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On Braids, Branched Covers and Transverse Invariants

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ON BRAIDS, BRANCHED COVERS AND TRANSVERSE INVARIANTS

A Dissertation

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Abstract

In this work, we present a brief survey of knot theory supported by contact 3-manifolds. We focus on transverse knots and explore different ways of studying transverse knots. We define a new family of transverse invariants, this is accomplished by considering n -fold cyclic branched covers branched along a transverse knot and we then extend the definition of the BRAID invariant t defined in [4] to the lift of the transverse knot. We call the new invariant the lift of the BRAID invariant and denote it by t_n .

We then go on to show that t_n satisfies a comultiplication formula and use this result to prove a vanishing theorem for t_n . We also re-prove a previously known result regarding the n -fold branched covers branched along stabilized transverse knot. We use this result to prove another vanishing result for t_n .

Chapter 1

Introduction

This dissertation will explore questions in low-dimensional topology, contact topology, and knot theory. The main problem in contact geometry is the classification of contact structures on a 3-manifold. One approach to studying the set of contact structures on a given 3-manifold is through the study of submanifolds that respect a given contact structure. The study of these submanifolds has become interesting in its own right and has gained the attention of low-dimensional topologists. A construction that has also played an important role in studying contact structure on 3-manifolds is branched coverings. Specifically, we will focus on studying transverse knots by defining a family of invariants with the aid of n -fold cyclic branched cover.

A *contact structure* ξ on an oriented 3-manifold Y is a 2-plane field satisfying a certain non-integrability condition. Two natural ways for a one dimensional submanifold of a contact 3-manifold to respect the contact structure is for the tangents of the submanifold to lie entirely in the contact planes in which case it is called a *Legendrian knot* or for the tangents to be transverse to the contact planes and the submanifold is called a *transverse knot*.

Legendrian knots come equipped with classical invariants tb and rot while transverse knots come equipped with sl . Knot types whose Legendrian or transverse representatives are classified by classical invariants are said to be *Legendrian or transversely simple*. There are few knot types that are known to be simple [9, 14], but most knot types appear to be non-simple. Therefore, a central question in

contact geometry is the development and understanding of non-classical invariants that are capable of distinguishing and classifying Legendrian and transverse knots.

We will pay special attention to transverse knots in this dissertation, due to the following description. Transverse knots and links can be represented by closed braids. If we consider the contact manifold (S^3, ξ_{std}) described in Chapter 2, then any closed braid around z -axis can be made transverse to the contact planes. Bennequin proved that any transverse link in (S^3, ξ_{std}) is transversely isotopic to a closed braid [5].

A natural construction in topology is branched covers, so it is only natural to consider branched covers of contact manifolds branched along a transverse knot. If $K \subset (Y, \xi)$ is a transverse knot or link and $p : \Sigma^n(K) \rightarrow Y$ is a n -fold cyclic branched covering with branch-locus K , then ξ naturally extends to a contact structure on $\Sigma^n(K)$. The contact manifold obtained by this process is denoted by $(\Sigma^n(K), \xi_n(K))$. Mathematicians have used this construction to develop tools to distinguish and classify transverse knots.

In [18], Harvey, Kawamuro, and Plamenevskaya studied contact manifolds that arise from cyclic branched covers with transverse knots as branch locus in (S^3, ξ_{std}) . The authors of that paper were interested in the following question:

Suppose that transverse knots K_1, K_2 are smoothly isotopic, and $sl(K_1) = sl(K_2)$. Fix $p \geq 2$. Are p -fold cyclic covers branched over K_1 and K_2 contactomorphic?

The authors of [18] were focused on studying the contact manifold $(\Sigma^n(K), \xi_n(K))$ in order to extract information about K . They were unable to find an example of transversely non-isotopic knots with non-contactomorphic branched covers. Furthermore, they were able to find several examples of smoothly isotopic transverse knots with the same self-linking number, with contactomorphic cyclic branch cov-

ers. The authors were able to show that the branched cyclic cover of all examples of Birman-Menasco [6, 7] were contactomorphic and also found that many of the examples of [23] are also contactomorphic. The authors had an excellent idea of studying transverse knots via n -fold cyclic branched covers. In this thesis, we will continue to explore the relationship between transverse knots and n -fold cyclic cover, but we will be interested in a different aspect.

Another approach taken by researchers has been to define invariants for transverse knots that take values in knot Floer Homology. Heegaard Floer homologies of closed 3-manifolds Y were introduced by Ozvath and Szabo. A refinement of the Heegaard Floer homologies for knots was introduced by Ozsvath and Szabo [27] and independently by Rasmussen [30]. The refinement to knots is known as knot Floer homology.

The first invariant of this type was defined by Ozsvath, Szabo and Thurston in [26]. This invariant is denoted by λ and is an invariant of Legendrian links in S^3 with the standard contact structure and takes values in the minus hat version of Knot Floer homology. The invariant λ is defined by special diagrams known as grid diagrams and is therefore combinatorial in nature. The authors were also able to define an invariant for transverse links from λ , which they denoted by θ . The invariants λ and θ are known as the GRID invariants.

The next invariant was defined by Lisca, Ozsvath, Stipsicz, and Szabo in [21]. This invariant is denoted by \mathcal{L} and is defined for Legendrian knots in arbitrary contact 3-manifolds. The invariant \mathcal{L} was defined by the use of open book decompositions and so it is related to the geometry of the Legendrian and transverse knot. This invariant also takes values in the minus version of knot Floer homology. The authors were also able to define an invariant for transverse knots \mathcal{T} from \mathcal{L} . The invariants \mathcal{L} and \mathcal{T} are known as the LOSS invariants.

Lastly, Baldwin, Vela-Vick and Vértesi in [4] defined the BRAID invariant denoted by t . This invariant was defined for any transverse knot in arbitrary 3-manifold and takes values in knot Floer homology. This invariant depends heavily on the fact that transverse knots can be thought of as braids braided with respect to an open book decomposition. The authors used this fact to construct special diagrams which they called BRAID diagram in order to define t . The BRAID invariant was used to prove that the GRID and LOSS invariants are equivalent for Legendrian and transverse knots in S^3 with the standard contact structure.

The approach of defining invariants that take values in knot Floer homology has proved to be very fruitful. Both the GRID and LOSS invariant have been successful in distinguishing new families of transversely non-simple knot types [23, 34, 3, 25].

The aim of this dissertation is to combine the idea of studying transverse knot through the construction of n -fold cyclic branched cover and defining an invariant similar in flavor to the ones described above. We will not just be defining one invariant, but actually a whole family of invariants. By combining these two ideas we will be extracting more information from both ideas. Specifically, we are interested in the following aspect of n -fold cyclic branched covers:

Suppose that K is a transverse knot in (S^3, ξ_{std}) . Can we obtain information about K from the lifted knot \tilde{K} in $(\Sigma^n(K), \xi_n)$?

Let K be a transverse knot in (S^3, ξ_{std}) , braided with respect to some open book decomposition (B, π) compatible with (S^3, ξ_{std}) . The open book decomposition for the contact manifold $(\Sigma^n(K), \xi_n)$ is obtain by branching the open book decomposition (B, π) about the intersections of its pages with K . From this construction we get that \tilde{K} is braided with respect to the new open book decomposition $(\tilde{B}, \tilde{\pi})$. Furthermore, we will see that if \mathcal{H} is BRAID diagram obtained from a pointed open book encoding K , then the lift $\tilde{\mathcal{H}}$ is a BRAID diagram encoding \tilde{K} . Since

we are able to construct a BRAID diagram $\tilde{\mathcal{H}}$ encoding \tilde{K} we can define the lift of the BRAID invariant t_n .

We show that the lift of the BRAID invariant behaves nicely under connect sums. We will prove the following property.

Theorem 1.1. $t_n(K_1) = 0$ or $t_n(K_2) = 0$ if and only if $t_n(K_1\#K_2) = 0$.

We prove this in two parts. In one direction we modify an argument from Baldwin [2], in order to define a comultiplication map which sends $t_n(K_1\#K_2)$ to $t_n(K_1) \otimes t_n(K_2)$. The second part of the proof is a diagrammatic analysis argument, which we use to define a map which sends $t_n(K_1) \otimes t_n(K_2)$ to $t_n(K_1\#K_2) = 0$.

Lastly, we will re-prove the following known result by using an elementary argument

Theorem 1.2. [18] *If K_{stab} is a negative braid stabilization of K , then $(\Sigma^n(K_{stab}) \setminus \widetilde{K_{stab}, \xi_n})$ is overtwisted.*

The following is a direct result of the previously known Theorem 1.2 and properties of the BRAID invariant.

Corollary 1.3. *If K_{stab} is a negative braid stabilization of K , then $t_n(K_{stab}) = 0$.*

This thesis is organized as follows: Chapter 2 reviews topics, definitions and theory that will be essential for this dissertation. In Chapter 3 we review the BRAID invariant and then define the lift the BRAID invariant. Lastly, in Chapter 4 we prove several useful properties of the lift of the BRAID invariant.

Chapter 2

Preliminaries

2.1 Contact Structures

In this section we will introduce contact structures and related terminology. We will also explore a few important results about contact structures. The goal of this section is to give an overview of contact geometry, for a more complete description of the theory the reader is referred to [10, 15].

Definition 2.1. *An oriented 2-plane field ξ on a 3-manifold Y is called a contact structure if there exists a 1-form $\alpha \wedge d\alpha \neq 0$. The pair (Y, ξ) is called a contact manifold.*

The condition $\alpha \wedge d\alpha \neq 0$ is known as a totally non-integrability condition. This condition ensures that there is no embedded surface in Y which is tangent to ξ on any open neighborhood. We will now explore three basic examples of contact manifolds.

Example 2.2. *Consider the 3-manifold $Y = \mathbb{R}^3$ with standard Cartesian coordinates (x, y, z) and the 1-form*

$$\alpha = dz + xdy$$

with a simple computation we can confirm that $\alpha \wedge d\alpha \neq 0$. Thus α is a contact form and $\xi_{std} = \ker(\alpha) = \ker(dz + xdy)$ is a contact structure on \mathbb{R}^3 .

Remark 2.3. *Example 2.2 is commonly referred to as the standard contact structure on \mathbb{R}^3 . At any point in the yz -plane ξ is horizontal and moving along a ray perpendicular to the xy -plane the plane field will always be tangent to this ray and rotate by $\pi/2$ in a left handed manner as move along the ray.*

Example 2.4. Consider the 3-manifold $Y = \mathbb{R}^3$ with cylindrical coordinates (r, θ, z) and the 1-form

$$\alpha = dz + r^2 d\theta.$$

Again with a simple computation we can check that $\alpha \wedge d\alpha \neq 0$. So $\xi_{sym} = \ker(\alpha) = \ker(dz + r^2 d\theta)$ is a contact structure on M .

Remark 2.5. In this contact manifold as you move along any ray perpendicular to the z -axis the contact planes twist clockwise. At the z -axis the contact planes are horizontal.

Example 2.6. Consider the 3-manifold $Y = \mathbb{R}^3$ with cylindrical coordinates (r, θ, z) and the 1-form

$$\alpha = \cos(r)dz + r \sin(r)d\theta.$$

Again with a simple computation we can check that $\alpha \wedge d\alpha \neq 0$. So $\xi_{OT} = \ker(\alpha) = \ker(\cos(r)dz + r \sin(r)d\theta)$ is a contact structure on M .

Remark 2.7. In this case we see that ξ_{OT} is horizontal along the z -axis and as you move out on any ray perpendicular to the z -axis the planes will twist in a clockwise manner, but the planes will make an infinitely many full twists as r goes towards infinity.

Contact structures on a 3-manifold fall into two disjoint classes: tight and overtwisted. A contact manifold (Y, ξ) is *overtwisted* if there exists an embedded disk $D \subset Y$ such that $T_p D = \xi_p$ for every $p \in \partial D$. On the other hand the contact manifold is said to be *tight* if it is not overtwisted. In general, topologists are interested in classification questions, so it is only natural to classify contact structures. We have the following notion of contact manifolds being equivalent.

Definition 2.8. *Two contact structures ξ_0 and ξ_1 on a manifold Y are called contactomorphic if there is a diffeomorphism $f : Y \rightarrow Y$ such that f send ξ_0 to ξ_1 :*

$$f_*(\xi_0) = \xi_1.$$

The contact structures in Example 2.2 and Example 2.4 are contactomorphic. The contact structure defined in Example 2.4 is not contactomorphic to the contact structures defined in Example 2.2 and Example 2.4. An explicit contactomorphism between Example 2.2 and Example 2.4 can be found in [15].

The following result illustrates that the theory of contact geometry is not interesting locally. Therefore, contact geometry is interested in global questions.

Theorem 2.9. *(Darboux's Theorem) Let (Y, ξ) be any contact 3-manifold and p any point in Y . Then there exists neighborhoods N of p in Y , and U of $(0, 0, 0)$ in \mathbb{R}^3 and a contactomorphism*

$$f : (N, \xi|_N \rightarrow (U, \xi_1|_U).$$

Darboux's theorem essentially says that every contact structure behaves the same near a point. Therefore, contact structures do not have interesting local structures, any interesting behavior in contact manifold will occur in terms of the global topology of the manifold.

2.2 Knot Theory and Contact Structures

The study of knots in 3-manifolds is a very important area of research for mathematicians. In this section, we will introduce a few basic definitions and results for knots in a contact 3-manifold which we will need in this paper. This section is not meant to be a complete survey on the subject, for a detail description of knot theory supported in a contact 3-manifold, see [11, 15]. We will also assume that the reader has some knowledge of topological knots and braids, for a detail treatment of these topics see [31].

There are two natural ways that knots can respect the geometry imposed by contact structures, therefore, there are two classes of knots: the Legendrian and transverse.

Definition 2.10. A Legendrian knot L in a contact 3-manifold (Y, ξ) is an embedded S^1 that is always tangent to ξ :

$$T_x L \in \xi_x, \quad x \in L.$$

Two Legendrian knots in $(\mathbb{R}^3, \xi_{std})$ are *Legendrian isotopic* if there is an isotopy through Legendrian knots between the two knots. Legendrian knots come equipped with two classical numerical invariants, the *Thurston-Bennequin number* tb and the *rotation number* r .

Definition 2.11. A transverse knot K in a contact manifold (Y, ξ) is an embedded S^1 that is always transverse to ξ :

$$T_x K \oplus \xi_x = T_x Y, \quad x \in K.$$

We classify transverse knots up to transverse isotopy. We say that two transverse knots are *transversely isotopic* if there is an isotopy taking one knot to the other while remaining transverse to the contact planes. Transverse knots come equipped with one classical invariant, the *self-linking number* sl .

A Legendrian knot L can be perturbed to a canonical (up to transverse isotopy) transverse link K called the *transverse pushoff*. Legendrian isotopic links give rise to transversely isotopic pushoffs. Also, any transverse link K will be transversely isotopic to the pushoff of a Legendrian link L , we say that L is a *Legendrian approximation* of K .

There is a local operation on Legendrian links called a *negative Legendrian stabilization*. The transverse pushoff a Legendrian link is transversely isotopic to the

pushoff of its negative stabilization. On the other hand, any two Legendrian approximations of a transverse link are Legendrian isotopic after a series of negative Legendrian stabilizations. For a complete description of the relationship between transverse and Legendrian knots, see [15]

The relationship between Legendrian links and transverse links gives us a way of studying transverse links by studying Legendrian links. Suppose that I is an invariant of Legendrian links (up to Legendrian isotopy) and I remains unchanged by negative stabilizations, then I gives rise to another invariant I' for transverse links.

A topological knot type is *Legendrian (resp. transversely) simple* if all Legendrian (transverse) knots in its class are determined up to Legendrian (transverse) isotopy by their classical invariants. Some knot types which are known to be Legendrian and transversely simple include the unknot [9], torus knots [14], and the figure eight knot [14]. The classical invariants for Legendrian and transverse knots are not complete invariants, for more information on non-simplicity we refer the reader to [6, 13].

2.3 Open Book Decomposition

Open book decompositions have become a primary tool for studying contact manifolds due to Giroux's correspondence. In this section we will present the basics of open book decompositions and conclude by stating Giroux's correspondence between open book decompositions and contact structures. For a more detailed description of open book decompositions and their role in studying contact 3-manifolds the reader is referred to [12].

Definition 2.12. *An open book decomposition of a contact 3-manifold (Y, ξ) is a pair (B, π) where*

1. B is an oriented, fibered, transverse link called the binding and
2. $\pi : (Y - B) \rightarrow S^1$ is a fibration of the complement of B by oriented surfaces S such that $\partial S = B$.

The transverse link B is called the binding of the open book (B, π) , while each of the fiber surfaces $S_\theta = \pi^{-1}(\theta)$ are called pages.

We can also abstractly define an open book decomposition in the following way.

Definition 2.13. *An abstract open book is a pair (S, ϕ) where*

1. S is an oriented compact surface with boundary and
2. $\phi : S \rightarrow S$ is a diffeomorphism such that ϕ is the identity in a neighborhood of ∂S . The map ϕ is called the monodromy.

We will now define an important operation on abstract open book decompositions that will play an important role in the Giroux's correspondence.

Definition 2.14. *A positive (negative) stabilization of an abstract open book (S, ϕ) is the open book*

1. with page $S' = S \cup 1\text{-handle}$ and
2. monodromy $\phi' = \phi \circ \tau_c$ where τ_c is a right- (left-)handed Dehn twist along a curve c in S' that intersects the co-core of the 1-handle exactly one time.

An open book decomposition is said to be *compatible* with a contact structure ξ if, after an isotopy of the contact structure, there exists a contact form α for ξ such that;

1. $\alpha(v) > 0$ for each nonzero oriented tangent vector v to B , and
2. $d\alpha$ restricts to a positive area form on each page of the open book.

Thurston and Winkelnkemper [33], proved that given an open book decomposition of a 3-manifold Y , one can produce a contact structure on Y compatible with the given open book decomposition. Giroux in [16], proved that two contact structures compatible with an open book decomposition must be isotopic as contact structures. The following is known as Giroux's correspondence.

Theorem 2.15. *Let Y be a closed oriented 3-manifold. Then there is a one to one correspondence between the set of oriented contact structures on Y up to isotopy and the set of open book decompositions of Y up to positive stabilization.*

The following example will play an important role throughout the dissertation.

Example 2.16. *Let (U, π_U) be the open book for S^3 , where U is the unknot and*

$$\pi_U : S^3 \setminus U \rightarrow S^1 : (r_1, \theta_1, r_2, \theta_2) \rightarrow \theta_1.$$

(We can think of S^3 as the unit sphere in \mathbb{C}^2 .) This open book is compatible with the standard contact structure $\xi_{std} = \ker(r_1^2 d\theta_1 + r_2^2 d\theta_2)$.

2.4 Transverse Knots and Braids

We will now shift our attention to transverse knots. In this section we will explore the relationship between transverse knots and braids with the aim of thinking of transverse knots as being braided with respect to a specific open book decomposition of (S^3, ξ_{std}) . This will be a key idea in order to construct pointed open book decompositions and BRAID diagrams. For a more complete description of the relationship between transverse knots and braids we refer the reader to [4, 11, 18, 29].

Recall that a *closed braid* is a knot or link in S^3 . In order to explore the connection between transverse knots and closed braids we will consider (S^3, ξ_{std}) . Bennequin in [5], proved the following:

Theorem 2.17. *Any transverse knot in (S^3, ξ_{std}) is transversely isotopic to a closed braid.*

It is clear that any closed braid in (S^3, ξ_{std}) with braid axis the z -axis, can be made transverse to the contact planes.

By fixing k points p_i , in a disk D^2 , an n -braid is an embedding of k arcs $\hat{\phi}_i : [0, 1] \rightarrow D^2 \times [0, 1]$ so that $\hat{\phi}_i(t) \in D^2 \times t$ and the endpoints of $\hat{\phi}$ corresponding to 0 and 1 as sets get map to $\{p_i\}$ in $D^2 \times 0$ or $D^2 \times 1$. The set of all n -braids B_n form a group. It is clear that the group is generated by σ_i , for $i = 1, \dots, k - 1$, where σ_i is the n braid with the i and $i + 1$ strands interchanging from a right handed manner and the rest of the strands are unchanged. The group B_k includes in B_{k_1} . Given a braid b in B_k the positive stabilization of b is $b\sigma_k$ in B_{k+1} .

Recall that the usual Markov Theorem states that two topological knots or links are equivalent to braids modulo conjugation and positive or negative braid stabilization. Orevkov and Shevchishin [24] and independently Wrinkle [35], proved the following transverse Markov theorem,

Theorem 2.18. *Two braids represent the same transverse knot if and only if they are related by positive stabilization and conjugation in the braid group.*

Remark 2.19. *The stabilization of a transverse link represented by a braid is equivalent to the negative braid stabilization, adding an extra strand and a negative kink to the braid.*

We will follow the convention and notation found in [29]. If K is a transverse knot in (Y, ξ) and K_{stab} is the stabilization of K , then

$$sl(L_{stab}) = sl(L) - 2.$$

Positive braid stabilization does not change the transverse type of the knot or link.

Suppose (B, π) is an open book decomposition compatible with the contact structure (Y, ξ) . A transverse link K in (Y, ξ) is said to be *braided with respect to* (B, π) if K is positively transverse to the pages of (B, π) . Two such braids are said

to be transversely isotopic with respect to (B, π) if they are transversely isotopic through links which are braided with respect to (B, π) . In [28], Pavelescu proved the following generalization of Bennequin’s theorem.

Theorem 2.20. *Suppose (B, π) is an open book compatible with (Y, ξ) . Then every transverse link in (Y, ξ) is transversely isotopic to a braid with respect to (B, π) .*

If K is braided with respect to (B, π) , then K must intersect each page of our open book decomposition the same number points. If K intersects each page at n points, then we say that K is an n -braid. Pavelescu defined a generalization of the standard positive Markov stabilization for braids in S^3 . This new operation is called a *positive Markov stabilization* and it replaces K with an $(n + 1)$ -braid K^+ with respect to (B, π) which is transversely isotopic to K .

Pavelescu in [28] also proved the following generalization of Wrinkel’s transverse Markov Theorem.

Theorem 2.21. *Suppose K_1 and K_2 are braids with respect to an open book (B, π) compatible with (Y, ξ) . Then K_1 and K_2 are transversely isotopic if and only if there exists positive Markov stabilizations K_1^+ and K_2^+ around the binding components of (B, π) such that K_1^+ and K_2^+ are transversely isotopic with respect to (B, π) .*

2.5 Pointed Open Book Encoding K

We will now describe how to encode braids in terms of abstract open books. The following description and construction can be found in [4]. Let (B, π) is an open book compatible with (Y, ξ) and K is transverse link in (Y, ξ) which is an n -braid with respect to (B, π) . Let (S, φ) be an abstract open book corresponding to (B, π) and let p_1, \dots, p_n be distinct points where K intersects the interior of S . Then φ is isotopic to a diffeomorphism $\hat{\varphi}$ of the pair $(S, \{p_1, \dots, p_n\})$ which fixes ∂S point-wise, such that the braid K corresponds to $(\{p_1, \dots, p_n\} \times [0, 1]) / \sim_{\hat{\varphi}}$ in

the identification of Y with $(S \times [0, 1]) / \sim_{\hat{\varphi}}$. We say that the braid K is encoded by the *pointed open book* $(S, \{p_1, \dots, p_n\}, \hat{\varphi})$.

Since we can encode braids by pointed open books we will like to describe what it means for two braids to be transversely isotopic with respect to a given open book. Let (S, φ_1) and (S, φ_2) are two abstract open books corresponding to (B, π) and K_1 and K_2 are braids with respect to (B, π) , encoded by the pointed open books $(S, \{p_1^1, \dots, p_n^1\}, \hat{\varphi}_1)$ and $(S, \{p_1^2, \dots, p_n^2\}, \hat{\varphi}_2)$

Definition 2.22. *K_1 and K_2 are transversely isotopic with respect to (B, π) if and only if $\hat{\varphi}_2$ is isotopic $h \circ \hat{\varphi}_1 \circ h^{-1}$ for some diffeomorphism h which sends $\{p_1^1, \dots, p_n^1\}$ to $\{p_1^2, \dots, p_n^2\}$.*

Remark 2.23. *It is possible to describe a positive Markov stabilization by a pointed open book, for more details [4].*

2.6 BRAID Diagram

In this section we will construct a diagram for the transverse knot K in (Y, ξ) from a pointed open book decomposition encoding the transverse knot K . Again this description and construction of BRAID diagrams can be found in [4]. Let $(S, \{p_1, \dots, p_n\}, \hat{\varphi})$ encode the transverse knot K . A *basis* for $(S, \{p_1, \dots, p_n\})$ is a set $\{a_1, \dots, a_k\}$ of properly embedded arcs in S such that $S - \{a_1, \dots, a_k\}$ is a union of n disks each of which contains exactly one point in $\{p_1, \dots, p_n\}$. Let $\{b_1, \dots, b_k\}$ be another such basis, where each b_i is obtained from a_i by shifting the endpoints of a_i slightly in the direction of the orientation on ∂S and isotoping to ensure that there is a single transverse intersection between b_i and a_i . Let Σ denote the surface $S_{1/2} \cup -S_0$. For $i = 1, \dots, k$, let $\alpha_i = a_i \times \{1/2\} \cup a_i \times \{0\}$ and $\beta_i = b_i \times \{1/2\} \cup \hat{\varphi}(b_i) \times \{0\}$, and let $w_i = p_i \times \{0\}$ and $z_i = p_i \times \{1/2\}$.

Definition 2.24. Let $\alpha = \{\alpha_1, \dots, \alpha_k\}$, $\beta = \{\beta_1, \dots, \beta_k\}$, $\mathbf{z}_K = \{z_1, \dots, z_k\}$ and $\mathbf{w}_K = \{w_1, \dots, w_k\}$. Then $(\Sigma, \alpha, \beta, \mathbf{w}_K, \mathbf{z}_K)$ is a BRAID diagram for $K \subset Y$.

2.7 Knot Floer Homology

The goal of this section is to give the reader a brief overview of knot Floer homology. Heegaard Floer homologies of closed 3-manifolds Y were introduced by Ozvath and Szabo. A refinement of the Heegaard floer homologies for knots were introduced by Ozsvath and Szabo [27] and independently by Rasmussen [30]. The refinement to knots is known as knot Floer homology. We will specifically follow the notation and description in [4].

Definition 2.25. A multi-pointed Heegaard diagram for an oriented link $L \subset Y$ is an ordered tuple $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{z}, \mathbf{w})$, where

1. Σ is a Riemann surface of genus $g \geq 0$, splitting Y into two handlebodies U_0 and U_1 , with Σ oriented as the boundary of U_0 .
2. $\alpha = \{\alpha_1, \dots, \alpha_{g+k-1}\}$ is a set of pairwise disjoint, simple closed curves on Σ such that each α_i bounds a properly embedded disk D_i^α in U_0 , and the complement of these disks in U_0 is a union of $g + k$ balls $B_1^\alpha, \dots, B_{g+k}^\alpha$,
3. $\beta = \{\beta_1, \dots, \beta_{g+k-1}\}$ with similar properties, bounding disks D_i^β in U_1 , such that their complement is a union of $g + k$ balls $B_1^\beta, \dots, B_{g+k}^\beta$.
4. Two collections of points on Σ , denote $\mathbf{w} = \{w_1, \dots, w_k\}$ and $\mathbf{z} = \{z_1, \dots, z_k\}$, all disjoint from each other and from the α and β curves.

A multi-pointed Heegaard diagram encodes both the manifold as well as the the link L . Specifically, (Σ, α, β) specifies a Heegaard diagram for Y and the link L is obtained by simply connecting the basepoints z to w -basepoints and w to z by

properly embedded arcs in the α and β handlebodies respectively which do not intersect the compression disk specified by the α and β curves.

Remark 2.26. *The diagram in Definition 2.24 is simply a specific multi-pointed Heegaard diagram for a transverse knot K in (Y, ξ) . It follows that*

$$\mathcal{H} = (\Sigma, \beta, \alpha, \mathbf{w}_K, \mathbf{z}_K)$$

is a BRAID diagram for $K \subset -Y$.

Furthermore, a *multi-pointed Heegaard triple* $\mathcal{H} = (\Sigma, \alpha, \beta, \gamma, \mathbf{z}, \mathbf{w})$ is a collection of three sets of curves α, β, γ such that each pair (α, β) , (α, γ) , and (β, γ) determines multi-pointed Heegaard diagrams. We will return to this topic later in this section.

From a multi-pointed Heegaard diagram for $L \subset Y$, we can obtain a chain complex $\text{CFK}^-(\mathcal{H})$. In the symmetric product $\text{Sym}^{g+k-1}(\Sigma)$ of the Heegaard surface, the multi-curves α, β determine $(g+k-1)$ -dimensional tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+k-1}$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+k-1}$. The knot Floer complex $\text{CFK}^-(\mathcal{H})$ is the free $\mathbb{F}[U_{w_1}, \dots, U_{w_k}]$ -module generated by elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. For each Whitney disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we let $n_{z_i}(\phi)$ and $n_{w_i}(\phi)$ denote the local multiplicity of ϕ at z_i and w_i respectively, We denote by $n_{\mathbf{z}}(\phi)$ and $n_{\mathbf{w}}(\phi)$ the sums of the local multiplicities at all of the \mathbf{z} and \mathbf{w} -basepoints respectively. The chain group has two gradings, the Maslov (homological) grading $M(\mathbf{x})$ and Alexander grading $A(\mathbf{x})$, which are determined up to an overall shift by the formulas

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2n_{\mathbf{w}}(\phi) \quad \text{and} \quad A(\mathbf{x}) - A(\mathbf{y}) = n_{\mathbf{z}}(\phi) - n_{\mathbf{w}}(\phi),$$

where $\phi = \pi_2(\mathbf{x}, \mathbf{y})$ and $\mu(\phi)$ denote the Maslov index of the Whitney disk ϕ . The differential on the complex $\text{CFK}^-(\mathcal{H})$ is defined by

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \\ n_{\mathbf{z}}(\phi)=0, \\ \mu(\phi)=1}} \# \widehat{\mathcal{M}}(\phi) \cdot U_{w_1}^{n_{w_1}(\phi)} \cdots U_{w_k}^{n_{w_k}(\phi)} \cdot \mathbf{y},$$

The minus version of knot Floer homology is the homology of the complex $(\text{CFK}^-(\mathcal{H}), \partial^-)$:

$$\text{HFK}_*^-(K) := \text{H}_*(\text{CFK}^-(\mathcal{H}), \partial^-).$$

When w_i and w_j are basepoints corresponding to the same component of the link K , then their associated formal variables U_{w_i} and U_{w_j} act identically on $\text{HFK}^-(K)$. Choose for each component K_i of the link K , a formal variable U_i associated to some basepoint for K_i . Then the knot Floer homology $\text{HFK}^-(K)$ is an invariant of the link $K \subset S^3$, which is well-defined up to graded $\mathbb{F}[U_1, \dots, U_l]$ -module isomorphism.

We will also describe two other variations of associated homology theories with which one commonly works. The first is known as the hat version of knot Floer homology and is obtained as follows. For each component $K_i \in K$, set exactly one of its associated formal variables $U_{w_i} = 0$ and denote the associated differential by $\widehat{\partial}$ on the quotient complex $(\widehat{\text{CFK}}(K), \widehat{\partial})$. It follows that the homology

$$\widehat{\text{HFK}}_*(K) := \text{H}_*(\widehat{\text{CFK}}(\mathcal{H}), \widehat{\partial}),$$

is an invariant of the link K up to \mathbb{F} -module isomorphism. Finally, it is often convenient to work with the further quotient of $(\widehat{\text{CFK}}(\mathcal{H}), \widehat{\partial})$ that is obtained by setting the remaining formal variables $U_{w_j} = 0$. The result is known as the tilde version of knot Floer homology and its complex is denoted by $(\widetilde{\text{CFK}}(\mathcal{H}), \widetilde{\partial})$. The associated homology

$$\widetilde{\text{HFK}}_*(K) := \text{H}_*(\widetilde{\text{CFK}}(\mathcal{H}), \widetilde{\partial})$$

is an invariant of the link K together with the number of z or w -basepoints. The relation of tilde version to the hat version of knot Floer homology is given by

$$\widetilde{\text{HFK}}(K) \cong \widehat{\text{HFK}}(K) \otimes V^{\otimes n}$$

where V_i is a rank 2 vector space supported in bi-gradings $(0, 0)$ and $(-1, -1)$.

Let $\Pi_{\alpha, \beta}$ denote the group of *periodic domains* is the Heegaard diagram $\mathcal{H} = (\Sigma_\alpha, \beta, \mathbf{z}, \mathbf{w})$. Recall that a 2-chain is periodic if its boundary is the union of some number of α and β curves. Let $\Pi_{\alpha, \beta}^0$ denote the subgroup of periodic domains that avoid $\mathbf{w} \cup \mathbf{z}$. As a group $\Pi_{\alpha, \beta}^0$ is isomorphic to $H^2(S^3 \setminus K; \mathbb{Z})$. The Heegaard diagram \mathcal{H} is *admissible* if every domain in $\Pi_{\alpha, \beta}^0$ has both positive and negative multiplicities. The groups $\Pi_{\alpha, \beta, \gamma}$, $\Pi_{\alpha, \beta, \gamma}^0$ of periodic domains and admissible are defined similarly for a triple α, β, γ .

The following construction can be found in [2]. We will modify Definition 2.24 to obtain a multi-pointed Heegaard triple diagram. We do this by constructing another set of disjoint properly embedded arc which we will denote by c_i . We obtain c_i from b_i by changing the arcs b_i via a small isotopy which moves the endpoints of the b_i along ∂S in the direction given by the orientation of ∂S . We will require that both a_i and b_i intersect c_i transversely in one point with a positive sign (where c_i inherits its orientation from b_i). For any two diffeomorphisms g and h , we construct three sets of attaching curves on the Heegaard surface $\Sigma = S_{1/2} \cup -S_0$:

$$\begin{aligned}\alpha_i &= a_i \times \{1/2\} \cup a_i \times \{0\} \\ \beta_i &= b_i \times \{1/2\} \cup g(b_i) \times \{0\} \\ \gamma_i &= c_i \times \{1/2\} \cup h(g(c_i)) \times \{0\}.\end{aligned}$$

Let \mathbf{w}_K and \mathbf{z}_K be the basepoints as defined in Definition 2.24 and

$$\begin{aligned}\alpha &= \{\alpha_1, \dots, \alpha_k\} \\ \beta &= \{\beta_1, \dots, \beta_k\} \\ \gamma &= \{\gamma_1, \dots, \gamma_k\}.\end{aligned}$$

Then $(\Sigma, \alpha, \beta, \gamma, \mathbf{z}_K, \mathbf{w}_K)$ is a multi-pointed Heegaard triple-diagram and can be used to construct a cobordism $X_{\alpha, \beta, \gamma}$ with

$$\partial X_{\alpha, \beta, \gamma} = -Y_{\alpha, \beta} - Y_{\beta, \gamma} + Y_{\alpha, \gamma}$$

where $Y_{\alpha, \beta}$ is the three manifold containing K_1 with Heegaard diagram $(\Sigma, \alpha, \beta, \mathbf{z}_{K_1}, \mathbf{w}_{K_1})$, $Y_{\beta, \gamma}$ is the three manifold containing K_2 with Heegaard diagram $(\Sigma, \beta, \gamma, \mathbf{z}_{K_2}, \mathbf{w}_{K_2})$, and $Y_{\alpha, \gamma}$ is the three manifold containing K_3 with Heegaard diagram $(\Sigma, \alpha, \gamma, \mathbf{z}_{K_3}, \mathbf{w}_{K_3})$.

Such a cobordism induces a chain map

$$\widehat{\text{CFK}}(Y_{\alpha, \beta}) \otimes_{\mathbb{Z}_2} \widehat{\text{CFK}}(Y_{\beta, \gamma}) \rightarrow \widehat{\text{CFK}}(Y_{\alpha, \gamma}).$$

By the description of the Heegaard diagram associated to an open book we have that

$$Y_{\alpha, \beta} = Y_g$$

$$Y_{\beta, \gamma} = Y_h$$

$$Y_{\alpha, \gamma} = Y_{hg}.$$

Thus, we have a chain map

$$m : \widehat{\text{CFK}}(Y_g) \otimes_{\mathbb{Z}_2} \widehat{\text{CFK}}(Y_h) \rightarrow \widehat{\text{CFK}}(Y_{hg}).$$

If the multi-pointed Heegaard triple diagram $(\Sigma, \alpha, \beta, \gamma, \mathbf{z}, \mathbf{w})$ is weakly-admissible then this map is defined on the generators of $\widehat{\text{CFK}}(Y_g) \otimes_{\mathbb{Z}_2} \widehat{\text{CFK}}(Y_h)$ by

$$m(\mathbf{a} \otimes \mathbf{b}) = \sum_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\substack{\phi \in \pi_2(\mathbf{a}, \mathbf{b}, \mathbf{x}), \\ n_z(\phi) = n_w(\phi) = 0, \\ \mu(\phi) = 0}} (\#\mathcal{M}(\phi)) \mathbf{x}$$

In this sum $\pi_2(\mathbf{a}, \mathbf{b}, \mathbf{x})$ is the set of homotopy classes of Whitney triangles connecting \mathbf{a} , \mathbf{b} , and \mathbf{x} ; $\mu(\phi)$ is the expected dimension of the holomorphic representative of ϕ ; $n_z(\phi)$ is the algebraic intersection number of ϕ with the subvariety $\{\mathbf{x}\} \times \text{Sym}^{g+k-2}(\Sigma) \subset \text{Sym}^{g+k-1}(\Sigma)$; and $\mathcal{M}(\phi)$ is the moduli space of the holomorphic representative of ϕ .

We apply the $\text{Hom}_{\mathbb{Z}_2}(-, \mathbb{Z}_2)$ functor to the expression above. If we represent each chain complex diagrammatically by drawing an arrow from x to y whenever y is a term in ∂x or when y is a term in the image of x under the map m , then applying the $\text{Hom}_{\mathbb{Z}_2}(-, \mathbb{Z}_2)$ functor corresponds to reversing the direction of every arrow. Doing so, we obtain a chain map

$$f : \widehat{CFK}(-Y_{hg}) \rightarrow \widehat{CFK}(-Y_g) \otimes_{\mathbb{Z}_2} \widehat{CFK}(-Y_h).$$

An element in $\widehat{CFK}(-Y_{hg})$ is a sum of intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. On such an intersection point the chain map μ is defined by

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, \mathbf{b} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma} \sum_{\substack{\phi \in \pi_2(\mathbf{a}, \mathbf{b}, \mathbf{x}) \\ \mu(\phi)=0 \\ n_z(\phi)=n_w(\phi)=0}} \#\mathcal{M}(\phi)(\mathbf{a} \otimes \mathbf{b})$$

as long as the multi-pointed Heegaard triple-diagram $(\Sigma, \alpha, \beta, \gamma, \mathbf{z}, \mathbf{w})$ is weakly admissible.

2.8 Branched Covers

The goal of this section is to introduce the basics definitions and constructions of branched covers of contact 3-manifolds. We will also give a description of open book decompositions of the 3-manifold that arises from the n -fold cyclic branched cover. For a more detail description of the theory of branched covers of 3-manifolds the reader is referred to [31] and for a more complete description of branched covers and contact 3-manifolds the reader is referred to [8, 18].

Definition 2.27. Let M and N be 3-manifolds. A map $p : M \rightarrow N$ is called a branched covering if there exists a dimension 1 complex K such that $p^{-1}(K)$ is a dimension 1 complex and $p|_{M-p^{-1}(K)}$ is a covering.

One can think of a branched covering as a map between manifolds such that away from the *branched locus* a codimension 2 set, p is a covering.

Branched coverings have been used to study 3-manifolds. The interest in the relationship between 3-manifolds and branched coverings was sparked by the following result by Alexander [1].

Theorem 2.28. Every closed orientable 3-manifold is a branched covering space of S^3 with branch locus a link in S^3 .

This result proved that branched covers is a tool for constructing every 3-manifold. There have been refinements to Alexander's result. One such refinement is due to Hilden and Montesinos, every closed orientable 3-manifold is a 3-fold branched covering branched along a knot [19] and [22]. Furthermore, there have also been restriction to branched covers looking at a fixed branch locus and still obtain all closed oriented 3 manifolds.

A link K is called *universal* if every 3-manifolds is obtained as a branched cover branching along K . In [32], Thurston proved the existence of a universal link. Since then it has been shown that the figure-eight knot, Borromean rings, and Whitehead link and 9_{46} are also universal [20].

Now, we will describe branched covers of contact manifolds. Let K be a transverse knot in (Y, ξ) and branched covering $p : \tilde{Y} := \Sigma^n(K) \rightarrow Y$ with branch locus K . How do we define the lift $\tilde{\xi}$ of ξ ?

We first construct $\Sigma^n(K)$ as normal. Let $\tilde{K} = p^{-1}(K)$. Then $p : (\Sigma^n(K) - \tilde{K}) \rightarrow Y - K$ is a true cover, and thus $\xi = p^*(\xi)$ on $(\Sigma^n(K) - \tilde{K})$. Gonzalo in [17], was

able to extend the contact structure by a careful analysis of the branched covering map near the branch locus K over Y . Therefore, we obtain a contact structure ξ_n on $\Sigma^n(K)$ and we will denote the contact manifold obtain from this construction by $(\Sigma^n(K), \xi_n)$.

2.9 Cyclic Branched Covers and Open Book Decomposition

Consider the open book decomposition in Example 2.16 for S^3 . In this set up we have that the pages of the open book are disk D^2 we will denote a page by $S_t = S \times \{t\}$ in $S \times [0, 1]$. If we have a knot K transversely braided through the pages of our open book, then we construct a pointed open book $(S, \{p_1, \dots, p_k\}, \hat{\phi})$ that encodes K as described in Section 2.4. In this case $S = D^2$, $\{p_1, \dots, p_k\}$ are k points in the interior of the disk, and $\hat{\phi}$ traces out the knot.

Let $p : \Sigma^n(K) \rightarrow S^3$ be a branched covering of S^3 branched along K . In order to construct the open book decomposition of the branched covering, we need to make branched cuts on the pages of our open book along a Seifert surface of K . We make the branched cuts so that traversing the boundary of S in the positive direction crosses the branch cuts in the order c_1, \dots, c_k .

We need to construct the lift of the page \tilde{S} of the open book for $\Sigma^n(K)$. Since S is a surface, we just need to glue n copies of S along the branch cuts to get \tilde{S} . Since we are only interested in cyclic branched covers, a formula is given in [18] for the monodromy of cyclic covers.

Recall that the open book decomposition (D^2, id) is compatible with the contact manifold (S^3, ξ_{std}) . We will now like to construct an open book decomposition compatible with the contact manifold $(\Sigma^n(K), \xi_n(K))$. The following theorem states how to construct such an open book

Theorem 2.29. [8] *Let K be a knot braided transversely through the pages of the open book decomposition (D^2, id) , which supports (S^3, ξ_{std}) . Let (M, ξ) be the*

covering contact manifold obtained by branching over K . The open book constructed as described above supports the contact manifold (M, ξ) .

Chapter 3

The BRAID Invariant in the n -Fold Cyclic Branch Cover

The BRAID invariant was defined by Baldwin, Vela-Vick, and Vertesi in [4]. This invariant was essential in proving that the GRID invariant [23] and the LOSS invariant [21] are equivalent in (S^3, ξ_{std}) . We will first introduce the BRAID invariant with the goal of generalizing the BRAID invariant to the lift of a transverse knots in the n -fold branched cover.

3.1 BRAID Invariant

Let K be a transverse knot in the contact manifold (Y, ξ) with compatible open book (S, ϕ) . Furthermore, let \mathcal{H} be a BRAID diagram for K as stated in Remark 2.26, then the BRAID invariant for K is defined as follows:

Definition 3.1. *Let $\mathbf{x}(\mathcal{H})$ denote the generator of $\text{CFK}^-(\mathcal{H})$ consisting of the intersection points on $S_{1/2}$ between α_i and β_i curves. We define*

$$t(K) := [\mathbf{x}(\mathcal{H})] \in \text{HFK}^-(-Y, K) = \text{HFK}^-(\mathcal{H}).$$

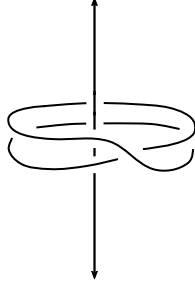
We define $\hat{t}(K) \in \widehat{\text{HFK}}(-Y, K)$ to be the image of $t(K)$ under the natural map $p_* : \text{HFK}^-(\mathcal{H}) \rightarrow \widehat{\text{HFK}}(\mathcal{H})$.

As mentioned above, the BRAID invariant is both equivalent to the LOSS invariant and the GRID invariant. In [21], the authors proved the following vanishing theorem for the LOSS invariant.

Theorem 3.2. *If $L \subset (Y, \xi)$ is an oriented, null-homologous Legendrian knot with an overtwisted complement, then $\mathcal{L}(L) = 0$.*

We should note that since the the BRAID invariant t is equivalent to the LOSS invariant \mathcal{L} . Therefore, t vanishes if the complement of a given knot is overtwisted.

Example 3.3. Let K be the unknot in (S^3, ξ_{std}) .



We can construct a BRAID diagram \mathcal{H} for $K \in -S^3$ with the standard contact structure.

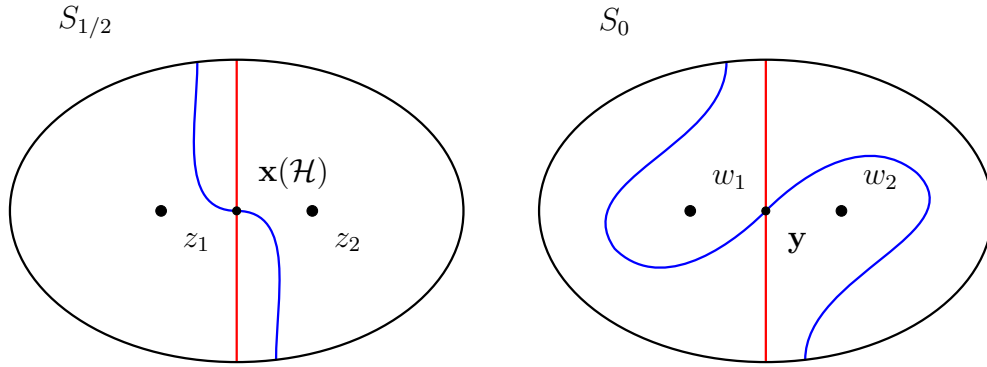


FIGURE 3.1. Pages 1/2 and 0 of the BRAID diagram \mathcal{H} .

By Definition 3.1 we are interested in the intersection of α and β on $S_{1/2}$, denoted by $\mathbf{x}(\mathcal{H}) \in \text{CFK}^-(\mathcal{H})$. Note that we have two disk from \mathbf{y} to $\mathbf{x}(\mathcal{H})$ but pass over \mathbf{w} , therefore, $\partial(\mathbf{y}) = U_{w_1}\mathbf{x}(\mathcal{H}) + U_{w_2}\mathbf{x}(\mathcal{H}) = 0$. Since $\mathbf{x}(\mathcal{H})$ is a cycle and is not a boundary, therefore, $t = [\mathbf{x}(\mathcal{H})]$ is non-trivial in $\text{HFK}^-(-S^3, K)$.

3.2 Lift of the BRAID Invariant

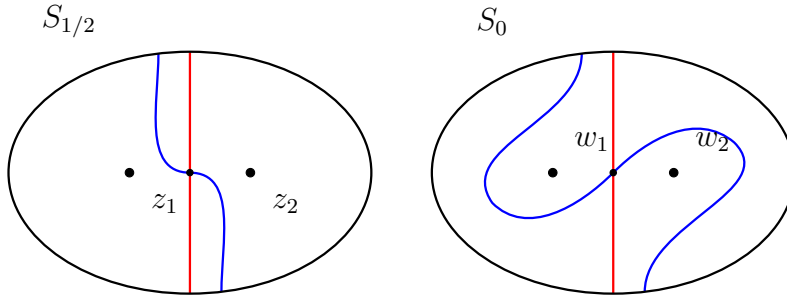
We will extend the BRAID invariant defined above to a braid invariant for the lift of a transverse knot K in a n -cyclic branch cover branched along K .

Let

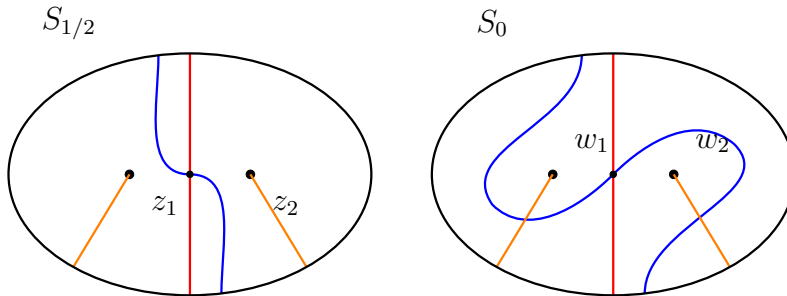
$$p : \Sigma^n(K) \rightarrow Y$$

be the n -fold cyclic covering branched along K . Since (S, ϕ) is an open book decomposition of Y compatible with ξ , the lift of $(\tilde{S}, \tilde{\phi})$ is open book decomposition of $\Sigma_n(K)$ compatible with $\xi_n(K)$ as constructed in the previous chapter. Furthermore, we can use p to lift the surfaces S_i for $i \in [0, 1/2]$ and we will denote the lift of S_t by \tilde{S}_t . We can also lift the α and β curves, and we will denote the lifts by $\tilde{\alpha}$ and $\tilde{\beta}$ respectively. In order to construct a Heegaard diagram for the lift of K in $\Sigma_n(K)$ we need to lift the basepoints \mathbf{w}_K and \mathbf{z}_K and we will denote the lifts by $\tilde{\mathbf{w}}_K$ and $\tilde{\mathbf{z}}_K$ respectively. Therefore, $\tilde{\mathcal{H}} = (\tilde{\Sigma}, \tilde{\beta}, \tilde{\alpha}, \tilde{\mathbf{w}}_K, \tilde{\mathbf{z}}_K)$ is BRAID diagram for $\tilde{K} \subset -\Sigma_n(K)$.

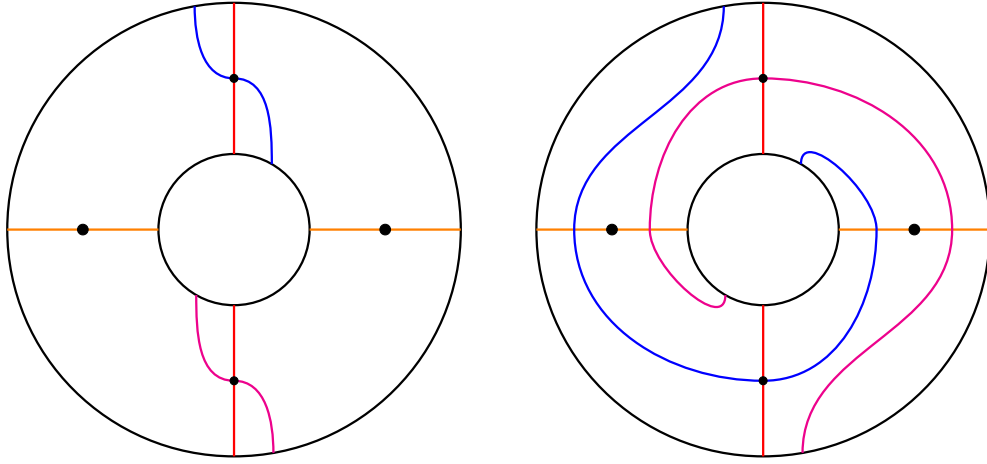
Example 3.4. Suppose K is the unknot in (S^3, ξ_{std}) , then a BRAID diagram \mathcal{H} is given by



Now, we take branched cuts along the Seifert surface of the unknot. The branch cuts is indicated by the orange line.



We can now produce the Heegaard diagram of the double branched cover of S^3 branched along the unknot.



Therefore, we have produced $\tilde{\mathcal{H}} = (\tilde{\Sigma}, \tilde{\beta}, \tilde{\alpha}, \tilde{\mathbf{w}}_K, \tilde{\mathbf{z}}_K)$ for $\tilde{K} \in -\Sigma_2(K)$ with contact structure ξ_2 .

Now, we have the tools to extend the definition of the BRAID invariant to the n -fold cyclic branch cover.

Definition 3.5. Let $\mathbf{x}(\tilde{\mathcal{H}})$ denote the generator of $\text{CFK}^-(\tilde{\mathcal{H}})$ consisting of the intersection points on $\tilde{S}_{1/2}$ between the $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ curves. Note that $\mathbf{x}(\tilde{\mathcal{H}})$ is a cycle in $\text{CFK}^-(\tilde{\mathcal{H}})$. We define

$$t_n(K) := [\mathbf{x}(\tilde{\mathcal{H}})] \in \text{HFK}^-(\tilde{\mathcal{H}}) = \text{HFK}^-(\Sigma^n(K), \tilde{K})$$

We define $\hat{t}_n(K) \in \widehat{\text{HFK}}(\Sigma^n(K), \tilde{K})$ to be the image of $t_n(K)$ under the natural map $p_*^1 : \text{HFK}^-(\tilde{\mathcal{H}}) \rightarrow \widehat{\text{HFK}}(\tilde{\mathcal{H}})$.

Chapter 4

Properties of t_n

The goal of this section is to prove the following theorem.

Theorem 4.1. $t_n(K_1) = 0$ or $t_n(K_2) = 0$ if and only if $t_n(K_1 \# K_2) = 0$.

In order to prove Theorem 4.1 we will need to define a pair of maps. In the process of defining the two maps we will see that to a certain extent t_n behaves nicely under connected sums. Before we can prove Theorem 4.1 we will need to prove several intermediate results.

4.1 Comultiplicativity

In this section we will prove that t_n satisfies a comultiplication formula. The following argument is a modification of John Baldwin's argument found in [2].

Suppose $(\Sigma, \alpha, \beta, \gamma, \mathbf{z}, \mathbf{w})$ is a multi-pointed Heegaard triple diagram. From Chapter 2, we know that this diagram encodes the following manifolds: $Y_{\alpha, \beta} = S^3$ containing the knot K_1 with corresponding BRAID diagram \mathcal{H}_1 , $Y_{\beta, \gamma} = S^3$ containing the knot K_2 with BRAID diagram \mathcal{H}_2 , and $Y_{\alpha, \gamma} = S^3$ containing the knot K_3 with BRAID diagram \mathcal{H}_3 .

First we will show that $(\Sigma, \alpha, \beta, \gamma, \mathbf{z}, \mathbf{w})$ is weakly-admissible. Then we show that there is only one pair $\{\mathbf{a} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, \mathbf{b} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma\}$ for which there exists a homotopy class $\phi \in \pi_2(\mathbf{a}, \mathbf{b}, \mathbf{x}(\mathcal{H}_3))$ with $n_{\mathbf{z}}(\phi) = 0$ and such that ϕ has a holomorphic representative. Moreover, $\mathbf{a} = \mathbf{x}(\mathcal{H}_1)$, $\mathbf{b} = \mathbf{x}(\mathcal{H}_2)$, and the number of holomorphic representatives of ϕ is one.

Lemma 4.2. $(\Sigma, \alpha, \beta, \gamma, \mathbf{z}, \mathbf{w})$ is weakly-admissible.

Proof. For each $i = 1, \dots, 2k + n - 1$ there curves α_i , β_i , and γ_i intersect on $S_{1/2}$ in the arrangement depicted in the figure below.

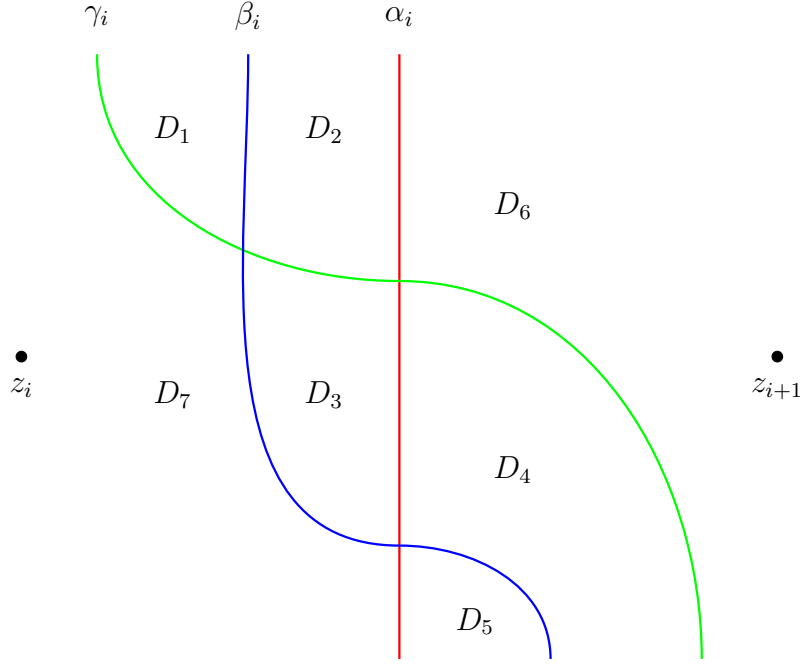


FIGURE 4.1. A triply-periodic domain.

If $\psi = \sum_j p_j D_j$ is a triply-periodic domain, then $p_6 = p_7 = 0$ since D_6 and D_7 contain the basepoint \mathbf{z} . Since $\partial\psi$ includes some number of complete α_i curves,

$$p_2 = p_3 - p_4 = -p_5.$$

Therefore, ψ has both positive and negative coefficients unless $p_2 = p_5 = 0$ and $p_3 = p_4$. Let us assume the latter. Since $\partial\psi$ includes some number of complete β_i curves,

$$p_1 = -p_3 = 0.$$

So, either ψ has both positive and negative coefficients or $p_1 = \dots = p_7 = 0$ and $\partial\psi$ includes no α_i , β_i , or γ_i curves. If we carry out this analysis for all $i = 1, \dots, 2k + n - 1$ we see that either ψ has both positive and negative coefficients or else it is trivial. Hence $(\Sigma, \alpha, \beta, \gamma, \mathbf{z}, \mathbf{w})$ is weakly-admissible. \square

Let Δ denote the 2-simplex and label its vertices clockwise $v_\alpha, v_\beta, v_\gamma$. Let e_α be the edge opposite v_α (and similarly for e_β and e_γ). The boundary of Δ inherits the standard counterclockwise orientation. Then

A map $u : \Delta \rightarrow \text{Sym}^{k-1}(\Sigma)$ satisfying $u(v_\gamma) = \mathbf{a}$, $u(v_\alpha) = \mathbf{b}$, and $u(v_\beta) = \mathbf{x}(\mathcal{H}_3)$, and $u(e_\alpha) \subset \mathbb{T}_\alpha$, $u(e_\beta) \subset \mathbb{T}_\beta$ and $u(e_\gamma) \subset \mathbb{T}_\gamma$ is called a *Whitney triangle* between \mathbf{a} , \mathbf{b} , and $\mathbf{x}(\mathcal{H}_3)$. This map u is represented schematically in the following figure.

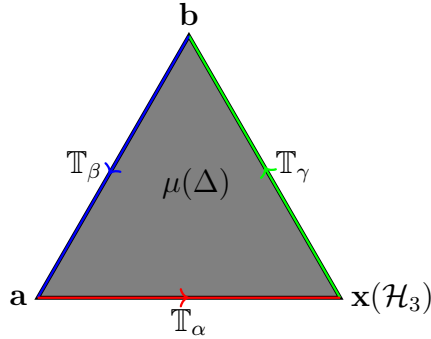


FIGURE 4.2. Whitney triangle between \mathbf{a} , \mathbf{b} and $\mathbf{x}(\mathcal{H}_3)$.

We can represent $\phi \in \pi_2(a, b, \mathbf{x}(\mathcal{H}_3))$ by a 2-chain $\hat{\phi} = \sum_j p_j D_j$ whose oriented boundary consists of α arcs from a to $\mathbf{x}(\mathcal{H}_3)$, β arcs from b to a , and γ arcs from $\mathbf{x}(\mathcal{H}_3)$ to b . Suppose $n_{\mathbf{z}}(\phi) = 0$ and ϕ has a holomorphic representative. Then $n_{\mathbf{z}}(\hat{\phi}) = 0$ and the p_j are all non-negative.

Lemma 4.3. *There is only one pair $\{\mathbf{a} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, \mathbf{b} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma\}$ for which there exists a homotopy class $\phi \in \pi_2(\mathbf{a}, \mathbf{b}, \mathbf{x}(\mathcal{H}_3))$ with $n_{\mathbf{z}}(\phi) = 0$ and such that ϕ has a holomorphic representative. Moreover, $\mathbf{a} = \mathbf{x}(\mathcal{H}_1)$, $\mathbf{b} = \mathbf{x}(\mathcal{H}_2)$, and the number of holomorphic representatives of ϕ is one.*

Proof. Refer to the Figure 4.3 below for the local picture near the i th component of the BRAID classes $\mathbf{x}(\mathcal{H}_1)$, $\mathbf{x}(\mathcal{H}_2)$, and $\mathbf{x}(\mathcal{H}_3)$. Write

$$\hat{\phi} = p_1 D_1 + \cdots + p_7 D_7 + \sum_{j>7} p_j D_j.$$

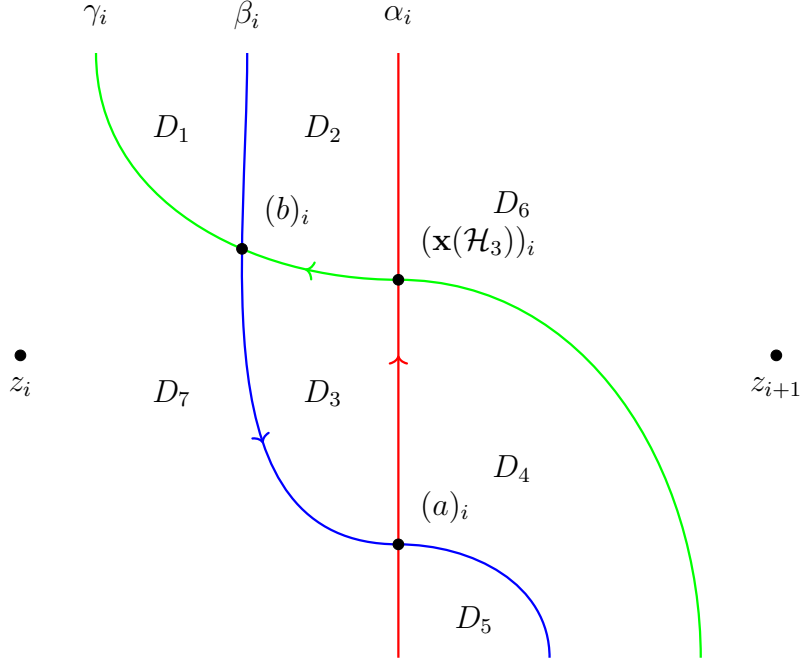


FIGURE 4.3. Local picture of the i th component of the BRAID classes of $\mathcal{H}_1, \mathcal{H}_2$ and \mathcal{H}_3 .

Now we can analyze the possibilities for p_1, \dots, p_7 given the boundary constraints on $\hat{\phi}$. $(\mathbf{x}(\mathcal{H}_3))_i$ must be a corner of the region defined by $\hat{\phi}$; moreover this corner is such that we enter $(\mathbf{x}(\mathcal{H}_3))_i$ along an arc of α_i and we leave along an arc of γ_i . Therefore, $p_6 + p_3 = p_2 + p_4 + 1$. If $(\mathbf{x}(\mathcal{H}_2))_i$ is not a corner, then $p_3 + p_1 = p_2 + p_7$. Note that $p_6 = p_7 = 0$ since $n_{\mathbf{z}}(\hat{\phi}) = 0$. Thus, these two equations become

$$p_3 = p_2 + p_4 + 1$$

$$p_3 + p_1 = p_2.$$

Subtracting the second equation from the first, we have

$$-p_1 = p_4 + 1$$

which implies either p_1 or p_4 is negative, which cannot happen since ϕ has a holomorphic representative. Therefore, $(\mathbf{x}(\mathcal{H}_2))_i$ is a corner. The same type of analysis shows that $(\mathbf{x}(\mathcal{H}_1))_i$ is a corner.

Since $(\mathbf{x}(\mathcal{H}_2))_i$ is a corner, either $p_1 + p_3 + 1 = p_2$ or $p_1 + p_3 = p_2 + 1$. Substituting $p_3 = p_2 + p_4 + 1$ into both expressions, we have the two possibilities $p_1 + p_2 + p_4 + 2 = p_2$ or $p_1 + p_2 + p_4 = p_2$. We can rule out the first possibility as it implies that either p_1 or p_4 is negative. And the second possibility holds if $p_1 = p_4 = 0$. So, to summarize what we know so far: $p_1 = 0$, $p_3 = p_2 + 1$, $p_4 = 0$, $p_6 = 0$ and $p_7 = 0$.

Since $((\mathbf{x}(\mathcal{H}_1))_i)$ is a corner, the either $p_5 + p_3 = p_4 + p_7 + 1$ or $p_5 + p_3 + 1 = p_4 + p_7$. Substituting what we know of p_3, p_4 , and p_7 into these two expressions, we obtain the two possibilities $p_5 + p_2 + 1 = 1$ or $p_5 + p_2 + 2 = 0$. We can rule out the second possibility as it implies that either p_5 or p_2 is negative. And the first possibility holds only if $p_5 = p_2 = 0$. Thus, we have determined that the only possibility for the values p_1, \dots, p_7 are

$$p_1 = p_2 = p_4 = p_5 = p_6 = p_7 = 0$$

$$p_3 = 1.$$

Because the same analysis works for every $i = 1, \dots, k - 1$ and because every component of $\partial\widehat{\phi}$ must contain some $(\mathbf{x}(\mathcal{H}_3))_i$ we can conclude that $\widehat{\phi}$ is the linear combination which is the sum of precisely one of the small triangular regions (D_3 in the figure above) for each i . Therefore, any holomorphic triangle ϕ between a, b , and $(\mathbf{x}(\mathcal{H}_3))_i$ with $n_{\mathbf{z}}(\phi) = 0$ is, in fact, a triangle between $(\mathbf{x}(\mathcal{H}_1))_i$, $(\mathbf{x}(\mathcal{H}_2))_i$, and $(\mathbf{x}(\mathcal{H}_3))_i$ and can be expressed as a product of these small triangles in our Heegaard diagram. Moreover, since each of these disjoint triangular regions is topologically a disk, and we have specified the image of three boundary points, $\#\mathcal{M}(\phi) = 1$ by the Riemann Mapping Theorem. \square

Theorem 4.4. *There exists a map*

$$f : \widehat{\text{HFK}}(\mathcal{H}_3) \rightarrow \widehat{\text{HFK}}(\mathcal{H}_1) \otimes \widehat{\text{HFK}}(\mathcal{H}_2)$$

which sends $\hat{t}(K_3)$ to $\hat{t}(K_1) \otimes \hat{t}(K_2)$

Proof. Applying Lemmas 4.2 & 4.3 to the multi-pointed Heegaard triple \mathcal{H} we obtain that there exists a map such that

$$f(\mathbf{x}(\mathcal{H}_3)) = \mathbf{x}(\mathcal{H}_1) \otimes \mathbf{x}(\mathcal{H}_2).$$

where $\mathbf{x}(\mathcal{H}_1)$, $\mathbf{x}(\mathcal{H}_2)$, and $\mathbf{x}(\mathcal{H}_3)$ are our BRAID classes. □

Corollary 4.5. *There exists a map*

$$f' : \text{HFK}^-(\mathcal{H}_3) \rightarrow \text{HFK}^-(\mathcal{H}_1) \otimes \text{HFK}^-(\mathcal{H}_2)$$

which sends $t(K_3)$ to $t(K_1) \otimes t(K_2)$.

Proof. The same argument as above works, since the second basepoint \mathbf{w} plays no role in the construction proof. □

Let (\tilde{S}, \tilde{g}) be the open book corresponding to $\Sigma^n(K_1)$, (\tilde{S}, \tilde{h}) be the open book corresponding to $\Sigma^n(K_2)$, and (\tilde{S}, \tilde{hg}) be the open book corresponding to $\Sigma_n(K_3)$.

Corollary 4.6. *There exists a map*

$$\tilde{f}' : \text{HFK}^-(-\Sigma^n(K_3), \tilde{K}_3) \rightarrow \text{HFK}^-(-\Sigma^n(K_1), \tilde{K}_1) \otimes_{\mathbb{Z}_2} \text{HFK}^-(-\Sigma^n(K_2), \tilde{K}_2)$$

which sends $t_n(K_3)$ to $t_n(K_1) \otimes t_n(K_2)$.

Proof. Since the branch cuts avoid the triangular regions of interest, we are simply multiplying the number of triangular regions by n . A similar argument applied to each sheet will proof our desired result. □

Corollary 4.7. *If $t_n(K_1) \neq 0$ and $t_n(K_2) \neq 0$, then $t(K_3) \neq 0$*

Proof. Suppose that $t_n(K_3) = 0$, then either $t_n(K_1) = 0$ or $t_n(K_2) = 0$, since a homomorphism maps 0 to 0. □

4.2 Construction of Map Sending $t_n(K_1) \otimes t_n(K_2)$ to $t_n(K_1 \# K_2)$.

Consider (S^3, ξ_{std}) with compatible open book decomposition (D^2, id) . Suppose that K_1 and K_2 be transverse knots in (S^3, ξ_{std}) , we know that K_1 and K_2 can be thought of as the closures of braids braided with respect to our open book decomposition. Therefore, let K_1 and K_2 correspond to the closure of the braids B_1 and B_2 , respectively. We can construct pointed Heegaard diagrams and BRAID diagram for both K_1 and K_2 , see Figure 4.4.

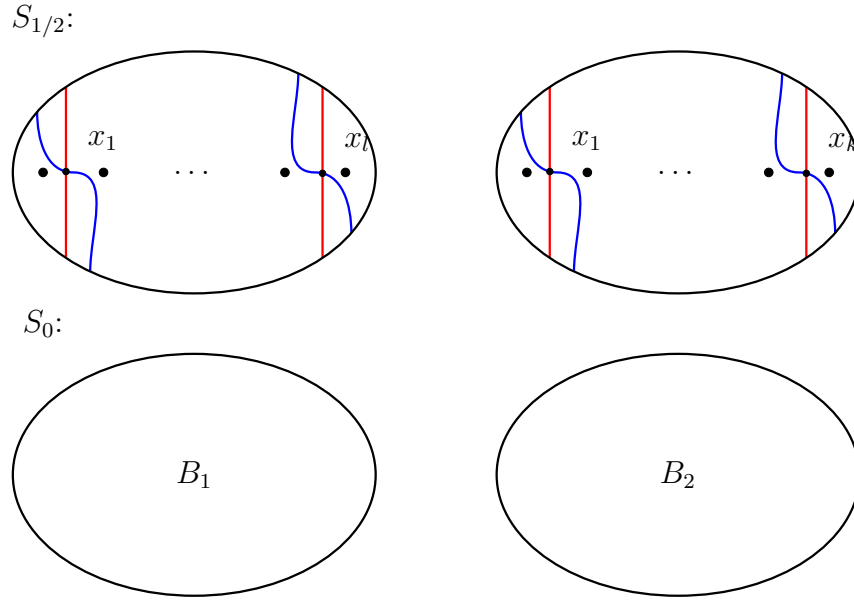


FIGURE 4.4. From the two figures on the left we obtain the BRAID diagram \mathcal{H}_1 for K_1 and from the two figures on the right we obtain the BRAID diagram \mathcal{H}_2 for K_2 .

Now if we consider the disjoint union $K_1 \sqcup K_2$ we will get the following knot that can be presented by a $l + k$ -braid braided about the the z -axis.

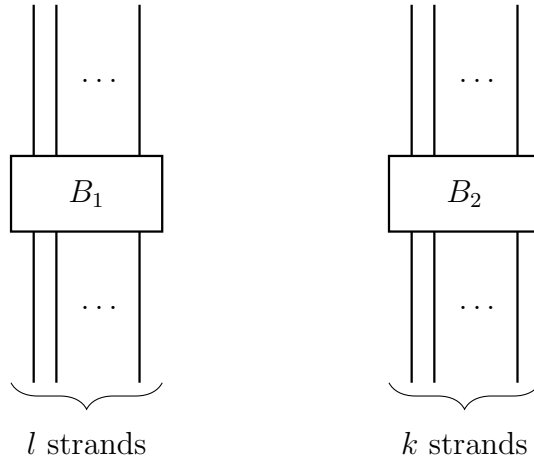


FIGURE 4.5. The closure of this $l + k$ -braid is $K_1 \sqcup K_2$.

We can construct pointed Heegaard diagram and BRAID diagrams encoding the $K_1 \sqcup K_2$. We can construct the pointed Heegaard diagram for $K_1 \sqcup K_2$ by simply joining the pages of the pointed open book encoding K_1 and K_2 and introducing a new pair of α and β arcs denoted by the red and blue curves in Figure 4.6.

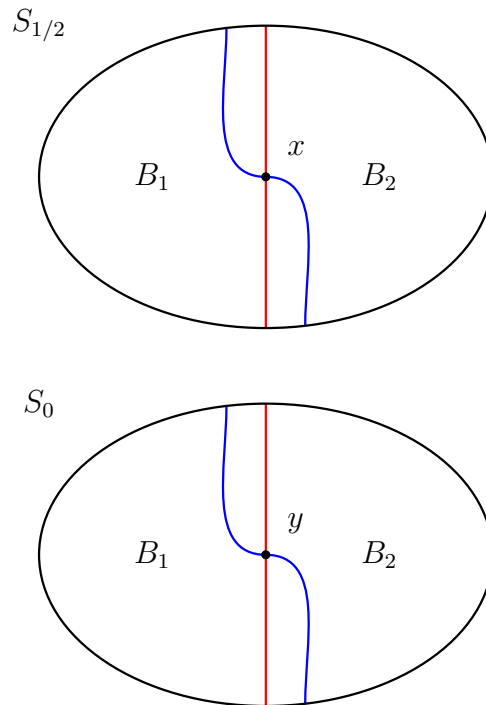


FIGURE 4.6. The page $S_{1/2}$ and S_0 of the pointed open book encoding $K_1 \sqcup K_2$.

As described in Chapter 2 we can construct the BRAID diagram \mathcal{H}_3 from $S_{1/2}$ and S_0 . We will now consider the stacking of the $l+k$ -braid $B_1 \sqcup B_2$ and σ_l the l -th generator of the braid group on $l+k$ strands.

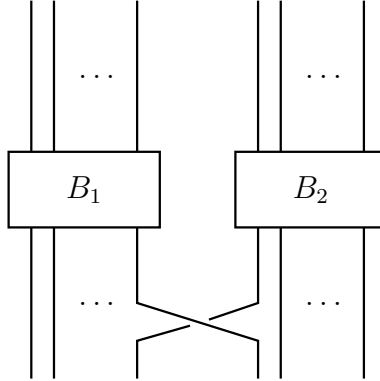


FIGURE 4.7. Stack of $B_1 \sqcup B_2$ and σ_l .

From this stacking we can obtain both a pointed Heegaard diagram and BRAID diagram. Note that the closure of the stacking described above is $K_1 \# K_2$. Therefore, we can obtain a BRAID diagram for the connected sum of K_1 and K_2 . We will denote this BRAID diagram by \mathcal{H} .

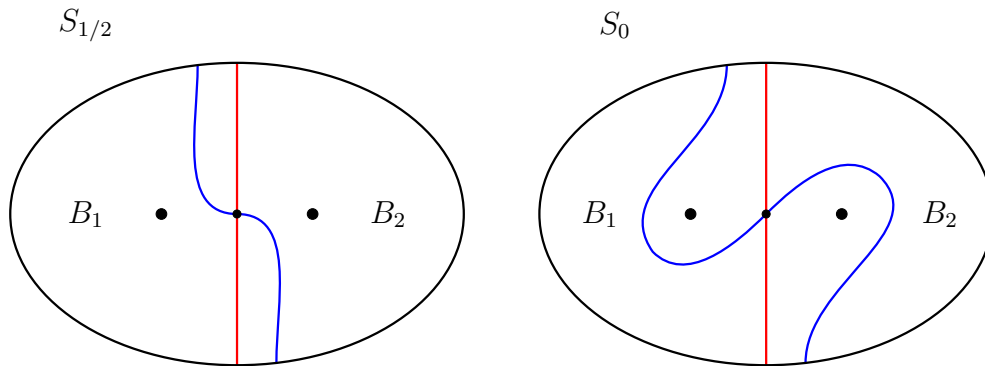


FIGURE 4.8. The BRAID diagram associated to the stack $(B_1 \sqcup B_2) \cdot \sigma_l$. The generators x and y are denoted by the purple dot. We can see that \mathcal{H}_1 and \mathcal{H}_2 are separated by an extra pair of α and β curves.

Lemma 4.8. *There exists chain maps $\iota : \text{CFK}^-(\mathcal{H}_i) \rightarrow \text{CFK}^-(\mathcal{H})$ for $i = 1, 2$ induced from inclusion maps.*

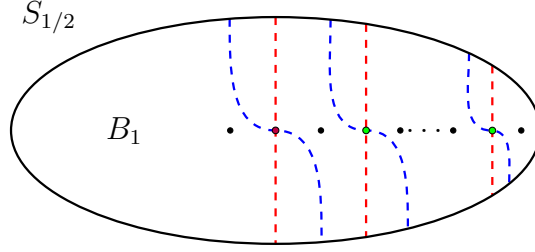


FIGURE 4.9. The generators x and $\mathbf{x}(\mathcal{H}_2) = (\mathbf{x}(\mathcal{H}_2)_1, \dots, \mathbf{x}(\mathcal{H}_2)_k)$ are represented by the purple dot and the green dots respectively.

Proof. Suppose we fix the intersection points between the α and β curves on page 1/2 from the BRAID diagram \mathcal{H}_2 and denote it $\mathbf{x}(\mathcal{H}_2)$ and fix the intersection between the extra set of α and β curves that we obtain from the union of \mathcal{H}_1 and \mathcal{H}_2 . In Figure 4.9 $\mathbf{x}(\mathcal{H}_2)$ and x are depicted by green and purple dots respectively. By fixing $(\mathbf{x}(\mathcal{H}_2))_k$ and x we are deleting the curves that contain these intersection points, the dashed lines in Figure 4.9. After deleting the dashed α and β curves we are essentially left with \mathcal{H}_1 , but changed by an isotopy so we have another BRAID diagram \mathcal{H}'_1 . Since the homology of the diagram is unaffected by isotopy, $\text{HFK}^-(\mathcal{H}_1)$ isomorphic to $\text{HFK}^-(\mathcal{H}'_1)$. Therefore, if \mathbf{a} and \mathbf{b} are in $\text{CFK}^-(\mathcal{H}'_1)$ such that $\partial(\mathbf{a}) = \mathbf{b}$, then $\partial((\mathbf{a}, \mathbf{x}(\mathcal{H}_2), x)) = (\mathbf{b}, \mathbf{x}(\mathcal{H}_2), x)$, since $\mathbf{x}(\mathcal{H}_2)$ and x must be mapped to themselves by the constant map. Therefore, this inclusion map is a chain map. A similar argument works for \mathcal{H}_2 . \square

We see that the differential sends elements of $\text{CFK}^-(\mathcal{H}_i)$ to elements of $\text{CFK}^-(\mathcal{H}_i)$ for $i = 1, 2$. Therefore, these complexes are subcomplexes of $\text{CFK}^-(\mathcal{H})$.

$$\begin{array}{ccc}
 \text{CFK}^-(\mathcal{H}_1) & \searrow & \\
 & & \text{CFK}^-(\mathcal{H}) \\
 & \swarrow & \\
 \text{CFK}^-(\mathcal{H}_2) & &
 \end{array}$$

These inclusion maps are defined on generators in the following way:

$$\begin{array}{ccc}
 \mathbf{a} & \searrow & (\mathbf{a}, \mathbf{x}(\mathcal{H}_2), x) \\
 & & \uparrow \\
 & & (\mathbf{x}(\mathcal{H}_1), \mathbf{b}, x) \\
 \mathbf{b} & \swarrow &
 \end{array}$$

These inclusions are chain maps and induce the following maps on homology.

$$\begin{array}{ccc}
 \mathrm{HFK}^-(-S^3, K_1) & \searrow & \mathrm{HFK}^-(-S^3, K_1 \# K_2) \\
 & & \uparrow \\
 \mathrm{HFK}^-(-S^3, K_2) & \swarrow &
 \end{array}$$

They send the BRAID invariant of each knot K_i to the BRAID invariant of $K_1 \# K_2$

$$\begin{array}{ccc}
 t(K_1) & \searrow & t(K_1 \# K_2) \\
 & & \uparrow \\
 t(K_2) & \swarrow &
 \end{array}$$

Therefore, we have a map that sends $t(K_1) \otimes t(K_2)$ to $t(K_1 \# K_2)$.

Remark 4.9. *Vértési in [34] proved that there is actually an isomorphism not just a map for the GRID invariants λ and θ . Since we know that the GRID invariant is equivalent to the BRAID invariant, the above result was already known. Vértési*

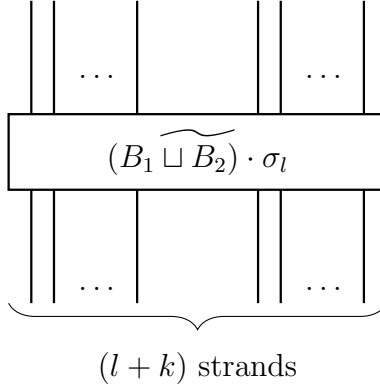


FIGURE 4.10. Lift of the stack $(B_1 \sqcup B_2) \cdot \sigma_l$.

took a different approach to prove the result for the GRID invariant, but we will need this diagrammatic approach in the next section.

4.2.1 Branched Cover

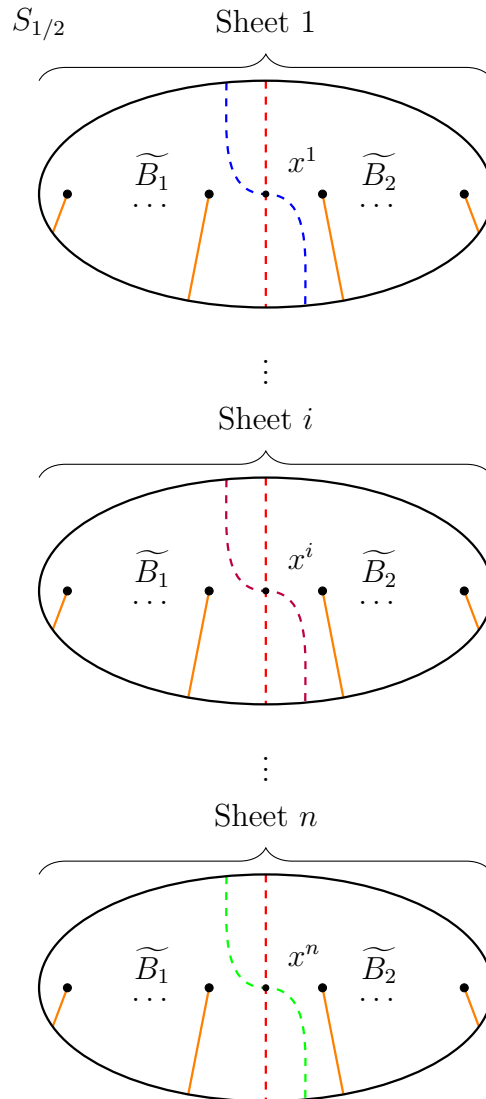
We produced a chain map that induced a map from $\text{HFK}^-(-S^3, K_1) \otimes \text{HFK}^-(-S^3, K_2)$ to $\text{HFK}^-(-S^3, K_1 \# K_2)$ which sends $t(K_1) \otimes t(K_2)$ to $t(K_1 \# K_2)$. We will now construct a chain map that induces a map from $\text{HFK}^-(-\Sigma^n(K_1), \tilde{K}_1) \otimes \text{HFK}^-(-\Sigma^n(K_2), \tilde{K}_2)$ to $\text{HFK}^-(-\Sigma(K_1 \# K_2), \widetilde{K_1 \# K_2})$ which sends $t_n(K_1) \otimes t_n(K_2)$ to $t_n(K_1 \# K_2)$. The goal of this section is to prove the following:

Theorem 4.10. *There is a map from $t_n(K_1) \otimes t_n(K_2)$ to $t_n(K_1 \# K_2)$.*

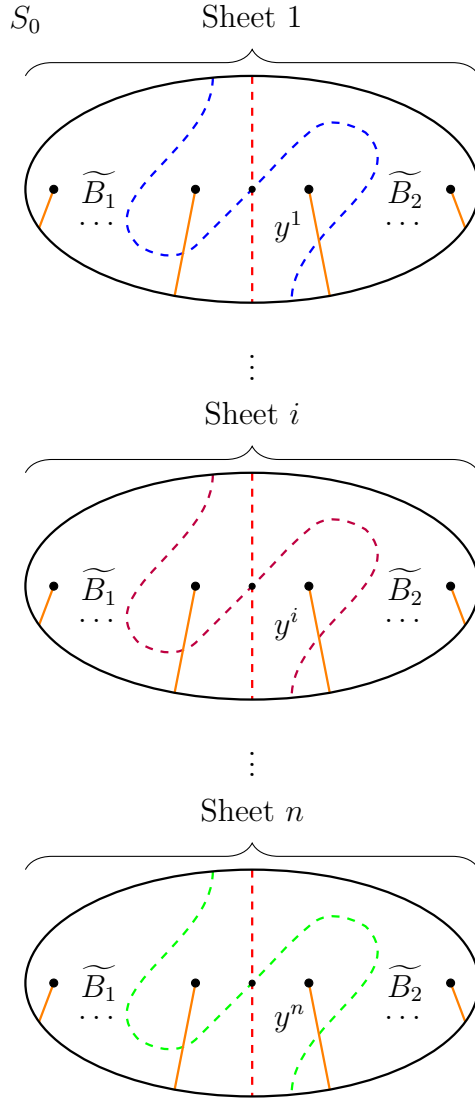
Proof. We will following the same construction as in the previous section. Consider (S^3, ξ_{std}) with compatible open book decomposition (D^2, id) . Suppose K_1 and K_2 are transverse knots in (S^3, ξ_{std}) , we can represent K_1 and K_2 by the closure of braids B_1 and B_2 braided with respect to our given open book decomposition. We will denote the BRAID diagram for K_1 by \mathcal{H}_1 and the BRAID diagram for K_2 by \mathcal{H}_2 . Furthermore, we can consider the disjoint union $K_1 \sqcup K_2$ and construct a BRAID diagram as described in the previous section, we will denote this BRAID diagram by \mathcal{H}_3 . We can now consider the stack of $B_1 \sqcup B_2 \cdot \sigma_l$ where σ_l is the l^{th} generator of the braid group on $l + k$ strands. The closure of this stack is the

connected sum of K_1 and K_2 , therefore, we can construct a BRAID diagram for $K_1 \# K_2$ which we denote by \mathcal{H} , see Figure 4.8.

We will now take the the n -fold cyclic branched cover with branch locus $K_1 \# K_2$. We can now construct a BRAID diagram for the lift $\widetilde{K_1 \# K_2}$ following the construction described in Chapter 3. Figure 4.11 and 4.12 describe the pages 0 and $1/2$ respectively the of the open book for $\widetilde{K_1 \# K_2}$.



Remark 4.11. Suppose we fix x^1 on sheet 1, then by the construction of the n -fold cyclic branched cover we are forced to choose x^2 on sheet 2 which forces us



to choose x^3 and so on. Therefore, we have $\tilde{x} = (x^1, \dots, x^n)$ or $\tilde{y} = (y^1, \dots, y^n)$. These are the only possible generators coming from the additional α and β curves, these curves are denoted by dashed curves in Figures 4.11 and 4.12

Lemma 4.12. *There exists chain maps $\iota : \text{CFK}^-(\tilde{\mathcal{H}}_i) \rightarrow \text{CFK}^-(\tilde{\mathcal{H}})$ for $i = 1, 2$ induced from inclusion maps.*

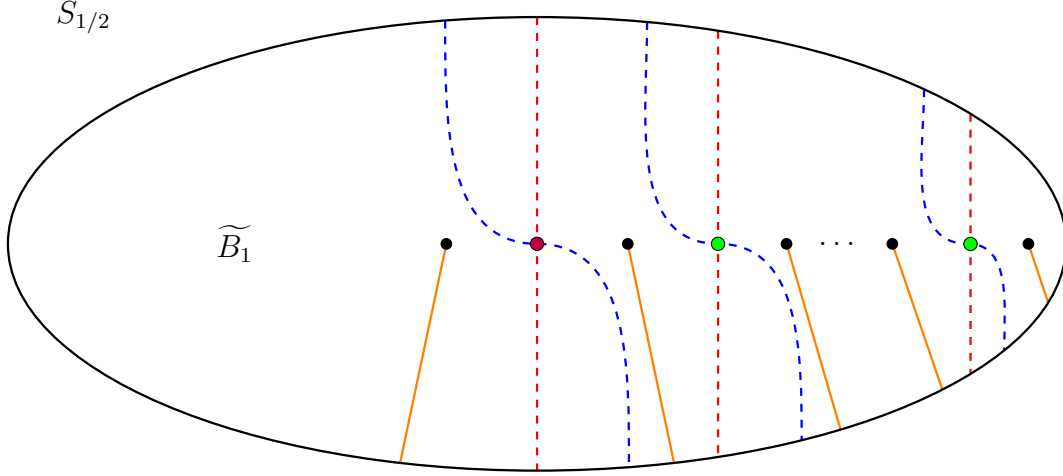
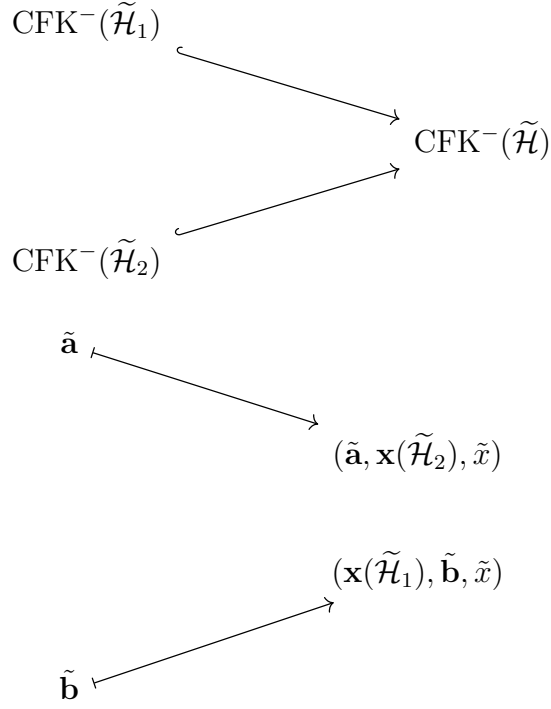


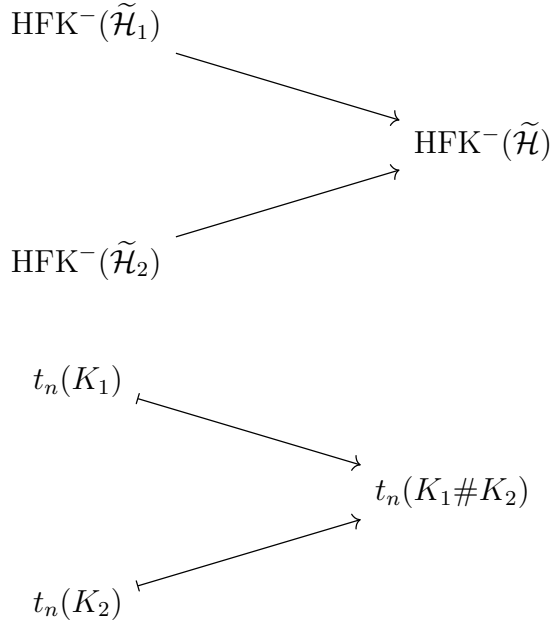
FIGURE 4.11. The generators \tilde{x} and $\mathbf{x}(\tilde{\mathcal{H}}_2) = (\mathbf{x}(\tilde{\mathcal{H}}_2)_1, \dots, \mathbf{x}(\tilde{\mathcal{H}}_2)_k)$ are represented by the purple dot and the green dots respectively.

Proof. Suppose we fix the intersection points between the $\tilde{\alpha}$ and $\tilde{\beta}$ curves on page 1/2 from the BRAID diagram $\tilde{\mathcal{H}}_2$ and denote it $\mathbf{x}(\tilde{\mathcal{H}}_2)$ and fix the intersection between the extra set of $\tilde{\alpha}$ and $\tilde{\beta}$ curves that we obtain from the union of $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$. In Figure 4.9 $\mathbf{x}(\tilde{\mathcal{H}}_2)$ and \tilde{x} are depicted by green and purple dots respectively. By fixing $(\mathbf{x}(\tilde{\mathcal{H}}_2))_k$ and x we are deleting the curves that contain these intersection points, the dashed lines in Figure 4.9. After deleting the dashed $\tilde{\alpha}$ and $\tilde{\beta}$ curves we are essentially left with $\tilde{\mathcal{H}}_1$, but changed by an isotopy so we have another BRAID diagram $\tilde{\mathcal{H}}'_1$. Since the homology of the diagram is unaffected by isotopy, $\text{HFK}^-(\tilde{\mathcal{H}}_1)$ isomorphic to $\text{HFK}^-(\tilde{\mathcal{H}}'_1)$. Therefore, if \mathbf{a} in $\text{CFK}^-(\tilde{\mathcal{H}}'_1)$ such that $\partial(\mathbf{a}) = \mathbf{b}$ in $\text{HFK}^-(\Sigma^n(K_1), \tilde{K}_1)$, then $\partial((\mathbf{a}, \mathbf{x}(\tilde{\mathcal{H}}_2), x)) = (\mathbf{b}, \mathbf{x}(\tilde{\mathcal{H}}_2), x)$, since $\mathbf{x}(\tilde{\mathcal{H}}_2)$ and x must be mapped to themselves by the constant map. Therefore, this inclusion map is a chain map. A similar argument works for $\tilde{\mathcal{H}}_2$. \square

Since we have that that the differential of $\text{CFK}^-(\mathcal{H}_1)$ and $\text{CFK}^-(\mathcal{H}_2)$ stay within these complexes, these chain complexes are actually subcomplex of $\text{CFK}^-(\mathcal{H})$, therefore, the inclusion maps are chain maps and they induce the following maps on homology.



This induces maps on the level of homology



Therefore, we have a map from $t_n(K_1) \otimes t_n(K_2)$ to $t_n(K_1 \# K_2)$.

□

Corollary 4.13. *If $t_n(K_1) = 0$ or $t_n(K_2) = 0$, then $t_n(K_1 \# K_2) = 0$.*

Now, we have all the components in order to prove our main result. Specifically, we have defined the two maps needed in order to prove Theorem 4.1.

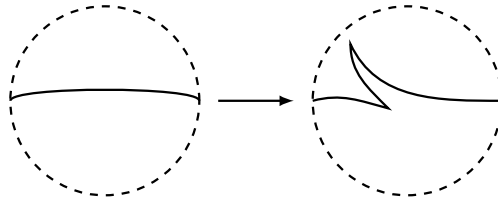
Proof for Theorem 4.1. By Corollary 4.7 & 4.13 we obtain our desired result. \square

4.3 Stabilization Result

In this section we show that the lift of the BRAID invariant t_n for $n > 1$ vanishes if the transverse knot is a stabilization of another transverse knot. We will show this vanishing condition by first reproving a known result about n -fold cyclic branched covers of such knots. The result was originally proved in [18] using an argument involving contact surgery. We will provide an elementary proof of the same result. We then use this result to prove that t_n vanishes under the conditions mentioned above.

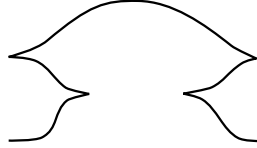
Theorem 4.14. *If K_{stab} is a negative braid stabilization of K , then $(\Sigma^n(K_{stab}) \setminus \widetilde{K_{stab}, \xi_n})$ is overtwisted.*

Proof. Given a transverse knot K in (S^3, ξ_{std}) , we can uniquely approximate K by a Legendrian knot L up to negative Legendrian stabilization. Given a Legendrian approximation of K we can locally perform a negative stabilization and still describe K .



Since transverse knots do not detect negative Legendrian stabilization, both of the above Legendrian knots are Legendrian approximations of the transverse knot K . The figure on left is the Legendrian approximation L of K , figure on the right is another Legendrian approximation L' after performing a negative Legendrian stabilization. Now, suppose that K_{stab} is the stabilization of the transverse knot

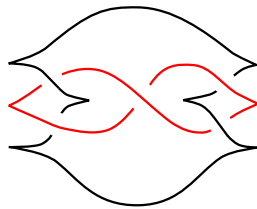
K . From the negatively stabilized Legendrian knot L' we can also obtain a Legendrian approximation of the stabilization of K_{stab} . In order to get a Legendrian approximation of K_{stab} we must perform a positive Legendrian stabilization to L' .



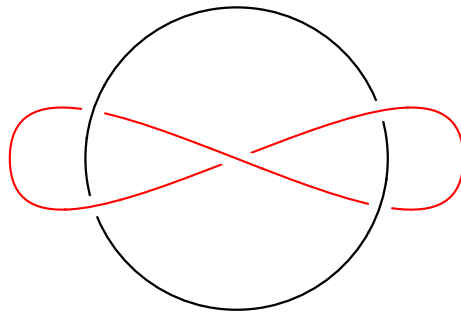
Above is a local picture of L'_+ , in the figure we see both the positive and negative Legendrian stabilizations. Now, from this we obtain another curve γ depicted below by the red curve.



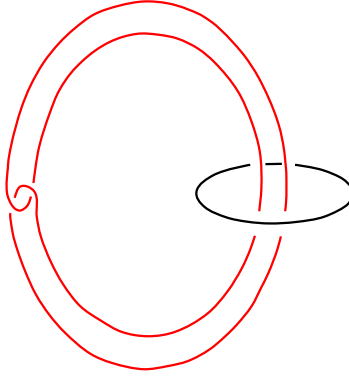
Since we can always perform the above local operations on a Legendrian approximation of K . We would like to think of these local picture separately, therefore, we can think of the appropriate Legendrian unknot depicted below and then connect sum the Legendrian unknot with any Legendrian knot to get the picture above.



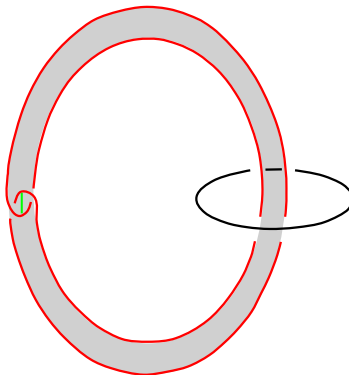
Note that γ has $\text{tb}(\gamma) = -2$ in S^3 . Topologically we have the following link.



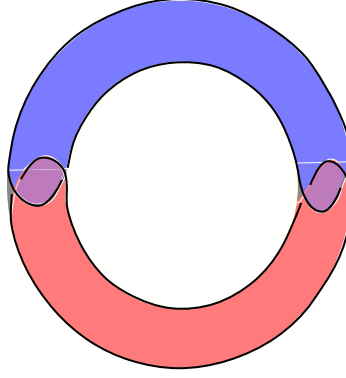
Furthermore, we can isotope γ to obtain the following link.



We see that γ bounds a disk D depicted in the following figure by the gray region. We have $lk(\gamma, \gamma') = -2$ where γ' is the pushoff of γ obtained by sliding γ into the interior of the Seifert surface. Hence, the twisting of D with respect to the contact planes is zero. Therefore, if D were to be embedded, it would be an overtwisted disk. We can see that D is a singular disk, since it self intersects at the green line in the following figure.



But if we consider the double branch cover $\pi : \Sigma^2(K_{stab}) \rightarrow S^3$



The disk D lifts to two embedded, overtwisted disk in $\Sigma^2(K_{stab})$. If we consider the n -fold cyclic branch cover branched along K_{stab} we will obtain n embedded disk with all of them being overtwisted. Therefore, any n -fold cyclic branch cover with branched locus k_{stab} is overtwisted. \square

Corollary 4.15. *If K_{stab} is a negative braid stabilization of K , then $t_n(K_{stab}) = 0$.*

Proof. Applying Theorem 4.14 we obtain that the complement of the lift \tilde{K}_{stab} in $(\Sigma^n(K_{stab}) \setminus \widehat{K_{stab}}, \xi_n)$ is overtwisted. From the discussion in Chapter 2 we know that t vanishes if the complement of the given knot is overtwisted. By the way that the lift of the BRAID invariant is defined we obtain that $t_n(K_{stab}) = 0$ for $n \geq 1$. \square

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Vita

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