=k_j}^{n_j} [X_i(T_2) - X_i(T_1)] \]

where \(k_j\) and \(n_j\) are sequence numbers of the first and the last airframe in lot \(j\). Thus, the model is restated as:
\[ \sum_{i=k_j}^{n_j} [X_i(T_2) - X_i(T_1)] \]
\[ = \sum_{i=k_j}^{n_j} \beta_0 (i - 1/2)^{-\gamma} \Gamma^{-\gamma} \left[ \rho(t_{di} - t_{si})/(\gamma-1), \beta_1 \right] \]
\[ V^{-\gamma}(T_1, T_2) \left[ \Gamma[\gamma \rho(t_{di} - T_1)/(\gamma-1), \beta_1] \right. \]
\[ \left. - \Gamma[\gamma \rho(t_{di} - T_2)/(\gamma-1), \beta_1] \right]. \quad (3.23) \]

The parameters in equation (3.23) are estimated using nonlinear programming. The value of the discount rate, \(\rho\), is assumed to be 0.0075. The following logic was used to arrive at this value. There is very little inflation in the rate of return since it is measured in terms of direct manufacturing hours. In this case the discount rate is
more like a real rate of return, therefore \( \rho \) is restricted to 3% per year. The data used for the estimation is quarterly data, so \( \rho = 0.03/4 = 0.0075 \) per quarter. The remaining parameters (\( \beta_0, \beta_1, \gamma, \delta, \) and \( \nu \)) are estimated from historical data.

2. The C-141 Program and Data

The C-141 program produced 284 aircrafts during the six year period from 1962 to 1968. Only one model of the aircraft was produced. The data for this analysis are drawn from two sources. Orsini (1970) reports direct man-hours per quarter for each of the twelve lots in the C-141 program. Orsini attributes these data to the C-141 Financial Management Reports maintained by the Air Force Plant Representative Office at the Lockheed-Georgia facility. The schedule of actual aircraft acceptances by month as reported in Acceptance Rates and Tooling Capacity for Selected Military Aircraft (1974) was used to check the Orsini delivery date.

This data, like much data on aircraft production, provides labor hours for a period of time (quarterly) and dates and quantities of deliveries. Preliminary data analysis revealed two problems. First, there were two instances, late in the program, where a small number of labor hours were expended on a production lot after the schedule indicated delivery. This probably is a situation
where deliveries were made out of sequence. To remedy this problem the labor hours for the last quarter of lots 9 and 10 were aggregated with those of the previous quarter. This reduced the number of observations by two.

The other problem is that in lots two through eight, delivery of aircraft seems to lag the last expenditure of labor hours by an average of four months. For the other five lots labor hours are expended up to the last month of delivery. To overcome this problem, the deliveries of aircraft in lots two through eight were advanced by four months.

With these adjustments eighty-nine observations on labor hours for twenty-four quarters for twelve lots were used. These observations, together with the number of aircraft delivered each month, constitute the data for the study. The C-141 database appears in Appendix A.

3. Unconstrained Optimization

In this section, the parameters $\beta_0$, $\beta_1$, $\gamma$, $\delta$, and $\nu$ in equation (3.23) are estimated. The nonlinear programming technique MINOS (1983) is used for the estimation.

MINOS stands for "Modular In-core Nonlinear Optimization System." It permits restricting the variables to some feasible region specified by a set of constraints and a set of upper and lower bounds. It is a Fortran based
computer system designed to solve large-scale nonlinear optimization problems.

There are several reasons for selecting MINOS as the tool for solving the optimization problem. Womer and Gulledge (1983) estimated the parameters in a similar model using the NLIN procedure of the Statistical Analysis System. These estimates were confirmed using independent FORTRAN code using Marquardt's Compromise (Marquardt, 1963). Either of these routines could be used to estimate the parameters in this research, but MINOS was selected for the following reason. The final hypothesized model in this dissertation requires estimating the parameters in a similar model, but subject to nonlinear constraints. Since none of the previously used systems have this capacity, MINOS was selected as a matter of convenience. The estimates, using MINOS, on the unconstrained model also permit comparison with the results obtained by Womer and Gulledge (1983).

In algebraic terms, the problem is:

Minimize \( f(x, \beta) = \sum_j [y_j - f_j(x, \beta)]^2 \)  

subject to: \( l < \beta < u \)

where \( y_j \) is the observed manhours, \( f_j(x, \beta) \) is the cost function [equation (3.23)] for \( j \)th observation and \( \beta \) is a vector of unknown parameters. Also, \( u \) and \( l \) are upper and lower bounds on the parameters respectively. The nonlinear objective function \( f(x, \beta) \) is a continuous differentiable
function, so that the gradient vector, $Vf(x, \beta) = \left[ \frac{\partial f}{\partial \beta} \right] = g(x, \beta)$, exists. The system uses the reduced gradient algorithm of Wolfe (1962) in conjunction with the quasi-Newton algorithm of Davidon (1959).

As previously noted, the learning parameter, $\delta$, is expected to be between zero and one. Since $x_i^{1/\gamma}(t)$ represents the effect of the speed of the production line, $\gamma$, which describes the return to variable resources is expected to be greater than one. This ensures diminishing returns to the variable composite resource. It is assumed that more airframes in the same facility result in a slight decrease in efficiency, so $\nu$ is expected to be negative and small. Since $\alpha$ represents the decrease in labor productivity when an airframe nears completion, $\alpha$ is expected to be positive. Based on these expectations, upper and lower bounds for all the parameters are set as presented in Table I.

<table>
<thead>
<tr>
<th>TABLE I</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BOUNDS ON THE PARAMETERS</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Upper Bound</th>
<th>Lower Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>20.00</td>
<td>0.25</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>10.00</td>
<td>0.001</td>
</tr>
<tr>
<td>$\delta$</td>
<td>1.00</td>
<td>0.001</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>2.00</td>
<td>0.00001</td>
</tr>
<tr>
<td>$\nu$</td>
<td>1.00</td>
<td>-2.00</td>
</tr>
</tbody>
</table>
FIGURE 2. RESIDUALS FROM THE UNCONSTRAINED OPTIMIZATION
TABLE II
PARAMETER ESTIMATES USING THE C-141 DATA

<table>
<thead>
<tr>
<th>Parameter Estimates</th>
<th>Parameter Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Using MINOS</td>
<td>Using NLIN</td>
</tr>
<tr>
<td>( \beta_0 )</td>
<td>1.136</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>3.080</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.503</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1.002</td>
</tr>
<tr>
<td>( \nu )</td>
<td>-0.452</td>
</tr>
</tbody>
</table>

The model is estimated and the results are presented in Table II. The estimated values of the parameters are consistent with a priori expectations. Table II also contains the parameter estimates from earlier work of Womer and Gulledge (1983), where estimates were obtained using nonlinear least squares as implemented by SAS's NLIN procedure.

The comparison of estimates shows that the values obtained by the two different models are very close. The residuals obtained from this analysis (see Figure 2) indicate that the error terms are independent. The asymptotic standard errors and confidence intervals are not presented for this analysis. It is noted that in the earlier work of Womer and Gulledge (1983), all of the estimates were significantly different from zero. Another point to note is that the estimate for \( \gamma \) was assumed to be greater than one, a restriction that has important theoretical implications. The estimate (\( \hat{\gamma} = 1.002 \)) is
close to one, but statistically it is very significantly different from one. The estimated standard error of \( \hat{\gamma} \) is very small. Since the estimates obtained in this analysis are extremely close to those previously obtained, we feel comfortable that the nonlinear optimization system is performing correctly.

The analysis is based on 89 observations and 38 of the residuals are found to have negative values. This shows that approximately forty-three percent of the observations lie below the fitted cost curve. Figure 3 illustrates the predicted time path of resource use for this program and the actual resources used. This is very similar to the time path of resource use (see Figure 4) obtained earlier by Womer and Gulledge (1983). Thus, the estimated cost function in this case represents an "average" cost curve like the previous analysis. The word "average" in this context does not mean average production costs. Since the objective function for the parameter estimation is the sum of the squared errors, by definition some points must lie above the estimated cost function, and some points must lie below. In this sense the function is indeed an average function. This observation motivates the next section of the dissertation. The cost function, as estimated above, is inconsistent with the definition of a neoclassical cost
FIGURE 3. TIMEPATH OF RESOURCE USE FOR UNCONSTRAINED PROBLEM (dashed line is the predicted)
FIGURE 4. THE PATH OF RESOURCE USE FOR WOMER AND GULLEDGE MODEL (DASHED LINE IS THE PREDICTED)
function. A cost function represents the minimum cost obtainable from given inputs, outputs and prices. This implies that all of the data points used in the estimation of equation (3.23) should lie on or above the estimated function. In this context, the estimated cost function is a cost frontier. In the next section, MINOS is used to estimate the parameters in equation (3.23) under the restriction that all data points lie on or above the frontier.

4. Estimation of a Nonlinear Frontier Cost Function

In this section, the cost function is estimated in a manner which recognizes that the function is a frontier representing the minimum cost that may be obtained from given inputs. A cost function should give the minimum level of cost at which it is possible to produce output, given inputs. The word frontier may meaningfully be applied here because the function sets a limit to the range of possible observations. Thus, the observed points should lie on or above the cost frontier, but no points should lie below the cost frontier.

In the airframe production program, the input $x$ is available at fixed prices to produce a single output, the airframe. The transformation of inputs into output is characterized by the production function [equation (3.1)] which by definition gives maximum output obtainable from
the input vector. An equivalent representation of this production technology is provided by the cost function [equation (3.23)], which shows the minimum expenditure required to produce output. We want to fit a cost frontier without assuming the form of the distribution of the one-sided error. Such a frontier is called a best-practice frontier, as used by Farrell (1957). This is slightly different from an absolute frontier, where an explicit distributional form of the one-sided error is assumed [Greene (1980)]. The practical importance of this distinction is not likely to be large, since the absolute and best-practice frontier necessarily converge asymptotically [Forsund; et al. (1980)]. This study uses eighty-nine observations which is fairly large, so the estimation of a best-practice frontier seems appropriate.

Equation (3.23) can be expressed as

\[ \hat{Y}(I) = F[x(I), \beta] \quad (3.25) \]

where \( \hat{Y}(I) \) is the predicted quantity, the direct man-hours; \( \beta \) is a vector of unknown parameters to be estimated, and \( x(I) \) represents the input for \( I \)th observation. For the minimum cost curve, the errors must all be non-negative. So, the problem is:

\[ \text{Minimize} \sum_{I} [Y(I) - F[x(I), \beta]]^2 \quad (3.26) \]

Subject to: \( Y(I) > F[x(I), \beta], I = 1, 2, \ldots 89 \)

or
Alternatively, we could consider the minimization of a linear loss function

\[ \sum_{I} |Y(I) - F[x(I), \beta]| \]  

subject to the same constraints given by the inequality (3.27). The estimation was performed using the objective function in equation (3.28), mainly for curiosity. There is a substantial body of statistical literature to support the quadratic loss function, particularly in reference to outlier penalties. The estimation did not converge when equation (3.28) was used, but at the time of failure the estimates were always very close to those obtained with the quadratic objective function. For these reasons equation (3.28) was abandoned, and the estimation was carried out with equation (3.26).

This is a nontrivial nonlinear programming problem, as \( F[x(I), \beta] \) is an extremely nonlinear function. In order to explain how the observed values lie above the cost frontier, a non-negative disturbance term has been implicitly assumed in equation (3.27). This implies the following model:

\[ Y(I) = F[x(I), \beta] + \psi(I) \]  

where \( \psi(I) > 0 \). This model is solved for \( \beta \) using MINOS.
The solution process consists of a sequence of "Major Iterations." At the start of each major iteration, the nonlinear constraints are linearized at the current point $\beta_k; i.e., F[x(I), \beta]$ is replaced by the approximation

$$F[x(I), \beta] = F[x(I), \beta] + J(\beta_k)(\beta - \beta_k)$$  \hspace{1cm} (3.30)

where $J(\beta_k)$ is the Jacobian matrix whose $(i,j)^{th}$ element is $\partial F(i)/\partial \beta_j$. The objective function is also modified to yield the following subproblem:

Minimize $\sum_I \{Y(I) - F(x(I), \beta)\}^2$

$$- \lambda_k^T(F - F) + \frac{1}{2} \mu(F - F)^T(F - F)$$  \hspace{1cm} (3.31)

subject to:  $Y(I) > F[x(I), \beta]$.

This objective function [equation (3.31)] is called the augmented Lagrangian function. The vector $\lambda_k$ is the estimate of the Lagrange multipliers for the nonlinear constraints, and the term involving $\mu$ is a modified quadratic penalty function. The vector $\lambda_k$ converges to the Lagrange multipliers for the original nonlinear constraints at the optimum. The penalty parameter, $\mu$, is essential to obtain convergence. That is, $\mu = 0$ gives the most rapid rate of convergence, but for highly nonlinear problems a positive value should be assigned to $\mu$. Since the function $F[x(I), \beta]$ is highly nonlinear, a penalty parameter of 0.1 is used in this estimation.
If \( \beta_k \) is an optimal solution to the \( k^{th} \) subproblem, and if it satisfies the nonlinear constraints within a specified tolerance, then \( \beta_{k+1} \), the solution to the next subproblem will probably be an optimal solution to the original nonlinear program. More precisely, let \((\beta_k, \lambda_k)\) be the solution that results from solving the \( k^{th} \) subproblem. The next subproblem is defined in terms of \( \beta_k \) and \( \lambda_k \), and will terminate at some point \((\beta_{k+1}, \lambda_{k+1})\). Convergence is assumed to have occurred if the following conditions are true:

1. \( \beta_k \) is an optimal solution to its subproblem;
2. \( \beta_k \) satisfies the nonlinear constraints to within a specified tolerance, 0.0001 in this research;
3. \( \lambda_k \) is not substantially different from \( \lambda_{k-1} \). A difference tolerance level of 0.00001 is used in this research;
4. \( \beta_{k+1} \) is an optimal solution to its subproblem;
5. a basis change did not occur during solution of subproblem \((k+1)\); and
6. the reduced gradient did not change significantly during solution of \((k+1)^{th} \) subproblem.

If all these conditions hold, \((\beta_{k+1}, \lambda_{k+1})\) will be accepted as an optimal solution to the original problem.

The solution, \( \beta_k \), is checked for feasibility and then the final point, \( \beta_{k+1} \), is checked for optimality. Very few minor iterations occur on the last subproblem. Hence the last two subproblem solutions, \( \beta_k \) and \( \beta_{k+1} \), will be virtually identical; therefore, the tests for feasibility
and optimality are being applied essentially at the same point.

The results of the estimation are presented in Table III. The upper and lower bounds on the parameters are the same as those presented in Table I. The value of the discount rate, $\rho$, is fixed as before at 0.0075.

The estimates in Table III agree with our expectations. The scale factor $\beta_0$ is different from the estimated value in Table II, but this is not unexpected. When the restriction is imposed, the predicted values are expected to be smaller in magnitude as compared to the values obtained from the unrestricted optimization. The estimate for $\beta_1 = 4.42007$ implies that $\alpha = 0.003$, a result which also satisfies our expectations. The minimum cost curve now lies below the observed values (see Figure 5), in agreement with economic theory.

The poor predictions for the time periods sixteen through twenty are difficult to explain, but there are two things that have some effect on these predictions. First, the model includes no factors expressing hiring or firing costs. Therefore, even though the model predicts that the workforce should decline, the firm correctly chose to maintain a higher workforce for that period. Secondly, the number of aircraft delivered during that period changed abruptly; twenty-six and twenty-seven aircraft were
FIGURE 5. PREDICTED AND ACTUAL TIME PATH OF RESOURCE USE (dashed line is the predicted)
### TABLE III
**NONLINEAR ESTIMATION OF PARAMETERS**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>0.82003</td>
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<tr>
<td>$\beta_1$</td>
<td>4.42007</td>
</tr>
<tr>
<td>$\delta$</td>
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</tr>
<tr>
<td>$\gamma$</td>
<td>1.00094</td>
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<tr>
<td>$\nu$</td>
<td>-0.60574</td>
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</tbody>
</table>

delivered respectively in time periods sixteen and seventeen. Then suddenly deliveries dropped to five in time period eighteen. In time period nineteen, only fifteen aircraft were delivered; then the deliveries increased to twenty-six in the next time period. Figures 6 through 11 are plots of required resources for some representative lots. Figures 6 through 9 show that the predictions for these individual production lots are quite good. Figures 10 and 11 for the above mentioned time periods show more variation predicted by the model than exhibited by the data. The predicted resource requirement function in these plots actually falls below the actual data curve. The spline fit algorithm makes the frontier appear to fall above the actual in Figures 10 and 11. Probably, a more appropriate delivery schedule could have reduced the program cost considerably.
FIGURE 6. ACTUAL VERSUS PREDICTED FOR THE FIFTH PRODUCTION LOT (dashed line is predicted)
FIGURE 7. ACTUAL VERSUS PREDICTED FOR THE SIXTH PRODUCTION LOT (dashed line is predicted)
FIGURE 8. ACTUAL VERSUS PREDICTED FOR THE SEVENTH PRODUCTION LOT (dashed line is predicted)
FIGURE 9. ACTUAL VERSUS PREDICTED FOR THE EIGHTH PRODUCTION LOT (dashed line is predicted)
FIGURE 10. ACTUAL VERSUS PREDICTED FOR THE TENTH PRODUCTION LOT (dashed line is predicted)
FIGURE 11. ACTUAL VERSUS PREDICTED FOR THE TWELFTH PRODUCTION LOT (dashed line is predicted)
FIGURE 12. RESIDUALS FROM THE ESTIMATION OF THE FRONTIER FUNCTION
The actual and predicted values from this model show that all constraints are satisfied at the optimal point, as the model provides (see Figure 12) all non-negative residuals. A few large residuals (circled values in Figure 12) are also observed, but it is hypothesized that these large values are associated with the variability in the accounting data. These values were treated as outliers (i.e., they were replaced with their predicted values) and the model was reestimated. Since there was only a slight change in the estimated values, the observations were replaced and retained at their original values.

The objective was to obtain an estimated cost function which acts as a lower boundary, which is well grounded in theory and estimated from actual data. In this chapter such a frontier cost function has been estimated. The next chapter contains sensitivity analysis on the model. Also, the model is used to test the major hypothesis of this research. Could an alternative procurement policy that results in a smooth production/delivery schedule result in reduced cost on this airframe program?
CHAPTER IV
MODEL SENSITIVITY AND DELIVERY SCHEDULE EFFICIENCY

It is important to illustrate the sensitivity of the model to changes that can occur during the production program. In this chapter several alternatives to the actual delivery schedule are considered. The objective is to determine whether alternative delivery schedules increase efficiency. This analysis is consistent with the central hypothesis of this dissertation. It is shown that a more even procurement policy, and hence a smoother delivery schedule, would have led to a more efficient production program for the C-141 airframe.

1. Measuring Efficiency

One of the primary motivations for estimating the frontier function is to study technical or cost efficiency. The study of production or cost frontier models is motivated in part by an interest in the structure of efficient production technologies, but also by an interest in the divergence between observed and frontier operation. The notions of frontier and efficiency are complementary. The cost frontier is obtained when inputs are being used in the most efficient manner possible, given the state of knowledge.

A production process can be inefficient in two ways. It can be technically inefficient, in the sense that it
fails to produce maximum output from a given input bundle; technical inefficiency results in an equiproportionate overutilization of all inputs. The production process can also be allocatively or cost inefficient in the sense that the marginal revenue product of an input might not be equal to the marginal cost of that input; cost inefficiency results in the utilization of inputs in the wrong proportions. A technically inefficient process operates beneath its production frontier, and an allocatively inefficient process operates above its least cost path.

In Chapter III, the cost function was estimated. The objective of this chapter is to draw inferences about allocative or cost inefficiency. The amount by which a firm lies above its minimum cost frontier can be regarded as a measure of inefficiency. The model estimated in Chapter III is of the form

\[ Y = f(x, \beta) + \phi \]

(4.1)

where \( \phi > 0 \). For each observation, we can obtain the individual estimates of the \( \phi \)'s, simply as the residuals of the fitted cost curve. These residuals provide specific estimates of the efficiency factor for each sample point.

If the sample contains a large number of observations, it is useful to have a summary measure of efficiency for the complete sample. Two measures of efficiency are considered in this study. The first measure of efficiency,
$E_1$, is based on the ratio between the observed value and the predicted value obtained using the frontier cost function. The average measure can be obtained by simply taking the average of the individual efficiency measures. Thus, $E_1$ is given by,

$$E_1 = \text{Mean Ratio Efficiency} = \frac{\sum Y(I)/f[x(I), \hat{\beta}]}{N}, \quad (4.2)$$

where $N$ is the total number of observations. The smaller the value of $E_1$, the more efficient is the process.

A second measure of efficiency, $E_2$, can be obtained by considering the average of the residuals obtained from the fitted frontier cost function. Thus, $E_2$ is given by,

$$E_2 = \text{Average efficiency} = \frac{\sum \hat{\phi}(I)}{N}, \quad (4.3)$$

where $\hat{\phi}(I)$ is the residual from the $I^{th}$ observation. Small values of $E_2$ represent a more efficient process. Table IV contains both measures of efficiency for the original data, without considering any change in the delivery schedule.

The next section contains alternatives to the actual delivery schedule. Each of the alternatives represents a small discrete change to the actual delivery schedule. In
TABLE IV
Efficiency of the Realized Production Program

<table>
<thead>
<tr>
<th>Measure</th>
<th>Efficiency</th>
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<tbody>
<tr>
<td>$E_1$</td>
<td>33.867</td>
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<tr>
<td>$E_2$</td>
<td>1.125</td>
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each case, efficiency is measured to determine whether a more efficient program exists.

2. Alternative Delivery Schedules

To illustrate the sensitivity of the model to changes in the delivery schedule, we now consider several alternatives to the actual delivery schedule. The C-141 program delivery schedule is presented in Table V. There were no deliveries made during time periods one through five. In all the figures in this section, the actual schedule is represented by a solid curve and alternative schedules by dashed curves.

The first sensitivity analysis involves three alternative delivery schedules. Schedule A, presented in Figure 13, represents an alternative to the actual delivery schedule with an equal number of deliveries for time periods seventeen and eighteen. Schedule B, presented in Figure 14, has equal number of deliveries for time periods seventeen through twenty-four and Schedule C, presented in Figure 15, has equal number of deliveries for time periods
TABLE V

The C-141 Delivery Schedule

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<th>Time Period (in quarters)</th>
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FIGURE 13. REALIZED AND CHANGED DELIVERY SCHEDULE A WITH EQUAL DELIVERIES FOR TIMEPERIODS SEVENTEEN AND EIGHTEEN (DASHED LINE REPRESENTS SCHEDULE A)
FIGURE 14. REALIZED AND CHANGED DELIVERY SCHEDULE B WITH EQUAL DELIVERIES FOR TIMEPERIODS SEVENTEEN THROUGH TWENTY-FOUR (DASHED LINE REPRESENTS SCHEDULE B)
FIGURE 15. REALIZED AND CHANGED DELIVERY SCHEDULE C WITH EQUAL DELIVERIES FOR TIMEPERIODS FIFTEEN THROUGH TWENTY-FOUR (DASHED LINE REPRESENTS SCHEDULE C)
fifteen through twenty-four. For alternative Schedule A, only lot eight is affected. The labor requirement for the realized program and Schedule A for the eighth lot are shown in Figure 16. The areas under each curve to the left of any point in time show the labor required up to that time to support the indicated delivery. Figure 16 shows that the labor requirement for Schedule A is slightly less than that of realized schedule. The time path of required labor of the complete program is the sum of these requirements for all the lots. The aggregated labor requirement for the realized schedule and Schedule A are presented in Figure 17. For Schedule A, the labor requirement is more than the realized requirement for the seventeenth and eighteenth quarters and less for quarters nineteen through twenty-three. There is not much difference in the total labor requirement for both of the schedules, an observation that is supported by the cost curves. Figure 18 shows the time path of discounted total cost for both Schedule A and the realized schedule. An enlargement of Figure 18 is presented in Figure 19. The cumulative cost of production for the changed Schedule A is slightly less than the cost of the realized schedule. Therefore, in terms of discounted cost, a schedule where equal number of deliveries are considered only for two time periods seventeen and eighteen has a relatively minor impact on cost.
FIGURE 16. TIME PATH OF LABOR USE FOR PRODUCTION LOT EIGHT
(DASHED LINE REPRESENTS SCHEDULE A)
FIGURE 17. AGGREGATED LABOR USE FOR THE REALIZED AND ALTERNATIVE SCHEDULE A  
(DASHED LINE REPRESENTS SCHEDULE A)
FIGURE 18. DISCOUNTED TOTAL COST FOR SCHEDULE A AND THE REALIZED SCHEDULE (DASHED LINE REPRESENTS SCHEDULE A)
FIGURE 19. DISCOUNTED TOTAL COST FOR THE REALIZED SCHEDULE AND SCHEDULE A (ENLARGED)
For Schedule B, where equal number of deliveries are considered for time periods seventeen through twenty-four, lots eight through twelve were affected. The labor use rate for these individual lots for both the realized and changed Schedule B are presented in Figures 20 through 24. For lot eight (see Figure 20), the labor use rate is the same for both the schedules. For lots nine and ten (see Figures 21 and 22) there is not much difference in the labor requirements, as the area under both curves are more or less equal. But for lots eleven and twelve (see Figures 23 and 24), clearly the requirement of labor hours is less for the changed Schedule B. The aggregated time path of labor use, presented in Figure 25 shows that more labor hours are required for the seventeenth and eighteenth quarters and less than the realized is required for nineteenth through twenty-fourth quarters. The discounted total cost for Schedule B, presented in Figure 26 and enlarged in Figure 27 shows that the total program cost associated with Schedule B is less than the realized program schedule, but the cost savings are not very substantial.

The last alternative, Schedule C, affects lots seven through twelve. The changes in labor use rate for these individual lots are presented in Figures 28 through 33. All these figures show clearly that the labor requirements for all the altered lots if Schedule C is adopted are less
FIGURE 20. TIME PATH OF LABOR USE FOR PRODUCTION LOT EIGHT
(DASHED LINE REPRESENTS SCHEDULE B)
FIGURE 21. TIME PATH OF LABOR USE FOR PRODUCTION LOT NINE
(DASHED LINE REPRESENTS SCHEDULE B)
FIGURE 22. TIME PATH OF LABOR USE FOR PRODUCTION LOT TEN
(DASHED LINE REPRESENTS SCHEDULE B)
FIGURE 23. TIME PATH OF LABOR USE FOR PRODUCTION LOT ELEVEN (DASHED LINE REPRESENTS SCHEDULE B)
FIGURE 24. TIME PATH OF LABOR USE FOR PRODUCTION LOT TWELVE
(DASHED LINE REPRESENTS SCHEDULE B)
FIGURE 25. AGGREGATED LABOR USE FOR THE REALIZED AND ALTERNATIVE SCHEDULE B (DASHED LINE REPRESENTS SCHEDULE B)
FIGURE 26. DISCOUNTED TOTAL COST FOR SCHEDULE B AND THE REALIZED SCHEDULE (DASHED LINE REPRESENTS SCHEDULE B)
FIGURE 27. DISCOUNTED TOTAL COST FOR THE REALIZED AND SCHEDULE B (ENLARGED)
than what actually was required. The aggregated labor use for Schedule C, presented in Figure 34, shows no break in resource use. It is fairly smooth for the time periods under consideration, and it shows that less labor hours are required for the same program if Schedule C is adopted. The time path of discounted total cost for Schedule C, presented in Figure 35, is consistent with the aggregate resource requirement. An even distribution for the number of deliveries for more time periods has a relatively larger impact on discounted total cost. Figure 36 is an enlargement of Figure 35 which shows that cost decreases considerably if Schedule C is adopted for this program.

A comparison of costs for the realized schedule and all three alternative schedules, A, B, and C; are presented in Figure 37. From this plot and the above discussion we can infer that the cost savings increase with a more even distribution of deliveries per time period, i.e., a more even procurement program results in larger cost savings.

The efficiencies, $E_1$ and $E_2$, are measured for all these schedules, and is presented in Table VI. These measures support the above hypothesis. Comparison of these efficiency values clearly indicates that the program becomes more efficient when airframes are delivered more evenly.
FIGURE 28. TIME PATH OF LABOR USE FOR PRODUCTION LOT SEVEN (DASHED LINE REPRESENTS SCHEDULE C)
FIGURE 29. TIME PATH OF LABOR USE FOR PRODUCTION LOT EIGHT
(DASHED LINE REPRESENTS SCHEDULE C)
FIGURE 30. TIME PATH OF LABOR USE FOR PRODUCTION LOT NINE
(DASHED LINE REPRESENTS SCHEDULE C)
FIGURE 31. TIME PATH OF LABOR USE FOR PRODUCTION LOT TEN  
(DASHED LINE REPRESENTS SCHEDULE C)
FIGURE 32. TIME PATH OF LABOR USE FOR PRODUCTION LOT ELEVEN (DASHED LINE REPRESENTS SCHEDULE C)
FIGURE 33. TIME PATH OF LABOR USE FOR PRODUCTION LOT TWELVE
(DASHED LINE REPRESENTS SCHEDULE C)
FIGURE 34. AGGREGATED LABOR USE FOR SCHEDULE C AND THE REALIZED SCHEDULE (DASHED LINE REPRESENTS SCHEDULE C)
FIGURE 35. DISCOUNTED TOTAL COST FOR SCHEDULE C AND THE REALIZED SCHEDULE (DASHED LINE REPRESENTS SCHEDULE C)
FIGURE 36. DISCOUNTED TOTAL COST FOR THE REALIZED SCHEDULE AND SCHEDULE C (ENLARGED)
FIGURE 37. DISCOUNTED TOTAL COST FOR SCHEDULES A, B AND C AND THE REALIZED SCHEDULE
TABLE VI

Efficiency of Alternative Delivery Schedules

<table>
<thead>
<tr>
<th>Efficiency Measure</th>
<th>Schedule A</th>
<th>Schedule B</th>
<th>Schedule C</th>
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<tr>
<td>E₁</td>
<td>33.820</td>
<td>32.281</td>
<td>30.218</td>
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<tr>
<td>E₂</td>
<td>1.068</td>
<td>1.045</td>
<td>1.013</td>
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</table>

The second sensitivity analysis illustrates the effect of compressing the delivery schedule at the end of the program. An alternative schedule D, where the effect of delivering the last six airframes one quarter early is considered. This results in reducing the time to work on these last six airframes and the time for learning. It also results in an increase in V during the period when these six airframes are completed. These changes suggest a decrease in efficiency as the facility becomes more crowded with increases in V. Efficiency measures for this alternative Schedule D are presented in Table VII.

With a decrease in efficiency, program cost is expected to be higher for this alternative. Figure 38 indicates that more resource is necessary for the time periods twenty through twenty-two and less is necessary for the last two quarters. Cost actually rises slightly due to this change in the schedule (see Figure 39) but this cost increase is not significant.
FIGURE 38. TIME PATH OF AGGREGATED LABOR USE FOR SCHEDULE D
WHERE DELIVERY OF LAST SIX AIRFRAMES IS ADVANCED ONE QUARTER
(DASHED LINE REPRESENTS SCHEDULE D)
FIGURE 39. DISCOUNTED TOTAL COST FOR SCHEDULE D WHERE THE DELIVERY OF LAST SIX AIRFRAMES IS ADVANCED ONE QUARTER (DASHED LINE REPRESENTS SCHEDULE D)
Finally, we consider the alternative Schedule E, where the first airframe is delivered one quarter later than reported in the actual delivery schedule. This causes the resource use rate to be lower early in the program but higher as the new delivery date, the end of quarter seven, is approached. The aggregate resource use rate for Schedule E for the first eight quarters is presented in Figure 40. The effect of delaying this delivery increases the learning applicable to the first unit by providing more time prior to delivery. It also reduces the planned speed of the production line. Both these effects tend to lower cost. They are offset by the fact that work on airframe one is delayed to a time when there are more airframes in the facility. This increases $V$ and the length of the production line. Thus, the cost of producing all the airframes in the facility is increased during the quarter, when delivery of airframe one is scheduled. The net effect is a rise in total program cost, indicated in
FIGURE 40. AGGREGATE RESOURCE USE RATE WHEN THE FIRST AIRFRAME IS DELIVERED ONE QUARTER LATER (DASHED LINE REPRESENTS SCHEDULE E)
FIGURE 41. DISCOUNTED TOTAL COST OF SCHEDULE E WHERE DELIVERY OF FIRST AIRFRAME IS DELAYED ONE QUARTER (DASHED LINE REPRESENTS SCHEDULE E)
TABLE VIII

Efficiency of Alternative Schedule E

<table>
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<th>Efficiency Measure</th>
<th>Efficiency of Realized Schedule</th>
<th>Efficiency of Schedule E</th>
</tr>
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<td>$E_1$</td>
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<tr>
<td>$E_2$</td>
<td>1.125</td>
<td>1.138</td>
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Figure 41. The efficiency measures of Schedule E, presented in Table VIII are consistent with the above findings.

The sensitivity analysis clearly imply that alternative schedules with even procurement quantities would have resulted in lower costs for the C-141 program.
CHAPTER V
CONCLUSION

The objective of this study is to provide a model of variable airframe production cost that is well grounded in theory, estimated from actual data and sensitive to exogeneous delivery schedule effects. A significantly new approach to estimating the cost of an airframe production program is developed and tested. In Chapter III, the rationale for the model is provided, the functional form is derived, the estimation procedure for the frontier cost function is developed, and the parameter estimates are reported. The sensitivity of the estimated model to delivery schedule alterations is examined in Chapter IV.

The usefulness of this research is apparent. The government is interested in predicting costs in the production phase of airframe programs. Since each contractor's cost accounting system is different, it is extremely difficult to examine the cost impact of changes using accounting techniques. A cost model is appropriate as it is particularly useful in determining the effect of changes that can occur during the production process.

In the sections of this chapter, the results are reviewed. In the first section, the role of the model in scheduling problems is discussed. In section 2, cost estimating technique and use of model to evaluate policy
options with respect to changes in delivery schedules is discussed. The essence of this research is to provide some methodology for obtaining better cost estimates in light of these changes. Finally, in the third section, the role of these results as a basis for further research is discussed.

1. The Model and Production Scheduling

The general approach is to augment a production function with a learning hypothesis. The discounted cost of production is minimized and the model is solved for optimal time paths of resource use rate and discounted cost. It is assumed throughout that cost minimization is the firm's prime objective. However, there may be other factors that motivate the contractor. Demong and Strayer (1981) summarized some of the possible alternative contractor objectives. These include "growth, new product lines, prestige, improved public image, social approval, national goals, potential for follow-up business, commercial applications and excelling for excellence". The firm may pursue any of these goals, but a firm must have more interest in profits and thus cost in order to remain in business. Therefore, the assumption of cost minimization seems appropriate.

The theoretical aspects of a dynamic factor augmented production function are emphasized. Learning and cumulative output are introduced as inputs in the
production process. An important aspect of the model is that it can be used to obtain updated cost estimates during the production period of an airframe program, because the resource use rate and discounted costs both are expressed as functions of time.

The four production cost drivers learning by doing, learning over time, the speed of the production line and the length of production line, incorporated in the model are discussed in Chapter III. These can be used to examine the consequences of altering a program delivery schedule.

The model provides a framework within which the various effects of alternative schedules can be addressed. It provides the means to forecast alternative cost profiles for different delivery schedules quickly and accurately. It can also be used to find a set of delivery schedules that fit a particular funding profile. This quick reaction capability of the model is demonstrated in Chapter IV where five alternative delivery schedules were evaluated for the C-141 program. Thus, increased understanding of production scheduling can lead to higher quality decisions about airframe programs.

The model can also be used for particular tasks in program management. The contractor's proposed costs can be compared to those forecasted by the model. The model's forecast can serve as a basis for negotiations and for understanding why a contractor proposes a particular
production rate.

There are some areas in which flaws in the model may be important. One is the lack of hiring and firing costs. In Chapter III, we have seen that in quarters sixteen through nineteen, the model predicts a much lower level of labor use than what was actually used. Introduction of these costs in the model probably would have provided more accurate estimates. The second is the incomplete interaction among the batches in the lots. The model permits work on a batch neither to start later than the lot release date nor to end sooner than the delivery date. The average number of airframes in the process at a particular time, V, is completely determined by the lot release date and the delivery schedule. If starting late or ending early could affect V, then from the point of view of the program, they may be attractive. Of course, more and better data might help in more accurate estimation.

2. Cost Estimation Technique and Policy Options

This research has provided a significantly new approach for estimating the variable cost of an airframe production program. Traditionally, cost estimation for military airframe programs has ignored important aspects of economic theory. The progress function has been the dominant analytical tool used in relating production quantities to airframe costs. It has only been recently
that cost researchers have attempted to integrate economic theory with traditional learning curve analysis. This research considers a dynamic cost function where the production process is modelled taking into account the effects of both learning and production rate. The model is solved to obtain an estimable cost relation. The parameters in this cost relation are then estimated using historical production data from the C-141 airframe production program under the restriction that the errors involved in the process are non-negative. The uniqueness of this research lies in the fact that this provides a minimum cost frontier which is consistent with economic theory, whereas previous cost estimation techniques derived for airframe production programs yielded "average" cost curves that violated the very definition of the cost function.

It was noted in Chapter IV that delivery schedules can have quite an impact on program costs. Even very small changes in the production schedule have an impact on the timing and magnitude of program costs. The relationship between program cost and delivery schedules may be straightforward for a single lot, but it can become complicated in the multiple lot situation as there is interaction among lots, so that one lots' delivery schedule affects the requirements of another. With the development of the continuous time model of Womer and Gulledge (1983)
and the model presented in this research, the cost impact of alternative schedules can be investigated, as in the case of five alternative schedules discussed in Chapter IV. When a delivery schedule forces a higher production rate than planned, increased cost results. This was demonstrated in the second sensitivity analysis where delivery of the last six airframes was advanced one quarter. Due to this schedule compression, the time to work on these six airframes decreased and production rate increased. The result was a decrease in efficiency and an increase in cost.

The most important result regarding delivery schedules is discussed in the first sensitivity analysis. It is shown through various alternative schedules that an alternative procurement policy with an equal number of deliveries for more time periods results in increased efficiency and a lower total program cost. This has important implications to the government. For any multiyear airframe production program (or any other weapon system) the government can adopt an even procurement policy and thereby achieve significant cost savings.

3. **Future Research**

This study is not the last word on airframe production planning and cost estimation, but it represents one more step in our understanding of the factors and forces that
determine costs of a production program. In this section, we list possible extensions and areas of future research, not necessarily to be applied to this model, but rather general ideas on future modelling efforts of similar situations.

There are two areas in which the model might be enhanced. First, the model should be expanded to include hiring and firing costs. This will tend to slow down and smooth out the model's reaction to delivery changes. It is important to know the effect of hiring and firing costs on low production rates or production gaps. Furthermore, the loss of learning which may occur from such a production gap needs to be modelled. Second, the model can be extended to include multiple product production functions.

A second area of further research is the application of the model to other products. Tanks and ships would be appropriate items for this type of modelling.

A third area in which more work needs to be done is the area of data consolidation. With more appropriate data the model's performance would be enhanced.

Furthermore, it is necessary to analyze contractor behavior when developing cost models. Models which regard cost as mechanically related to other variables are destined to have problems in explaining real world data.
## THE C-14 DATA BASE

<p>| Column 1 | Column 2 | Column 3 | Column 4 | Column 5 | Column 6 | Column 7 | Column 8 | Column 9 | Column 10 | Column 11 | Column 12 | Column 13 | Column 14 | Column 15 | Column 16 | Column 17 | Column 18 | Column 19 | Column 20 | Column 21 | Column 22 | Column 23 | Column 24 | Column 25 | Column 26 | Column 27 | Column 28 | Column 29 | Column 30 | Column 31 | Column 32 | Column 33 | Column 34 | Column 35 | Column 36 | Column 37 | Column 38 | Column 39 | Column 40 | Column 41 | Column 42 | Column 43 | Column 44 | Column 45 | Column 46 | Column 47 | Column 48 | Column 49 | Column 50 | Column 51 | Column 52 | Column 53 | Column 54 | Column 55 | Column 56 | Column 57 | Column 58 | Column 59 | Column 60 | Column 61 | Column 62 | Column 63 | Column 64 | Column 65 | Column 66 | Column 67 | Column 68 | Column 69 | Column 70 | Column 71 | Column 72 | Column 73 | Column 74 | Column 75 | Column 76 | Column 77 | Column 78 | Column 79 | Column 80 | Column 81 | Column 82 | Column 83 | Column 84 | Column 85 | Column 86 | Column 87 | Column 88 | Column 89 | Column 90 | Column 91 | Column 92 | Column 93 | Column 94 | Column 95 | Column 96 | Column 97 | Column 98 | Column 99 | Column 100 | Column 101 | Column 102 | Column 103 | Column 104 | Column 105 | Column 106 | Column 107 | Column 108 | Column 109 | Column 110 | Column 111 | Column 112 | Column 113 | Column 114 | Column 115 | Column 116 | Column 117 | Column 118 | Column 119 | Column 120 |</p>
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REFERENCES


VITA

Banani Dhar was born on September 1, 1949 in Shillong (INDIA). She obtained her Master of Science (Mathematics) degree in 1972 from Delhi University, INDIA. She worked as Research Associate in CARE-Delhi from 1974 to 1975. She worked as Lecturer of Mathematics in St. Anthony's College, Shillong from 1976 to 1980. In 1981, she joined the Louisiana State University at Baton Rouge. She received a Master of Science in Quantitative Business Analysis in 1983 from LSU. She is now a candidate for the Degree of Doctor of Philosophy in the Department of Business Administration (QBA) at LSU.

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Major Field: Business Administration (Quantitative Business Analysis)

Title of Dissertation: Estimation of Nonlinear Frontier Functions and Efficiency Measurements for Airframe Production Programs

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