2014

Distributed Stabilization of Nonlinear Multi-Agent Systems

Jianming Zhou
Louisiana State University and Agricultural and Mechanical College, jzhou15@tigers.lsu.edu

DISTRIBUTED STABILIZATION OF NONLINEAR MULTI-AGENT SYSTEMS

A Thesis

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Master of Science in Electrical Engineering

in

The Department of Electrical and Computer Engineering

by
Jianming Zhou
Bachelor of Science in Engineering, Shanghai Jiao Tong University, 2013
May 2015

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor and committee chair, Dr. Guoxiang Gu, for his helpful academic suggestions and patient guidance throughout my research and the preparation of this thesis. His technical advice and support helped me overcome obstacles, kept me motivated, and made this research a meaningful learning experience.

Meanwhile, I would like to thank my committee members, Dr. Kemin Zhou and Dr. Shuangqing Wei, for taking time out of their busy schedule and consenting to be a part of my committee, and for their valuable feedback.

In addition, I would like to thank the faculty, staff, and students in the Department of Electrical and Computer Engineering for all the help and support I have received.

I would like to deeply thank my dear parents, who sacrifice so much to make my dream come true. Thanks also go to my family, who have always been by my side to help me go through all the difficulty and pain during the past two years.

Finally, I would like to thank Bixiang Tang and Abhishek Pandey for their generous academic and moral support throughout my study at LSU and my life in the United States. A special thank goes to my friend, Tian Zhang, for always being my source of courage and inspiration.

I am sure I forget to mention a lot of other people who made this possible for me. I express my sincere gratitude to everyone who has been with me thoughout this tough but wonderful journey.

TABLE OF CONTENTS

ABSTRACT

The study of multi-agent systems (MASs) is focused on systems in which many autonomous agents interact and operate within a limited communication environment. The general goal of the MAS research is to design interconnection control laws such that all the dynamic agents in the group are synchronized to a desired common trajectory by exchanging information with adjacent agents over certain constrained communication networks.

Based on the review and modification of existing results concerning the consensus control of linear heterogeneous MASs in Moreau (2004) [21], Scardovi and Sepulchre (2009) [25], Wieland et al (2011) [30], and Alvergue et al. (2013) [1], this thesis investigates the distributed stabilization of the heterogeneous MAS, consisting of N different continuous-time nonlinear dynamic systems, under connected communication graphs. The conditions for a nonlinear dynamic agent to be feedback equivalent to a strictly passive system are derived along with the feedback law. A distributed stabilization control protocol using state feedback is then proposed under the idea of feedback connection of two passive systems. It proves to be sufficient for only one or a few agents to have access to the reference signal for the MAS to achieve stability, which lowers the communication overhead from the reference to different agents. The result can be interpreted as an extension of the stabilizing law for linear MASs introduced in [1], and considered as a fundamental preliminary for the consensus research for nonlinear MASs in the future.

CHAPTER 1
INTRODUCTION

This chapter introduces the consensus control problem and the existing research work. The contribution of this thesis is discussed briefly, followed by an overview of the main content in each chapter.

1.1 Motivation

Consensus control of multi-agent systems (MASs) has drawn great attention in recent years due to its broad applications in various areas, such as sensor networks [22, 23], robots networks [32, 33], unmanned air vehicles [3, 4, 31], and other practical biological and social systems [7, 8, 26], etc. The general goal of MAS research is to find methods that allow us to build a group of complex systems composed of autonomous agents which, while communicating and operating on local information under limited processing abilities, are nonetheless capable of enacting the desired global behaviors. Consensus control of MASs differs from traditional output regulation of control systems because both individual dynamics and communication constraints have to be taken into consideration. If no cooperative control is involved, each agent will run separately, utilizing more resources and increasing the implementation cost, without being able to achieve consensus.

Many of the existing consensus control studies impose restrictive assumptions on the communication graph, such as bidirectional and time-invariant communication patterns, which may inevitably increase the load and cost of the whole communication network. However, unidirectional communication is typically more common and practical in real world applications and can be easily implemented, for example, via broadcasting. In addition, sensed information flow, which plays an important role in flocking, is usually non-bidirectional. Furthermore, many practical communication topologies tend to be time-varying since link failure, link creation, network reconfiguration, and other conditions may accidentally occur during the operating process. Therefore, there is a need to look for other possible connectivity assumptions which are less restrictive, but still ensure the stabilization and the consensus of the MASs.

Another challenge in consensus control research is that most of the existing work only focuses on homogeneous MASs, where individual agents have the same dynamics, while in real world, individual systems are hardly exactly identical. The consensus control of heterogeneous MASs is in general more difficult than that of homogeneous MASs. State synchronization among all agents is not applicable any more due to the fact that individual systems may have different states and state dimensions. Hence the consensus objective should be switched to output synchronization. It is also worth noting that system dynamics may change due to aging and operating environments. As a result, it is important to study the more complex consensus problem of heterogeneous MASs, such as heterogeneous MASs with time-delays, heterogeneous MASs under time-varying topologies, discrete-time heterogeneous MASs, etc.

Consensus control concerning heterogeneous nonlinear MASs is also a challenging issue since the output regulation problem of nonlinear systems itself is rather difficult. Various types of nonlinearities make it hard to find a general control law or condition to achieve synchronization. Yet such research should be useful and promising since most of the practical systems are nonlinear in our real life.

1.2 Research Work on Consensus Control

Early research work in the field of coordination and synchronization of MASs includes [21] where each individual system in the network is assumed to have simple integrator, and [13] where the agents' dynamics are modeled as linear switched systems. The motions of a group of vehicles are represented by double integrators in [26]. Results regarding integrator chains more than two can be found in [24].

More recently some of the research [18, 19] turned to the investigation of homogeneous MASs with state-space model as the system representation. Such more generalized MASs can be considered as an extension of the aforementioned integrator dynamics as a special case. In [25] the authors deal with the synchronization problem in the homogeneous MASs case, and the result can be interpreted as a generalization of the classical consensus algorithms corresponding to the special case introduced in [21].

Inspired by the fact that individual systems in a communication network are hardly exactly identical, the consensus problem has been extended to the study of synchronization of heterogeneous MASs. A high-gain approach is proposed in [9] to dominate the heterogeneity of MASs. In [16] a homogeneous reference model is included in each agent's distributed controller to overcome the difficulty caused by the non-identical dynamics of different agents. The results in [21] and [25] are modified and developed in [30] to show the asymptotic synchronization of MASs over time-varying directed graphs satisfying a uniform connectivity condition. A consensus law that uses relative information only and requires rather low communication overhead is discussed in [1], with its synthesis based on $\mathcal{H}_\infty$ loop shaping [20] and LQG/LTR [2] methods. The main results in [21, 25, 30, 1] will be introduced in a detailed way in Chapter 2 and Chapter 3.

The sufficient and necessary connectivity conditions that communication graphs have to satisfy to reach consensus has attracted much attention as well. Frequently, the communication graphs are assumed to be time-independent, undirected, or even balanced, for instance, in [17]. However, it has been shown early in [27] and later in [21] that very mild assumptions on graph connectivity are sufficient to uniformly exponentially achieve consensus. The case of switching interconnection topology is discussed in [29]. The cases of frequently connected and jointly connected communication graphs are considered in [28] and [15] respectively, where a slow switching condition and a fast switching condition are introduced.

Finally, heterogeneous nonlinear MASs are also studied, for example, in [6, 10, 34], which use the concepts and properties of passive and dissipative systems. Such nonlinear agents exclude unstable dynamical systems, which is hardly the case in real applications. Yet the results are instrumental to future work concerning nonlinear MASs. In Chapter 4, we will study the distributed stabilization of heterogeneous nonlinear MASs, which will serve as a fundamental preliminary to the corresponding consensus control problem.

1.3 Thesis Contribution

Based on the review of several solid results concerning the distributed stabilization and synchronization of heterogeneous linear MASs in Alvergue et al. (2013) [1], this thesis shows that the rank condition assumption for the graph Laplacian in [1] can be removed provided that the connectedness of the communication graph holds. Thus improved versions of the stabilization and consensus results in [1] are derived. Furthermore, nonlinear systems are also studied. The thesis exploits the feedback equivalence and the properties of passive systems, extends the results for linear MASs in [1], and develops a state feedback control protocol for the distributed stabilization of heterogeneous nonlinear MASs, which will be an instrumental preliminary to the corresponding future research on the consensus control problem.

1.4 Thesis Organization

The thesis is organized as follows. Chapter 1 briefly discusses the background knowledge and applications of consensus control, as well as the previous literatures and work that motivate our research. Chapter 2 reviews the main results concerning the synchronization of homogeneous and heterogeneous MASs in Moreau (2004) [21], Scardovi and Sepulchre (2009) [25] and Wieland et al (2011) [30]. In Chapter 3, the fundamental lemma in Alvergue et al. (2013) [1] is modified. The corresponding improved version of the solution to consensus problem in [1] is derived as well. In Chapter 4, based on the concepts and stability properties of strictly passive systems, a state feedback control law is proposed, as an extension of the results in Chapter 3, for the distributed stabilization of heterogeneous nonlinear MASs. Chapter 5 concludes the whole thesis and presents some points that can be considered as possible future work.

CHAPTER 2
CONSENSUS CONTROL OF HETEROGENEOUS LINEAR MULTI-AGENT SYSTEMS I

This chapter summarizes the main results in Moreau (2004) [21], Scardovi and Sepulchre (2009) [25] and Wieland et al (2011) [30].

2.1 Preliminaries

Given N vectors $\{x_k\}_{k=1}^N$, x is denoted as the stacking of the vectors, i.e.,

$$x = \text{vec}(x_1, \ldots, x_N) = [x_1^T, \ldots, x_N^T]^T.$$

The $n-$dimensional identity matrix is denoted by I_n, and

$$1_N \triangleq [1, \ldots, 1]^T \in \mathbb{R}^N, \quad \mathcal{N} = \{1, \ldots, N\}.$$

In addition, $A \otimes B$ denotes the Kronecker product of two matrices A and B, and $\sigma(A)$ stands for the spectrum of the square matrix A. The closed left-half and right-half complex plane are denoted as $\overline{\mathbb{C}}_-$ and $\overline{\mathbb{C}}_+$ respectively, and the imaginary axis as $j\mathbb{R}$.

2.1.1 Communication Graph

The interconnections between the individual dynamic systems are encoded through a communication graph. Let $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), A_{\mathcal{G}}(t)\}$ be a time-varying weighted digraph (directed graph) with the vertex set $\mathcal{V} = \{v_1, \ldots, v_N\}$, edge set $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$, and adjacency matrix $A_{\mathcal{G}}(t)$. The vertex v_k in $\mathcal{V}$ represents the kth system and the directed edges in $\mathcal{E}(t)$ show the information flows, i.e. $\{v_j, v_k\} \in \mathcal{E}(t)$ if and only if the kth system receives information from the jth system at time t. The adjacency matrix $A_{\mathcal{G}}(t)$ encodes the edge weights with $\{v_j, v_k\} \in \mathcal{E}(t)$ if and only if $a_{kj}(t) \geq \gamma$ for some positive threshold γ, where

$a_{kj}(t)$ is the entry of $A_{\mathcal{G}}(t)$ on the kth row and the jth column. In this chapter, we assume $A_{\mathcal{G}}(t)$ is piecewise continuous and bounded and $a_{kk}(t) = 0, \forall k \in \mathcal{N}$ for all t. A path from v_{i_1} to v_{i_l} at time t is a sequence of distinct vertices $\{v_{i_1}, \ldots, v_{i_l}\}$ such that $\{v_{i_k}, v_{i_{k+1}}\} \in \mathcal{E}(t), k = 1, \ldots, l-1$. If there is a path from v_i to v_j, then v_j is said to be reachable from v_i, which is denoted as $v_i \to v_j$.

The Laplacian matrix $\mathcal{L}(t)$ associated to the digraph $\mathcal{G}(t)$ is defined as

$$l_{kj}(t) = \begin{cases} \sum_{i=1}^{N} a_{ki}(t), & j = k \\ -a_{kj}(t), & j \neq k \end{cases}.$$

Also we recall the following definitions concerning the concept and features of connectivity for time-varying digraphs.

Definition 1. *The digraph $\mathcal{G}(t)$ is connected at time t if there exists a vertex v_j such that every other vertex in the graph is reachable from v_j at time t.*

Definition 2. *A vertex v_k in digraph $\mathcal{G}(t)$ is connected to vertex v_j $(j \neq k)$ in the time interval $I = [t_1, t_2]$ if there is a path from v_j to v_k which respects the direction of the edges of the digraph $(\mathcal{V},\ \cup_{t \in I}\mathcal{E}(t),\ \frac{1}{|I|}\int_I A_{\mathcal{G}}(\tau)d\tau)$.*

Definition 3. *The digraph $\mathcal{G}(t)$ is uniformly connected if there exists a time horizon $T > 0$ and a vertex v_j such that for all t all the vertices v_k $(k \neq j)$ are connected to v_j across time interval $[t, t+T]$.*

2.1.2 A Fundamental Consensus Protocol

Consider N systems exchanging information about their state vector $\{x_k\}_{k=1}^{N}$ over the communication graph $\mathcal{G}(t)$. A widely-studied consensus algorithm for continuous-time MASs is presented as

$$\dot{x}_k(t) = \sum_{j=1}^{N} a_{kj}(t)(x_j(t) - x_k(t)) \tag{2.1}$$

for all $k \in \mathcal{N}$, where $x_k(t) \in \mathbb{R}^n$. In a compact form, (2.1) can be equivalently written as

$$\dot{x}(t) = -(\mathcal{L}(t) \otimes I_n)x(t). \tag{2.2}$$

The following theorem summarizes the main results in Moreau (2004) [21].

Theorem 1. *Assume that the communication graph $\mathcal{G}(t)$ is uniformly connected and the corresponding Laplacian matrix $\mathcal{L}(t)$ is a bounded and piecewise continuous function of time. Then the equilibrium set of consensus states of* (2.2) *is uniformly and exponentially stable. In particular, the N solutions of* (2.1) *asymptotically converge to a common value $\alpha \in \mathbb{R}^n$ as $t \to \infty$.*

The proof of Theorem 1 in [21] considers only the scalar case, i.e., $n = 1$, where $V(x) = \max\{x_1, \ldots, x_N\} - \min\{x_1, \ldots, x_N\}$ can be taken as a candidate Lyapunov function. That is, V is positive definite with respect to the desired equilibrium set $\{x : x_1 = \cdots = x_N\}$ and non-increasing along the solutions of (2.2). Generally V may not decrease at every time instant. However, it can be shown that V decreases over time intervals of sufficient length.

Consensus law (2.1) deals with a very simple case where each individual system only has trivial integrator dynamics. It is indicated in Theorem 1 that only very mild restrictions have to be imposed on the communication topology to ensure the stability and the convergence to a consensus state for such MASs. Uniform connectivity allows the interconnection to be directional and time-varying. In the following sections, the work in Theorem 1 will be extended to more general systems beyond the simple integrators.

2.2 Synchronization of Homogeneous MASs

Consider N identical linear time-invariant (LTI) dynamical systems

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t), \tag{2.3a}$$

$$y_k(t) = Cx_k(t) \tag{2.3b}$$

for all $k \in \mathcal{N}$, with state vector $x_k(t) \in \mathbb{R}^n$, control input $u_k(t) \in \mathbb{R}^p$, and output vector $y_k(t) \in \mathbb{R}^q$. We assume that no additional common references have to be embraced as leaders to synchronize the whole MAS. The objective of synchronization is to find a control law to ensure that $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for all $i, j \in \mathcal{N}$, leading to the following problem statement.

Problem 1. *Given N identical LTI systems defined by* (2.3) *and a uniformly connected communication graph $\mathcal{G}(t)$, find a distributed control protocol over the topological graph $\mathcal{G}(t)$, such that the states of the closed-loop systems asymptotically synchronize.*

Solutions to the aforementioned synchronization problem for homogeneous MASs have been extensively studied in literatures. This section will present the dynamic control laws proposed by Scardovi and Sepulchre (2009) [25] and Wieland et al. (2011) [30] respectively. Before we introduce the main results, it is necessary to mention the following lemma given in [25] as a direct extension of Theorem 1, which is fundamental for the justification of the two solutions.

Lemma 1. *Consider N LTI systems given in* (2.3) *with B and C $n \times n$ nonsingular matrices and $\sigma(A) \subset \overline{\mathbb{C}}_-$. Assume that the communication graph $\mathcal{G}(t)$ is uniformly connected and the corresponding Laplacian matrix $\mathcal{L}(t)$ piecewise continuous and bounded. Then the N solutions of* (2.3) *with the static controller*

$$u_k = B^{-1}C^{-1}\sum_{j=1}^{N} a_{kj}(t)(y_j - y_k), \quad \forall k \in \mathcal{N}, \tag{2.4}$$

uniformly and exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$.

Proof. Applying control protocol (2.4) to (2.3) yields the closed-loop system

$$\dot{x}_k = Ax_k + \sum_{j=1}^{N} a_{kj}(t)(x_j - x_k), \quad \forall k \in \mathcal{N}. \tag{2.5}$$

The change of variable

$$z_k = \mathrm{e}^{-A(t-t_0)} x_k, \quad \forall k \in \mathcal{N},$$

yields

$$\begin{aligned} \dot{z}_k &= -A\mathrm{e}^{-A(t-t_0)} x_k + \mathrm{e}^{-A(t-t_0)} A x_k + \mathrm{e}^{-A(t-t_0)} \sum_{j=1}^{N} a_{kj}(t)(x_j - x_k) \\ &= \sum_{j=1}^{N} a_{kj}(t)(z_j - z_k), \quad \forall k \in \mathcal{N}, \end{aligned}$$

which can be equivalently expressed in the compact form

$$\dot{z} = -(\mathcal{L}(t) \otimes I_n) z. \tag{2.6}$$

According to Theorem 1, the solutions $\{z_k(t)\}_{k=1}^N$ of (2.6) converge exponentially to a common value $x_0 \in \mathbb{R}^n$ as $t \to \infty$, i.e., there exist constants $\varphi_1 > 0$ and $\varphi_2 > 0$ such that for all t_0,

$$\| z_k(t) - x_0 \| \leq \varphi_1 \mathrm{e}^{-\varphi_2 (t-t_0)} \| z_k(t_0) - x_0 \|, \quad \forall t > t_0.$$

Changing back to the original coordinates leads to

$$\| x_k(t) - \mathrm{e}^{A(t-t_0)} x_0 \| \leq \varphi_1 \mathrm{e}^{-\varphi_2 (t-t_0)} \| \mathrm{e}^{A(t-t_0)} \| \times \| x_k(t_0) - x_0 \|, \quad \forall t > t_0.$$

Since all the eigenvalues of the matrix A lie on the closed left-half complex plane, there exists a constant $\varphi_3 > 0$ such that

$$\| x_k(t) - \mathrm{e}^{A(t-t_0)} x_0 \| \leq \varphi_1 \mathrm{e}^{-\varphi_3 (t-t_0)} \| x_k(t_0) - x_0 \|, \quad \forall t > t_0,$$

which proves that all systems exponentially synchronize to a solution of $\dot{x}_0 = A x_0$. □

In Lemma 1, the system matrix A is required to have all eigenvalues on the closed left-half complex plane. In the case where A has eigenvalues with positive real parts, synchronization can be addressed is a similar way. However the graph connectivity should be sufficiently strong to dominate the divergence caused by the unstable modes of the system, which is easy to see from the last part of the proof. It is also worth noting that in the special case where $A = 0$ and $B = C = I_n$, the synchronization problem is simplified to the consensus problem mentioned in Section 2.1.2.

Now consider the following dynamic control protocol

$$\dot{\eta}_k = (A + BK)\eta_k + \sum_{j=1}^{N} a_{kj}(t)(\eta_j - \eta_k + \hat{x}_k - \hat{x}_j), \tag{2.7a}$$

$$\dot{\hat{x}}_k = A\hat{x}_k + Bu_k + H(\hat{y}_k - y_k), \tag{2.7b}$$

$$u_k = K\eta_k \tag{2.7c}$$

with $\hat{y}_k = C\hat{x}_k$, $\eta_k \in \mathbb{R}^n$ and estimated states $\hat{x}_k \in \mathbb{R}^n$ for all $k \in \mathcal{N}$. The following theorem summarizes the solution given in [25].

Theorem 2. *Consider N identical LTI systems given in* (2.3) *with (A, B) stabilizable, (A, C) detectable, and $\sigma(A) \subset \overline{\mathbb{C}}_-$. Assume that the communication graph $\mathcal{G}(t)$ is uniformly connected and the corresponding Laplacian matrix $\mathcal{L}(t)$ piecewise continuous and bounded. Then for any feedback gain matrix K and observer gain matrix H such that $A+BK$ and $A+HC$ are Hurwitz, the N solutions of* (2.3) *with the dynamic controller* (2.7) *uniformly and exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$.*

Proof. Let the estimation errors $e_k = x_k - \hat{x}_k$ and the consensus dynamics $s_k = \hat{x}_k - \eta_k$ for all $k \in \mathcal{N}$. There hold

$$\dot{e}_k = (A + HC)e_k,$$

$$\dot{s}_k = As_k + \sum_{j=1}^{N} a_{kj}(t)(s_j - s_k) - HCe_k,$$

$$\dot{\eta}_k = (A + BK)\eta_k + \sum_{j=1}^{N} a_{kj}(t)(s_k - s_j)$$

for all $k \in \mathcal{N}$. Since H is chosen such that $A + HC$ is Hurwitz, $e_k \to 0$, $\forall k \in \mathcal{N}$ as $t \to \infty$ uniformly exponentially. Furthermore, $\{s_k\}_{k=1}^N$ conform to the dynamics (2.5) discussed in the proof of Lemma 1 with $-HCe_k$ as an extra input that vanishes exponentially. Hence, according to Lemma 1, $s_i - s_j \to 0, \forall i, j \in \mathcal{N}$ as $t \to \infty$ uniformly exponentially. Since $\{s_k\}_{k=1}^N$ are uniformly exponentially synchronized to a solution of $\dot{s}_0 = As_0$, and K is stabilizing, i.e., $A + BK$ is Hurwitz, there holds $\eta_k \to 0$, $\forall k \in \mathcal{N}$ as $t \to \infty$ uniformly exponentially. As a result, it can be concluded that $x_k = s_k + e_k + \eta_k$ exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$ for all $k \in \mathcal{N}$. □

It is easy to see that no absolute reference frame is required in the dynamic control law (2.7). A slight modification to protocol (2.7) is proposed in Wieland et al. (2011) [30] to improve the existing results. Consider

$$\dot{\eta}_k = (A + BK)\eta_k + \sum_{j=1}^{N} a_{kj}(t)(\eta_j - \eta_k + \hat{x}_k - \hat{x}_j) + H(\hat{y}_k - y_k), \tag{2.8a}$$

$$\dot{\hat{x}}_k = A\hat{x}_k + Bu_k + H(\hat{y}_k - y_k), \tag{2.8b}$$

$$u_k = K\eta_k, \tag{2.8c}$$

which is the same to (2.7) except for the term $H(\hat{y}_k - y_k)$ being added to (2.8a). The reason for this minor revision is made clear by the change of coordinates $\zeta_k = \hat{x}_k - \eta_k$, which leads to

$$\dot{\zeta}_k = A\zeta_k + \sum_{j=1}^{N} a_{kj}(t)(\zeta_j - \zeta_k), \tag{2.9a}$$

$$\dot{\hat{x}}_k = A\hat{x}_k + Bu_k + H(\hat{y}_k - y_k), \tag{2.9b}$$

$$u_k = K(\hat{x}_k - \zeta_k) \tag{2.9c}$$

with $\hat{y}_k = C\hat{x}_k$, controller states $\zeta_k \in \mathbb{R}^n$ and estimated states $\hat{x}_k \in \mathbb{R}^n$ for all $k \in \mathcal{N}$. One important feature of the protocol (2.9) is that it can be generally interpreted as three parts:

(2.9a) as synchronized reference generator, (2.9b) as state observer and (2.9c) as static output regulator. The next theorem given by [30] can be regarded as a modification of Theorem 2.

Theorem 3. *Consider N identical LTI systems given in* (2.3) *with* (A, B) *stabilizable,* (A, C) *detectable, and* $\sigma(A) \subset \overline{\mathbb{C}}_-$. *Assume that the communication graph* $\mathcal{G}(t)$ *is uniformly connected and the corresponding Laplacian matrix* $\mathcal{L}(t)$ *piecewise continuous and bounded. Then for any feedback gain matrix* K *and observer gain matrix* H *such that* $A+BK$ *and* $A+HC$ *are Hurwitz, the* N *solutions of* (2.3) *with the dynamic controller* (2.9) *uniformly and exponentially synchronize to a solution of* $\dot{x}_0 = Ax_0$.

Proof. Let the estimation errors $e_k = x_k - \hat{x}_k$ and the tracking errors $\delta_k = x_k - \zeta_k$ for all $k \in \mathcal{N}$. There hold

$$\begin{aligned} \dot{e}_k &= (A + HC)e_k, \\ \dot{\delta}_k &= (A + BK)\delta_k - BKe_k - \sum_{j=1}^{N} a_{kj}(t)(\zeta_j - \zeta_k) \end{aligned}$$

for all $k \in \mathcal{N}$. Since H is chosen such that $A + HC$ is Hurwitz, $e_k \to 0, \forall k \in \mathcal{N}$ as $t \to \infty$ uniformly exponentially. Since $\{\zeta_k\}_{k=1}^{N}$ conform to the dynamics (2.5) discussed in the proof of Lemma 1, according to Lemma 1, $\zeta_i - \zeta_j \to 0, \forall i, j \in \mathcal{N}$ as $t \to \infty$ uniformly exponentially. Furthermore, K is stabilizing, i.e., $A + BK$ is Hurwitz, hence there holds $\delta_k \to 0, \forall k \in \mathcal{N}$ as $t \to \infty$ uniformly exponentially. As a result, it can be concluded that $x_k = \zeta_k + \delta_k$ exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$ for all $k \in \mathcal{N}$. □

The fundamental principle of the protocol (2.9) is to build a reference system model for each individual system and meanwhile synchronize these reference models by Lemma 1. A Similar idea can be developed for the case of heterogeneous MASs as in the next section. However, each local controller has an order twice that of the corresponding local dynamic agent.

2.3 Synchronization of Heterogeneous MASs

Due to the fact that individual systems in a communication network are hardly exactly identical, it is natural to study the synchronization in heterogeneous networks. Consider N heterogeneous LTI dynamical systems

$$\dot{x}_k(t) = A_k x_k(t) + B_k u_k(t), \tag{2.10a}$$

$$y_k(t) = C_k x_k(t) \tag{2.10b}$$

for all $k \in \mathcal{N}$, with state vector $x_k(t) \in \mathbb{R}^{n_k}$, control input $u_k(t) \in \mathbb{R}^{p_k}$, and output vector $y_k(t) \in \mathbb{R}^q$. Most of the assumptions and constraints discussed in problem 1 remain unchanged for the heterogeneous synchronization problem. However, since individual systems are no longer identical and may have different states and state dimensions, state synchronization among all agents is not applicable any more. Thus the consensus is switched to output synchronization, i.e. $y_i(t) - y_j(t) \to 0$ as $t \to \infty$ for all $i, j \in \mathcal{N}$. In addition, the trivial case is excluded where the closed-loop system has an asymptotically stable equilibrium set, because no protocol is needed to obtain synchronization in that case. The following problem statement will be the focus of this section.

Problem 2. *Given N heterogeneous LTI systems defined by* (2.10) *and a uniformly connected communication graph $\mathcal{G}(t)$, find a distributed control protocol over the topological graph $\mathcal{G}(t)$, such that the outputs of the closed-loop systems asymptotically synchronize to some non-trivial common trajectory.*

Apparently when we set $A_k = A$, $B_k = B$ and $C_k = C$ for all $k \in \mathcal{N}$, Problem 2 simply reduces to Problem 1, to which solutions are given in both Theorem 2 and Theorem 3. Before the solution to Problem 2 is presented, the following internal model principle proposed by [30] is introduced first.

Consider a general controller model described by

$$\dot{\xi}_k = D_k\xi_k + E_k y_k + F_k v_k, \tag{2.11a}$$

$$\zeta_k = P_k\xi_k + Q y_k, \tag{2.11b}$$

$$v_k = \sum_{j=1}^{N} a_{kj}(t)(\zeta_k - \zeta_j), \tag{2.11c}$$

$$u_k = G_k\xi_k + M_k y_k + O_k v_k \tag{2.11d}$$

for all $k \in \mathcal{N}$, with controller states $\xi_k \in \mathbb{R}^{\mu_k}$, controller inputs $y_k \in \mathbb{R}^q$, and controller outputs $\zeta_k \in \mathbb{R}^{\mu}$. Protocol (2.11) can be regarded as a general LTI dynamic controller driven by the system outputs y_k and the relative controller output signals v_k. Matrix Q can be the same for all $k \in \mathcal{N}$ since $\{y_k\}_{k=1}^N$ have the same dimension and the same physical meaning. The internal model principle is elaborated in the following lemma.

Lemma 2. *Consider N heterogeneous linear systems defined by* (2.10) *and coupled with dynamic controllers* (2.11). *Assume that the closed-loop system has no asymptotically stable equilibrium set where $y_k = 0, \forall k \in \mathcal{N}$. If $y_i - y_j \to 0$ and $\zeta_i - \zeta_j \to 0$ for all $i, j \in \mathcal{N}$ uniformly and exponentially as $t \to \infty$, then there exist an integer m, matrices $S \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{q\times m}$ with $\sigma(S) \subset \overline{\mathbb{C}}_+$ and (S, R) observable, and matrices $\Pi_k \in \mathbb{R}^{n_k\times m}$, $\Gamma_k \in \mathbb{R}^{p_k\times m}$ such that*

$$A_k\Pi_k + B_k\Gamma_k = \Pi_k S, \tag{2.12a}$$

$$C_k\Pi_k = R \tag{2.12b}$$

for $k = 1, \ldots, N$. Furthermore, there exists $\zeta_0 \in \mathbb{R}^m$ such that the system outputs $y_k(t)$ uniformly and exponentially synchronize to $Re^{-St}\zeta_0$ for all $k \in \mathcal{N}$.

Proof. The closed-loop system has the compact form

$$\dot{\underline{x}} = \underline{A}\ \underline{x} - \underline{B}(\mathcal{L}(t) \otimes I_\mu)\zeta,$$
$$y = \underline{C}\ \underline{x},$$
$$\zeta = \underline{D}\ \underline{x},$$

where $\underline{x} = \text{vec}(x_1, \xi_1, \ldots, x_N, \xi_N)$, and

$$\underline{A} = \text{diag}\left(\begin{bmatrix} A_1+B_1M_1C_1 & B_1G_1 \\ E_1C_1 & D_1 \end{bmatrix}, \quad \ldots \quad, \begin{bmatrix} A_N+B_NM_NC_N & B_NG_N \\ E_NC_N & D_N \end{bmatrix} \right),$$
$$\underline{B} = \text{diag}\left(\begin{bmatrix} B_1O_1 \\ F_1 \end{bmatrix}, \quad \ldots \quad, \begin{bmatrix} B_NO_N \\ F_N \end{bmatrix} \right),$$
$$\underline{C} = \text{diag}\left(\begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad \ldots \quad, \begin{bmatrix} C_N & 0 \end{bmatrix} \right),$$
$$\underline{D} = \text{diag}\left(\begin{bmatrix} QC_1 & P_1 \end{bmatrix}, \quad \ldots \quad, \begin{bmatrix} QC_N & P_N \end{bmatrix} \right).$$

From the given conditions, the state space of the closed-loop system has an asymptotically attractive invariant subspace χ where $y_i = y_j$ and $\zeta_i = \zeta_j$ for all $i, j \in \mathcal{N}$. This implies $(\mathcal{L}(t) \otimes I_\mu)\zeta = 0$ on χ for all t, and thus the closed-loop system can be given by $\dot{\underline{x}} = \underline{A}\ \underline{x}$. Here χ is chosen to contain no stable modes, possess only modes that are observable at the outputs, and be non-trivial with dimension $m > 0$. Denote Span$(\cdot)$ as a collection of all linear combinations of its column vectors. Hence, there exist matrices $\Psi \in \mathbb{R}^{\dim(\underline{x}) \times m}$ and $S \in \mathbb{R}^{m \times m}$ such that

$$\chi = \text{Span}(\Psi), \tag{2.13}$$

$$\underline{A}\Psi = \Psi S, \tag{2.14}$$

where S depicts the dynamics of the closed-loop system regarding the subspace χ. Write Ψ as $\Psi = \left[\Pi_1^T, \Sigma_1^T, \ldots, \Pi_N^T, \Sigma_N^T\right]^T$ with $\Pi_k \in \mathbb{R}^{n_k \times m}$ and $\Sigma_k \in \mathbb{R}^{\mu_k \times m}$ for all $k \in \mathcal{N}$. Then (2.14) becomes equivalent to (2.12a) with $\Gamma_k = M_kC_k\Pi_k + G_k\Sigma_k$. Since we have $y_i = y_j$ and thus $C_i\Pi_i = C_j\Pi_j$ for all $i, j \in \mathcal{N}$, it follows that $C_k\Pi_k = R$ for some $R \in \mathbb{R}^{q \times m}$ and all

$k \in \mathcal{N}$. Moreover, the constraint that all modes in χ are required to be observable at the output y implies the observability of (S, R). As χ is exponentially attractive, all trajectories converge to a trajectory restricted to χ, i.e., $y_k(t)$ uniformly exponentially synchronize to $Re^{-St}\zeta_0$ for all $k \in \mathcal{N}$. □

It is worth noting that the conditions in (2.12) of Lemma 2 is quite similar to the well-known regulator equations appearing in the output regulation of linear systems. Therefore, generally speaking, the physical meaning of those conditions is that all models of individual systems together with their local controllers contain an internal model of a virtual exosystem defined by the dynamics matrix S and output matrix R, i.e.,

$$\dot{x}_0(t) = Sx_0(t),$$
$$y_0(t) = Rx_0(t),$$

and all individual systems are able to track this virtual exosystem to achieve output synchronization.

Lemma 2 presents the necessary conditions for the synchronization of heterogeneous MASs. As will be shown in the sequel, under some mild assumptions, those conditions are also sufficient. Consider protocol

$$\dot{\zeta}_k = S\zeta_k + \sum_{j=1}^{N} a_{kj}(t)(\zeta_j - \zeta_k), \tag{2.15a}$$

$$\dot{\hat{x}}_k = A_k\hat{x}_k + B_k u_k + H_k(\hat{y}_k - y_k), \tag{2.15b}$$

$$u_k = K_k(\hat{x}_k - \Pi_k\zeta_k) + \Gamma_k\zeta_k \tag{2.15c}$$

with $\hat{y}_k = C_k\hat{x}_k$, controller states $\zeta_k \in \mathbb{R}^m$ and estimated states $\hat{x}_k \in \mathbb{R}^{n_k}$ for all $k \in \mathcal{N}$. Similar to (2.9), protocol (2.15) can also be decomposed into three parts: (2.15a) as synchronized reference generator, (2.15b) as state observer and (2.15c) as static output regulator. The next theorem and corollary given by [30] state the main results.

Theorem 4. *Consider N heterogeneous LTI systems given in* (2.10) *with* (A_k, B_k) *stabilizable and* (A_k, C_k) *detectable for all* $k \in \mathcal{N}$*. Assume that there exist an integer* m*, matrices* $S \in \mathbb{R}^{m\times m}$ *and* $R \in \mathbb{R}^{q\times m}$ *with* $\sigma(S) \subset j\mathbb{R}$ *and* (S, R) *observable, and matrices* $\Pi_k \in \mathbb{R}^{n_k\times m}$, $\Gamma_k \in \mathbb{R}^{p_k\times m}$ *satisfying equations* (2.12) *for all* $k \in \mathcal{N}$*. Assume that the communication graph* $\mathcal{G}(t)$ *is uniformly connected and the corresponding Laplacian matrix* $\mathcal{L}(t)$ *piecewise continuous and bounded. Then for any feedback gain matrix* K_k *and observer gain matrix* H_k *such that* $A_k + B_kK_k$ *and* $A_k + H_kC_k$ *are Hurwitz for all* $k \in \mathcal{N}$*, there exists* $\zeta_0 \in \mathbb{R}^m$ *such that the outputs* $y_k(t)$ *of the system* (2.10) *with the dynamic controllers* (2.15) *uniformly and exponentially synchronize to* $Re^{-St}\zeta_0$ *for all* $k \in \mathcal{N}$*.*

Proof. Let the estimation errors $e_k = x_k - \hat{x}_k$ and the tracking errors $\delta_k = x_k - \Pi_k\zeta_k$ for all $k \in \mathcal{N}$. There hold

$$\begin{aligned}\dot{e}_k &= (A_k + H_kC_k)e_k,\\ \dot{\delta}_k &= (A_k + B_kK_k)\delta_k - B_kK_ke_k - \Pi_k\sum_{j=1}^{N} a_{kj}(t)(\zeta_j - \zeta_k)\end{aligned}$$

for all $k \in \mathcal{N}$. The remainder of the proof is the same as the proof of Theorem 3. □

Corollary 1. *Consider N heterogeneous LTI systems given in* (2.10) *with* (A_k, B_k) *stabilizable and* (A_k, C_k) *detectable for all* $k \in \mathcal{N}$*. Assume that the communication graph* $\mathcal{G}(t)$ *is uniformly connected and the corresponding Laplacian matrix* $\mathcal{L}(t)$ *piecewise continuous and bounded. A solution to Problem* 2 *with exponential convergence and bounded outputs exists if and only if there exist an integer* m*, matrices* $S \in \mathbb{R}^{m\times m}$ *and* $R \in \mathbb{R}^{q\times m}$ *with* $\sigma(S) \subset j\mathbb{R}$ *and* (S, R) *observable, and matrices* $\Pi_k \in \mathbb{R}^{n_k\times m}$, $\Gamma_k \in \mathbb{R}^{p_k\times m}$ *satisfying equations* (2.12) *for all* $k \in \mathcal{N}$*.*

Proof. Necessity is implied by Lemma 2 and sufficiency is implied by Theorem 4. □

Note that there are no restrictions on system matrices $\{A_k\}_{k=1}^N$, which allows $\{A_k\}_{k=1}^N$ to have eigenvalues on the right-half complex plane. However, the spectrum of the matrix S, which determines the dynamics of the virtual exosystem, is required to be purely imaginary.

Each local controller has a local reference model. Synchronization of the heterogeneous MAS is replaced by synchronization of the local reference models that are homogeneous. Their outputs serve as reference signals for each heterogeneous agent in the MAS.

CHAPTER 3
CONSENSUS CONTROL OF HETEROGENEOUS LINEAR MULTI-AGENT SYSTEMS II

A modified version of the main results in Alvergue et al. (2013) [1] is presented in this chapter.

3.1 Preliminaries

Most of the notations and concepts employed in this chapter can be referred to § 2.1. In addition, some new concepts and facts are introduced as follows. Denote $e_i \in \mathbb{R}^N$ as a column vector with 1 in its ith entry and 0 elsewhere. Let $M = [\ \mu_{ij}\ ]$ be a matrix with μ_{ij} the (i, j)th entry. The ith singular value of M is denoted by $\sigma_i(M)$ arranged in descending order with $\overline{\sigma}(M) = \sigma_1(M)$. The ith eigenvalue of M is denoted by $\lambda_i(M)$ if M is square. A real square matrix M is called *row dominant* if $|\mu_{ii}| \geq \sum_{j \neq i} |\mu_{ij}|$, *column dominant* if $|\mu_{jj}| \geq \sum_{i \neq j} |\mu_{ij}|$, and doubly dominant if it is both row and column dominant. M is called strictly row or column or doubly dominant if these inequalities are strict.

3.1.1 Graph Connectivity and Laplacian Matrix

Recall the definitions and notations introduced in § 2.1.1.

Definition 4. *The digraph $\mathcal{G}$ is strongly connected if $v_i \to v_j$ and $v_j \to v_i$ for all $i \neq j$ and $i, j \in \mathcal{N}$.*

To prepare the results in later sections, it is necessary to include the following properties concerning the Laplacian matrices associated to connected and strongly connected digraphs.

Lemma 3. *Let $\mathcal{L}$ be the Laplacian matrix associated with the digraph $\mathcal{G}$. If $\mathcal{G}$ is strongly connected, then* 0 *is a simple eigenvalue of $\mathcal{L}$.*

Lemma 4. *Let $\mathcal{L}$ be the Laplacian matrix associated with the digraph $\mathcal{G}$. Then $\mathcal{G}$ is connected if and only if* 0 *is a simple eigenvalue of $\mathcal{L}$.*

3.1.2 Problem Formulation

Consider N heterogeneous LTI dynamical systems

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \tag{3.1a}$$

$$y_i(t) = C_i x_i(t) \tag{3.1b}$$

for all $i \in \mathcal{N}$, with state vector $x_i(t) \in \mathbb{R}^{n_i}$, control input $u_i(t) \in \mathbb{R}^m$, and output vector $y_i(t) \in \mathbb{R}^p$. Clearly the transfer matrix of the ith agent is given by $P_i(s) = C_i(sI_{n_i} - A_i)^{-1}B_i$. Individual systems may have different state dimensions. However, they admit the same number of inputs and the same number of outputs. Hence the consensus problem is concerned with output synchronization, i.e. $y_i(t) - y_j(t) \to 0$ as $t \to \infty$ for all $i, j \in \mathcal{N}$. In particular, the N outputs of the MAS are required to track the output of some exosystem or reference model described by

$$\dot{x}_0(t) = A_0 x_0(t), \tag{3.2a}$$

$$y_0(t) = C_0 x_0(t), \tag{3.2b}$$

where $\sigma(A_0) \subset j\mathbb{R}$, with zero steady-state error. A real-time reference trajectory may not be exactly from this exosystem, but consists of piece-wise step, ramp, sinusoidal signals, etc., whose poles coincide with the eigenvalues of A_0. Moreover, the reference information is required to be transmitted to only one or a few of the N agents in order to reduce the communication overhead. Thus the consensus problem can be summarized as follows.

Problem 3. *Given N heterogeneous LTI systems defined by* (3.1) *over the communication graph $\mathcal{G}$, find the conditions that $\mathcal{G}$ has to satisfy, such that there exist distributed stabilizing controllers and consensus control protocols ensuring that the N outputs of the MAS asymptotically synchronize to some common trajectory given in* (3.2) *under low communication overhead, and at mean time, elaborate these control protocols.*

3.1.3 Useful Facts

Definition 5. *A square matrix M is called an M-matrix (semi M-matrix), if all its off-diagonal elements are either negative or zero, and all its principal minors are positive (non-negative).*

Obviously the Laplacian matrix is a semi M-matrix. The following facts are useful.

Fact 1. *Suppose that all the off-diagonal elements of a square matrix M are either negative or zero. Then the following are equivalent:*

1. *M is an M-matrix;* 2. *$-M$ is Hurwitz;*

3. *The leading principal minors of M are all positive;*

4. *There exists $D = \operatorname{diag}\{d_1, \cdots, d_N\} > 0$ such that MD (DM) is strictly row (column) dominant.*

Fact 2. *Let superscript * denote conjugate transpose. If two square matrices M_1 and M_2 satisfy*

$$M_1 + M_1^* \geq 0, \quad M_2 + M_2^* > 0,$$

then $\det(I + M_1 M_2) \neq 0$. Note that $M_1 = (I + R_1)^{-1}(I - R_1)$, $M_2 = (I + R_2)^{-1}(I - R_2)$ for some (R_1, R_2) satisfying $\overline{\sigma}(R_1) \leq 1$ and $\overline{\sigma}(R_2) < 1$.

Fact 3. *Let $X_\mathrm{a} \geq 0$ be the stabilizing solution to the following algebraic Riccati equation (ARE)*

$$A_\mathrm{a}^T X_\mathrm{a} + X_\mathrm{a} A_\mathrm{a} - X_\mathrm{a} B_\mathrm{a} R_\mathrm{a}^{-1} B_\mathrm{a}^T X_\mathrm{a} + Q_\mathrm{a} = 0, \tag{3.3}$$

where $Q_\mathrm{a} \geq 0$ and $R_\mathrm{a} > 0$. Then with $F_\mathrm{a} = R_\mathrm{a}^{-1} B_\mathrm{a}^T X_\mathrm{a}$, $(A_\mathrm{a} - B_\mathrm{a} F_\mathrm{a})$ is Hurwitz, and the transfer matrix

$$T_{F_\mathrm{a}}(s) = R_\mathrm{a} F_\mathrm{a} (sI - A_\mathrm{a} + B_\mathrm{a} F_\mathrm{a})^{-1} B_\mathrm{a} \tag{3.4}$$

is positive real (PR). That is,

$$T_{F_\mathrm{a}}(s) + T_{F_\mathrm{a}}(s)^* \geq 0 \ \ \forall \ \mathrm{Re[s]} \geq 0. \tag{3.5}$$

Let $A_{F_{\rm a}} = A_{\rm a} - B_{\rm a}F_{\rm a}$ *and* $s = \frac{1}{2}\sigma + j\omega$, $\sigma \geq 0$. *Then ARE* (3.3) *can be written as*

$$(sI - A_{F_{\rm a}})^* X_{\rm a} + X_{\rm a}(sI - A_{F_{\rm a}}) = Q_{\rm a} + F_{\rm a}^T R_{\rm a} F_{\rm a} + \sigma X_{\rm a} \geq 0. \tag{3.6}$$

Multiplying (3.6) *by* $B_{\rm a}^T(sI - A_{F_{\rm a}})^{*-1}$ *from left, by* $(sI - A_{F_{\rm a}})^{-1}B_{\rm a}$ *from right, and using the relation* $R_{\rm a}F_{\rm a} = B_{\rm a}^T X_{\rm a}$ *leads to* (3.5)*, which concludes the PR property.*

3.2 A Modified Fundamental Lemma

The following lemma proposed in Alvergue et al. (2013) [1] is instrumental to the main results in later sections.

Lemma 5. *Let* $\mathcal{L}$ *be the Laplacian matrix associated with the communication digraph* $\mathcal{G}$. *There exist diagonal matrices* $D > 0$ *and* $G \geq 0$ *with* $\text{rank}(G) = 1$, *such that*

$$(D\mathcal{L} + G) + (D\mathcal{L} + G)^T > 0, \tag{3.7}$$

if and only if $\mathcal{G}$ *is connected, and*

$$\text{rank}\left\{\begin{bmatrix} \mathcal{L} & e_i \\ -e_i^T & 0 \end{bmatrix}\right\} = N + 1 \tag{3.8}$$

for at least one index $i \in \mathcal{N}$.

The complete proof for Lemma 5 can be found in [1]. It is worth noting that, given condition (3.8), $i = N$ can be taken without loss of generality and $-(\mathcal{L} + ge_N e_N^T)$ can be proven Hurwitz for some $g > 0$. Since $(\mathcal{L} + ge_N e_N^T)$ possesses either negative or zero off diagonal elements, Fact 1 can be employed to conclude that there exists a diagonal matrix $D > 0$ such that

$$\mathcal{M} = D\mathcal{L} + G = D(\mathcal{L} + ge_N e_N^T) \tag{3.9}$$

is strictly column dominant where $G = gDe_N e_N^T$ is diagonal with only one nonzero element.

Since $D\mathcal{L}$ remains row dominant,

$$\mathcal{M} + \mathcal{M}^T = (D\mathcal{L} + G) + (D\mathcal{L} + G)^T \tag{3.10}$$

is strictly doubly dominant, and thus (3.7) holds true.

It is implied in Lemma 5 that both graph connectedness and rank condition (3.8) have to be satisfied to achieve condition (3.7). However, our further analysis indicates that graph connectedness and condition (3.8) are actually equivalent to each other, which is elaborated in the next lemma.

Lemma 6. *Let $\mathcal{L}$ be the Laplacian matrix associated with the communication digraph $\mathcal{G}$. Then*

$$\text{rank}\left\{\begin{bmatrix} \mathcal{L} & e_i \\ -e_i^T & 0 \end{bmatrix}\right\} = N + 1 \tag{3.11}$$

for at least one index $i \in \mathcal{N}$, if and only if $\mathcal{G}$ is connected.

Proof. For convenience, let

$$\mathcal{H} = \begin{bmatrix} \mathcal{L} & e_i \\ -e_i^T & 0 \end{bmatrix}.$$

For sufficiency: Denote $\mathcal{V}$ as the set containing all vertices and $\mathcal{V}'$ as the set containing all connected vertices. Since $\mathcal{G}$ is connected, we assume $\mathcal{V}'$ contains r, $1 \leq r \leq N$, vertices and $u \to v$ and $v \nrightarrow u$ for all $u \in \mathcal{V}'$, $v \in \mathcal{V} \backslash \mathcal{V}'$. Without loss of generality, the vertices of $\mathcal{G}$ can be renumbered such that $\mathcal{V}' = \{1, 2, \ldots, r\}$. Therefore $\mathcal{L}$ has the block partition form

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}, \tag{3.12}$$

where $\mathcal{L}_{11} \in \mathbb{R}^{r \times r}$ has 0 as a simple eigenvalue, and $\mathcal{L}_{22}$ is nonsingular by Lemma 3 and Lemma 4. Since $\text{rank}(\mathcal{L}_{11}) = r - 1$, we can always choose an index $i \in \mathcal{N}' := \{1, \ldots, r\}$, such that the ith row of $\mathcal{L}_{11}$ can be expressed as a linear combination of the other $r-1$ rows of $\mathcal{L}_{11}$. Then rewrite $\mathcal{H}$ as

$$\mathcal{H} = \begin{bmatrix} \mathcal{L}_{11} & 0 & e_i \\ \mathcal{L}_{21} & \mathcal{L}_{22} & 0 \\ -e_i^T & 0 & 0 \end{bmatrix}.$$

Transformations of matrix lead to

$$\mathcal{H}_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & 0 & e_i \\ \mathcal{L}_{21} & \mathcal{L}_{22} & 0 \\ -e_i^T & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11} & e_i & 0 \\ -e_i^T & 0 & 0 \\ \mathcal{L}_{21} & 0 & \mathcal{L}_{22} \end{bmatrix}$$

and

$$\mathcal{H}_2 = \begin{bmatrix} \mathcal{L}_{11} & e_i \\ -e_i^T & 0 \end{bmatrix} \begin{bmatrix} I & 1_r \\ 1_r^T & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11} + e_i 1_r^T & 0 \\ -e_i^T & -1 \end{bmatrix}.$$

Assume $\mathcal{L}_{11} + e_i 1_r^T$ is singular. Then there exists a nonzero column vector q of dimension r, i.e. $q = [q_1, \dots, q_i, \dots, q_r]^T \neq 0$, such that

$$q^T(\mathcal{L}_{11} + e_i 1_r^T) = 0. \tag{3.13}$$

Since $\mathcal{L}_{11} 1_r = 0$, multiplying 1_r to (3.13) from the right yields $q^T(\mathcal{L}_{11} + e_i 1_r^T)1_r = q^T e_i r = 0$, which indicates $q_i = 0$. Rewrite $\mathcal{L}_{11}$ as

$$\mathcal{L}_{11} = \left[\mathcal{L}_{11}^{(1)T}, \quad \dots \quad , \mathcal{L}_{11}^{(i)T}, \quad \dots \quad , \mathcal{L}_{11}^{(r)T} \right]^T,$$

where $\mathcal{L}_{11}^{(k)}$ represents the kth row of $\mathcal{L}_{11}$ for all $k \in \mathcal{N}'$. Then (3.13) can be written as

$$q_1 \mathcal{L}_{11}^{(1)} + \cdots + q_r \mathcal{L}_{11}^{(r)} + q_i 1_r^T = q_1 \mathcal{L}_{11}^{(1)} + \cdots + q_{i-1} \mathcal{L}_{11}^{(i-1)} + q_{i+1} \mathcal{L}_{11}^{(i+1)} + \cdots + q_r \mathcal{L}_{11}^{(r)} = 0.$$

Since $\text{rank}(\mathcal{L}_{11}) = r - 1$ and $\mathcal{L}_{11}^{(i)}$ is selected as a linear combination of the other $r - 1$

rows, $\mathcal{L}_{11}^{(k)}, \forall k \in \{1, \ldots, i-1, i+1, \ldots, r\}$ have to be linearly independent. As a result, $q_k = 0, \forall k \in \{1, \ldots, i-1, i+1, \ldots, r\}$, and thus vector q has to be 0 to satisfy (3.13), which contradicts to the assumption. Therefore, $\mathcal{L}_{11} + e_i 1_r^T$ and $\left[\begin{smallmatrix} \mathcal{L}_{11} & e_i \\ -e_i^T & 0 \end{smallmatrix}\right]$ are nonsingular. Because $\mathcal{L}_{22}$ is also nonsingular, we obtain rank($\mathcal{H}$) =rank($\mathcal{H}_1$) $= N+1$ for at least one $i \in \mathcal{N}$, which concludes the proof for sufficiency.

For necessity: We use the contrapositive argument to show that if $\mathcal{G}$ is not connected then $\mathcal{H}$ does not have full rank. Since $\mathcal{G}$ is not connected, the vertices can be renumbered to obtain the following block partition form for $\mathcal{L}$,

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & 0 & 0 \\ 0 & \mathcal{L}_{22} & 0 \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}.$$

By Lemma 3, both $\mathcal{L}_{11}$ and $\mathcal{L}_{22}$ have 0 as a simple eigenvalue. Hence it is easy to see that rank($\mathcal{L}$) $\leq N-2$, and consequently rank($\mathcal{H}$) $\leq N$ for all $i \in \mathcal{N}$. □

With the presence of Lemma 6, Lemma 5 can be revised as follows.

Lemma 7. *Let $\mathcal{L}$ be the Laplacian matrix associated with the communication digraph $\mathcal{G}$. There exist diagonal matrices $D > 0$ and $G \geq 0$ with* rank$(G) = 1$*, such that*

$$(D\mathcal{L} + G) + (D\mathcal{L} + G)^T > 0, \tag{3.14}$$

if and only if $\mathcal{G}$ is connected.

Apparently condition (3.14) in Lemma 7 (condition (3.7) in Lemma 5) can be reformulated as

$$\mathcal{M} + \mathcal{M}^T = (D\mathcal{L} + G) + (D\mathcal{L} + G)^T > 2\kappa I \tag{3.15}$$

for some $\kappa > 0$. In fact $\kappa = 1$ can be taken with no loss of generality. Those i satisfying (3.8) can be selected to calculate G as $G = g_i e_i e_i^T$ with some $g_i > 0$. Moreover, algorithms for linear matrix inequality (LMI) can be used to search for D and G. Hence the computation of the required D and G is not an issue.

For the case where MIMO agents have m inputs and p outputs, Lemma 5 holds true if condition (3.8) is extended to

$$\operatorname{rank}\left\{\begin{bmatrix} \widetilde{\mathcal{L}} & e_i \otimes I_q \\ -e_i^T \otimes I_q & 0 \end{bmatrix}\right\} = (N+1)q, \tag{3.16}$$

and condition (3.7) is extended to

$$\left(D\widetilde{\mathcal{L}} + G\right) + \left(D\widetilde{\mathcal{L}} + G\right)^T > 0, \tag{3.17}$$

where

$$\widetilde{\mathcal{L}} = \mathcal{L} \otimes I_q, \tag{3.18a}$$

$$D = \operatorname{diag}(d_1 I_q, \ldots, d_N I_q), \tag{3.18b}$$

$$G = \operatorname{diag}(g_1 I_q, \ldots, g_N I_q) \tag{3.18c}$$

with only one nonzero $g_i > 0$ and $q = m$ or $q = p$. In addition, it is easy to verify that Lemma 6 remains applicable when condition (3.11) is replaced by (3.16). Therefore, Lemma 7 holds true as well for MIMO cases when condition (3.14) is extended to (3.17).

Intuitively (3.11) should be true for any index i as long as the ith node in the connected graph $\mathcal{G}$ is a connected node. In that case, when we try to find a possible index i to determine the rank 1 diagonal matrix G, we can randomly choose one node which can reach all the other nodes and simply set i as the index corresponding to that specific node. Efforts are still made to verify this conjecture.

3.3 Distributed Stabilization

In light of Lemma 7, this section studies the distributed stabilization control protocols using state feedback and output feedback.

3.3.1 State Feedback

Consider the following control protocol

$$u_i = g_i(r - F_i x_i) - d_i \sum_{j=1}^{N} a_{ij}(F_i x_i - F_j x_j) \tag{3.19}$$

with $d_i > 0$, $g_i \geq 0$, and F_i the state feedback gain for all $i \in \mathcal{N}$. Basically the control input for the ith agent consists of two parts: the tracking error with respect to the reference model and the error signals with respect to the adjacent agents. In order to minimize the communication overhead, only one of $\{g_i\}_{i=1}^N$ is required to be nonzero. Substituting (3.19) into (3.1) leads to the closed-loop dynamics

$$\dot{x}_i = A_i x_i - B_i d_i \sum_{j=1}^{N} a_{ij}(F_i x_i - F_j x_j) - B_i g_i(F_i x_i - r). \tag{3.20}$$

Let $\widetilde{\mathcal{L}}$, D and G be in (3.18) with $q = m$. Then (3.20) can be equivalently expressed in the compact form

$$\begin{aligned} \dot{x} &= \left[A - B(D\widetilde{\mathcal{L}} + G)F\right] x + BG\,[1_N \otimes r] \\ &= [A - B\mathcal{M}F]\, x + B\mathcal{M}\,[1_N \otimes r] \end{aligned} \tag{3.21}$$

with $A = \text{diag}(A_1, \ldots, A_N)$, $B = \text{diag}(B_1, \ldots, B_N)$, $F = \text{diag}(F_1, \ldots, F_N)$, and $\mathcal{M} = D\widetilde{\mathcal{L}} + G$. The following theorem can be regarded as a modified version of the main result presented in [1] concerning the stabilization of the MAS via state feedback.

Theorem 5. *Consider the heterogeneous* MAS *given in* (3.1) *with* (A_i, B_i) *stabilizable for all* $i \in \mathcal{N}$. *There exists a stabilizing state feedback control protocol* (3.19) *for the underlying* MAS *over the communication digraph* $\mathcal{G}$, *if* $\mathcal{G}$ *is connected.*

Proof. If $\mathcal{G}$ is connected, then Lemma 7 implies the existence of required D and G, such that (3.15) holds for some $\kappa > 0$. Thus taking $Z = (D\widetilde{\mathcal{L}} + G)/\kappa - I$ yields $Z + Z^T > 0$. Stability of the closed-loop dynamics (3.21) indicates that

$$\det\left[sI - A + B(D\widetilde{\mathcal{L}} + G)F\right] \neq 0 \quad \forall \ \mathrm{Re}\{s\} \geq 0. \tag{3.22}$$

Taking $\kappa = 1$ with no loss of generality and substituting $(D\widetilde{\mathcal{L}} + G) = (Z + I)$ into (3.22) yield

$$\det\left(sI - A + BF + BZF\right) \neq 0 \quad \forall \ \mathrm{Re}\{s\} \geq 0,$$

which is equivalent to the following inequality by simple manipulations,

$$\det\left[I + T_F(s)Z\right] \neq 0 \quad \forall \ \mathrm{Re}\{s\} \geq 0, \tag{3.23}$$

where $T_F(s) = F(sI - A + BF)^{-1}B$. Stabilizability of (A_i, B_i) assures the existence of a stabilizing state feedback control gain F_i such that for each $i \in \mathcal{N}$,

$$T_{F_i}(s) = F_i(sI - A_i + B_iF_i)^{-1}B_i \tag{3.24}$$

is PR on the basis of Fact 3 with $R_{\mathrm{a}} = I > 0$. As a result,

$$T_F(s) + T_F(s)^* \geq 0 \quad \forall \ \mathrm{Re}\{s\} \geq 0$$

with $T_F(s) = \mathrm{diag}\left\{T_{F_1}(s), \ldots, T_{F_N}(s)\right\}$. It follows that inequality (3.23) holds by $Z + Z^T > 0$ and in light of Fact 2. □

Theorem 5 provides a sufficient condition for the stabilization under state feedback control. This sufficient condition becomes necessary for the two special cases as stated below.

Corollary 2. *Consider state feedback control for the* MAS *over the communication digraph* $\mathcal{G}$. *If feedback stability holds for the* MAS *consisting of either* (i) *homogeneous multi-input un-*

stable agents or (ii) *heterogeneous single input unstable agents with* $\{A_i\}_{i=1}^N$ *having a common unstable eigenvalue, then* $\mathcal{G}$ *is connected.*

Now recall the property of uniformly connected graphs in Definition 3. Apparently, if a time-varying digraph $\mathcal{G}(t)$ with Laplacian matrix $\mathcal{L}(t)$ is uniformly connected with a time interval $f > 0$, then

$$\overline{\mathcal{L}}_f(t) = \frac{1}{f}\int_t^{t+f} \mathcal{L}(\tau)\, d\tau$$

is a Laplacian matrix associated with some connected graph for all t, moreover,

$$\text{rank}\left\{\begin{bmatrix} \overline{\mathcal{L}}_f(t) & e_{i(t)} \\ -e_{i(t)}^T & 0 \end{bmatrix}\right\} = N + 1 \tag{3.25}$$

for at least one index $i(t) \in \mathcal{N}$ for all t. The next result extends Theorem 5 to the case of time-varying graphs.

Corollary 3. *Consider the heterogeneous* MAS *given in* (3.1) *with* (A_i, B_i) *stabilizable for all* $i \in \mathcal{N}$. *There exists a stabilizing state feedback control protocol*

$$u_i(t) = g_i(r - F_i x_i) - d_i \sum_{j=1}^N a_{ij}(t)(F_i x_i - F_j x_j) \tag{3.26}$$

for the underlying MAS *over the communication digraph* $\mathcal{G}(t)$, *if* $\mathcal{G}(t)$ *is uniformly connected with a sufficiently small time interval* $f > 0$.

3.3.2 Output Feedback

Observers have to be employed to estimate the states of the MASs when they are not available for feedback control. Two specific observers are introduced in Alvergue et al. (2013) [1] for the design of distributed output feedback controllers for heterogeneous MASs. The local observer is described by

$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_iu_i - L_i(\hat{y}_i - y_i) = (A_i - L_iC_i)\hat{x}_i + L_iC_ix_i + B_iu_i \tag{3.27}$$

with $\hat{y}_i = C_i\hat{x}_i$ and estimated states $\hat{x}_i \in \mathbb{R}^{n_i}$ for all $i \in \mathcal{N}$. It is worth noting that no communication between individual agents is involved in the output estimation part. Let the estimation errors $e_{x_i} = x_i - \hat{x}_i$. There holds

$$\dot{e}_{x_i} = (A_i - L_iC_i)e_{x_i} \tag{3.28}$$

for all $i \in \mathcal{N}$. Replacing the states x_i with the estimated states $\hat{x}_i$ in (3.19) leads to the new control input

$$u_i = g_i(r - F_i\hat{x}_i) - d_i\sum_{j=1}^{N} a_{ij}(F_i\hat{x}_i - F_j\hat{x}_j). \tag{3.29}$$

Combining (3.1), (3.28) and (3.29) together in a compact form leads to the state space equation for the overall MAS

$$\begin{bmatrix} \dot{x} \\ \dot{e}_x \end{bmatrix} = \begin{bmatrix} A - B\mathcal{M}F & B\mathcal{M}F \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} + \begin{bmatrix} B\mathcal{M} \\ 0 \end{bmatrix} (1_N \otimes r) \tag{3.30}$$

with e_x as the stacking of $\{e_{x_i}\}_{i=1}^N$, $L = \operatorname{diag}(L_1, \ldots, L_N)$, and A, B, F and $\mathcal{M}$ the same as in (3.21). It follows that the feedback stability holds, if and only if $(A - B\mathcal{M}F)$ and $(A - LC)$ are both Hurwitz.

In many practical MASs, $y_i(t)$ may not be available to the ith agent for feedback except the one with $g_i \neq 0$. Instead, only relative information can be conveyed between adjacent agents. Hence the following neighborhood observer is proposed as

$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_iu_i + g_i[L_i(\hat{y}_i - y_i) - L_0(\hat{y}_0 - y_0)] + d_iL_i\sum_{j=1}^{N} a_{ij}[(\hat{y}_i - \hat{y}_j) - (y_i - y_j)] \tag{3.31}$$

with $\hat{y}_i = C_i\hat{x}_i$ for all $i \in \{0, \ldots, N\}$. Notice that this time communication between adjacent agents is involved in the output estimation part. In addition, the state of the reference model $x_0(t)$ also requires estimation at the ith agent whenever $g_i \neq 0$ due to possible noise corrupted in the received reference signal $r(t) = C_0 x_0(t)$. As a result, the state space equation for the overall MAS is given by

$$\begin{bmatrix} \dot{x} \\ \dot{e}_x \end{bmatrix} = \begin{bmatrix} A - B\mathcal{M}F & B\mathcal{M}F \\ 0 & A - L\mathcal{M}C \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} + \begin{bmatrix} B\mathcal{M}\left(1_N \otimes C_0 \hat{x}_0\right) \\ \left(I_N \otimes L_0\right)\mathcal{M}\left(1_N \otimes C_0 e_{x_0}\right) \end{bmatrix} \tag{3.32}$$

with $e_{x_0} = x_0 - \hat{x}_0$. For both observers, the separation principle for stabilization holds true as illustrated in (3.30) and (3.32). The main results for stabilization via output feedback are presented in the following theorem.

Theorem 6. *Consider the heterogeneous* MAS *given in* (3.1) *with* (A_i, B_i) *stabilizable and* (A_k, C_k) *detectable for all* $i \in \mathcal{N}$*.* (i) *There exists a stabilizing output feedback control protocol* (3.29) *with observer* (3.27) *or* (3.31) *for the underlying* MAS *over the digraph* $\mathcal{G}$*, if* $\mathcal{G}$ *is connected.* (ii) *There exists a stabilizing output feedback control protocol for the underlying* MAS *over the time-varying digraph* $\mathcal{G}(t)$*, if* $\mathcal{G}(t)$ *is uniformly connected with a sufficiently small time interval* $f > 0$*.*

The proof is omitted here. Many observer-based output feedback controllers satisfy the required PR property, including the controllers designed using $\mathcal{H}_\infty$ loop shaping [20] and LQG/LTR [2] methods. More details can be found in [1].

3.4 Output Consensus

Prior to the study of output consensus, a known result from [11] is introduced first.

Lemma 8. *Let the agent model be given by*

$$\dot{x}_{\mathrm{a}}(t) = A_{\mathrm{a}} x(t) + B_{\mathrm{a}} u_{\mathrm{a}}(t), \quad y_{\mathrm{a}}(t) = C_{\mathrm{a}} x_{\mathrm{a}}(t)$$

with $A_\mathrm{a} \in \mathbb{R}^{n_\mathrm{a}\times n_\mathrm{a}}$*,* $B_\mathrm{a} \in \mathbb{R}^{n_\mathrm{a}\times m_\mathrm{a}}$*,* $C_\mathrm{a} \in \mathbb{R}^{p_\mathrm{a}\times n_\mathrm{a}}$*, and* $(A_\mathrm{a}, B_\mathrm{a})$ *stabilizable, and the reference model be given by* (3.2) *with* $A_0 \in \mathbb{R}^{n_0\times n_0}$*,* $C_0 = I_p$*, and* $p = p_\mathrm{a}$*. Consider the control protocol* $u_\mathrm{a}(t) = -F_\mathrm{a}x_\mathrm{a}(t) + F_{0\mathrm{a}}r(t)$*. Then for each stabilizing state feedback gain* $F_\mathrm{a} \in \mathbb{R}^{m_\mathrm{a}\times n_0}$*, there exists a reference feed-forward gain* $F_{0\mathrm{a}} \in \mathbb{R}^{m_\mathrm{a}\times p}$ *such that* $\lim_{t\to\infty}[y_\mathrm{a}(t) - r(t)] = 0$*, i.e., the output of the agent model tracks the reference with zero steady-state error, if and only if*

$$\mathrm{rank}\left\{\begin{bmatrix} \lambda I - A_\mathrm{a} & B_\mathrm{a} \\ C_\mathrm{a} & 0 \end{bmatrix}\right\} = n_\mathrm{a} + p_\mathrm{a} \tag{3.33}$$

at $\lambda = \lambda_\ell(A_0)$ *for all* $\ell \in \{1, \dots, n_0\}$*.*

Given a stabilizing F_a, computation of $F_{0\mathrm{a}}$ requires first computing the solution $(W_\mathrm{a}, U_\mathrm{a})$ to the equation

$$\begin{bmatrix} I_{n_\mathrm{a}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_\mathrm{a} \\ U_\mathrm{a} \end{bmatrix} A_0 - \begin{bmatrix} A_\mathrm{a} & B_\mathrm{a} \\ C_\mathrm{a} & 0 \end{bmatrix}\begin{bmatrix} W_\mathrm{a} \\ U_\mathrm{a} \end{bmatrix} = \begin{bmatrix} 0 \\ C_0 \end{bmatrix}$$

and then letting $F_{0\mathrm{a}} = U_\mathrm{a} - F_\mathrm{a}W_\mathrm{a}$. Although the agent model is not required to contain the modes $\lambda_\ell(A_0)$ due to the presence of $F_{0\mathrm{a}}$, the inclusion of these modes in A_a, in fact, can improve the performance of both tracking and disturbance rejection, which motivates the following assumption.

Assumption 1. *Each eigenvalue of* A_0*, i.e.,* $\lambda(A_0) \in \{\lambda_\ell(A_0)\}_{\ell=1}^{n_0}$*, is a pole of* $P_i(s)$ *and* $\mathrm{rank}\left\{\lim_{s\to\lambda(A_0)}[s - \lambda(A_0)]P_i(s)\right\}$ *is full for all* $i \in \mathcal{N}$*.*

If Assumption 1 is not satisfied, then dynamic weighting functions $\{W_i(s)\}_{i=1}^N$, which have poles at the missing modes of $\{A_i\}_{i=1}^N$ respectively, can be applied so that Assumption 1 holds for weighted model $P_{W_i}(s) = P_i(s)W_i(s)$, $\forall\, i \in \mathcal{N}$. The design of controller can then proceed with $P_{W_i}(s)$, and $W_i(s)$ should be taken as a part of the controller at last. The next result explains the output consensusability conditions for heterogeneous MASs.

Theorem 7. *Consider the heterogeneous* MAS *given in* (3.1) *with* (A_i, B_i) *stabilizable,* (A_k, C_k) *detectable, and equal number of inputs and outputs, i.e.,* $p = m$*, for all* $i \in \mathcal{N}$*. Let the ref-*

erence model be described by (3.2) *with* $C_0 = I_p$. *Under* Assumption 1*, the underlying* MAS *over the communication digraph* $\mathcal{G}$ *is output consensusable, if* $\mathcal{G}$ *is connected and* (3.33) *holds true for all* $\mathrm{a} = i \in \mathcal{N}$.

It is worth noting that, for output consensus, the control protocol (3.29) is revised to $u_i = F_{0i}\widehat{u}_i$, where

$$\widehat{u}_i = \widehat{G}_i(r - \widehat{F}_i\hat{x}_i) - \widehat{D}_i \sum_{j=1}^{N} a_{ij}(\widehat{F}_i\hat{x}_i - \widehat{F}_j\hat{x}_j) \tag{3.34}$$

with $\widehat{G}_i = R_{0i}^{-1}g_i$, $\widehat{D}_i = R_{0i}^{-1}d_i$, $R_{0i} = F_{0i}'F_{0i}$ and $\widehat{F}_i = F_{0i}^{-1}F_i$ for all $i \in \mathcal{N}$. Theorem 7 only discusses the case of $p = m$. The methods to deal with the issue of $p \neq m$ are elaborated in [1], thus omitted here. In addition, for the case concerning time-varying graphs, the output consensusability condition in Theorem 7 can be easily extended to the uniformly connected graph with sufficiently small $f > 0$.

CHAPTER 4
DISTRIBUTED STABILIZATION OF HETEROGENEOUS NONLINEAR MULTI-AGENT SYSTEMS

In this chapter, we review the properties of strictly passive systems in [5], and combine them with the fundamental Lemma 7 in Chapter 3 to extend the results for distributed stabilization of heterogeneous linear MASs to the case of heterogeneous nonlinear MASs.

4.1 Preliminaries

Suppose λ is a real-valued function and $f = \text{vec}(f_1, \ldots, f_n)$ is a vector field, both defined on an open set $X \subseteq \mathbb{R}^n$. Function $\lambda = \lambda(x) = \lambda(x_1, \ldots, x_n)$ is said to be $C^k, k \geq 0$ if its partial derivatives of order i with respect to $x_1, \ldots, x_n$ exist and are continuous for all $i \in \{0, \ldots, k\}$. In addition, λ is said to be C^∞ (or smooth) if its partial derivatives of order i exist and are continuous for all $i \geq 0$. Let $\langle a, b \rangle$ denote the inner product of a and b. The following differential operations will be used throughout this chapter. The differential of λ and f with respect to x are defined as

$$d\lambda(x) = \frac{\partial \lambda}{\partial x} = \begin{bmatrix} \frac{\partial \lambda}{\partial x_1} & \cdots & \frac{\partial \lambda}{\partial x_n} \end{bmatrix}$$

and

$$df(x) = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

respectively. The derivative of λ along f is defined as

$$L_f \lambda(x) = \langle d\lambda(x), f(x) \rangle = \frac{\partial \lambda}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x).$$

The notation $L_f^k \lambda$ is used to denote λ being differentiated k times along f, which is given by

$$L_f^k\lambda(x) = \frac{\partial(L_f^{k-1}\lambda)}{\partial x} f(x)$$

with $L_f^0\lambda(x) = \lambda(x)$. Suppose g is also a vector field defined on an open set $X \subseteq \mathbb{R}^n$, then

$$L_g L_f \lambda(x) = \frac{\partial(L_f\lambda)}{\partial x} g(x).$$

4.1.1 Passivity

Consider a nonlinear system described in the form

$$\dot{x} = f(x) + g(x)u, \tag{4.1a}$$

$$y = h(x) \tag{4.1b}$$

with state vector $x \in X = \mathbb{R}^n$, control input $u \in U = \mathbb{R}^m$, and output vector $y \in Y = \mathbb{R}^m$. The dimensions of the input and output are the same. The functions $f(x)$, $g(x)$, and $h(x)$ are assumed to be smooth. In addition, $g(x)$ and $h(x)$ can be written as

$$g(x) = [\ g_1(x) \dots g_m(x)\ ] \quad \text{and} \quad h(x) = \text{vec}(h_1(x), \dots, h_m(x))$$

respectively, where $g_i(x),\ i \in \{1, \dots, m\}$ is the ith column of $g(x)$ and $h_i(x),\ i \in \{1, \dots, m\}$ is the ith entry of $h(x)$. Suppose that f has at least one equilibrium. Hence it can be assumed without loss of generality that $f(0) = 0$ and $h(0) = 0$.

A number of concepts will be introduced as the section proceeds. The supply rate is denoted as w, which is a real-valued function defined on $U \times Y$. Assume that for any $u \in U$ and $x(0) = x^\circ \in X$, the output of (4.1) is $y(t) = h(\Phi(t, x^\circ, u))$, and the corresponding $w(s) = w(u(s), y(s))$ satisfies

$$\int_0^t |w(s)| ds < \infty \quad \forall t \geq 0.$$

Definition 6. *A nonlinear system* Σ *of the form* (4.1) *with supply rate* w *is dissipative if there exists a* C^0 *nonnegative function* $V : X \to \mathbb{R}$*, called the storage function, such that for all* $u \in U, x^\circ \in X, t \geq 0$,

$$V(x) - V(x^\circ) \leq \int_0^t w(s)ds$$

with $x = \Phi(t, x^\circ, u)$.

Definition 7. *A nonlinear system* Σ *of the form* (4.1) *is passive if it is dissipative with supply rate given by the inner product, i.e.,* $w = \langle u, y \rangle$*, and storage function* V *satisfying* $V(0) = 0$*. In other words, a system* Σ *is passive if there exists a* C^0 *nonnegative function* $V : X \to \mathbb{R}$ *satisfying* $V(0) = 0$*, such that*

$$V(x) - V(x^\circ) \leq \int_0^t y^T(s)u(s)ds. \tag{4.2}$$

If (4.2) *becomes a strict inequality, then system* Σ *is strictly passive.*

A fundamental property related to passive systems is the Kalman-Yacubovitch-Popov (KYP) property.

Definition 8. *A nonlinear system* Σ *has the* KYP *property if there exists a* C^1 *nonnegative function* $V : X \to \mathbb{R}$ *satisfying* $V(0) = 0$*, such that for all* $x \in X$,

$$L_f V(x) \leq 0, \tag{4.3a}$$

$$L_g V(x) = h^T(x). \tag{4.3b}$$

The relations between being passive and having KYP property are elaborated in the following lemma.

Lemma 9. *A nonlinear system* Σ *having the* KYP *property is passive; A nonlinear passive system* Σ *with a* C^1 *storage function has the* KYP *property.*

Proof. If Σ has the KYP property, then it holds that

$$\begin{aligned}\frac{dV(x(t))}{dt} &= \frac{\partial V(x(t))}{\partial x(t)} \frac{dx(t)}{dt} = \frac{\partial V(x(t))}{\partial x(t)} \left[f(x(t)) + g(x(t))u(t)\right] \\ &= L_f V(x(t)) + L_g V(x(t))u(t) \leq y^T(t)u(t).\end{aligned} \tag{4.4}$$

The integration of (4.4) from 0 to t yields (4.2) with V as the storage function, which implies the passivity of Σ. On the other hand, if Σ is passive with V as the C^1 storage function, taking the derivative of (4.2) with respect to t leads to (4.4), which implies that (4.3) holds true. □

Note that for a strictly passive system, (4.3a) becomes a strict inequality, i.e., $L_f V(x)$ is negative.

4.1.2 Properties of Nonlinear System

Before we proceed to study the issue of feedback equivalence to a passive system, it is necessary to understand the concepts of relative degree, normal form, zero dynamics, minimum phase, etc.

Definition 9. *A nonlinear system Σ of the form* (4.1) *has a relative degree $\{r_1, \ldots, r_m\}$ at $x = x^\circ$ if* (i)

$$L_{g_j} L_f^k h_i(x) = 0 \tag{4.5}$$

for all $j \in \{1, \ldots, m\}$, for all $0 \leq k < r_i - 1$, for all $i \in \{1, \ldots, m\}$, and for all x in a neighborhood of x°, (ii) *the $m \times m$ matrix*

$$\begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \ldots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \ldots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} \tag{4.6}$$

is nonsingular at $x = x^\circ$.

In fact, a linear system given by

$$\dot{x} = Ax + Bu,$$
$$y = Cx$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$, can be regarded as a special case of the nonlinear system (4.1) with $f(x) = Ax$, $g(x) = B$, and $h(x) = Cx$. Hence, for linear systems, equation (4.5) becomes $c_i A^k b_j = 0$ and matrix (4.6) becomes

$$\begin{bmatrix} c_1 A^{r_1-1} b_1 & \dots & c_1 A^{r_1-1} b_m \\ \vdots & \ddots & \vdots \\ c_m A^{r_m-1} b_1 & \dots & c_m A^{r_m-1} b_m \end{bmatrix},$$

where $b_j,\ j \in \{1, \dots, m\}$ is the jth column of B and $c_i,\ i \in \{1, \dots, m\}$ is the ith row of C. Obviously, a linear system has relative degree $\{1, \dots, 1\}$ if matrix

$$\begin{bmatrix} c_1 b_1 & \dots & c_1 b_m \\ \vdots & \ddots & \vdots \\ c_m b_1 & \dots & c_m b_m \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix} = CB$$

is nonsingular, which is equivalent to the condition $\det(CB) \neq 0$ mentioned in many existing literatures.

By definition 9, it is easy to see that a system Σ of the form (4.1) has relative degree $\{1, \dots, 1\}$ at $x = 0$ if matrix

$$L_g h(0) = \begin{bmatrix} L_{g_1} h_1(0) & \dots & L_{g_m} h_1(0) \\ \vdots & \ddots & \vdots \\ L_{g_1} h_m(0) & \dots & L_{g_m} h_m(0) \end{bmatrix}$$

is nonsingular. If this is the case and if the distribution spanned by the m columns of $g(x)$

is involutive [12], $n-m$ real-valued functions $\{z_i(x)\}_{i=1}^{n-m}$, locally defined near $x = 0$ and vanishing at $x = 0$, can be found to qualify as a new set of coordinates, along with the m components of $y = h(x)$. Under this new coordinates (z, y), the system Σ has the following structure, which is called the normal form:

$$\dot{z} = q(z, y), \tag{4.7a}$$

$$\dot{y} = b(z, y) + a(z, y)u. \tag{4.7b}$$

Note that based on form (4.1),

$$\dot{y} = \frac{\partial h(x)}{\partial x}\dot{x} = L_f h(x) + L_g h(x)u.$$

Hence $b(z, y) = L_f h(x)$, $a(z, y) = L_g h(x)$, and matrix $a(z, y)$ is nonsingular for all (z, y) near $(0, 0)$ due to the assumption that Σ has relative degree $\{1, \ldots, 1\}$ at $x = 0$.

Definition 10. *The zero dynamics of a nonlinear system Σ correspond to the dynamics describing the internal behavior of the system when inputs and initial conditions are chosen such that the outputs remain identically zero, i.e., $y = 0,\ \forall t \geq 0$.*

If system Σ has relative degree $\{1, \ldots, 1\}$ at $x = 0$, its zero dynamics locally exist in a neighborhood U of $x = 0$, evolve on the smooth zero dynamics manifold

$$Z^* = \{x \in U : h(x) = 0\},$$

and are characterized by

$$\dot{x} = f^*(x) \quad x \in Z^*,$$

where $f^*(x)$ denotes the restriction to Z^* of the vector field

$$f^*(x) = f(x) + g(x)u^*(x) \tag{4.8}$$

with $u^*(x) = -[L_g h(x)]^{-1} L_f h(x)$.

If the system is given in the normal form (4.7), then the zero dynamics are governed by

$$\dot{z} = q(z, 0).$$

Thus $q(z, y)$ can be written as

$$q(z, y) = f^*(z) + p(z, y)y,$$

where $f^*(z) = q(z, 0)$ and $p(z, y)$ is a smooth function.

Conditions for the existence of a globally defined normal form of the type (4.7) have also been investigated. Readers can refer to [5] for more details. Hence, concepts concerning the minimum phase system can be presented as follows.

Definition 11. *Suppose $L_g h(x)$ is nonsingular at $x = 0$, and the normal form* (4.7) *exists for system Σ. Then Σ is*

(i) *minimum phase if $z = 0$ is an asymptotically stable equilibrium of $f^*(z)$;*

(ii) *weakly minimum phase if there exists a $C^r, r \geq 2$, positive definite function $W^*(z)$, locally defined near $z = 0$ with $W^*(0) = 0$, such that $L_{f^*} W^*(z) \leq 0$ for all z near $z = 0$.*

Suppose $L_g h(x)$ is nonsingular at $x = 0$, and the globally defined normal form (4.7) *exists for system Σ. Then Σ is*

(iii) *globally minimum phase if $z = 0$ is a globally asymptotically stable equilibrium of $f^*(z)$;*

(iv) *globally weakly minimum phase if there exists a $C^r, r \geq 2$, positive definite and proper function $W^*(z)$, defined for all z with $W^*(0) = 0$, such that $L_{f^*} W^*(z) \leq 0$ for all z.*

If the output y in the form (4.7) is replaced by

$$y_F = y + F(z), \tag{4.9}$$

where $F(z)$ is a C^1 function, then the normal form becomes

$$\dot{z} = q(z, y), \tag{4.10a}$$

$$\dot{y} = b(z, y) + a(z, y)u, \tag{4.10b}$$

$$y_F = y + F(z). \tag{4.10c}$$

If a nonlinear system Σ of the form (4.7) is not (globally) minimum phase, we can always try to find a C^1 function $F(z)$, such that the system Σ of the modified form (4.10) with output y_F is (globally) minimum phase, i.e., $z = 0$ is an (globally) asymptotically stable equilibrium of

$$\dot{z} = q(z, -F(z)).$$

4.2 Feedback Equivalence to Passivity

In this section, conditions are derived, under which a given nonlinear system is feedback equivalent to a strictly passive system.

Lemma 10. *There exist a feedback law $\alpha(x)$ and an output map $h_F(x)$ such that*

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)u, \tag{4.11a}$$

$$y_F = h_F(x) \tag{4.11b}$$

is strictly passive with a positive definite and proper C^1 storage function, if the nonlinear system (4.1) *is globally asymptotically stabilizable by state feedback, with a positive definite and proper C^1 Lyapunov function V satisfying $\lim\limits_{x\to\infty} \frac{|L_f V(x)|}{\|L_g V(x)\|^2} \leq M \in \mathbb{R}_+$.*

Proof. Suppose (4.1) is globally asymptotically stabilizable via $u = \alpha(x)$ and V is a positive definite and proper C^1 Lyapunov function for the resulting feedback system

$$\dot{x} = f_F(x)$$

with $f_F(x) = f(x) + g(x)\alpha(x)$. The continuity of $L_f V(x)$ and $L_g V(x)$ is assured since $V(x)$ is C^1 and $f(x)$ and $g(x)$ are both smooth. Setting $\alpha(x) = -K(L_g V(x))^T$ yields

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} = \frac{\partial V(x)}{\partial x} f_F(x) = L_f V(x) - K L_g V(x)(L_g V(x))^T = L_f V(x) - K\|L_g V(x)\|^2.$$

Obviously there exists a scalar $K > 0$ such that $\dot{V}(x) < 0$ for $\forall x$ in a compact set of $\mathbb{R}^n$. For $\forall x \to \infty$, $\dot{V}(x) < 0$ still holds if $\lim_{x\to\infty} \frac{|L_f V(x)|}{\|L_g V(x)\|^2} \leq M$ and K is sufficiently large. Thus the feedback system is made globally asymptotically stable. Then set $h_F(x) = (L_g V(x))^T$. Calculation shows that

$$L_g V(x) = h_F^T(x) \quad \text{and} \quad L_{f_F} V(x) = \frac{\partial V(x)}{\partial x} f_F(x) = \dot{V}(x) < 0,$$

which, in view of Lemma 9, imply the strict passivity of the feedback system (4.11). □

Corollary 4. *There exists a feedback law $\alpha(z, y)$ such that*

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = f_F(z, y) + g(z, y)u, \tag{4.12a}$$

$$y_F = y + F(z) \tag{4.12b}$$

with $f_F(z, y) = \begin{bmatrix} q(z, y) \\ b(z, y) + a(z, y)\alpha(z, y) \end{bmatrix}$ and $g(z, y) = \begin{bmatrix} 0 \\ I \end{bmatrix}$, is strictly passive with a positive definite and proper C^1 storage function, if the nonlinear system (4.10) *has relative degree $\{1, \ldots, 1\}$ at $x = 0$, is globally minimum phase, and is globally asymptotically stabilizable* *by state feedback, with a positive definite and proper C^1 Lyapunov function V satisfying* $\lim_{z,y\to\infty} \frac{|\frac{\partial V(z,y)}{\partial z} q(z,y) + y_F^T b(z,y)|}{\|y_F\|^2} \leq M \in \mathbb{R}_+$.

Proof. Let $W(z)$ be a C^1 Lyapunov function for $\dot{z} = q(z, -F(z))$. Suppose (4.10) is globally asymptotically stabilizable via $u = \alpha(z, y)$ and let $V = W(z) + \frac{1}{2} y_F^T y_F$ be a positive definite and proper C^1 Lyapunov function for the resulting feedback system

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = f_F(z, y).$$

Setting $\alpha(z, y) = -Ka(z, y)^{-1} y_F$ yields

$$\begin{aligned} \dot{V}(z,y) &= \frac{\partial V(z,y)}{\partial \left[z^T \ y^T\right]^T} \begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{\partial V(z,y)}{\partial z} & y_F^T \end{bmatrix} f_F(z,y) = \frac{\partial V(z,y)}{\partial z} q(z,y) + y_F^T b(z,y) - K y_F^T y_F \\ &= \frac{\partial V(z,y)}{\partial z} q(z,y) + y_F^T b(z,y) - K \|y_F\|^2. \end{aligned}$$

Obviously there exists a scalar $K > 0$ such that $\dot{V}(z, y) < 0$ for $\forall \text{vec}(z, y)$ in a compact set of $\mathbb{R}^n$. For $\forall z, y \to \infty$, $\dot{V}(z, y) < 0$ still holds if $\lim_{z,y\to\infty} \frac{|\frac{\partial V(z,y)}{\partial z} q(z,y) + y_F^T b(z,y)|}{\|y_F\|^2} \leq M$ and K is sufficiently large. Thus the feedback system is made globally asymptotically stable. Calculation shows that

$$L_g V(z,y) = \begin{bmatrix} \frac{\partial V(z,y)}{\partial z} & y_F^T \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = y_F^T \quad \text{and} \quad L_{f_F} V(z,y) = \dot{V}(z,y) < 0,$$

which, in view of Lemma 9, imply the strict passivity of the feedback system (4.12). □

4.3 Distributed Stabilization via State Feedback

This section studies the distributed stabilization control protocol for heterogeneous nonlinear MASs using state feedback over the communication graph $\mathcal{G}$, represented by its Laplacian matrix $\mathcal{L}$. All the notations, concepts and results concerning the communication graphs can be referred to previous chapters. Consider N heterogeneous nonlinear systems

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \tag{4.13a}$$

$$y_i = h_i(x_i) \tag{4.13b}$$

with state vector $x_i \in \mathbb{R}^{n_i}$, control input $u_i \in \mathbb{R}$, and output $y_i \in \mathbb{R}$.

Consider the following control protocol

$$u_i = \tilde{g}_i(r - F_i(x_i)) - d_i \sum_{j=1}^{N} a_{ij}(F_i(x_i) - F_j(x_j)) \tag{4.14}$$

with $d_i > 0$ and $\tilde{g}_i \geq 0$ for all $i \in \mathcal{N}$. Basically the control input for the ith agent consists of two parts: the tracking error with respect to the reference model and the error signals with respect to the adjacent agents. In order to minimize the communication overhead, only one of $\{\tilde{g}_i\}_{i=1}^{N}$ is required to be nonzero. Substituting (4.14) into (4.13a) leads to the closed-loop dynamics

$$\dot{x}_i = f_i(x_i) - g_i(x_i)d_i \sum_{j=1}^{N} a_{ij}(F_i(x_i) - F_j(x_j)) - g_i(x_i)\tilde{g}_i(F_i(x_i) - r), \tag{4.15}$$

which can be equivalently expressed in the compact form

$$\begin{aligned} \dot{x} &= f(x) - g(x)\left[D\mathcal{L} + G\right]F(x) + g(x)G\left[1_N \otimes r\right] \\ &= f(x) - g(x)\mathcal{M}F(x) + g(x)\mathcal{M}\left[1_N \otimes r\right] \end{aligned} \tag{4.16}$$

with $x = \text{vec}(x_1, \ldots, x_N)$, $F(x) = \text{vec}(F_1(x_1), \ldots, F_N(x_N))$, $f(x) = \text{vec}(f_1(x_1), \ldots, f_N(x_N))$, $g(x) = \text{diag}(g_1(x_1), \ldots, g_N(x_N))$, $D = \text{diag}(d_1, \ldots, d_N)$, $G = \text{diag}(\tilde{g}_1, \ldots, \tilde{g}_N)$, and $\mathcal{M} = D\mathcal{L} + G$.

Theorem 8. *Consider the heterogeneous nonlinear* MAS *given in* (4.13) *and assume the N individual systems are all globally asymptotically stabilizable by state feedback, with C^1 Lya-*

punov function V_i satisfying $\lim_{x_i \to \infty} \frac{|L_{f_i} V_i(x_i)|}{\|L_{g_i} V_i(x_i)\|^2} \leq M_i \in \mathbb{R}_+$ *for all* $i \in \mathcal{N}$*. There exists a stabilizing state feedback control protocol* (4.14) *for the underlying* MAS *over the communication digraph* $\mathcal{G}$*, if* $\mathcal{G}$ *is connected.*

Proof. Since the N individual systems are all globally asymptotically stabilizable, by Lemma 10 and its proof, there exist a feedback law $F_i(x_i)$, an output map $h_{Fi}(x_i)$, and a scalar $K > 0$, satisfying $F_i(x_i) = K h_{Fi}(x_i)$, such that for all $i \in \mathcal{N}$,

$$\begin{aligned} \dot{x}_i &= f_{Fi}(x_i) + g_i(x_i) u_i, \\ y_{Fi} &= h_{Fi}(x_i) \end{aligned}$$

with $f_{Fi}(x_i) = f_i(x_i) - g_i(x_i) F_i(x_i)$ is strictly passive with a positive definite and proper C^1 storage function $V_i(x_i)$, i.e.,

$$L_{f_{Fi}(x_i)} V_i(x_i) < 0 \quad \text{and} \quad L_{g_i(x_i)} V_i(x_i) = h_{Fi}^T(x_i).$$

As a result, the compact system

$$\dot{x} = f_F(x) + g(x) u, \tag{4.17a}$$

$$y_F = h_F(x) \tag{4.17b}$$

with $y_F = \text{vec}(y_{F1}, \ldots, y_{FN})$, $u = \text{vec}(u_1, \ldots, u_N)$, $f_F(x) = \text{vec}(f_{F1}(x_1), \ldots, f_{FN}(x_N))$ and $h_F(x) = \text{vec}(h_{F1}(x_1), \ldots, h_{FN}(x_N))$, is strictly passive as well. In addition, by Lemma 7, the connectivity of $\mathcal{G}$ implies the existence of required D and G, such that (3.15) holds for some $\kappa > 0$. Thus taking $Z = (D\mathcal{L} + G)/\kappa - I$ yields

$$Z + Z^T > 0. \tag{4.18}$$

Then taking $\kappa = 1$ with no loss of generality and substituting $(D\mathcal{L} + G) = (Z + I)$ into

(4.16) lead to

$$\dot{x} = f_F(x) - g(x)ZF(x) + g(x)\tilde{r}, \tag{4.19a}$$

$$y_F = h_F(x) \tag{4.19b}$$

with input $\tilde{r} = \mathcal{M}\left[1_N \otimes r\right]$. Thus setting $V(x) = \sum_{i=1}^{N} V_i(x_i)$ yields

$$\begin{aligned} L_{g(x)}V(x) &= \left[\frac{\partial V_1(x_1)}{\partial x_1}, \ldots, \frac{\partial V_N(x_N)}{\partial x_N}\right] g(x) \\ &= \left[L_{g_1(x_1)}V_1(x_1), \ldots, L_{g_N(x_N)}V_N(x_N)\right] \\ &= \left[h_{F1}^T(x_1), \ldots, h_{FN}^T(x_N)\right] = h_F^T(x), \end{aligned}$$

$$\begin{aligned} L_{f_F(x)-g(x)ZF(x)}V(x) &= L_{f_F(x)}V(x) - \left[L_{g(x)}V(x)\right] ZF(x) \\ &= \left[\frac{\partial V_1(x_1)}{\partial x_1}, \ldots, \frac{\partial V_N(x_N)}{\partial x_N}\right] f_F(x) - Kh_F^T(x)Zh_F(x) \\ &= \sum_{i=1}^{N} L_{f_{Fi}(x_i)}V_i(x_i) - \frac{1}{2}Kh_F^T(x)(Z + Z^T)h_F(x) < 0, \end{aligned}$$

which, in view of Lemma 9, imply the strict passivity of system (4.19). Hence, by Lemma 6.7 in [14], the unforced closed-loop system $\dot{x} = f_F(x) - g(x)ZF(x)$ is globally asymptotically stable. In fact, system (4.19) can be regarded as a feedback connection of the strictly passive system (4.17) and a memoryless function KZ satisfying (4.18). Hence, by Lemma 6.8 in [14], system (4.19) is also finite-gain $\mathcal{L}_2$ stable, which completes the proof. □

It is worth noting that if the N individual systems (4.13) are already strictly passive, then feedback law $F_i(x_i)$ in (4.14) can be simply chosen as $F_i(x_i) = h_i(x_i) = y_i$ to achieve stability. In other words, output feedback stabilizes the closed-loop system (4.16).

CHAPTER 5
CONCLUSION AND FUTURE WORK

5.1 Conclusion

Chapter 2 summarizes the main results for consensus control of MASs presented in Moreau (2004) [21], Scardovi and Sepulchre (2009) [25] and Wieland et al (2011) [30]. In [25], the authors develop a dynamic output feedback control law that ensures the exponential synchronization of the homogeneous linear MASs, which can be regarded as a generalization of the classical consensus protocol studied in [21]. Building on the results in [21] and [25], [30] turns to the case of heterogeneous MASs and proposes the necessary and sufficient conditions for exponential synchronizability of the MASs over uniformly connected communication graphs. However, since synchronized reference generator, state observer and output regulator are all embedded in the dynamic control protocol, each distributed controller has very high order, which increases the implementation complexity and difficulty.

The modified work presented in Chapter 3 has several advantages over many existing results. One of the major distinctions is that it proves to be sufficient for only one agent to have access to the reference trajectory for the whole MAS to achieve consensus, which significantly lowers the communication overhead between different agents. In addition, the absence of a local reference model at each agent eliminates the need for additional synchronization of the local reference models, thus remarkably reducing the dimensions and complexity of the distributed controllers. Moreover, many existing well-developed design methods such as $\mathcal{H}_\infty$ loop shaping [20] and LQG/LTR [2] can be efficiently used to implement the consensus control law with required performance and robustness. However, the results are applicable only to the cases of connected graph and uniformly connected graph with sufficiently small time interval. Once the time interval becomes larger, the MAS may fail to reach output consensus under proposed control protocol.

Chapter 4 studies the problem of distributed stabilization for heterogeneous nonlinear MASs over connected graphs. The N individual agents are assumed to be single-input single-output (SISO) and admit different dynamical models. The conditions for a nonlinear system being feedback equivalent to a passive system are derived along with the feedback law. A distributed stabilization control protocol is then proposed using state feedback. The properties of connected communication graphs and the idea of feedback connection of two passive systems prove to be extremely crucial for the design process. The result can be interpreted as an extension of the stabilizing control protocol for linear MASs introduced in Chapter 3, and will serve as an instrumental preliminary to the corresponding consensus control problem.

5.2 Future Work

For future research, the distributed stabilization problem needs to be studied for multi-input multi-output (MIMO) nonlinear MASs. However, this problem should be easy to solve since the result for MIMO linear MASs indicates that distributed stability holds as long as condition (3.14) is replaced by (3.17).

In many practical MASs, the state information is usually not available for feedback. In that case, output feedback can be applied and distributed observers have to be employed to estimate the states of individual agents. It is mentioned at the end of Chapter 4 that direct output feedback stabilizes the closed-loop MAS if the N individual agents in the network are all strictly passive. However, how to stabilize the MAS using output feedback control laws remains to be a difficult problem when the N agents are not necessarily strictly passive, especially since the design of observers for nonlinear systems are much more complex than the case of linear systems.

Just as we claimed before, the distributed stabilization is just a fundamental part in the consensus problem. Our final goal is to find a distributed control law such that the outputs of the heterogeneous nonlinear MAS asymptotically synchronize to some desired common trajectory. In [6], coupling control laws are derived for the output synchronization of homogeneous nonlinear MAS with relative degree one and weakly minimum phase, while the results are limited to the case when the interconnection graph is balanced. In [11], the author considers the output regulation problem for regular nonlinear systems and proposes to embed an internal model to the controller, which happens to be quite similar to the internal model principle introduced in Chapter 2.

Future work can also be focused on some more complex consensus problems of heterogeneous nonlinear MASs, such as heterogeneous nonlinear MASs with time-delays, heterogeneous nonlinear MASs under time-varying topologies or random networks, discrete-time heterogeneous nonlinear MASs, etc.

REFERENCES

[1] Luis Alvergue, Abhishek Pandey, Guoxiang Gu, and Xiang Chen. Output consensus control for heterogeneous multi-agent systems. In *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*, pages 1502–1507. IEEE, 2013.

[2] Brian DO Anderson and John B Moore. *Optimal control: linear quadratic methods*. Courier Dover Publications, 2007.

[3] Randal W Beard, Timothy W McLain, Michael A Goodrich, and Erik P Anderson. Coordinated target assignment and intercept for unmanned air vehicles. *Robotics and Automation, IEEE Transactions on*, 18(6):911–922, 2002.

[4] Louis E Buzogany, M Pachter, and JJ dAzzo. Automated control of aircraft in formation flight. In *AIAA Guidance, Navigation, and Control Conference*, volume 3, pages 1349–1370, 1993.

[5] Christopher I Byrnes, Alberto Isidori, and Jan C Willems. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *Automatic Control, IEEE Transactions on*, 36(11):1228–1240, 1991.

[6] Nikhil Chopra and Mark W Spong. Output synchronization of nonlinear systems with relative degree one. In *Recent advances in learning and control*, pages 51–64. Springer, 2008.

[7] Nikhil Chopra and Mark W Spong. On exponential synchronization of kuramoto oscillators. *Automatic Control, IEEE Transactions on*, 54(2):353–357, 2009.

[8] Jorge Cortés, Sonia Martínez, and Francesco Bullo. Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *Automatic Control, IEEE Transactions on*, 51(8):1289–1298, 2006.

[9] Håvard Fjær Grip, Tao Yang, Ali Saberi, and Anton A Stoorvogel. Output synchronization for heterogeneous networks of non-introspective agents. *Automatica*, 48(10):2444–2453, 2012.

[10] Sandra Hirche and Shinji Hara. Stabilizing interconnection characterization for multi-agent systems with dissipative properties. In *Proc. IFAC world congress*, pages 1571–1577. Citeseer, 2008.

[11] Jie Huang. *Nonlinear output regulation: Theory and applications*, volume 8. SIAM, 2004.

[12] Alberto Isidori. *Nonlinear control systems*, volume 1. Springer, 1995.

[13] Ali Jadbabaie, Jie Lin, and A Stephen Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *Automatic Control, IEEE Transactions on*, 48(6):988–1001, 2003.

[14] Hassan K Khalil and JW Grizzle. *Nonlinear systems*, volume 3. Prentice hall Upper Saddle River, 2002.

[15] Hongkeun Kim, Hyungbo Shim, Juhoon Back, and Jin Heon Seo. Consensus of output-coupled linear multi-agent systems under fast switching network: Averaging approach. *Automatica*, 49(1):267–272, 2013.

[16] Hongkeun Kim, Hyungbo Shim, and Jin Heon Seo. Output consensus of heterogeneous uncertain linear multi-agent systems. *Automatic Control, IEEE Transactions on*, 56(1):200–206, 2011.

[17] Yoshiki Kuramoto. Self-entrainment of a population of coupled non-linear oscillators. In *International symposium on mathematical problems in theoretical physics*, pages 420–422. Springer, 1975.

[18] Zhongkui Li, Zhisheng Duan, Guanrong Chen, and Lin Huang. Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint. *Circuits and Systems I: Regular Papers, IEEE Transactions on*, 57(1):213–224, 2010.

[19] Cui-Qin Ma and Ji-Feng Zhang. Necessary and sufficient conditions for consensusability of linear multi-agent systems. *Automatic Control, IEEE Transactions on*, 55(5):1263–1268, 2010.

[20] Duncan C McFarlane and Keith Glover. Robust controller design using normalized coprime factor plant descriptions. 1990.

[21] Luc Moreau. Stability of continuous-time distributed consensus algorithms. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 4, pages 3998–4003. IEEE, 2004.

[22] Reza Olfati-Saber and Parisa Jalalkamali. Coupled distributed estimation and control for mobile sensor networks. *Automatic Control, IEEE Transactions on*, 57(10):2609–2614, 2012.

[23] Reza Olfati-Saber and Jeff S Shamma. Consensus filters for sensor networks and distributed sensor fusion. In *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pages 6698–6703. IEEE, 2005.

[24] Wei Ren, Kevin Moore, and Yang Quan Chen. High-order consensus algorithms in cooperative vehicle systems. In *Networking, Sensing and Control, 2006. ICNSC'06. Proceedings of the 2006 IEEE International Conference on*, pages 457–462. IEEE, 2006.

[25] Luca Scardovi and Rodolphe Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557–2562, 2009.

[26] Herbert G Tanner, Ali Jadbabaie, and George J Pappas. Flocking in fixed and switching networks. *Automatic Control, IEEE Transactions on*, 52(5):863–868, 2007.

[27] John Nikolas Tsitsiklis. Problems in decentralized decision making and computation. Technical report, DTIC Document, 1984.

[28] Dmitry Vengertsev, Hongkeun Kim, Hyungbo Shim, and Jin Heon Seo. Consensus of output-coupled linear multi-agent systems under frequently connected network. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 4559–4564. IEEE, 2010.

[29] Jinhuan Wang, Daizhan Cheng, and Xiaoming Hu. Consensus of multi-agent linear dynamic systems. *Asian Journal of Control*, 10(2):144–155, 2008.

[30] Peter Wieland, Rodolphe Sepulchre, and Frank Allgöwer. An internal model principle is necessary and sufficient for linear output synchronization. *Automatica*, 47(5):1068–1074, 2011.

[31] JD Wolfe, DF Chichka, and JL Speyer. Decentralized controllers for unmanned aerial vehicle formation flight. *AIAA Paper*, pages 96–3833, 1996.

[32] Hiroaki Yamaguchi. A cooperative hunting behavior by mobile-robot troops. *The International Journal of Robotics Research*, 18(9):931–940, 1999.

[33] Hiroaki Yamaguchi, Tamio Arai, and Gerardo Beni. A distributed control scheme for multiple robotic vehicles to make group formations. *Robotics and Autonomous systems*, 36(4):125–147, 2001.

[34] Jun Zhao, David J Hill, and Tao Liu. Synchronization of dynamical networks with nonidentical nodes: Criteria and control. *Circuits and Systems I: Regular Papers, IEEE Transactions on*, 58(3):584–594, 2011.

VITA

Jianming Zhou was born in Shanghai, China. He is the son of Zemin Zhou and Wenying Zhang. He completed his undergraduate program of Automation and obtained his Bachelor's Degree of Science in Engineering from Shanghai Jiao Tong University, Shanghai, China, in June 2013. In Fall 2013, he came to the United States and joined the Department of Electrical and Computer Engineering at Louisiana State University, Baton Rouge, in pursuit of his graduate study, focusing on Systems Control. He anticipates being awarded the Master's Degree of Science in Electrical Engineering in May 2015.