12-13-2017

Characterizations of Some Classes of Graphs That Are Nearly Series-Parallel

Victoria Fontaine
Louisiana State University and Agricultural and Mechanical College

CHARACTERIZATIONS OF SOME CLASSES OF GRAPHS THAT ARE NEARLY SERIES-PARALLEL

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Victoria Fontaine
B.S. in Mathematics, Seton Hall University, 2012
M.S. in Mathematics, Louisiana State University, 2014
M.M. in Organ Performance, Louisiana State University, 2016
May 2018

Acknowledgments

My advisor, James Oxley, gave me the benefit of his profound and patient guidance on all things mathematical (and some non-mathematical) as I researched and wrote this dissertation. For all of it, I am intensely grateful. Thank you, James.

I also offer my thanks to Cecilia Philippine and William Broad, whose support was instrumental in helping me complete my graduate school journey.

This dissertation is dedicated to my first, best, and greatest teacher: my father, Robert Fontaine.

Table of Contents

Abstract

A series-parallel graph can be built from a single-edge graph by a sequence of series and parallel extensions. The class of such graphs coincides with the class of graphs that do not have the complete graph K_4 as a minor. This dissertation considers a class $\mathcal{M}_1$ of graphs that are close to being series-parallel. In particular, every member of the class has the property that one can obtain a series-parallel graph by adding a new edge and contracting it out, or by splitting a vertex into two vertices whose neighbor sets partition the neighbor set of the original vertex. The class $\mathcal{M}_1$ is minor-closed. The goal of this dissertation is to show that $\mathcal{M}_1$ has exactly twelve excluded minors, including K_5, the cube, and the octahedron.

Chapter 1

Introduction

This initial chapter presents the commonly known definitions and theorems that we will use in subsequent chapters. For graph theory, we largely follow Diestel [7], with nontrivial reference to Chartrand and Lesniak [3]. For matroid theory, we follow Oxley [17].

1.1 Basic Graph Theory Definitions

A *graph* G is a pair that consists of a set $V(G)$ and a multiset $E(G)$. The elements of the *vertex set* $V(G)$ are *vertices* of G, and the elements of the *edge set* $E(G)$, called *edges*, are unordered pairs of (possibly identical) vertices. We will simply write V and E for $V(G)$ and $E(G)$ when there is no ambiguity about which graph we mean. If V is the empty set, then G is the *empty* graph; if E is the empty set, then G is *trivial*. We say a graph with vertex set V is a graph *on* V. The *order* of G is the number of vertices in G, denoted $|G|$. In a similar fashion, the number of edges, called the *size* of G, is denoted by $||G||$. All graphs in this dissertation have finite order and size.

An edge $e = (u, v)$ is *incident* with its *endpoints* u and v, while vertices u and v are *adjacent vertices* or, alternatively, *neighbors*. The *set of neighbors* of a vertex v of $V(G)$ is written $N_G(v)$, or $N(v)$, if no ambiguity will result. The number of edges incident with vertex v is the *degree* of v, denoted by $deg_G(v)$. Again, we abbreviate our notation to $deg(v)$ if the context makes clear which graph is under consideration. If $deg(v) = 0$, then we say v is an *isolated vertex*. We define the *minimum degree* of G as $\delta(G) = min\{deg(v) : v \in V(G)\}$, while we define the *maximum degree* of G as $\Delta(G) = max\{deg(v) : v \in V(G)\}$.

An edge e that is only incident with one vertex is a *loop*. Note that a loop adds 2 to a vertex's degree. If edges e and f both have the same pair of endpoints, they are *parallel edges*. The set of all edges incident with a given pair of distinct endpoints is a *parallel class*; given some non-loop edge e, the *parallel class of* e specifies the parallel class to which e belongs. A graph G is *simple* if it has no parallel edges and no loops.

Two graphs G_1 and G_2 are *isomorphic* if there are bijections $\phi : V(G_1) \to V(G_2)$ and $\theta : E(G_1) \to (G_2)$ such that a vertex v of G_1 is incident with an edge e of G_1 if and only if $\phi(v)$ is incident with $\theta(e)$. We write $G_1 \cong G_2$ when G_1 and G_2 are isomorphic. A graph H is a *subgraph* of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say H is a *spanning subgraph* of G if $V(G) = V(H)$. Graph H is an *induced subgraph* of G if $V(H)$ is a nontrivial subset of $V(G)$, and every edge in $E(G)$ that has both endpoints in $V(H)$ is in $E(H)$. Alternatively, we say $U = V(H)$ *induces* a subgraph of G and write $G[U]$.

A simple graph is *complete* if every two distinct vertices are adjacent. We denote the complete graph on n vertices by K_n. A graph G is *k-partite* if there is a partition $V_1, V_2, \ldots, V_k$ of $V(G)$ into k subsets, known as *partite sets*, where every edge of $E(G)$ has one endpoint in V_i and one in V_j, for $1 \leq i, j \leq k$, with $i \neq j$ and $k \geq 2$. If $k = 2$, the graph is *bipartite*. A simple graph is a *complete k-partite graph* if it is k-partite with one additional property: for any two vertices u and v that are in distinct partite sets, the edge (u, v) is in the graph's edge set. We denote a complete k-partite graph by $K_{n_1,n_2,\ldots,n_k}$, where n_i is the cardinality of partite set V_i. Thus, the notation for a complete bipartite graph is K_{n_1,n_2}. The graph K_{1,n_2} is a *star*.

Say u and v are vertices, not necessarily distinct, of a graph G. A $u - v$ *walk* is a finite, alternating sequence of vertices and edges

$$u = v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n = v$$

of G, with $e_i = (v_{i-1}, v_i)$ for each $i \in \{1, 2, \dots, n\}$. The *length* of the walk is n, the number of edges. Vertices $v_1, v_2, \dots, v_{n-1}$ are *internal vertices*. A $u - v$ *trail* is a $u - v$ walk where no edge is repeated. A $u - v$ walk or $u - v$ trail is *closed* if vertex u is equal to vertex v. A $u - v$ *path* is a $u - v$ walk where no vertex is repeated. A walk, path, or trail is *trivial* if its length is zero – that is, if it has no edges and consists of a single vertex. Given two or more $u - v$ paths (or walks or trails) in graph G, we call them *internally disjoint* if they have no internal vertices in common.

A *cycle* of graph G is nontrivial closed trail where no internal vertex is repeated. A cycle of length k is a k-cycle. If a graph has no cycles, we say it is *acyclic*. We denote a graph of order n that is a cycle by C_n; the length of C_n is n, as well. A graph of order $n + 1$ that consists of a cycle of length n (the *rim*) together with a vertex (the *hub*) that is adjacent to every rim vertex is called a *wheel*, denoted W_n. The edges incident with the hub vertex are called *spokes*.

A graph G is *connnected* if there exists a path between any pair of its vertices. A *forest* is an acyclic graph, and a *tree* is an acyclic connected graph. In a tree, a vertex with degree 1 is a *leaf*.

The following theorem about trees is well known (see, for example, Chartrand and Lesniak [3]).

Theorem 1.1. *Every nontrivial tree has at least two leaves.*

1.2 Graph Minors

Given subset U of $V(G)$, we denote by $G - U$ the graph that results from *deleting* from G all the vertices of U and their incident edges. For the sake of brevity, when we are deleting a single vertex v from G, we will simply write $G - v$. Given subset F' of $E(G)$, the notation $G\backslash F'$ indicates the graph that results from *deleting* from G all the edges of F'. We write $G\backslash e$ when we are deleting a single edge e from G. With G/e, we indicate the *contraction* of edge $e = (u, v)$, where we delete e and identify its endpoints into a new *conglomerate vertex* w. Observe that the contraction of a loop is equal to its deletion. Given a subset F of $E(G)$, for G/F, we contract each edge of F. Note that G/F is well-defined, since an easy check establishes that $(G/e)/f = (G/f)e$ for any edges e and f of G.

A graph H is a *minor* of graph G if we can obtain H from G through a sequence of vertex deletions, edge deletions, and edge contractions, such that $H = (G - U)\backslash F'/F$, with F' and F disjoint. We say that G has an H-minor, or that G has H as a minor; more informally, G contains H as a minor, or G contains H. Graph H is a *proper* minor of G if H is a minor of G, but $G \neq G$. Observe that any subgraph of G is also a minor of G, and G is a minor of itself – an *improper minor*.

A class of graphs is closed under the taking of minors, or *minor-closed*, if any minor of a member of the class is also a member of the class. An *excluded minor* of a minor-closed class of graphs is not in the class itself, but each of its proper minors are. A minor-closed class of graphs can be characterized by a list of its excluded minors.

One of the most widely known excluded-minor theorems in graph theory, due to Wagner [27], gives the two excluded minors for the class of planar graphs, K_5 and $K_{3.3}$. A *planar graph* is a graph that can be drawn in the Euclidean plane so that the vertices of G correspond to distinct points of the plane; each edge of G

corresponds to a simple curve that connects the ends of the edge but meets no other vertices; and each point of intersection of two such simple curves is an end of both edges. A graph that is so drawn in the plane is a *plane graph.*

Theorem 1.2. *A graph G is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.*

1.3 Connectivity in Graphs

Recall that a graph G is connnected if there exists a path between any pair of its vertices. When a graph is not connected, we say it is *disconnected.* Given a subset U of $V(G)$, if the induced subgraph $G[U]$ is connected, then we consider U to be connected in G. A *component* of G is a maximal connected subgraph of G.

In a graph G, a *separating set* is a subset U of $V(G)$ such that $G - U$ is disconnected; *i.e.*, there is more than one component in $G - U$. If $G - v$ has more components than G, we say vertex v is a *cut vertex.* For $k \geq 1$, a graph G is *k-connected* if $|G| > k$ and $G - X$ is connected for all subsets X of $V(G)$ where X has fewer than k vertices. In other words, the cardinality of any separating set of G is at least k. We say a graph is *minimally k-connected* if G is k-connected but $G\backslash e$ fails to be k-connected for every edge e in $E(G)$.

The next lemma, which is straightforward to prove directly, follows from a matroid result of Tutte [26].

Lemma 1.3. *If G is a 2-connected loopless graph with at least four vertices, and e is an edge of G, then at least one of G/e and $G\backslash e$ is 2-connected and loopless.*

Given graph G, a subset F' of $E(G)$ is a *disconnecting set* if G is connected and $G\backslash F'$ is disconnected. An *edge cut* is a subset F of $E(G)$ such that $G\backslash F$ has more components than G. If F contains exactly one edge e, then we say e is a *cut edge.* A minimal edge cut is a *bond* or *cocycle* of G. For $l \geq 0$, a graph G is *l-edge-connected*

if $|G| > 1$ and $G - Y$ is connected for all subsets Y of $E(G)$ where Y has fewer than l edges. In other words, the cardinality of any edge cut of G is at least l.

Diestel [7] presents a Global Version of Menger's Theorem [15], which was first proved by Whitney [30] .

Theorem 1.4. *Let G be a graph.*

(i) G is k-connected if and only if it contains k internally disjoint paths between any two vertices.

(ii) G is k-edge-connected if and only if it contains k edge-disjoint paths between any two vertices.

For a graph G, its *(vertex-)connectivity* $\kappa(G)$ is equal to the greatest k for which G is k-connected. The connectivity of a disconnected graph is 0, and the connectivity of a complete graph on n vertices is $n - 1$. The *edge-connectivity* $\lambda(G)$ is equal to the greatest l for which G is l-edge-connected. The edge-connectivity of a disconnected graph is zero.

The following theorem on the relationships among a graph's connectivity, edge connectivity, and minimum degree is a result of Whitney [30].

Theorem 1.5. *In any loopless, nontrivial graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

The following is Tutte's Wheels Theorem [25].

Theorem 1.6. *If G is a 3-connected simple graph on at least four vertices that is not a wheel, then there is an edge e of G such that at least one of G/e and $G\backslash e$ is also 3-connected and simple.*

In view of the last result, throughout this dissertation, when we state that a graph G is 3-connected, we will mean it is both 3-connected and simple, and

has at least four vertices; equivalently, cycle matroid $M(G)$ is 3-connected (see Proposition 1.12 in Section 1.7).

1.4 Selected Graph Operations

For edge e of a graph G, we call G an *undeletion* of $G\backslash e$, and an *uncontraction* of G/e. Suppose $v \in V(G/e)$ is the conglomerate vertex resulting from the contraction of edge $e = (v_1, v_2)$ in G. Then *splitting* vertex v results in graph $G\backslash e$. Observe that splitting vertex v is equivalent to uncontracting and then deleting e.

For an edge $e = (u, v)$ of a graph G, we *subdivide* e by deleting it and replacing it with a $u - v$ path of length 2 or more. Note that we also allow a loop to be subdivided, where this consists of replacing the loop by a cycle of length two or more. A graph G' is a *subdivision* of G if we can produce G' by subdividing one or more edges of G. To obtain the *simplification* $si(G)$ of graph G, we delete all but one edge from every parallel class of G and delete every loop of G. The graph $si(G)$ is sometimes called the *underlying simple graph* of G.

A plane graph G partitions the Euclidean plane into regions that we call *faces.* More formally, let P be the set of points in the plane that are neither vertices of G nor lie on edges of G. We consider two points of P to be in the same face of G so long as there exists a simple curve joining them, such that every point on that simple curve is in P. The *dual graph* G^* of G has its vertex set in one-to-one correspondence with the set of faces of G; the edge $e^* = (u, v)$ is in $E(G^*)$ if and only if u and v correspond to faces of G that share a common edge, and there is exactly one edge of G^* for every common edge of G between two faces.

A graph G' is a *parallel extension* of a graph G if G' has a 2-cycle consisting of edges $\{e, f\}$ such that $G'\backslash f = G$. We may also say that G is a *parallel deletion* of G', and we speak of edges e and f being *in parallel* in G'. Graph G'' is a *series extension* of G if edges e and f form a bond in G'', such that $G''/f = G$. We may

equivalently call G a *series contraction* of G'', and we say edges e and f are *in series* in G''. Let H be a graph such that the deletion of all loops from H is a forest. If we can recursively construct a graph G from such a graph H by the operations of parallel extension and series extension, then G is a *series-parallel graph*.

We present two theorems that characterize series-parallel graphs. The first is an excluded-minor theorem; the second employs bounded tree-width.

Theorem 1.7. *A graph G is a series-parallel graph if and only if it has no minor isomorphic to K_4.*

Let G be a graph, let T be a tree, and let $\mathcal{V} = \{V_t\}_{t \in V(T)}$ be a family of vertex sets where each V_t is a subset of $V(G)$. The pair $(T, \mathcal{V})$ is a *tree decomposition* of G if it satisfies three conditions:

(i) $V(G) = \bigcup_{t \in T} V_t$;

(ii) for every edge e in $E(G)$, there exists some t such that both endpoints of e are in V_t; and

(iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever vertices t_1, t_2, and t_3 of T have a $t_1 - t_3$ path in T where t_2 is an internal vertex.

We define the *width* of $(T, \mathcal{V})$ as $max\{|V_t| : t \in V(T)\}$, and take the *tree-width* $tw(G)$ of G to be the least width of all tree decompositions of G.

We are concerned with tree-width because it can be used to characterize series-parallel graphs, as stated in the following theorem, which is given in Diestel [7].

Theorem 1.8. *A graph has tree-width at most two if and only if it has no K_4-minor.*

Let G_1 and G_2 be graphs with edges e_1 and e_2, respectively. Arbitrarily assign a direction to e_i for each i in $\{1, 2\}$; label the tail u_i and the head v_i. We obtain the

series connection $S(G_1, G_2)$ of G_1 and G_2 in the following way. Delete edges e_1 and e_2, identify vertices u_1 and u_2, and add a new edge with endpoints v_1 and v_2. We obtain the *parallel connection* $P(G_1, G_2)$ of G_1 and G_2 in a slightly different way, as follows. Delete edges e_1 and e_2, identify u_1 and u_2 as vertex u, identify vertices v_1 and v_2 as vertex v, and add a new edge (u, v). Unless exactly one of e_1 and e_2 is a loop, then, we can simply view the parallel connection as the identification of e_1 and e_2 such that their directions agree.

A *clique* in a graph G is a subgraph of G that is complete. For two graphs G_1 and G_2, we obtain the *k-sum* $G_1 \oplus_k G_2$ by identifying a clique of G_1 having order k with a clique of G_2 of the same order, and then deleting the identified edges. Thus, $G_1 \oplus_0 G_2$ is the disjoint union of G_1 and G_2, while $G_1 \oplus_1 G_2$ identifies one vertex from G_1 with one vertex from G_2.

1.5 Matroid Definitions

Although the objects we work with are almost exclusively graphs, in this dissertation, the motivation for and background of the research lies in matroid theory. As we will see in the subsequent section, the class of graphic matroids is derived from graphs. There are matroid analogues for many graph concepts, operations, and theorems. For us, it is sometimes more straightforward to approach a question about graphs through matroids; for example, in Section 2.9, the proof of Theorem 2.81 on the structure of certain graphs can be accomplished quickly and cleanly by taking duals of the cycle matroids associated with those graphs. The cycle matroids have duals – but the graphs we begin with are not necessarily plane graphs. So we make judicious use of matroid theory to solve certain graph problems throughout the dissertation. We now present a brief introduction to some basic matroid theory, following Oxley [17].

There are several equivalent definitions of matroids; we give two here. A *matroid* M is an ordered pair $(E, \mathcal{I})$, where E is a finite *ground set* and $\mathcal{I}$ is a collection of subsets of E, called *independent sets*, satisfying three axioms:

(I) $\emptyset \in \mathcal{I}$.

(II) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(III) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

We say $M = (E, \mathcal{I})$ is a matroid *on* E. To specify the ground set or set of independent sets of a particular matroid M, we write $E(M)$ and $\mathcal{I}(M)$, respectively. Any subset of E that does not appear in $\mathcal{I}$ is a *dependent set* of M. A *circuit* of M is a minimal dependent set, and $\mathcal{C}(M)$ is the set of all circuits of M. We will simply write $\mathcal{C}$ when there is no risk of confusion about which matroid is under consideration. The set $\mathcal{C}$ of circuits uniquely determines a matroid. An alternative definition of a matroid consists of the following three axioms:

(I) $\emptyset \notin \mathcal{C}$.

(II) If C_1 and C_1 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(III) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Let M_1 and M_2 be two matroids. If there is a bijection ϕ from $E(M_1)$ to $E(M_2)$ such that $\phi(X)$ is in $\mathcal{I}(M_2)$ if and only if X is in $\mathcal{I}(M_1)$ for any subset X of $E(M_1)$, then M_1 and M_2 are *isomorphic*. We write $M_1 \cong M_2$ and call ϕ an *isomorphism* from M_1 to M_2.

For some graph G, let $E = E(G)$. If we take $\mathcal{C}$ to be the set of edge sets of cycles of G, then $\mathcal{C}$ is the set of circuits of a matroid on E, the *cycle matroid* of G, denoted $M(G)$. If a matroid M is isomorphic to the cycle matroid of a graph, then we say M is a *graphic matroid*.

Among the terms matroid theory borrows from graph theory are loops, parallel classes, and simple matroids. Let M be a matroid and e, f, and g be elements of M. Element e is a *loop* if it is a one-element circuit of M. Elements f and g are *parallel* in M if they form a circuit of M. A *parallel class* of M is a maximal subset of $E(M)$ such that every pair of elements in the subset are parallel, with none of them being loops. We say a parallel class is *trivial* if there is only one element in it. A matroid M is *simple* if it has no loops and all its parallel classes are trivial.

Suppose A is an $m \times n$ matrix over a field $\mathbb{F}$. Let E be the set of column labels of A, and take $\mathcal{I}$ to be the set of subsets of E that are linearly independent in the vector space $V(m, \mathbb{F})$. The pair $(E, \mathcal{I})$ is a matroid, called the *vector matroid of* A, and we denote it by $M[A]$. If an arbitrary matroid M is isomorphic to $M[A]$ for some matrix A over a field $\mathbb{F}$, we say M is *representable over* $\mathbb{F}$ or $\mathbb{F}$-*representable*, and A is a *representation for* M *over* $\mathbb{F}$. A matroid that is representable over the 2-element field is *binary*.

A maximal independent set in a matroid M is a *basis* of M. The notation $\mathcal{B}(M)$ indicates the set of bases of a matroid M. Every basis of a given matroid has the same cardinality. Let M be the matroid $(E, \mathcal{I})$, and let X be some subset of ground set E. We define $\mathcal{I}|X$ as $\{I \subseteq X : I \in \mathcal{I}\}$, and observe that the pair $(X, \mathcal{I}|X)$ is also a matroid, which we call the *restriction of* M *to* X, denoted $M|X$; alternatively, this matroid is the *deletion of* $E - X$ *from* M, denoted $M \backslash (E - X)$. The *rank function* r of matroid M maps the power set 2^E into the non-negative integers, with $r(X)$ defined as the cardinality of a basis of $M|X$. Thus, the rank

of an independent set of M is equal to its cardinality. We will often write $r(M)$ instead of $r(E(M))$. We note that the set of bases or the rank function can both be used to give definitions of a matroid, equivalent to those we have given using independent sets and circuits.

Given a matroid M on E with rank function r, the *closure function* cl maps 2^E into itself and is defined for any subset X of E as

$$cl(X) = \{x \in E : r(X \cup x) = r(X)\}.$$

When $cl(X) = X$, we call X a *flat* of M. Two flats X and Y of M form a *modular pair* if $r(X)+r(Y) = r(X \cup Y)+r(X \cap Y)$. A flat of rank $r(M)-1$ is a *hyperplane* of M. A *spanning set* of M is a subset X of the ground set E such that $cl(X) = E$. For two subsets X and Y of E, we say X *spans* Y if $Y \subseteq cl(X)$.

1.6 Selected Matroid Operations

In a matroid M, let X be a subset of $E(M)$ that is both a circuit and hyperplane, which we call a *circuit-hyperplane*. The *relaxation* of M is the matroid whose set of bases is $\mathcal{B} \cup \{X\}$.

Suppose M_1 and M_2 are two arbitrary matroids with disjoint ground sets. The *direct sum* of these matroids, $M_1 \oplus M_2$, is the matroid having ground set $E(M_1) \cup E(M_2)$ and the collection of independent sets $\{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1), I_2 \in \mathcal{I}(M_2)\}$.

Let M be a matroid. Define $\mathcal{B}^*(M)$ as the set $\{E(M) - B : B \in \mathcal{B}(M)\}$. Then $\mathcal{B}^*(M)$ is the set of bases of a matroid, the *dual* of M, for which we write M^*. The ground set of M^* is $E(M)$. A basis of M^* is a *cobasis* of M; a circuit of M^* is a *cocircuit* of M. We refer to a 3-element circuit of M as a *triangle*, and a 3-element cocircuit of M as a *triad*. A single-element cocircuit of M is a *coloop*.

For a graph G, we write $M^*(G)$ for the dual of its cycle matroid, which is the *bond matroid* of G. If an arbitrary matroid M is isomorphic to the bond matroid of

some graph, then we say M is *cographic*. Notice that, while only certain graphs have duals, all graphic matroids have duals, though they are not necessarily graphic.

Suppose M is a matroid, and T is a subset of $E(M)$. In the last section, Section 1.5, we defined the *deletion* of T from M, denoted by $M\backslash T$ or $M|(E-T)$. The ground set of the resulting matroid is $E(M) - T$, and the set of circuits is $\mathcal{C}(M\backslash T) = \{C \subseteq E(M) - T : C \in \mathcal{C}(M)\}$. Deletion of elements of a matroid extends the operation of deletion of edges of a graph, and $M(G\backslash T) = M(G)\backslash T$, when G is a graph and T is a set of edges of G.

Again, suppose M is a matroid, and T is a subset of $E(M)$. The *contraction* of T from M produces a matroid M/T on ground set $E(M)-T$, whose set $\mathcal{C}(M/T)$ of circuits consists of the minimal non-empty members of $\{C - T : C \in \mathcal{C}(M)\}$. Alternatively, we can think of contraction as the dual operation of deletion, with $M/T = (M^*\backslash T)^*$. In addition, if G is a graph and T is a set of edges of G, then $M(G/T) = M(G)/T$.

Given disjoint, possibly empty subsets X and Y of $E(M)$, we can express any sequence of deletions and contractions from M by $M\backslash X/Y$, a *minor* of M. If at least one of X and Y is nonempty, then $M\backslash X/Y$ is a *proper minor* of M. A class of matroids such that all minors of any member of the class are also in the class is said to be *closed under minors* or, equivalently, *minor-closed*. The class of graphic matroids and the class of $\mathbb{F}$-representable matroids are two examples of minor-closed classes.

Suppose M_1 and M_2 are matroids that both have ground sets of at least three elements, such that $E(M_1) \cap E(M_2) = \{p\}$ and p is not a loop or coloop of either matroid. The 2-*sum* of matroids M_1 and M_2, denoted $M_1 \oplus_2 M_2$, is the matroid having ground set $(E(M_1) \cup E(M_2)) - \{p\}$ and set of circuits that consists of *(i)*

the circuits of $M_1\backslash p$, *(ii)* the circuits of $M_2\backslash p$, and *(iii)* all sets $(C_1 \cup C_2) - \{p\}$, where $p \in C_1 \in \mathcal{C}(M_1)$ and $p \in C_2 \in \mathcal{C}(M_2)$.

The *simplification* of an arbitrary matroid M is the simple matroid $si(M)$ that results from deleting any loops and all elements save one from every nontrivial parallel class in M. We obtain the *cosimplification* of M by taking the dual of the simplification of M^*; that is, $co(M) = (si(M^*))^*$.

If a matroid M is obtained from a matroid N by deleting a nonempty subset T of $E(N)$, then N is an *extension* of M. If we wish to emphasize that $|T| = 1$, we say N is a *single-element extension* of M. If N^* is an extension of M^*, then we call N a *coextension* of M. Thus N is a coextension of M if $M = N/T$ for some subset T of $E(N)$. A *modular cut* is an arbitrary set of flats of a matroid M if it satisfies conditions (i) and (ii) of the next lemma.

Lemma 1.9. *Let N be an extension of a matroid M by an element e, and let $\mathcal{M}$ be the set of flats F of M such that $F \cup e$ is a flat of N with the same rank as F. Then $\mathcal{M}$ has the following properties:*

(i) If $F \in \mathcal{M}$ and F' is a flat of M containing F, then $F' \in \mathcal{M}$.

(ii) If F_1 and F_2 are in $\mathcal{M}$ and (F_1, F_2) is a modular pair of flats, then $F_1 \cap F_2 \in \mathcal{M}$.

Every single-element extension of a matroid gives rise to a modular cut, and every modular cut gives rise to a unique extension; see Oxley [17] for a detailed argument. If $M = N\backslash e$ and $\mathcal{M}$ is the modular cut corresponding to the extension N, write N as $M +_{\mathcal{M}} e$. A modular cut of matroid M is *proper* if it is not the set of all flats of M. Let $\mathcal{M}$ be a nonempty proper modular cut of M. The *elementary quotient* of M with respect to $\mathcal{M}$ is the matroid $(M +_{\mathcal{M}} e)/e$. If matroid M_2 is an elementary quotient of M_1, then M_1 is an *elementary lift* of M_2.

Lemma 1.10. *Let $\mathcal{N}$ be a minor-closed class of matroids. Let $\mathcal{M}$ be the class of matroids M such that either there is an elementary quotient or an elementary lift of M that is in $\mathcal{N}$. Then $\mathcal{M}$ is closed under the taking of minors.*

Proof. Let $M \in \mathcal{M}$ and let the extension of M by the element e_1 be M_1; let $M_1/e_1 \in \mathcal{N}$. Suppose $f \in E(M)$. Since $M = M_1\backslash e_1$, we have

$$M\backslash f = M_1\backslash f\backslash e_1 \qquad \text{and} \qquad M/f = M_1/f\backslash e_1.$$

Thus $M_1\backslash f$ is an extension of $M\backslash f$ by e_1, and M_1/f is an extension of M/f by e_1. Since $\mathcal{N}$ is minor-closed, both

$$M_1/e_1\backslash f \qquad \text{and} \qquad M_1/e_1/f$$

are in $\mathcal{N}$. However, we know that

$$M_1/e_1\backslash f = M_1\backslash f/e_1 \qquad \text{and} \qquad M_1/e_1/f = M_1/f/e_1.$$

As $(M_1\backslash f)/e_1$ and $(M_1/f)/e_1$ are in $\mathcal{N}$, we deduce that each of $M\backslash f$ and M/f has an elementary quotient in $\mathcal{N}$. Therefore, both $M\backslash f$ and M/f are in $\mathcal{M}$.

A dual argument treats the case when $M_2\backslash e_2 \in \mathcal{N}$, where M_2 is the coextension of M by the element e_2. Thus, $\mathcal{M}$ is closed under the taking of minors. $\square$

1.7 Connectivity in Matroids

A matroid M with ground set E is 2-*connected* if and only if any two distinct elements of E lie on a circuit of M. We will usually simply say that M is *connected* instead of 2-connected.

Proposition 1.11. *Let G be a loopless graph with no isolated vertices, and suppose $|G| \geq 3$. Then $M(G)$ is a connected matroid if and only if G is a 2-connected graph.*

Our definition of matroid k-connectivity comes from Tutte [26]. Let M be a matroid whose ground set is E. We define the *connectivity function* λ_M for a

subset X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. If $\lambda_M(X) < k$ for a positive integer k, then X and $(E - X)$ are *k-separating*. If X and $E - X$ satisfy $min\{|X|, |E - X|\} \geq k$, they constitute a *$k$-separation* of M. This k-separation is *minimal* if $min\{|X|, |E - X|\} = k$. According to Tutte, for an integer $n \geq 2$, a matroid M is *n-connected* if it has no k-separations, for all $k \in \{1, 2, ..., n - 1\}$. Note that if M is n-connected, then so is M^*.

The next proposition motivated our decision in Section 1.3 to require the 3-connected graphs in this dissertation to be both 3-connected and simple.

Proposition 1.12. *Let G be a graph with no isolated vertices, and suppose $||G|| \geq 4$. Then $M(G)$ is 3-connected if and only if G is 3-connected and simple.*

In a matroid context, a *wheel* is the cycle matroid $M(W_n)$, where W_n is a wheel graph with $n \geq 3$. A *whirl* is the matroid W^n obtained from wheel $M(W_n)$ by relaxing its solitary circuit-hyperplane.

The following theorem is due to Tutte [26]. The graph analog, Tutte's Wheels Theorem, was stated earlier as Theorem 1.6. Tutte's Triangle Lemma, while given as a part of Tutte's proof of the Wheels and Whirls Theorem, is useful in its own right.

Theorem 1.13 (Tutte's Wheels and Whirls Theorem)**.** *The following are equivalent for a 3-connected matroid M having at least one element:*

(i) For every element e of M, neither $M \backslash e$ nor M/e is 3-connected.

(ii) M has rank at least three and is isomorphic to a wheel or a whirl.

Lemma 1.14 (Tutte's Triangle Lemma)**.** *Let M be a 3-connected matroid having at least four elements, and suppose that $\{e, f, g\}$ is a triangle of M such that neither $M \backslash e$ nor $M \backslash f$ is 3-connected. Then M has a triad that contains e and exactly one of f and g.*

The next lemma is a well-known and widely used result of Bixby [1].

Lemma 1.15 (Bixby's Lemma)**.** *Let e be an element of a 3-connected matroid M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations. Moreover, in the first case, $co(M\backslash e)$ is 3-connected, while, in the second case, $si(M/e)$ is 3-connected.*

Let M_1 and M_2 be matroids and e be an element that is in both $E(M_1)$ and $E(M_2)$. Then M_1 and M_2 are *e-isomorphic* if there is an isomorphism between M_1 and M_2 under which e is fixed. The following theorem is due to Tseng and Truemper [24].

Theorem 1.16. *Let N be a 3-connected proper minor of a 3-connected matroid M. Suppose that $|E(N)| \geq 4$ and e is an element of N. Then M has a 3-connected minor M_1 having a minor N_1 that is e-isomorphic to N such that either*

(i) $|E(M_1) - E(N_1)| = 1$; *or*

(ii) $N_1 \cong M(W_n)$ *for some* $n \geq 3$ *and* $M_1 \cong M(W_{n+1})$; *or*

(iii) $N_1 \cong W^n$ *for some* $n \geq 2$ *and* $M_1 \cong W^{n+1}$.

Let k be a positive integer. A *matroid-labeled tree* is a tree T where $V(T) = \{M_1, M_2, \ldots, M_k\}$, with each *vertex label* M_i being a matroid. There are two conditions on these vertex labels. First, if M_{j_1} and M_{j_2} are incident with edge e_i of T, then the intersection of ground sets $E(M_{j_1})$ and $E(M_{j_2})$ is $\{e_i\}$, and $\{e_i\}$ is not a 1-separating set of either vertex label. Second, if M_{j_1} and M_{j_2} are nonadjacent in T, then the intersection of their ground sets is empty. When we contract an edge e of T whose endpoints were labeled by vertex labels N_1 and N_2, we label the resulting conglomerate vertex by $N_1 \oplus_2 N_2$.

A *tree decomposition* of a 2-connected matroid M is a matroid-labeled tree T such that if $V(T) = \{M_1, M_2, \ldots, M_k\}$ and $E(T) = \{e_1, e_2, \ldots, e_{k-1}\}$, then

(i) $E(M) = (E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k)) - \{e_1, e_2, \ldots, e_{k-1}\}$;

(ii) $|E(M_i)| \geq 3$ for all i unless $|E(M)| < 3$, in which case $k = 1$ and $M_1 = M$; and

(iii) M is the matroid that labels the single vertex of $T/\{e_1, e_2, \ldots, e_{k-1}\}$.

The next theorem describes the *canonical tree decomposition* of a 2-connected matroid, from Cunningham and Edmonds [5] as found in Oxley [17]. This tree decomposition of a matroid is unique, and we can recover the matroid by taking 2-sums of the vertex labels of the decomposition.

Theorem 1.17. *Let M be a 2-connected matroid. Then M has a tree decomposition T in which every vertex label is 3-connected, a circuit, or a cocircuit, and there are no two adjacent vertices that are both labeled by circuits or are both labeled by cocircuits. Moreover, T is unique to within relabeling of its edges.*

The following generally known lemma is taken from Oxley and Taylor [19]:

Lemma 1.18. *Let M_1 and M_2 label vertices in a tree decomposition T of a connected matroid M. Let P be the path in T joining M_1 and M_2, and let p_1 and p_2 be edges of P meeting M_1 and M_2, respectively. In other words, p_1 and p_2 are basepoints for 2-sums in the reconstruction of M. Then M has a minor isomorphic to the 2-sum of M_1 and M_2, where $p_1 = p_2$ is the basepoint of the 2-sum.*

Lemma 1.19. *If T is a tree decomposition for a connected matroid M and every vertex label is replaced by its dual, then the resulting matroid-labeled tree T^* is a tree decomposition for M^*.*

Our final definition in this chapter is due to Seymour [23]. For a positive integer t, a class $\mathcal{N}$ of matroids is *t-rounded* if two conditions hold. First, every member of $\mathcal{N}$ is $(t+1)$-connected. Second, if M is a $(t+1)$-connected matroid having an

$\mathcal{N}$ minor and X is a subset of $E(M)$ with at most t elements, then M has an $\mathcal{N}$-minor using X. Seymour also presented a theorem that can be used to verify whether a class of matroids is t-rounded, if t is 1 or 2.

Theorem 1.20. *Let t be 1 or 2 and $\mathcal{N}$ be a collection of $(t+1)$-connected matroids. Then $\mathcal{N}$ is t-rounded if and only if the following condition holds: If M is a $(t+1)$-connected matroid having an $\mathcal{N}$-minor N such that $|E(M)-E(N)|=1$, and X is a subset of $E(M)$ with at most t elements, then M has an $\mathcal{N}$-minor using X.*

Chapter 2

Nearly Series-Parallel Graphs

In their work on the structure of matroids, Geelen, Gerards, and Whittle [11] define the *distance*, $dist(N_1, N_2)$, between matroids N_1 and N_2 as the minimum number of elementary quotients and elementary lifts that must be performed on N_1 to produce N_2.

Let $M_2 \in \mathcal{M}_2$ where $\mathcal{M}_2$ is the class of cycle matroids of series-parallel graphs; let M_0, an extension or coextension of M_1, be in the class $\mathcal{M}_0$ of graphic matroids. We will consider the class $\mathcal{M}_1$ of all graphic matroids M_1 such that $dist(M_1, M_2) \leq 1$ and $M_0 \in \mathcal{M}_0$, as shown in Figure 2.1.

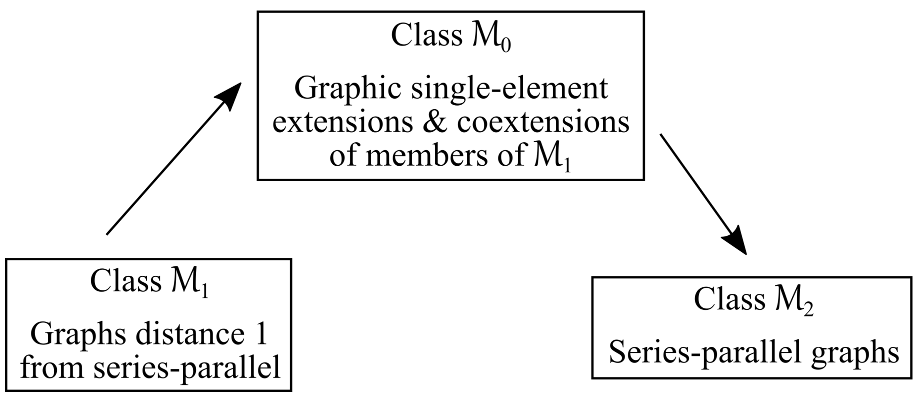


FIGURE 2.1: $dist(M_1, M_2) \leq 1$

Let $\mathcal{S}$ be the class of *almost series-parallel graphs*, defined by Warshauer [28] as the graphs to which adding some edge (an undeletion) and contracting it out will produce a series-parallel graph. Warshauer [28] also defines the class $\mathcal{S}^*$ to be those graphs G such that splitting some vertex v of $V(G)$ results in a series-parallel graph. Alternatively, we may view the graphs in $\mathcal{S}^*$ as those containing a vertex v such that splitting v results in a series-parallel graph. Observe that $\mathcal{M}_1 = \mathcal{S} \cup \mathcal{S}^*$.

Warshauer showed that each of the classes $\mathcal{S}$ and $\mathcal{S}^*$ is minor-closed, found their excluded minors, and proved several results about the structure of their member graphs. The following are due to Warshauer [28]. The graphs in these results that are not already defined are shown in Figure 2.2.

Theorem 2.1. *The excluded minors for the class $\mathcal{S}$ are the following eleven graphs: $K_4 \oplus_0 K_4$, $K_4 \oplus_1 K_4$, $S(K_4, K_4)$, K_5, $K_{2,2,2}$, R, U, H_8, Q_3, S, and V.*

Theorem 2.2. *The excluded minors for the class $\mathcal{S}^*$ consist of the following nine graphs: $K_4 \oplus_0 K_4$, $K_4 \oplus_1 K_4$, $P(K_4, K_4)$, K_5, $K_{3,3}$, $K_{2,2,2}$, R, U, and Q_3.*

In this chapter, our focus is on the class $\mathcal{M}_1 = \mathcal{S} \cup \mathcal{S}^*$. The following is an immediate consequence of Lemma 1.10.

Corollary 2.3. *The class $\mathcal{M}_1$ of matroids is closed under the taking of minors.*

The purpose of this chapter is to prove the following excluded-minor characterization of the class $\mathcal{M}_1$; the graphs listed in this theorem are depicted in Figure 2.2.

Theorem 2.4. *The twelve excluded minors of class $\mathcal{M}_1$ are $K_4 \oplus_0 K_4$, $K_4 \oplus_1 K_4$, S_1, S_v, K_5, $K_{2,2,2}$, Q_3, R, U, H_8, S, and V.*

Our proof of Theorem 2.4 will break into cases based on the connectivity of the excluded minors. While the arguments for $\kappa \leq 2$ and $\kappa \geq 4$ are comparatively easy, the argument for $\kappa = 3$ is more involved and is broken into four lengthy subcases.

Lemma 2.5. *Each of the twelve graphs shown in Figure 2.2 is an excluded minor of $\mathcal{M}_1$.*

Each graph can be established as an excluded minor via a straightforward but tedious case check, which can be done, for example, in SageMath. Code to facilitate that process is given in Appendix B.

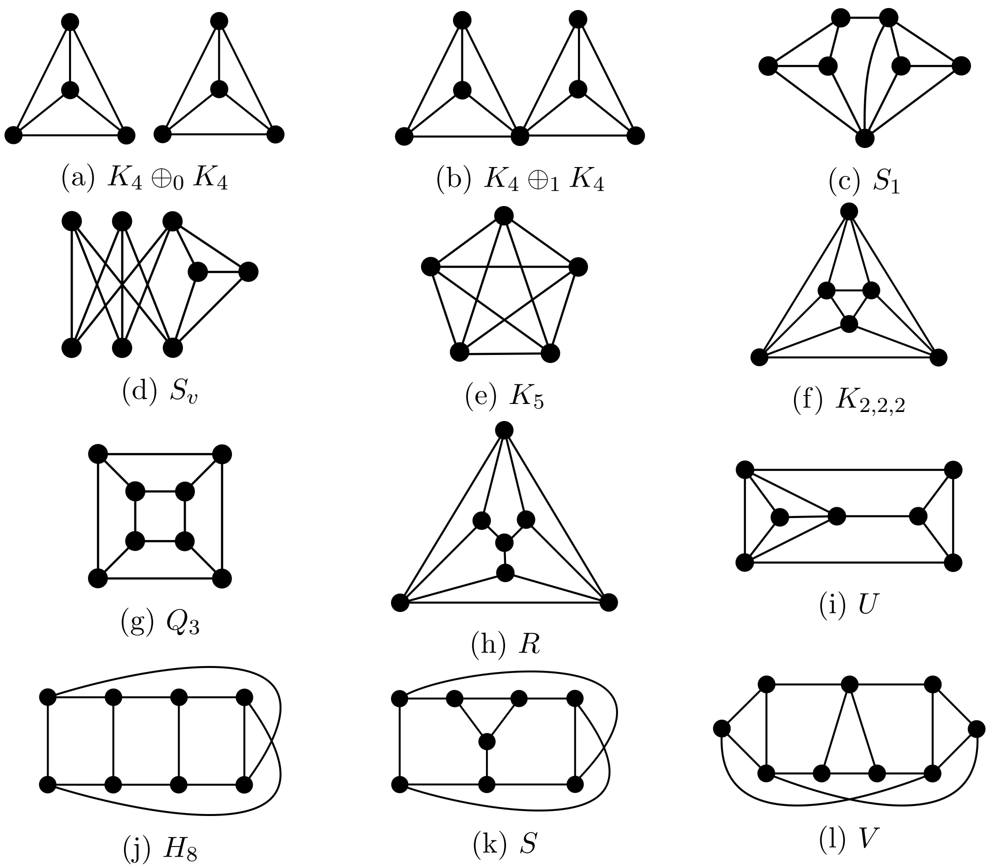


FIGURE 2.2: The excluded minors of $\mathcal{M}_1$.

2.1 Excluded Minors of $\mathcal{M}_1$ with Connectivity at Least Four

The following result of Halin and Jung [14] is key to specifying the excluded minors of $\mathcal{M}_1$ that have connectivity at least four.

Lemma 2.6. *A simple graph whose minimum degree is at least four has K_5 or $K_{2,2,2}$ as a minor.*

Since both K_5 and $K_{2,2,2}$ are excluded minors for the class $\mathcal{M}_1$, we have the following result.

Proposition 2.7. *The excluded minors of $\mathcal{M}_1$ having connectivity at least four are K_5 and $K_{2,2,2}$.*

The next result is obtained by combining Lemma 2.6 and Proposition 2.7.

Proposition 2.8. *$\mathcal{M}_1$ does not contain any graphs with connectivity four or greater.*

2.2 Preliminaries

The following theorem of Tutte [25] tells us that every 3-connected graph has a K_4-minor.

Theorem 2.9. *A graph G is 3-connected if and only if there is a sequence $G_0, G_1, \ldots, G_n$ of graphs with the following two properties:*

(i) $G_0 \cong K_4$ and $G_n = G$;

(ii) G_{i+1} has an edge e such that both its endpoints have at least three neighbors, and $G_i = si(G_{i+1}/e)$, for every $i < n$.

Lemma 2.10. *Every excluded minor of $\mathcal{M}_1$ is simple and has no degree-2 vertices.*

Proof. Let G be an excluded minor of $\mathcal{M}_1$. Clearly G has no loops. Assume to the contrary that G has a 2-cycle with edges $\{e, f\}$. Since G is an excluded minor,

$G\backslash e$ is in $\mathcal{M}_1$. Thus, we can obtain a series-parallel graph from $G\backslash e$ by a vertex identification or a vertex split. However, by replacing e in parallel with f, we can then obtain a series-parallel graph from G by a vertex identification, or a vertex split, a contradiction. We deduce that G is simple.

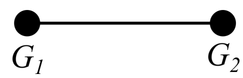
Now assume to the contrary that G has a degree-2 vertex, and let the edges incident with that vertex be e and f. As G/e is a member of $\mathcal{M}_1$, we can obtain a series-parallel graph from G/e by a vertex identification or a vertex split. Then, the same operation can be performed on G to produce a series-parallel graph, a contradiction. Hence, G has no degree-2 vertices. □

The next lemma will be crucial in determining the excluded minors of $\mathcal{M}_1$ whose connectivity is two. Recall that, in the canonical tree decomposition of a 2-connected graph, each vertex is labeled by a cycle, a bond, or a 3-connected graph.

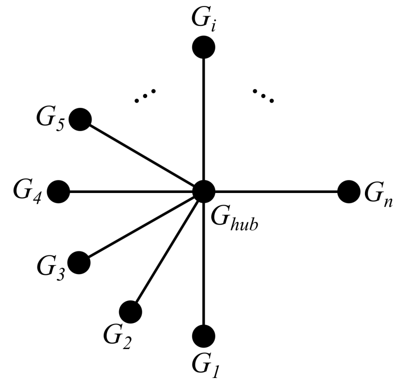
Lemma 2.11. *Let G be a simple, 2-connected excluded minor of $\mathcal{M}_1$. Let T be the canonical tree decomposition of G. Either the tree T consists of two vertices labeled by 3-connected graphs; or T is a star such that the hub vertex is labeled by a k-edge cycle or bond for some $k \geq 3$, there are at least $k-1$ leaves, and each leaf is labeled by a 3-connected graph.*

Proof. Every 3-connected vertex label has a K_4-minor, by Theorem 2.9. Since G is simple, for each vertex G_j of $V(T)$ that is labeled by a bond, at most one edge of G_j is not the basepoint of a 2-sum. Moreover, by Lemma 2.10, the leaves of T cannot be bonds or cycles.

2.11.1. *T has no path in which three of the vertex labels are 3-connected.*



(a) Two 3-connected vertex labels



(b) A star with 3-connected leaves; G_{hub} is a cycle or bond.

FIGURE 2.3: Possible forms of T

Let G_1, G_2, and G_3 be the labels of three vertices of T, appearing in that order on a path, where each is a 3-connected graph. Then, by repeated application of Lemma 1.18, G has as a minor a graph K, for which the canonical tree decomposition T_H is a 3-edge path with vertices labeled by G_1, G_2, and G_3. The only alterations to these labels from their appearance in T is that the edge labels on basepoint edges may change. For each $i \in \{1, 2\}$, let e_i be the edge of G_{i+1} joining G_i and G_{i+1}. By a roundedness result due to Seymour [21] (see also Oxley [17, Section 12.3]), $M(K_4)$ is 2-rounded in the collection of graphic matroids. Thus, G_1 and G_3 have K_4-minors using e_1 and e_2, respectively, while G_2 has a K_4-minor using $\{e_1, e_2\}$. Hence, G has as a minor the graph shown in Figure 2.4, which illustrates the proper minor isomorphic to S_1 that exists in T_H. As S_1 is an excluded minor for $\mathcal{M}_1$, we deduce that 2.11.1 holds.

Next, we show the following.

2.11.2. *If T contains no vertex labeled by a cycle, then T consists of two vertices labeled by 3-connected graphs, or T is a star with the hub labeled by a bond B and the leaves labeled by 3-connected graphs, where the number of such leaves is at least $|E(B)| - 1$.*

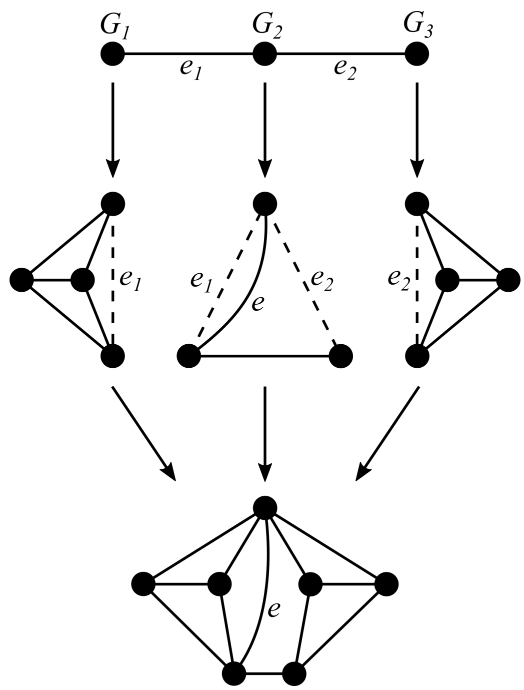


FIGURE 2.4: A path of three 3-connected labels in T leads to an S_1-minor from G.

If T contains no vertices labeled by either cycles or bonds, then 2.11.1 forces T to be the single-edge graph K_2, with both vertices labeled by 3-connected graphs.

Suppose T has a vertex G_{co} labeled by a bond. Vertices adjacent to G_{co} can only be labeled by 3-connected graphs. Since bonds and cycles cannot label leaves and 2.11.1 prevents any path from having three vertices labeled by 3-connected graphs, T must be a star where G_{co} is the hub and the leaves are labeled by 3-connected graphs. Finally, observe that, since G is simple, $|E(G_{co}) \cap E(T)| \geq |E(G_{co})| - 1$. Thus, 2.11.2 holds.

2.11.3. *If T contains a vertex labeled by a cycle, then T is a star with a cycle at the hub and* 3*-connected labels on the leaves.*

Suppose the vertex G_c of T is labeled by a cycle. By Lemma 2.10, there are at least two neighbors of G_c, so T contains paths from G_c to distinct leaves G_1 and G_2 of T that have only the vertex G_c in common. Clearly G_1 and G_2 are labeled by 3-connected graphs. The neighbors of G_c cannot be labeled by cycles, by definition

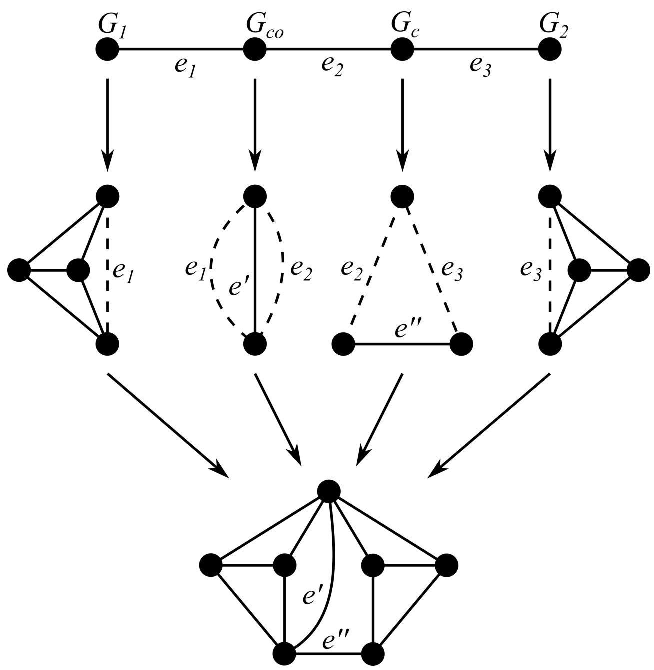


FIGURE 2.5: G_c cannot have a neighbor labeled by a bond.

of the canonical tree decomposition. In addition, its neighbors cannot be labeled by bonds, either, because then G has an S_1-minor constructed by using K_4-minors of each of G_1 and G_2, an edge e' from the bond, and an edge e'' from G_c (see Figure 2.5). Then, since the leaves of T must have 3-connected labels but 2.11.1 forbids a path with three 3-connected vertex labels, G_c must be adjacent to every leaf of T. Hence, T is a star, as desired. It remains to show that, if G_c is a k-edge cycle, that T has at least $k-1$ leaves.

Assume to the contrary that two edges e and f of G_c do not function as basepoints of a 2-sum with a leaf. Since G is an excluded minor of $\mathcal{M}_1$, it follows that G/f is a member of $\mathcal{S}$ or $\mathcal{S}^*$. Suppose $G/f \in \mathcal{S}$. Then we can identify two vertices u and v of G/f and obtain a series-parallel graph. Recall that, by our roundedness result from Seymour, each of the 3-connected graphs labeling a leaf of our star T has a K_4-minor that uses its basepoint as an edge. We have at least two such leaves, which implies u and v are not both endpoints of (possibly distinct)

basepoints, or identifying them leaves a K_4-minor behind. Perhaps exactly one of u or v, say u, is an endpoint of a basepoint of some leaf, say G_1. However, if v is a non-basepoint vertex in G_1, then after the identification there is a still a path between the endpoints of basepoint e_2 of G_2 that avoids all edges of G_2. Similarly, if v is a non-basepoint vertex of G_2, then after the identification we still have a path between the endpoints of basepoint e_1 of G_1 that avoids all edges of G_1. So when only one of u and v is the endpoint of a basepoint of a 2-sum, there is a K_4-minor that persists after their identification. If u and v to be vertices of some leaf, say G_1, with neither of them being an endpoint of e_1, their identification leaves a path between the endpoints of e_2 that avoids all edges of G_2, and there remains a K_4-minor. Similarly, if u and v are vertices of G_1 and G_2, respectively, with neither of e_1 nor e_2 incident with them, there is again a K_4 minor after the identification. Therefore, G/f is not a member of $\mathcal{S}$.

Hence, G/f must be a member of $\mathcal{S}^*$. Then, we can split some vertex of G/f and obtain a series-parallel graph. This vertex must be f', the conglomerate vertex formed by contracting f, or G would be a member of $\mathcal{S}^*$. Observe that if f had at most one of its endpoints in common with a 3-connected leaf, then we have an obvious split of G that results in a series-parallel graph. So we may assume f has one endpoint in G_1 and another in G_2. We call the vertices resulting from this split f_1 and f_2. Observe that if $N(f_i) \cap V(G_1) = \emptyset$ for $i \in \{1, 2\}$, then we again have a split of some endpoint of f in G that results in a series-parallel graph. Thus, each f_i is adjacent to some vertex from G_1 and from G_2. Let $\{g_1, g_2\} \subseteq N(f_1)$ and let $\{g_1', g_2'\} \subseteq N(f_2)$, where $g_1, g_1' \in V(G_1)$ and $g_2, g_2' \in V(G_2)$. Let x be the endpoint of e_1 not incident with f, and let y be the endpoint of e_2 not incident with f. Notice that $g_1, g_1' \neq x$ and $g_2, g_2' \neq y$, as this either means one of G_1 or G_2 has an edge parallel to its basepoint, or we have a contradiction to the star structure of

T. Now, since G_1 is 3-connected, by Menger's Theorem there is a path between g_1 and g_1' that avoids x and the endpoint of f; moreover, this path is preserved in $G_1\backslash e_1$. There is a similar path between g_2 and g_2' in G_2 that avoids y and the other endpoint of f. So, after the split of f' in G/f, we have path P_1 between g_1 and g_1' that avoids x, f_1, and f_2; and path P_2 between g_2 and g_2' that avoids y, f_1, and f_2. Let C be the cycle consisting of path P_1, edges (g_1, f_1) and (f_1, g_2), path P_2, and edges (g_2', f_2) and (f_2, g_1'). Again by Menger's Theorem, there are two internally disjoint paths from x to C using edges of G_1, and two internally disjoint paths from y to C using edges of G_2. Contract so that x and y are in one conglomerate vertex xy, such that no edges of G_1 or G_2 are contracted. Then we have a vertex xy with three neighbors (whatever three-element subset we choose of the four neighbors x and y had on C) on a cycle. Whence, G/f has a K_4-minor after the vertex split, which is a contradiction.

Thus, if G_c is a k-cycle, then T has at least $k-1$ leaves. This concludes the proof of Lemma 2.11. □

2.3 Daisy Chains

This section explores the graphs of connectivity three that are members of class $\mathcal{S}^*$. We begin with two preliminary results. The first is due to Dirac [8]; see also Oxley [17, Lemma 5.4.11].

Lemma 2.12. *A simple, 2-connected graph G in which the degree of every vertex is at least three has a subgraph that is a subdivision of K_4.*

The second result places a lower bound on the number of degree-2 vertices found in a 2-connected series-parallel graph.

Lemma 2.13. *Let G be a simple, 2-connected, series-parallel graph. If $|V(G)| \geq 4$, then G has at least two degree-2 vertices.*

Proof. By Lemma 2.12, G has a vertex v such that $deg(v) = 2$. Let $N(v) = \{u_1, u_2\}$, and let the order of G be k. We proceed by induction on k. Suppose $k = 4$, and let $V(G) = \{v, u_1, u_2, q\}$. By Theorem 1.5, since q is not adjacent to v, we must have $N(q) = \{u_1, u_2\}$; hence, $deg(q) = 2$. Assume, for $4 < k < n$, that G has at least two degree-2 vertices. Now suppose $k = n \geq 5$. Let edge $f = (v, u_1)$. By Lemma 1.3, the graph G/f is 2-connected, since $deg_{G\setminus f}(v) = 1$. As G/f is series-parallel, by hypothesis, G/f has a degree-2 vertex, q, and performing a series extension on G/f will not change the degree of q. Therefore, G has two degree-2 vertices, q and v. □

Graph G admits an *open ear decomposition* if it can be constructed, as follows, from a sequence of paths $P_0, P_1, \ldots, P_k$ (the *ears* of the decomposition), where P_0 is a single edge, the sets of internal vertices from distinct ears are pairwise disjoint, and $k \geq 1$. For ear P_j having endpoints a and b, with $1 \leq j \leq k$, we identify a with vertex v_a on ear P_i and b with v_b on P_h, where $i, h < j$; we allow $i = h$, but vertices v_a and v_b must be distinct.

If $i = h$, then P_j is *nested* on P_i. The *nest interval* of P_j in P_i is the subpath of P_i between P_j's endpoints. A *nested open ear decomposition* is an open ear decomposition that satisfies two additional conditions:

(a) given $j > 1$, there is an $i < j$ such that P_j is nested on P_i; and

(b) if two ears P_j and $P_{j'}$ are nested on the same ear P_i, then either the nest interval of P_j contains that of $P_{j'}$ (or vice versa), or the two nest intervals are edge disjoint.

We can break down a nested open ear decomposition into *nested subsequences*, each of which is maximal and of the form $P_0, P_1, P_{q_1}, \ldots, P_{q_m}$, where P_{q_1} is nested

on P_1, and $P_{q_{i+1}}$ is nested on P_{q_i} for $1 \leq i \leq m-1$. Two nested subsequences are equal if their sets of ears are equal.

The next theorem from Whitney [29] states that any 2-connected graph has an open ear decomposition.

Theorem 2.14 (Whitney's Ear Decomposition)**.** *A simple graph G is 2-connected if and only if G has an open ear decomposition.*

Oporowski [16] explicitly states the following lemma, which originated with Whitney [29].

Lemma 2.15. *Every edge in a loopless 2-connected graph is the initial ear P_0 of some open ear decomposition.*

Eppstein [10] characterizes series-parallel graphs by their ear decompositions, and Goodall et. al. [12] refine this result for identifying 2-connected series-parallel graphs by the presence of a nested open ear decomposition:

Lemma 2.16. *A loopless 2-connected graph G is series-parallel if and only if every open ear decomposition of G is nested.*

Consider a loopless, 2-connected, series-parallel graph H that has exactly two degree-2 vertices s and t. Assume also that H has a nested open ear decomposition such that

(i) s is an endpoint of initial ear P_0;

(ii) t is the internal vertex of length-2 terminal ear P_k;

(iii) $k \geq 2$;

(iv) P_1 is length-2 (so that the initial cycle is a 3-cycle);

(v) the decomposition has only one nested subsequence; and

(vi) P_j has length 2 or 3 for $1 < j < k$.

A *daisy chain* is a simple graph that can be formed from such a graph H either by adding an edge e between the two degree-2 vertices s and t of H; or by identifying s and t. To distinguish between daisy chains in which s and t are identified and those in which they are not, we write G for when e exists and $\widetilde{G}$ for when s and t are identified into vertex st. Call $\widetilde{G}$ the *identification* of G. The cycle formed by identifying the endpoints of ear P_1 with the vertices of P_0 is the *initial cycle*, and the cycle formed by identifying the endpoints of P_k with vertices of P_{k-1} is the *terminal cycle.* A cycle formed by identifying the endpoints of ear P_j with vertices of ear P_{j-1}, where $1 < j < k$, is an *inside cycle* of the daisy chain; we say P_j is the ear associated with this inside cycle. The *inside vertices* of daisy chain G are the members of $V(G) - \{s,t\}$, or, for an identified daisy chain $\widetilde{G}$, the members of $V(\widetilde{G}) - \{st\}$.

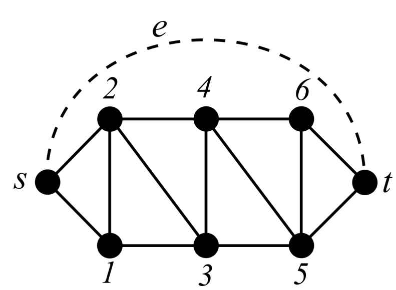


FIGURE 2.6: Example of two possible daisy chains. Edge e may be present, or it may be contracted so that s and t are identified.

For an alternative way to view the structure of a daisy chain G, we can construct G from a series-parallel graph H. Suppose the canonical tree decomposition T of H is a path whose vertex labels are alternating cycles and bonds. The leaves of T are both labeled by 3-cycles; an internal vertex of T may be labeled by a 3-cycle, a 4-cycle, or a bond of size three. Moreover, for each 4-cycle, the two basepoints do not share any endpoints. We get G from H by either identifying the two degree-2

vertices of G that belong to the initial and final cycles of T but that do not meet basepoint edges; or by adding an edge between these two degree-2 vertices. This is a faster way to define a daisy chain, but an ear decomposition allows us to more readily specify aspects of a daisy chain's structure upon which parts of our proof of Theorem 2.4 rely, such as $s-t$ paths, or subpaths that are subtended by inside cycles (see Section 2.7).

Any daisy chain can be associated with at least one daisy chain whose inside cycles are all 3-cycles, in the following way. Take a daisy chain G having only inside 3-cycles. Select one of these inside cycles, which corresponds to some ear P_i of a nested open ear decomposition of G. Replace P_i by a length-3 ear P_{i3}, so that the endpoints of P_{i3} are the same as those of P_i, and the subsequent ears in the sequence that had endpoints identified with an endpoint of P_i and one of its internal vertices now have both their endpoints identified with the internal vertices of P_{i3}. The resulting daisy chain now has an inside cycle of length 4 and is called a *first cousin* of G. A daisy chain consisting of G with n ears replaced is an n^{th} *cousin* of G.

Say $\mathcal{G}$ is the set of all daisy chains G, such that G has k ears in its nested open ear decomposition, and G has only inside 3-cycles. Note this means G has $k-2$ inside cycles. Consider the set $\mathcal{G}'$ of each G from $\mathcal{G}$ and all its x^{th} cousins, for each x between 1 and $k-2$; if cousins from distinct members of $\mathcal{G}$ are isomorphic, we keep only one of the cousins in $\mathcal{G}'$. Then $\mathcal{G}'$ is the set of all daisy chains with $k-2$ inside cycles.

Let graph G be a daisy chain. Let edge $(s,2)$ be ear P_0 of a nested open ear decomposition of G, and let $(s,1)$ be the edge of ear P_1 that is not incident with endpoint 2 of P_0 (so the edges of P_1 are $(s,1)$ and $(1,2)$). We define two *distinguished $s-t$ paths* of G, calling them Q_0 and Q_1; they both have s as initial vertex

and t as terminal vertex. The first edge of Q_0 is $(s,2)$, but Q_0 may not contain any edge of ear P_1; the initial edge of Q_1 is $(s,1)$, and Q_1 avoids ear P_0. Thereafter, each path includes at most one edge and exactly one endpoint from every ear P_i where $2 \leq i \leq k$. For an identified daisy chain $\widetilde{G}$, the $s-t$ paths become *st cycles*, which we call C_0 and C_1.

Lemma 2.17. *The sets* $V(Q_0) - \{s,t\}$ *and* $V(Q_1) - \{s,t\}$ *partition the inside vertices of a daisy chain into two sets.*

Proof. Assume to the contrary that $V(Q_0)$ and $V(Q_1)$ have nonempty intersection over the set of inside vertices of G. Suppose v is such a vertex in that intersection, and moreover is on an inside cycle whose associated ear P_i has the lowest index of any ear associated with a vertex in that intersection. So the inside vertices on ears whose indices are smaller than i are partitioned between Q_0 and Q_1, and we know there are such vertices because of how the definition of $s-t$ paths assigns edges from P_0 and P_1 to Q_0 and Q_1. Since v is an inside vertex, $P_k \neq P_i$. Then, v and another vertex p of P_i are where the endpoints of P_{i+1} are attached, and p is on one of the $s-t$ paths, say Q_0. So Q_0 contains both v and p, but they are the two endpoints of P_{i+1}. So we have a contradiction, since each of Q_0 and Q_1 can only have exactly one endpoint from any ear. □

Notice that Lemma 2.17 may be restated for an identified daisy chain: the sets $V(C_0) - \{st\}$ and $V(C_1) - \{st\}$ partition the inside vertices of a daisy chain into two sets. The proof holds when similarly modified.

Lemma 2.18. *Every daisy chain is* 3*-connected.*

Proof. Let G be a daisy chain. To demonstrate that G is 3-connected, we will construct a sequence of graphs to satisfy Theorem 2.9. Suppose first that vertices

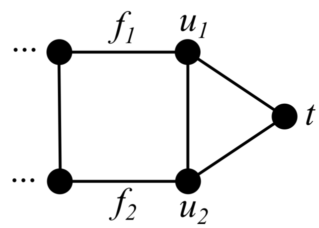


FIGURE 2.7: Ear P_{k-1} of daisy chain G has length 3.

s and t of G are joined by edge e; that is, G is an unidentified daisy chain. Let $G_n = G$. Consider ear P_{k-1}. If P_{k-1} is length-3, shown in Figure 2.7, choose one of the ear's internal vertices, say u_1, and contract the edge f_1 between it and the endpoint of P_{k-1}. We take $G_{n-1} = G/f_1$. Then, in this case, G_{n-2} will be $si(G_{n-1}/f_2)$. If P_{k-1} has length 2, contract edge f between the internal vertex and the endpoint which is not on a triangle with t. We take G_{n-1} to be $si(G_n/f)$. Repeat the process for each ear P_j, where $1 < j \leq k-1$. Simplifying the last contraction of an edge of ear P_2 yields $G_0 = K_4$, as required. Hence, G is a 3-connected graph.

Now suppose that we have an identified daisy chain; then $\widetilde{G} \cong G/e$, and, by definition, $\widetilde{G}$ is simple. Consider the cycle matroids $M(\widetilde{G})$ and $M(G)$. We have shown that $M(G)$ is 3-connected, since our definition of graph 3-connectivity requires the graph be simple. By Bixby's Lemma 1.15, as $co(G\backslash e)$ is not simple, we have $si(M(G/e))$ is 3-connected. However, $si(M(G/e))$ is equal to $M(G/e)$, so $M(G/e)$ is 3-connected, and, therefore, G/e is 3-connected. Thus, every daisy chain is 3-connected. $\square$

Proposition 2.19. *The daisy chains are precisely the graphs of connectivity three that are members of class $\mathcal{S}^*$.*

Proof. Let G be a 3-connected member of $\mathcal{S}^*$. Then G has a vertex v that can be split into vertices v_1 and v_2 to produce a series-parallel graph $\widehat{G}$. Thus, we have $V(\widehat{G}) = (V(G) - \{v\}) \cup \{v_1, v_2\}$. Note that $\widehat{G}$ will have connectivity one or two.

Suppose $\kappa(\widehat{G}) = 2$. By Lemma 2.13, it must have at least two degree-2 vertices, and vertices v_1 and v_2 are the only possible degree-2 vertices, since G is 3-connected. By Lemmas 2.15 and 2.16, $\widehat{G}$ has a nested open ear decomposition $P_0, P_1, \ldots, P_k$ whose initial edge P_0 has v_1 as an endpoint. As $\widehat{G}$ has only two degree-2 vertices, v_2 is an internal vertex of terminal ear P_k, and P_k has length 2 (that is, P_k will have three vertices).

Since G is simple, every ear other than P_0 must have length at least two. The last ear in every nested subsequence of the decomposition results in at least one degree-2 vertex of $\widehat{G}$. Therefore, our nested open ear decomposition of $\widehat{G}$ will only have one nested subsequence, namely, the decomposition's complete sequence of ears. However, the internal vertices of P_j for $1 < j < k$ must have degree at least three in $\widehat{G}$. Thus every such vertex is the endpoint of some subsequent ear. Additionally, the length of every P_j must be two or three; otherwise, $\widehat{G}$ has some degree-2 vertex other than s and t.

The length of ear P_1 must be precisely two. The degree of every vertex in the initial cycle formed by P_0 and P_1 is two, and since there is only one nested subsequence, the only ear nested on P_1 will be P_2. Thus, identifying the endpoints of P_2 with vertices of P_1 must raise the degree of every vertex on the initial cycle except v_1, but this is only possible if ear P_1 has length two.

We now know that $\widehat{G}$ satisfies conditions (i) – (vi) of the definition of a daisy chain. Since $\widehat{G}$ was the result of a vertex split of G that produced two vertices of degree two, we conclude that G is a daisy chain.

It remains to consider the case when $\kappa(\widehat{G}) = 1$. Then $\widehat{G}$ has a cut vertex x. Observe that $x \notin \{v_1, v_2\}$, or G would have v as a cut vertex. If the deletion of x results in a component containing vertices besides just v_1 or just v_2, this implies $G - \{x, v\}$ is disconnected, but G was 3-connected. Thus, $\widehat{G} - \{x\}$ has two

components, one of which is the trivial graph on v_1 or v_2; let us take it to be v_2. Let edge $f = (x, v) \in E(G)$ and edge $f' = (x, v_2) \in E(\widehat{G})$, with this latter being a pendant edge in $\widehat{G}$. Notice the connectivity of $\widehat{G}/f'$ is two, since that of G is three and $\widehat{G}/f' \cong G \backslash f$. Given this, we can repeat the proof done above when $\kappa(\widehat{G}) = 2$, simply substituting $G \backslash f$ for $\widehat{G}$. The only modification that we have to make is the two degree-2 vertices guaranteed by Lemma 2.13 are now x and v. Hence, $\widehat{G}/f'$ satisfies conditions (i) – (vi) of the definition of a daisy chain. Adding the edge f shows that G is a daisy chain.

Conversely, let H be a daisy chain. By Lemma 2.18, H is 3-connected. By definition, H can be viewed as a series-parallel graph whose two vertices s and t have been either joined by an edge, or identified. Therefore, if H is an unidentified daisy chain, there is an obvious split of either vertex s or t that produces a series-parallel graph. Likewise, if H is an identified daisy chain, there is an obvious split of the identified vertex st that produces a series-parallel graph. Thus, H is a member of $\mathcal{S}^*$ having connectivity three. $\square$

Lemma 2.20. *Let G be a daisy chain. Every inside 4-cycle of G contributes precisely one edge to both Q_0 and Q_1 (or C_0 and C_1).*

Proof. Suppose G has an inside 4-cycle. Take a nested open ear decomposition of G; let P_i be the length-3 ear associated with this inside 4-cycle. By the definition of an $s-t$ path and Lemma 2.17, the endpoints v_1 and v_4 of P_i are, without loss of generality, on the $s-t$ paths Q_0 and Q_1, respectively. Since G is 3-connected and can have only one nested subsequence, the endpoints of ear P_{i+1} are identified with the internal vertices v_2 and v_3 of P_i. Each of Q_0 and Q_1 includes exactly one endpoint of P_{i+1}, by definition; thus, edge (v_1, v_2) is in Q_0, and edge (v_4, v_3) is in

Q_1. Hence, the 4-cycle associated with ear P_i contributed one edge to each of Q_0 and Q_1. □

Corollary 2.21. *Let G be a daisy chain. If the $s-t$ path Q_i of G (or st cycle, if G is an identified daisy chain) has length n, then G has at most $n-2$ inside 4-cycles.*

Proof. Take a nested open ear decomposition of G. Both distinguished $s-t$ paths (or cycles) of G have one edge that lies on the initial cycle of the decomposition, and one edge that lies on the terminal ear. So, excluding these edges that do not lie on inside cycles of Q_i, there are $n-2$ edges left. By Lemma 2.20, any inside 4-cycle of G adds one edge to Q_i. Thus, G has at most $n-2$ inside 4-cycles. □

2.4 Excluded Minors of $\mathcal{M}_1$ with Connectivity at Most Two

We consider separately the cases where $\kappa < 2$ and $\kappa = 2$.

Proposition 2.22. *Let G be a simple graph with $\kappa(G) \in \{0,1\}$. Then G is an excluded minor for the class $\mathcal{M}_1$ if and only if G is isomorphic to $K_4 \oplus_0 K_4$ or $K_4 \oplus_1 K_4$.*

Proof. From Lemma 2.5, we know that $K_4 \oplus_0 K_4$ and $K_4 \oplus_1 K_4$ are excluded minors of $\mathcal{M}_1$.

For the converse, let G be an excluded minor for $\mathcal{M}_1$. Then $G \cong G_1 \oplus_{\kappa(G)} G_2$ for some nontrivial graphs G_1 and G_2. As G cannot be a series-parallel graph, it must have a K_4-minor, which must furthermore be a minor of G_1, G_2, or both. If both G_1 and G_2 have K_4-minors, then, as $M(K_4)$ is 2-rounded for graphic matroids, G has $K_4 \oplus_{\kappa(G)} K_4$ as a minor, so G must be isomorphic to $K_4 \oplus_0 K_4$ or $K_4 \oplus_1 K_4$.

Now assume that only one of G_1 and G_2 has a K_4-minor, say G_2 has a K_4-minor but G_1 does not. Let $f \in E(G_1)$. Since G is an excluded minor, we must have $G\backslash f \in \mathcal{M}_1$. The vertex identification or split that transforms $G\backslash f$ into a

series-parallel graph must eliminate all K_4-minors of G_2. Hence this identification or split is being performed entirely in G_2. Performing this same operation on G_2 in G shows that $G \in \mathcal{M}_1$, but this is a contradiction. $\square$

Before proving that the only 2-connected minors of $\mathcal{M}_1$ are S_1 and S_v, we give a few preliminary results. The first is is due to Brylawski [2] and Seymour [21], as given in Oxley [17, Section 4.3].

Theorem 2.23. *Let N be a connected minor of a connected matroid M, and suppose that $e \in E(M) - E(N)$. Then at least one of $M \backslash e$ and M/e is connected, having N as a minor.*

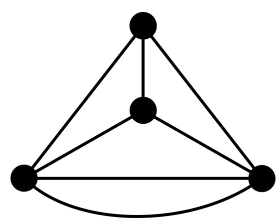


FIGURE 2.8: The graph K_4^+, which appears in Lemma 2.24.

Lemma 2.24. *Let M be a regular 3-connected matroid with element e. Suppose $|E(M) \geq 4|$. Then M has no $M(K_4^+)$-minor using e in the 2-circuit if and only if $M \backslash e$ has no $M(K_4)$-minor.*

Proof. If M has an $M(K_4^+)$-minor using e in the 2-circuit, then clearly $M \backslash e$ has $M(K_4)$ as a minor.

Suppose that $M \backslash e$ has an $M(K_4)$-minor, which we call N. Since M is 3-connected, $M \backslash e$ is connected. We argue by induction on $|E(M) - E(N)| = n$ that M has an $M(K_4^+)$-minor using e in the 2-circuit. This is vacuously true if $n = 1$, as M is regular and simple.

Assume the hypothesis holds for $n < k$. Let $n = k \geq 2$. By Theorem 2.23, there is an element e_1 of $E(M \backslash e) - E(N)$ such that $M \backslash e \backslash e_1$ or $M \backslash e / e_1$ is connected

and has N as a minor. In the first case, since $M\backslash e\backslash e_1 = M\backslash e_1\backslash e$, it follows by induction that $M\backslash e_1$, and hence, M, has an $M(K_4^+)$-minor in which e is in the 2-circuit of $M(K_4^+)$.

The result follows similarly in the second case, provided M/e_1 is connected. Thus, we may assume that M/e_1 is disconnected. Since M and $M/e_1\backslash e$ are connected, it follows that $\{e, e_1\}$ is a circuit of M. As $M(K_4)$ is 1-rounded in the class of binary matroids, M has an $M(K_4)$-minor N' using e. In producing N', we cannot contract e_1, otherwise e becomes a loop. Thus, e_1 must be deleted to produce N'. Whence, $N' = M\backslash(X\dot{\cup}e)/Y$ for some sets X and Y. It follows that $M\backslash X/Y$ is an $M(K_4^+)$-minor of M, in which e is in the 2-circuit.

$\square$

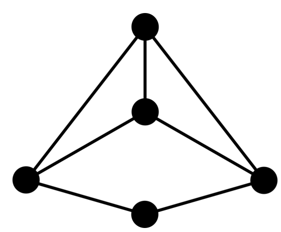


FIGURE 2.9: The graph $K_4^\times$, which appears in Corollary 2.25.

Corollary 2.25. *Let M be a regular 3-connected matroid with element e. Suppose M has an $M(K_4)$-minor. Then M has no $M(K_4^\times)$-minor using e in the 2-cocircuit if and only if M/e has no $M(K_4)$-minor.*

Proof. This is the dual statement of Lemma 2.24. $\square$

We shall need two results from Dirac [9]. Dirac's original statement of the next result was for multigraphs [9, Theorem 2]; we amend it here to apply only to simple graphs. Before we give the result, however, we must define a few extensions of the bipartite graph $K_{3,p}$, with $p \geq 3$. Let the bipartition of $K_{3,p}$ be the pair (X, Y),

where $X = \{v_1, v_2, v_3\}$. Then we take $K'_{3,p} = K_{3,p} + \{f_1\}$ where $f_1 = (v_1, v_2)$; $K''_{3,p} = K_{3,p} + \{f_1, f_2\}$ where $f_2 = (v_2, v_3)$; and, finally, $K'''_{3,p} = K_{3,p} + \{f_1, f_2, f_3\}$ where $f_3 = (v_3, v_1)$.

Theorem 2.26. *The only* 3*-connected graphs with at least four vertices that do not contain two vertex-disjoint cycles are* W_k *for* $k \geq 3$*,* K_5*,* $K_5 \backslash f$ *for any edge* f*,* $K_{3,p}$*,* $K'_{3,p}$*,* $K''_{3,p}$*, and* $K'''_{3,p}$*, for* $p \geq 3$.

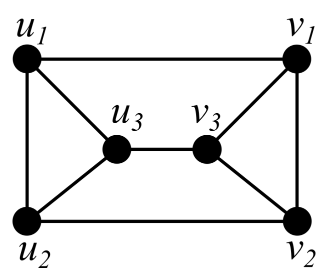


FIGURE 2.10: A prism graph. The two vertex-disjoint triangles u_1, u_2, u_3 and v_1, v_2, v_3 are sometimes called the *ends* of the prism.

Theorem 2.27. *Let* H *be a* 3*-connected graph with at least six vertices. If* H *is not* $K_{3,p}$*,* $K'_{3,p}$*,* $K''_{3,p}$*,* $K'''_{3,p}$*, or a wheel, then* H *has a prism-minor.*

Proof. By Theorem 2.26, H has two vertex-disjoint cycles C_1 and C_2. Moreover, as H is 3-connected, by Menger's Theorem (Theorem 1.4), H has three disjoint paths, each having one endpoint in C_1 and the other endpoint in C_2. It follows that H has a prism-minor. □

We shall also use the following result.

Lemma 2.28. *Let* H *be a* 3*-connected simple graph, and let* e *be an edge of* H*. Then either* $H \cong K_4$*, or* H *has a* K_4^+*-minor having* e *in the parallel pair, or* H *has a* $K_4^\times$*-minor having* e *in the series pair.*

Proof. Since H is 3-connected, it follows by Tutte's Wheels Theorem (Theorem 1.6) that $H \cong K_4$ or H has a W_4-minor. Clearly we are done if $H \cong K_4$; so we may

assume that H has a W_4-minor. By a result of Oxley and Reid [18], $\{M(W_4)\}$ is 2-rounded within the class of binary matroids. Therefore, H has a W_4-minor using e. If e occurs as a spoke of the wheel, then H has a K_4^+-minor with e in the 2-cycle. If e occurs as a rim element of the wheel, then H has a $K_4^\times$-minor using e in the 2-element bond. Thus, the statement holds. □

Lemma 2.29. *Let G be a 2-connected plane graph. Then G is a member of $\mathcal{S}$ if and only if G^* is a member of $\mathcal{S}^*$.*

Proof. Warshauer [28, Lemma 2.8.4] proves that if G^* is a member of $\mathcal{S}^*$, then G is a member of $\mathcal{S}$. Now suppose that G is a member of $\mathcal{S}$. Then there exists a graph H with edge f such that $H\backslash f = G$, and H/f is series-parallel. If H is planar, let H^* be a planar dual of it. Then, by duality, $H^*/f = G^*$, and $H^*\backslash f$ is series-parallel; thus, G^* is a member of $\mathcal{S}^*$.

If H is not planar, then it has a K_5-minor or a $K_{3,3}$-minor, by Wagner's Theorem (Theorem 1.2). This leads to a contradiction, as follows. Since G is a plane graph, we must have H be 2-connected. Moreover, $H\backslash X/Y$ is K_5 or $K_{3,3}$, where X and Y are subsets of $E(H)$. Now, we cannot have $f \in X$, because $H\backslash f$ is planar; but we cannot have $f \in Y$, either, because H/f is series-parallel. Thus, f is an edge in the K_5-minor or $K_{3,3}$-minor of H. Therefore, K_5/f or $K_{3,3}/f$ is a minor of H/f; but each of K_5/f and $K_{3,3}/f$ has a K_4-minor. It follows that K_4 is a minor of H/f. This is a contradiction, since H/f is series-parallel. □

Corollary 2.30. *Let $G = G_1 \oplus_2 K_4$, where G_1 is 3-connected and planar. Then G is an excluded minor for $\mathcal{M}_1$ if and only if G^* is an excluded minor for $\mathcal{M}_1$.*

Proof. Since G_1 is planar and $G = G_1 \oplus_2 K_4$, the graph G must be planar, as well. First suppose that G is an excluded minor for $\mathcal{M}_1$, and let e be an edge of G. Then $G\backslash e$ is a member of $\mathcal{S}$ or $\mathcal{S}^*$. Thus, by Lemma 2.29, we also have G^*/e is a member

of $\mathcal{S}^*$ or $\mathcal{S}$. Using this fact along with symmetry, we see that every single-edge deletion and every single-edge contraction of G^* is a member of $\mathcal{M}_1 = \mathcal{S} \cup \mathcal{S}^*$. However, we cannot have G^* in $\mathcal{M}_1$, or else $G^{**} = G$ is a member of $\mathcal{M}_1$. A dual argument establishes the converse. $\square$

Corollary 2.31. *If $\mathcal{M}_1$ has an excluded minor of the form $G_1 \oplus_2 K_4$, where G_1 is 3-connected and planar, then $G_1 \in \mathcal{S}^*$, or $\mathcal{M}_1$ has an excluded minor of the form $G_1^* \oplus_2 K_4$ where G_1^* is a member of $\mathcal{S}^*$.*

Proof. As G_1 is a proper minor of G, it is in $\mathcal{S}$ or $\mathcal{S}^*$. If G_1 is in $\mathcal{S}^*$, we are done. If G_1 is not in $\mathcal{S}^*$, then it is in $\mathcal{S}$. By Lemma 2.29, we have $G_1^* \in \mathcal{S}^*$, and by Corollary 2.30,

$$G^* = (G_1 \oplus_2 K_4)^* = G_1^* \oplus_2 K_4$$

is an excluded minor of $\mathcal{M}_1$. $\square$

Theorem 2.32. *Let the graph G be simple and 2-connected. Then G is an excluded minor of class $\mathcal{M}_1$ if and only if G is isomorphic to S_1 or S_v.*

Proof. By Lemma 2.5, we have S_1 and S_v are excluded minors of $\mathcal{M}_1$.

Conversely, let G be an excluded minor of $\mathcal{M}_1$. Then the structure of the canonical tree decomposition T of G was determined in Lemma 2.11. Suppose first that

2.33. *T is a star whose hub is labeled by a bond G_b.*

From Lemma 2.11, we know G_b is a k-edge bond, for $k \geq 3$, and each of the at least $k-1$ leaves is labeled by a 3-connected graph.

Suppose no 3-connected leaf G_i of T has a $K_4^{\times}$-minor with its basepoint e_i in the series pair. Then by Corollary 2.25, each G_i/e_i has no K_4-minor. Thus, if we add an edge e in G between the two vertices from G_b, we find G/e has no K_4-minors. Thus, G is a member of $\mathcal{S}$, which is a contradiction.

We may now suppose some leaf G_1 of T has a $K_4^{\times}$-minor with its basepoint e_1 in the series pair. Let G_2 be another leaf with basepoint e_2. Then G_2 has a K_4-minor that uses e_2. Now, either G has another leaf G_3 with basepoint e_3, or G_b contains an edge e_b that is also an edge of G. If the latter occurs, we immediately see G has an S_1-minor. In the first case, let e_c be an edge of G_3 that is different from e_3; so G_3 has a 2-cycle $\{e_3, e_b\}$ as a minor. It follows that G has as a minor the graph whose canonical tree decomposition consists of a star with a degree-2 hub with edge set $\{e_1, e_2, e_c\}$, with the neighbors of this hub being a copy of $K_4^{\times}$ having e_1 in the series pair and a copy of K_4 having e_2 as basepoint. This graph is S_1.

Lemma 2.11 gives us a second possibility for the structure of the tree decomposition of excluded minor G, namely,

2.34. *T is a star whose hub is labeled by a cycle G_c.*

So G_c is a k-edge cycle, where $k \geq 3$ and the star has at least $k-1$ leaves labeled by 3-connected graphs. Suppose no leaf G_i or T has a K_4^{+}-minor with its basepoint e_i in the parallel pair. Then, by Lemma 2.24, G_i/e_i has no K_4-minor. Thus, if we split a vertex of G_c into two vertices such that no G_j has edges meeting both vertices, we find a graph with no K_4-minor. Therefore, G is in $\mathcal{S}^*$, which is a contradiction.

We may now suppose that some leaf G_1 of T has a K_4^{+}-minor with its basepoint e_1 in the parallel pair. Then, by the dual argument to that used in 2.33, we deduce that G is isomorphic to S_1.

There is one final structure of T, by Lemma 2.11:

2.35. *T consists of two vertices labeled by 3-connected graphs.*

Let the 3-connected labels be G_1 and G_2, with e the basepoint for their 2-sum. We begin by discussing the presence of K_4^{+}-minors and $K_4^{\times}$-minors in G_1 and G_2.

Suppose that G_1 has a $K_4^{\times}$-minor with e in the series pair, and G_2 has a K_4^{+}-minor with e in the parallel pair. Then G is either isomorphic to S_1, as we wished to prove; or G has a proper S_1-minor, which is a contradiction.

Suppose neither G_1 nor G_2 has a K_4^{+}-minor with e in the parallel pair. By Lemma 2.24, neither of $G_1 \backslash e$ and $G_2 \backslash e$ has a K_4-minor. We can split either endpoint of e and get a series-parallel graph from G. Thus, G is a member of $\mathcal{S}^*$, which is a contradiction.

Next, suppose neither G_1 nor G_2 has a $K_4^{\times}$-minor with e in the series pair. By Corollary 2.25, neither of G_1/e and G_2/e has a K_4-minor. Thus, we can identify the endpoints of e and get a series-parallel graph from G; so G is a member of $\mathcal{S}$, which is a contradiction.

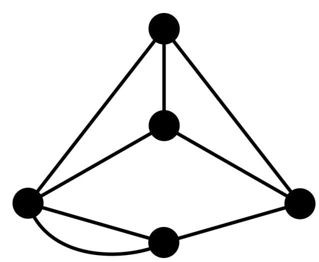


FIGURE 2.11: Graph $K_4^{\times +}$.

We now know that G_1 or G_2, say G_1, has both a K_4^{+}-minor with e in the parallel pair and a $K_4^{\times}$-minor with e in the series pair. Then G_2 has neither of these minors. Thus, by Lemma 2.28, $G_2 \cong K_4$. If G_1 has a $K_4^{\times +}$-minor (see Figure 2.11), where e is one of the parallel edges, then G has an S_1-minor, perhaps an improper one. However, the K_4^{+}-minors and $K_4^{\times}$-minors of G_1 may not result in the desired $K_4^{\times +}$-minor, and in this case, we must do more work. We first show the following.

2.35.1. G_1 *has a prism-minor.*

Assume to the contrary that G_1 has no prism-minor; then G_1 must be one of the graphs in Theorem 2.26. Clearly G_1 and G_2 cannot both be K_4, or G would

be a member of $\mathcal{M}_1$. Neither can G_1 be K_5, since K_5 is an excluded minor of $\mathcal{M}_1$. If G_1 is $K_5\backslash f$, then either G has an S_1-minor (when e and f share an endpoint) or G is a member of $\mathcal{S}$ (we can identify the endpoints of e and get a series-parallel graph from G). If G_1 is a wheel W_n for $n > 3$ and e is a rim element, then G is a member of $\mathcal{S}^*$, since we can split either endpoint of e in G to get a series-parallel graph. If e is a spoke edge, however, G is a member of $\mathcal{S}$, since we can identify the endpoints of e in G to get a series-parallel graph.

That leaves us with G_1 being one of $K_{3,p}$, $K'_{3,p}$, $K''_{3,p}$, and $K'''_{3,p}$, for $p \geq 3$. Recall that the bipartition of $K_{3,p}$ is the pair (X, Y), where $X = \{v_1, v_2, v_3\}$. Then we take $K'_{3,p} = K_{3,p} + \{f_1\}$ where $f_1 = (v_1, v_2)$; $K''_{3,p} = K_{3,p} + \{f_1, f_2\}$ where $f_2 = (v_2, v_3)$; and, finally, $K'''_{3,p} = K_{3,p} + \{f_1, f_2, f_3\}$ where $f_3 = (v_3, v_1)$. Observe that if G_1 is any of $K_{3,p}$, $K'_{3,p}$, $K''_{3,p}$, and $K'''_{3,p}$ and $p \geq 4$, then G must have an S_v-minor. We must be somewhat careful when $p = 3$. If G_1 is $K_{3,3}$, then G is isomorphic to S_v, as we wished to prove. If G_1 is any of $K'_{3,3}$, $K''_{3,3}$, or $K'''_{3,3}$ and $e \notin \{f_1, f_2, f_3\}$, then G has an S_v-minor, a contradiction. However, if e is one of f_1, f_2, and f_3, then G is a member of $\mathcal{S}$, since we can identify the endpoints of e and get a series-parallel graph from G. We conclude that 2.35.1 holds.

We now know that G_1 has a prism-minor. If e is an edge in one of the triangles of some prism minor, then G has an S_1-minor, since G_1 has a $K_4^{\times +}$-minor with e in the parallel pair. If e is not in a triangle of any prism-minor of G_1, however, we must pursue a more detailed analysis of the situation.

2.35.2. *The graph G_1 does not have two vertex-disjoint cycles such that e is in one of these cycles.*

Assume to the contrary that G_1 has two vertex-disjoint cycles and e is in one of these cycles. As G_1 is 3-connected, it follows by Menger's Theorem 1.4 that G_1 has a prism-minor in which e occurs in one of the ends.

Now $G = G_1 \oplus_2 K_4$. Suppose G_1 is planar. Then, by Corollary 2.31, either $G_1 \in \mathcal{S}^*$, or $G_1^* \oplus_2 K_4$ is an excluded minor of M_1 and $G_1^* \in \mathcal{S}^*$. In the latter case, G_1^* has a prism-minor; otherwise, since G_1^* is planar, $G_1^* \cong K_4$ or $G_1^* \cong K_5 \backslash f$ for some element f. Then we obtain the contradiction that $G_1^* \oplus_2 K_4$ is in $\mathcal{M}_1$. We deduce that we may assume we have an excluded minor of $\mathcal{M}_1$ of the form $G_1 \oplus_2 K_4$, where $G_1 \in \mathcal{S}^*$ and G_1 has a prism-minor. By 2.35.2, G_1 has no prism-minor in which the basepoint e is in one of the triangles. So G_1 has no vertex-disjoint cycles such that e is in one of them.

As G_1 is a member of $\mathcal{S}^*$, Theorem 2.19 implies that it is a daisy chain. We perform a case analysis based on whether G_1 is an identified or unidentified daisy chain, and where e is placed in the structure of that daisy chain.

Let G_1 be an unidentified daisy chain, and consider the location of e in the daisy chain structure. We cannot have $e = (s,t)$, since then there is a vertex split on either endpoint of e in G that makes a series-parallel graph. We make the following observation about the initial and terminal cycles of an unidentified daisy chain.

2.35.3. *Let H be an unidentified daisy chain that is not a wheel. The initial and terminal cycles of H are vertex-disjoint.*

Let $N_H(s) = \{t, 1, 2\}$ and $N_H(t) = \{s, 4, 5\}$. Since H is not a wheel, both of its $s-t$ paths have length at least three. Therefore, vertices 1 and 2 are distinct from vertices 4 and 5. Thus, the initial and terminal cycles of H are vertex-disjoint.

Thus, e is not an edge of either the initial cycle or the terminal cycle of G_1. Consider an inside cycle C that contains e. It must contain one vertex (but not

s) of the initial cycle and one vertex (not t) of the terminal cycle, to prevent C from being one of a pair of vertex-disjoint cycles in G_1. This inside cycle C may be a 3-cycle, in which case we label its edges α, β, and γ; or it may be a 4-cycle, in which case we label its edges α, β, γ, and δ. We will assume α and δ are edges on the $s-t$ paths of G_1. The possibilities for C in G_1 are shown in Figure 2.12. Note that, in the figure, dotted edges in an inside cycle may be subdivided, with the resulting new vertices made adjacent to the original vertex of the inside cycle that was not incident with the dotted edge.

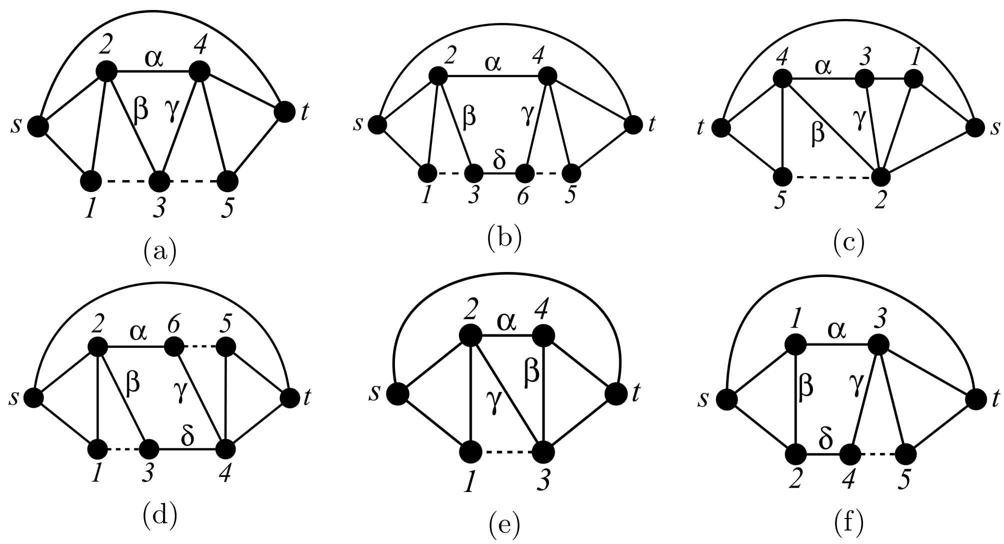


FIGURE 2.12: Possible locations of cycle C containing e.

Observe that e cannot be β or γ in Figures 2.12a, 2.12b, 2.12d, or 2.12f, since then there would be two vertex-disjoint cycles in G, one of them containing e. Likewise, e cannot be γ in Figure 2.12c, nor β in Figure 2.12e.

If $e = \alpha$ in any of the daisy chains shown in Figure 2.12, then G has an S_1-minor. If $e = \delta$ in a daisy chain with the structure shown in Figure 2.12b, then G is a member of $\mathcal{S}^*$. We can split either endpoint of δ in G to get a series-parallel graph. If $e = \beta$ in a daisy chain with the structure shown in Figure 2.12c, then G is a member of $\mathcal{S}$. We can identify the endpoints of β in G and produce a series-parallel

graph. If $e = \delta$ in a daisy chain with the structure shown in Figure 2.12d, however, then G has an S_1-minor. If $e = \gamma$ in a daisy chain with the structure shown in Figure 2.12e, then G is a member of $\mathcal{S}$, since identifying the endpoints of γ in G makes a series-parallel graph. Lastly, if $e = \delta$ in a daisy chain with the structure shown in Figure 2.12f, then G is a member of $\mathcal{S}^*$, as we can split either endpoint of δ in G and get a series-parallel graph.

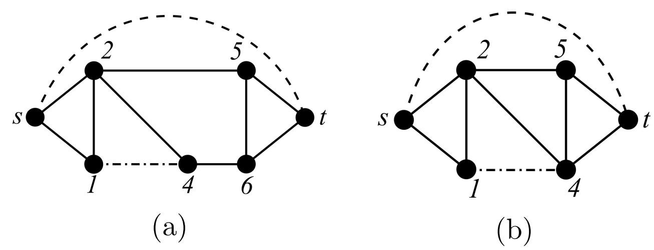


FIGURE 2.13: Possible structures of identified daisy chain G_1 when e is in the initial or terminal cycle. Edge $(1, 4)$ may be subdivided, with the resulting vertices made adjacent to vertex 2.

Now let G_1 be an identified daisy chain. If e is an edge of the initial cycle, then the terminal cycle and all the inside cycles of G_1 would share a vertex with the initial cycle. So, assume e is an edge of the initial or terminal cycle of G_1. Recall that the st cycles Q_0 and Q_1 of an identified daisy chain must have length at least three. If both of Q_0 or Q_1 have length greater than three, however, G_1 has an inside cycle that is vertex disjoint with the initial cycle. Then, by Corollary 2.35.2, G_1 has a prism-minor with e in one of the triangles, and, thus, G has an S_1-minor. This is a contradiction. Therefore, we may assume that one of Q_0 and Q_1, say Q_0, has length exactly three. Observe that if both Q_0 and Q_1 have length three and there is only one inside cycle, necessarily a 4-cycle, then G_1 is isomorphic to $\widetilde{RW_{3c1}}$, which is W_4. So in addition to assuming that Q_0 has length exactly three, we also assume there are at least two inside cycles in G_1. By Lemma 2.20, G_1 has at most one inside 4-cycle. Thus, cycle Q_0 consists of three vertices, st, x, and y, where

$deg_G(x) \geq 4$ and $deg_G(y) = 3$. The structure of G_1 is one of the two illustrated in Figure 2.13.

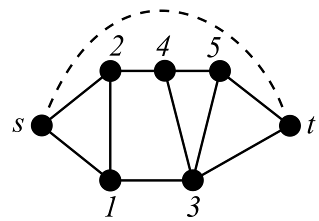


FIGURE 2.14: $\widetilde{AW_{4c1}}$

Let e be incident with vertex st. Observe that G_1 has an $\widetilde{AW_4}$-minor that keeps e on an initial or terminal cycle, incident with st; and that e will be, up to isomorphism, one of $(st, 1)$ or $(st, 2)$. When $e = (st, 1)$ is the basepoint, $(\widetilde{AW_4} \oplus_2 K_4) \backslash (1, 2)$ is isomorphic to S_1. When $e = (st, 2)$ is the basepoint, $(\widetilde{AW_4} \oplus_2 K_4) \backslash (2, 3)$ is isomorphic to S_1. Thus, G has an S_1-minor.

Let e be on the initial or terminal cycle of G_1, but not incident with st. The structure of G_1 is still one of the two illustrated in Figure 2.13. Again, G_1 has an $\widetilde{AW_4}$-minor that preserves e as an edge of the initial or terminal cycle, not incident with st. So, up to isomorphism, $e = (1, 2)$. Observe that $(\widetilde{AW_4} \oplus_2 K_4) \backslash (1, 3)$ is isomorphic to S_1. Thus, G has an S_1-minor.

Now suppose e is not an edge of either the initial or terminal cycle of G_1. Then e is on some inside cycle C of G_1 that shares vertices (but not st) with both the initial and terminal cycles. The possibilities for C in G_1 are shown in Figure 2.12, but now with s and t identified.

Once again, if $e = \alpha$ in any identified daisy chain with one of these six structures, then G contains an S_1-minor. Since $\mathcal{S}$ and $\mathcal{S}^*$ are minor-closed classes of graphs, taking $e = \delta$ in Figure 2.12b, or in Figure 2.12f, still leaves G in $\mathcal{S}^*$. Likewise, taking $e = \beta$ in Figure 2.12c, or $e = \gamma$ in Figure 2.12e still leaves G in $\mathcal{S}$. If $e = \delta$

in a daisy chain with the structure shown in Figure 2.12d, then G has an S_1-minor. If $e = \beta$ in Figure 2.12e, then G once more has an S_1-minor.

Observe that e cannot be β or γ in Figures 2.12a, 2.12b, 2.12d, or 2.12f, because then there would be two vertex-disjoint cycles in G, one of them containing e. For the same reason, e cannot be γ in Figure 2.12c.

We have now complete the proof when $G = G_1 \oplus_2 K_4$ and G_1 is planar. It remains to consider the case when G_1 is non-planar. Within the class of graphic matroids, the set $\{M(K_5), M(K_{3,3}), M(K'_{3,3})\}$ is 1-rounded, by a result of Seymour [21]. Clearly G_1 having a K_5-minor using e is a contradiction. If G_1 has a $K_{3,3}$-minor using edge e, then G has S_v as a minor.

Supposing G_1 has neither a K_5-minor nor a $K_{3,3}$-minor that uses e, this leaves us with G_1 having e as the edge f_1 of a $K'_{3,3}$-minor. If $G_1 \cong K'_{3,3}$, then G is a member of $\mathcal{S}$, since we can identify the endpoints of e in G and get a series-parallel graph. Thus, we assume $K'_{3,3}$ is a proper minor of G_1. By Tseng and Truemper (Theorem 1.16), there is an edge f of G_1, such that some H_1 in $\{G_1\backslash f, G_1/f\}$ is 3-connected and has a $K'_{3,3}$-minor with $e = f_1$.

Let H be G/f or $G\backslash f$, depending on whether H_1 is G_1/f or $G_1\backslash f$. Then H is a member of $\mathcal{S} \cup \mathcal{S}^*$. If H is a member of $\mathcal{S}^*$, then we can split some vertex of H and get a series-parallel graph. The split cannot occur on a vertex of G_2 that is not an endpoint of e, since G_2 is isomorphic to K_4; splitting either vertex that is not an endpoint of e fails to touch the $K'_{3,3}$-minor of H_1, and therefore the split leaves a graph with a K_4-minor, a contradiction. Say we split a vertex in H_1 that is not an endpoint of e. Then a path contained in $H_1\backslash e$ between the endpoints of e remains after the split; this path and $G_2\backslash e$ yield a K_4-minor, a contradiction. So this split must occur on an endpoint of e, and moreover the split must leave the two resulting new vertices adjacent only to vertices of $H_1\backslash e$ and

$G_2\backslash e$, respectively. Otherwise, we would again be able to form a K_4-minor from $G_2\backslash e$ and a path in $H_1\backslash e$ between the endpoints of e. Thus, we can split a vertex of H and get a series-parallel graph. However, this implies $H_1\backslash e$, which still has a $K_{3,3}$-minor, has no K_4-minor, a contradiction. Thus, H is a member of $\mathcal{S}$. The only vertex identification that can remove all K_4-minors from H is the one which identifies the endpoints of e, call them u and v. By Warshauer [28, proof of Lemma 2.5.2], $H_1 - \{u, v\}$ is a tree.

Let $H_1 = G_1\backslash f$. We cannot have f incident with either of the endpoints u and v of edge e, or G_1 and hence G would be a member of $\mathcal{S}$ as well, since identifying u and v destroys all K_4-minors. Consider $H_1 - \{u, v\}$. Suppose this tree has a vertex of degree at least three. Then G_1 has two disjoint cycles, one containing f and the other containing e. By Corollary 2.35.2, then, G has a prism-minor with e in one of the triangles. Therefore, G has an S_1-minor.

Now suppose the tree $H_1 - \{u, v\}$ is a path, with endpoints w_1 and w_2. In G_1, then, both of w_1 and w_2 are neighbors of u as well as v. Therefore, G_1 will have two disjoint cycles, one containing f and the other containing e, unless the endpoints of f are w_1 and w_2. Since H_1 is 3-connected, every internal vertex of $H_1 - \{u, v\}$ is adjacent to some member of $\{u, v\}$. Furthermore, since H_1 has a $K'_{3,3}$-minor, tree $H_1 - \{u, v\}$ must contain at least four vertices. If each of u and v is adjacent to at least one internal vertex of the tree, then G_1 has a K_5-minor. We can get this K_5 by contracting all edges of the $H_1 - \{u, v\}$ path in G_1 that are not incident with w_1 or w_2. If only one of u and v, say u, is adjacent to any internal vertices of the tree, then H_1 is planar, which is a contradiction.

Let $H_1 = G_1/f$. We must have f incident with one of u or v, say u. Otherwise, $G_1 - \{u, v\}$ is a tree; so identifying u and v in G destroys all K_4-minors, a contradiction. Let $f = (u', u'')$, and suppose u is the conglomerate vertex that results

from contracting f. Assume $e = (u', v)$ in G_1. Now, u'' has at least two neighbors other than u', since deleting u' and v does not leave a tree. Thus G_1 has a cycle C_1 that uses the vertex u'' but neither of vertices u' and v.

Suppose u' is adjacent to some vertex w that is a leaf of the tree $H_1 - \{u, v\}$. Then G_1 has a cycle with edges e, (v, w), and (w, u'). This cycle is vertex disjoint from C_1, so we obtain a contradiction. We deduce that no leaf of $H_1 - \{u, v\}$ is adjacent to u'. Hence, every such leaf is adjacent to u''. Moreover, u' has degree at least three, and is therefore adjacent to at least one vertex that remains in $H_1 - \{u, v\}$.

Now, H_1 has a $K'_{3,3}$-minor using e. We cannot delete two vertices from $K'_{3,3}$ and obtain a path. Thus, $H_1 - \{u, v\}$ is not a path, and therefore it has a vertex z of degree at least three. Then either u' is adjacent only to z, u'', and v; or u' has some other neighbor, t. In the latter case, there are two paths from z to leaves of $H_1 - \{u, v\}$ that avoid t. Using these two paths and edges from u'' to the leaves at the ends of these paths, we construct a cycle of G_1. This cycle is vertex disjoint from one that uses e, (u', t), and a path from t to a tree leaf that avoids z and is adjacent to v. We conclude that the only neighbors of u' are z, u'', and v.

Therefore, $H_1 - \{u, v\}$ has a unique vertex of degree exceeding two. Moreover, the only vertices of $H_1 - \{u, v\}$ to which u'' is adjacent are its leaves. Otherwise, G_1 has two vertex-disjoint cycles with one containing e. Now, v is adjacent to all the leaves of the tree. Choose two paths from z to ends of $H_1 - \{u, v\}$. Let the leaves at the ends of these paths be t_1 and t_2. Contract each of these paths down to a single edge, and contract all the other edges of $H_1 - \{u, v\}$. Performing these same contractions in G_1, we see that every vertex in $\{u'', v, z\}$ is adjacent to every vertex of $\{u', t_1, t_2\}$, so G_1 has a $K_{3,3}$-minor using e, which is a contradiction. $\square$

2.5 Excluded minors of $\mathcal{M}_1$ with Connectivity Three: Proof Strategy

In the following sections, we will make repeated use of the next theorem, which is from Warshauer [28].

Theorem 2.36. *Let G be a simple graph with $\kappa(G) = 3$. Then G has two vertices u and v such that $G - \{u, v\}$ is a tree if and only if $G \in \mathcal{S}$.*

Corollary 2.37. *The class of wheel graphs is contained in $\mathcal{S}$. Moreover, the wheel graphs are contained in $\mathcal{S}^*$, as well.*

Proof. Let W_n be a wheel of order $n + 1$, for integer $n \geq 3$. Let h be the hub vertex of W_n, and let r be some rim vertex. Then $W_n - \{h, r\}$ is a tree, and, by Theorem 2.36, W_n is a member of $\mathcal{S}$.

Now, split vertex r into vertices r_1 and r_2, such that r_1 is adjacent to h and one of the rim vertex neighbors of r, while r_2 is only adjacent to the other rim vertex neighbor of r. The resulting graph is series-parallel. Thus, W_n is also a member of $\mathcal{S}^*$; that is, wheels are daisy chains. $\square$

Consider an excluded minor G of $\mathcal{M}_1$, having $\kappa(G) = 3$. Let e be an edge in $E(G)$. Then, by Tutte's Wheels-and-Whirls Theorem 1.13, at least one of $G\backslash e$ and G/e is 3-connected. Furthermore, since $G\backslash e$ and G/e are members of $\mathcal{M}_1$, each of them must be a member of at least one of $\mathcal{S}$ and $\mathcal{S}^*$. Therefore, we break down the proof of Theorem 2.4 when $\kappa(G) = 3$ into four main cases:

(1) G/e is 3-connected and in $\mathcal{S}^*$ (Section 2.6);

(2) $G\backslash e$ is 3-connected and in $\mathcal{S}^*$ (Section 2.7);

(3) $G\backslash e$ is 3-connected and in $\mathcal{S}$ (Section 2.8);

(4) G/e is 3-connected and in $\mathcal{S}$ (Section 2.9).

2.38. *Case 1, when G/e is 3-connected and in $\mathcal{S}^*$*

For Case 1, G/e is a daisy chain, and we can recover the excluded minor G via an uncontraction of some vertex v from $V(G/e)$ into vertices v_1 and v_2. The uncontraction must leave both v_1 and v_2 with degree of three or higher, since G is 3-connected. Thus, we can only uncontract vertices where $deg_{G/e}(v) \geq 4$. The distinguished vertices s and t of an unidentified daisy chain, therefore, cannot be selected for uncontraction. However, distinguished vertex st of an identified daisy chain is precisely degree-4; we consider this case last, and, unless explicitly stated, assume that v is an inside vertex of G/e, not st.

We begin Case 1 with Lemmas 2.42 and 2.44, along with definitions of the terms they use. These results exploit the structure of daisy chains and allow us to conclude that, if G/e has as a minor some daisy chain H, then an uncontraction of G/e has an uncontraction of H as a minor, under specific circumstances. Thus, we can perform a case analysis of the uncontractions of a finite number of small daisy chains, and draw conclusions about the uncontractions of larger daisy chains that have one of these as a minor.

The analysis of uncontractions in Case 1 breaks down into subcases based on the length of Q_a, the $s-t$ path of G/e (or st cycle, if G/e is an identified daisy chain) containing v, and the degrees of any other inside vertices on Q_a. The main subcases are as follows:

(A) length of Q_a is two;

(B) length of Q_a is three;

(C) length of Q_a is at least four;

(D) vertex v is the vertex st.

For Subcase A, we assume Q_a has length two. The daisy chains that have an $s - t$ path of length two are the wheels, and thus v is the hub vertex of a wheel. No identified daisy chains can have an st cycle of length two – that would create a parallel edge. We examine the possible uncontractions of the hub vertex of a wheel, and conclude they always result in either a member of $\mathcal{S}$, or a graph with a K_5-minor.

In Subcase B, we let Q_a have length three. Therefore, one other internal vertex besides v is on Q_a. We call this vertex w. We break our argument down based on the degrees of v and w in G/e.

(a) $deg_{G/e}(v) = 4$ and $deg_{G/e}(w) = 3$;

(b) $deg_{G/e}(v) = 5$ and $deg_{G/e}(w) = 3$;

(c) $deg_{G/e}(v) \geq 6$ and $deg_{G/e}(w) = 3$;

  (i) G/e has no inside 4-cycle;

  (ii) G/e has an inside 4-cycle;

(d) $deg_{G/e}(v), deg_{G/e}(w) \geq 4$.

In Subcase C, we let Q_a have length at least four. Its internal vertices are v and w_j, where $1 \leq j \leq i$. Then $i \geq 2$, since the length of Q_a is at least four. This is by far the most lengthy of our four main subcases of Case 1. Once again, we break our argument down based on the degrees of v and the other internal vertices of Q_a.

(a) $deg_{G/e}(v) = 4$, $deg_{G/e}(w_j) = 3$ for all j, and the length of Q_a is four;

(b) $deg_{G/e}(v) = 4$, $deg_{G/e}(w_j) = 3$ for all j, and the length of Q_a is at least five;

  (i) length of Q_a is exactly five;

(ii) length of Q_a is greater than five;

(c) $deg_{G/e}(v) \geq 5$, $deg_{G/e}(w_j) = 3$, and the length of Q_a is at least four;

(i) length of Q_a is exactly four, $deg_{G/e}(v) = 5$, and G/e has no inside 4-cycle;

(ii) length of Q_a is exactly four, $deg_{G/e}(v) = 5$, and G/e has an inside 4-cycle;

(iii) length of Q_a exceeds four, $deg_{G/e}(v) \geq 5$, and G/e has an inside 4-cycle;

(iv) length of Q_a exceeds four, $deg_{G/e}(v) \geq 5$, and G/e has no inside 4-cycle;

(d) $deg_{G/e}(v) \geq 4$, $deg_{G/e}(w_1) \geq 4$, $deg_{G/e}(w_j) = 3$ when $j \neq 1$, and the length of Q_a is at least four;

(i) length of Q_a is exactly four, $deg_{G/e}(v) = deg_{G/e}(w_1) = 4$, and G/e has no inside 4-cycles;

(ii) length of Q_a is exactly four, $deg_{G/e}(v) = deg_{G/e}(w_1) = 4$, and G/e has an inside 4-cycle;

(iii) length of Q_a is four or more, $deg_{G/e}(v) \geq 4$, $deg_{G/e}(w_1) \geq 4$, $deg_{G/e}(w_j) = 3$ when $j \neq 1$, and G/e has an inside 4-cycle;

(iv) length of Q_a is four, $deg_{G/e}(v) \geq 4$, $deg_{G/e}(w_1) \geq 4$, $deg_{G/e}(w_j) = 3$ when $j \neq 1$, and G/e has no inside 4-cycle;

(e) $deg_{G/e}(v), deg_{G/e}(w_1), deg_{G/e}(w_2) \geq 4$, $deg_{G/e}(w_j) \geq 3$ when $j \geq 3$, and the length of Q_a is at least four;

(i) length of Q_a is four, G/e has no inside 4-cycles, and $deg_{G/e}(v) = 4 = deg_{G/e}(w_1) = deg_{G/e}(w_2)$;

(ii) $deg_{G/e}(v), deg_{G/e}(w_1)$, and $deg_{G/e}(w_2)$ exceeds 4, $deg_{G/e}(w_j) \geq 3$ when $j \geq 3$, and the length of Q_a is at least four.

In Subcase D, we consider uncontractions of the distinguished vertex st, when G/e is an identified daisy chain. Let C_a be one of the st cycles of G/e, whose inside vertices are w and w_j, for $j \geq 1$. The minimum length of an st cycle is three, since a daisy chain cannot have parallel edges. We break the case analysis down based on the length of C_a, as well as the degrees of its vertices w and w_j.

(a) length of C_a is three;

  (i) $deg_{G/e}(w) = deg_{G/e}(w_1) = 3$;

  (ii) $deg_{G/e}(w) \geq 4$ and $deg_{G/e}(w_1) = 3$;

  (iii) $deg_{G/e}(w) = deg_{G/e}(w_1) = 4$;

  (iv) $deg_{G/e}(w), deg_{G/e}(w_1) \geq 4$;

(b) length of C_a is at least four;

  (i) $deg_{G/e}(w) = deg_{G/e}(w_j) = 3$ for all j;

  (ii) $deg_{G/e}(w) \geq 4$ and $deg_{G/e}(w_j) = 3$ for all j;

    - w is not adjacent to st;
    - w is adjacent to st;

  (iii) $deg_{G/e}(w) \geq 4$ and $deg_{G/e}(w_j) \geq 4$ for at least one j;

2.39. *Case 2, when $G \backslash e$ is 3-connected and in $\mathcal{S}^*$*

In Case 2, since $G \backslash e$ is 3-connected and is a member of $\mathcal{S}^*$, we know $G \backslash e$ is a daisy chain. Therefore, we can recover G by undeleting an edge from a daisy chain. The three main subcases of Case 2 are based on our options for the endpoints of e; we classify three types of undeleted edges:

(A) an e_1-undeletion;

(B) an e_2-undeletion;

(C) an e_3-undeletion.

No undeletion that creates a parallel edge is considered. Since $G\backslash e$ may be either an identified or unidentified daisy chain, we define the different types of undeletions for both kinds of daisy chains. For the e_1-undeletion, the undeleted edge e_1 has for its endpoints two inside vertices on distinct $s-t$ paths [st cycles] of daisy chain $G\backslash e$. However, these endpoints cannot be members of the same inside cycle of the daisy chain – otherwise, the undeletion just produces a new daisy chain. For the e_2-undeletion, the endpoints of the undeleted edge e_2 are nonadjacent inside vertices of the same $s-t$ path [st cycle] of $G\backslash e$. Finally, the e_3-undeletion adds edge e_3 between an inside vertex and vertex s or t, or vertex st (depending on whether $G\backslash e$ is an unidentified or identified daisy chain). When $G\backslash e$ is an unidentified daisy chain, we add the restriction that the inside vertex serving as an endpoint of e cannot be adjacent to s or t; otherwise, the undeletion produces a new daisy chain, this one identified.

2.39.1. *$G\backslash e$ is a wheel*

One family of graphs in the daisy chains, the wheels, have to be considered apart from the main subcases due to their structure. We find that wheels only permit e_2- and e_3-undeletions, and any undeletion we can perform results in a graph G that is a member of $\mathcal{S}$, by Theorem 2.36.

2.39.2. *Case 2, Subcase A*

Subcase A covers the e_1-undeletions (see Figure 2.15 for an example), and consists of a short lemma, Lemma 2.69. The lemma establishes that any e_1-undeletion of a daisy chain produces a graph with a K_5-minor, which is a contradiction.

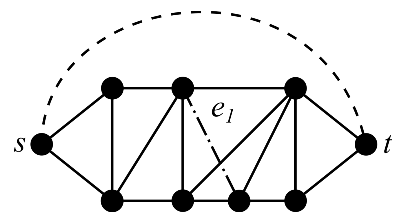


FIGURE 2.15: Example of an e_1-undeletion of a daisy chain.

2.39.3. *Case 2, Subcase B*

With the e_2-undeletions, both endpoints of the undeleted edge e_2 are on the same $s-t$ path [or st cycle], which we call Q_i; the subpath of Q_i between the endpoints of e_2 is L. Call the other $s-t$ path [st cycle] Q_j. (See Figure 2.16.) The set Z of inside cycles consists of all inside cycles such that, for each cycle in Z, all its vertices that are on $s-t$ path Q_i are also on L. We break down the argument based on how many vertices are both in $V(Z)$ and in $V(Q_j)$:

(a) $|V(Z) \cap V(Q_j)| = 1$;

(b) $|V(Z) \cap V(Q_j)| = 2$;

(c) $|V(Z) \cap V(Q_j)| \geq 3$.

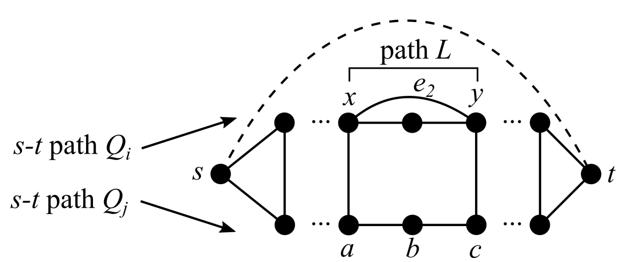


FIGURE 2.16: Example of an e_2-undeletion. Some edges between vertices of Q_i and Q_j are omitted.

In Subcase B.a, the set Z of inside cycles and $s-t$ path Q_j have just one vertex in common, namely, a. If a is the only inside vertex of Q_j, then $G\backslash e$ is a wheel, and we already considered wheels. So there is at least one other inside cycle on Q_j.

We use this to establish that G is isomorphic to U, as we wished to show; or that G contains U as a proper minor, which is a contradiction.

In Subcase B.b, the set Z of inside cycles and $s-t$ path Q_j have two vertices in common, a and b. Recall that the length of L is at least two, so it has an internal vertex; and due to the structure of a daisy chain, any internal vertex of L must be adjacent to at least one of a and b. We divide the argument as follows:

(i) each a and b is adjacent to one or more internal vertices of L;

(ii) only one of a and b is adjacent to internal vertices of L, and Z contains all the inside cycles of $G\backslash e$;

(iii) only one of a and b is adjacent to internal vertices of L, and Z does not contain all the inside cycles of $G\backslash e$.

Both B.b.i and B.b.ii lead to immediate contradictions. For B.b.iii, we make one further division of the argument, considering whether one of the non-Z inside cycles is a 4-cycle, or whether all the non-Z inside cycles are 3-cycles.

In Subcase B.c, the set Z of inside cycles and $s-t$ path Q_j have at least three vertices in common. We label three such vertices a, b, and c, and take a and c to be endpoints of a subpath in Q_j containing all such common vertices. We break the argument down based on how many inside 4-cycles are in Z.

(i) Z has two or more inside 4-cycles;

(ii) Z has one or no inside 4-cycles.

Subcase B.c.i directly results in G either being isomorphic to R, as we wished to prove; or G properly containing an R-minor, which is a contradiction. In Subcase B.c.ii, we further break down the argument based on whether either of paths L

and $a - c$ has an internal vertex of degree at least four, or whether all internal vertices of these paths have degree at most three.

2.39.4. *Case 2, Subcase C*

Subcase C covers the final type of undeletion we can perform on $G \backslash e$, the e_3-undeletions. See Figure 2.17. The undeleted edge is $e_3 = (s, y)$ [or $e_3 = (st, y)$ if we are considering an identified daisy chain], where we choose s instead of t without loss of generality. Endpoint y is on $s - t$ path [st cycle] Q_i, and it is not adjacent to either s or t [or st, in an identified daisy chain]. Thus there are vertices x and r on Q_i, as well. We take L to be the $x - y$ subpath of Q_i; note that length of L is at least one. The set Z of inside cycles consists of all inside cycles such that, for each cycle in Z, all its vertices that are on $s - t$ path Q_i are also on L. Let Q_j be the $s - t$ path [st cycle] that does not contain y. Our first means of breaking down the argument in Subcase C is to consider the length of L:

(1) length of L is one;

(2) length of L is at least two.

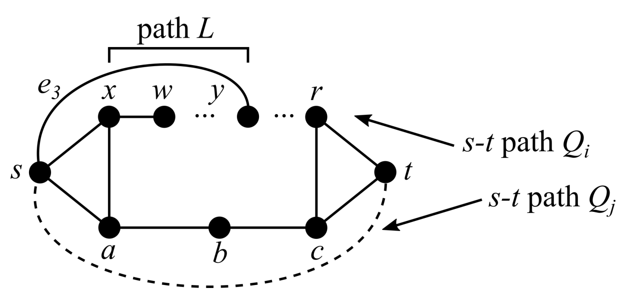


FIGURE 2.17: Example of an e_3-undeletion. Some edges between vertices of Q_i and Q_j are omitted.

When the length of L is one, we break down our case analysis based on how many vertices are in both Z and Q_j. We may also consider whether $G \backslash e$ is an unidentified

or identified daisy chain, the number and lengths of non-Z inside cycles in $G\backslash e$, and the lengths of the cycles in Z. This proof structure is outlined in the following:

(a) $V(Z) \cap V(Q_j) = \{a\}$;

  (i) $G\backslash e$ is an unidentified daisy chain;

  - the only non-Z inside cycle is a 3-cycle;
  - the non-Z inside cycle(s) include a 4-cycle;
  - there are at least two non-Z inside cycles, but none of them is a 4-cycle.

  (ii) $G\backslash e$ is an identified daisy chain;

  - there is one non-Z inside cycle, a 3-cycle;
  - there is one non-Z inside cycle, a 4-cycle;
  - there are two non-Z inside cycles, both 3-cycles;
  - there are two non-Z inside cycles, one of them a 4-cycle that contains edge (y, a);
  - there are two non-Z inside cycles, neither of them a 4-cycle that contains edge (y, a);
  - there are at least three non-Z inside cycles.

(b) $V(Z) \cap V(Q_j) = \{a, b\}$;

  (i) Z consists of one 4-cycle;

  - there is one non-Z inside cycle, a 3-cycle;
  - there is one non-Z inside cycle, a 4-cycle;
  - there are two non-Z inside cycles, both of them 3-cycles;
  - there are two non-Z inside cycles, a 4-cycle and a 3-cycle;

- there are at least three non-Z inside cycles, and all are 3-cycles that are incident with vertex b;
- there are at least three non-Z inside cycles, all 3-cycles, but one of them does not meet vertex b;
- there are at least three non-Z inside cycles, one of them a 4-cycle;

(ii) Z consists of two 3-cycles, with edge (x, b);

- this falls out from our work on (i);

(iii) Z consists of two 3-cycles, with edge (y, a);

- there is one non-Z inside cycle, a 3-cycle;
- there is one non-Z inside cycle, a 4-cycle;
- there are two or more non-Z inside cycles;

(c) $\{a, b, c\} \subseteq V(Z) \cap V(Q_j)$;

(i) Z has only inside 3-cycles; and $deg_{G \backslash e}(x) \geq 5$;

(ii) Z has only inside 3-cycles; and $deg_{G \backslash e}(x), deg_{G \backslash e}(y) \geq 4$;

(iii) Z has only inside 3-cycles; and $deg_{G \backslash e}(y) \geq 5$;

- there is one non-Z inside cycle in $G \backslash e$;
- there are at least two non-Z inside cycles in $G \backslash e$;

(iv) Z has one inside 4-cycle; and $deg_{G \backslash e}(x) \geq 4$;

(v) Z has one inside 4-cycle; and $deg_{G \backslash e}(y) \geq 4$;

- there is one non-Z inside cycle in $G \backslash e$;
- there are at least two non-Z inside cycles in $G \backslash e$.

When the length of L is two or more, we break the argument down very similarly to the way we did for the length of L being one. The major consideration, once again, is the number of vertices common to Z and Q_j:

(a) $V(Z) \cap V(Q_j) = \{a\}$;

  (i) $G \backslash e$ is an unidentified daisy chain;

    - there is one non-Z inside cycle;
    - there are two or more non-Z inside cycles;

  (ii) $G \backslash e$ is an identified daisy chain;

    - there is one non-Z inside cycle;
    - there are two or more non-Z inside cycles, and they are all incident with a;
    - there are two or more non-Z inside cycles, and at least one of them is not incident with a;

(b) $V(Z) \cap V(Q_j) = \{a, b\}$;

  (i) Z has only inside 3-cycles; and $deg_{G\backslash e}(a) = 3$ but $deg_{G\backslash e}(b) \geq 5$;

    - there is one non-Z inside cycle, a 3-cycle;
    - there is one non-Z inside cycle, a 4-cycle;
    - there are at least two non-Z inside cycles, and at least one of them is not incident with b;
    - there are at least two non-Z inside cycles, and all of them are incident with b;

  (ii) Z has one inside 4-cycle; and $deg_{G\backslash e}(a) = 3$ but $deg_{G\backslash e}(b) \geq 4$;

    - there is one non-Z inside cycle, a 3-cycle;

- there is one non-Z inside cycle, a 4-cycle;
- there are at least two non-Z inside cycles, and at least one of them is not incident with b;
- there are at least two non-Z inside cycles, and all of them are incident with b;

(iii) Z has only inside 3-cycles; and $deg_{G\backslash e}(a) \geq 5$;

- there is at least one non-Z inside cycle;

(iv) Z has one inside 4-cycle; and $deg_{G\backslash e}(a) \geq 4$;

- there is at least one non-Z inside cycle;

(v) Z has only inside 3-cycles; and $deg_{G\backslash e}(a), deg_{G\backslash e}(b) \geq 4$;

- there is at least one non-Z inside cycle;

(c) $\{a, b, c\} \subseteq V(Z) \cap V(Q_j)$;

(i) Z has two inside 4-cycles;

(ii) Z has one inside 4-cycle, and $deg_{G\backslash e}(x) = 3$;

(iii) Z has one inside 4-cycle, and $deg_{G\backslash e}(y) = 3$.

While it may appear in the outline that Subcase C.2.c misses the situations when Z has no inside 4-cycles (that is, when Z consists solely of inside 3-cycles), we establish in the proof that all the 3-cycle situations are subsumed by the three subcases listed in C.2.c.i - iii.

2.40. *Case 3, when $G\backslash e$ is 3-connected and in $\mathcal{S}$*

For this case, we have $G' = G\backslash e$ is a member of $\mathcal{S}$. By Theorem 2.36, there are two vertices in G', call them u and v, whose deletion produces a tree T. Moreover, neither u nor v is an endpoint of e; otherwise, $G - \{u, v\}$ is also a tree and G is

therefore a member of $\mathcal{S}$, which is a contradiction. So, if we begin with tree T, then add back edge e and vertices u and v, we can retrieve our excluded minor G. The three major subcases of Case 3 are based on the number of leaves in T:

(A) T has two leaves;

(B) T has three leaves;

(C) T has at least four leaves.

Each of the main subcases can be further divided based on the structure of $T+e$.

2.40.1. *Case 3, Subcase A*

For Subcase A, T has two leaves – that is, T is a path. So $T+e$ has one cycle C, and any edges of T not in C hang off either end of C in a *tail*. Depending on whether the endpoints of e coincide with both, neither, or one of the leaves of T, we get $T+e$ with two, none, or one tail, respectively. These constitute the three subcases of A:

(a) $T+e$ has two tails;

(b) $T+e$ has no tails;

(c) $T+e$ has one tail.

Once the subcase is broken down into Subcase A.a., Subcase A.b, and Subcase A.c, it becomes feasible to consider the results of adding vertices u and v to $T+e$, considering their possible adjacencies, and thus reconstructing excluded minor G. Since G' is 3-connected, one thing we know about these adjacencies is that every vertex of T must be adjacent to at least one of u and v. In Subcase A.a, we assume an edge exists between u and a vertex on the cycle C of $T+e$. We then examine

the results when v is adjacent to vertices on both tails of $T + e$, neither tail, or exactly one tail.

In Subcase A.b, because we assume $T + e$ has no tails, the cycle C formed by adding edge e to path T is precisely $T + e$. So we break down our argument based on the length of C, considering when C is a 3-cycle, a 4-cycle or an n-cycle with $n \geq 5$. To finish reconstructing G, for each length of C, we add vertices u and v, and consider their possible adjacencies.

To complete our analysis of this subcase, we turn to its third part, Subcase A.c. Here, $T + e$ has only one tail. We once more break down our argument by the length of C. When C is a 4-cycle or an n-cycle with $n \geq 5$, we are able add u and v to $T + e$ and begin analyzing the results for G based on their adjacencies. Unfortunately, the smallest length of C is also the longest part of the subcase. When C is a 3-cycle, we further break down the argument based on the length of the tail of $T + e$, and consider separately when that tail has length one, two, three, or at least four. For each of those tail lengths, we add u and v, consider their adjacencies, and determine the excluded minor G.

2.40.2. *Case 3, Subcase B*

Subcase B, where tree T has three leaves, is the most straightforward subcase of Case 3. With three leaves, we know there is a degree-3 vertex in T, which we call r. Since T is a tree, there is a unique path from r to each leaf. We consider the structure of $T + e$, depending on the endpoints of e; these endpoints can be r, the leaves of T, or vertices on the paths between r and the leaves. Up to isomorphism, there are seven possibilities for the structure of $T + e$ that arise from our choices for the endpoints of e. With each $T + e$, we add u and v, and consider their possible adjacencies so that we can reconstruct excluded minor G.

2.40.3. *Case 3, Subcase C*

This is the final subcase of Case 3, where T has at least four leaves. Thus, when we pick four leaves we have two options for the structure of T with respect to those leaves. Delete vertices until we are left with a tree that has only those four leaves; either that tree has a degree-4 vertex, or two degree-3 vertices. That is, in T, either any two paths between a pair of the four leaves meets in one vertex, r; or some paths meet in vertex r_1, vertex r_2, or both. We again break the analysis down based on whether the endpoints of e are leaves, r or r_i, or some other internal vertices of T. For each possible structure of $T + e$, we add vertices u and v and reconstruct excluded minor G by considering their adjacencies.

2.41. *Case 4, when G/e is 3-connected and in $\mathcal{S}$*

To avoid overlap with Case 3, we assume that excluded minor G of $\mathcal{M}_1$ is minimally 3-connected. We explicitly prove a theorem (Theorem 2.81) implicit in Warshauer [28] about the structure of any such excluded minor; in particular, G must have one of three structures. Call these three possible structures G_A, G_B, and G_C. The main subcases of Case 4 are determined by these three possible structures of G that we identified:

(A) the structure of G is that of G_A;

(B) the structure of G is that of G_B;

(C) the structure of G is that of G_C.

Each of G_A, G_B, and G_C shows a nonessential edge that exists in G. Since these are nonessential edges in a graph that is minimally 3-connected, we know that their contraction results in a 3-connected graph. Thus, we have a specific edge e such that G/e is 3-connected, whether the structure of G is that of G_A, G_B, or

G_C. We assume that G/e is a member of $\mathcal{S}$, since Case 1 in Section 2.6 has already handled the case where G/e is a member of $\mathcal{S}^*$.

By Theorem 2.36, then, there are two vertices of G/e, call them x and y, whose deletion produces a tree T. Now, one of these vertices, say x, must be the conglomerate vertex formed by the contraction of e, or else $G - \{x, y\}$ is a tree – but then G would be a member of $\mathcal{S}$, which is a contradiction. Each of the three structures G_A, G_B, and G_C specifies the endpoints of e, including all the neighbors of one of the endpoints, and also provides considerable information about the other vertices of G. This allows us to extrapolate a limited number of possible structures for T, depending on the vertex we choose to be y. Therefore, in each of Subcases A - C, we reconstruct G by beginning with a tree T; adding back the endpoints of e, the edge e, and vertex y; and assessing possible adjacencies for y and one endpoint of e.

2.41.1. *Case 4, Subcase A*

In Subcase A, we assume that our excluded minor G has the G_A structure. This structure has nonessential edge (u, u_1), and we take $e = (u, u_1)$. We begin with a lemma to establish a lower bound on the order of G, namely, eight vertices. We know that $G/e - \{x, y\}$ is a tree T, and x is the conglomerate vertex u' formed by contracting e. So a lower bound on the order of G also gives us a lower bound of five vertices on the order of T. The G_A structure specifies all the neighbors of u, and gives us information on the adjacencies we can expect to find in T. This tree T may have two leaves, or have at least three leaves. Additionally, the vertex of G labeled w by G_A can be degree-3, or be degree-4 or greater. So we break down Subcase A into four smaller subcases:

(a) $deg_G(w) = 3$ and T has at least three leaves;

(b) $deg_G(w) = 3$ and T has exactly two leaves;

(c) $deg_G(w) \geq 4$ and T has at least three leaves;

(d) $deg_G(w) \geq 4$ and T has exactly two leaves.

We handle each of Subcases A.a-d in the same way. Recall that $G/e - \{x, y\}$ is a tree T, and x is the conglomerate vertex u' formed by contracting e. Based on the information we have about G due to its having the G_A structure, we must consider when y is one of the vertices about which G_A gives us information (these are v, w, v_1, or w_1), or y is none of those. For each choice of y, the structure of G_A determines the possible trees that can be T. So, by beginning with a tree T; adding back the endpoints u and u_1 of e, the edge e, and vertex y; and assessing possible adjacencies for y and u_1, we can reconstruct G.

2.41.2. *Case 4, Subcase B*

We now suppose excluded minor G has the G_B structure. With this structure, we know $e = (v, v_1)$ is a nonessential edge of G. We know that $G/e - \{x, y\}$ is a tree T, and x is the conglomerate vertex v' formed by contracting e. We give a lemma that puts a lower bound of seven on the order of G; therefore, T has order at least four. The G_B structure specifies all the neighbors of v, and gives us information on the adjacencies we can expect to find in T. Our proof breaks into two smaller subcases:

(a) T has at least three leaves;

(b) T has exactly two leaves.

Our analysis of Subcases B.a and B.b proceed in the same way as that of Subcases A.a-d. We know $G/e - \{x, y\}$ is a tree T, and $x = v'$. So we must choose the vertex

that is y in G, and it can either be one of the vertices named by the G_B structure, or not. For each choice of y, the structure of G_B determines the possible trees that can be T. Thus, we add vertices v, v_1, and y to T, along with edge e; then we consider possible adjacencies of v_1 and y, to reconstruct G.

2.41.3. *Case 4, Subcase C*

We handle Subcase C exactly as we did Subcase B, except that G_C gives the structure of G, our lemma on the order of G gives a lower bound of eight, and our nonessential edge is $e = (u, u_2)$. The subdivisions of the argument are the same, as are their analyses:

(a) T has at least three leaves;

(b) T has exactly two leaves.

2.6 Excluded minors of $\mathcal{M}_1$ with Connectivity Three: Case 1

This section of the proof covers Case 1. So G/e is a daisy chain, and we can recover the excluded minor G via an uncontraction of some vertex $v \in V(G/e)$, resulting in new vertices v_1 and v_2. Observe that $deg(v)$ is at least four; otherwise, the uncontraction indicates there is a vertex of G with degree at most two. By Theorem 1.5, however, this would force the connectivity of G to be at most two, a contradiction.

We now state a few definitions and two lemmas (Lemmas 2.42 and 2.44) that we will use repeatedly in Case 1. For these definitions and lemmas, we only talk about unidentified daisy chains, but by substituting "*st* cycle" for every occurrence of "$s - t$ path," we would get analogous definitions and results for identified daisy chains.

Suppose daisy chain G_c is a cousin of daisy chain G. Then we can *cousin-recover* G from G_c by contractions of edges in the distinguished $s{-}t$ paths of G_c, specifically

edges in the $s-t$ paths that also belong to inside 4-cycles of G_c. Let v be a vertex of G, and let v_c be a vertex of G_c, such that $deg_G(v) = deg_{G_c}(v_c)$. We say v and v_c are *constant under cousins* if G can be cousin-recovered from G_c via contractions of edges that are not incident with v_c.

Lemma 2.42. *Suppose daisy chain G_c is a cousin of daisy chain G, and their vertices v_c and v, respectively, are constant under cousins. Then any uncontraction of v_c in G_c results in a graph H_c that contains H as a minor, where H is a graph that results from an uncontraction of v in G.*

Proof. Let F be the set of edges contracted from G_c to cousin-recover G. Suppose an uncontraction of v_c in G_c produces graph H_c, whose vertex set is $(V(G_c)-v_c) \cup \{v_1, v_2\}$. Observe that F is a subset of $E(H_c)$, since no edge of F was incident with v_c. Thus H_c/F is isomorphic to H, a graph produced by an uncontraction of vertex v in G. □

The next lemma describes an aspect of the structure of daisy chains with respect to inside vertices that have degree at least four.

Lemma 2.43. *Suppose v is an inside vertex of daisy chain G, and that $deg(v) \geq 4$. Then v lies on at least one inside 3-cycle of G, and this inside 3-cycle contributes an edge to the distinguished $s-t$ path Q_j not containing v.*

Proof. Assume to the contrary that v lies only on inside 4-cycles of G. Since a nested open ear decomposition of G can only have one nested subsequence and G is 3-connected, v then lies on precisely two inside 4-cycles, being an internal vertex of one length-4 ear and identified with an endpoint of another length-4 ear. Therefore, $deg(v) = 3$, a contradiction.

So v lies on at least one inside 3-cycle of G. If the only inside 3-cycles meeting v do not contribute edges to Q_j, then v again can only lie on precisely two inside

cycles of G. That is, since a nested open ear decomposition of G can only have one nested subsequence, v is the internal vertex of one ear and identified with the endpoint of another ear. Once more, $deg(v) = 3$. Hence, v must lie on at least one inside 3-cycle that has an edge in Q_j. $\square$

Let G and G' be two daisy chains. Suppose $s-t$ path $Q_i = s, v_1, v_2, \ldots, v_k, t$ of G and $s-t$ path $Q'_i = s', v'_1, v'_2, \ldots, v'_k, t'$ of G' have the same length, $k+1$. Further, suppose that $deg_G(v_i) = deg_{G'}(v'_i)$ for all values of i in $1, 2, \ldots, k$ except for one, say l. We assume $4 \leq deg_G(v_l) < deg_{G'}(v'_l)$. Then we say G' is an *ear extension* of G, and vertices v'_l and v_l are the *ear extension pair.*

We give two equivalent methods of constructing G' from G. The first makes use of the fact that every daisy chain can be viewed as a nested open ear decomposition – thus the terminology of an ear extension. The second accomplishes an identical construction, but describes it more succinctly, by means of subdivisions and edge additions, without reference to a daisy chain's ear decomposition.

Suppose G' is an ear extension of G, and we have ear extension pair v' and v. Let $deg_{G'}(v') - deg_G(v) = n$. By Lemma 2.43, v is on at least one inside 3-cycle of G that contributes an edge to the $s-t$ path not containing v. Given a nested open ear decomposition of G, let P_m be the length-2 ear whose endpoint is identified with v and corresponds to this inside 3-cycle. We extend the ear decomposition of G by replacing P_m in the decomposition by a sequence of length-2 ears $P_{m_0}, P_{m_1}, \ldots, P_{m_n}$, where each ear P_{m_i} for $i \geq 1$ has one endpoint identified with v, and the other endpoint identified with the internal vertex of ear $P_{m_{i-1}}$. The endpoints of ear P_{m_0} are identified with the same vertices on P_{m-1} as ear P_m was. Ear P_{m_n} takes the place of P_m in the old decomposition, with respect to how the endpoints of P_{m+1} are identified with its vertices. That is, if P_{m+1} has

its endpoints identified with v and the internal vertex v_i of P_m, it now has its endpoints identified with v and the internal vertex of P_{m_n}, and similarly if P_{m+1} has its endpoints identified with v_i and the endpoint of P_m that was not v.

Alternatively, since Lemma 2.43 guarantees v is on an inside 3-cycle that contributes an edge to the distinguished $s-t$ path Q_j not containing v, we know v has at least two neighbors on Q_j. Pick two of these neighbors on Q_j that are adjacent to each other, say y_1 and y_2. Subdivide edge (y_1, y_2) so that it is a $y_1 - y_2$ path with n internal vertices. Add an edge between each of the n internal vertices of $y_1 - y_2$ and vertex v. The resulting graph is isomorphic to G'.

To retrieve G from G', then, we can simply select two neighbors of v' on the distinguished $s-t$ path Q'_j that does not contain v', such that the subpath of Q'_j between these two neighbors of v' has n internal vertices. Contracting the subpath down to length one and simplifying leaves a graph isomorphic to G. We say G has been *ear-recovered* from G'.

We note that it is possible to create an ear extension of G by picking several pairs of neighbors of v that are adjacent on Q_j, and subdividing their edges to make a total of n new vertices. Likewise, to retrieve G from an ear extension of G', we can select several pairs of neighbors of v', such that the subpaths in Q'_j having each pair as endpoints have a total of n internal vertices. These modified constructions may be substituted in the proof of Lemma 2.44 that follows; however, for the sake of clarity, we use the method of subdividing one edge (y_1, y_2) to make an ear extension, and contracting down one subpath to perform an ear-recovery.

Lemma 2.44. *Suppose daisy chain G' is an ear extension of daisy chain G, and their vertices v' and v, respectively, are the ear extension pair. Consider uncontractions that leave every vertex with degree at least three. Then any uncontraction*

of v' in G' results in a graph H' that contains H as a minor, where H is some graph that results from an uncontraction of v in G.

Proof. Let $deg_{G'}(v') - deg_G(v) = n$. Let Q_j be the distinguished $s-t$ path in G' that does not contain v'. Select two neighbors y_1 and y_2 of v' that are the endpoints of a subpath Y of Q_j, having n internal vertices. Contracting out all internal vertices of Y and then simplifying ear-recovers a graph isomorphic to G. Let the subset N' of $N_{G'}(v')$ consist of all vertices of $N_{G'}(v')$ except the internal vertices of Y. Thus, N' has the same cardinality as $N_G(v)$, which has cardinality at least four.

Suppose an uncontraction of v' in G' produces graph H', whose vertex set is $(V(G') - v') \cup \{v_1, v_2\}$. Observe that N' is a subset of $V(H')$, and that every vertex in N' must be assigned as a neighbor of either v_1 or v_2 in the uncontraction. By the pigeonhole principle, at least one of vertices v_1 and v_2 has two or more vertices from N' in its neighbor set. Let this be v_1. Meanwhile, $N(v_2)$ may have at least two, exactly one, or no vertices of N'.

If $N(v_2)$ has two or more vertices from N', then delete set X of all edges between v_1 or v_2 and the internal vertices of Y. Contract Y to a path of length one; collect these contracted edges in set X_1. Thus, $H' \backslash X / X_1$ is isomorphic to H, a graph produced by an uncontraction of vertex v of G.

If $N(v_2)$ has no vertices from N', its only neighbors, aside from v_1, are internal vertices of path Y, and there are at least two of them, say y_a and y_b. One of the subpaths $y_a - y_1$ and $y_a - y_2$ in Y avoids y_b. Suppose this is $y_a - y_1$; then, contract every edge of subpaths $y_a - y_1$ and $y_b - y_2$; if the subpath $y_a - y_b$ in Y had length greater than one, contract all edges but one. Note that, with these contractions, we have contracted out every edge of Y except one; collect these edges in set

X_1. Simplify, then delete edges (v_1, y_1) and (v_1, y_2); collect all these simplified and deleted edges in set X. Thus, $H' \backslash X / X_1$ is isomorphic to H.

If $N(v_2)$ has exactly one vertex from N', then v_2 must also be adjacent to at least one internal vertex of Y, call it y_a. Since $N(v_2)$ has exactly one vertex from N', at least one of y_1 and y_2 is not in $N(v_2)$, say y_1. Delete any edges between v_1 or v_2 and internal vertices of subpaths $y_1 - y_a$ and $y_2 - y_a$ of Y, and delete edge (v_1, y_1); collect these deleted edges in set X. Contract every edge of the $y_1 - y_a$ subpath of Y, and contract all but one edge of the $y_2 - y_a$ subpath of Y. Collect these contracted edges in set X_1. Thus, $H' \backslash X / X_1$ is isomorphic to H, as desired. $\square$

Now we are ready to proceed with the main subcases of Case 1. Recall that G/e is a daisy chain, and we can recover the excluded minor G via an uncontraction of some vertex $v \in V(G/e)$ resulting in new vertices v_1 and v_2. We characterize uncontractions of v by the subset of $N(v)$ that is assigned to $N(v_1)$. A graph that is an uncontraction is denoted by a superscript. For us, an uncontraction yields a graph whose vertices retain their pre-uncontraction labels, except for the two new vertices resulting from the uncontracted vertex.

Assume, if G/e is an identified daisy chain, that $v \neq st$; this subcase is considered at the end of our proof, in 2.66.

2.44.1. *Throughout the proof of Case 1, ignore uncontractions of v that produce either of the following:*

- *a graph whose connectivity is below three;*
- *a graph that is a daisy chain cousin of G/e.*

Thus, the degree of v will always be at least four. We also dismiss without consideration any uncontraction that is simply a cousin of G/e, since clearly our excluded minor G cannot be a daisy chain.

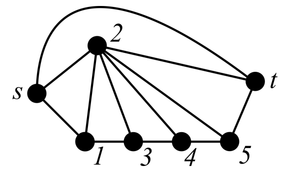


FIGURE 2.18: W_6

2.45. *vertex v is on an $s-t$ path of length two*

Suppose v is the only internal vertex on an $s-t$ path; then, G/e is a wheel. There are no identified daisy chains with st cycles of length two, since this would be a daisy chain with a parallel edge, which is forbidden by definition. The wheel of smallest order, W_3, has no vertices whose degree is four or more, so we cannot perform any uncontraction on it.

Every uncontraction of W_4, W_5, and W_6 remains in $\mathcal{M}_1$. We demonstrate how to reach this conclusion with W_6. The only vertex in W_6 (or any wheel) with high enough degree for an uncontraction is the hub vertex, labeled vertex 2, so we take $v = 2$. Due to the symmetry in a wheel, up to isomorphism there are four uncontractions satisfying our restrictions from 2.44.1. In these four uncontractions, the subset of $N(v)$ assigned to $N(v_1)$ is one of the following: $\{s, 3\}$, $\{s, 4\}$, $\{s, 1, 4\}$, and $\{s, 3, 5\}$. The uncontractions that assign $\{s, 3\}$ and $\{s, 4\}$ to $N(v_1)$ both result in graphs that are members of $\mathcal{S}$ by Theorem 2.36, since $G - \{v_2, 3\}$ and $G - \{v_2, 4\}$, respectively, are trees. On the other hand, the uncontractions that assign $\{s, 1, 4\}$ and $\{s, 3, 5\}$ to $N(v_1)$ both result in graphs, call them W_6^1 and W_6^2, that properly contain a K_5-minor. In each of these situations, we have created a contradiction to G being an excluded minor of $\mathcal{M}_1$.

In general, for a wheel W_n with $n > 6$, we again observe that the only vertex with high enough degree for a permissible uncontraction is the hub vertex. Therefore, we take the hub vertex to be v. Recall that our uncontraction must obey the restrictions given in 2.44.1. If the uncontraction assigns exactly two vertices $\{u_1, u_2\}$

of $N(v)$ to $N(v_1)$, then the resulting graph G is a member of $\mathcal{S}$ by Theorem 2.36, since $G - \{v_2, u_2\}$ is a tree.

So suppose we have an uncontraction that assigns at least three vertices of $N(v)$ to v_1. Note that v_2 must also be assigned at least three vertices of $N(v)$, or the uncontraction is isomorphic to the previous case considered, where the uncontraction produced a graph that was a member of $\mathcal{S}$. By the restrictions from 2.44.1, we further know that the set $N(v_1) - v_2$ does not constitute the vertices of a subpath of $s-t$ path Q_j (the $s-t$ path that does not contain v). If $N(v_1) - v_2$ were the set of vertices of such a subpath, then the graph produced by the uncontraction is a daisy chain cousin of G/e. Thus, some of the edges incident with v_1 and v_2 "cross"; more formally, any subpath of Q_j that contains all of the vertices of $N(v_1) - v_2$ must also contain a vertex from $N(v_2) - v_1$. We know by Lemma 2.44 that the uncontraction produces a graph G with some uncontraction of W_6 as a minor. If no vertices of $N(v_1) - v_2$ are adjacent, then we readily see G has W_6^2 as a minor. If any two vertices of $N(v_1) - v_2$ are adjacent, then G has W_6^1 as a minor. Either way, G contains a K_5-minor, which is a contradiction.

2.46. *vertex v is on an $s-t$ path or st cycle of length three*

We note that if one $s-t$ path of a daisy chain is length three, the daisy chain can have at most one inside cycle of length 4, by Lemma 2.20. Suppose v and w are the internal vertices on the $s-t$ path of length three. We break our argument into five smaller subcases, based on the degrees of v and w, and, where necessary, whether G/e has an inside 4-cycle or not.

2.46.1. *$deg(v) = 4$ and $deg(w) = 3$*

Under these conditions for the degrees of v and w, there are four daisy chains that can be G/e, namely, AW_4 and AW_{4c1}, along with their identifications $\widetilde{AW_4}$

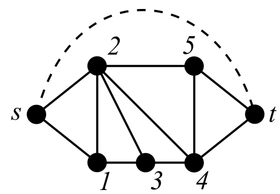
(a) AW_4 or $\widetilde{AW_4}$ (b) AW_{4c1} or $\widetilde{AW_{4c1}}$

FIGURE 2.19: Subcase with $deg(v) = 4$ and $deg(w) = 3$.

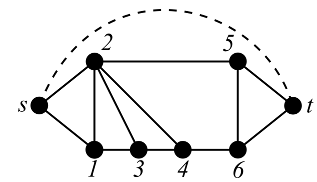
and $\widetilde{AW_{4c1}}$. See Figure 2.19. All uncontractions of AW_4 or $\widetilde{AW_4}$ produce a graph G that remains in the class $\mathcal{M}_1$, which is a contradiction. To verify this, note that due to the symmetry of AW_4 and $\widetilde{AW_4}$, it suffices to analyze uncontractions of vertex 3. Take $v = 3$. Under the restrictions of 2.44.1 and up to isomorphism, there are two uncontractions to consider, those that assign $\{1, 4\}$ or $\{2, 4\}$ to $N(v_1)$. Performing the $\{1, 4\}$ uncontraction in either AW_4 or $\widetilde{AW_4}$, we get $G - \{2, t\}$ is a tree, so G is in $\mathcal{S}$ by Theorem 2.36. Performing the $\{2, 4\}$ uncontraction in either AW_4 or $\widetilde{AW_4}$, we can now split vertex 2 and produce a series-parallel graph, meaning G is in $\mathcal{S}^*$. These are all contradictions.

Now consider AW_{4c1} and $\widetilde{AW_{4c1}}$. By the restrictions in 2.44.1, the only vertex we can uncontract in either of these is 3; and up to isomorphism, there are only two allowable uncontractions of $v = 3$ in either AW_{4c1} or $\widetilde{AW_{4c1}}$ to consider: $\{1, 5\}$ and $\{4, 5\}$. First consider these uncontractions in AW_{4c1}. The $\{4, 5\}$ uncontraction produces a graph G that is another daisy chain, since we can split vertex 4 of G and the result is a series-parallel graph – a contradiction. However, the $\{1, 5\}$ uncontraction gives a graph G which is isomorphic to the excluded minor S, as we wished to prove. With $\widetilde{AW_{4c1}}$, though, both uncontractions remain in $\mathcal{M}_1$. After either the $\{1, 5\}$ or $\{4, 5\}$ uncontraction, $G - \{st, 4\}$ is a tree and therefore a member of $\mathcal{S}$, by Theorem 2.36, a contradiction.

2.46.2. *$deg(v) = 5$ and $deg(w) = 3$*



(a) K or $\widetilde{K}$



(b) K_{c1} or $\widetilde{K_{c1}}$

FIGURE 2.20: Subcase with $deg(v) = 5$ and $deg(w) = 3$.

Under these conditions for the degrees of v and w, there are four daisy chain candidates for G/e, namely, K, K_{c1}, and their identifications $\widetilde{K}$ and $\widetilde{K_{c1}}$. See Figure 2.20. In each of these daisy chains, the vertex corresponding to v is labeled 2. Given the restrictions in 2.44.1, there are eight uncontractions of $v = 2$ up to isomorphism that we need to examine for K (and again for $\widetilde{K}$), shown in Table 2.1. All of these either produce another member of $\mathcal{M}_1$ or a graph properly containing a known excluded minor, except for one. The uncontraction of $\widetilde{K}$ that assigns $\{1, 3\}$ to $N(v_1)$ results in graph $\widetilde{K}^9$, which is isomorphic to U, as we wanted to prove.

TABLE 2.1: The eight uncontractions of $v = 2$ in K or $\widetilde{K}$.

Vertices in $N(v)$ assigned to $N(v_1)$	Label of resulting G	Resulting G	Remarks
$\{3, 5\}$	K^3 or $\widetilde{K}^3$	has a K_5-minor	
$\{1, 5\}$	K^4 or $\widetilde{K}^4$	has a K_5-minor	
$\{s, 5\}$ or $\{st, 5\}$	K^5 or $\widetilde{K}^5$	member of $\mathcal{S}^*$	split vertex 4 for a series-parallel graph
$\{1, 4\}$	K^6 or $\widetilde{K}^6$	member of $\mathcal{S}$	$G - \{4, v_2\}$ is a tree
$\{s, 4\}$ or $\{st, 4\}$	K^7 or $\widetilde{K}^7$	member of $\mathcal{S}$	$G - \{4, v_2\}$ is a tree
$\{3, 4\}$	K^8 or $\widetilde{K}^8$	member of $\mathcal{S}$	$G - \{4, v_2\}$ is a tree
$\{1, 3\}$	K^9 or $\widetilde{K}^9$	has a U-minor	$\widetilde{K}^9 \cong U$
$\{s, 3\}$ or $\{st, 3\}$	K^{10} or $\widetilde{K}^{10}$	K^{10} has an S-minor; $\widetilde{K}^{10}$ is a member of $\mathcal{S}$	$\widetilde{K}^{10} - \{st, v_2\}$ is a tree

Likewise, up to isomorphism, there are eight uncontractions to check on $v = 2$ of K_{c1}, and again on its identification, after we apply the restrictions in 2.44.1. Each of these results in a graph that properly contains K_5, U, or S; or is still a member of $\mathcal{M}_1$. The results are shown in Table 2.2.

TABLE 2.2: The eight uncontractions of $v = 2$ in K_{c1} or $\widetilde{K_{c1}}$.

Vertices in $N(v)$ assigned to $N(v_1)$	Label of resulting G	Resulting G	Remarks
$\{3, 5\}$	K_{c1}^3 or $\widetilde{K_{c1}}^3$	has a K_5-minor	
$\{1, 5\}$	K_{c1}^4 or $\widetilde{K_{c1}}^4$	has a K_5-minor	
$\{s, 5\}$ or $\{st, 5\}$	K_{c1}^5 or $\widetilde{K_{c1}}^5$	member of $\mathcal{S}^*$	split vertex 4 for a series-parallel graph
$\{1, 4\}$	K_{c1}^6 or $\widetilde{K_{c1}}^6$	has an S-minor	
$\{s, 4\}$ or $\{st, 4\}$	K_{c1}^7 or $\widetilde{K_{c1}}^7$	K_{c1}^7 has an S-minor; $\widetilde{K_{c1}}^7$ is a member of $\mathcal{S}$	$\widetilde{K_{c1}}^7 - \{st, v_2\}$ is a tree
$\{3, 4\}$	K_{c1}^8 or $\widetilde{K_{c1}}^8$	member of $\mathcal{S}^*$	split vertex 4 for a series-parallel graph
$\{1, 3\}$	K_{c1}^9 or $\widetilde{K_{c1}}^9$	has a U-minor	$\widetilde{K}_{c1}^9 \cong U$
$\{s, 3\}$ or $\{st, 3\}$	K_{c1}^{10} or $\widetilde{K_{c1}}^{10}$	K_{c1}^{10} has an S-minor; $\widetilde{K}_{c1}^{10}$ is a member of $\mathcal{S}$	$\widetilde{K}_{c1}^{10} - \{st, v_2\}$ is a tree

2.46.3. *deg*$(v) \geq 6$ *and deg*$(w) = 3$*; G/e has no inside* 4*-cycle*

We assume v is adjacent to s, if G/e is an unidentified daisy chain. For this subcase, we want to generalize from our analysis of the uncontractions of K and $\widetilde{K}$. By Lemma 2.44, any uncontraction of v in G/e results in a graph G that has as a minor some graph that is an uncontraction of vertex 2 in K or $\widetilde{K}$. Let Q_j be the $s - t$ path (or st cycle) in G/e that does not contain v. Observe that if $N(v_1) - v_2$ is the set of vertices of a subpath in Q_j and contains s (or st), then G is a daisy

chain cousin of G/e. The restrictions in 2.44.1 do not allow such an uncontraction. So either $N(v_1) - v_2$ is the set of vertices of a subpath in Q_j and does not contain s (or st), or any subpath in Q_j that has all the vertices of $N(v_1) - v_2$ must also contain vertices from $N(v_2) - v_1$. Whichever of these two situations occurs, G has as a minor one of the sixteen graphs listed in Table 2.1.

If the uncontraction G of v in G/e has as a minor one of K^3, $\widetilde{K}^3$, K^4, $\widetilde{K}^4$, K^9, $\widetilde{K}^9$, or K^{10}, then G properly contains an excluded minor, by transitivity.

Now suppose G only contained one of the other uncontractions of K and $\widetilde{K}$, namely, K^5, $\widetilde{K}^5$, K^6, $\widetilde{K}^6$, K^7, $\widetilde{K}^7$, K^8, $\widetilde{K}^8$, or $\widetilde{K}^{10}$. We know G/e is an ear extension of K or $\widetilde{K}$. Therefore, based on the constructions given for ear extensions and in the proof of Lemma 2.44, we can retrieve G from one of those uncontractions of K or $\widetilde{K}$ by subdividing one or both of edges $(1,3)$ and $(3,4)$, and making the new vertices adjacent to v_1 or v_2.

Start with K^5 and $\widetilde{K}^5$. No matter which of edges $(1,3)$ and $(3,4)$ we subdivide and make adjacent to v_1, G has a U-minor. If the new vertices are only adjacent to v_2, then G remains a daisy chain; splitting vertex 4 creates a series-parallel graph from G.

Next, consider K^6 and $\widetilde{K}^6$. Increasing the degree of v_1 by adding an edge whose other endpoint subdivides edge $(1,3)$ gives G a K^9-minor or a $\widetilde{K^9}$-minor, and therefore a U-minor. Increasing the degree of v_1 by adding an edge whose other endpoint subdivides edge $(3,4)$ produces a G that has a K_5-minor. Increasing the degree of v_2, however, leaves a graph that is still a member of $\mathcal{S}$, by Theorem 2.36, since deleting v_2 and 4 from G leaves a tree.

Now consider K^7 and $\widetilde{K}^7$. Once again, increasing only the degree of v_2 keeps G in $\mathcal{S}$, since $G - \{v_2, 4\}$ is a tree. Increasing the degree of v_1 makes for a G that has a K_5. We can readily see this since increasing the degree of v_1 forces G

to also have other uncontractions of K or $\widetilde{K}$ as minors: subdividing edge $(1,3)$ gives G a K^3-minor or $\widetilde{K}^3$-minor, while subdividing edge $(3,4)$ gives a K^4-minor or $\widetilde{K}^4$-minor.

Consider K^8 and $\widetilde{K}^8$. Again, a degree increase for just v_2 leaves G still in $\mathcal{S}$ by Theorem 2.36, since $G-\{v_2,4\}$ is a tree. Increasing the degree of only v_1 maintains the structure of G as a daisy chain; we can split vertex 4 and get a series-parallel graph. So, we must determine what happens if we simultaneously increase degrees of v_1 and v_2, in this case. If v_1 is adjacent to subdivisions of $(3,4)$ and v_2 to subdivisions of $(1,3)$, then G is still a daisy chain, and we can split vertex 4 to get a series-parallel graph; and likewise, if v_1 and v_2 are both adjacent to subdivisions of $(1,3)$ so that G is planar. If v_1 and v_2 are both adjacent to subdivisions of $(1,3)$ so that G is non-planar, then G has a K_5-minor. If v_1 is adjacent to subdivisions of $(1,3)$ and v_2 to subdivisions of $(3,4)$, then G has K^9 or $\widetilde{K}^9$ as a minor and, therefore, a K_5-minor. If both v_1 and v_2 are adjacent to subdivisions of $(3,4)$, then G has a K_5-minor or a U-minor.

Finally, consider $\widetilde{K}^{10}$. Increasing only the degree of v_2 keeps G in $\mathcal{S}$, since $G-\{v_2,st\}$ is a tree. Any increase in the degree of v_1, meanwhile, gives G a $\widetilde{K}^4$-minor, and therefore a K_5-minor.

2.46.4. *$deg(v)\geq 6$ and $deg(w)=3$; G/e has one inside 4-cycle*

We assume v is adjacent to s, if G/e is an unidentified daisy chain. For this subcase, our analysis is based on what happened with the uncontractions of K_{c1} and $\widetilde{K_{c1}}$. As with the previous subcase, by Lemma 2.44, any uncontraction of v in G/e results in a graph that contains as a minor an uncontraction of vertex 2 of K_{c1} or $\widetilde{K_{c1}}$. Let Q_j be the $s-t$ path (or st cycle) in G/e that does not contain v. Observe that if $N(v_1)-v_2$ is the set of vertices of a subpath in Q_j and contains

s (or st), then G is a daisy chain cousin of G/e. The restrictions in 2.44.1 do not allow such an uncontraction. So either $N(v_1) - v_2$ is the set of vertices of a subpath in Q_j and does not contain s (or st), or any subpath in Q_j that has all the vertices of $N(v_1) - v_2$ must also contain vertices from $N(v_2) - v_1$. Whichever of these two situations occurs, G has as a minor one of the sixteen graphs listed in Table 2.2.

Suppose G has as a minor one of K_{c1}^3, $\widetilde{K_{c1}}^3$, K_{c1}^4, $\widetilde{K_{c1}}^4$, K_{c1}^6, $\widetilde{K_{c1}}^6$, K_{c1}^7, K_{c1}^9, $\widetilde{K_{c1}}^9$, or K_{c1}^{10}. Then by transitivity, G has as a minor one of K_5, S and U.

So we need to consider when G only contains as a minor one of K_{c1}^5, $\widetilde{K_{c1}}^5$, $\widetilde{K_{c1}}^7$, K_{c1}^8, $\widetilde{K_{c1}}^8$, and $\widetilde{K_{c1}}^{10}$. We know G/e is an ear extension of K_{c1} or $\widetilde{K_{c1}}$. Therefore, based on the constructions given for ear extensions and in the proof of Lemma 2.44, we can retrieve G from one of those uncontractions of K_{c1} or $\widetilde{K_{c1}}$ by subdividing one or both of edges $(1, 3)$ and $(3, 4)$, and making the new vertices adjacent to v_1 or v_2.

For K_{c1}^5 and $\widetilde{K_{c1}}^5$, if we increase the degree of v_1 by making it adjacent to at least one vertex in a subdivision of either $(1, 3)$ or $(3, 4)$, then G will have a K_{c1}^6- or $\widetilde{K_{c1}}^6$-minor and therefore an S-minor. If only v_2 is made adjacent to the new vertices, however, G retains a daisy chain structure; we can split vertex 4 to produce a series-parallel graph from G.

Now, consider $\widetilde{K_{c1}}^7$. If only v_2 is made adjacent to the vertices resulting from subdividing $(1, 3)$ and $(3, 4)$, we have G in $\mathcal{S}$ by Theorem 2.36, since $G - \{v_2, st\}$ is a tree. Making v_1 adjacent to any of these subdivision vertices gives G a K_5-minor. We can see this through transitivity, because making v_1 adjacent to a subdivision of $(3, 4)$ gives G a $\widetilde{K_{c1}}^4$-minor, while v_1 adjacent to a subdivision of $(1, 3)$ gives G a $\widetilde{K_{c1}}^3$-minor.

Next, consider K_{c1}^8 and $\widetilde{K_{c1}}^8$. If v_1 is made adjacent to any subdivisions of $(1, 3)$ and $(3, 4)$ (but v_2 is not made adjacent to any), this leaves G with a daisy chain

structure – we can split vertex 4 and get a series-parallel graph from G. If v_2 gets an edge to a subdivision of $(3,4)$, then graph G will have a K_{c1}^6- or $\widetilde{K_{c1}}^6$-minor and thus an S-minor. If we just subdivide edge $(1,3)$ and make the new vertices adjacent to v_2, however, then G is still a daisy chain; we can split vertex 3 to get a series-parallel graph. So, we need to also check when v_2 is adjacent to vertices in the subdivision of $(1,3)$, while v_1 is adjacent to vertices in either subdivided edge. As long as we preserve planarity, G is a daisy chain; we can split it at vertex 1 to form a series-parallel graph. If we break planarity by the way we make v_1 adjacent to a subdivision of $(1,3)$, then G has a K_{c1}^6-minor or $\widetilde{K_{c1}}^6$-minor, and thus an S-minor.

Lastly, consider $\widetilde{K_{c1}}^{10}$. If only v_2 is made adjacent to the vertices in subdivided edges $(1,3)$ and $(3,4)$, then G is in $\mathcal{S}$ by Theorem 2.36, since $G - \{v_2, st\}$ is a tree. If v_1 is made adjacent to any vertex in a subdivision of $(1,3)$ or $(3,4)$, meanwhile, this leads to G having a $\widetilde{K_{c1}}^4$-minor, and, therefore, a K_5-minor.

2.46.5. $deg(v), deg(w) \geq 4$

If we take $deg(v) = 4 = deg(w)$, then there are four daisy chains satisfying these conditions on the degrees of v and w, namely, AW_5, $\widetilde{AW_5}$, AW_{5c2}, and $\widetilde{AW_{5c2}}$. See Figure 2.21. We will study the uncontractions of these daisy chains, and use those results to analyze the situation where $deg(v), deg(w) > 4$.

(a) AW_5 or $\widetilde{AW_5}$ (b) AW_{5c2} or $\widetilde{AW_{5c2}}$

FIGURE 2.21: Subcase with $deg(v) = 4 = deg(w)$.

First consider AW_5 and $\widetilde{AW_5}$. Due to the internal symmetry of each of these graphs, it makes no difference whether we choose v to be vertex 2 or 4. Take vertex $v = 2$. Under the restrictions of 2.44.1, there are two uncontractions of v to consider, up to isomorphism. These are the uncontractions that assign one of the following subsets of $N(v)$ to $N(v_1)$: $\{s, 3\}$ (or $\{st, 3\}$) and $\{1, 3\}$. The $\{s, 3\}$ (or $\{st, 3\}$) uncontraction, on AW_5 and $\widetilde{AW_5}$, results in graphs we label AW_5^1 and $\widetilde{AW_5}^1$, respectively. Graph AW_5^1 has an S minor, while $\widetilde{AW_5}^1$ is in $\mathcal{S}$ by Theorem 2.36, since $G - \{st, 3\}$ is a tree. The $\{1, 3\}$ uncontraction results in graphs we label AW_5^2 and $\widetilde{AW_5}^2$. Both are members of $\mathcal{S}^*$, since splitting vertex 3 leaves us with a series-parallel graph.

For AW_{5c2} and $\widetilde{AW_{5c2}}$, there are once again two uncontractions of $v = 2$ to consider, $\{s, 3\}$ (or $\{st, 3\}$) and $\{1, 3\}$. We call the graphs resulting from the $\{s, 3\}$ (or $\{st, 3\}$) uncontraction AW_{5c2}^1 and $\widetilde{AW_{5c2}}^1$; the graphs resulting from the $\{1, 3\}$ uncontraction are AW_{5c2}^2, and $\widetilde{AW_{5c2}}^2$. Both AW_{5c2}^1 and $\widetilde{AW_{5c2}}^1$ have an S-minor. However, both AW_{5c2}^2 and $\widetilde{AW_{5c2}}^2$ are members of $\mathcal{S}^*$; we can split either of them at vertex 3 to get series-parallel graphs.

Now we need to examine what will occur when one or both of $deg(v)$ and $deg(w)$ in G/e is at least five. We assume v is adjacent to s, if G/e is an unidentified daisy chain. If $deg(w) \geq 5$, then G/e is an ear extension of a daisy chain H that had a length-3 $s - t$ path (or st cycle) with two vertices v' and w', where w' was a degree-4 vertex, and w and w' were the ear extension pair. Assume we ear-recover H from G/e by contracting the set of edges X and deleting set of edges Y. So if G is the graph resulting from an uncontraction of v in G/e, then $G/X\backslash Y$ is the graph resulting from an uncontraction of v'.

When G/e has $deg(v) \geq 5$ and $deg(w) = 4$, by Lemma 2.44 the uncontraction of v from G/e results in a graph G that contains as a minor an uncontraction of

vertex 2 from AW_5, $\widetilde{AW_5}$, AW_{5c2}, or $\widetilde{AW_{5c2}}$. Whether G/e has an uncontraction of AW_5, $\widetilde{AW_5}$, AW_{5c2}, or $\widetilde{AW_{5c2}}$ depends upon whether G/e is an identified or unidentified daisy chain, and if G/e has an inside 4-cycle or not. Let Q_j again be the $s-t$ path (or st cycle) in G/e that does not have v. Observe that if $N(v_1)-v_2$ is the set of vertices of a subpath in Q_j and contains s (or st), then G is a daisy chain cousin of G/e. The restrictions in 2.44.1 do not allow such an uncontraction. So either $N(v_1)-v_2$ is the set of vertices of a subpath in Q_j and does not contain s (or st), or any subpath in Q_j that has all the vertices of $N(v_1)-v_2$ must also contain vertices from $N(v_2)-v_1$. Whichever of these two situations occurs, G has as a minor one of the eight uncontractions of $v=2$ in AW_5, $\widetilde{AW_5}$, AW_{5c2}, or $\widetilde{AW_{5c2}}$ discussed above.

Therefore, when one or both of $deg(v)$ and $deg(w)$ in G/e is at least five, we can recover G, the uncontraction of v in G/e, from one of these eight uncontractions by subdividing edge $(1,3)$ and making the new vertices adjacent to v_1 or v_2; or subdividing $(3,5)$ and making the new vertices adjacent to vertex $w=4$.

Suppose G contains as a minor an uncontraction of AW_5 or $\widetilde{AW_5}$. If G contains AW_5^1, we are done, because AW_5^1 has an S-minor. We must be somewhat more careful if G does not have AW_5^1, but instead contains one of $\widetilde{AW_5}^1$, AW_5^2, and $\widetilde{AW_5}^2$. We have to reconstruct G as described above, by subdividing edge $(1,3)$ and making the new vertices adjacent to v_1 or v_2; or subdividing $(3,5)$ and making the new vertices adjacent to vertex $w=4$. Start with G containing $\widetilde{AW_5}^1$. If we subdivide and v_1 adjacent to the new vertices, then G has an S-minor. If we subdivide and make v_2 adjacent to the new vertices, then G will have a U-minor. If we subdivide $(3,5)$ instead so that $w=4$ is adjacent to the new vertices, then G has an S-minor. Next, consider AW_5^2 and $\widetilde{AW_5}^2$. Subdividing and making the new vertices adjacent to v_1 or $w=4$ keeps G a daisy chain, since splitting vertex

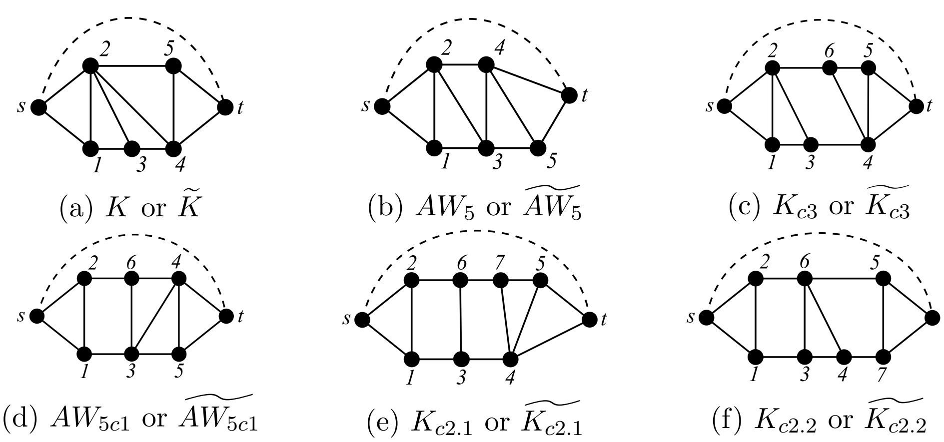
3 leaves a series-parallel graph. If we make v_2 adjacent to some new vertex of subdivided $(1, 3)$, however, then G has an S_v-minor.

If G contains as a minor either of AW^1_{5c2} and $\widetilde{AW_{5c2}}^1$, we are done, since these both have S-minors. If G does not contain either of these, it must have one of AW^2_{5c2}, and $\widetilde{AW_{5c2}}^2$. We reconstruct G as described above, by subdividing edge $(1, 3)$ and making the new vertices adjacent to v_1 or v_2; or subdividing $(3, 5)$ and making the new vertices adjacent to vertex $w = 4$. Subdividing and making the new vertices adjacent to v_1 or w leaves G with a daisy chain structure, such that splitting vertex 3 produces a series-parallel graph. If we make v_2 adjacent to some new vertex of subdivided $(1, 3)$, though, this puts an S_v-minor in G.

2.47. *vertex v is on an $s - t$ path or st cycle of length at least four*

Let Q_a be the $s - t$ path (or st cycle) that has vertex v. Let the other internal vertices of Q_a be w_j, where $1 \leq j \leq i$ and $i \geq 2$. Say the length of Q_a is n; then, by Corollary 2.21, G/e has at most $n - 2$ inside 4-cycles. We break our analysis down based on the degrees of v and vertices w_j, as well as whether the length of Q_a is exactly four or greater than four.

2.47.1. *$deg(v) = 4$, $deg(w_j) = 3$ for all j, and length of Q_a is four*



(a) K or $\widetilde{K}$ (b) AW_5 or $\widetilde{AW_5}$ (c) K_{c3} or $\widetilde{K_{c3}}$

(d) AW_{5c1} or $\widetilde{AW_{5c1}}$ (e) $K_{c2.1}$ or $\widetilde{K_{c2.1}}$ (f) $K_{c2.2}$ or $\widetilde{K_{c2.2}}$

FIGURE 2.22: Daisy chains with $deg(v) = 4$, $deg(w_j) = 3$, and length of Q_a is four.

Suppose the length of Q_a is four, and that v, w_1, and w_2 are its internal vertices. Then the daisy chains satisfying our conditions are K, AW_5; their cousins K_{c3}, $K_{c2.1}$, AW_{5c1}, and $K_{c2.2}$; and the identifications of all these daisy chains. See Figure 2.22. Take $v = 4$ in K, $\widetilde{K}$, and their cousins K_{c3}, $\widetilde{K_{c3}}$, $K_{c2.1}$, and $\widetilde{K_{c2.1}}$. Take $v = 6$ in $K_{c2.2}$ and $\widetilde{K_{c2.2}}$. Take $v = 3$ in AW_5 and $\widetilde{AW_5}$; and, additionally, consider both $v = 3$ and $v = 4$ in AW_{5c1} and its identification. Careful and tedious checks of every uncontraction of v allowed by the restrictions of 2.44.1 in these graphs shows they result in a G that is a member of $\mathcal{S}$; is a member of $\mathcal{S}^*$; or has an S-, U-, K_5-, or R-minor.

2.47.2. *$deg(v) = 4$, $deg(w_j) = 3$ for all j, and length of Q_a is at least five*

We proceed in this subcase by considering the position of v in Q_a, relative to the distinguished vertices s and t (or st, if G/e is an identified daisy chain). We begin by establishing that if v is not adjacent to s, t, or st, as in B and $\widetilde{B}$, then the graph G resulting from the uncontraction of v in G/e has an R-minor or a K_5-minor. We then consider the situation when v is adjacent to one of s, t, or st, breaking down into subcases based on the presence of inside 4-cycles in G/e.

Start with B and $\widetilde{B}$, taking v to be vertex 3; see Figure 2.23. These are the two daisy chains of smallest order that have $deg(v) = 4$, $deg(w_j) = 3$, and length of Q_a is at least five, along with v not adjacent to s, t, or st. The two uncontractions of $v = 3$ that respect the restrictions in 2.44.1 are those that assign $\{2, 4\}$ or $\{1, 4\}$ to $N(v_1)$. The $\{2, 4\}$ uncontraction of $v = 3$ in B or $\widetilde{B}$ yields a graph G that has a proper R-minor. The $\{1, 4\}$ uncontraction on either of these graphs produces a graph G that has a proper K_5-minor.

Lemma 2.48. *Let G/e have deg$(v) = 4$, deg$(w_j) = 3$ for all j, and length of Q_a is at least five. If G/e contains B or $\widetilde{B}$ as a proper minor, then an uncontraction of vertex v in G/e will properly contain an R- or K_5-minor.*

Proof. Split G/e and B at their distinguished vertices, and consider the nested open ear decompositions of the resulting series-parallel graphs. Either they have the same number of ears, or G/e has more ears. If they have the same number of ears, then G/e and B or $\widetilde{B}$ are cousins. Then vertex v of G/e and vertex 3 of B or $\widetilde{B}$ are constant under cousins. By Lemma 2.42, any uncontraction of v in G/e results in a graph G that has as a minor some graph that resulted from an uncontraction of 3 in B or $\widetilde{B}$. Moreover, since the uncontraction of v in G/e must follow the restrictions of 2.44.1, there are exactly two uncontractions, up to isomorphism, that are allowed; these correspond to the two uncontractions of vertex 3 discussed above for B or $\widetilde{B}$. Thus, G properly contains R or K_5 as a minor.

If G/e has more ears in its decomposition, then G/e may be an ear extension of B or $\widetilde{B}$, with the ear-extension pair including neither v in G/e nor vertex 3 in B or $\widetilde{B}$. The other possibility is that G/e is a cousin of such an ear extension of B or $\widetilde{B}$. Suppose G/e is such an ear extension of B or $\widetilde{B}$. Then contracting a set of edges X and deleting a set of edges Y from G/e leaves a graph isomorphic to B or $\widetilde{B}$. Moreover, the sets X and Y are in the edge set of graph G that results from an uncontraction of v in G/e. Thus, $G/X\backslash Y$ is isomorphic to some uncontraction of vertex 3 in B or $\widetilde{B}$. Now, since the uncontraction of v in G/e must follow the restrictions of 2.44.1, there are exactly two uncontractions, up to isomorphism, that could have produced G; these correspond to the two uncontractions of vertex 3 discussed above for B or $\widetilde{B}$. Thus, G contains as a minor one of those two

graphs resulting from the uncontractions $\{2,4\}$ or $\{1,4\}$, and, therefore, G properly contains R or K_5 as a minor.

We observe that if G/e is a cousin of such an ear extension of B or $\widetilde{B}$, then combining the preceding paragraph with Lemma 2.42 will again show G properly contains R or K_5 as a minor. $\square$

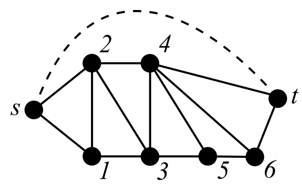
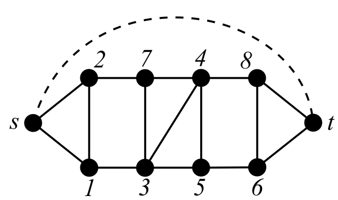


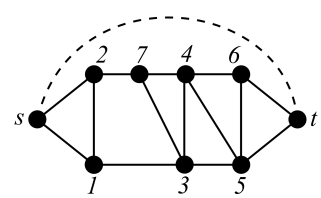
FIGURE 2.23: B or $\widetilde{B}$

Corollary 2.49. *Let G/e have $deg(v) = 4$, $deg(w_j) = 3$ for all j, and length of Q_a is at least five. If v is not adjacent to one of the distinguished vertices s, t, or st, then an uncontraction of vertex v in G/e will properly contain an R- or K_5-minor.*

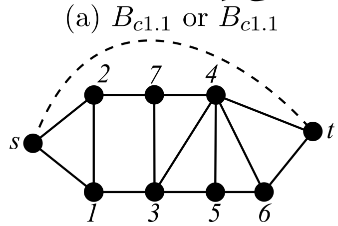
Proof. Contracting edges in either $s-t$ path (or st cycle) of G/e, such that these edges are not incident with v, yields a B-minor or $\widetilde{B}$-minor. The result follows immediately by Lemma 2.48. $\square$



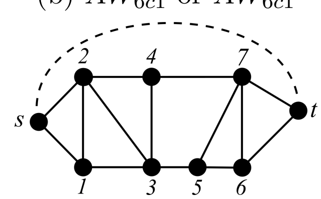
(a) $B_{c1.1}$ or $\widetilde{B_{c1.1}}$



(b) AW_{6c1} or $\widetilde{AW_{6c1}}$



(c) B_{c1} or $\widetilde{B_{c1}}$

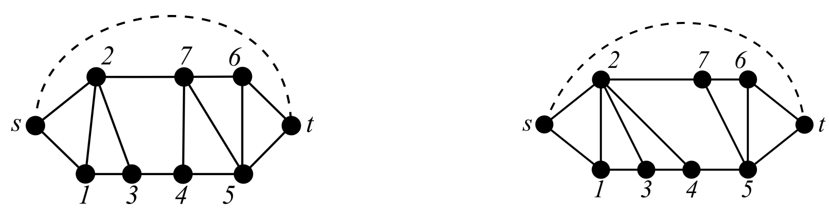


(d) B_{c3} or $\widetilde{B_{c3}}$

FIGURE 2.24: Daisy chains with $deg(v) = 4$, $deg(w_j) = 3$, and length of Q_a is five.

We briefly present a few examples of daisy chains on which Corollary 2.49 can be used. Consider $B_{c1.1}$ and $\widetilde{B_{c1.1}}$; see Figure 2.24a. Observe there is an automorphism that maps vertex 3 to vertex 4 for both daisy chains. Thus, we can simply consider the two uncontractions of $v = 4$ that meet the restrictions of 2.44.1 and use Corollary 2.49. With AW_{6c1} and its identification, we take $v = 4$. See Figure 2.24b. For B_{c1}, B_{c3}, and their identifications, see Figure 2.24. We take vertex $v = 3$ in each of these graphs.

Our next task is to sort out the subcase where v is adjacent to s or t (or st). Start with A_{c3}, A_{c4}, and their identifications. These are the daisy chains of smallest order such that $deg(v) = 4$, $deg(w_j) = 3$, and length of Q_a is at least five, along with v on the initial or terminal cycle, and an inside 4-cycle that does not share an edge with the initial or terminal cycle. For each of A_{c3}, A_{c4}, and their identifications, we take $v = 5$. See Figure 2.25. In each graph, there are two uncontractions of v that follow the restrictions of 2.44.1. As usual, we characterize them by the neighbors of $N(v)$ that they assign to $N(v_1)$, namely, $\{4, 6\}$ and $\{7, 6\}$. Performing either of these uncontractions on any of A_{c3}, A_{c4}, and their identifications results in a graph G that has either an S- or U-minor.



(a) A_{c3} or $\widetilde{A_{c3}}$ (b) A_{c4} or $\widetilde{A_{c4}}$

FIGURE 2.25: Daisy chains with $deg(v) = 4$, $deg(w_j) = 3$, and length of Q_a is five.

Lemma 2.50. *Let G/e have $deg(v) = 4$, $deg(w_j) = 3$ for all j, and length of Q_a is at least five. If G/e contains A_{c3}, A_{c4}, or one of their identifications as a proper*

minor, then an uncontraction of vertex v in G/e will properly contain an S- or U-minor.

Proof. Split G/e, A_{c3}, and A_{c4} at their distinguished vertices, and consider the nested open ear decompositions of the resulting series-parallel graphs. Either they have the same number of ears, or G/e has more ears. If they have the same number of ears, then G/e and one of A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$ are cousins. Then vertex v of G/e and vertex 5 of A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$ are constant under cousins. By Lemma 2.42, any uncontraction of v in G/e results in a graph G that has as a minor some graph that resulted from an uncontraction of vertex 5 in A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$. Moreover, since the uncontraction of v in G/e must follow the restrictions of 2.44.1, there are exactly two uncontractions, up to isomorphism, that are allowed; these correspond to the two uncontractions of vertex 5 discussed above for A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$. Thus, G properly contains S or U as a minor.

If G/e has more ears in its decomposition, then G/e may be an ear extension of A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$, with the ear-extension pair including neither v in G/e nor vertex 5 in A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$. The other possibility is that G/e is a cousin of such an ear extension of A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$. Suppose G/e is such an ear extension of A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$. Then contracting a set of edges X and deleting a set of edges Y from G/e leaves a graph isomorphic to A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$. Moreover, the sets X and Y are in the edge set of graph G that results from an uncontraction of v in G/e. Thus, $G/X\backslash Y$ is isomorphic to some uncontraction of vertex 5 in A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$. Now, since the uncontraction of v in G/e must follow the restrictions of 2.44.1, there are exactly two uncontractions, up to isomorphism, that could have produced G; these correspond to the two uncontractions of vertex 5 discussed above for A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$. Thus, G contains as a minor one of those

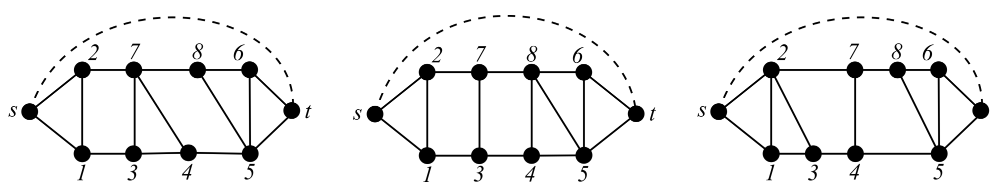
the graphs resulting from the uncontractions $\{4, 6\}$ and $\{7, 6\}$, and, therefore, G properly contains S or U as a minor.

We observe that if G/e is a cousin of such an ear extension of A_{c3}, A_{c4}, $\widetilde{A_{c3}}$, or $\widetilde{A_{c4}}$, then combining the preceding paragraph with Lemma 2.42 will again show G properly contains S or U as a minor.

□

Corollary 2.51. *Let G/e have $deg(v) = 4$, $deg(w_j) = 3$ for all j, and length of Q_a is at least five. Let v be adjacent to s, t, or st. If G/e has an inside 4-cycle that does not share an edge with either the initial or terminal cycle, then an uncontraction of vertex v in G/e will properly contain an S- or U-minor.*

Proof. Contracting edges in either $s - t$ path (or st cycle) of G/e, such that these edges are not on the specified inside 4-cycle and are not incident with v, yields as a minor A_{c3}, A_{c4}, or one of their identifications. The result now follows immediately by Lemma 2.50. □



(a) $A_{c2.2}$ or $\widetilde{A_{c2.2}}$ (b) $A_{c2.3}$ or $\widetilde{A_{c2.3}}$ (c) $A_{c3.2}$ or $\widetilde{A_{c3.2}}$

FIGURE 2.26: Daisy chains with $deg(v) = 4$, $deg(w_j) = 3$, and length of Q_a is five.

We briefly present a few examples of daisy chains on which Corollary 2.51 can be used. With the daisy chains $A_{c2.2}$, $A_{c2.3}$, $A_{c3.2}$, and their identifications, we can take $v = 5$. See Figure 2.26. By Corollary 2.51, any uncontraction of $v = 5$ in one these graphs that satisfies the restrictions of 2.44.1, results in a graph G that has an S-minor or a U-minor. For $A_{c3.2}$ and its identification, the allowable uncontractions of $v = 2$ also result in a graph G that has an S-minor or a U-minor,

by Corollary 2.51. As an aside, we note that Corollary 2.49 gives an R-minor or K_5-minor for $A_{c2.2}$ and its identification, when we uncontract $v = 7$; or for $A_{c2.3}$ and its identification, when we uncontract $v = 8$.

Recall that we are assuming G/e has $deg(v) = 4$, $deg(w_j) = 3$, and length of Q_a is at least five. There are now only two structures of G/e that remain for us to consider: G/e has no inside 4-cycles; or G/e has exactly one inside 4-cycle, which shares an edge with the initial or terminal cycle. The daisy chains of smallest order that meet all these conditions are A, A_{c2}, and their identifications, where we take $v = 5$. Under the restrictions of 2.44.1, there are two uncontractions of $v = 5$ for each of these graphs: $\{4, 6\}$ and $\{7, 6\}$.

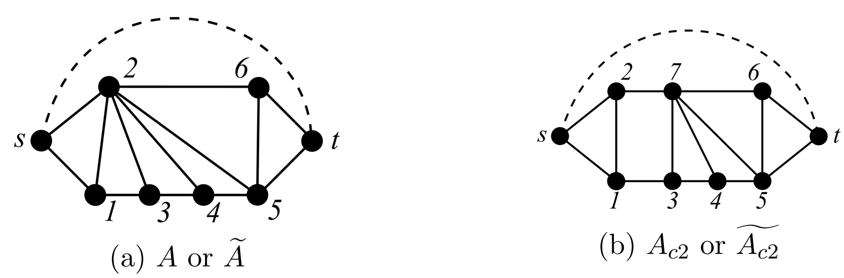


FIGURE 2.27: Daisy chains with $deg(v) = 4$, $deg(w_j) = 3$, and length of Q_a is five.

The two uncontractions $\{4, 6\}$ and $\{7, 6\}$ of A_{c2}, which we call A^1_{c2} and A^2_{c2}, respectively, both contain a proper U-minor. So we have a lemma and corollary pair similar to Lemma 2.48 and Corollary 2.49, or Lemma 2.50 and Corollary 2.51, and which is easily proved following the patterns either of those provide.

Lemma 2.52. *Let G/e have $deg(v) = 4$, $deg(w_j) = 3$ for all j, and length of Q_a is at least five. If G/e contains A_{c2} as a proper minor, then an uncontraction of vertex v in G/e will properly contain a U-minor.*

Corollary 2.53. *Let G/e have $deg(v) = 4$, $deg(w_j) = 3$ for all j, and length of Q_a is at least five. Let v be adjacent to s or t. If G/e is an unidentified daisy chain*

that has an inside 4*-cycle sharing an edge with either the initial or terminal cycle, then an uncontraction of vertex* v *in* G/e *will properly contain a* U*-minor.*

The uncontractions $\{4,6\}$ and $\{2,6\}$ of A and $\widetilde{A}$, whose resulting graphs we call A^{21}, $\widetilde{A}^{21}$, A^{22}, and $\widetilde{A}^{22}$, are all members of $\mathcal{S}$ by Theorem 2.36. We can see this since $G-\{2,t\}$ or $G-\{2,st\}$ are trees. The two uncontractions $\{4,6\}$ and $\{7,6\}$ of $\widetilde{A_{c2}}$, whose resulting graphs we call $\widetilde{A_{c2}}^1$ and $\widetilde{A_{c2}}^2$, are also members of $\mathcal{S}$ by Theorem 2.36, since $G-\{st,7\}$ is a tree.

Consider the structure of G/e, when it has no inside 4-cycles. In order to meet the conditions that $deg(v)=4$ and $deg(w_j)=3$ for all j, we must have G/e be an ear extension A or $\widetilde{A}$, such that vertex 2 is the vertex of A or $\widetilde{A}$ that is in the ear extension pair; any other structure leads to a daisy chain that has an inside 4-cycle, or w_j for some j that has degree not three. As the degree of v is four, and we must obey the restrictions of 2.44.1 on uncontractions of v, there are two allowable uncontractions, up to isomorphism, and they correspond in a natural way to the uncontractions $\{4,6\}$ and $\{2,6\}$ of A and $\widetilde{A}$. We can therefore reconstruct G, the graph resulting from such an uncontraction of v in G/e, from A^{21}, $\widetilde{A}^{21}$, A^{22}, and $\widetilde{A}^{22}$. Specifically, we can subdivide the edges $(1,3)$, $(3,4)$, or $(4,5)$ and make the new vertices adjacent to vertex 2. We will always have $G-\{v_2,2\}$ is a tree, and thus by Theorem 2.36, G is a member of $\mathcal{S}$.

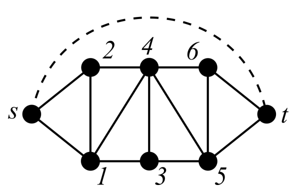
Now consider the structure of G/e, when it is an identified daisy chain that has exactly one inside 4-cycle, and this 4-cycle shares an edge with the initial or terminal cycle. In order to meet the conditions that $deg(v)=4$ and $deg(w_j)=3$ for all j, we must have G/e be an ear extension of $\widetilde{A_{c2}}$, with vertex 7 being the vertex of $\widetilde{A_{c2}}$ that is in the ear extension pair. Similar to the argument above, since the degree of v is four, and we must obey the restrictions of 2.44.1 on uncontractions of v, there are two allowable uncontractions, up to isomorphism, and they correspond

in a natural way to the uncontractions $\{4,6\}$ and $\{2,6\}$ of $\widetilde{A_{c2}}$. We can therefore reconstruct G from $\widetilde{A_{c2}}$ by subdividing $(1,3)$, $(3,4)$, or $(4,5)$ and making the new vertices adjacent to vertex 7. However, we still have $G - \{st, 7\}$ as a tree.

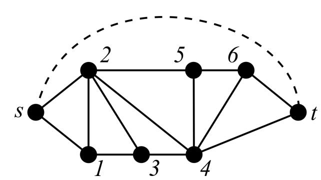
2.53.1. *$deg(v) = 5$, $deg(w_j) = 3$ for all j, and length of Q_a is at least four*

In G/e, let $deg(v) \geq 5$ and $deg(w_j) = 3$ for $1 \leq j \leq i$, where $i \geq 2$. We will break our analysis down based on whether v is adjacent to one of the distinguished vertices s, t, or st in G/e, and on whether G/e contains an inside 4-cycle or not.

If G/e does not have v adjacent to s, t, or st, then the daisy chains of minimal order (and Q_a of length exactly four) satisfying all our conditions for G/e are C and $\widetilde{C}$. Now say G does have v adjacent to s, t, or st. If G/e has no inside 3-cycles, then D and $\widetilde{D}$ are the daisy chains of minimal order (and, again, length of Q_a exactly four) that meet all the conditions for G/e. If G/e has an inside 4-cycle, then A_{c4}, B_{c1}, and their identifications are the daisy chains of minimal order and length of Q_a exactly four that meet our conditions for G/e. We begin with a case analysis for the uncontractions of v in each of these graphs, and then state results for G/e when the length of Q_a is longer than four.



(a) C or $\widetilde{C}$



(b) D or $\widetilde{D}$

FIGURE 2.28: Daisy chains with $deg(v) = 5$, $deg(w_j) = 3$, length of Q_a is four, and no inside 4-cycles.

For C and $\widetilde{C}$, the uncontractions are performed on vertex $v = 4$. Note that vertex 4 is not adjacent to the distinguished vertices s or t (or st). There are eight uncontractions, up to isomorphism, that meet the restrictions of 2.44.1. Each

of these uncontractions of v yields a graph G that properly contains one of the excluded minors K_5, R, and U. See Table 2.3.

TABLE 2.3: The eight uncontractions of $v = 4$ in C or $\widetilde{C}$.

Vertices in $N(v)$ assigned to $N(v_1)$	Label of resulting G	Resulting G
$\{1,5\}$	C^{24} or $\widetilde{C}^{24}$	has a K_5-minor
$\{2,5\}$	C^{25} or $\widetilde{C}^{25}$	has a K_5-minor
$\{2,6\}$	C^{26} or $\widetilde{C}^{26}$	has an R-minor
$\{1,3\}$	C^{27} or $\widetilde{C}^{27}$	has a U-minor
$\{1,5\}$	C^{28} or $\widetilde{C}^{28}$	has an R-minor
$\{1,6\}$	C^{29} or $\widetilde{C}^{29}$	has a K_5-minor
$\{3,5\}$	C^{30} or $\widetilde{C}^{30}$	has a U-minor
$\{3,6\}$	C^{31} or $\widetilde{C}^{31}$	has a K_5-minor

We now give a lemma and corollary pair, in the same fashion as Lemma 2.48 and Corollary 2.49, or Lemma 2.50 and Corollary 2.51. They are easily proved by adapting the proofs of these previous pairs.

Lemma 2.54. *Let G/e have deg$(v) = 5$, deg$(w_j) = 3$ for all j, and length of Q_a is at least four. If G/e contains C or $\widetilde{C}$ as a minor, then an uncontraction of v in G/e has a K_5-, R-, or U-minor.*

Corollary 2.55. *Let G/e have deg$(v) = 5$, deg$(w_j) = 3$ for all j, and length of Q_a is at least four. If v is not adjacent to a distinguished vertex s, t, or st in G/e, then an uncontraction of v in G/e has a K_5-, R-, or U-minor.*

Note that this lemma and corollary are somewhat more generalized than one might initially assume; we have not restricted G/e to having only inside 3-cycles. So long as v is not adjacent to one of the distinguished vertices s, t, or st of the

daisy chain, regardless of the presence or absence of inside 4-cycles in G/e, we are done.

(a) A_{c4} or $\widetilde{A_{c4}}$ (b) B_{c1} or $\widetilde{B_{c1}}$

FIGURE 2.29: Daisy chains with $deg(v) = 5$, $deg(w_j) = 3$, length of Q_a is four, and one inside 4-cycle.

We take $v = 2$ for A_{c4} and its identification, and $v = 4$ for B_{c1} and its identification. The eight uncontractions of $v = 2$ in A_{c4} and $\widetilde{A_{c4}}$ which respect the restrictions in 2.44.1 all produce graphs that properly contain U or S; we get the same with the eight permissible uncontractions of $v = 4$ in B_{c1} and $\widetilde{B_{c1}}$.

We give another lemma and corollary pair in the style of Lemma 2.48 and Corollary 2.49, or Lemma 2.50 and Corollary 2.51. The proofs are readily adapted from those provided for the prior results.

Lemma 2.56. *Let G/e have $deg(v) = 5$, $deg(w_j) = 3$ for all j, and length of Q_a is at least four. Let G/e have some inside 4-cycle. If G/e contains A_{c4}, $\widetilde{A_{c4}}$, B_{c1}, or $\widetilde{B_{c1}}$ as a minor, then an uncontraction of v in G/e has a U-minor or S-minor.*

Corollary 2.57. *Let G/e have $deg(v) = 5$, $deg(w_j) = 3$ for all j, and length of Q_a at least four. Let G/e have some inside 4-cycle. If v is adjacent to s, t, or st, then an uncontraction of v in G/e has a U-minor or S-minor.*

For D and $\widetilde{D}$, we take $v = 4$, and up to isomorphism, under the restrictions given in 2.44.1 there are eight uncontractions to consider for each. These are listed in Table 2.4. Performing the uncontractions results in a graph G that has as a proper minor one of S, U, and K_5, with three exceptions. The uncontractions

characterized by assigning $\{2,5\}$, $\{2,6\}$, or $\{2,t\}$ (or $\{2,st\}$) to $N(v_1)$ result in the graphs we label D^4, $\widetilde{D}^4$, D^5, $\widetilde{D}^5$, D^6, and $\widetilde{D}^6$. If G is one of these uncontractions, then $G - \{2, v_2\}$ is a tree, and by Theorem 2.36, G is a member of $\mathcal{S}$.

TABLE 2.4: The eight uncontractions of $v = 4$ in D or $\widetilde{D}$.

Vertices in $N(v)$ assigned to $N(v_1)$	Label of resulting G	Resulting G	Remarks
$\{3,5\}$	D^1 or $\widetilde{D}^1$	has a K_5-minor	
$\{3,6\}$	D^2 or $\widetilde{D}^2$	has a K_5-minor	
$\{3,t\}$ or $\{3,st\}$	D^3 or $\widetilde{D}^3$	has a U-minor	
$\{2,5\}$	D^4 or $\widetilde{D}^4$	is a member of $\mathcal{S}$	$G - \{2, v_2\}$ is a tree
$\{2,6\}$	D^5 or $\widetilde{D}^5$	is a member of $\mathcal{S}$	$G - \{2, v_2\}$ is a tree
$\{2,t\}$ or $\{2,st\}$	D^6 or $\widetilde{D}^6$	is a member of $\mathcal{S}$	$G - \{2, v_2\}$ is a tree
$\{5,6\}$	D^7 or $\widetilde{D}^7$	has a U-minor	
$\{5,t\}$ or $\{5,st\}$	D^8 or $\widetilde{D}^8$	has an S-minor	

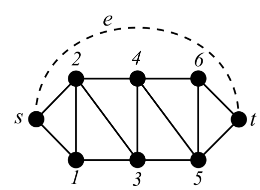
Now focus on the structure of G/e, when the length of Q_a is greater than four; we keep G/e free of inside 4-cycles, and v is still adjacent to s, t, or st. Under these restrictions, however, we can only have G/e be an ear extension of D or $\widetilde{D}$, where the vertex of D or $\widetilde{D}$ in the ear extension pair is vertex 2. Thus, by the definition of ear extensions, we can ear-recover D or $\widetilde{D}$ from G/e by contracting edges X and deleting edges Y. Moreover, graph G, which results from an uncontraction of v in G/e, retains the edges in X and Y. Thus, $G/X\backslash Y$ is isomorphic to an uncontraction of vertex 4 in D or $\widetilde{D}$; not only that, but since the uncontraction of v in G/e had to also obey the restrictions of 2.44.1, there were only eight allowable uncontractions of v, all of them corresponding naturally to the allowable uncontractions of vertex 4 in D or $\widetilde{D}$.

Thus, an uncontraction of v from G/e contains as a minor an uncontraction of D or $\widetilde{D}$. If this is one of the uncontractions listed in Table 2.4 that has a K_5-, U-,

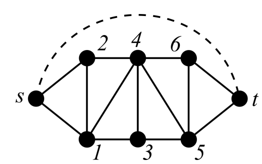
or S-minor, then we are done. So we assume that the only uncontractions of D or $\widetilde{D}$ that are minors of G are D^4, D^5, D^6, $\widetilde{D}^4$, $\widetilde{D}^5$, or $\widetilde{D}^6$. Then $G - \{v_2, 2\}$ is a tree, and G is in $\mathcal{S}$ by Theorem 2.36.

2.57.1. *$deg(v) \geq 4$, $deg(w_1) \geq 4$, $deg(w_j) = 3$ when $j \neq 1$, and length of Q_a is at least four*

Note that the subscript of vertex w_1 or w_j makes no indication of the position of the vertex on path Q_a. We break up our argument based on whether v and w_1 are adjacent or not; whether v and w_1 are adjacent to s, t, or st; and, when needed, whether G/e contains an inside 4-cycle or not. We perform case analyses onuncontractions of v in the daisy chains of smallest order that meet all the conditions we impose on G/e, and then make statements about uncontractions in G/e when the order is higher.



(a) AW_6 or $\widetilde{AW_6}$



(b) C or $\widetilde{C}$

FIGURE 2.30: Daisy chains with $deg(v) = 4$, $deg(w_j) = 4$, length of Q_a is four, and no inside 4-cycles.

Let v and w_1 be adjacent. Then daisy chains having smallest order that satisfy all our conditions for G/e are AW_6 and $\widetilde{AW_6}$, where we have $deg(v) = deg(w_1) = 4$. We need to check uncontractions of $v = 2$ as well as $v = 3$, since 2 is adjacent to s (or st), but 3 is not adjacent to any distinguished vertex s, t, or st. Observe that, due to the symmetry of these daisy chains, we could just as well take $v = 4$ and $v = 5$. There are two uncontractions, up to isomorphism, for each of vertices 2 and 3 that satisfy the restrictions in 2.44.1. With vertex 2, these are $\{s, 3\}$ (or $\{st, 3\}$) and $\{1, 3\}$; with vertex 3, these are $\{2, 5\}$ and $\{2, 4\}$. For both AW_6 and

$\widetilde{AW_6}$, every one of these uncontractions results in a graph G that has a U-, S-, or R-minor.

Lemma 2.58. *Let G/e have $deg(v), deg(w_1) \geq 4$, $deg(w_j) = 3$ for all $j \neq 1$, and length of Q_a is at least four. If G/e contains AW_6 or $\widetilde{AW_6}$ as a proper minor, then an uncontraction of v in G/e has a U-minor, S-minor, or R-minor.*

Proof. Compare nested open ear decompositions from the series-parallel graphs that result from splits of G/e and AW_6. Either they have the same number of ears, or the split of G/e has more ears.

If the number of ears are the same, then G/e and one of AW_6 or $\widetilde{AW_6}$ are cousins, depending on whether G/e is an unidentified or identified daisy chain, and $deg(v) = deg(w_1) = 4$. Vertex v and vertex 2 of AW_6 or $\widetilde{AW_6}$ are constant under cousins if v is adjacent to s, t, or st; or v and vertex 3 of AW_6 or $\widetilde{AW_6}$ are constant under cousins if v is not adjacent to a distinguished vertex. Whichever pair of vertices is constant under cousins, by Lemma 2.42, an uncontraction of v in G/e produces a graph G that contains as a minor an uncontraction of 2 (or 3) in AW_6 or $\widetilde{AW_6}$. Now since our uncontraction of v is subject to the restrictions of 2.44.1, there are exactly two uncontractions allowable, up to isomorphism. They correspond to the two uncontractions of vertex 2 (or 3) discussed for AW_6 and $\widetilde{AW_6}$ above. Hence, G has a a U-minor, S-minor, or R-minor.

On the other hand, the nested open ear decomposition associated with G/e can have more ears. Let G be the graph that results from an uncontraction of v. Then G/e is the result of one or more ear extensions of AW_6 or $\widetilde{AW_6}$, or is a cousin of such a daisy chain.

To start, fix $deg(v) = 4$, while we continue to assume $deg(w_1) \geq 4$. This way, we can assume that v and 2 (or v and 3, if w_1 is adjacent to one of the distinguished

vertices of G/e) is not one of the ear extension pairs. We can therefore ear-recover AW_6 or $\widetilde{AW_6}$ from G/e via contraction of a set of edges X and simplification of a set of edges Y, where the sets X and Y are edges of graph G as well. So we have $G/X\backslash Y$ isomorphic to an uncontraction of vertex 2 (or vertex 3) of AW_6 or $\widetilde{AW_6}$. Since G is the result of an uncontraction of v that had to obey the restrictions of 2.44.1, there are only two uncontractions allowed, up to isomorphism, call them n_1 and n_2. We can readily see that these correspond to the two uncontractions of vertex 2 (or of vertex 3) in AW_6 or $\widetilde{AW_6}$. Therefore, G has a U-minor, S-minor, or R-minor.

Now let $deg(v) > 4$. Let H be a daisy chain satisfying the conditions that we placed on G/e in the previous paragraph – that is, when we fixed the degree of v at four. We take v' to be this degree-4 vertex in H. Then G/e is an ear extension of H, with the ear extension pair being v in G/e and v' in H. By Lemma 2.44, graph G has as a minor an uncontraction of v' in H. We want to show that the uncontraction is either n_1 or n_2. Recall that our uncontraction of v in G/e must obey the restrictions of 2.44.1. Let the two new vertices produced by uncontracting v be v_1 and v_2.

Suppose v is adjacent to a distinguished vertex of G/e, without loss of generality s or st. Then 2.44.1 prevents us from having an uncontraction of v such that $N(v_1) - v_2$ form the set of vertices of a subpath of Q_b, with s or st in $N(v_1) - v_2$. So either $N(v_1) - v_2$ is the set of vertices of a subpath of Q_b that does not contain a distinguished vertex, or any subpath of Q_b whose set of vertices contains $N(v_1) - v_2$ also has some vertex from $N(v_2) - v_1$. Thus, G has as a minor a graph which resulted from either the n_1 or n_2 uncontraction of v' in H.

So we are forced to have v not adjacent to a distinguished vertex of G/e. The restrictions from 2.44.1 preclude an uncontraction of v such that exactly one vertex

of $N(v_1) - v_2$ is on Q_a, and there is some subsequence of the ears in the nested open ear decomposition we associate with G/e that contains all the vertices of $N(v_1) - v_2$ but none of $N(v_2) - v_1$. So we have two options. All of the vertices of $N(v_1) - v_2$ are on Q_b; or exactly one vertex of $N(v_1) - v_2$ is on Q_a, but any subsequence of ears whose vertices contain $N(v_1) - v_2$ also has at least one vertex from $N(v_2) - v_1$. Thus, G has as a minor a graph which resulted from either the n_1 or n_2 uncontraction of v' in H. We conclude G has a U-minor, S-minor, or R-minor. □

Corollary 2.59. *Let G/e have $deg(v), deg(w_1) \geq 4$, $deg(w_j) = 3$ for all $j \neq 1$, and length of Q_a is at least four. If v is adjacent to w_1, or if one of v and w_1 is not adjacent to s, t, or st, then an uncontraction of v in G/e has a U-minor, S-minor, or R-minor.*

Proof. Notice that AW_6 or $\widetilde{AW_6}$ is a minor of G/e. We produce this AW_6- or $\widetilde{AW_6}$-minor in the following way. Contract all but four edges in Q_a so that v and w_1 are adjacent, and one of v and w_1 is adjacent to a distinguished vertex of G/e. Collect these edges in set X_1. Simplify $G/e/X_1$; collect these edges in X_2. Let Q_b be the $s - t$ path (or st cycle) that does not contain v. If $G/e/X_1 \backslash X_2$ has any inside 4-cycles, contract their edges in Q_b, producing a daisy chain with only inside 3-cycles. Collect these edges in X_3. Now, if $deg(v) = deg(w_1) = 4$, then $G/e/X_1\backslash X_2/X_3$ is isomorphic to AW_6 or $\widetilde{AW_6}$. If either of v or w_1 has degree above four, however, $G/e/X_1\backslash X_2/X_3$ is an ear extension of AW_6 or $\widetilde{AW_6}$. The ear extension pairs are v and 2, and w_1 and 3; or v and 3, and w_1 and 5. Thus, we can ear-recover AW_6 or $\widetilde{AW_6}$ from $G/e/X_1\backslash X_2/X_3$ in the usual way. Therefore, AW_6 or $\widetilde{AW_6}$ is a minor of G/e. The result follows by Lemma 2.58. □

Let v and w_1 be nonadjacent, but each of v and w_1 is adjacent to s, t, or st, and let G/e have some inside 4-cycle. The daisy chains of smallest order that satisfy all these conditions for G/e are B_{c3} and its identification.

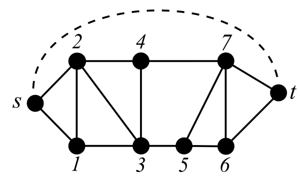


FIGURE 2.31: B_{c3} or $\widetilde{B_{c3}}$

For B_{c3} and its identification, we can take vertex $v = 2$ or $v = 7$; that is, we may have v on an inside 4-cycle, or not. Recall that all uncontractions in this section follow the restrictions of 2.44.1. Then there are two uncontractions up to isomorphism that are allowed on $v = 2$, those that assign $\{1, 4\}$ or $\{1, 3\}$ to $N(v_1)$; likewise, when $v = 7$, we again have two uncontractions, $\{4, 6\}$ and $\{5, 6\}$. Every one of these uncontractions on B_{c3} or $\widetilde{B_{c3}}$ results in a graph G that has a proper U-minor or a proper S-minor.

The subsequent lemma and its corollary are patterned directly after Lemma 2.58 and Corollary 2.59. Their proofs may be used as a template for these next results, needing only the slightest of alterations.

Lemma 2.60. *Let G/e have $deg(v), deg(w_1) \geq 4$, $deg(w_j) = 3$ for all $j \neq 1$, and length of Q_a is at least four. If G/e contains B_{c3} or $\widetilde{B_{c3}}$ as a minor, then an uncontraction of v in G/e has a U-minor or S-minor.*

Corollary 2.61. *Let G/e have $deg(v), deg(w_1) \geq 4$, $deg(w_j) = 3$ for all $j \neq 1$, and length of Q_a is at least four. Suppose v is not adjacent to w_1, and that each of v and w_1 is adjacent to s, t, or st. If G/e has an inside 4-cycle, then an uncontraction of v in G/e has a U-minor or S-minor.*

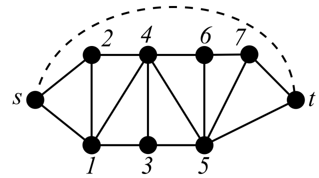


FIGURE 2.32: C_1 or $\widetilde{C_1}$

Next, suppose G/e has no inside 4-cycles, and that one of v and w_1 has degree at least five; we still let v and w_1 be nonadjacent, with each of v and w_1 adjacent to s, t, or st. The daisy chains of smallest order that satisfy all these conditions on G/e are C_1 and $\widetilde{C_1}$. We take $v = 1$ or $v = 5$, depending on which of v and w_1 we want to be degree-5. Since the uncontractions we can perform are restricted by 2.44.1, there are two allowable uncontractions of vertex 1, and eight of vertex 5. Whichever of these uncontractions we perform in C_1 or $\widetilde{C_1}$, the result is a graph G that has an S- or U-minor.

We present a lemma and corollaries on the generalized situation, with one of v and w_1 having degree at least five and G/e having no inside 4-cycles. The proofs do not differ significantly from those given for Lemma 2.58 and Corollary 2.59, and are therefore omitted.

Lemma 2.62. *Let G/e have deg$(v) \geq 4$ and deg$(w_1) \geq 5$, or deg$(w_1) \geq 4$ and deg$(v) \geq 5$; deg$(w_j) = 3$ for all $j \neq 1$; and length of Q_a is at least four. If G/e contains C_1 or $\widetilde{C_1}$ as a minor, then an uncontraction of v in G/e has a U-minor or S-minor.*

Corollary 2.63. *Let G/e have deg$(v) \geq 4$ and deg$(w_1) \geq 5$, or deg$(w_1) \geq 4$ and deg$(v) \geq 5$; deg$(w_j) = 3$ for all $j \neq 1$; and length of Q_a is at least four. Suppose v is not adjacent to w_1, and that each of v and w_1 is adjacent to s, t, or st. If G/e has no inside 4-cycle, then an uncontraction of v in G/e has a U-minor or S-minor.*

Our final subcase is where G/e has no inside 4-cycles, $deg(v) = deg(w_1) = 4$, v and w_1 are not adjacent, and each of v and w_1 is adjacent to s, t, or st. Now, the daisy chains of smallest order that satisfy all these conditions on G/e are C and $\widetilde{C}$, with $v = 5$. (Due to symmetry, we could also take $v = 1$.) There are two uncontractions of $v = 5$ that meet all the requirements in 2.44.1, those that assign $\{3, 6\}$ or $\{3, t\}$ to $N(v_1)$. We call the graphs resulting from the $\{3, 6\}$ uncontractions C^{32}, $\widetilde{C}^{32}$; and from the $\{3, t\}$ or $\{3, st\}$ uncontractions C^{33} and $\widetilde{C}^{33}$. While C^{32} and C^{33} have U-minors, $\widetilde{C}^{32}$ and $\widetilde{C}^{33}$ are members of $\mathcal{S}$ by Theorem 2.36; for both, $G - \{st, 4\}$ is a tree.

When G/e has higher order, however, it is forced to be an ear extension of C or $\widetilde{C}$, with vertex 4 of C or $\widetilde{C}$ in the ear extension pair. So we can ear-recover C or $\widetilde{C}$ from G/e by contracting a set of edges X and simplifying a set of edges Y. Let G be the graph that results from an uncontraction of v in G/e. Since G retains the edges in X and Y, we have $G/X\backslash Y$ is isomorphic to an uncontraction of vertex 5 in C or $\widetilde{C}$. In addition, the restrictions of 2.44.1 left two possible uncontractions, up to isomorphism, of v in G/e, which correspond to the two allowable uncontractions of vertex 5 in C or $\widetilde{C}$.

Thus, an uncontraction of v in G/e has as a minor one of the uncontractions of C or $\widetilde{C}$ that we just discussed. If G has as a minor either of the uncontractions C^{32} or C^{33}, then we are done, since by transitivity G has a U-minor. Thus G only has either $\widetilde{C}^{32}$ and $\widetilde{C}^{33}$. This, however, leaves us with $G - \{4, st\}$ a tree, and therefore G is in $\mathcal{S}$ by Theorem 2.36.

2.63.1. *$deg(v), deg(w_1), deg(w_2) \geq 4$ and $deg(w_j) \geq 3$ for any $j \geq 3$, and length of Q_a is at least four*

Now we assume there are at least three vertices on Q_a whose degree is four or greater, namely, v, w_1, and w_2. Any other vertex w_j for $j \geq 3$ on Q_a has degree at least three. Note that the subscripts on vertices w_1, w_2, and w_j make no indication of the position of the vertex on path Q_a.

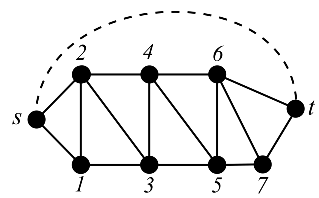


FIGURE 2.33: AW_7 or $\widetilde{AW_7}$

The daisy chains of smallest order that satisfy all our conditions for G/e are AW_7 and $\widetilde{AW_7}$. Take $deg(v) = deg(w_1) = deg(w_2) = 4$. With AW_7 and $\widetilde{AW_7}$, we need to consider both $v = 2$ and $v = 4$, since v may or may not be adjacent to a distinguished vertex s, t, or st. There are two uncontractions of $v = 2$, those that assign $\{1, 3\}$ or $\{1, 4\}$ to $N(v_1)$; and there are two uncontractions of $v = 4$ to consider, namely, those that assign to $N(v_1)$ the sets $\{2, 5\}$ and $\{3, 5\}$. Every one of these uncontractions on AW_7 and $\widetilde{AW_7}$ produces a graph G that has a U-, S-, or R-minor.

We give the following results, for larger G/e that satisfy this subsection's conditions. Note that these are consequences of Lemma 2.58 and Corollary 2.58.

Lemma 2.64. *Let G/e have $deg(v), deg(w_1), deg(w_2) \geq 4$ and $deg(w_j) \geq 3$ for any $j \geq 3$; and length of Q_a is at least four. If G/e contains AW_7 or $\widetilde{AW_7}$ as a minor, then an uncontraction of v in G/e has a U-minor, S-minor, or R-minor.*

Corollary 2.65. *Let G/e have $deg(v), deg(w_1), deg(w_2) \geq 4$ and $deg(w_j) \geq 3$ for any $j \geq 3$; and length of Q_a is at least four. An uncontraction of v in G/e has a U-minor, S-minor, or R-minor.*

2.66. $v = st$

Now let $v = st$. Note that because $deg(st) = 4$, there are only two uncontractions of st, up to isomorphism, that do not automatically return a graph isomorphic to the original daisy chain. Let s' and t' be the new vertices produced by uncontracting st. We will characterize uncontractions of st by the set of neighbors the uncontraction assigns to vertex s'. We consider subcases based on the length of one st cycle, call it C_a, whose inside vertices are w and w_j with $j \geq 1$. Note that the minimum length of C_a is three, since we cannot identify the distinguished vertices s and t of a wheel and have a daisy chain. We further divide up the argument based on the degrees of the vertices of C_a.

2.66.1. *The length of st cycle C_a is three*

Suppose the length of C_a is three, and that the vertices on the cycle are st, w, and w_1.

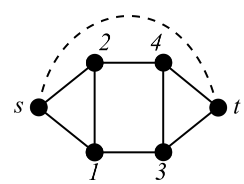
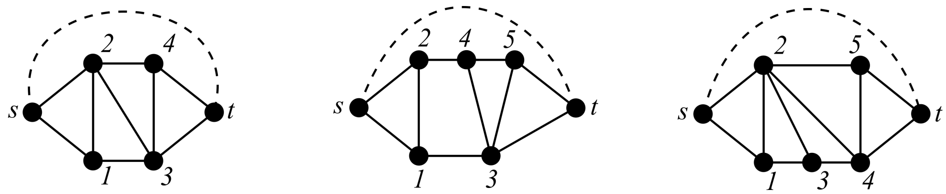


FIGURE 2.34: $\widetilde{RW}_{3c1}$

Let $deg(w) = deg(w_1) = 3$. The only daisy chain that meets this criterion is $\widetilde{RW}_{3c_1}$. Due to this graph's particular structure, the uncontraction that assigns $\{2, 4\}$ to $N(s')$ results in a graph G that is isomorphic to $\widetilde{RW}_{3c_1}$. The other uncontraction, which assigns $\{2, 3\}$ to $N(s')$, produces a graph G that is a member of $\mathcal{S}$ by Theorem 2.36, since $G - \{3, t'\}$ is a tree.

Let $deg(w) \geq 4$ and $deg(w_1) = 3$. Small examples of daisy chains that satisfy this condition are $\widetilde{AW}_4$, $\widetilde{AW}_{4c1}$, $\widetilde{K}$, $\widetilde{K}_{c1}$, $\widetilde{A}$, and $\widetilde{A}_{c1}$. In general, both of the un-



(a) $\widetilde{AW_4}$ (b) $\widetilde{AW_{4c1}}$ (c) $\widetilde{K}$

contractions of $v = st$ from G/e produce a graph G that is a member of $\mathcal{S}$ by Theorem 2.36, since $G - \{w, t'\}$ or $G - \{w, s'\}$ is a tree.

There are precisely two daisy chains where $deg(w) = deg(w_1) = 4$, namely, $\widetilde{AW_5}$ and $\widetilde{AW_{5c2}}$. For the uncontraction of $v = st$ in $\widetilde{AW_5}$ that assigns $\{2, 4\}$ to $N(s')$, the graph G is isomorphic to R, as we wished to show. For the other uncontraction in $\widetilde{AW_5}$, namely, where we assign $\{2, 5\}$ to $N(s')$, we find that G contains a K_5-minor. Observe that either uncontraction of $\widetilde{AW_{5c2}}$ contains one of the uncontractions of $\widetilde{AW_5}$, and, therefore, properly contains a known minor of $\mathcal{M}_1$.

(a) AW_5 (b) AW_{5c2}

Let $deg(w), deg(w_1) \geq 4$. Then G/e contains $\widetilde{AW_5}$ as a minor. Thus, an uncontraction of $v = st$ in G/e properly contains R or K_5 as a minor.

2.66.2. *The length of st cycle C_a is at least four*

Now we suppose the length of cycle C_a is at least four. The inside vertices of C_a are w and w_j, where $1 \leq j \leq i$ and $i \geq 2$.

Let $deg(w) = deg(w_j) = 3$ for $1 \leq j \leq i$ and $i \geq 2$. Consider the other st-cycle, C_b. Its length must be at least three, since G/e cannot be a wheel. If its length is exactly three, then this is the subcase when C_a is length three and $deg(w) \geq 4$ and $deg(w_1) = 3$, which we considered above, and G is a member of $\mathcal{S}$. If the length of

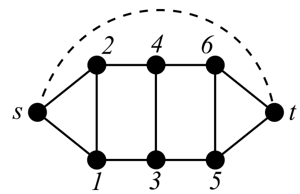
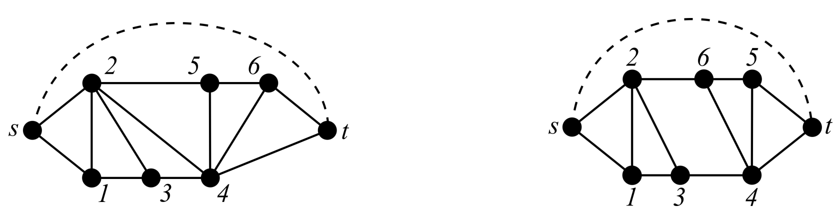


FIGURE 2.37: $\widetilde{AW}_{4c1.1}$

C_b is at least four, then the structure of a daisy chain forces G/e to contain $\widetilde{AW}_{4c1.1}$ as a minor. The uncontraction of $v = st$ in $\widetilde{AW}_{4c1.1}$ that assigns $\{2, 6\}$ to $N(s')$ is isomorphic to Q_3, while the other uncontraction $\{2, 5\}$ is isomorphic to H_8. So the uncontractions of $v = st$ in $\widetilde{AW}_{4c1.1}$ do result in a graph that is an excluded minor of $\mathcal{M}_1$, as we wanted to show. Thus, any uncontraction of $v = st$ in a G/e that has $\widetilde{AW}_{4c1.1}$ as a proper minor will result in a graph G that properly contains a known excluded minor of $\mathcal{M}_1$.

Let $deg(w) \geq 4$, but $deg(w_j) = 3$ for every j. Suppose w is adjacent to both w_1 and w_2 (that is, w is not adjacent to st). Since G/e cannot be a wheel, the other st cycle C_b must be of length at least 3; if the length is exactly 3, and $deg(w_1) = deg(w_2) = 3$, then G/e is isomorphic to the daisy chain $\widetilde{AW}_5$. See above for discussion of the uncontractions of st. We conclude that when $deg(w) \geq 4$, but $deg(w_j) = 3$ and w is not adjacent to st, then any uncontraction of $v = st$ of G/e properly contains an excluded minor of $\mathcal{M}_1$.



(a) D

(b) K_{c3}

We continue to let $deg(w) \geq 4$ and $deg(w_j) = 3$ for all j, but now suppose w is adjacent to w_1 and st on their st cycle C_a. Consider the inside 4-cycles of our daisy chain G/e. Should there be no inside 4-cycles, then there are only two vertices

aside from st that have degree greater than three: vertex w, and some vertex x from the other st-cycle C_b. A small example is $\widetilde{D}$, with $w = 2$ and $x = 4$. Thus, after either st uncontraction, $G - \{w, x\}$ will be a tree, and therefore a member of $\mathcal{S}$. Next, suppose there is one inside 4-cycle; either this inside 4-cycle contains exactly one w_j vertex or exactly two. There are still just two vertices aside from st that have degree greater than three: vertex w, and some vertex x from C_b. If G/e only contains exactly one w_j vertex, then $G - \{w, x\}$ is a tree, after either uncontraction of st. A small example would be $\widetilde{K_{c3}}$, where we take $w = 2$, $w_1 = 6$, and $x = 4$. However, if there are two w_j vertices on this inside 4-cycle, then G has a $\widetilde{AW_{4c1.1}}$-minor, whose st uncontractions were discussed above, and G therefore properly contains a known excluded minor of $\mathcal{M}_1$. Of course, once G/e has two inside 4-cycles, there is a clear $\widetilde{AW_{4c1.1}}$ minor.

Let $deg(w) \geq 4$ and $deg(w_j) \geq 4$ for at least one j. Then G/e has $\widetilde{AW_5}$ as a minor. The uncontractions of $\widetilde{AW_5}$ have already been discussed. Therefore any uncontraction of $v = st$ in G/e will properly contain a known excluded minor of $\mathcal{M}_1$.

2.7 Proof of Case 2

This section covers Case 2 of the $\kappa(G) = 3$ portion of the proof of Theorem 2.4, where $G\backslash e$ is 3-connected and a member of $\mathcal{S}^*$. Thus, we can recover the excluded minor G by undeleting an edge in a daisy chain. We classify three types of undeleted edges, based on their endpoints. Each type of undeletion is considered as a separate subcase.

We require any undeletion to produce a simple graph G whose connectivity is three. An e_1-*undeletion* adds an edge $e_1 = (u, v)$ to $G\backslash e$, where inside vertices u and v are on distinct $s - t$ paths (or st-cycles), but they are not both members of the same inside cycle. An e_2-*undeletion* adds an edge $e_2 = (u, v)$ to $G\backslash e$ such that inside

vertices u and v are on the same $s-t$ path (or st-cycle) but are not neighbors. An e_3-*undeletion* adds an edge $e_3 = (u, v)$ between $u \in \{s, t, st\}$ and an inside vertex v, where v is not adjacent to any of $\{s, t, st\}$. This last restriction avoids parallel edges and, if s and t are distinct, the possibility of G being isomorphic to an identified daisy chain.

By Corollary 2.37, the class of wheels is contained in both $\mathcal{S}$ and $\mathcal{S}^*$. Their unique structure makes it preferable to first consider undeletions of wheels. Thereafter, we will consider the e_1-, e_2-, and e_3-undeletions in turn on all non-wheel daisy chains. Note that the four daisy chains W_3, W_4, RW_{3c1}, and $\widetilde{RW_{3c1}}$ are too small for any kind of undeletion.

2.67. $G\backslash e$ *is a wheel*

Wheels W_3 and W_4 are too small to have any allowable undeletions, but once $i \geq 5$, we are able to perform undeletions of W_i. Due to the structure of a wheel, any undeletion will be either an e_2-undeletion or an e_3-undeletion, and therefore the undeleted edge will have rim vertices for both endpoints. Let one of these endpoints be r. Then, the graph G that results from the undeletion is a member of $\mathcal{S}$ by Theorem 2.36, since $G - \{2, r\}$ is a tree, which is a contradiction.

2.68. e_1-*undeletions*

Lemma 2.69. *Let H be a daisy chain that is not a wheel. Any e_1-undeletion of H results in a graph G that has a K_5-minor.*

Proof. Assume H is an unidentified daisy chain. For the distinguished vertices s and t of H, let their neighbor sets be $N(s) = \{1, 2, t\}$ and $N(t) = \{n-1, n, s\}$, with vertices 2 and n on $s-t$ path Q_0, while 1 and $n-1$ are on Q_1. We consider undeletion $e_1 = (u, v)$, where, without loss of generality, u is on Q_0 and v is therefore on Q_1. So in a nested open ear decomposition of $H\backslash(s, t)$ that has k ears,

vertices u and v are endpoints of distinct ears P_i and P_j, respectively. We observe that $2 \leq i, j \leq k$ and, without loss of generality, we assume that $i < j$. Label vertices u' and v', such that u' is the endpoint of P_i on Q_1, and v' is the endpoint of P_j on Q_0. See Figure 2.39 for an example. Contract the subpaths $2 - u$ and $v' - n$ in Q_0, and the subpaths $1 - u'$ and $v - (n-1)$ in Q_1 down to single vertices. If subpath $u - v'$ in Q_0 or subpath $u' - v$ in Q_1 is nontrivial, contract so that the internal vertices form a conglomerate vertex with v' or u', respectively. Contract edge (s, t). This results in a K_5-minor.

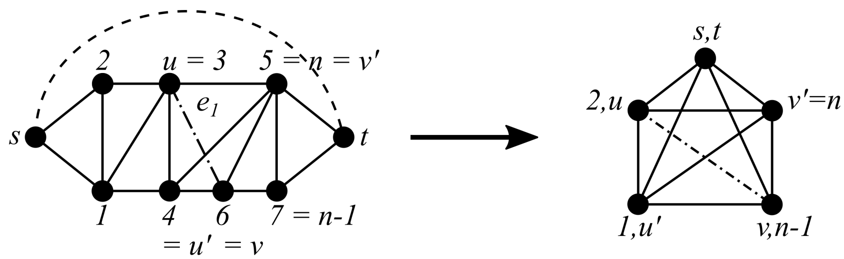


FIGURE 2.39: An example of Lemma 2.69, showing how an e_1-undeletion of daisy chain B_2 or $\widetilde{B_2}$ has a K_5-minor.

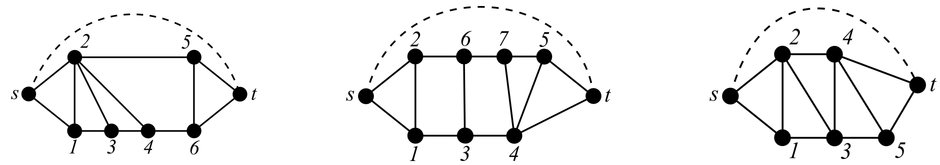
Note that, since we contract edge (s, t) to produce the K_5-minor from H, an e_1-undeletion of an identified daisy chain will also have a K_5-minor. □

2.70. *e_2-undeletions of selected daisy chains*

We briefly consider a few e_2-undeletions of particular daisy chains. For each, we identify specific known excluded minors of $\mathcal{M}_1$. These identifications will enable us to eliminate particular e_2-undeletions of arbitrary daisy chains from further consideration.

The daisy chain $\widetilde{K_{c1}}$ is shown in Figure 2.40a. Consider $\widetilde{K_{c1}} + e_{2i}$, where $e_{2i} = (1, 4)$. The graph $\widetilde{K_{c1}} + e_{2i}$ is isomorphic to U.

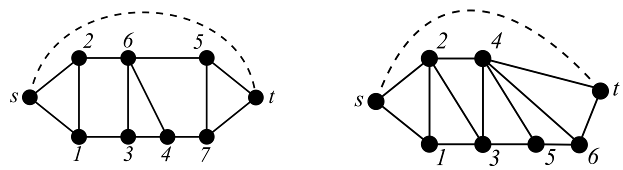
The daisy chain $\widetilde{K_{c2.1}}$ is shown in Figure 2.40b. Consider $\widetilde{K_{c2.1}} + e_{2iii}$, where $e_{2iii} = (6, 5)$. Graph $\widetilde{K_{c2.1}} + e_{2iii}$ has a U-minor.



(a) $\widetilde{K_{c1}}$ (b) $\widetilde{K_{c2.1}}$ (c) $\widetilde{AW_5}$

FIGURE 2.40: Selected daisy chains.

The daisy chain $\widetilde{AW_5}$ is shown in Figure 2.40c. Consider the e_2-undeletion $\widetilde{AW_5}+e_{2i}$, where $e_{2i} = (1,5)$. This graph has a $K_{2,2,2}$-minor.

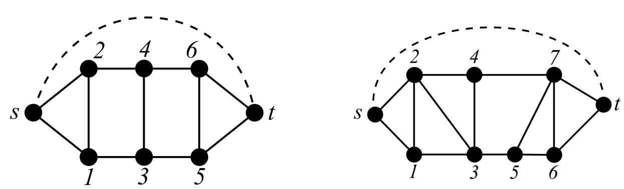


(a) $\widetilde{K_{c2.2}}$ (b) $\widetilde{B}$

FIGURE 2.41: Selected daisy chains.

The daisy chain $\widetilde{K_{c2.2}}$ is shown in Figure 2.41a. Consider $\widetilde{K_{c2.2}} + e_{2ii}$, where $e_{2ii} = (1,4)$. The graph resulting from this undeletion has an S_1-minor.

The daisy chain $\widetilde{B}$ is shown in Figure 2.41b. We want to consider the undeletion $\widetilde{B} + e_{2iii}$, where we take $e_{2iii} = (3,6)$. The graph resulting from this undeletion contains an S_1-minor.



(a) $\widetilde{AW_{4c1.1}}$ (b) $\widetilde{B_{c3}}$

FIGURE 2.42: Selected daisy chains.

The daisy chain $\widetilde{AW_{4c1.1}}$ is shown in Figure 2.42a. Consider the undeletion $\widetilde{AW_{4c1.1}} + e_{2i}$, where $e_{2i} = (2,6)$. The graph resulting from this undeletion contains an R-minor.

The daisy chain $\widetilde{B_{c3}}$ is shown in Figure 2.42b. Consider the e_2-undeletion $\widetilde{B_{c3}}+e_2$, where $e_2 = (3, 6)$. We find that $\widetilde{B_{c3}} + e_2$ has a U-minor.

2.71. *e_2-undeletions*

We now consider performing an e_2-undeletion on daisy chain $G\backslash e$, where $G\backslash e$ is not a wheel. Let the undeleted edge be $e_2 = (x, y)$, and assume x and y are on $s - t$ path Q_i (or *st* cycle Q_i, if $G\backslash e$ is an identified daisy chain). Let L be the subpath of Q_i that has endpoints x and y; the length of L must be at least two. See Figure 2.43 for an illustration. Let Z be the set of inside cycles such that, for each cycle z in Z, all of the vertices of z that are on Q_i are also vertices of L. We say e_2 *subtends* path L and the cycles of Z. So Q_j is the $s - t$ path (or the *st* cycle) that e_2 does not meet. We will break down our case analysis of e_2-undeletions based on whether there are one, two, or at least three vertices of $G\backslash e$ that are in both $V(Z)$ and $V(Q_j)$.

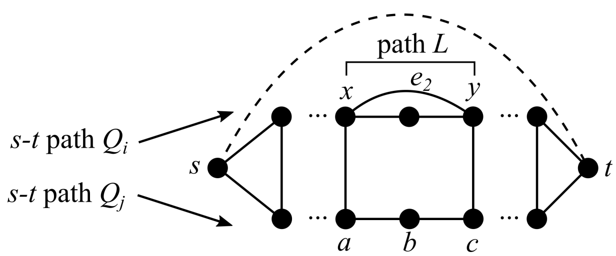


FIGURE 2.43: Example of an e_2-undeletion. Edges between vertices of L and Q_j are omitted.

2.71.1. $V(Z) \cap V(Q_j) = \{a\}$

Suppose all the cycles of Z have only one point in common with Q_j, call it a. All the cycles e_2 subtends, then, must be inside 3-cycles that meet vertex a. If there are no other internal cycles in $G\backslash e$ besides those in the set Z, then $G\backslash e$ is a wheel. That case was covered in 2.67 above.

We may now assume there is at least one inside cycle in $G\backslash e$ that is not a member of Z. If one of the non-Z inside cycles is a 4-cycle, the undeletion of $G\backslash e$ results in a graph that is either isomorphic to $\widetilde{K_{c1}} + e_{2i}$ or contains it as a proper minor; thus, G has a U-minor. This U-minor is improper if $G\backslash e$ is isomorphic to $\widetilde{K_{c1}}$.

If all the non-Z inside cycles are 3-cycles, choose one that contains edge (x, a) or (y, a), say without loss of generality (y, a). Let this inside cycle be associated with ear P_h. Its third vertex besides y and a, call it v, must belong to $s-t$ path Q_i. (If v is on Q_j, then Z shares two vertices with Q_j.) In order to keep $G\backslash e$ from being a wheel, there must be another non-Z inside 3-cycle, associated with ear P_{h+1}. Then deleting their shared edge turns these two 3-cycles into one 4-cycle, so the undeletion of $G\backslash e$ contains $\widetilde{K_{c1}} + e_{2i}$ as a minor, and, therefore, G has a proper U-minor.

2.71.2. $V(Z) \cap V(Q_j) = \{a, b\}$

Suppose the cycles of Z have exactly two vertices in common with Q_j, call them a and b. We note Z cannot have more than one inside 4-cycle, or, by Lemma 2.20, it would share three or more vertices with Q_j. Recall that the length of L is at least two, to avoid e being a parallel edge. Any inside cycles of $G\backslash e$ that contain an internal vertex of L are members of Z. Thus, in order for these internal vertices of L to have degree three or greater, they must be adjacent to at least one of a and b. We must therefore consider each of the following:

(i) each of a and b is adjacent to one or more internal vertices of L;

(ii) only one of a and b is adjacent to internal vertices of L; and Z contains all the inside cycles of $G\backslash e$; or

(iii) only one of a and b is adjacent to internal vertices of L; and Z does not contain all the inside cycles of of $G\backslash e$.

Suppose we have subcase (i), where both a and b are adjacent to internal vertices of L. If Z has no inside 4-cycle, then one internal vertex of L is adjacent to both a and b. If Z contains an inside 4-cycle, contracting the edge in Q_i that is on that 4-cycle leaves an internal vertex of L that is adjacent to both a and b. With this vertex that is adjacent to both a and b, it is easy to see that the e_2-undeletion of $G\backslash e$ is isomorphic to or contains as a minor $\widetilde{AW_5} + e_{2i}$, meaning G has $K_{2,2,2}$ as a minor.

Suppose we have subcase (ii), and, without loss of generality, we assume b is adjacent only to one vertex of L, namely y, and not to any internal vertices of L. If Z contains all the inside cycles of $G\backslash e$, then $G - \{a, y\}$ is a tree, and $G \in \mathcal{S}$. Examples of this situation use daisy chains K, A, and their identifications.

Finally, suppose we have subcase (iii), where once again b is adjacent only to one vertex y of L, but now there is some non-Z inside cycle in $G\backslash e$. Say there is some non-Z inside cycle that is a 4-cycle. The undeletion of e_2 results in a graph G that has $\widetilde{K_{c2.1}} + e_{2iii}$ or $\widetilde{K_{c2.2}} + e_{2ii}$ as a minor, depending upon whether we can contract until the 4-cycle contains edge (x, a) or edge (b, y). Therefore S_1 or U is a minor of G. So it only remains to consider when all the non-Z inside cycles are 3-cycles. We note that any inside 3-cycle with edge (x, a) or (y, b) cannot have its third vertex fall on Q_j, or Z would meet Q_j in more than two vertices.

Say there is one non-Z inside cycle, a 3-cycle, and it contains edge (x, a); then $G - \{a, y\}$ is a tree, so G is a member of $\mathcal{S}$, by Theorem 2.36. Notice $G - \{a, y\}$ remains a tree for $G\backslash e$ with any number of non-Z inside 3-cycles, so long as they all meet a and $Q_j = \{s, a, b, t\}$ (or $\{st, a, b\}$, if $G\backslash e$ is an identified daisy chain). Some daisy chains that can exhibit these kinds of undeletions are A, A_{c2}, RW_{6c1}, and their identifications.

Say there is one non-Z inside 3-cycle, and it contains edge (y, b). If Z has an inside 4-cycle and a is incident with only one inside 3-cycle of Z, then G is another daisy chain. We can split vertex y and produce a series-parallel graph. Notice that this holds for any number of non-Z inside 3-cycles, so long as they all meet b and $Q_j = \{s, a, b, t\}$ (or $\{st, a, b\}$). If the edge (a, y) is present (that is, Z has no inside 4-cycle) or we can contract to get that edge (that is, a is adjacent to more than one inside 3-cycle contained in Z), then G contains as a minor $\widetilde{B} + e_{2iii}$, and therefore G contains an S_1-minor.

We have already covered two situations where there is more than one non-Z inside 3-cycle and no non-Z inside 4-cycle; there are two more. Suppose that any non-Z inside 3-cycle, not containing (x, a) or (y, b), has a vertex that is not from $\{s, a, b, t\}$ (or $\{st, a, b\}$) but does lie on Q_j. Then, we can delete the cycle's edge that has an endpoint on both Q_i and Q_j and create an inside 4-cycle. We have already seen that a non-Z inside 4-cycle results in G properly containing a known excluded minor of $\mathcal{M}_1$. Alternatively, we may have inside 3-cycles on both sides of Z (*i.e.*, these inside 3-cycles are associated with ears whose indices are greater and lesser than those associated with the cycles in Z) that only meet vertices of Q_i, aside from a and b. Graph G then has a daisy chain structure, so long as a meets at most one inside 3-cycle of Z. Once a meets more than one inside 3-cycle of Z, this is the previous case where G contains $\widetilde{B} + e_{2iii}$, and therefore G has both an S_1-minor.

2.71.3. $V(Z) \cap V(Q_j) \subseteq \{a, b, c\}$

Now, suppose the cycles of Z share at least three vertices with Q_j, including a, b, and c. Let a and c be the endpoints of the longest subpath in Q_j between two of the common inside vertices, having internal vertex b. By Lemma 2.20, since L

and $a-c$ both are at least length-3, it is possible for Z to have two inside 4-cycles. We break our argument down based on the number of inside 4-cycles in Z:

(i) Z has two or more inside 4-cycles;

(ii) Z has one or no inside 4-cycles.

This first subcase (i) is very short. Let Z have two inside 4-cycles. Then our e_2-undeletion of $G \backslash e$ results in a G that has $\widetilde{AW_{4c1.1}} + e_{2i}$ as a minor. If $G \backslash e$ is isomorphic to $\widetilde{AW_{4c1.1}}$, then G is isomorphic to R. Otherwise, G has a proper R-minor.

The second subcase (ii), when Z has at most one inside 4-cycles, requires a bit more care. We further break down the argument based on whether either of paths L and $a-c$ has a degree-4 internal vertex, or not. Begin by assuming at least one of these paths has such an internal degree-4 vertex.

When Z has only inside 3-cycles, note that we cannot have only one of L and $a-c$ with an internal vertex that is degree four. As both L and $a-c$ have length two or greater, this would lead to Z containing an inside 4-cycle. So, when Z has only inside 3-cycles, some internal vertex of either L or $a-c$ has degree at least five, or one internal vertex on each path has degree at least four. Then, two edge deletions result in a daisy chain where Z has two inside 4-cycles. Thus, the e_2-undeletion of $G \backslash e$ results in a G that has $\widetilde{AW_{4c1.1}} + e_{2i}$ as a minor. Therefore, G has an R-minor.

If Z has exactly one inside 4-cycle and there is an internal vertex of L or $a-c$ with degree greater than four, deleting the edge with endpoints on Q_i and Q_j that is common to some adjacent pair of 3-cycles gives us two 4-cycles in Z. So again, the e_2-undeletion of $G \backslash e$ results in a G that has $\widetilde{AW_{4c1.1}} + e_{2i}$ as a minor. Thus, G has an R-minor.

Next, we need to consider when all internal vertices of L and $a-c$ are degree-3. Observe that if any pair of internal vertices, one each from L and $a-c$, are adjacent, then G has an $\widetilde{AW_{4c1.1}} + e_{2i}$ minor, and therefore an R-minor. So we assume otherwise; internal vertices of L and $a-c$ are only adjacent to an endpoint of the other path. Notice that, since L and $a-c$ have lengths of two or greater, each of these paths must have an endpoint of degree at least four, with at least three neighbors of such an endpoint being in $V(Z)$. Furthermore, these endpoints must be nonadjacent, or $G\backslash e$ is not a daisy chain.

Suppose only nonadjacent endpoints of L and $a-c$, without loss of generality a and y, have degree at least four, where three or more neighbors each of a and y are from $V(Z)$. There may be one or no inside 4-cycle in Z. If Z has all the inside cycles of $G\backslash e$, then $G-\{a,y\}$ is a tree. Thus, by Theorem 2.36, G is a member of $\mathcal{S}$. So assume $G\backslash e$ does have a non-Z inside cycle. If any non-Z cycle is an inside 4-cycle, then G has $\widetilde{K_{c2.1}} + e_{2iii}$ or $\widetilde{K_{c2.2}} + e_{2ii}$ as a minor, which means G has an S_1- or U-minor, a contradiction. So say the non-Z inside cycles are all 3-cycles. If all the non-Z inside 3-cycles meet a and have their other vertices on Q_i, then $G-\{a,y\}$ is still a tree. However, if one of the non-Z inside 3-cycles meeting a has a second vertex on Q_j, then there is an edge deletion that turns two non-Z 3-cycles into a 4-cycle, meaning G has $\widetilde{K_{c2.1}} + e_{2iii}$ or $\widetilde{K_{c2.2}} + e_{2ii}$ as a minor. If there are non-Z inside 3-cycles incident with y, then one of them must have a second vertex on Q_i, so G contains $\widetilde{B_{c3}} + e_2$, and therefore, G has a U-minor.

2.72. e_3*-undeletions*

There is one final kind of undeletion to examine, the e_3-undeletions of $G\backslash e$. Let the undeleted edge be $e_3 = (s,y)$; note that, depending on how we have labeled the vertices of $G\backslash e$, that edge $e_3 = (t,y)$ is also an option (and, we take $e_3 = (st,y)$,

if we have an identified daisy chain). Let y be on $s-t$ path (or st cycle, in an identified daisy chain) Q_i, where it is not adjacent to either s or t (or, y is not adjacent to st). Thus, there are vertices x and r on Q_i, as shown in Figure 2.44. So, Q_i must have length at least four. Let L be the subpath of Q_i that has endpoints x and y; observe that the length of L is at least one. Let Z be the set of inside cycles in $G\backslash e$ such that, for each cycle z in Z, all of the vertices of z that are on Q_i are also vertices of L. We say e_3 *subtends* path L and the cycles of Z. Observe that Z does not contain all inside cycles of $G\backslash e$, due to the existence of vertex r on Q_i. Let Q_j be the $s-t$ path (or st cycle) that does not contain y.

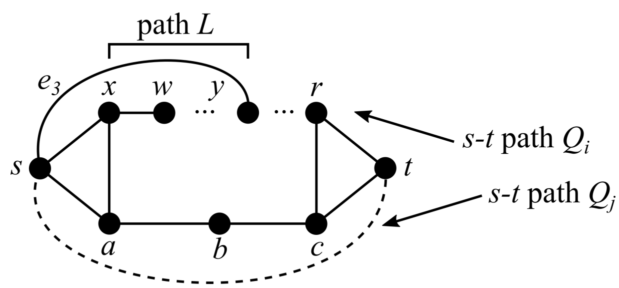


FIGURE 2.44: Example of an e_3-undeletion. Edges between vertices of L and Q_j are omitted.

Before we delve into the e_3-undeletion case analysis, we present a lemma describing a situation that will frequently arise in the analysis.

Lemma 2.73. *Suppose an e_3-undeletion is performed on a daisy chain $G\backslash e$ where*

(1) $Q_j = s, a, b, t$, graph $G\backslash e$ is an unidentified daisy chain, $deg(a) = 3$, and $deg(b) \geq 4$;

(2) $Q_j = st, a, b, st$, graph $G\backslash e$ is an identified daisy chain, $deg(a) = 3$, and $deg(b) \geq 4$; or

(3) $Q_j = st, a, b, c, st$, graph $G\backslash e$ is an identified daisy chain, $deg(a) = deg(c) = 3$, and $deg(b) \geq 3$.

Then the resulting graph G is in $\mathcal{S}$. Note that we may reverse the degrees of a and b in (1) or (2), and the result holds.

Proof. For (1), $G - \{s, b\}$ is a tree. For (2) and (3), $G - \{st, b\}$ is a tree. Hence, by Theorem 2.36, G is in $\mathcal{S}$. □

Consideration of the e_3-undeletions involves a lengthy case analysis. We break the analysis into two parts, depending on the length of L:

(1) length of L is one; or

(2) length of L is at least two.

2.73.1. *The length of L is one*

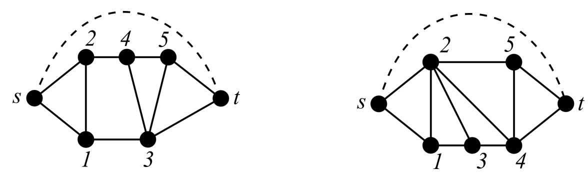
We begin by considering when the length of L is one, and we will break this down into subcases depending on how many inside vertices of $G\backslash e$ are in both $V(Z)$ and $V(Q_j)$. Each of these subcases is broken down further by considerations such as whether $G\backslash e$ is identified or unidentified, the lengths of the inside cycles in Z, and the lengths of the non-Z inside cycles of $G\backslash e$.

2.73.2. $V(Z) \cap V(Q_i) = \{a\}$ *and the length of L is one*

Suppose all the cycles of Z have only one inside vertex a in common with Q_j, and the length of L is one. Then Z consists of exactly one inside 3-cycle, on vertices x, y, and a. We perform a case analysis based on the non-Z inside cycles that appear in $G\backslash e$. Observe that any non-Z inside 3-cycle that uses edge (y, a) must have its third vertex on Q_i, or we contradict our assumption about a being the only Q_j vertex also in $V(Z)$.

If there is only one non-Z inside cycle and it is a 3-cycle, then $G\backslash e$ is a wheel; see 2.67 for the discussion of undeletions of wheels. If the non-Z inside cycle is a

4-cycle, the daisy chain $G\backslash e$ is AW_{4c1} or its identification, and our undeleted edge is $e_{3i} = (t, 4)$. We see $AW_{4c1} + e_{3i}$ is isomorphic to U. By Lemma 2.73, $\widetilde{AW_{4c1}} + e_{3i}$ is in $\mathcal{S}$.



(a) AW_{4c1} or $\widetilde{AW_{4c1}}$ (b) K or $\widetilde{K}$

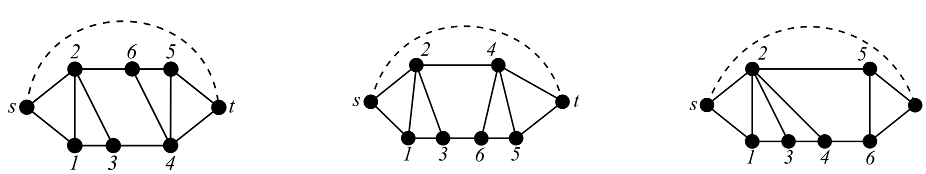
FIGURE 2.45: Selected daisy chains.

Now assume there are two non-Z inside cycles of $G\backslash e$. For two 3-cycles, since we do not wish $G\backslash e$ to be a wheel, we get $G\backslash e$ is isomorphic to K or its identification, and the undeleted edge is $e_{3i} = (s, 3)$. While $K + e_{3i}$ has a U-minor, undeletion $\widetilde{K} + e_{3i}$ is in $\mathcal{S}$ by Lemma 2.73.

Thus, we know that

2.73.3. *Any graph G that results from an e_3-undeletion of an unidentified $G\backslash e$ with at least two non-Z inside cycles, will contain the known excluded minor U.*

This is due to the fact that we can start with G, contract all the inside cycles in Z down to just one inside 3-cycle, contract out out all but two of the non-Z inside cycles, and contract the remaining two down to 3-cycles if necessary, leaving $K + e_{3i}$.

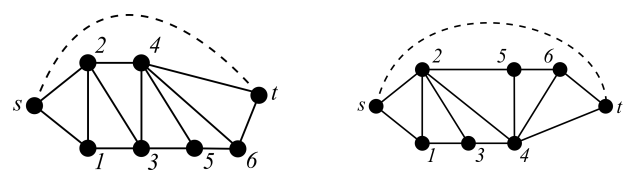


(a) $\widetilde{K_{c3}}$ (b) $\widetilde{AW_{5c2}}$ (c) $\widetilde{K_{c1}}$

FIGURE 2.46: Selected daisy chains.

We must still do more work when $G\backslash e$ is an identified daisy chain. Suppose the two non-Z inside cycles of $G\backslash e$ are a 3-cycle and a 4-cycle. Then $G\backslash e$ is isomorphic to $\widetilde{K_{c3}}$, $\widetilde{AW_{5c2}}$, or $\widetilde{K_{c1}}$, and, for all of these, the undeleted edge is $e_{3i} = (st, 3)$. Now, $\widetilde{K_{c3}} + e_{3i}$ has a U-minor, and $\widetilde{AW_{5c2}} + e_{3i}$ has an S_1-minor. However, $\widetilde{K_{c1}} + e_{3i}$ is a member of $\mathcal{S}$, by Lemma 2.73.

From this, we can draw some conclusions. Say there are k ears in a nested open ear decomposition of the series-parallel graph we get via splitting st in $G\backslash e$. Keep the length of L one, and $V(Z) \cap V(Q_j) = \{a\}$, as they have been for this section. If $G\backslash e$ has two or more non-Z inside cycles, and one of those cycles is associated with an ear whose index is strictly less than $k-1$, then the undeletion results in a graph G that has $\widetilde{K_{c3}} + e_{3i}$ or $\widetilde{AW_{5c2}} + e_{3i}$ as a minor. Therefore, G has a U-minor or an S_1-minor.



(a) $\widetilde{B}$ (b) $\widetilde{D}$

FIGURE 2.47: Selected daisy chains.

Therefore, at this point, we still do not know what happens with an e_3-undeletion when $G\backslash e$ an identified daisy chain, and there are three or more non-Z inside cycles that are either all 3-cycles, or all 3-cycles except for the inside cycle associated with ear P_{k-1}, which is a 4-cycle. Now, if all of these non-Z inside cycles up through the one associated with ear P_{k-2} only have vertices on Q_i besides a, the e_3-undeletion G is in $\mathcal{S}$ by Lemma 2.73.

So, say one of the non-Z inside 3-cycles associated with an ear whose index is strictly less than $k-1$ has some vertex besides a on Q_j. Then the graph G that results from the undeletion has as a minor one of $\widetilde{B} + e_{3i}$ or $\widetilde{D} + e_{3i}$, where

$e_{3i} = (st, 5)$. This is significant, since $\widetilde{B} + e_{3i}$ has an S_1-minor, and $\widetilde{D} + e_{3i}$ has a U-minor. Thus, G contains an S_1-minor or a U-minor.

2.73.4. *$V(Z) \cap V(Q_i) = \{a, b\}$ and the length of L is one*

We next suppose the cycles of Z meet exactly two inside vertices of Q_j, namely, a and b. The length of L remains one. We will form subcases based on the inside cycles contained in Z. Since we know the length of L is one, by Lemma 2.20, the set Z may consist of

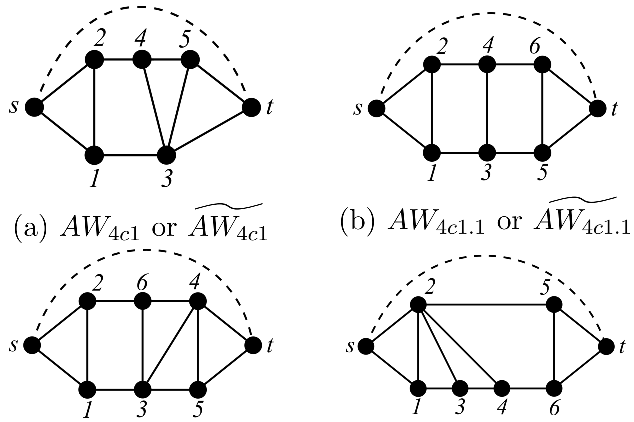
(i) one 4-cycle;

(ii) two 3-cycles, such that $G\backslash e$ has edge (x, b); or

(iii) two 3-cycles, such that $G\backslash e$ has edge (y, a).

Start with subcase (i). We will proceed based on the non-Z inside cycles of $G\backslash e$. Say there is only one non-Z inside cycle; one of its edges is therefore (y, b). If this non-Z inside cycle is a 3-cycle, it must have its third vertex on Q_i, or we contradict a and b being the only vertices of Z that lie on Q_j. So, $G\backslash e$ is isomorphic to AW_{4c1} or $\widetilde{AW_{4c1}}$, and the undeleted edge is $e_{3ii} = (s, 4)$ (or $e_{3ii} = (st, 4)$). By Lemma 2.73, either undeletion produces a G that is a member of $\mathcal{S}$. If the non-Z inside cycle is a 4-cycle, an unidentified $G\backslash e$ is isomorphic to $AW_{4c1.1}$, where we take the undeleted edge to be $e_{3i} = (s, 4)$, and the undeletion has an S_1-minor. An identified $G\backslash e$ is isomorphic to $\widetilde{AW_{4c1.1}}$, where we take the undeleted edge to be $e_{3ii} = (st, 4)$. The G that results from this undeletion is in $\mathcal{S}$, since $G - \{st, 3\}$ is a tree.

Next, let there be exactly two non-Z inside cycles in $G\backslash e$. If these are two inside 3-cycles, then $G\backslash e$ is isomorphic to AW_{5c1} where we undelete $e_{3ii} = (s, 6)$, $\widetilde{AW_{5c1}}$ where we undelete $e_{3ii} = (st, 6)$, K_{c1} where we undelete $e_{3ii} = (t, 4)$, or $\widetilde{K_{c1}}$ where we undelete $e_{3ii} = (st, 4)$. Both $K_{c1} + e_{3ii}$ and $\widetilde{K_{c1}} + e_{3ii}$ are in $\mathcal{S}$, by



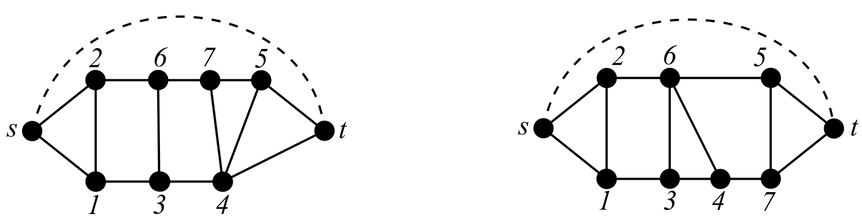
(a) AW_{4c1} or $\widetilde{AW_{4c1}}$ (b) $AW_{4c1.1}$ or $\widetilde{AW_{4c1.1}}$

(c) AW_{5c1} or $\widetilde{AW_{5c1}}$ (d) K_{c1} or $\widetilde{K_{c1}}$

FIGURE 2.48: Selected daisy chains.

Lemma 2.73. While $AW_{5c1} + e_{3ii}$ has an S_1-minor, the undeletion $\widetilde{AW_{5c1}} + e_{3ii}$ is in $\mathcal{S}$ by Lemma 2.73.

If one of the two non-Z inside cycles is a 4-cycle, then $G \backslash e$ is $K_{c2.1}$ where we undelete edge $e_{3ii} = (s, 3)$ or $e_{3ii} = (s, 6)$, $K_{c2.2}$ where we undelete edge $e_{3ii} = (s, 3)$, or the identification of one of these two graphs. The undeletion $K_{c2.2} + e_{3ii}$ has an S_1-minor, but $\widetilde{K_{c2.2}} + e_{3ii}$ is a member of $\mathcal{S}$ by Lemma 2.73. Both of $K_{c2.1} + e_{3ii}$ and $\widetilde{K_{c2.1}} + e_{3ii}$ have an S_1-minor, whether we take $e_{3ii} = (s, 3)$ or $e_{3ii} = (s, 6)$ (or $(st, 3)$ and $(st, 6)$, for $\widetilde{K_{c2.1}}$).



(a) $K_{c2.1}$ or $\widetilde{K_{c2.1}}$ (b) $K_{c2.2}$ or $\widetilde{K_{c2.2}}$

FIGURE 2.49: Selected daisy chains.

To complete the analysis of this subcase, we need to consider when there are three or more non-Z cycles in $G \backslash e$. When all the non-Z inside cycles are 3-cycles that meet vertex b, the graph G resulting from the e_3-undeletion of $G \backslash e$ is in $\mathcal{S}$

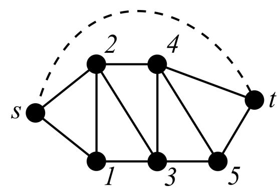
by Lemma 2.73. When an inside 3-cycle associated with P_{k-2} or an ear having a lower index has a vertex on Q_j besides b (and, therefore, the cycle associated with the next ear does not meet b), the undeletion has $K_{c2.1} + e_{3ii}$ (or $\widetilde{K_{c2.1}} + e_{3ii}$) as a minor, and therefore has an S_1-minor.

If one of the non-Z inside cycles is a 4-cycle associated with P_{k-2} or an ear having a lower index, the e_3-undeletion has $K_{c2.1}+e_{3ii}$ (or $\widetilde{K_{c2.1}}+e_{3ii}$) as a minor, and therefore has an S_1-minor. So long as $G\backslash e$ is not an identified daisy chain, in fact, we may relax the restriction on the location of the non-Z inside 4-cycle, and simply say that if there is one, then G has an S_1-minor, since G contains $K_{c2.2}+e_{3ii}$. However, we must be somewhat more careful when $G\backslash e$ is an identified daisy chain. If there is an inside 4-cycle associated with ear P_{k-1}, where one endpoint of P_{k-1} is identified with b, and all the other non-Z inside cycles (which are 3-cycles) meet vertex b, then, by Lemma 2.73, G is in $\mathcal{S}$. If, on the other hand, neither endpoint of P_{k-1} is identified with b, then G has a $\widetilde{K_{c2.1}} + e_{3ii}$ minor, and, therefore, an S_1-minor.

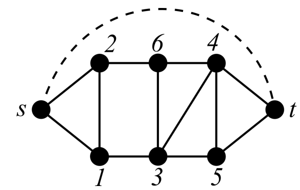
The next subcase (ii) considers when Z consists of two 3-cycles, such that $G\backslash e$ has edge (x, b). This subcase falls out from the work we did above for (i). The change from $G\backslash e$ in (i) to $G\backslash e$ in (ii) is the addition of edge (x, b), meaning Lemma 2.73 still applies for the same non-Z cycle arrangements, while the S_1-minors of the undeletions from the remaining non-Z cycle arrangements remain intact.

Finally, consider subcase (iii), where Z consists of two 3-cycles, but now such that $G\backslash e$ has edge (y, a). Say there is only one non-Z inside cycle. If this is a 3-cycle, it must have its third vertex on Q_i, or Z and Q_j have more than two inside vertices in common. So, we have AW_5 or $\widetilde{AW_5}$ as $G\backslash e$, and we take $e_{3ii} = (s, 3)$ or $e_{3ii} = (st, 3)$. While $AW_5 + e_{3ii}$ has an S_1-minor, we find $\widetilde{AW_5} + e_{3ii}$ is a member

of $\mathcal{S}$, since $G - \{st, 3\}$ is a tree. If the one non-Z inside cycle is a 4-cycle, we get similar results from considering undeletions of AW_{5c1} and $\widetilde{AW_{5c1}}$ with $e_{3ii} = (t, 3)$ or $e_{3ii} = (st, 3)$, namely, a G with an S_1-minor and a G that is a member of $\mathcal{S}$, respectively.



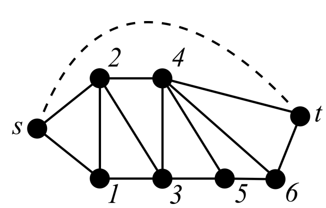
(a) AW_5 or $\widetilde{AW_5}$



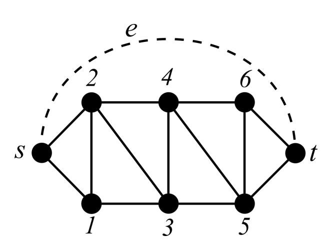
(b) AW_{5c1} or $\widetilde{AW_{5c1}}$

FIGURE 2.50: Selected daisy chains.

Say there are two or more non-Z inside cycles; it will suffice to examine two non-Z inside 3-cycles. If b is the only vertex of Q_j that either of these inside 3-cycles meets, then $G \backslash e$ is B or its identification, where we take the undeleted edge to be $e_{3ii} = (s, 3)$ (or $(st, 3)$, in $\widetilde{B}$). If the cycle associated with ear P_{k-1} meets b and some other vertex from Q_j, then $G \backslash e$ is AW_6 or its identification, and we take $e_{3ii} = (s, 3)$ (or, $(st, 3)$). Both $B + e_{3ii}$ and $\widetilde{B} + e_{3ii}$ have an S_1-minor, while $AW_6 + e_{3ii}$ and $\widetilde{AW_6} + e_{3ii}$ have an S_1-minor. Once $G \backslash e$ has at least two non-Z inside cycles, we know the undeletion will result in a graph G that has as a minor one of $B + e_{3ii}$, $\widetilde{B} + e_{3ii}$, $AW_6 + e_{3ii}$, and $\widetilde{AW_6} + e_{3ii}$, and, therefore, G has a proper S_1-minor.



(a) B or $\widetilde{B}$



(b) AW_6 or $\widetilde{AW_6}$

FIGURE 2.51: Selected daisy chains.

The following lemma is very useful in limiting the number of checks one must run on e_3-undeletions of daisy chains. We have already seen that, once we have selected which of the distinguished vertices of an unidentified daisy chain are labeled s and t, the cycles in Z can change depending on whether our e_3-undeletion uses edge (s, y) or (t, y). This lemma allows us to just check if undeleting edge (st, y) of the daisy chain's identification results in a graph that contains a known excluded minor of $\mathcal{M}_1$.

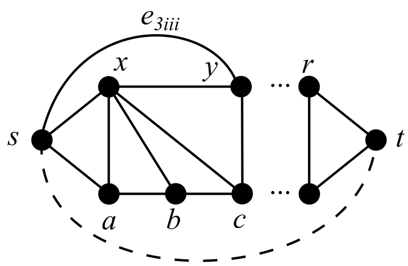
Lemma 2.74. *Let H be a daisy chain, and let and its identification be $\widetilde{H}$. Consider the e_3-undeletions $e_{3i} = (s, y)$ and $e_{3ii} = (t, y)$ of H, and the e_3 undeletion $e_{3iii} = (st, y)$ of $\widetilde{H}$. If $\widetilde{H} + e_{3iii}$ has an excluded minor of $\mathcal{M}_1$, then so do $H + e_{3i}$ and $H + e_{3ii}$.*

Proof. Both $H + e_{3i}$ and $H + e_{3ii}$ contain $\widetilde{H} + e_{3iii}$ as a minor; we simply need to contract edge (s, t). Thus, by transitivity, the statement holds. □

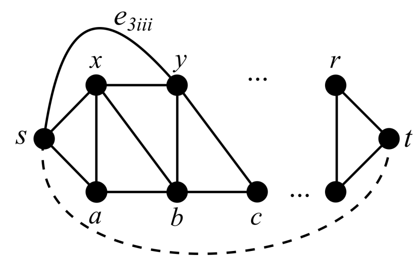
2.74.1. *$V(Z) \cap V(Q_i) \subseteq \{a, b, c\}$ and the length of L is one*

Lastly, assume the cycles of Z meet at least three inside vertices of Q_j, including vertices a, b, and c. Let a and c be the endpoints of the longest subpath in Q_j between two of the common inside vertices, with internal vertex b. Now, either Z contains only inside 3-cycles, or Z has exactly one inside 4-cycle; this follows by Lemma 2.20 because the length of L in one. Since Z meets Q_j in at least three internal vertices but L is length-1, when Z contains only inside 3-cycles, either $G\backslash e$ has one of x and y with degree at least five, or it has both vertices with degree at least four. When we allow Z to contain an inside 4-cycle, at least one of x and y must have degree at least four. We show the five base cases with x and y having minimal degree and $V(Z) \cap V(Q_i) = \{a, b, c\}$ in Figure 2.52. We will break our

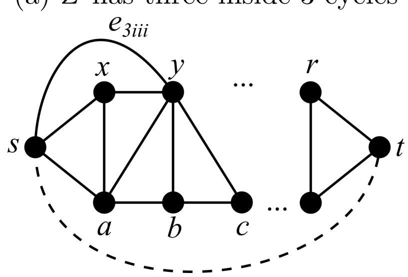
argument down into subcases based on the degrees of x and y, and whether Z has an inside 4-cycle or not.



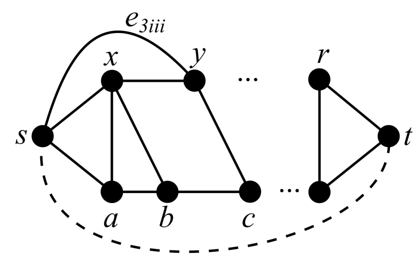
(a) Z has three inside 3-cycles



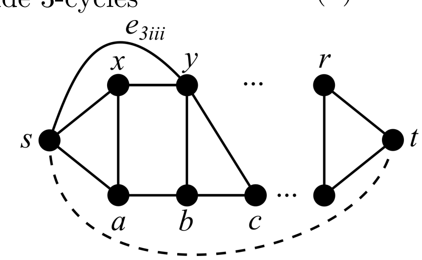
(b) Z has three inside 3-cycles



(c) Z has three inside 3-cycles



(d) Z has one inside 4-cycle

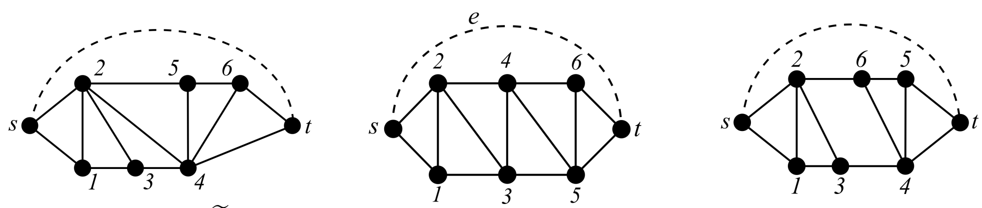


(e) Z has one inside 4-cycle

FIGURE 2.52: Arrangements of Z when x and y have minimal degree, for length of L being one and Z having inside vertices a, b, and c in common with Q_j.

Suppose Z has only inside 3-cycles, and $deg_{G\backslash e}(x) \geq 5$. An example of this situation is sketched in Figure 2.52a. The smallest daisy chain satisfying our conditions for $G\backslash e$ is D or $\widetilde{D}$, where we are undeleting edge $e_{3iii} = (s, 5)$ (or, $(st, 5)$); then $deg_{G\backslash e}(x) = 5$, $deg_{G\backslash e}(y) = 3$, and the one non-$Z$ inside cycle is a 3-cycle. We have previously considered the undeletion of $\widetilde{D}$ that added edge $e_{3i} = (st, 5)$, where we found a U-minor. By Lemma 2.74, then, $D + e_{3iii}$ and $\widetilde{D} + e_{3iii}$ have a U-minor. We observe that the graph G resulting from an e_3-undeletion of $G\backslash e$ where Z has

only inside 3-cycles, and $deg_{G\backslash e}(x) \geq 5$, will always have $D + e_{3iii}$ or $\widetilde{D} + e_{3iii}$ as a minor, and, thus, G has a U-minor.



(a) D or $\widetilde{D}$ (b) AW_6 or $\widetilde{AW_6}$ (c) K_{c3} or $\widetilde{K_{c3}}$

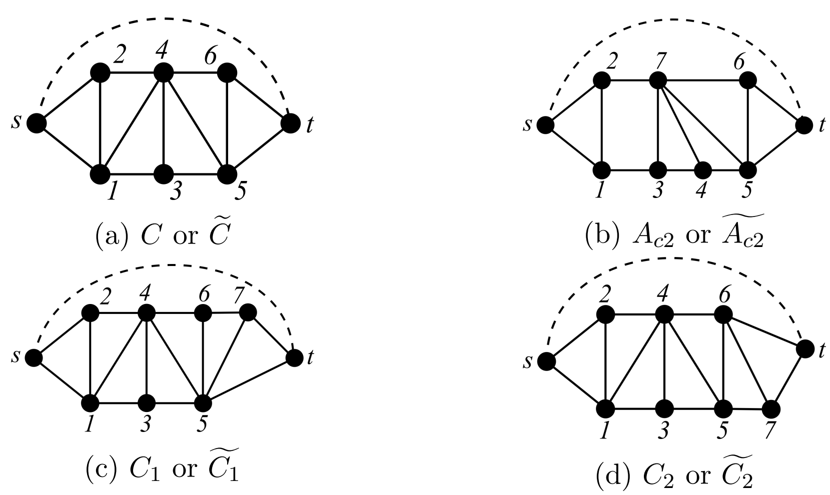
FIGURE 2.53: Selected daisy chains.

Suppose Z has only inside 3-cycles, and $deg_{G\backslash e}(x), deg_{G\backslash e}(y) \geq 4$. An example of this situation is sketched in Figure 2.52b. The smallest daisy chain satisfying our conditions for $G\backslash e$ is AW_6 or $\widetilde{AW_6}$, where the undeleted edge is $e_{3iii} = (t, 3)$ or $(st, 3)$. Again, $AW_6 + e_{3iii}$ or $\widetilde{AW_6} + e_{3iii}$ has an S_1-minor, by Lemma 2.74, since we previously found $\widetilde{AW_6} + e_{3ii}$ with $e_{3ii} = (st, 3)$ has an S_1-minor. We observe that the graph G resulting from an e_3-undeletion of $G\backslash e$ where Z has only inside 3-cycles, and $deg_{G\backslash e}(x), deg_{G\backslash e}(y) \geq 4$, will always have $AW_6 + e_{3iii}$ or $\widetilde{AW_6} + e_{3iii}$ as a minor, and, thus, G has an S_1-minor.

We delay our discussion of Z having only inside 3-cycles and $deg_{G\backslash e}(y) \geq 5$ until after the next paragraph.

Suppose Z has one inside 4-cycle, and $deg_{G\backslash e}(x) \geq 4$. An example of this situation is sketched in Figure 2.52d. The smallest daisy chain satisfying our conditions for $G\backslash e$ is K_{c3} or $\widetilde{K_{c3}}$, where the undeleted edge is $e_{3iii} = (t, 3)$ or $(st, 3)$. Again, $K_{c3} + e_{3iii}$ or $\widetilde{K_{c3}} + e_{3iii}$ has a U-minor, by Lemma 2.74, since we previously found $\widetilde{K_{c3}} + e_{3i}$ with $e_{3i} = (st, 3)$ has a U-minor. We observe that the graph G resulting from an e_3-undeletion of $G\backslash e$ where Z has an inside 4-cycle, and $deg_{G\backslash e}(x) \geq 4$, will always have $K_{c3} + e_{3iii}$ or $\widetilde{K_{c3}} + e_{3iii}$ as a minor, and, thus, G has a U-minor.

Suppose Z has only inside 3-cycles, and $deg_{G\backslash e}(y) \geq 5$. We impose the additional restriction that $deg_{G\backslash e}(x) = 3$, or we have already found contradictions for the G that results from our undeletion. An example of the situation is sketched in Figure 2.52c. We break our analysis down based on the non-Z inside cycles in $G\backslash e$. If there is exactly one non-Z inside cycle and $deg_{G\backslash e}(y) = 5$, then $G\backslash e$ is isomorphic to C with $e_{3iii} = (s, 4)$, to A_{c2} with $e_{3iii} = (t, 7)$, or to one of their identifications, depending on whether the non-Z inside cycle is a 3-cycle or a 4-cycle. Both $C + e_{3iii}$ and $A_{c2} + e_{3iii}$ have a U-minor. So we conclude that, as long as $G\backslash e$ is an unidentified daisy chain in this subcase of Z having only inside 3-cycles, and $deg_{G\backslash e}(y) \geq 5$, the graph G that results from the e_3-undeletion contains a U-minor. However, $\widetilde{C} + e_{3iii}$ and $\widetilde{A_{c2}} + e_{3iii}$ are both members of $\mathcal{S}$ by Theorem 2.36, since deleting st and y leaves a tree, with either of them. Observe that if y has a higher degree (that is, Z and Q_j meet in more than three inside vertices), the graph G resulting from the undeletion still has $G - \{st, y\}$ a tree. Thus, we must consider when there is more than one non-Z inside cycle.



(a) C or $\widetilde{C}$ (b) A_{c2} or $\widetilde{A_{c2}}$

(c) C_1 or $\widetilde{C_1}$ (d) C_2 or $\widetilde{C_2}$

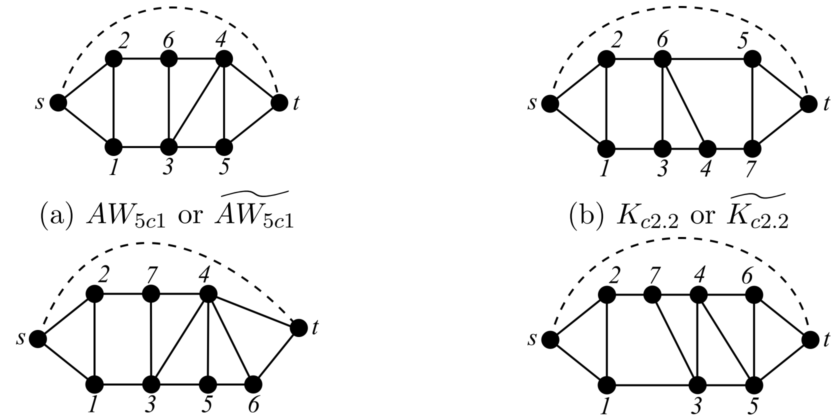
FIGURE 2.54: Selected daisy chains.

Say there are at least two non-Z inside cycles; it will be sufficient for us to consider two 3-cycles. Either these two non-Z inside 3-cycles only meet the vertex

c on Q_j, or one of them meets both c and some vertex on Q_j that is not in $a - c$. If we take $deg_{G\backslash e}(y) = 5$, then $G\backslash e$ is isomorphic to one of $\widetilde{C_1}$ and $\widetilde{C_2}$ with the undeleted edge being $e_{3iii} = (st, 4)$. Both $\widetilde{C_1} + e_{3iii}$ and $\widetilde{C_2} + e_{3iii}$ have an S_1-minor. So, if there are at least two non-Z inside cycles, the undeletion results in a G that properly contains a known excluded minor.

Suppose Z has one inside 4-cycle, and $deg_{G\backslash e}(y) \geq 4$. We impose the additional restriction that $deg_{G\backslash e}(x) = 3$, or we have already found contradictions for the G that results from our undeletion. An example of the situation is sketched in Figure 2.52e. We break our analysis down based on the non-Z inside cycles in $G\backslash e$. If there is exactly one non-Z inside cycle and $deg_{G\backslash e}(y) = 4$, then $G\backslash e$ is isomorphic to AW_{5c1} with $e_{3iii} = (s, 3)$, to $K_{c2.2}$ with $e_{3iii} = (s, 6)$, or to one of their identifications, depending on whether the non-Z inside cycle is a 3-cycle or a 4-cycle. Both $AW_{5c1} + e_{3iii}$ and $K_{c2.2} + e_{3iii}$ have a U-minor. So we conclude that, as long as $G\backslash e$ is an unidentified daisy chain in this subcase of Z having one inside 4-cycle, and $deg_{G\backslash e}(y) \geq 4$, the graph G that results from the e_3-undeletion contains a U-minor. However, $\widetilde{AW_{5c1}} + e_{3iii}$ and $\widetilde{K_{c2.2}} + e_{3iii}$ are both members of $\mathcal{S}$ by Theorem 2.36, since deleting st and y leaves a tree, for either graph. Observe that if y has a higher degree (that is, Z and Q_j meet in more than three inside vertices), the graph G resulting from the undeletion still has $G - \{st, y\}$ as a tree. Thus, we must consider when there is more than one non-Z inside cycle.

Say there are at least two non-Z inside cycles; it will be sufficient for us to consider two 3-cycles. Either these two non-Z inside 3-cycles only meet the vertex c on Q_j, or one of them meets both c and some vertex on Q_j that is not in $a - c$. If we take $deg_{G\backslash e}(y) = 4$, then $G\backslash e$ is isomorphic to one of $\widetilde{B_{c1}}$ and $\widetilde{AW_{6c1}}$ with the edge we undelete being $e_{3iii} = (st, 3)$. Both $\widetilde{B_{c1}} + e_{3iii}$ and $\widetilde{AW_{6c1}} + e_{3iii}$ have an



(a) AW_{5c1} or $\widetilde{AW_{5c1}}$ (b) $K_{c2.2}$ or $\widetilde{K_{c2.2}}$

(c) B_{c1} or $\widetilde{B_{c1}}$ (d) AW_{6c1} or $\widetilde{AW_{6c1}}$

FIGURE 2.55: Selected daisy chains.

S_1-minor. So, if there are at least two non-Z inside cycles, the undeletion results in a G that properly contains a known excluded minor.

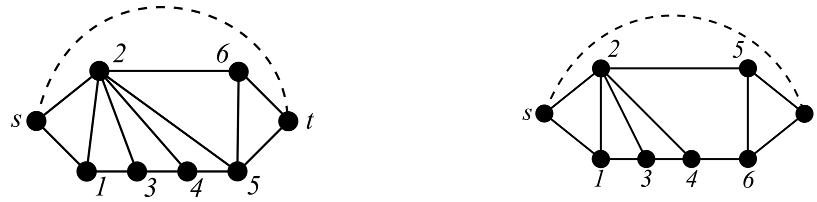
2.74.2. *The length of L is at least two*

There are at least three vertices on L; designate one of the interior vertices w. We break the argument down very similarly to the way we did for the length of L being one. The major consideration is the number of inside vertices that are in both Z and Q_j.

2.74.3. $V(Z) \cap V(Q_j) = \{a\}$

Suppose the cycles of Z meet exactly one inside vertex of Q_j, and call the vertex in this intersection a. If the length of Q_j is two, then $G\backslash e$ is a wheel; see 2.67 for an analysis of this case. So we may assume there is another vertex on Q_j, call it b, and we let b be adjacent to r. Then the non-Z inside cycles of $G\backslash e$ either include a 4-cycle, or include two 3-cycles where one of them is incident with b.

If these are precisely the non-Z inside cycles and L is exactly length two, then we are considering $G\backslash e$ that is isomorphic to A, to K_{c1}, or to their identifications, with the undeleted edge being $e_{3iii} = (s, 4)$ or $(st, 4)$. Both $A+e_{3iii}$ and $K_{c1}+e_{3iii}$ contain



(a) A or $\widetilde{A}$ (b) K_{c1} or $\widetilde{K_{c1}}$

FIGURE 2.56: Selected daisy chains.

U as a minor. Thus, an e_3-undeletion of $G\backslash e$ that is an unidentified daisy chain, with L length at least two, will yield a graph G that contains one of $A + e_{3iii}$ and $K_{c1} + e_{3iii}$ as a minor, and, therefore, G has a U-minor. However, by Lemma 2.73, the undeletions $\widetilde{A} + e_{3iii}$ and $\widetilde{K_{c1}} + e_{3iii}$ are in $\mathcal{S}$.

(a) RW_{6c2} or $\widetilde{RW_{6c2}}$ (b) A_{c4} or $\widetilde{A_{c4}}$

FIGURE 2.57: Selected daisy chains.

Thus, when $G\backslash e$ is an identified daisy chain, we need to consider the presence of additional non-Z inside cycles. Let there be k ears in a nested open ear decomposition of the series-parallel graph obtained from splitting $G\backslash e$ at st. Observe that, so long as all the non-Z inside cycles are incident with a, then by Lemma 2.73 the undeletion G is a member of $\mathcal{S}$. Thus, it only remains to consider when there is a non-Z inside cycle whose associated ear has index at most $k-2$, such that some vertex of the cycle, aside from a, is on Q_j. It will suffice to consider when the non-Z inside cycles of $G\backslash e$ are a 4-cycle incident with a and a 3-cycle not incident with a. Any set of non-Z inside cycles, where one of the cycles is not incident with a, will have such a 4-cycle and 3-cycle pair after an edge deletion. If the length of L is

exactly two, the daisy chain $G\backslash e$ is isomorphic to $\widetilde{RW_{6c2}}$ or $\widetilde{A_{c4}}$, with $e_{3iii} = (st, 4)$. The undeletion $\widetilde{RW_{6c2}} + e_{3iii}$ has an S_1-minor, while $\widetilde{A_{c4}} + e_{3iii}$ has a U-minor.

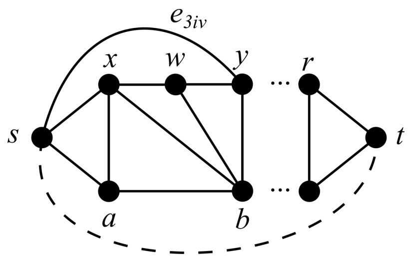
Letting the length of L be at least two, we see that the graph G resulting from an e_3-undeletion of unidentified daisy chain $G\backslash e$ will contain an excluded minor, by virtue of having $A + e_{3iii}$ or $K_{c1} + e_{3iii}$ as a minor. When $G\backslash e$ is an identified daisy chain, the graph G will be in $\mathcal{S}$ by Lemma 2.73 when all the non-Z inside cycles are are incident with a. Otherwise, $\widetilde{RW_{6c2}} + e_{3iii}$ or $\widetilde{A_{c4}} + e_{3iii}$ is a minor of G, and, therefore, so is U or S_1.

2.74.4. $V(Z) \cap V(Q_j) = \{a, b\}$

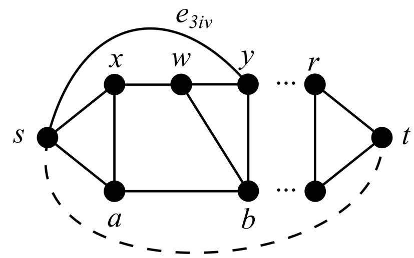
Next, suppose the set of cycles Z meets Q_j in exactly two inside vertices, call them a and b. Then, by Lemma 2.20, we can have at most one inside 4-cycle in Z. We use the presence or absence of an inside 4-cycle in Z to divide the analysis into smaller subcases, and we along with the degrees of a and b in $G\backslash e$.

Either Z contains an inside 4-cycle, or not. If Z consists of only inside 3-cycles, then either $G\backslash e$ has one of a and b has with at least five, or $G\backslash e$ has both a and b with degree at least four. If there is an inside 4-cycle in Z, then one of a and b has degree at least four. We show the five base cases with a and b having minimal degree for each of these situations in Figure 2.58.

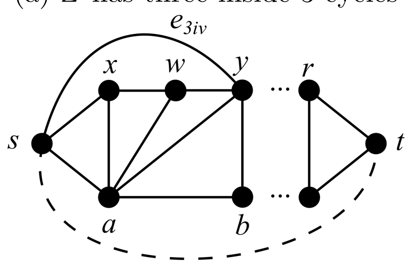
Suppose Z only contains inside 3-cycles and $deg_{G\backslash e}(a) \geq 5$; an example of this situation is sketched in Figure 2.58c. Suppose Z contains an inside 4-cycle and $deg_{G\backslash e}(a) \geq 4$; an example of this situation is sketched in Figure 2.58d. Suppose Z only contains inside 3-cycles and $deg_{G\backslash e}(a), deg_{G\backslash e}(b) \geq 4$; an example of this situation is sketched in Figure 2.58e. Let G be the graph resulting from the undeletion of $G\backslash e$ in any of these three cases. Then G has as a minor $AW_{5c2} + e_{3ii}$ or $\widetilde{AW_{5c2}} + e_{3ii}$, where $e_{3i} = (t, 3)$ or $e_{3i} = (st, 3)$. We have previously found that



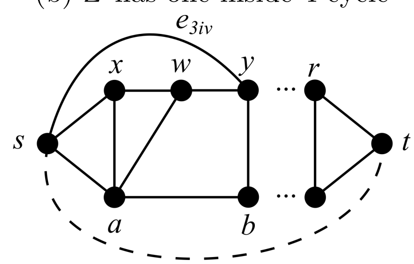
(a) Z has three inside 3-cycles



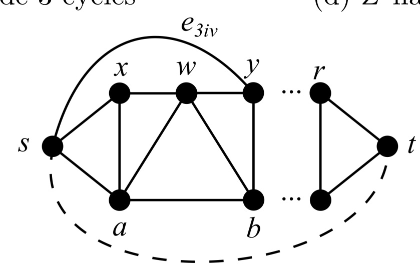
(b) Z has one inside 4-cycle



(c) Z has three inside 3-cycles



(d) Z has one inside 4-cycle



(e) Z has three inside 3-cycles

FIGURE 2.58: Arrangements of Z when a and b have minimal degree, for length of L being exactly two and Z having inside vertices a and b in common with Q_j.

$\widetilde{AW_{5c2}} + e_{3i}$, where $e_{3i} = (st, 3)$, has an S_1-minor. Thus, by Lemma 2.74, the graph G has an S_1-minor.

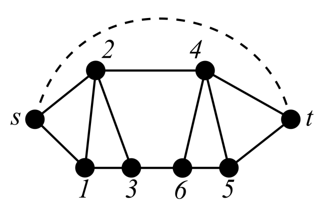
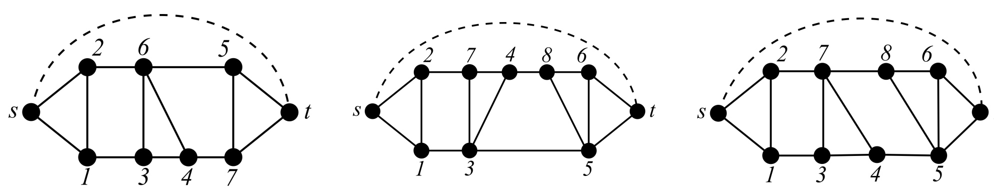


FIGURE 2.59: AW_{5c2} or $\widetilde{AW_{5c2}}$

Now assume Z has an inside 4-cycle, and $deg_{G\setminus e}(b) \geq 4$. We impose the additional restriction that $deg_{G\setminus e}(a) = 3$; otherwise, we have already found a contradiction in the G that results from the e_3-undeletion. An example is sketched in Figure 2.58b.

We will break down our argument based on the non-Z inside cycles of $G\backslash e$, and whether $G\backslash e$ is an identified or unidentified daisy chain.

If $G\backslash e$ is an unidentified daisy chain and all the non-Z inside cycles are 3-cycles that meet Q_j only in vertex b, then G is a member of $\mathcal{S}$ by Lemma 2.73. However, if one of the non-Z inside cycles is a 4-cycle, or if one of the non-Z inside cycles meets Q_j in a vertex besides b, then G contains as a minor $K_{c2.2} + e_{iv}$, where $e_{iv} = (t, 3)$. The graph $K_{c2.2} + e_{iv}$ has an S_1-minor, and, therefore, G has an S_1-minor, as well.



(a) $K_{c2.2}$ or $\widetilde{K_{c2.2}}$ (b) $AW_{6c1.2}$ or $\widetilde{AW_{6c1.2}}$ (c) $A_{c2.2}$ or $\widetilde{A_{c2.2}}$

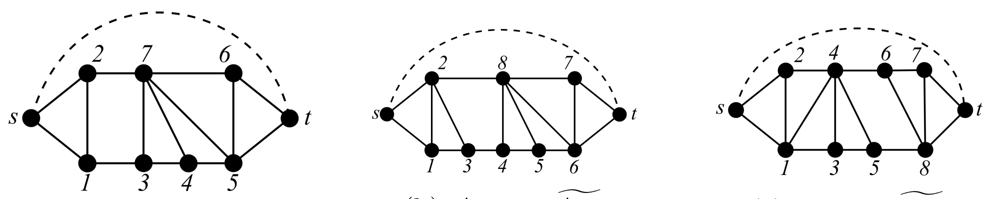
FIGURE 2.60: Selected daisy chains.

If $G\backslash e$ is an identified daisy chain and all the non-Z inside cycles are incident with vertex b, then G is a member of $\mathcal{S}$ by Lemma 2.73. However, suppose one of the non-Z inside cycles is not incident with b (that is, some non-Z inside cycle associated with an ear whose index was at most $k - 2$ has a vertex on Q_j that is not b). Then the graph G resulting from the undeletion has $\widetilde{AW_{6c1.2}} + e_{3iv}$ or $\widetilde{A_{c2.2}} + e_{3iv}$ as a minor, with $e_{3iv} = (st, 4)$ in either graph. The graph $\widetilde{AW_{6c1.2}} + e_{3iv}$ has an S_1-minor and $\widetilde{A_{c2.2}} + e_{3iv}$ has a U-minor. Thus, G has as a proper minor a known excluded minor of $\mathcal{M}_1$.

Now assume Z has only inside 3-cycles, and $deg_{G\backslash e}(b) \geq 5$. We impose the additional restriction that $deg_{G\backslash e}(a) = 3$, as otherwise we have already produced a contradiction for the graph G that results from the e_3-undeletion. An example is sketched in Figure 2.58a. We will again break down our argument based on the

non-Z inside cycles of $G\backslash e$, and whether $G\backslash e$ is an identified or unidentified daisy chain.

If $G\backslash e$ is an unidentified daisy chain and all the non-Z inside cycles are 3-cycles that meet Q_j only in vertex b, then G is a member of $\mathcal{S}$ by Lemma 2.73. However, if one of the non-Z inside cycles is a 4-cycle, or if one of the non-Z inside 3-cycles meets Q_j in a vertex besides b, then G contains as a minor $A_{c2} + e_{iv}$, where $e_{iv} = (t, 3)$, and $A_{c2} + e_{iv}$ has an S_1-minor. Thus, G has an S_1-minor.



(a) A_{c2} or $\widetilde{A_{c2}}$ (b) A_{1c2} or $\widetilde{A_{1c2}}$ (c) C_{1c4} or $\widetilde{C_{1c4}}$

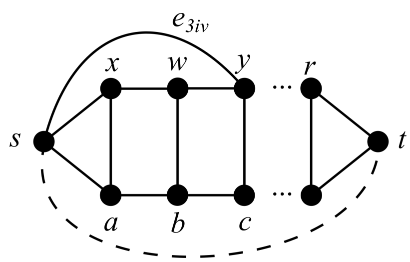
FIGURE 2.61: Selected daisy chains.

If $G\backslash e$ is an identified daisy chain and all its non-Z inside cycles are incident with vertex b, then G is a member of $\mathcal{S}$ by Lemma 2.73. However, suppose one of the non-Z inside cycles is not incident with b (that is, some non-Z inside cycle associated with an ear whose index was at most $k-2$ has a vertex on Q_j that is not b). Then the graph G resulting from the undeletion has as a minor $\widetilde{A_{1c2}} + e_{3iv}$ with $e_{3iv} = (st, 4)$, or $\widetilde{C_{1c4}} + e_{3iv}$ with $e_{3iv} = (st, 5)$. The graph $\widetilde{A_{1c2}} + e_{3iv}$ has an S_1-minor, and $\widetilde{C_{1c4}} + e_{3iv}$ has a U-minor. Thus, G has as a proper minor a known excluded minor of $\mathcal{M}_1$.

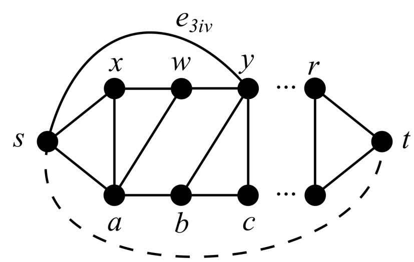
2.74.5. $V(Z) \cap V(Q_j) \subseteq \{a, b, c\}$

Finally, assume the cycles of Z meet at least three inside vertices of Q_j, including a, b, and c. Let a and c be the endpoints of the longest subpath in Q_j between two of the common inside vertices, with $a - c$ having internal vertex b. Given the lengths of L and Q_j, we know from Lemma 2.20 that there may be two or more

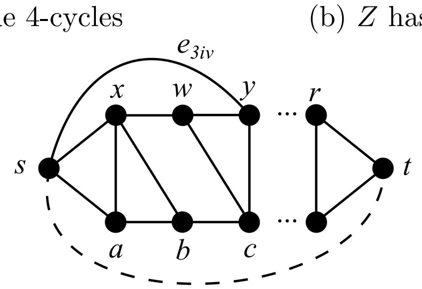
inside 4-cycles in Z. Of course, there may also be one or none. To streamline the proof, we observe that any graph G resulting from our undeletion of $G\backslash e$ will have the structure shown by one of the sketches in Figure 2.62, perhaps after deletions and contractions of inside cycles of Z. We show each of these three situations leads to G having a U-minor or an S_1-minor.



(a) Z has two inside 4-cycles

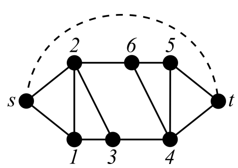


(b) Z has one inside 4-cycle

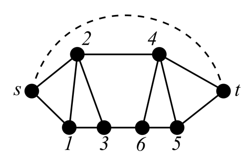


(c) Z has one inside 4-cycle

FIGURE 2.62: Arrangements of the minimal number of cycles Z contains, for length of L being two and Z having vertices a, b, and c in common with Q_j.



(a) K_{c3} or $\widetilde{K_{c3}}$



(b) AW_{5c2} or $\widetilde{AW_{5c2}}$

FIGURE 2.63: Selected daisy chains.

We know there is a non-Z inside cycle in $G\backslash e$. Then, any G with the structure shown in Figure 2.62a or Figure 2.62c has as a minor $\widetilde{K_{c3}} + e_{3i}$ with $e_{3i} = (st, 3)$. We have previously found that $\widetilde{K_{c3}} + e_{3i}$ has a U-minor, and thus, by Lemma 2.74, G also has a U-minor. If G has the structure shown in Figure 2.62b, then G has

as a minor $\widetilde{AW_{5c2}} + e_{3i}$ with $e_{3i} = (st, 3)$. We saw previously that $\widetilde{AW_{5c2}} + e_{3i}$ has an S_1-minor. Hence, by Lemma 2.74, G has an S_1-minor.

This concludes our proof of Case 2.

2.8 Proof of Case 3

This proof closely follows one originally presented by Warshauer [28] for excluded minors of $\mathcal{S}$.

Let G be an excluded minor of $\mathcal{M}_1$. Suppose $G' = G\backslash e$ is 3-connected and a member of $\mathcal{S}$. We may assume G' is not in $\mathcal{S}^*$, as this was covered by Case 2 in Section 2.7. By Theorem 2.36, there exist vertices u and v in $V(G')$ such that $G' - \{u, v\} \cong T$, where T is a tree. Neither u nor v is an endpoint of e, or else $G - \{u, v\}$ would be a tree and therefore a member of $\mathcal{S}$, which is a contradiction. Thus, $T + e$ has a cycle C as a subgraph. Therefore, by adding edge e and vertices u and v to T, we can rebuild excluded minor G. We break the remainder of the proof into subcases based on how many leaves T has, and then attempt to reconstruct G from $T + e$ by considering the different adjacencies possible for u and v. The main subcases are as follows:

(A) T has two leaves;

(B) T has three leaves; and

(C) T has at least four leaves.

2.75. *Subcase A: tree T has two leaves*

We let $e = (r, t)$, where $G\backslash e$ is 3-connected and a member of $\mathcal{S}$. The only degree-3 vertices in $T + e$ are r and t, since T is a path. There is one cycle subgraph of $T + e$, which is C. Each of r and t in $T + e$ meets a unique maximal path that has no edge from C; such a path is a *tail* of C.

See Figure 2.64 for a sketch of $T+e$. Let l_j and m_k be the leaves of $T+e$, where $1 \leq j, k$. We partition the non-leaf vertices of $T+e$ into three sets:

$$L = \{l_i : 1 \leq i \leq j-1\} \cup \{r\},$$

$$M = \{m_i : 1 \leq i \leq k-1\} \cup \{t\}, \text{and}$$

$$S = \{s_i : 1 \leq i \leq n\}.$$

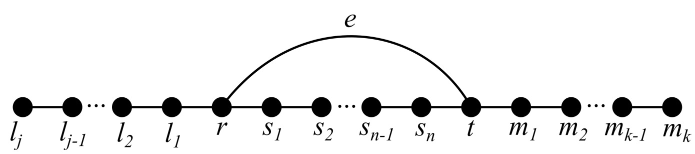


FIGURE 2.64: $T+e$ with 2 tails.

Now, it is possible for $T+e$ to have two tails, no tails, or only one tail. These three situations arise depending on whether neither, both, or one of the endpoints of e coincide with the leaves l_j and m_k of T. We will separately consider each situation for the endpoints of e in the following three subsections. For each, we analyze the adjacencies of u and v, and thus reconstruct the possible excluded minor G.

2.75.1. *$T+e$ has two tails*

It is possible that either of our two tails to have exactly one vertex – that is, $L = \{r\}$ or $M = \{t\}$, since the leaves l_j and m_k of T may be neighbors of the endpoints of e in $T+e$. Hence, $|L|, |M| \geq 1$ in this subcase. Furthermore, to avoid e being a parallel edge, we must have $|S| \geq 1$.

Now, we attempt to reconstruct G by adding u and v to $T+e$. As $G\backslash e$ is 3-connected, we know u and v are each adjacent to both leaves l_j and m_k of T. Any l_i, m_i, or s_i that is in T must be adjacent to at least one of u and v, as well. Without loss of generality, let us assume edge (u, s_1) exists.

We consider the possible adjacencies of v. Either v is adjacent to vertices from both tails, neither tail, or just one tail. If v is adjacent to one or more vertices from each of L and M, then G is isomorphic to R or contains it as a proper minor.

If v is adjacent to no vertices of $L \cup M$, then, since $G \backslash e$ is 3-connected, u must be adjacent to r, t, and any other vertices in $L \cup M$. In addition, v must be adjacent to either u or some $s_i \in S$. Suppose we have edge (v, s_i) for some $i \geq 1$; then, once again, G is isomorphic to or properly contains R (and K_5, as well). On the other hand, if v is not adjacent to any s_i, then $G - \{u, r\}$ is a tree, even if we include edge (v, u). Therefore G is a member of $\mathcal{S}$ by Theorem 2.36, a contradiction.

Finally, if v has a neighbor in exactly one tail, without loss of generality L, then u must be adjacent to every vertex in M, thanks to the connectivity of $G \backslash e$. Consider the other adjacencies possible for v. Should v have a neighbor in S, then G will have a K_5-minor, which is a contradiction. Thus v is not adjacent to any s_i in S, and we already assumed v is not adjacent to any vertex in M. So v is adjacent only to vertices in L, and perhaps to u. The structure of the graph G we construct will change depending on the number of vertices in L.

Say $|L| = 1$, forcing v to be adjacent to r; then $G - \{u, r\}$ is a tree − a contradiction. Let $|L| \geq 2$. If r remains the only neighbor of v in L (and therefore u is adjacent to all l_i for $i \leq j$), we still have $G - \{u, r\}$ is a tree. Thus we must have some edge (v, l_i) for $i \geq 1$. If u is also adjacent to some vertex in L, then G has a proper S_1-minor. If u is not adjacent to any vertex of L, then $G \in \mathcal{S}^*$, whether edge (u, v) is included or not. We can see $G \in \mathcal{S}^*$ by splitting r into two vertices whose neighbors are $\{s_1, t\}$ and $\{v, l_1\}$, respectively, as this produces a series-parallel graph.

2.75.2. *$T + e$ has no tails*

Recall that C is the cycle we form by adding e to T. For this subcase, $T + e$ is precisely C, because $T + e$ has no tails. We will break down our argument based on the length of C: if C is a 3-cycle, a 4-cycle, or a p-cycle where $p \geq 5$. For each length of C, we add back u and v to $T + e$ and consider their possible adjacencies, to reconstruct excluded minor G. As $G \backslash e$ is 3-connected, we always have u and v each adjacent to both leaves r and t of T.

Suppose C is a 3-cycle. Then the only vertices in $T + e$ are r, t, and s_1. We know u and v are both adjacent to r and t because $G \backslash e$ is 3-connected, and, for the same reason, s_1 must be adjacent to at least one of u and v. Assume we have edge (u, s_1), without loss of generality. There are two other edges to we can consider adding, (v, u) and (v, s_1), and we must use at least one of them so that the degee of v in G is three or greater. If we add only one of these edges, then $G - \{r, t\}$ is a tree. However, if we add both of edges (v, u) and (v, s_1), we have $G \cong K_5$, a contradiction, since K_5 is an excluded minor of $\mathcal{M}_1$ whose connectivity is four.

Suppose C is a 4-cycle. Then $S = \{s_1, s_2\}$, and the vertices of $T + e$ are r, t, s_1, and s_2. Again, due to the 3-connectedness of $G \backslash e$, the vertices u and v must both be adjacent to the two endpoints of T, that is, r and t. Moreover, each of s_1 and s_2 must be adjacent to at least one of u and v. Without loss of generality, let s_1 be adjacent to u. Now, either u and v are adjacent, or not.

Assume first that we have edge (u, v). We still must make s_2 adjacent to one of u and v; and we observe that v must be adjacent to s_1 or s_2, or else $G - \{u, t\}$ is a tree. Let v be adjacent to s_2; then G has a K_5-minor. If v is not adjacent to s_2, we must have edges (v, s_1) and (u, s_2); however, $G/(s_1, s_2) \cong K_5$.

So assume there is no edge (u, v). We still need s_2 adjacent to one of u and v; and we still have to make v adjacent to s_1 or s_2, so that v has degree at least three. Let v be adjacent to s_2. Then $G - \{s_2, t\}$ and $G - \{r, s_1\}$ are trees, unless

we also include both of edges (u, s_2) and (v, s_1). This leaves $G \cong K_{2,2,2}$, which is a contradiction, since $K_{2,2,2}$ is an excluded minor of $\mathcal{M}_1$ with connectivity four. Thus, assume v is not adjacent to s_2. Then we have edges (v, s_1) and (u, s_2), as these are the only edges we can use to get v and s_2 up to degree three. There are no other edges we can add, but $G - \{r, s_1\}$ is a tree, so we have a contradiction.

Suppose C is a p-cycle, where $p \geq 5$. So $S = \{s_i : 1 \leq i \leq n, n \geq 3\}$, and $V(T+e) = \{r, t\} \cup S$. As usual, u and v are each adjacent to both of r and t, since $G \backslash e$ is 3-connected. If no vertex in S is adjacent to v, all vertices in S are adjacent to u. Then $G - \{u, t\}$ is a tree, and, by Theorem 2.36, G is a member of $\mathcal{S}$ – a contradiction. By symmetry, $G - \{v, t\}$ is a tree if no vertex in S is adjacent to u. Therefore, each of u and v is adjacent to one or more vertices of S.

In addition, since G is 3-connected, every vertex in S must be adjacent to at least one of u and v. Thus, at least one of u and v, say u, is adjacent to two or more vertices of S. Suppose u is adjacent to s_{u_1} and s_{u_2}. Meanwhile, suppose v is adjacent to s_v. If the indices satisfy $u_1 < v < u_2$, then G is isomorphic to R or contains it as a proper minor. Observe that having the indices satisfy $v < u_1, u_2$ is equivalent to having the indices satisfy $u_1, u_2 < v$, by the symmetry of $T + e$. Hence, it will suffice to consider when the indices satisfy $u_1, u_2 < v$ to complete the proof.

So, suppose the indices satisfy $u_1, u_2 < v$. If $s_v = s_n$ and v is not adjacent to other vertices of S or to u, then G is a daisy chain. To see this, split s_n into vertices s_{n_a} and s_{n_b} that have neighbor sets $N(s_{n_a}) = \{t, v\}$ and $N(s_{n_b}) = \{s_{n-1}, u\}$ (or $\{s_{n-1}\}$, if u not adjacent to s_n). If $s_v = s_n$ and v is adjacent to no other vertices of S, but we include (v, u), then G has a K_5-minor. If v is adjacent to s_n and some other s_i, then we either have the $u_1 < v < u_2$ case, or the following. Say the only members of S to which v is adjacent are $s_{v_1}, s_{v_2}, \ldots, s_{v_m} = s_n$, which form a

subpath in T. If u is not adjacent to v or any s_{v_i} other than possibly s_{v_1}, then G is a daisy chain; we can split s_{v_1} and obtain a series-parallel graph. If u is adjacent to v, then G has a proper K_5-minor. If u is adjacent to any s_{v_i} where $i \neq 1$, then G has a proper $K_{2,2,2}$-minor.

2.75.3. *$T + e$ has one tail*

Without loss of generality, let the one tail be M. Recall that C is the cycle in $T + e$. We will once again break the analysis of this subcase down by the length of cycle C, considering separately when C is a 3-cycle, a 4-cycle, and a j-cycle with $j \geq 5$. For each length of C, we add back u and v to $T + e$, considering their possible adjacencies, to reconstruct excluded minor G. As $G \backslash e$ is 3-connected, we always have that u and v are each adjacent to both leaves r and m_k of T.

2.75.4. *$T + e$ has one tail and C is a 3-cycle*

Let C be a 3-cycle; then $S = \{s_1\}$. We divide the argument based on the cardinality of the tail M; we check $|M| = 1, 2, 3$, and k, where $k \geq 4$. Assume the tail has cardinality one; that is, $M = \{t\}$. Due to the 3-connectedness of $G \backslash e$, the vertices u and v must both be adjacent to the endpoints of T, namely, r and m_k. One of u and v, let us say u, is adjacent to t, since $G \backslash e$ is 3-connected. Furthermore, for this case, we must have edge (v, s_1), or else $G - \{u, t\}$ is a tree; we also have edge (u, s_1), or $G - \{v, t\}$ is a tree. Thus, G has a K_5-minor, which is a contradiction.

When $|M| \geq 2$, some adjacencies for u and v cannot occur, and other adjacencies must be present. Knowing these before we look at the remaining choices for $|M|$ will greatly shorten our case analysis. We know u and v are both adjacent to the endpoints of T, namely, r and m_k. At least one of u and v is adjacent to s_1, and likewise at least one of u and v is adjacent to t, by the 3-connectedness of $G \backslash e$. Pick edge (u, s_1). If both u and v are adjacent to s_1, however, G has a K_5-minor.

So, we are forbidden from using (v, s_1) now. If v is also not adjacent to any m_i for $1 \leq i < k$, then $G - \{u, t\}$ is a tree, meaning $G \in \mathcal{S}$. Hence, there is some edge (v, m_i). So long as we have edge (u, t), we get G has a U-minor (see Figure 2.65 for an example). So assume we have (v, t), but not (u, t).

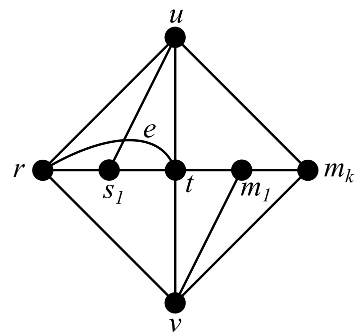


FIGURE 2.65: Delete edge (v, t) for a U-minor.

To summarize, when $|M| \geq 2$, the edges we know to be present, aside from u and v being adjacent to the endpoints, are as follows: (u, s_1), (v, t), and (v, m_i) for some i. We do not have the following edges: (v, s_1) and (u, t). It remains to check the consequences for G with the remaining possible edges incident with u and v.

Let $|M| = 2$. Then $M = \{m_1, t\}$. We are forced to have edge (v, m_1), since $i = 1$ when $|M| = 2$; it remains to check what happens with G when edge (u, m_1) or (u, v) exist. Having both (u, m_1) and (u, v) as edges, the graph G has a proper S_1-minor, a contradiction. With at most one of (u, m_1) and (u, v) as an edge, G is a daisy chain. We can split vertex u into two vertices u_1 and u_2 where $N(u_1) = \{r, s_1\}$ and $N(u_2) = \{m_2\}$ or $\{m_2, v\}$, and thus produce a series-parallel graph from G.

Now, let $|M| = 3$. Then $M = \{m_1, m_2, t\}$. We know G must have at least one of edges (v, m_1) and (v, m_2); and it may have any of edges (u, v), (u, m_1), and (u, m_2). The options are listed in Table 2.5

Assume $|M| = k \geq 4$. Then $M = \{m_i : 1 \leq i \leq k - 1\} \cup \{t\}$. We know there exists an edge (v, m_y) for $1 \leq y \leq k - 1$; we may also have any or none of edges

TABLE 2.5: Possible edges in G when $|M| = 3$.

Select which of edges (v, m_i) are in G	Decide if any edge (u, m_i) is in G	Resulting G	G if (u, v) is also present
(v, m_1)	(u, m_2)	daisy chain	has S_1- and U-minors
(v, m_1)	(u, m_1), (u, m_2)	has proper S_1-minor	still has proper S_1-minor
(v, m_2)	(u, m_1)	has S_1-minor	has proper S_1-minor
(v, m_2)	(u, m_1), (u, m_2)	has S_1-minor	has proper S_1-minor
(v, m_1), (v, m_2)	(u, m_1), (u, m_2)	has S_1-minor	has proper S_1-minor
(v, m_1), (v, m_2)	(u, m_1)	has an S_1-minor	has proper S_1-minor
(v, m_1), (v, m_2)	(u, m_2)	daisy chain	has S_1- and U-minors
(v, m_1), (v, m_2)	none	daisy chain	still a daisy chain

(u, v) and (u, m_x), for $1 \leq x \leq k-1$. Notice that if there is no edge (u, m_x) for any x, then G is a daisy chain, whether (u, v) is present or not. We can split t into two vertices t_1 and t_2 having $N(t_1) = \{r, s_1\}$ and $N(t_2) = \{v, m_1\}$, and the graph resulting from this split of G is series-parallel.

Thus, there is an edge (u, m_x) for some x. Suppose the only such edge is (u, m_{k-1}); as Table 2.5 indicates, this also leaves us with G a daisy chain when (u, v) is not present. We can split u into vertices u_1 and u_2 whose neighbor sets are $\{m_k, m_{k-1}\}$ and $\{s_1, r\}$. If edge (u, v) is present, then G has both S_1- and U-minors.

So, assume that u is adjacent to m_x for $1 \leq x < k-1$, with the possibility for an additional edge between u and m_{k-1}. If we have edges (u, m_x) and (v, m_y), and the indices satisfy $x < y$, then G has the graph in Figure 2.66 as a minor, which itself contains an S_1-minor. Suppose instead that $y < x$ for every pair of edges

(u, m_x) and (v, m_y). Then G has as a minor the graph in Figure 2.67, which has an S_1-minor.

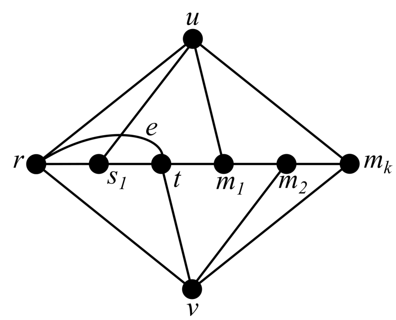


FIGURE 2.66: When $x < y$, the graph G contains the above graph as a minor. Contracting edge (u, m_1) leads to an S_1-minor.

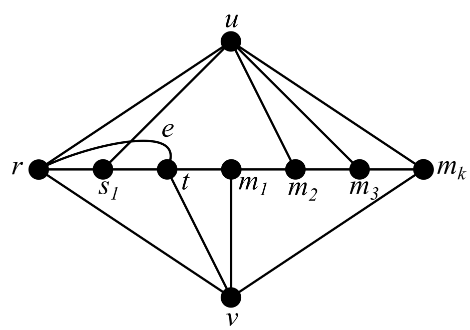


FIGURE 2.67: When $y < x$, the graph G contains the above graph as a minor. Contracting edges (m_1, m_2) and (v, m_k) leads to an S_1-minor.

2.75.5. *$T + e$ has one tail and C is a 4-cycle*

Let C be a 4-cycle. Then $S = \{s_1, s_2\}$. As usual, $G\backslash e$ being 3-connected forces us to have each of u and v adjacent to both endpoints r and m_k of T. Further, at least one of u and v must be adjacent to t, say u. We proceed by considering which of u and v are adjacent to s_1 and s_2.

Suppose v is adjacent to both s_1 and s_2. Then $G - \{v, t\}$ is a tree, unless we have another edge that is incident with u. Note that adding edge (u, v) does not change $G - \{v, t\}$ being a tree. If u is adjacent to one of $\{s_1, s_2\}$, then G has a K_5-minor. If u is adjacent to some $m_i \in M$, then G has an S_1-minor.

Next, suppose we have edges (v, s_1) and (u, s_2); then, G has a K_5-minor. Similarly, when (v, s_2) and (u, s_1) are edges, we find G has a K_5-minor.

Lastly, let u be adjacent to both s_1 and s_2. Then $G - \{u, r\}$ is a tree unless we have an additional edge that is incident with v. If v is adjacent to s_1 or s_2, then G has a K_5-minor. If v is adjacent to $m_i \in M$, then G has a U-minor. If v is not adjacent to any s_i or m_i, then the only options are edges (u, v) and (v, t); however, both of these leave $G - \{u, t\}$ as a tree, and therefore G is a member of $\mathcal{S}$ by Theorem 2.36.

2.75.6. *$T + e$ has one tail, and C is a j-cycle with $j \geq 5$*

Let C be a j-cycle with $j \geq 5$. So $S = \{s_i : 1 \leq i \leq n \text{ and } n \geq 3\}$. As usual, u and v are each adjacent to both of r and m_k, since $G \backslash e$ is 3-connected. Since G is 3-connected, one of u and v, say u, is adjacent to t. We proceed by considering whether one of u and v is adjacent to all the vertices in S or not, and then how u and v are adjacent to vertices in M.

Suppose every vertex in S is adjacent to u, but none of the vertices of S are adjacent to v. If no vertex in M is adjacent to v either (and, therefore, all vertices in M are adjacent to u), then $G - \{u, t\}$ is a tree. So, by Theorem 2.36, G is a member of $\mathcal{S}$ – a contradiction. If v is adjacent to at least one member of $M - \{t\}$, then G has a proper U-minor. (It does not matter if v is adjacent to t or not.) On the other hand, suppose every vertex in S is adjacent to v but none of them are adjacent to u. If u is also not adjacent to any vertices of $M - \{t\}$, we get that $G - \{v, t\}$ is a tree. If u is adjacent to some vertex in $M - \{t\}$, then G has a proper U-minor.

Now assume that neither u nor v is adjacent to every vertex in S. However, since $G \backslash e$ is 3-connected, every vertex in S must be adjacent to at least one of u and v.

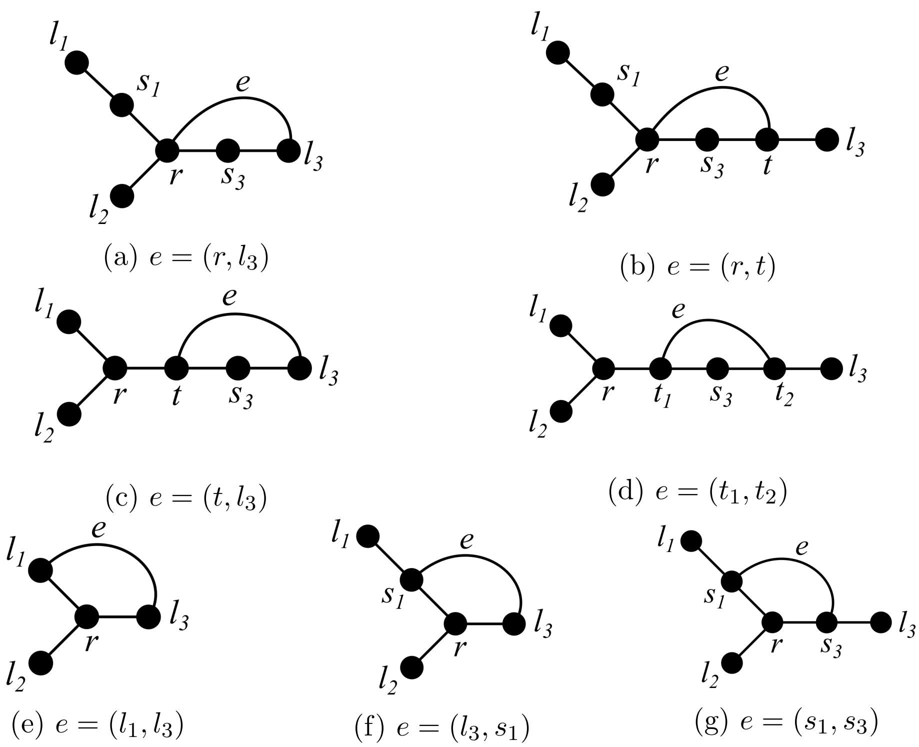
Therefore, at least one of u and v is adjacent to two or more vertices of S. Suppose it is u, adjacent to s_{x_1} and s_{x_2}. Meanwhile, v is adjacent to s_y. If the indices satisfy $x_1 < y < x_2$, then G has R as a proper minor. If $s_y = s_n$ (or the only vertices of S to which v is adjacent are $s_{y_1}, s_{y_2}, \ldots, s_{y_h} = s_n$, which form a path), then G has a proper K_5-minor. Similarly, if $s_y = s_1$ (or the only vertices of S to which v is adjacent are $s_1 = s_{y_1}, s_{y_2}, \ldots, s_{y_h}$, which form a path), then G again has a proper K_5-minor. Alternatively, suppose v is adjacent to at least two vertices of S, call them s_{y_1} and s_{y_2}, while u is adjacent to s_x. Then G has a proper K_5-minor.

2.76. *Subcase B: tree T has three leaves*

Now that T has precisely three leaves, it will have exactly one degree-3 vertex, which we call r. Label the leaves l_1, l_2, and l_3. There is a unique path L_i from r to each leaf l_i. Up to isomorphism, there are seven choices on T for the endpoints of e among the leaves, the internal vertices of the paths L_i, and r. Examples of these seven possibilities for $T+e$ are shown in Figure 2.68. For the proof of this subcase, we take each $T+e$ and consider the possible adjacencies of u and v, when we add them to $T+e$ to reconstruct excluded minor G.

2.76.1. $e = (r, l_3)$

To avoid e being a parallel edge, there has to be at least one internal vertex s_3 on L_3. See Figure 2.68a. Since $G \backslash e$ is 3-connected, each of u and v is adjacent to all members of $\{l_1, l_2, l_3\}$. In addition, any internal vertices on the L_i paths of T are adjacent to at least one of u and v. Without loss of generality, let u be adjacent to s_3. Notice that $G - \{u, r\}$ will be a tree unless there is another edge incident with v. So there has to be an edge from v to an internal vertex of some L_i. Suppose v is adjacent to an internal vertex of L_3, such as s_3 in Figure 2.68a, or some other internal vertex of L_3. Then G has a proper K_5-minor. If v is instead



(a) $e = (r, l_3)$ (b) $e = (r, t)$ (c) $e = (t, l_3)$ (d) $e = (t_1, t_2)$ (e) $e = (l_1, l_3)$ (f) $e = (l_3, s_1)$ (g) $e = (s_1, s_3)$

FIGURE 2.68: If T has three leaves, then one of the above graphs illustrates the selection of the endpoints of e. All edges pictured, except e, represent possible paths of length at least one.

adjacent to s_1, an internal vertex of L_1, then G has a proper S_1-minor. Note that, by symmetry, v being adjacent to an internal vertex of L_2 also leads to G having a proper S_1-minor.

2.76.2. $e = (r, t)$

To avoid e being a parallel edge, there has to be at least one internal vertex s_3 on the $r - t$ subpath of L_3. See Figure 2.68b. As always, each of u and v must be adjacent to all members of $\{l_1, l_2, l_3\}$, due to $G \backslash e$ being 3-connected. For the same reason, s_3 must be adjacent to one of u and v, and we again choose edge (u, s_3). Now $G - \{u, r\}$ will be a tree, unless we have an edge between v and an internal vertex of some L_i. Suppose we have v adjacent to some internal vertex of $r - t$, such as edge (v, s_3); then G has a K_5-minor. Suppose we have edge (v, s_1), where s_1 is an internal vertex on L_1; then G has an S_1-minor. Notice that, by symmetry,

G also has an S_1-minor if we make v adjacent to an internal vertex of L_2. Lastly, suppose we have edge (v, t), or v adjacent to any internal vertex on the $t - l_3$ subpath of L_3. Then G has an R-minor.

2.76.3. $e = (t, l_3)$

This case is sketched in Figure 2.68c. Since $G\backslash e$ is 3-connected, each of u and v is adjacent to all members of $\{l_1, l_2, l_3\}$. Moreover, s_3 must be adjacent to one of u and v, as must t. Without loss of generality, say we have (u, s_3). If we also have edge (u, t), then G has a proper S_1-minor. If we have (v, t) instead, then G properly contains an S-minor.

2.76.4. $e = (t_1, t_2)$

See Figure 2.68d for an illustration. To prevent e being a parallel edge, there must be a vertex s_3 on the $t_1 - t_2$ subpath of L_3. The 3-connectedness of $G\backslash e$ results in the usual adjacencies. Each of u and v is adjacent to all members of $\{l_1, l_2, l_3\}$, and vertex s_3 must be adjacent to one of u and v, as must t_1. Without loss of generality, let u be adjacent to s_3. No matter which of u and v is adjacent to t_1, then, we deduce from our work in 2.76.3 that G has an S_1-minor or an S-minor.

2.76.5. $e = (l_1, l_3)$

See Figure 2.68e for an illustration of $T+e$. Despite the fact that e has two leaves of T as endpoints, we still have to make sure $G\backslash e$ is 3-connected, and therefore u and v are each adjacent to all the leaves. Observe that $G - \{l_1, l_3\}$ will be a tree, unless there is another edge in G, one of whose endpoints is u or v, say u. The options for the neighbor of u are r, an internal vertex of L_i (in which case, we can contract edges so that u is adjacent to the conglomerate vertex r), or v. Any of these results in a graph G that has a K_5-minor.

2.76.6. $e = (l_3, s_1)$

A sketch of this subcase is shown in Figure 2.68f. Since $G \backslash e$ is 3-connected, each of u and v is adjacent to all the leaves of T. Additionally, at least one of u and v must be adjacent to s_1. Without loss of generality, say we have edge (u, s_1); then G has a K_5-minor, a contradiction.

2.76.7. $e = (s_1, s_3)$

See Figure 2.68g. Since $G \backslash e$ is 3-connected, each of u and v is adjacent to all the leaves of T. Additionally, at least one of (u, s_1) and (v, s_1), say the former, must be an edge of G. Then G has a K_5-minor, a contradiction.

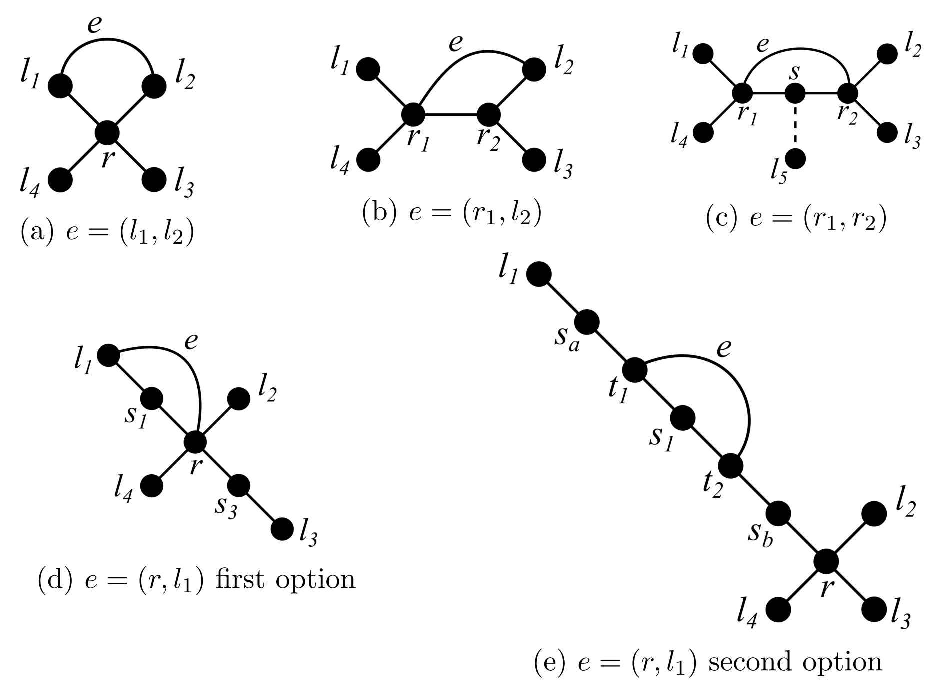
2.77. *Subcase C: T has at least four leaves*

Figure 2.69 shows examples of the two situations about to be described. Let l_1, l_2, l_3, and l_4 be four of the leaves of T. Then T may have a vertex r whose degree is at least four, such that all paths L_i from r to l_i in T have precisely $\{r\}$ for their pairwise vertex intersections. Alternatively, T may have two vertices r_1 and r_2 of degree at least three such that paths $L_1 = r_1 - l_1$ and $L_2 = r_1 - l_2$ intersect on $\{r_1\}$, while paths $L_3 = r_2 - l_3$ and $L_4 = r_2 - l_4$ intersect on $\{r_2\}$. All other pairwise intersections among the L_i paths are empty. In this situation, let P_r be the path in T between r_1 and r_2.

(a) T_r with degree-4 vertex r (b) $T_{r_1 r_2}$ with degree-3 vertices r_1 and r_2

FIGURE 2.69: Examples of the two structures of T with four leaves. Note that these are the smallest possible examples of T.

So, in this subcase, we have two distinct structures possible for our tree T. We must also choose the endpoints of e from among the leaves of T, the internal vertices of legs L_i, and r or the vertices of P_r. The main situations are illustrated in Figure 2.70. We streamline our proof by considering $T_r + e$ and $T_{r_1r_2} + e$ structures together, when contracting P_r to a single vertex does not alter the excluded minor contained in G. Thus, we have the five subcases that follow, where we consider the adjacencies of u and v when we add them back to $T + e$ to reconstruct excluded minor G.



(a) $e = (l_1, l_2)$

(b) $e = (r_1, l_2)$

(c) $e = (r_1, r_2)$

(d) $e = (r, l_1)$ first option

(e) $e = (r, l_1)$ second option

FIGURE 2.70: The five possibilities for $T + e$ in Subcase C, where T has at least four leaves. All edges except e represent paths of length at least one, and, for clarity, all but four leaves have been deleted from T.

2.77.1. $e = (l_1, l_2)$

This subcase covers both a T_r or a $T_{r_1r_2}$ structure for T. The endpoints of e may both be leaves of T, as shown in Figure 2.70a; a leaf l_h and an internal vertex of L_i where $i \neq h$; or internal vertices of distinct paths L_i and L_h. All of these

possibilities for $T + e$ have the structure shown in Figure 2.70a. Since $G\backslash e$ is 3-connected, vertices u and v are each adjacent to all of the leaves of T. Thus, G has a proper K_5-minor.

2.77.2. $e = (r_1, l_2)$

This subcase only arises when T has a $T_{r_1 r_2}$ structure. One endpoint of e is r_1 or an internal vertex of P_r, and the other endpoint is a leaf or an internal vertex of some L_i, such that r_1 is not a vertex of L_i. So $T + e$ has the structure shown in Figure 2.70b. Since $G\backslash e$ is 3-connected, each of u and v is adjacent to all the leaves of T. Therefore G has K_5 as a minor.

2.77.3. $e = (r_1, r_2)$

Again, this subcase only arises when T has a $T_{r_1 r_2}$ structure. Both endpoints of e are vertices of P_r. There must be an internal vertex s of P_r that is on the subpath between those endpoints of e, or e is a parallel edge. So $T + e$ has the structure shown in Figure 2.70c. Since $G\backslash e$ is 3-connected, each of u and v is adjacent to all the leaves of T. If $deg_T(s) = 2$, then s must be adjacent to one of u and v, since $G\backslash e$ is 3-connected. We can choose (s, u) without loss of generality, and then G has an R-minor. If $deg_T(s) \geq 3$, then there is a path from s to leaf l_5 that avoids both r_1 and r_2. Since all leaves are adjacent to both u and v by $G\backslash e$, we can contract the path from s to u down to a single edge, and, once again, G has an R-minor.

2.77.4. $e = (r, l_1)$

All that remains is the subcase where the endpoints of e both fall on one L_i, and it will therefore be irrelevant whether T has a T_r or a $T_{r_1 r_2}$ structure. We will assume we have contracted P_r down to a single vertex labeled r.

In this subcase, one endpoint of e is leaf l_1 or an internal vertex of L_1, and the other endpoint of e is an internal vertex of L_1 or r. Since e cannot be a parallel

edge, there is an internal vertex s_1 on subpath of L_1 between the two endpoints of e. See Figure 2.70d. Note s_1 has a path to u or v, say to u, that avoids r. However, $G - \{u, r\}$ will be a tree unless there is an additional edge in G that is incident with v. The options for the other endpoint of this edge are an internal vertex of L_1 or an internal vertex of L_i where $i \neq 1$. Consider the latter; s_3 is an internal vertex on some other L_i path, say L_3, and we have edge (v, s_3). Then G has an S_1-minor.

Now, consider when we have an additional edge incident with v, and the other endpoint is an internal vertex of L_1. There are three possible placements for this internal vertex. See Figure 2.70e. Let s_a be an internal vertex on L_1, on a shortest path from l_1 to an endpoint of e. Let s_b be an internal vertex of L_1, on a shortest path from r to an endpoint of e. If we have edge (v, s_1) (or any edge from v to a vertex on the subpath $t_1 - t_2$), then G has a K_5-minor. So G must have one of edges (v, s_a) and (v, s_b). Then G has a U-minor.

2.9 Proof of Case 4

We now assume that excluded minor G of $\mathcal{M}_1$ is minimally 3-connected – so there is some edge e such that G/e is 3-connected. We may also assume that G/e is a member of $\mathcal{S}$, since Section 2.6 dealt with Case 1, where $G/e \in \mathcal{S}^*$. We present a result on the structure of such an excluded minor G in Theorem 2.81. Then we use this information to show via detailed case analysis that G must be one of S, U, R, H_8, or Q_3. We begin with some results and definitions that will help us establish the structure of G. The first theorem we give is due to Halin [13].

Theorem 2.78. *In any minimally 3-connected graph H, each cycle of H contains at least two degree-3 vertices.*

A matroid M is *vertically* 3*-connected* if and only if $si(M)$ is 3-connected. An element e of M is a *vertically contractible element* if M/e is vertically 3-connected. Also, in a 3-connected matroid M, we say an element e of M is *essential* if neither $M\backslash e$ nor M/e is 3-connected. In [31], Wu states the next theorem, which is equivalent to a result independently proven by Cunningham [4] and Seymour [22].

Theorem 2.79. *Let M be a 3-connected matroid with at least one element. Then M contains at least one vertically contractible element.*

We require one additional definition and lemma, taken from Oxley and Wu [20]. In a simple, cosimple matroid M, let S be a subset of $E(M)$ that has at least three elements. Subset S is a *fan* in M if there is an ordering $(s_1, s_2, \ldots, s_n)$ of the elements of S such that, for all i in $\{1, 2, \ldots, n-2\}$,

(i) $\{s_i, s_{i+1}, s_{i+2}\}$ is a triange or a triad; and,

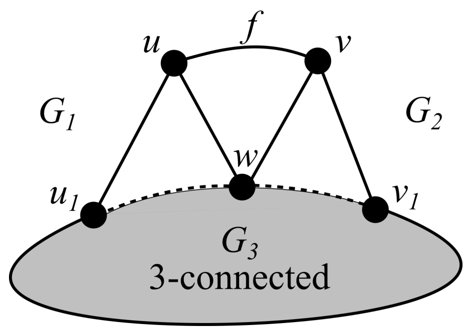
(ii) when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad; and when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

Lemma 2.80. *Let M be a 3-connected matroid with at least four elements, and suppose that M is not a wheel or a whirl. Let $(s_1, s_2, \ldots, s_n)$ be a maximal fan in M. If $n \geq 4$, then s_1 and s_n are non-essential. If $n = 3$, then, after a possible reordering, s_1 and s_3 are non-essential.*

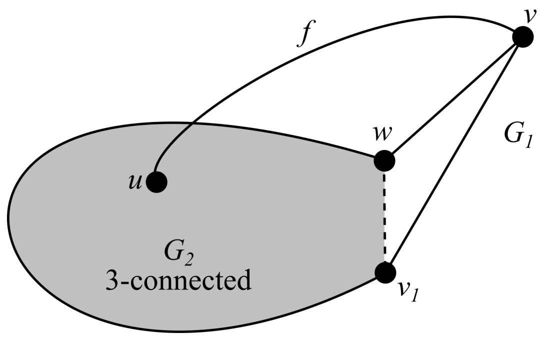
The following lemma is taken from Warshauer [28, pp. 42-43].

Theorem 2.81. *Let G be a minimally 3-connected excluded minor of $\mathcal{M}_1$. The structure of G is one of the three possibilities G_A, G_B, or G_C shown in Figure 2.71.*

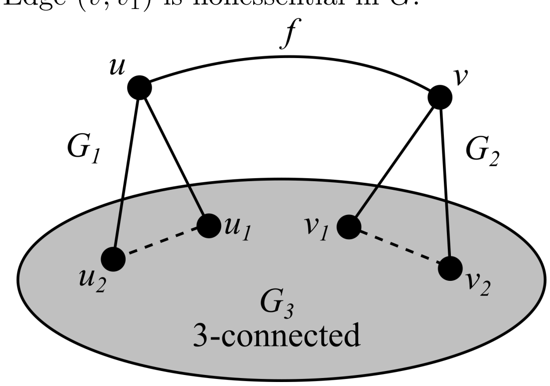
Proof. Consider the cycle matroid $M(G)$. By Theorem 2.79, there is a vertically contractible element in the dual $M^*(G)$, which we call f. Then the simplification



(a) The G_A structure of G. The shaded region represents 3-connected graph G_3, while G_1 and G_2 are 3-cycles on the vertices $\{u, u_1, w\}$ and $\{v, v_1, w\}$. Dotted edges are basepoints of the 2-sums between G_3 and G_1 or G_2. Edges (u, u_1) and (v, v_1) are nonessential in G.



(b) The G_B structure of G. The shaded region represents 3-connected graph G_2, while G_1 is a 3-cycle on the vertices $\{v, v_1, w\}$. The dotted edge is a basepoint of the 2-sum between G_2 and G_1. Edge (v, v_1) is nonessential in G.



(c) The G_C structure of G. The shaded region represents 3-connected graph G_3, while G_1 and G_2 are 3-cycles on the vertices $\{u, u_1, u_2\}$ and $\{v, v_1, v_2\}$. Dotted edges are basepoints of the 2-sums between G_3 and G_1 or G_2. Edges (u, u_2) and (v, v_2) are nonessential in G.

FIGURE 2.71: The three possible structures for a minimally 3-connected excluded minor G of $\mathcal{M}_1$. Dotted edges do not exist in G.

$si(M^*(G)/f) = si(M^*(G/f))$ is 3-connected. Taking the dual, the cosimplification $co(M(G\backslash f)) = co(M(G)\backslash f)$ is 3-connected. Now, $M(G)\backslash f$ has no coloops, since $G\backslash f$ is 2-connected. Then $G\backslash f$ is 2-connected up to series pairs, and it has at most two degree-2 vertices. These degree-2 vertices result from the deletion of f. We consider the canonical tree decomposition T of $G\backslash f$. We show that T must be a path, then restrict the locations of vertices with cycle labels.

By Oxley [17, Proposition 8.3.16], every 2-separation of a 2-connected matroid is displayed in the canonical tree decomposition by an edge of T or by a vertex of T that is labeled by a circuit or cocircuit. Now, if T has a cycle vertex label of length greater than three, there is no way to undelete f that will leave G free of 2-separations. Then G is not 3-connected, a contradiction. Thus, any cycle label of T has length three. Moreover, any cycle label of length three must be a leaf of tree.

If T has a vertex labeled by a bond, with at least three neighbor vertices having 3-connected labels, we once more cannot undelete f in a way that avoids leaving G with a 2-separation. So any bond labeling a vertex of T has size at most three. Therefore, the tree decomposition for $G\backslash f$ is a path. Note that G cannot be a daisy chain, since it is an excluded minor of $\mathcal{M}_1$.

Observe that there cannot be a subpath of T having two 3-connected vertex labels, possibly with a bond label between them; otherwise, the cosimplification of $G\backslash f$ is not 3-connected. The vertex labels of T that contain the former endpoints of f must be two triangles or a triangle and a 3-connected label.

So we have established that G has one of the structures shown in Figure 2.71. It remains to show that the dotted edges in these pictures do not occur, and that each of the nominated edges is non-essential. The first of these follows because, in each case, deleting f and cosimplifying must produce a simple graph. In the cases of

the G_A and G_C structures, it is a straightforward consequence of Tutte's Triangle Lemma (Lemma 1.14) that each of the nominated edges is non-essential. The same argument establishes the same conclusion in the case of the G_B structure, possibly after interchanging the labels on w and v_1, unless G has both (w, u) and (u, v_1) as edges.

Suppose G does have both (w, u) and (u, v_1) as edges. In $M(G)$, we have $((v, w), (w, u), f, (u, v_1), (v, v_1))$ as a fan. This fan is not maximal, as (v, v_1) is essential. Extend this fan to a maximal fan $(s_1, s_2, \ldots, s_n)$. Then $n \geq 6$, and each of $\{s_1, s_2, s_3\}$ and $\{s_{n-2}, s_{n-1}, s_n\}$ is a triad in $M(G)$. Since $M(G)/s_2$ has $\{s_3, s_4, s_5\}$ as a 2-separating set, Bixby's Lemma implies that $co(M(G)\backslash s_2)$ is 3-connected. By letting $s_1 = e$ and relabeling s_2 as f, we see that G has the structure specified in G_B.

□

Now that we have identified the three possible structures G_A, G_B, and G_C that our minimally 3-connected excluded minor G can have, we need to show that every such excluded minor is listed in Theorem 2.4. We break the argument into three main parts, based on the structure of G:

(A) the structure of G is that of G_A;

(B) the structure of G is that of G_B; and

(C) the structure of G is that of G_C.

In each of these cases, we have identified a particular nonessential edge e in G. Since G is minimally 3-connected, we know G/e is 3-connected. Moreover, we assume G/e is a member of $\mathcal{S}$, since Case 1 in Section 2.6 already covered G/e being in $\mathcal{S}^*$.

By Theorem 2.36, then, there are two vertices of G/e, call them x and y, whose deletion produces a tree T. One of these vertices, say x, must be the conglomerate vertex formed by the contraction of e, or else $G - \{x, y\}$ is a tree – but then G would be a member of $\mathcal{S}$, which is a contradiction. Each of the three structures G_A, G_B, and G_C specifies the endpoints of e, including all the neighbors of one of the endpoints, and also provides considerable information about the other vertices of G. This gives us a limited number of possible structures for T, depending on the vertex we choose to be y. Therefore, in each of Subcases A - C, we reconstruct G by beginning with a tree T; adding back the endpoints of e, the edge e, and vertex y; and assessing possible adjacencies for y and one endpoint of e.

The following result places a lower bound on the number of vertices in G, when G has the G_A structure.

2.81.1. *If G has the G_A structure, then $|G| \geq 8$.*

From Figure 2.71a, we can see G must already contain six vertices, namely, u, v, w, u_1, v_1, and w_1. Vertex w_1 is in G_3, where it is a neighbor of w. We assume to the contrary that G has exactly seven vertices; let this seventh vertex be a. Consider the cosimplification of $G \backslash f$, where we contract edges (u, u_1) and (v, v_1), labeling the resulting conglomerate vertices by u' and v', respectively. Thus, $co(G \backslash f) \cong G_3$. Since G_3 is 3-connected, every vertex in G_3 has minimum degree of three, by Theorem 1.5. So a must be adjacent to all of u', v', and w_1 or w.

Suppose a is adjacent to u', v', and w_1; in other words, $deg_G(w) = 3$. However, G is also 3-connected, so we still need the degrees of u_1, v_1, and w_1 to be at least three in G. There are only three more edges possible: (u_1, w_1), (w_1, v_1), and (u_1, v_1). We would not add all three edges in, since then we would lose minimal 3-connectivity in G, but suppose we did. Then G is isomorphic to U. So we know that G is a

minor of U when there are exactly seven vertices in G. This is a clear contradiction, since G is an excluded minor of $\mathcal{M}_1$.

Now suppose a is adjacent to u', v', and w; so both a and w_1 are neighbors of w. Both G_3 and G are 3-connected, with G having a G_A structure, so w_1 must be adjacent to u_1 and v_1 in G. There are no more edges available to us, but $G - \{w, v_1\}$ is a tree, so G is a member of $\mathcal{S}$ by Theorem 2.36. Thus, there are at least eight vertices in $V(G)$.

2.81.2. *If G has the G_A structure, then $|T| \geq 5$, where T is the tree resulting from deleting two vertices of G/e.*

Recall that by Theorem 2.36, there are two vertices of G/e, call them x and y, whose deletion produces a tree T. Also recall that x must be u' (or v'). Thus, T has three vertices fewer than G.

We now consider the first case.

2.82. *G has the G_A structure.*

Assume that our excluded minor G has a G_A structure, as shown in Figure 2.71a. We saw in the proof of Theorem 2.81 that both of (u, u_1) and (v, v_1) are nonessential edges in G. Let $e = (u, u_1)$, and we will consider G/e, which is 3-connected and in $\mathcal{S}$. Let u' be the vertex resulting from the contraction of e. By Theorem 2.36, G/e has two vertices, call them x and y, we can delete to create a tree T. One of these vertices must be u', or else G would have been in $\mathcal{M}_1$. Without loss of generality, we let $x = u'$.

2.82.1. *The graph G can be reconstructed from tree T by doing the following:*

(i) add vertices u and u_1 with their edge e,

(ii) add vertex y,

(iii) add edges (u, v) *and* (u, w)*,*

(iv) add edges between u_1 *and any leaves of* T *that are not* v *or* w*,*

(v) add edges between y *and each of the leaves of* T*,*

(vi) consider possible additional edges incident with u_1 *and* y*.*

Note the edges in (iii) are required by the G_A structure of G. We add the edges in (iv) and (v) because G is 3-connected, so by Theorem 1.5 it follows that every vertex in G has minimum degree of three. However, we are careful in (iv) to not make any vertex adjacent to both u and u_1.

We will break our argument down based on the degree of w in G, and whether T is a path or not:

(a) $deg_G(w) = 3$ and T has at least three leaves;

(b) $deg_G(w) = 3$ and T has exactly two leaves;

(c) $deg_G(w) \geq 4$ and T has at least three leaves;

(d) $deg_G(w) \geq 4$ and T has exactly two leaves.

In each of these four subcases, we proceed the same way. We consider whether y is one of the vertices that the G_A structure specifies (namely, v, w, v_1, or a neighbor of w), or y is some vertex about whose adjacencies we know nothing. For each choice of y, the G_A structure of G places limits on the trees that can be T. We then reconstruct G from T as outlined in 2.82.1.

2.82.2. *Let* T *have at least three leaves, and* $deg_G(w) = 3$*.*

So we have G with a G_A structure, where $deg(u) = deg(v) = deg(w) = 3$. Then w has exactly one other neighbor besides u and v; call it w_1.

Suppose $y = v$; from Figure 2.71a, we see that the only neighbors of v in G are u, w, and v_1. We observed in 2.82.1(v) that y must be adjacent to each leaf of T, in order for every vertex of G to have high enough degree; but u cannot be a leaf of T. Thus, there is some leaf of T to which $y = v$ is not adjacent, meaning G has a degree-2 vertex, a contradiction. We have a similar difficulty if $y = w$. In this case, the only neighbors of w in G are u, v, and w_1. Once more, there is a leaf in T that is forced to have degree-2 in G, which is a contradiction. Thus, y is neither v nor w.

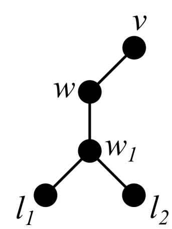


FIGURE 2.72: T when $y = v_1$ and T has exactly five vertices and three leaves.

Suppose $y = v_1$. Recall G has the G_A structure, $deg_G(w) = 3$, and T has at least three leaves. So v must be a leaf of T. Also w_1 must have a path to every leaf in T that avoids w, aside from leaf v. Then T has the structure sketched in Figure 2.72, possibly with $w_1 - l_1$ and $w_1 - l_2$ representing paths of length greater than one, or with T having additional leaves, such that the path between w_1 and such a leaf avoids w and v. Since $|T| \geq 5$ by 2.81.2, Figure 2.72 shows the smallest tree that can be T, and any tree that is T contains Figure 2.72 as minor. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Adding these vertices and edges to the tree in Figure 2.72 produces $G \cong S$, as we wished to prove.

Suppose $y = w_1$. Since G has the G_A structure and $deg_G(w) = 3$, we know w must be a leaf of T. Moreover, every leaf in T, aside from w, must a have a path to v_1 that avoids v. So T has the structure sketched in Figure 2.73, possibly with $v_1 - l_1$ and $v_1 - l_2$ representing paths of length greater than one, or with T

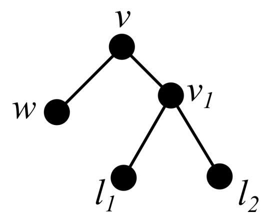


FIGURE 2.73: T when $y = w_1$ and T has exactly five vertices and three leaves.

having additional leaves that have paths to v_1 that avoid v and w. Since $|T| \geq 5$ by 2.81.2, Figure 2.73 shows the smallest tree that can be T, and any tree that is T contains Figure 2.73 as minor. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Adding these vertices and edges to the tree in Figure 2.73 produces $G \cong S$, as we wished to prove.

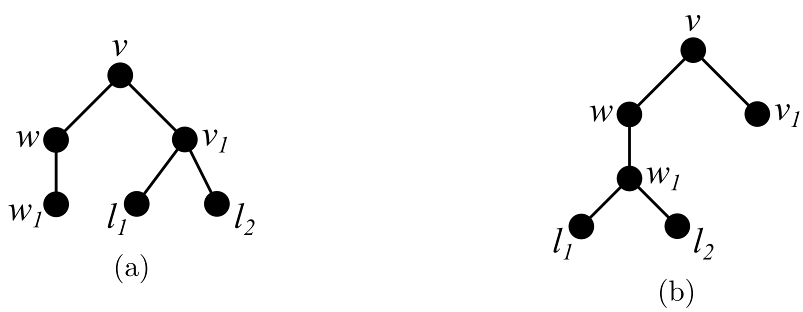


FIGURE 2.74: The two possible structures of T when $y = a$ and T has exactly six vertices and three leaves.

Suppose $y = a$, where $a \notin \{v, v_1, u, u_1, w_1, w\}$. For T to have at least three leaves, it must contain at least six vertices, due to the G_A structure of G. In addition, at least one of v_1 and w_1 will have paths to two or more leaves of T, such that these paths avoid v and w. So Figure 2.74 shows the two possibilities for T with exactly six vertices, and we observe that any T with additional vertices will have one of the trees in Figure 2.74 as a minor. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Adding these vertices and edges to either of the trees in Figure 2.74 produces a graph G that properly contains excluded minor S. This is a contradiction.

2.82.3. *Let T have exactly two leaves, and $deg_G(w) = 3$.*

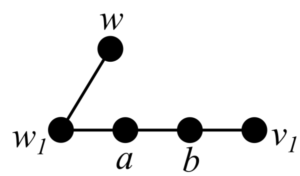


FIGURE 2.75: T when $y = v$; and T has exactly two leaves and five vertices.

Let $y = v$. Since we are assuming G has the G_A structure, the only neighbors of v in G are u, w, and v_1. Thus, the leaves of T must be w and v_1. Figure 2.75 sketches T; note $a - b$ may be a path of length greater than one. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Then we still have vertices w_1, a, b, and any internal vertices on subpath $a - b$ with insufficient degree. Since the only neighbors of v are w and v_1, our only option is to make u_1 adjacent to w_1, a, b, and any internal vertices of $a - b$. At this point we have exhausted all possible edges we can add in the reconstruction of G, but we are still left with $G - \{u_1, v\}$ being a tree. Thus, by Theorem 2.36, G is a member of $\mathcal{S}$.

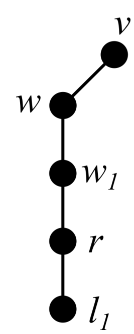


FIGURE 2.76: T when $y = v_1$, and T has exactly two leaves and its order is five.

Let $y = v_1$. Since G has a G_A structure, one of the leaves of T must be v; the other leaf must be on a path to w_1 in T that avoids w. Figure 2.76 shows a sketch of the structure of T, where the length of $w_1 - r$ may be one or more. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Observe that, at this point in our reconstruction, w_1 needs another neighbor to be a degree-3 vertex.

We can choose either u_1 or v_1 to be a neighbor of w_1, and these are currently symmetric. So, without loss of generality, we include edge (u_1, w_1). Table 2.6 lists the remaining edges we choose to add, and the graph G that is produced. To satisfy our minimum-degree requirement of three, we must have r and any internal vertices of $w_1 - r$ adjacent to one of u_1 and v_1; and v_1 must be adjacent to w_1, r, a vertex on $w_1 - r$, or u_1. Any vertices of $w_1 - r$ that are not explicitly given as adjacent to v_1 in Table 2.6 are assumed to be adjacent to u_1, to make their degree high enough.

TABLE 2.6: Additional edges added, from the $y = v_1$ subcase.

Edges added	Resulting G	Remarks
v_1 is adjacent to an internal vertex of $w_1 - r$	has a U-minor	
v_1 is adjacent to any two vertices of $w_1 - r$	has a U-minor	
(u_1, v_1) and v_1 is adjacent to no vertex of $w_1 - r$	G is in $\mathcal{S}^*$	split v or v_1 for a series-parallel graph
(v_1, w_1) and v_1 is adjacent to no other vertices of $w_1 - r$	G is not minimally 3-connected	there is a 3-cycle in G that fails Theorem 2.78
(v_1, r) and v_1 is adjacent to no other vertices of $w_1 - r$	G is in $\mathcal{S}^*$	split v or v_1 for a series-parallel graph
(v_1, r) and (u_1, v_1)	has a U-minor	

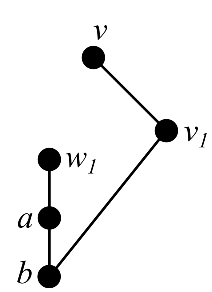


FIGURE 2.77: T when $y = w$, and T has exactly two leaves and five vertices.

Suppose $y = w$. We know the only neighbors of w in G are u, v, and w_1, since $deg_G(w) = 3$ and G has the G_A structure. This means the leaves of T have to be v and w_1. We sketch T in Figure 2.77, where $a - b$ may be a path of length one or more. To reconstruct G, we add to T the vertices and edges described in 2.82.1. After doing this, every vertex on subpath $a - v_1$ of T still only has degree two, which is too low for G. The only way to give these vertices higher degree, based on the restrictions imposed by the G_A structure, is to make them adjacent to u_1. So, we make u_1 adjacent to each vertex on $a - v_1$. Thus, $G \in \mathcal{S}^*$, since splitting v or v_1 creates a series-parallel graph from G.

(a) $|T| = 5$

(b) $|T| = 6$

FIGURE 2.78: T when $y = w_1$ and T has exactly two leaves.

Suppose $y = w_1$. As G has the G_A structure, we know that w must be one of the leaves of T, and, moreover, the other leaf must be on a path with v_1 that avoids v and w. We know from 2.81.2 that the order of T is at least five. If we proceed with T having exactly five vertices, as shown in Figure 2.78a, the result will be either a graph G that is a member of $\mathcal{S}^*$, or a G that is no longer minimally 3-connected, by Theorem 2.78.

So T has at least six vertices, as sketched in Figure 2.78b; note that if T has higher order, then the vertices are all on the path $r_1 - r_2$. To reconstruct G, we add to T the vertices and edges described in 2.82.1. At this point, however,

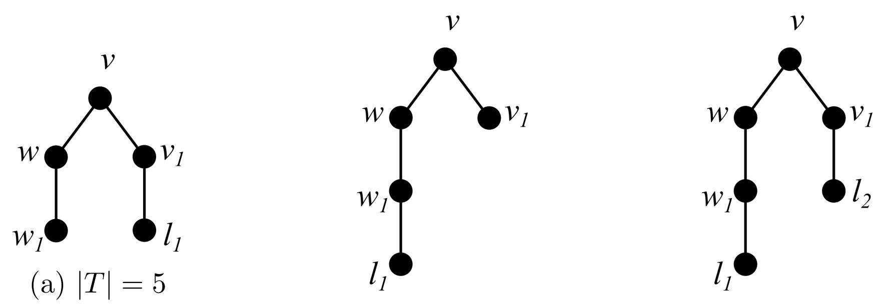
u_1, w_1, v_1, and all vertices on subpath $r_1 - r_2$ need higher degree, in order for G to be 3-connected. How we arrange getting the degrees of u_1, w_1, v_1, r_1, and r_2 high enough by additional edges is detailed in Table 2.7, for $|T| = 6$. Note that, by Theorem 2.78, in order for G to be minimally 3-connected, each of its cycles must have at least two degree-3 vertices. Thus, we have to choose between having edge (u_1, w_1), and letting both of u_1 and w_1 be adjacent to the same member of $\{v_1, r_1, r_2\}$.

TABLE 2.7: Additional edges added, from the $y = w_1$ subcase, when $|T| = 6$.

Edges added	Resulting G	Remarks
w_1 adjacent to $\{u_1, v_1, r_1, r_2, l_1\}$	$G \in \mathcal{S}^*$	split u or u_1 to get a series-parallel graph from G
u_1 adjacent to $\{w_1, v_1, r_1, r_2, l_1\}$	$G \in \mathcal{S}^*$	split w or w_1 to get a series-parallel graph from G
(w_1, v_1), and u_1 adjacent to $\{r_1, r_2\}$	has a U-minor	if have edge (u_1, w_1) as well, G still has a U-minor
(w_1, r_1) and u_1 adjacent to $\{v_1, r_2\}$	has a U-minor	if have edge (u_1, w_1) as well, G still has a U-minor
(w_1, r_2) and u_1 adjacent to $\{v_1, r_1\}$	$G \in \mathcal{S}^*$	split w_1 or w to get a series-parallel graph from G
(w_1, r_2), and u_1 adjacent to $\{w_1, v_1, r_1\}$	has a U-minor	
(u_1, v_1), and w_1 adjacent to $\{r_1, r_2\}$	has a U-minor	if have edge (u_1, w_1) as well, G still has a U-minor
(u_1, r_1) and w_1 adjacent to $\{v_1, r_2\}$	has a U-minor	if have edge (u_1, w_1) as well, G still has a U-minor
(u_1, r_2) and w_1 adjacent to $\{v_1, r_1\}$	$G \in \mathcal{S}^*$	split u or u_1 to get a series-parallel graph from G
(u_1, r_2), and w_1 adjacent to $\{u_1, v_1, r_1\}$	contains U	

In general, for T having six or more vertices, filling up the degrees of u_1, w_1, v_1, and all vertices on subpath $r_1 - r_2$ produces results that can be described as follows. If exactly one of u_1 and w_1 is adjacent to all the vertices of $v_1 - r_2$, then edge (u_1, w_1) must also be included, and G is in $\mathcal{S}^*$. If u_1 and w_1 are both adjacent to some of

the vertices on subpath $v_1 - r_2$ (but with neither of u_1 and w_1 only adjacent to r_2), and possibly to each other, then G contains U. If one of u_1 and w_1 is adjacent to all vertices of $v_1 - r_2$ but the other is adjacent to r_2, then $G \in \mathcal{S}^*$. However, if one of u_1 and w_1 is adjacent to all of $v_1 - r_2$, the other is adjacent to r_2, and edge (u_1, w_1) exists, then G contains U. All of these are contradictions.



(a) $|T| = 5$ (b) $|T| = 5$ (c) $|T| = 6$

FIGURE 2.79: T when $y = a$ and T has exactly two leaves.

Suppose $y = a$, where $a \notin \{u, v, w, u_1, v_1, w_1\}$. Since we know G has a G_A structure, T has exactly two leaves, and $|T| \geq 5$, there are three possible structures for T, depending on what vertices are leaves. Tree T may have only one of w_1 and v_1 as a leaf, with the other vertex on a path to a leaf l_1 that avoids v and w; or both of w_1 and v_1 may be on paths to the leaves of T, which avoid v and w. These three structures are sketched in Figure 2.79 with the lowest order of T possible. We note that T may have additional vertices besides those shown. In Figure 2.79a, the path $v_1 - l_1$ may have length one or more; in Figure 2.79b, the path $w_1 - l_1$ may have length one or more; and in Figure 2.79c, paths $w_1 - l_1$ and $v_1 - l_2$ may have length one or more.

Say we have T with w_1 as a leaf, like the tree in Figure 2.79a, although possibly with $v_1 - l_1$ being a path of length one or more. To reconstruct G, we add to T the vertices and edges described in 2.82.1. We still have v_1, a, and any internal vertices of $v_1 - l_1$ with insufficient degree for G to be 3-connected, so we consider

what edges we can add to a and u_1. If v_1 and the internal vertices of $v_1 - l_1$ are all adjacent to a (but not u_1), and we may or may not have edge (a, u_1), then G is in $\mathcal{S}^*$. We can split either of v or v_1 and get a series-parallel graph. Similarly, if u_1 is adjacent to v_1 and all the internal vertices of $v_1 - l_1$, with edge (a, u_1), then G is in $\mathcal{S}^*$. We can split either of w or w_1 and get a series-parallel graph. However, if both a and u_1 are adjacent to one or more of v_1 and the internal vertices of $v_1 - l_1$, then G has a proper U-minor.

Say we have T with v_1 as a leaf, like the tree in Figure 2.79b, but possibly with $w_1 - l_1$ being a path of length one or more. To reconstruct G, we add to T the vertices and edges described in 2.82.1. We still have w_1, a, and any internal vertices of $w_1 - l_1$ with insufficient degree for G to be 3-connected, so we consider what edges we can add to a and u_1. If w_1 and the internal vertices of $w_1 - l_1$ are all adjacent to a (but not u_1), and we may or may not have edge (a, u_1), then G is in $\mathcal{S}^*$. We can split either of w or w_1 and get a series-parallel graph. Similarly, if u_1 is adjacent to w_1 and to all the internal vertices of $w_1 - l_1$, and we have edge (a, u_1), then G is in $\mathcal{S}^*$. We can split either of v or v_1 and get a series-parallel graph. However, if both a and u_1 are adjacent to one or more of w_1 and the internal vertices of $w_1 - l_1$, then G has a proper U-minor.

Finally, say we have T with neither v_1 nor w_1 as a leaf, like the tree in Figure 2.79c, possibly with $w_1 - l_1$ or $v_1 - l_2$ being paths of length more than one. To reconstruct G, we add to T the vertices and edges described in 2.82.1. We still have w_1, v_1, a, and any internal vertices of $w_1 - l_1$ and $v_1 - l_2$ with insufficient degree for G to be 3-connected, so we consider what edges we can add incident with a and u_1. If we have a adjacent to all internal vertices of $w - l_1$, and u_1 adjacent to all internal vertices of $v - l_2$, then G is a daisy chain. We can split either w or w_1 to get a series-parallel graph from G. Likewise, if u_1 is adjacent to all internal vertices

of $w - l_1$, and a adjacent to all internal vertices of $v - l_2$, then G is a daisy chain. We can split either v or v_1 to get a series-parallel graph. Our previous work with the other two structures of T tells us that if each a and u_1 have neighbors among the internal vertices on $w - l_1$, or among the internal vertices on $v - l_2$, then G has a U-minor. If only u_1 is adjacent to the internal vertices of $w - l_1$ and $v - l_2$, and so we are forced to have edge (u_1, a), then $G - \{u_1, v\}$ is a tree. By Theorem 2.36, then, G is a member of $\mathcal{S}$. The final situation to consider is that only a and not u_1 is adjacent to internal vertices on subpaths $w - l_1$ and $v - l_2$ of T. Then G has an R-minor.

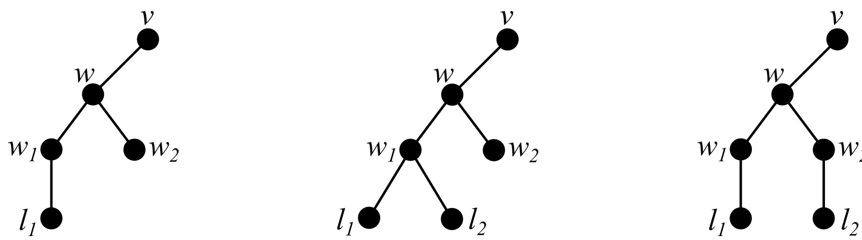
2.83. *G has the G_A structure and $deg_G(w) \geq 4$.*

We continue assuming G has the G_A structure. However, we now consider when w has at least two neighbors besides u and v; call them w_1 and w_2. We let $\{w_i\} = N(w) - \{u, v\}$, for $1 \leq i \leq n$ where $n \geq 2$; in other words, $\{w_i\}$ is the set of neighbors of w that are not u and v. Once again, we subdivide our analysis based on whether T is a path or not.

2.83.1. *Let T have at least three leaves, and $deg_G(w) \geq 4$.*

Suppose $y = v$; from Figure 2.71a, we see that the only neighbors of v in G are u, w, and v_1. We observed in 2.82.1(v) that y must be adjacent to each leaf of T, in order for every vertex of G to have high enough degree; but u cannot be a leaf of T. Thus, there is some leaf of T to which $y = v$ is not adjacent, meaning G has a degree-2 vertex, a contradiction.

Suppose $y = v_1$. Then v is one of the leaves of T, since G has the G_A structure. Moreover, the remaining leaves of T are either the vertices of $\{w_i\}$, or are on paths to vertices in $\{w_i\}$ that avoid w. We can view the structure of T as having v for a leaf and one of the following:



(a) $|T| = 5$ (b) $|T| = 6$ (c) $|T| = 6$

FIGURE 2.80: T when $y = v_1$ and T has three or four leaves.

- the remaining leaves of T are the set of vertices $\{w_i\}$, except possibly one leaf on a path to some w_i that avoids w;

- at least two of the remaining leaves of T are not members of $\{w_i\}$, but are on paths to some w_i that avoid w; or,

- at least two of the remaining leaves of T, say l_r and l_s, are not in $\{w_i\}$, but each of l_r and l_s is on a path to a distinct member of $\{w_i\}$ that avoids w.

Assume the leaves of T are v and the set of vertices $\{w_i\}$, except possibly one leaf on a path to some w_i that avoids w. We observe that if the leaves of T are precisely v and the set $\{w_i\}$, then $G - \{u_1, w\}$ is a tree. Thus, G is a member of $\mathcal{S}$ by Theorem 2.36, which is a contradiction. So we assume one neighbor of w, say w_1, has a path to a leaf of T that avoids w. This structure of T is sketched in Figure 2.80a, where, of course, there may be additional vertices w_i, and $w_1 - l_1$ may be a path with length greater than one. To reconstruct G, we add to T the vertices and edges described in 2.82.1. This leaves w_1 and any other internal vertices of $w - l_1$ with insufficient degree for G to be 3-connected. We must add more edges that are incident with u_1 or v_1. If we make all the internal vertices of $w - l_1$ adjacent to u_1 but none adjacent to v_1 (or vice versa), then, by Theorem 2.36, G is a member of $\mathcal{S}$, since $G - \{u_1, w\}$ or $G - \{v_1, w\}$ is a tree, respectively. However, if both u_1

and v_1 have neighbors among the internal vertices of $w - l_1$, then G has a proper U-minor.

Next, assume the leaves of T are v, and at least two vertices l_1 and l_2 that are not members of $\{w_i\}$, but are on paths that avoid w to some w_i, say w_1. This structure of T is sketched in Figure 2.80b; note T may have more than two w_i neighbors of w, and the paths $w_1 - l_1$ and $w_1 - l_2$ are not necessarily of length one, nor internally disjoint. The tree shown in Figure 2.80b has the minimal order that T can possess, given our assumptions about its structure. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Then G contains S as a minor, which is a contradiction.

Finally, assume the leaves of T are v, and at least two vertices l_1 and l_2 that are not members of $\{w_i\}$, but are on paths that avoid w to distinct vertices in $\{w_i\}$, say w_1 and w_2. This structure of T is sketched in Figure 2.80c; note T may have more than two w_i neighbors of w, and the paths $w_1 - l_1$ and $w_2 - l_2$ are not necessarily of length one. The tree shown in Figure 2.80c has the minimal order of T, given our assumptions about its structure. To reconstruct G, we add to T the vertices and edges described in 2.82.1. This leaves any internal vertices on paths $w - l_1$ and $w - l_2$ (and $w - l_i$) with insufficient degree for G to be 3-connected. We can change this by including more edges incident with v_1 or u_1. Let L be the set of vertices that are internal vertices on some path $w - l_i$, for all i. Now, so long as u_1 but not v_1 (or vice versa) is adjacent to every internal vertex in set L, the graph G remains in $\mathcal{S}$, since either $G - \{u_1, w\}$ or $G - \{v_1, w\}$ is a tree. So, it must be that both u_1 and v_1 have neighbors in L. Then G has V as a minor. In fact, if T is the tree shown in Figure 2.80c, then G is isomorphic to V, as we wished to prove.

Suppose $y = w$. Then all the leaves of T must be neighbors of w, an example of which is shown in Figure 2.81. Observe that $G - \{u_1, w\}$ creates a tree that

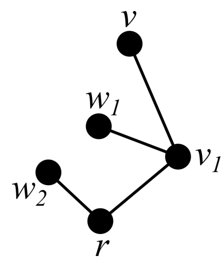


FIGURE 2.81: T when $y = w$, and T has exactly three leaves and five vertices.

is simply T plus a new leaf, u. Therefore G is in $\mathcal{S}$ by Theorem 2.36, which is a contradiction.

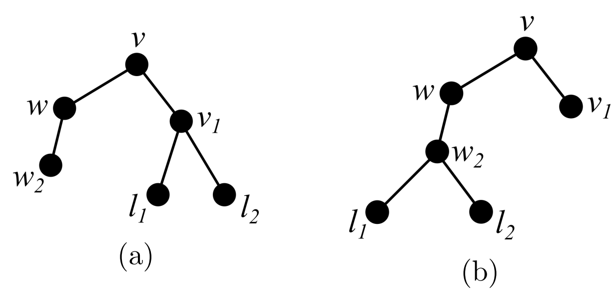
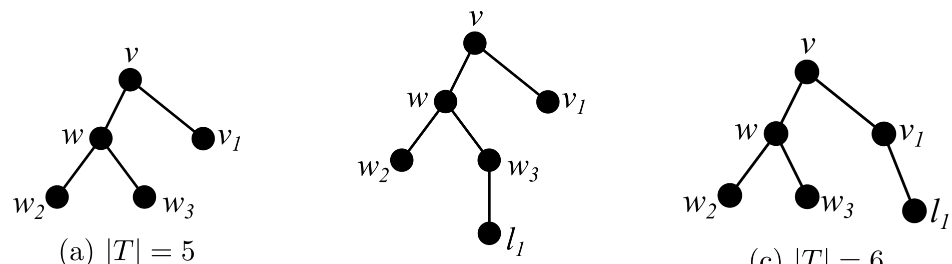


FIGURE 2.82: T when $y = w_1$ and T has exactly three leaves, with $deg_G(w) = 4$.

Suppose $y = w_1$. We consider separately the situations where $deg_G(w) = 4$ and $deg_G(w) \geq 5$. Begin with the more straightforward of these, $deg_G(w) = 4$; thus, aside from w_1 and u, the neighbors of w in G are v and w_2. Given that G has the G_A structure, there are two possible structures for T, under all our restrictions, depending on whether w_2 or v_1 has paths that avoid w to two or more leaves of T. These are sketched in Figure 2.82 for the minimum possible order of T under the restrictions, which is six. Note that for Figure 2.82a, there may be additional leaves on paths to w_2 or v_1, and that $v_1 - l_1$ and $v_1 - l_2$ may have length greater than one. Likewise, note that for Figure 2.82b, there may be additional leaves on paths to w_2 or v_1, and that $w_1 - l_1$ and $w_1 - l_2$ may have length greater than one. Whichever structure T has, to reconstruct G, we add to T the vertices and edges described in 2.82.1. Then we can see that G must have a U-minor, a contradiction.



(a) $|T| = 5$ (b) $|T| = 6$ (c) $|T| = 6$

FIGURE 2.83: The possible structure of T when $y = w_1$ and T has three leaves, with $deg_G(w) = 5$.

Now take $deg_G(w) \geq 5$. We make an observation based on our work in the $deg_G(w) = 4$ subcase. Any T with either v_1 or some w_i having paths that avoid w to two or more leaves of T must have one of the trees in Figure 2.82 as a minor. Thus, any such T results in a graph G that contains a proper U-minor. We restrict our analysis to trees T with one of the following structures:

- the leaves of T are v_1 and the members of $\{w_i\}$ for all $i \neq 1$;
- the leaves of T are v_1, possibly some members of $\{w_i\}$, and one or more vertices $l_1, l_2, \ldots, l_m$ where each l_j has a path to a distinct w_j that avoids w; or
- the leaves of T are the members of $\{w_i\}$ for all $i \neq 1$, and vertex l_1 that is on a path $v_1 - l_1$ which avoids w.

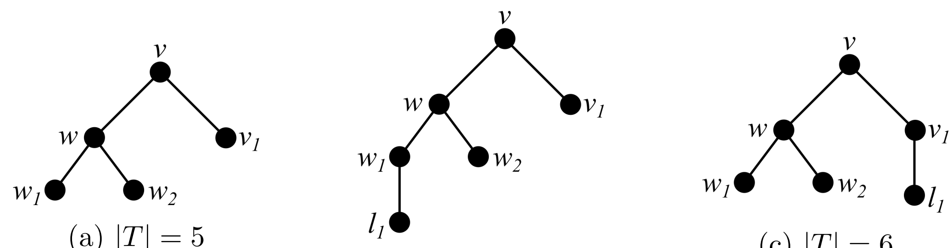
Assume the leaves of T are v_1 and the members of $\{w_i\}$ for all $i \neq 1$. A sketch of this situation is given in Figure 2.83a; note that w may have additional neighbors from $\{w_i\}$ that are not shown. To reconstruct G, we add to T the vertices and edges described in 2.82.1. There is nothing more to be done to reconstruct G, but $G - \{u_1, w\}$ is a tree, and, therefore, G is a member of $\mathcal{S}$ by Theorem 2.36. This is a contradiction.

Assume the leaves of T are v_1, possibly some members of $\{w_i\}$, and one or more vertices $l_1, l_2, \ldots, l_m$ where each l_j has a path to a distinct w_j that avoids w. A sketch of T with this structure is shown in Figure 2.83b; note there may be additional neighbors of w in T, and that there may be more leaves l_j on paths $w_j - l_j$ that avoid w. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Then any vertex w_j on a path $w_j - l_j$ to leaf l_j, as well as any internal vertices on such a path $w_j - l_j$, have insufficient degree for G to be 3-connected. Collect these vertices in set L. There must be additional edges incident with the vertices of L and u_1 or w_1 in G. If u_1 but not w_1 is incident with all the vertices of L, then $G - \{u_1, w\}$ is a tree, and G is a member of $\mathcal{S}$ by Theorem 2.36. However, if w_1 is adjacent to some vertex in L, then G has a U-minor.

Assume the leaves of T are the members of $\{w_i\}$ for all $i \neq 1$, and vertex l_1 that is on a path $v_1 - l_1$ avoiding w. A sketch of T with this structure is shown in Figure 2.83c. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Then v_1 and any internal vertices of $v_1 - l_1$ have insufficient degree for G to be 3-connected. So there must be additional edges incident with u_1 or w_1 in G. If u_1 but not w_1 is incident with v_1 and any internal vertices of $v_1 - l_1$, then $G - \{u_1, w\}$ is a tree, and G is a member of $\mathcal{S}$ by Theorem 2.36. However, if w_1 is adjacent to any one of v_1 and the internal vertices of $v_1 - l_1$, then G has an R-minor.

We may get a combination of the two previous subcases, with a tree T that has leaves on paths that avoid w to both some w_i and v_1. Observe that v_1, any internal vertices of $v_1 - l_1$, any w_i that is not a leaf, and any internal vertices of paths $w_i - l_i$ that avoid w all have only degree-2 after we go through the rebuilding process outlined in 2.82.1. Collect these degree-2 vertices in set L; they must be incident with u_1 or w_1 in G. Suppose both u_1 and w_1 have neighbors in set L; we

get a U- or R-minor in G. If w_1 is not a neighbor of any the vertices of L (and, therefore, u_1 is adjacent to all vertices in L), then $G - \{u_1, w\}$ is a tree and G is in $\mathcal{S}$.



(a) $|T| = 5$ (b) $|T| = 6$ (c) $|T| = 6$

FIGURE 2.84: T when $y = a$ and T has three leaves, with order of T five or six.

Finally, suppose $y = a$, where $a \notin \{u, u_1, v, v_1, w, w_i\}$. Considering the following three structures for tree T is sufficient, as we will see. Observe that these are the same structures we had when $y = w_1$, up to re-indexing the w_i vertices in T. However, the reconstruction of G is slightly different, since $N(a)$ is not restricted by the G_A structure of G, as $N(w_1)$ is.

- the leaves of T include l_1 and l_2, each of them having paths that avoid w to v_1, or to the same w_i;
- the leaves of T are v_1 and the vertices of $\{w_i\}$;
- the leaves of T are v_1, possibly some members of $\{w_i\}$, and one or more vertices $l_1, l_2, \ldots, l_m$ where each l_j has a path to a distinct w_j that avoids w; or
- the leaves of T are the members of $\{w_i\}$, and vertex l_1 that is on a path $v_1 - l_1$ which avoids w.

Assume the leaves of T include l_1 and l_2, each of them having paths (not necessarily internally disjoint) that avoid w to v_1, or to the same w_i. To reconstruct G,

we add to T the vertices and edges described in 2.82.1. Then G has as a U-minor. Therefore, in the remaining three subcases, we must have v_1 on a path that avoids w to at most one leaf; and, similarly, if some neighbor w_i of w has a path to a leaf that avoids w, it has only one such path.

Assume the leaves of T are v_1 and the members of $\{w_i\}$. This structure of T is sketched in Figure 2.84a. To reconstruct G, we add to T the vertices and edges described in 2.82.1. There are no remaining edges that can be added to reconstruct G, but we are left with $G - \{u_1, w\}$ being a tree. So when T has this structure, G is always a member of $\mathcal{S}$, by Theorem 2.36.

Assume the leaves of T are v_1, possibly some members of $\{w_i\}$, and one or more vertices $l_1, l_2, \ldots, l_m$ where each l_j has a path to a distinct w_j that avoids w. A sketch of T with this structure is shown in Figure 2.84b; note there may be additional neighbors of w in T, and that there may be more leaves l_j on paths $w_j - l_j$ that avoid w. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Then any vertex w_j on a path $w_j - l_j$ to leaf l_j, as well as any internal vertices on such a path $w_j - l_j$, have insufficient degree for G to be 3-connected. Collect these vertices in set L. So there must be additional edges incident with the vertices of L and u_1 or a in G. If u_1 but not a is incident with all the vertices of L, then $G - \{u_1, w\}$ is a tree, and G is a member of $\mathcal{S}$ by Theorem 2.36. However, if a is adjacent to some vertex in L, then G has a U-minor.

Assume the leaves of T are the members of $\{w_i\}$ for all $i \neq 1$, and vertex l_1 that is on a path $v_1 - l_1$ avoiding w. A sketch of T with this structure is shown in Figure 2.84c. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Then v_1 and any internal vertices of $v_1 - l_1$ have insufficient degree for G to be 3-connected. So there must be additional edges incident with u_1 or a in G. If u_1 but not a is incident with v_1 and any internal vertices of $v_1 - l_1$, then $G - \{u_1, w\}$

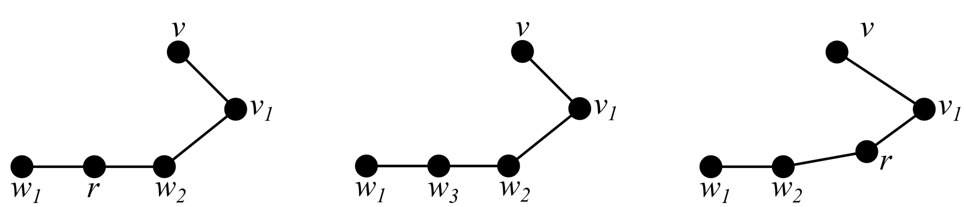
is a tree, and G is a member of $\mathcal{S}$ by Theorem 2.36. However, if a is adjacent to any one of those vertices, then G has an R-minor.

The tree T may have leaves that satisfy the previous two subcases; that is, T has leaves on paths that avoid w to both v_1 and some w_i. Go through the steps in 2.82.1 to rebuild G from T. We collect the vertices that are still degree-2 in set L. These vertices consist of v_1 and any internal vertices of $v_1 - l_1$; and any w_i that is not a leaf, and any internal vertices on paths $w_i - l_i$ that avoid w. If both u_1 and a have neighbors in L, we get a U- or R-minor in G. If a is not adjacent to any vertex in L, then we are forced to have u_1 adjacent to all the vertices in L, and $G - \{u_1, w\}$ is a tree. Thus, G is in $\mathcal{S}$.

2.83.2. *Let T have exactly two leaves, and $deg_G(w) \geq 4$.*

Suppose $y = v$; since G has the G_A structure, we know the only neighbors of v in G are u, w, and v_1. We observed in 2.82.1(v) that y must be adjacent to each leaf of T, in order for every vertex of G to have high enough degree. Now u cannot be a leaf of T, and since, $deg_G(w) \geq 4$, neither can w. Thus, there is a leaf of T whose degree in G is only two, which is a contradiction. So y is not v.

The contradiction is even more readily apparent if we set $y = v_1$ or $y = a$, where $a \notin \{u, u_1, v, v_1, w, w_i\}$. We know that $deg_T(w) \geq 3$ since w is adjacent to at least v, w_1, and w_2, which implies there are at least three leaves in T, a contradiction. Thus, y is neither v_1 nor a.



(a) (b) (c)

FIGURE 2.85: T when $y = w$, and T has exactly two leaves and order five.

Suppose $y = w$. Observe that, since G has the G_A structure, v must be a leaf of T; the other leaf must be a neighbor of w, say w_1. Consider the path $w_1 - v_1$ in T; let $w_1 - w_2$ be the longest subpath of $w_1 - v_1$ that has two neighbors of w as endpoints. We show in Figure 2.85 the three trees that can be T when T has order five; vertex r is not a neighbor of w. In general, tree T has subpath $w_1 - w_2$ of length one or more, with internal vertices that are neighbors of w or not; and subpath $w_2 - v_1$ of length one or more, with internal vertices that are not neighbors of w. To reconstruct G, we add to T the vertices and edges described in 2.82.1. In addition, we make w adjacent to all members of $\{w_i\}$. Any vertices whose degree is still too low must then be made adjacent to u_1. The result is that $G - \{u_1, w\}$ is a tree, specifically T with new leaf u appended to v. Thus, $G \in \mathcal{S}$ by Theorem 2.36, which is a contradiction.

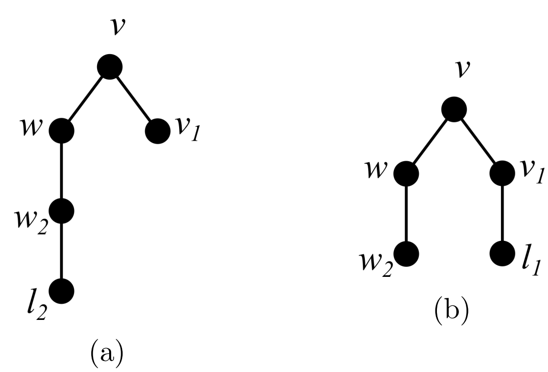


FIGURE 2.86: T when $y = w_1$, and T has two leaves, with order of T five.

Suppose $y = w_1$. Using the fact that G has a G_A structure and that $deg_G(w) \geq 4$, we find that $deg_G(w) = 4$, or T cannot be a path. In addition, T either has one of v_1 and w_2 as a leaf – see sketches in Figure 2.86 – or both leaves l_1 and l_2 are on paths $w_2 - l_2$ and $v_1 - l_1$ that avoid w and v. Note that paths $w_2 - l_2$ and $v_1 - l_1$ can have length greater than one.

Assume T has v_1 and l_2 as leaves, where path $w_2 - l_2$ avoids w. An example of this structure of T is sketched in Figure 2.86. To reconstruct G, we add to T the

vertices and edges described in 2.82.1. Also, we add edge (w_1, w). At this point, all internal vertices of $w - l_2$ have insufficient degree for G to be 3-connected. So there must be more edges that are incident with u_1 or w_1. If we make u_1 adjacent to all these internal vertices, but w_1 adjacent to none of them, then $G - \{w, u_1\}$ is a tree. If we make w_1 adjacent to all these vertices and u adjacent to none of them, then G is not minimally 3-connected, by Theorem 2.78. However, let r_1 and r_2 be internal vertices, not necessarily distinct, of $w - l_2$; if u_1 is a neighbor of r_1 and w_1 is a neighbor of r_2, then G has a U-minor.

Assume T has w_2 and l_1 as leaves, where path $v_1 - l_1$ avoids v and w. An example of this structure of T is sketched in Figure 2.86b. To reconstruct G, we add to T the vertices and edges described in 2.82.1. Also, we add edge (w_1, w). At this point, all internal vertices of $v - l_1$ have insufficient degree for G to be 3-connected. So there must be more edges that are incident with u_1 or w_1. If we make u_1 adjacent to all these vertices, but w_1 adjacent to none of them, then $G - \{w, u_1\}$ is a tree. However, if w_1 is adjacent to some internal vetex on $v - l_1$, then G has an R-minor.

Finally, assume T has l_1 and l_2 as leaves, where paths $w_2 - l_2$ and $v_1 - l_1$ avoid v and w. If u_1 is adjacent to all internal vertices on subpaths $w - l_2$ and $v - l_1$, but w_1 is not adjacent to any of these vertices, then $G - \{u_1, w\}$ is a tree. We know if w_1 is adjacent to any vertex on subpath $v - l_2$, we get G with an R-minor. Lastly, if w_1 is adjacent to some internal vertex on subpath $w - l_1$ (but not every vertex, or else G will fail to be minimally 3-connected, by Theorem 2.78), G has a U-minor.

2.84. *G has G_B structure*

The following result places a lower bound on the number of vertices in G, when G has the G_B structure.

2.84.1. *If G has the G_B structure, then $|G| \geq 7$.*

In establishing this result, we refer to the illustration of G with a G_B structure from Figure 2.71b. Consider the cosimplification of $G\backslash f$, where we contract the edge (v, v_1) and label the resulting conglomerate vertex v'. Observe that the 3-connected graph G_2 is isomorphic to $co(G\backslash f)$. We can see G_2 has three or more vertices, but by the definition of 3-connected, $|G_2| > 3$. There must be some other vertex a, and we have $\{w, v, u, a\} \subseteq V(G_2)$. If these four vertices are the only members of $V(G_2)$, then $G_2 \cong K_4$. This leads to a contradiction, because $G - \{u, a\}$ is a tree, and, by Theorem 2.36, we have $G \in \mathcal{S}$. Thus, we add another vertex, say b, to $V(G_2)$. So G has at least six vertices, namely, $\{u, v, v_1, w, a, b\}$. We assume that G has exactly six vertices and produce a contradiction.

If G has six vertices, then G_2 has five. By Theorem 2.9, we can therefore obtain G_2 by subdividing an edge of K_4, and adding edges between the resulting vertex x and the vertices of K_4. Without loss of generality, x is adjacent to vertex 4; see Figure 2.87. The graph G_2 may also have the edge $(x, 3)$. We choose a vertex of this subdivided K_4 to be v' of G_2, and then uncontract v' to retrieve G.

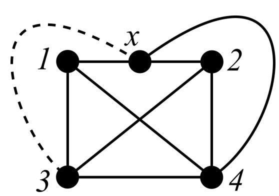


FIGURE 2.87: Complete graph K_4 with an edge subdivided, and the resulting vertex x adjacent to vertex 4; edge $(x, 3)$ may or may not be present.

When we uncontract v', we want the resulting vertices v and v_1 to have minimum degree of three so that G can be 3-connected, which only occurs if we split a vertex of degree at least four. The only vertices that can be v' in Figure 2.87, up to isomorphism, are x (if edge $(x, 3)$ exists) and 4. Suppose we uncontract 4; no matter how we do so, the resulting G is in $\mathcal{S}$, since $G - \{x, 3\}$ is a tree. Similarly,

if we uncontract x, we find $G - \{3, 4\}$ is a tree. We conclude G has at least seven vertices.

2.84.2. *If G has the G_B structure, then $|T| \geq 4$, where T is the tree resulting from deleting two vertices of G/e.*

Recall our assumption in this section, that for some edge e, the graph G/e is a 3-connected member of $\mathcal{S}$. Thus, by Theorem 2.36, there are two vertices x and y whose deletion produces a tree T. Also recall that x must be v'. Thus, T has three vertices fewer than G.

We now consider the second case of this section.

In this second main case, we assume that G has the G_B structure, as shown in Figure 2.71b. By the proof of Theorem 2.81, the edge (v, v_1) is nonessential in G. Let $e = (v, v_1)$, and let v' be the conglomerate vertex resulting from the contraction of e. Since we know G is minimally 3-connected, graph G/e is a 3-connected member of $\mathcal{S}$. By Theorem 2.36, G/e has two vertices x and y, whose deletion gives us a tree T. Observe that v' must be one of these vertices, or else G would have been a member of $\mathcal{S}$. We let $x = v'$.

2.84.3. *The graph G can be reconstructed from tree T by doing the following:*

(i) *add vertices v and v_1 with their incident edge e,*

(ii) *add vertex y,*

(iii) *add edges (u, v) and (v, w),*

(iv) *add edges between v_1 and any leaves of T that are not u or w,*

(v) *add edges between y and each of the leaves of T,*

(vi) *consider possible additional edges incident with v_1 or y.*

Note the edges in (iii) are required by the G_B structure of G. We add the edges in (iv) and (v) because G is 3-connected, so by Theorem 1.5 it follows that every vertex in G has minimum degree of three. However, we are careful in (iv) to not make any vertex adjacent to both v and v_1, since then G/f would contain parallel edges.

We will break our argument down based on whether T is a path or not:

(a) T has at least three leaves; or

(b) T has exactly two leaves.

In both of these subcases, we proceed the same way. We consider whether y is one of the vertices that the G_B structure specifies (namely, w or u), or y is some vertex about whose adjacencies we know nothing. For each choice of y, the G_B structure of G places limits on the trees that can be T. We then reconstruct G from T as outlined in 2.84.3.

2.84.4. *Let T have at least three leaves.*

Suppose $y = w$. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Any additional edges we add are incident with one of w or v_1. However, since G has the G_B structure, $G - \{v_1, w\}$ is a tree (specifically, T with a new leaf, v). By Theorem 2.36, this means G is a member of $\mathcal{S}$.

Suppose $y = u$. Again, after reconstructing G from T, we find $G - \{u, v_1\}$ is a tree; this tree is, in fact, simply T with an additional leaf, v. By Theorem 2.36, this means G is a member of $\mathcal{S}$.

Now, our only choice for y is $y = a$, where $a \notin \{u, v, v_1, w\}$. We make the following observation:

2.84.5. *G does not contain edge (a, v_1).*

Assume to the contrary that we let G contain edge (a, v_1). Then any of the 3-cycles in G that involve a, v_1, and a vertex that was a leaf of T, will fail to contain two degree-3 vertices. By Theorem 2.78, G is not minimally 3-connected, which is a contradiction.

Since u and w are adjacent only to v but not v_1, for the remainder of the G_B case, we will consider the structure of T up to interchanging the u and w labels.

Since T has at least three leaves, there is a vertex of degree at least three in T. Either this vertex is one of $\{u, w\}$, say w; or it is some other vertex not named by the G_B structure shown in Figure 2.71b, say b, where where $b \notin \{u, v, v_1, w, a\}$. There are three possible relationships for this high-degree vertex to have with the leaves of T:

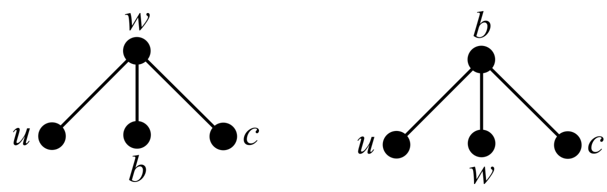
(i) this vertex is on a path of length one to every leaf of T,

(ii) there is exactly one path from this vertex to a leaf of T that has length greater than one, or

(iii) at least two leaves of T are on paths of length greater than one to this vertex.

Assume we have subcase (i), where T is a star. If the order of T is exactly four, then T is one of the two trees shown in Figure 2.88; for T with higher order, we still have one of w and b as the hub vertex, but there are more leaves. To reconstruct G, we add to T the vertices and edges described in 2.84.3. There are no other edges we can add, but this yields a G where either $G - \{w, v_1\}$ or $G - \{a, b\}$ is a tree, depending on which of w and b is the hub of T, respectively. Thus, G is a member of $\mathcal{S}$ by Theorem 2.36.

Assume we have subcase (ii), where w is our degree-3 vertex of T; so there is exactly one path from w to a leaf that has length greater than one. The possible structures of T, given these restrictions, are illustrated in Figure 2.89. Vertex u

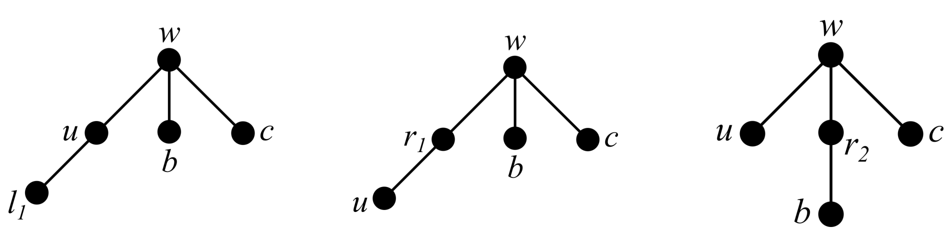


(a) T_i (b) T_{ii}

FIGURE 2.88: T when $y = a$, with three leaves and with order four; illustrates subcase (i).

may be a leaf of T, or not; and, if u is a leaf, we must furthermore decide if it is the leaf on the long path to w.

For the situation when u is not a leaf, the tree T is sketched in Figure 2.89a. Note that either of paths $w - u$ or $u - l_1$ may be of length greater than one, and w may be adjacent to additional leaves that are not pictured. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then any of the internal vertices of $w - l_1$, besides u, still have insufficient degree for G to be 3-connected; if u is the only internal vertex, then $G - \{v_1, w\}$ is a tree. So we must add additional edges that are incident with v_1 or a. If some internal vertex on $w - l_1$ is adjacent to a, then G has an R-minor (this is regardless of the adjacencies or lack therefore between these internal vertices and v_1). However, if none of the internal vertices of $w - l_1$ are adjacent to a, and we are therefore forced to have all these vertices adjacent to v_1, then $G - \{v_1, w\}$ is a tree.



(a) T_1 (b) T_2 (c) T_3

FIGURE 2.89: T when $y = a$, with three leaves and order five; illustrates subcase (ii) when $deg_T(w) \geq 3$.

For the situation when u is a leaf such that $u - w$ has length greater than one, the tree T is sketched in Figure 2.89b. Note that path $w - u$ may have length exceeding two, and w may be adjacent to additional leaves that are not shown. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then the internal vertices of $w - u$ are of insufficient degree for G to be 3-connected. So we must add additional edges that are incident with v_1 or a. If some internal vertex on $w - u$ (but not all of the internal vertices) is adjacent to a, then G has an R-minor (this is regardless of the adjacencies or lack therefore between these internal vertices and v_1). However, if all the internal vertices of $w - u$ are adjacent to v_1 but not to a, or vice versa, then $G - \{v_1, w\}$ or $G - \{a, w\}$ is a tree.

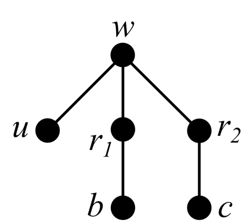
For the situation when u a leaf such that $u - w$ has length one, the tree T is sketched in Figure 2.89c; note that path $w - b$ may have length exceeding two, and w may be adjacent to leaves not shown. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then the internal vertices of $w - b$ are of insufficient degree for G to be 3-connected. So we must add additional edges that are incident with v_1 or a. If all the internal vertices of $w - b$ are adjacent to v_1 but not to a, or vice versa, then $G - \{w, v_1\}$ or $G - \{w, a\}$ is a tree, respectively. So G is a member of $\mathcal{S}$ by Theorem 2.36. On the other hand, if both a and v_1 have some neighbor among the internal vertices of $w - b$ (with the neighbors of a and v_1 not necessarily coinciding), then G has a U-minor, with one of the vertices being the conglomerate of all internal vertices of $w - b$.

Assume we have subcase (iii), where w is our vertex of degree at least three in T; so at least two leaves of T are on paths of length greater than one to w. These paths of length greater than one may be internally disjoint, or not.

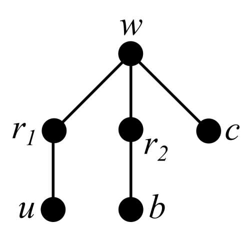
Suppose two leaves l_1 and l_2 have paths to w that are of length at least two, but not internally disjoint. Then paths $l_1 - w$ and $l_2 - w$ meet in another vertex besides

w that has degree three or more in T. To reconstruct G, we add to T the vertices and edges described in 2.84.3. We already see that, although the reconstruction is incomplete, G has an R-minor or a U-minor. We deduce that all the paths from w to leaves of T are internally disjoint.

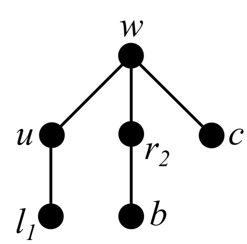
When u is a leaf such that $u - w$ has length one, a sketch of T is shown in Figure 2.90a; note that there may be additional leaves of T, not pictured, whose paths to w are of length one or more. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then, all internal vertices on paths $w - l_i$ still have insufficient degree for G to be 3-connected, where l_i is a leaf. Collect all these degree-2 vertices in set L. So we must add additional edges that are incident with L and v_1 or a. If all the vertices of L are adjacent to v_1 but none of them to a, or vice versa, then $G - \{w, v_1\}$ or $G - \{w, a\}$ is a tree, respectively. So G is a member of $\mathcal{S}$ by Theorem 2.36. On the other hand, if both a and v_1 are incident with internal vertices of one particular path $w - l_1$, then we know G has a U-minor from our work on subcase (ii). Lastly, if a is adjacent to some internal vertex of $w - l_{i_1}$ and v_1 is adjacent to an internal vertex of $w - l_{i_2}$, where the leaves l_{i_1} and l_{i_2} are not the same, then G has a V-minor.



(a) T_4



(b) T_5



(c) T_6

FIGURE 2.90: T when $y = a$, with three leaves and order six; illustrates subcase (iii) when$deg_T(w) \geq 3$.

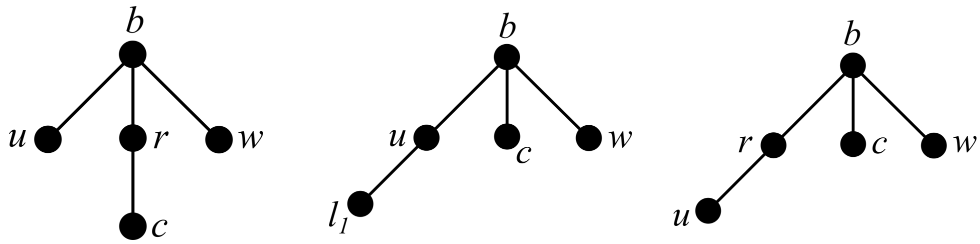
When u is a leaf such that $u - w$ has length greater than one, a sketch of T is shown in Figure 2.90b; note that the length of $u - w$ and $b - w$ may be greater

than two, and there may be additional leaves whose paths to w are of length one or more. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then, all internal vertices on paths $w - l_i$ still have insufficient degree for G to be 3-connected, where l_i is a leaf. Collect these vertices in set L. So we must add additional edges that are incident with v_1 or a. If all the vertices in L are adjacent to v_1 but none of them to a, or vice versa, then $G - \{w, v_1\}$ or $G - \{w, a\}$ is a tree, respectively. If a has a neighbor r_1 and v_1 has a neighbor r_2 such that $r_1, r_2 \in L$, with r_1 and r_2 not necessarily distinct nor necessarily on different $w - l_i$ paths, then G has a U-minor.

When u is an internal vertex on a path between w and a leaf of T, a sketch of T is shown in Figure 2.90c; note that the lengths of $w - l_1$ and $w - b$ may be greater than two, and there may be additional leaves whose paths to w are of length at least one. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then, all internal vertices on paths $w - l_i$, besides u, still have insufficient degree for G to be 3-connected, where l_i is a leaf. Collect these vertices in set L. We must add additional edges that are incident with the vertices of L and v_1 or a. If all the vertices in L are adjacent to v_1 but none of them to a, or vice versa, then $G - \{w, v_1\}$ or $G - \{w, a\}$ is a tree, respectively. If a has a neighbor r_1 and v_1 has a neighbor r_2 such that $r_1, r_2 \in L$, with r_1 and r_2 not necessarily distinct nor necessarily on different $w - l_i$ paths, then G has a U-minor.

Now we look at subcases (ii) and (iii) when our vertex b of degree at least three in T is not w (or u).

Assume we have subcase (ii), where b has degree at least three in T; so there is exactly one path from b to a leaf that has length greater than one. The possible structures of T, given these restrictions, are illustrated in Figure 2.91. One or both



(a) T_1 (b) T_2 (c) T_3

FIGURE 2.91: T when $y = a$, with three leaves and order five; illustrates subcase (ii) when $deg_T(b) \geq 3$.

of u and w may be leaves; and, if both of them are leaves, we must also decide if one of them is on a path to b that has length greater than one.

Say u is not a leaf of T; then it is on path $b - l_1$ between b and leaf l_1. This structure of T is illustrated in Figure 2.91b; note there may be additional leaves of T that are neighbors of b, not shown, and the length of $b - l_1$ can be greater than two. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then G has an H_8-minor. This H_8-minor is proper, unless T is isomorphic to the tree shown in Figure 2.91b; in this case, G is isomorphic to H_8.

Say both of u and w are leaves of T, and both are neighbors of b. A sketch of T with this structure is given in Figure 2.91a; note that there may be additional leaves of T that are neighbors of b, not shown, and the length of $b - c$ can be greater than two. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then the internal vertices of $b - c$ still have insufficient degree for G to be 3-connected; and, if T has only three leaves, vertex v_1 has insufficient degree, as well. So we must add additional edges that are incident with v_1 or a. Should T have only three leaves, then there must be an edge from v_1 to some internal vertex of $b - c$, since 2.84.5 forbids edge (a, v_1). Once we add this edge from v_1 to an internal vertex of $b - c$, the graph G will have a U-minor. So let T have at least four leaves. If all the internal vertices of $b - c$ are adjacent to a but none of these

vertices are adjacent to v_1, then $G - \{a, b\}$ is a tree, and, therefore, G is a member of $\mathcal{S}$ by Theorem 2.36. If some internal vertex of $b - c$ is adjacent to v_1, however, G has an S-minor.

Say both of u and w are leaves of T, but only w is a neighbor of b. A sketch of T with this structure is given in Figure 2.91c; note that there may be additional leaves of T that are neighbors of b, not shown, and the length of $b - u$ can be greater than two. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then the internal vertices of $b - u$ still have insufficient degree for G to be 3-connected. If all the internal vertices of $b - u$ are adjacent to a but none of them are adjacent to v_1, then $G - \{a, b\}$ is a tree, and, therefore, G is a member of $\mathcal{S}$ by Theorem 2.36. If some internal vertex of $b - u$ is adjacent to v_1, however, G has an H_8-minor.

For subcase (iii), where b has degree at least three in T, we have at least two leaves of T on paths to b with length greater than one. If one or both of u and w is an internal vertex on one of these nontrivial paths, then G has an H_8-minor. We are left with both u and w being leaves of T. Should there be two such nontrivial paths that are not internally disjoint, then G has an H_8-minor if one or both of u and w are leaves on these paths; and an S-minor if neither of u and w are leaves on these paths. So we now are left not only with u and w being leaves of T, but with every path from b to a leaf of T being internally disjoint. If a is adjacent to all the internal vertices on these paths, but v_1 is not adjacent to any of them, then $G - \{a, b\}$ is a tree; if v_1 is adjacent to some internal vertex on one of these paths, then G has a U-minor or an S-minor (and this is regardless of whether there are also internal vertices adjacent to a or not).

2.84.6. *Let T have at exactly two leaves.*

Suppose $y = w$ or $y = u$. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Any additional edges we add are incident with one of w or v_1. However, since G has the G_B structure, $G - \{v_1, w\}$ or $G - \{v_1, u\}$ is a tree. To be specific, it is the tree T with added leaf v. Then, by Theorem 2.36, G is a member of $\mathcal{S}$. Thus, y is not w or u.

(a) T_a (b) T_b (c) T_c (d) T_d

FIGURE 2.92: T has two leaves and four vertices.

Thus, let $y = a$, where $a \notin \{u, v, v_1, w\}$. We make subcases based on the vertices that are leaves of T – that is, whether both, neither, or one of w and u are leaves. When u and w are not both leaves, we must also consider whether u is adjacent to w or not. The smallest order trees for each of these subcases is pictured in Figure 2.92. Recall that, since u and w are adjacent only to v but not v_1 with the G_B structure, we consider the structure of T up to interchanging the u and w labels.

Assume the leaves of T are b and c, neither of which is u or w. We first consider T under these restrictions and having exactly four vertices; this is T_a shown in Figure 2.92a. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then a is a degree-2 vertex, which is too low a degree for G to be 3-connected; so there is another edge or edges incident with a. The available edges are (a, v_1), (a, u), and (a, w). If we add only one of these edges, then G is a member of $\mathcal{S}$ by Theorem 2.36, since one of $G - \{u, v_1\}$ or $G - \{w, v_1\}$ is a tree. Adding any pair of edges from (a, v_1), (a, u), and (a, w) gives G a 3-cycle with two degree-4 vertices which, by Theorem 2.78, causes G to fail to be minimally 3-connected.

(We note, however, that if we add edges (a, u), and (a, w), then G has an R-minor; this information will be useful in spotting R-minors when T has higher order.)

When T has higher order, with vertices b and c its leaves, the tree T_a shown in Figure 2.92a is still a sketch of its structure; now, however, $b - u$, $u - w$, and $w - c$ may be paths with lengths greater than one. In other words, we now take the edges of T_a to represent paths of length at least one. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then a is a degree-2 vertex, as are any internal vertices of $b - u$, $u - w$, and $w - c$. Thus, G has additional edges that are incident with these vertices.

2.84.7. *None of the following can occur:*

(i) G has both of the edges (a, u) and (a, w);

(ii) G has both of the edges (a, x_1) and (a, x_2), where x_1 is an internal vertex of $b - u$, $u - w$, or $w - c$, and x_2 is an internal vertex on one of $b - u$, $u - w$, or $w - c$ that does not contain x_1;

(iii) G has both of edges (a, u) and (a, x), where x is an internal vertex from $w - c$; or G has both of edges (a, w) and (a, y), where y is an internal vertex of $b - u$;

(iv) G has an edge whose endpoints are a and an internal vertex of $u - w$.

We have already seen that putting in edges (a, u) and (a, w) gives G an R-minor when T had only four vertices. Parts (ii) and (iii) follow from the observation that a adjacent as listed means G has a minor where a is adjacent to u and w, and, thus, G has an R-minor. Finally, if G has an edge as described in (iv), then either G is isomorphic to Q_3, as we wished to prove (this occurs for T having order five, as shown in Figure 2.93b); or G has a proper Q_3-minor, a contradiction.

FIGURE 2.93: T is a path with order 5, where neither u nor w is a leaf.

Therefore, now we know a can only be adjacent to vertices in one of the paths $b-u$ and $w-c$, and possibly to v_1.

2.84.8. *In addition, neither of the following can occur:*

(i) G has no edges (a,x), where x is an internal vertex of $b-u$, $u-w$, or $w-c$;

(ii) G has no edges (a,x) as in (i), but G does have one of the following:

- *edge (a,v_1),*
- *edge (a,u),*
- *edge (a,w), or*
- *edge (a,v_1) and one of edges (a,u) or (a,w);*

(iii) G has both of edges (a,u) [or (a,w)] and (a,u_1), where u_1 is a neighbor of u [w] in T

For (i) and (ii), we get that $G-\{v_1,u\}$ or $G-\{v_1,w\}$ is a tree. The edges in (iii) cause G to contain a 3-cycle lacking two degree-3 vertices, meaning G is not minimally 3-connected, by Theorem 2.78.

The items in 2.84.7 and 2.84.8 limit us to a handful of options for the additional edges we use to reconstruct G.

2.84.9. *These are the only other possible adjacencies:*

(i) both a and v_1 have neighbors among the internal vertices of $b-u$ [or of $w-c$] (note that u and w may or may not be adjacent);

(ii) *u and w are adjacent; G has a adjacent to all internal vertices of $b-u$ [$w-c$], but v_1 adjacent to none of them; and v_1 adjacent to any internal vertices of $w-c$ [$b-u$], but a adjacent to none of them;*

(iii) *there are internal vertices of $u-w$ (all adjacent to v_1 but none of them adjacent to a), and a is adjacent to some internal vertex in $b-u$ or $w-c$.*

If (i) or (iii) holds, we find G has a U-minor. If (ii) holds, we find G is a member of $\mathcal{S}^*$, since splitting u [or w] creates a series-parallel graph from G.

Now, we assume w is a leaf of T, but u is not a leaf. (This is equivalent to having u but not w be a leaf.) We must consider separately when u and w are adjacent in T, and when they are not. We begin with u and w adjacent in T.

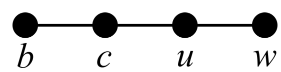


FIGURE 2.94: T_b

Start by considering T having the smallest order. When T follows all these restrictions and has exactly four vertices, it is tree T_b shown in Figure 2.94. To reconstruct G, we add to T the vertices and edges described in 2.84.3. However, we still have vertices v_1, a, and c with insufficient degree for G to be 3-connected. To increase the degrees of these vertices but avoid G failing to be minimally 3-connected by Theorem 2.78, we can choose from four pairs of edges. If G has edges (v_1, c) and (a, c), then $G - \{c, w\}$ is a tree. If G has edges (v_1, c) and (a, v_1), then $G - \{v_1, w\}$ is a tree. If G has edges (v_1, a) and (a, c), then $G - \{a, v\}$ is a tree. Finally, if G has edges (v_1, c) and (a, u), then G is a member of $\mathcal{S}^*$, since we can split any of vertices a, b, v, and v_1 to get a series-parallel graph from G.

When T has higher order, with vertices b and w as leaves, and u and w adjacent, the tree T_b shown in Figure 2.94 is still a sketch of the structure of T. Now, though, $c-u$ is a path of length greater than one. To reconstruct G, we add to T the vertices

and edges described in 2.84.3. Then, vertices a, v_1, c, and the internal vertices of $c-u$ have insufficient degree for G to be 3-connected; alternatively, we can say a, v_1, and the internal vertices of $b-u$ have insufficient degree. We keep c fixed as the neighbor of b.

2.84.10. *None of the following occurs:*

(i) aside from the edges specified by 2.84.3, G has v_1 adjacent to all the internal vertices of $b-u$, possibly edge (a,u), and no other edges;

(ii) aside from the edges specified by 2.84.3, G has a adjacent to all the internal vertices of $b-u$, and possibly edge (a,u), and no other edges;

(iii) G has edge (a,v_1).

Both (i) and (ii) lead to contradictions because $G-\{v_1,u\}$ and $G-\{a,u\}$ are trees, so $G \in \mathcal{S}$. If (a,v_1) is an edge of G, then only one of a and v_1 is adjacent to the internal vertices of subpath $b-u$ in T. Otherwise, G fails to be minimally 3-connected by Theorem 2.78, due to the 3-cycle formed by a, v_1, and b. Thus, for (iii), one of $G-\{v_1,u\}$ and $G-\{a,u\}$ is a tree.

Thus, both a and v_1 must be adjacent to internal vertices of $b-u$. We find, however, that it is not enough for one of them to be adjacent only to c but no other internal vertices of $b-u$.

2.84.11. *Neither of the following occurs:*

(i) aside from the edges specified by 2.84.3, G has edge (a,c), has v_1 adjacent to the internal edges of $c-u$, and has no other edges;

(ii) aside from the edges specified by 2.84.3,, G has edge (v_1,c), has a adjacent to the internal edges of $c-u$, and has no other edges.

If (i) occurs, then G is a daisy chain, since we can split either a or w to produce a series-parallel graph from G. If (ii) occurs, then G is again a daisy chain, since we can split either of vertices v and v_1 to get a series-parallel graph from G.

Based on 2.84.10 and 2.84.11, the only additional edges we can add to G to increase the degrees of a, v_1, and the internal vertices of $b - u$ must make both v_1 and a adjacent to internal vertices of $c - u$. Then G has a U-minor, which is a contradiction.

Next, assume the leaves of T are b and w, but that u is not a neighbor of w. This structure for T is sketched in Figure 2.95a; note that either of paths $b-u$ and $u-c$ may have length greater than one. We will assume that c is the neighbor of w in T, no matter the order of T. To reconstruct G, we add to T the vertices and edges described in 2.84.3. Then vertices a, v_1, and any internal vertices of $b-u$ and $u-c$ have insufficient degree for G to be 3-connected. So there are additional edges in G, and they are incident with v_1 or a. We make a few observations about some of the ways we can add these edges.

2.84.12. *None of the following occurs:*

(i) aside from the edges specified by 2.84.3, G has a adjacent to all internal vertices of T, vertex v_1 not adjacent to any internal vertex of T, possibly edge (a, u), and no other edges;

(ii) aside from the edges specified by 2.84.3, G has v_1 adjacent to any internal vertices of $b-u$ and $u-c$, but a not adjacent to any of such internal vertices, possibly edge (a, u), and no other edges;

(iii) G has edge (a, v_1).

If G has the edges described in (i), then $G - \{a, u\}$ is a tree; by Theorem 2.36, G is a member of $\mathcal{S}$. Similarly, if G has the edges described in (ii), then we get

$G - \{v_1, u\}$ is a tree; again, G is a member of $\mathcal{S}$ by Theorem 2.36. Finally, if (a, v_1) is an edge of G, then only one of a and v_1 can be adjacent to internal vertices of T. Otherwise, G fails to be minimally 3-connected by Theorem 2.78, due to the 3-cycle formed by a, v_1, and b. As in (i) and (ii), $G - \{a, u\}$ or $G - \{v_1, u\}$ is a tree.

FIGURE 2.95: Vertices u and w are not adjacent.

We separately consider T with leaves b and w but with u and w nonadjacent, and having exactly four vertices; this is T_c with edges as shown, in Figure 2.95a. Given the information in 2.84.12, there is only one situation to check. If we increase the degrees of a, v_1, and c by adding the edges (a, c) and (v_1, c), then $G - \{b, c\}$ is a tree, and, by Theorem 2.36, G is a member of $\mathcal{S}$.

We will make a few more observations to help us with the analysis when the order of T is at least five.

2.84.13. *None of the following occurs:*

(i) G has edges (a, r_1) and (v_1, r_2), where r_1 and r_2 are distinct internal vertex of $u - w$;

(ii) G has a adjacent to some internal vertex of $b - u$, and v_1 adjacent to some internal vertex of $u - w$;

(iii) G has a adjacent to two or more internal vertices of $u - w$, and v_1 adjacent to some internal vertex of $b - u$.

When G has the edges from (i) and $|T| = 5$, the graph G is isomorphic to S or H_8, depending whether a or v_1 is adjacent to c, respectively. For higher order T, we find G properly contains an S-minor or H_8-minor. If G has the edges from (ii)

and $|T| = 5$, then G is isomorphic to S. For higher order T, then, we get G has a proper S-minor. Lastly, if G has the edges from (iii), then G has a U-minor.

Based on our analyses of the subcases in 2.84.13 and 2.84.12, our only remaining considerations are when G is being reconstructed from a tree T with the length of its $u - w$ path being exactly two. The only internal vertex of $u - w$ is c, then. This situation is sketched in Figure 2.95b. Note that by 2.84.13.ii, we must have edge (a, c). However, if we also have (v_1, c), then G will fail to be minimally 3-connected, by Theorem 2.78, since one of v_1 and a is adjacent to internal vertices of $b - u$, thus making one of the 3-cycles v_1, c, w and a, c, w fail to have two vertices that are degree-3.

2.84.14. *Assume $u - w$ in T has length two, and that G has edge (a, c), but not (v_1, c). Let d be the neighbor of b in T. None of the following occurs:*

(i) G has all internal vertices of $b - u$ adjacent to v_1 but none of them adjacent to a;

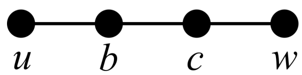
(ii) the only internal vertex of $b - u$ that is adjacent to a is d;

(iii) a is adjacent to some internal vertex of $b - u$ that is not d.

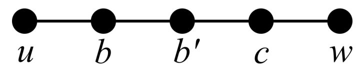
When G has edges as described in (i), then G is a member of $\mathcal{S}^*$; we can split either a or b and get a series-parallel graph from G. When G has edge (a, d) as described in (ii),then G again is a member of $\mathcal{S}^*$; we can split vertex a and get a series-parallel graph. If (iii) holds, then G has a U-minor.

To close our analysis of G having a G_B structure, we consider when T has both u and w as leaves. This structure of T is sketched in Figure 2.96. Note that path $b - c$ may be of length greater than one, but we will always assume that $N_T(u) = \{b\}$ and $N_T(w) = \{c\}$. To reconstruct G, we add to T the vertices and edges described

in 2.84.3. This makes a and the vertices of the path $b - c$ into degree-2 vertices, and v_1 is a degree-1 vertex. Clearly there must be other edges in G, or it is not minimally 3-connected, by Theorem 2.78. Observe that at least one of these edges will have v_1 and a vertex of $b - c$ as endpoints.



(a) T_d



(b) T_{d1}

FIGURE 2.96: Vertices u and w are leaves of T.

We make a few observations to start.

2.84.15. *None of the following occurs:*

(i) aside from the edges indicated by 2.84.3, G has v_1 adjacent to all vertices of $b - c$, vertex a adjacent to none of these vertices, edge (a, v_1), and no other edges;

(ii) aside from the edges indicated by 2.84.3, G has edges (v_1, a), (v_1, x) where x is a vertex on $b - c$, vertex a adjacent to every vertex of $b - c$ except possibly x, and no other vertices.

For (i), G is a member of $\mathcal{S}$ by Theorem 2.36, since $G - \{v_1, w\}$ is a tree. For (ii), G is again a member of $\mathcal{S}$, since $G - \{a, v\}$ is a tree. Thus, from 2.84.15, we can conclude that G has v_1 adjacent to at least two vertices of $b - c$, while a must be adjacent to at least one vertex of $b - c$.

We consider separately when T has order exactly four; this means T is the tree T_d shown in Figure 2.96a. As usual, to reconstruct G, we add to T the vertices and edges described in 2.84.3. Given our work in 2.84.15 and that we have to avoid a cycle forbidden by Theorem 2.78, there is only one set of additional edges we can

add to get the degrees of b, c, a, and v_1 higher, up to isomorphism. However, if G has these edges (v_1, b), (v_1, c), and (a, b), then $G - \{b, w\}$ is a tree.

So, we consider the additional edges we can add in reconstructing G from a tree T with order at least five, using what we know from 2.84.15. The tree with minimal such order is shown in Figure 2.96b. Recall that $N_T(u) = \{b\}$ and $N_T(w) = \{c\}$; we will furthermore insist that the other neighbor of b is vertex b' in any tree T.

2.84.16. *We supporess subcases that cause G to fail to be minimally 3-connected under Theorem 2.78, and also subcases that are isomorphic to one already listed.*

(i) G has edges (v_1, b) and (v_1, c), and a is adjacent to some internal vertex of $b - c$;

(ii) G has edges (a, b) and (a, x), where x is an internal vertex of $b - c$ and $x \neq b'$, and also v_1 is adjacent to all other vertices of $b - c$;

(iii) G has edge (a, b), vertex v_1 is adjacent to all other vertices of $b - c$, and a is not adjacent to any of these;

(iv) G has edges (a, b) and (a, b'), vertex v_1 is adjacent to all other vertices of $b - c$ (and possibly to b'), and vertex a has no other neighbors on $b - c$;

(v) G has edges (a, b) and (a, v_1), and v_1 is adjacent to c and an internal vertex of $b - c$;

(vi) G has edges (a, b) and (a, c), and v_1 is adjacent to b and an internal vertex of $b - c$;

(vii) G has edges (a, b) and (a, c), and v_1 is adjacent to two internal vertices of $b - c$.

Suppose G has the edges in (i); if T is isomorphic to T_{d1} (see Figure 2.96b), then G is isomorphic to Q_3. For T with higher order, G clearly has a proper Q_3-minor. Suppose G has the edges in (ii); then G has a proper Q_3-minor. Suppose G has the edges in (iii); then G is in $\mathcal{S}^*$, since we can split b or b' and get a series-parallel graph. Likewise, if G has the edges in (iv), then then G is in $\mathcal{S}^*$, since we can split b'. If G has the edges in (v), then G has a K_5-minor. Lastly, if G has the edges in (vi) or (vii), then G has an R-minor.

2.85. *G has G_C structure*

We begin this subcase with a lemma that provides a lower bound on the order of G.

2.85.1. *If G has the G_C structure, then $|G| \geq 8$.*

From Figure 2.71c, we can see G must already contain six vertices, namely, u, v, u_1, v_1, u_2, and v_2. We assume to the contrary that G has exactly seven vertices; let this seventh vertex be a. Consider the cosimplification of $G\backslash f$, where we contract edges (u, u_2) and (v, v_2), labeling the resulting conglomerate vertices by u' and v', respectively. Thus, $co(G\backslash f) \cong G_3$. Since G_3 is 3-connected, every vertex in G_3 has a minimum degree of three. So a must be adjacent to at least three of u_1, u_2, v_1, and v_2. Note that, since we are considering G having a G_C structure, G_3 has edges (u_1, u_2) and (v_1, v_2).

Say a is adjacent to exactly three of u_1, u_2, v_1, and v_2, and, without loss of generality, that these are v_1, v_2, and u_1. Then we must make u_2 adjacent to both v_1 and v_2 to get its degree up to three, and, without loss of generality, we add (u_1, v_1) to raise the degree of u_1. Observe that any additional edges are either forbidden by the G_C structure or result in a G that is not minimally 3-connected. So this is our candidate for G, where G_3 is isomorphic to $K_5\backslash e$ for any edge e of

K_5. Thus, G is a member of $\mathcal{S}$ by Theorem 2.36, since $G - \{v_1, v_2\}$ is a tree, which is a contradiction.

Now let a be adjacent to all four of u_1, v_1, u_2 and v_2. We must raise the degrees of u_1 and u_2. To accomplish this with u_2, without loss of generality, we add edge (u_2, v_1). There are no more edges incident with u_2 that can be in G – either the potential edges cannot exist with a G_C structure, or the edges make G fail to be minimally 3-connected, by Theorem 2.78. Now, to raise the degree of u_1, our only option is to add edge (u_1, v_2), since (u_1, v_1) makes G fail Theorem 2.78. However, this means G_3 is again isomorphic to $K_5 \backslash e$, and G is in $\mathcal{S}$, as $G - \{a, v\}$ is a tree. We conclude that G must contain at least eight vertices.

2.85.2. *If G has the G_C structure, then $|T| \geq 5$, where T is the tree resulting from deleting two vertices of G/e.*

Recall our assumption in this section that, for some edge e, the graph G/e is a 3-connected member of $\mathcal{S}$. Thus, by Theorem 2.36, there are two vertices of G/e, call them x and y, whose deletion produces a tree T. Also recall that x must be u' (or v'). Thus, T has three vertices fewer than G.

In this third and final main subcase of the section, we assume that G has the G_C structure, as shown in Figure 2.71c. By the proof of Theorem 2.81, the edges (u, u_2) and (v, v_2) are nonessential in G. Let $e = (u, u_2)$, and let u' be the conglomerate vertex resulting from the contraction of e. Since we know G is minimally 3-connected, graph G/e is a 3-connected member of $\mathcal{S}$. By Theorem 2.36, G/e has two vertices x and y whose deletion creates a tree T. Observe that u' must be one of these vertices, or else G would be a member of $\mathcal{S}$. We take $x = u'$.

2.85.3. *The graph G can be reconstructed from T by doing the following:*

(i) add vertices u and u_1 with their edge e,

(ii) add vertex y,

(iii) add edges (u, v) and (u, u_2),

(iv) add edges between u_1 and any leaves of T that are not v or u_2,

(v) add edges between y and each of the leaves of T,

(vi) consider possible additional edges incident with u_1 or y.

Note the edges in (iii) are required by the G_C structure of G. We add the edges in (iv) and (v) because G is 3-connected, so by Theorem 1.5 it follows that every vertex in G has minimum degree of three. However, we are careful in (iv) to not make any vertex adjacent to both u and u_1.

We will break our argument down based on whether T is a path or not:

(a) T has at least three leaves;

(b) T has exactly two leaves.

In both of these subcases, we proceed the same way. We consider whether y is one of the vertices that the G_C structure specifies (namely, u_2, v, v_1, or v_2), or y is some vertex about whose adjacencies we know nothing. For each choice of y, the G_C structure of G places limits on the trees that can be T. We then reconstruct G from T as outlined in 2.85.3.

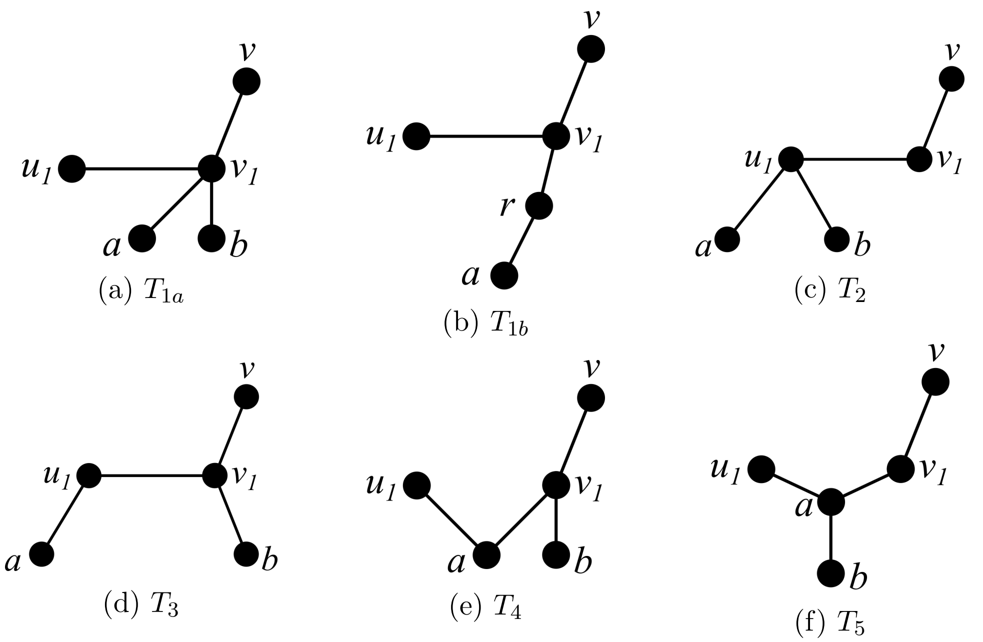
2.85.4. *Let T have at least three leaves.*

Suppose $y = v$. From Figure 2.71c, we can see that $N_G(v) = \{u, v_1, v_2\}$. We observed in 2.85.3(v) that y must be adjacent to every leaf of T, in order for every vertex of G to have high enough degree for G to be 3-connected; but u is not a leaf of T. Thus, there is some leaf of T to which $y = v$ is not adjacent. This means G has a degree-2 vertex, a contradiction.

Suppose $y = v_2$. Note that this is equivalent to taking $y = v_1$. Since $N_G(v) = \{u, v_1, v_2\}$, we know v must be a leaf of T; however, since edge (v_1, v_2) cannot exist in G, we also know v_1 cannot be a leaf of T. We may or may not have u_1 as a leaf of T, the vertices u_1 and v_1 may or may not be neighbors in T, and the other leaves of T that are not v or u_1 may have paths to u_1 that avoid v_1, or to v_1 that avoid u_1. If $|T| = 5$, then T can be one of five trees; see Figure 2.97. We also show a few of the structures possible for T when T is a tree with six vertices, in Figure 2.98. We break the analysis of the $y = v_2$ subcase down based on the structure of T.

Let edge (u_1, v_1) be in T. Suppose u_1 is not a leaf of T; and that u_1 has a path to a leaf that avoids v_1, and likewise v_1 has a path to a leaf (aside from v) that avoids u_1. An example of T with this structure is T_3, shown in Figure 2.97d. Should T have higher order than five, then paths $u_1 - a$ and $v_1 - b$ may have length greater than one, or there may be additional leaves of T that are on paths of length at least one to u_1 that avoid v_1, or to v_1 that avoid u_1. To reconstruct G, we add to T the vertices and edges described in 2.85.3. If T is T_3, then G is isomorphic to H_8, one of our known excluded minors. If T has higher order, then we can see G has a proper H_8-minor.

Let the $u_1 - v_1$ path in T be of length greater than one. Suppose u_1 is a leaf of T, and that any leaf of T, besides u_1 and v, has a path to each of u_1 and v_1 that avoids v_1 and u_1, respectively. An example of T with this structure is T_5, shown in Figure 2.97f. Should T have higher order than five, then path $a - b$ may have length greater than one, path $u_1 - v_1$ may have length greater than two, or there may be additional leaves of T on paths to u_1 and v_1 that avoid v_1 and u_1, respectively. To reconstruct G, we add to T the vertices and edges described in 2.85.3. If T is T_5, then G is isomorphic to H_8. If T has higher order, then we can see G has a



(a) T_{1a} (b) T_{1b} (c) T_2

(d) T_3 (e) T_4 (f) T_5

FIGURE 2.97: T has three leaves and order five, $y = v_2$.

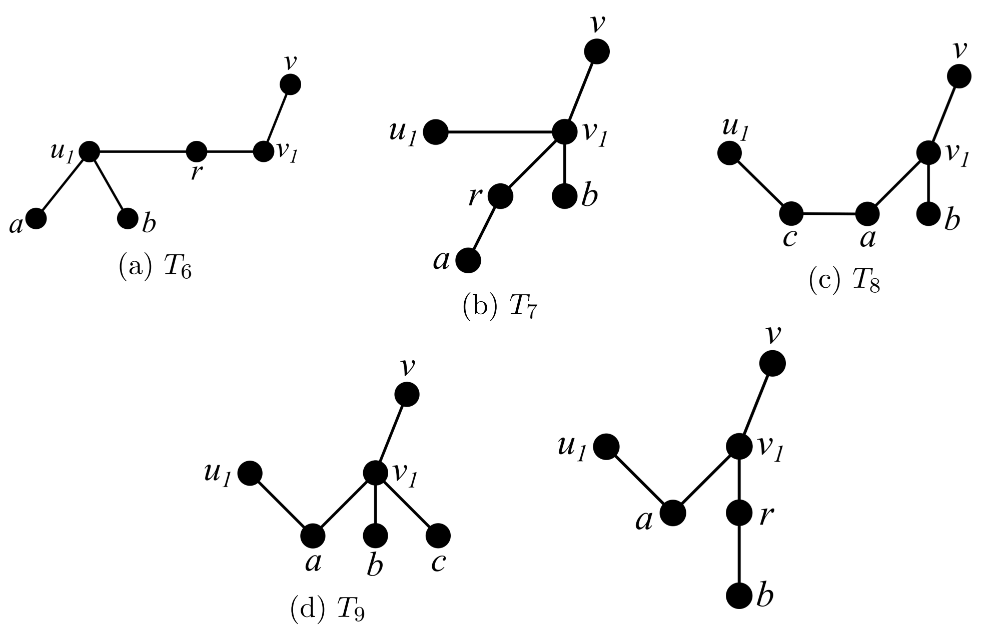
proper H_8-minor. Note that if we alter the structure of T so that u_1 is not a leaf, but otherwise keep everything the same, then G still has an H_8-minor.

Let edge (u_1, v_1) be in T. Suppose u_1 is a leaf of T, and that any leaf of T, besides u_1 and v, is on a path to v_1 that avoids u_1. Examples of T with this structure are T_{1a} and T_{1b}, Figures 2.97a and 2.97b, which show the two possible trees with five vertices; and T_7 in Figure 2.98b, which is one example of T having six vertices. First, say T has v_1 is a neighbor of every leaf of T, such as we see in T_{1a}. To reconstruct G, we add to T the vertices and edges described in 2.85.3. There are no more edges we can add to reconstruct G, without creating a cycle that causes G to fail to be minimally 3-connected by Theorem 2.78. So we have completed G, but $G - \{v_1, v_2\}$ is a tree; thus, by Theorem 2.36, G is a member of $\mathcal{S}$, a contradiction.

Say the length of at least one path from v_1 to a leaf of T, call it $v_1 - a$, is at least two, such as we see in T_{1b} or T_7. To reconstruct G, we add to T the vertices and edges described in 2.85.3. The internal vertices of $v_1 - a$ and any other paths from v_1 to leaves that have length at least two are still degree-2 vertices. Since G is

3-connected, there must be other edges in G, incident with these internal vertices. If v_2 is adjacent to all these internal vertices and u_2 is adjacent to none, then $G - \{v_1, v_2\}$ is a tree, and, by Theorem 2.36, G is a member of $\mathcal{S}$. However, if u_2 is adjacent to any internal vertex of a v_1 - leaf path of T, then G has an S-minor. In fact, if T is T_{1b} and we add edge (u_2, r), then G is isomorphic to S.

Observe that if all leaves of T besides v have paths to u_1 that avoid v_1, due to the G_C structure of G, we will produce the same graphs for G that we did when we said all leaves except u_1 had paths avoiding u_1 to v_1. That is, reconstructing G from tree T_2 in Figure 2.97c will get us the same graphs for G as reconstructing from T_{1a}; as another example, reconstructing from T_6 or T_9 will produce isomorphic graphs for G. So we will just consider T with leaf u_1, and all other leaves on paths to v_1 that avoid u_1, to keep from redundant analysis.



(a) T_6 (b) T_7 (c) T_8 (d) T_9 (e) T_{10}

FIGURE 2.98: Selected examples of when T has three leaves and order six, $y = v_2$.

We still must consider what occurs when T has path $u_1 - v_1$ with length at least two, vertex u_1 is a leaf, and vertex v_1 has a path to every leaf, besides u_1,

that avoids u_1. Examples of T exhibiting this structure are T_4 in Figure 2.97e; and T_8, T_9, and T_{10}, shown in Figure 2.98. To reconstruct G, we add to T the vertices and edges described in 2.85.3. After doing this, though, we still have some degree-2 vertices: the internal vertices of path $u_1 - v_1$, any internal vertices on paths from v_1 to leaves of T, and possibly u_2, if T only has three leaves (see T_4 or T_8 for examples). So G certainly has more edges that are incident with v_2 or u_2. If G has an edge incident with u_2 and some internal vertex of $u_1 - v_1$, then G has an H_8-minor. In fact, if T is T_4 and G has edge (u_2, a), then G is isomorphic to H_8.

So we assume that v_2 is adjacent to all internal vertices of $u_1 - v_1$, but that u_2 is not adjacent to any of these vertices. Assume that the paths between v_1 and all leaves except u_1 are length one. Examples of T with this structure include T_4, T_8, and T_9. Then $G - \{v_1, v_2\}$ is a tree. This is a contradiction, since Theorem 2.36 tells us G is a member of $\mathcal{S}$. So assume that at least one path from v_1 to a leaf of T besides u_1 has length two or more. An example of T with this structure is T_{10}. We know from our work on T_{1b} that if u_2 is adjacent to any internal vertex of such a path, then G will have an S-minor. On the other hand, if u_2 is not adjacent to any internal vertices of paths between v_1 and leaves of T, then $G - \{v_1, v_2\}$ is again a tree.

Suppose $y = u_1$. We know that $N_G(u) = \{v, u_1, u_2\}$. So, after we reconstruct G as outlined in 2.85.3, we have $G - \{u_1, u_2\}$ is a tree; to be precise, the tree is T plus a new leaf, v. Therefore, G is a member of $\mathcal{S}$, by Theorem 2.36. This is a contradiction.

Suppose $y = a$, where $a \notin \{u, v, u_1, v_1, u_2, v_2\}$. So vertex v cannot be a leaf of T, and at most one of v_1 and v_2 can be a leaf of T, say v_2, without loss of generality. Vertex u_1 may or may not be a leaf, and may or may not be adjacent to v_1. There is only one tree that meets all our requirements for T and has order exactly five,

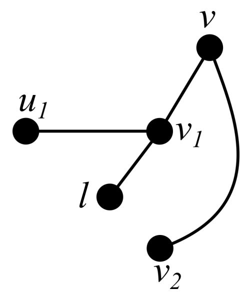
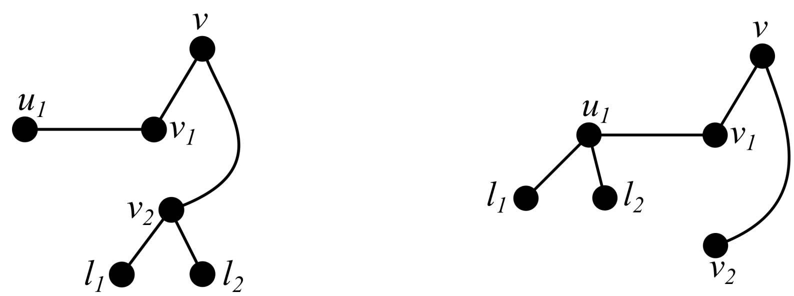


FIGURE 2.99: T when $y = a$ and $|T| = 5$



(a) T_1 (b) T_2

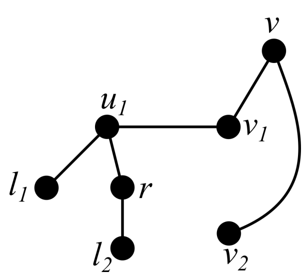
FIGURE 2.100: Examples of T having three leaves, when $y = a$ and $|T| = 6$.

shown in Figure 2.99. Examples of trees with the required structure for T and higher order are shown in Figures 2.100 and 2.101. We will break this subcase down based on the structure of T.

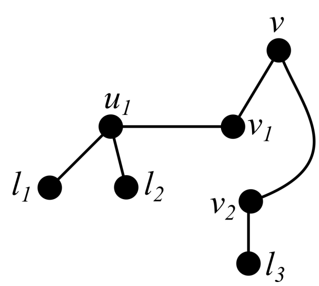
Say both u_1 and v_2 are leaves of T, the length of $u_1 - v_1$ is one or more, and the other leaf or leaves of T have paths to v_1 that avoid u_1, v, and v_2. An example of this structure for T is the 5-vertex case, in Figure 2.99. To reconstruct G, we add to T the vertices and edges described in 2.85.3. We can then see that G has an H_8-minor. When T is of order exactly five, we get that G is isomorphic to H_8; otherwise, G has a proper H_8-minor. Thus, we now need to consider the various structures of T where no leaf has a path to v_1 in T that avoids all of u_1, v_2, and v; that is, at most one of u_1 and v_2 is a leaf in T.

Say u_1 is a leaf of T but not v_2, the length of $u_1 - v_1$ is one or more, and every leaf of T, besides u_1, has a path to v_2 that avoids u_1, v_1, and v. An example of this structure of T is shown in Figure 2.100a. To reconstruct G, we add to T the

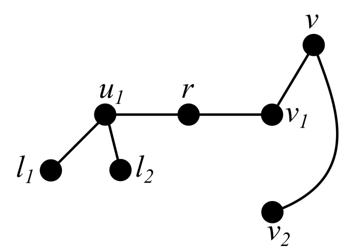
vertices and edges described in 2.85.3. Any internal vertices of $u_1 - v_1$, vertex v_1, and any internal vertices of paths $v_2 - l_i$, where l_i is a leaf of T and these internal vertices were degree-2 in T, still are degree-2. Since G is 3-connected, there must be other edges that are incident with these vertices and a or u_2. In particular, G must include one of edges (u_2, v_1) or (a, v_1); and now we can see that G must have an S-minor. Thus, we can restrict the structures of T that we still need to consider even more, ruling out T where two or more leaves have paths to v_2 that avoid v, v_1, and u_1 – this means G has an S-minor.



(a) T_a



(b) T_b



(c) T_c

FIGURE 2.101: Examples of T having three leaves, when $y = a$ and $|T| = 7$.

Say v_2 is a leaf of T but not u_1, the length of $u_1 - v_1$ is one or more, and every leaf l_i of T, besides v_2, has a path to u_1 that avoids v_2, v, and v_1. Examples of T with this structure are shown in Figures 2.100b, 2.101a, and 2.101c. To reconstruct G, we add to T the vertices and edges described in 2.85.3. Then any internal vertices of $u_1 - v_1$, any internal vertices of $u_1 - l_i$ such that these internal vertices were degree-2 in T, and vertex v_1 are still degree-2 vertices. So, since G is 3-connected, there must be other edges, incident with these vertices and a or u_2. We note that it is also possible for a or u_2 to be adjacent to internal vertices on paths $u_1 - l_i$ whose degree is three or more in T. If a is adjacent to v_1 or any internal vertex of $u_1 - v_1$, then G has a Q_3-minor. If a is adjacent to any internal vertex of a $u_1 - l_i$ path, then G has an S-minor. However, if a is adjacent to none of these,

it follows that u_2 must be adjacent to all the vertices that had insufficient degree (and possibly to internal vertices of $u_1 - l_i$ whose degree in T was already three); then $G - \{u_1, u_2\}$ is a tree. By Theorem 2.36, G is a member of $\mathcal{S}$.

Say neither of u_1 and v_2 is a leaf of T, the length of $u_1 - v_1$ is one or more, and every leaf l_i of T either has path $u_1 - l_i$ that avoids v, v_1, and v_2, or path $v_2 - l_i$ that avoids u_1, v_1, and v. An example of a tree exhibiting this structure is shown in Figure 2.101b. To reconstruct G, we add to T the vertices and edges described in 2.85.3. The following vertices are still degree-2: vertex v_2, vertex v_1, any internal vertices of $u_1 - v_1$, and any internal vertices of $u_1 - l_i$ or $v_2 - l_j$ that were degree-2 in T. So, since G is 3-connected, there must be other edges, incident with these vertices and a or u_2. Note that G may also contain edges incident with a or u_2 and internal vertices of paths $u_1 - l_i$ or $v_2 - l_j$ that had degree at least three in T. If a is adjacent to v_1 or any internal vertex of $u_1 - v_1$, then G has a Q_3-minor. If a is adjacent to v_2, any internal vertex of a $u_1 - l_i$ path, or any internal vertex of a $v_2 - l_j$ path, then G has an S-minor. Lastly, if a is not adjacent to any of these vertices, it follows that u_2 must be adjacent to the vertices that had insufficient degree (and possibly to internal vertices of $u_1 - l_i$ or $v_2 - l_j$ whose degree in T was already three), and then $G - \{u_1, u_2\}$ is a tree. By Theorem 2.36, we find G is a member of $\mathcal{S}$.

2.85.5. *Let T have exactly two leaves.*

Suppose $y = v$. Since G has a G_C structure, we know $N_G(v) = \{u, v_1, v_2\}$. We observed in 2.85.3(v) that y must be adjacent to each leaf of T, in order for every vertex of G to have high enough degree that G is 3-connected. So the leaves of T must be v_1 and v_2. However, the G_C structure tells us that edge (u_1, u_2) cannot

exist; but then either T is disconnected, or it has more than two leaves, both of which are clear contradictions. Thus, y cannot be v.

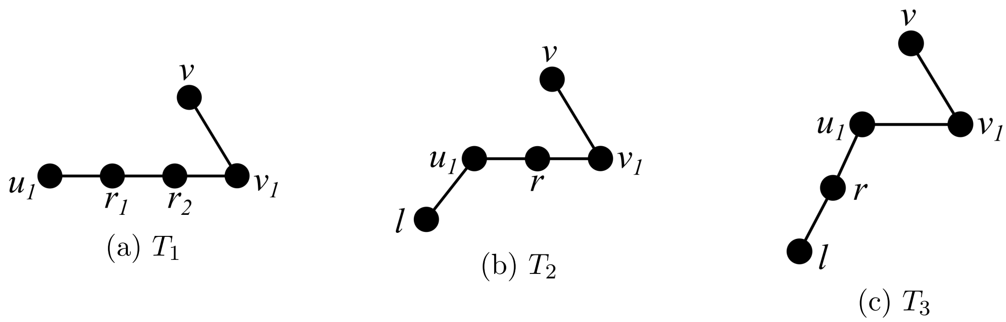


FIGURE 2.102: T when $y = v_2$, with exactly two leaves and $|T| = 5$.

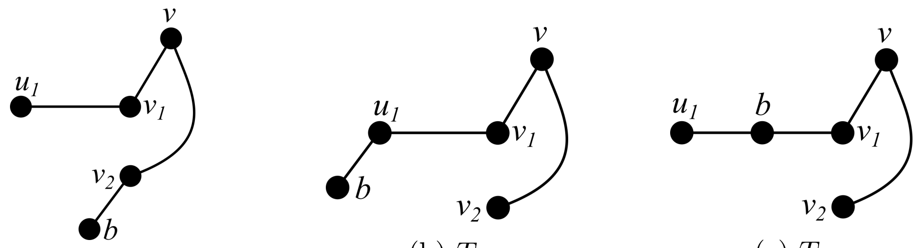
Suppose $y = v_2$. Note that this is equivalent to considering $y = v_1$, since G has a G_C structure. We cannot have v_1 as a leaf of T, or we should be forced to have edge (v_1, v_2) in G, which G_C forbids. Also, since $N_G(v) = \{u, v_1, v_2\}$, vertex v must be a leaf of T. The other leaf may be u_1, or a vertex not specified by the G_C structure. Examples of T with order five are shown in Figure 2.102. We will subdivide our argument based on the other leaf of T, and whether u_1 and v_1 are adjacent in T.

Assume u_1 and v are the leaves of T. Then path $u_1 - v_1$ has length at least two; the tree T with order exactly five is shown in Figure 2.102a. To reconstruct G, we add to T the vertices and edges described in 2.85.3. Then v_1, v_2, all internal vertices of $u_1 - v_1$ are degree-2; and u_2 is a degree-1 vertex. As G is 3-connected, there must be other edges, incident with these vertices and v_2 or u_2. Thus, G must include edge (u_2, v_1), since the G_C structure forbids (v_1, v_2). If u_2 is adjacent to all the internal vertices of $u_1 - v_1$ (but none of them is adjacent v_2), then $G - \{u_1, u_2\}$ is a tree. Similarly, if v_2 is adjacent to all the internal vertices of $u_1 - v_1$ and u_2 is adjacent to none of them, then $G - \{v_2, v_1\}$ is a tree. By Theorem 2.36, therefore, G is a member of $\mathcal{S}$, a contradiction. So, each of v_2 and u_2 is adjacent to some internal vertex of $u_1 - v_1$.

Let the vertices of subpath $u_1 - v_1$ of T be $u_1, r_1, r_2, \ldots, r_m, v_1$. If G has an edge (v_2, r_i) such that there is also edge (u_2, r_j) where $i > j$, then G has a Q_3-minor. In fact, if T has order five, which is illustrated in Figure 2.102a, then G is isomorphic to Q_3. If there is no edge (u_2, r_j) where $i > j$ but G does have edge (u_2, v_2), then G has a K_5-minor. Suppose there is no (u_2, r_j) where $i > j$ and G does not have edge (u_2, v_2). In the path $u_1 - v_1$, let i' be the greatest index i such that vertex r_i is adjacent to v_2, and let j' be the least index j such that vertex r_j is adjacent to u_2. Then we can split either of vertices $r_{i'}$ and $r_{j'}$ to get a series-parallel graph from G. Thus, G is a member of $\mathcal{S}^*$, a contradiction. Note that $r_{i'}$ and $r_{j'}$ may be the same vertex.

Now assume that the leaves of T are v and l, where the path $u_1 - l$ has length at least one. Recall that by 2.85.2 the order of T is at least five. Therefore one (or both) of paths $u_1 - l$ and $u_1 - v_1$ must be of length at least two. To reconstruct G, we add to T the vertices and edges described in 2.85.3. Then any internal vertices of $u_1 - v_1$ and $u_1 - l$, as well as v_1, u_2, and v_2, are degree-2 vertices. So, since G is 3-connected, there must be other edges, incident with these vertices and v_2 or u_2. Note that G must contain edge (v_1, u_2), since v_1 and v_2 cannot be adjacent in a G_C structure. If v_2 is adjacent to any internal vertex of $u_1 - v_1$, then G has an H_8-minor. In fact, if T has exactly five vertices, it is the tree pictured in Figure 2.102b, and the resulting G is isomorphic to H_8. If v_2 is adjacent to any internal vertex of $u_1 - l$, then G has an S-minor. In particular, if T has exactly five vertices, it is the tree shown in Figure 2.102c, and G is isomorphic to S. Finally, if v_2 is not adjacent to any internal vertex of either $u_1 - v_1$ or $u_1 - l$, it follows that u_2 is adjacent to all those vertices (and v_2 is adjacent to u_1 or u_2). Then $G - \{u_1, u_2\}$ is a tree, and G is a member of $\mathcal{S}$ by Theorem 2.36.

Suppose $y = u_1$. We know that $N_G(u) = \{v, u_1, u_2\}$. So, after we reconstruct G as outlined in 2.85.3, we have $G - \{u_1, u_2\}$ is a tree; specifically, it is T with the added leaf v. Thus, y is not u_1.



(a) T_1 (b) T_2 (c) T_3

FIGURE 2.103: T with exactly two leaves, for $y = a$ and $|T| = 5$.

Suppose $y = a$. Since G has a G_C structure, v cannot be a leaf of T. Vertex u_1 may or may not be a leaf of T, and likewise v_2. Note that v_2 being a leaf or not is equivalent to v_1 being a leaf or not. The subpath $u_1 - v_1$ of T has length at least one. Examples of trees with this structure are shown in Figures 2.103 and 2.104. We subdivide our argument based on whether or not u_1 and v_2 are leaves of T.

Say both u_1 and v_2 are leaves of T. This structure of T is shown for a tree with five vertices in Figure 2.103c. We note that, for higher order T, the path $u_1 - v_1$ has length greater than two. To reconstruct G, we add to T the vertices and edges described in 2.85.3. Then vertices v_1, u_2, and a, and any internal vertices of $u_1 - v_1$ are still degree-2. So there must be other edges in G, incident with these vertices and a or u_2. If u_2 is adjacent to v_1 and to every internal vertex of $u_1 - v_1$, but a is adjacent to none of them, G must have edge (a, u_2), as well; but then $G - \{u_1, u_2\}$ is a tree. So G is a member of $\mathcal{S}$ by Theorem 2.36. Likewise, if a is adjacent to v_1 and to every internal vertex of $u_1 - v_1$, but u_2 is adjacent to none of them, G must have edge (a, u_2); and now $G - \{u, a\}$ is a tree.

However, if a is adjacent to v_1 or an internal vertex of $u_1 - v_1$, and u_2 is also adjacent to at least one of these vertices, then G has an S-minor or an H_8-minor.

In particular, if T has minimum order, then T is T_3 in Figure 2.103c. If G, reconstructed from T_3, has edges (b, u_2) and (v_1, a), then $G \cong H_8$, and if G has edges (b, a) and (v_1, u_2), then $G \cong S$.

Based on this last subcase, we now make an observation that is helpful for the upcoming subcases where we assume T has at least one of u_1 and v_2 not a leaf.

2.85.6. *(i) If G has vertex a adjacent to v_1, and u_2 adjacent to some internal vertex of $u_1 - v_1$, or vice versa, then G must have an H_8-minor or an S-minor.*

(ii) If G has both a and v_1 adjacent to internal vertices of $u_1 - v_1$, then G has an H_8-minor or an S-minor.

Thus, we are forced to have u_2 adjacent to v_1 and all internal vertices of $u_1 - v_1$, while a is not adjacent to any of these vertices; or the reverse, that a is adjacent to v_1 and all internal vertices of $u_1 - v_1$, but u_2 is not adjacent to any of them. Moreover, we observe the following:

2.85.7. *(i) If u_2 is adjacent to internal vertices of T, but a is not adjacent to any, then $G - \{u_1, u_2\}$ is a tree.*

(ii) If u_1 is a leaf of T, and a is adjacent to internal vertices of T, but u_2 is not adjacent to any, then $G - \{u, a\}$ is a tree.

Say u_1 but not v_2 is a leaf of T. One or both of paths $u_1 - v_1$ and $v_2 - l$, where l is the other leaf of T, may be of length greater than one. Examples of T with this structure are shown in Figures 2.103a, 2.104a, and 2.104c. To reconstruct G, we add to T the vertices and edges described in 2.85.3. Then any internal vertices of $u_1 - v_1$ and $v_2 - l$, and vertices v_1, v_2, u_2, and a remain degree-2. Now since G is 3-connected, there must be other edges, incident with a or u_2. If u_2 is adjacent to v_1 or some internal vertex of $u_1 - v_1$, and a is adjacent to v_2 or some internal

vertex of $v_2 - l$, then G has an S-minor. In particular, $G \cong S$ when T is T_1, that is, when T has exactly five vertices.

For the remainder of our analysis with u_1 but not v_2 a leaf, we will further divide our argument based on the lengths of paths $u_1 - v_1$ and $v_2 - l$. Say both of these paths have length one. Then T is T_1 shown in Figure 2.103a. We do not consider sets of edges which, if added, cause G to fail to be minimally 3-connected by Theorem 2.78. Then the other edges in G, aside from those described in 2.85.3, are (a, v_1), (u_2, v_2), and possibly (a, u_2), and G is a member of $\mathcal{S}^*$. We can split a to get a series-parallel graph from G.

Next, assume subpath $v_2 - l$ of T has length one, but $u_1 - v_1$ has length at least two. An example of T would be Figure 2.104a. By 2.85.6, we know that all of the internal vertices of $u_1 - v_1$ and v_1 are adjacent to a and none of them to u_2, or they are all adjacent to $u - 2$ and none of them to a. If a is adjacent to all of the internal vertices of $u_1 - v_1$ and v_1, we must have, based on 2.85.7, that G has edge (u_2, v_2); then G has a U-minor. If u_2 is adjacent to all of the internal vertices of $u_1 - v_1$ and v_1, and G thus has edge (a, v_2), then G has an S-minor.

Assume $u_1 - v_1$ has length one, but $v_2 - l$ has length at least two. An example of T would be Figure 2.104c. Note that G cannot have both (u_2, v_1) and (a, v_1), or G has a 3-cycle that causes G to not be minimally 3-connected, by Theorem 2.36. By 2.85.7, we know u_2 and a must both be adjacent to internal vertices of T; and we know already that if u_2 is adjacent to v_1 and a is adjacent to some internal vertex of $v_2 - l$ or v_2, then G has an S-minor. Therefore, G has edge (a, v_1) but not (u_2, v_1). If G has only u_2 adjacent to v_2 and all internal vertices of $v_2 - l$, while a is adjacent to none of these, then G is a member of $\mathcal{S}^*$; we can split a and get a series-parallel graph from G. Let l' be the neighbor of leaf l in T. If G has u_2 adjacent to v_2 and all internal vertices of $v_2 - l$ except possibly l', and a is adjacent

to only l' of the internal vertices of $v_2 - l$, then G is still a member of $\mathcal{S}^*$, and we can split a. If a is adjacent to an internal vertex of $v_2 - l$ that is not l', then G has a U-minor.

Finally, suppose both $u_1 - v_1$ and $v_2 - l$ have length at least two. This entire subcase falls out by the prior subcase, where $v_2 - l$ had length one. We review the results. By 2.85.6, we know that v_1 and all of the internal vertices of $u_1 - v_1$ are adjacent to a and none of them to u_2, or vice versa. Also, 2.85.7 tells us that whichever of a and u_2 is not adjacent to all of the internal vertices of $u_1 - v_1$ and v_1 must be adjacent to v_2 or an internal vertex of $v_2 - l$. If u_2 is adjacent to all of the internal vertices of $u_1 - v_1$ and v_1, then we already know G has an S-minor. If a is adjacent to all of the internal vertices of $u_1 - v_1$ and v_1, then we already know G has a U-minor. So we are done with the subcase where u_1 is a leaf of T and v_2 is not.

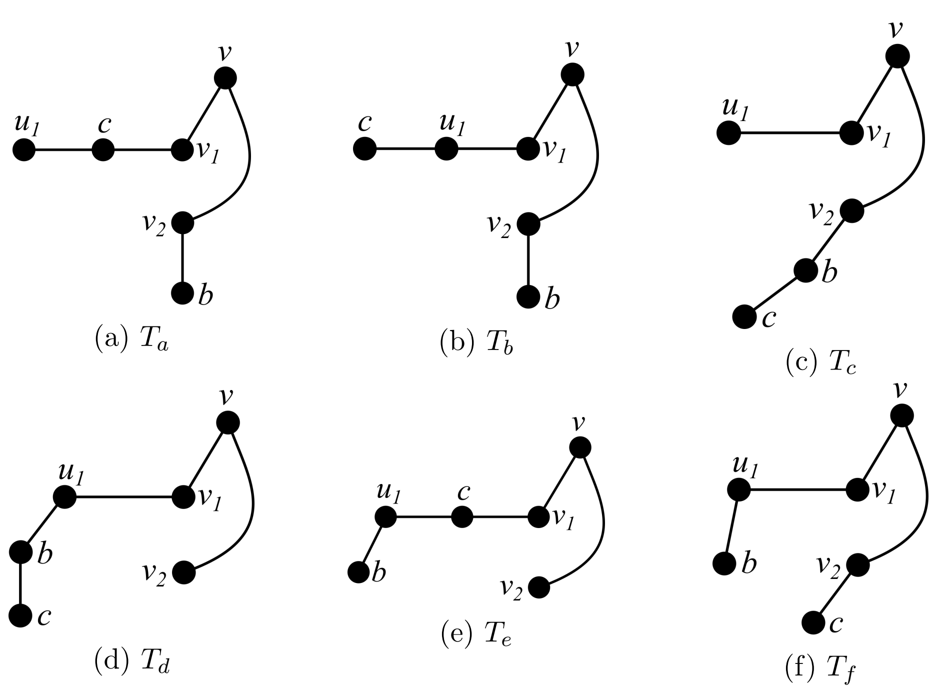


FIGURE 2.104: T when $y = a$, with exactly two leaves and order six.

Say v_2 but not u_1 is a leaf of T. One or both of paths $u_1 - v_1$ and $u_1 - l$, where l is the other leaf of T, may be of length greater than one. Examples of T with this structure are shown in Figures 2.103b, 2.104d, and 2.104e. To reconstruct G, we add to T the vertices and edges described in 2.85.3. Then any internal vertices of $u_1 - v_1$ and $u_1 - l$, and vertices v_1 and a are degree-2. So, since G is 3-connected, there must be other edges, incident with a or u_2. If a is adjacent to v_1 or an internal vertex of $u_1 - v_1$, then G has a Q_3-minor. In particular, if T has minimum order, then we are rebuilding G from T_2; the graph G has edge (a, v_1), and G is actually isomorphic to Q_3. If a is adjacent to some internal vertex of $u_1 - l$, then G has a U-minor. If u is adjacent to all the internal vertices of $u_1 - v_1$ and $u_1 - l$, and vertex v_1, then G must also have (a, u_1) or (a, u_2); however, $G - \{u_1, u_2\}$ is a tree.

Our very last consideration is when neither u_1 nor v_2 is a leaf of T. Let the leaves of T be l_1, where $u_1 - l_1$ avoids v_2; and l_2, where $v_2 - l_2$ avoids u_1. The paths $u_1 - v_1$, $u_1 - l_1$, and $v_2 - l_2$ may all have length one or greater in T. Examples of T with this structure are shown in Figures 2.104b and 2.104f. To reconstruct G, we add to T the vertices and edges described in 2.85.3. Then any internal vertices of $u_1 - v_1$, $u_1 - l_1$, and $v_2 - l_2$; and vertices v_1, v_2, and a are degree-2 vertices still. So, since G is 3-connected, there must be other edges, incident with a or u_2. All the following conclusions can be deduced from the structures of T where at least one of u_1 and v_2 is a leaf. We summarize them briefly. If a is adjacent to v_1 or any internal vertex of $u_1 - v_1$, then G has a Q_3-minor. If a is adjacent to v_2 or any internal vertex of $v_2 - l_2$, then G has an S-minor. If a is adjacent to an internal vertex of $u_1 - l_1$, then G has a U-minor. Lastly, if u_2 is adjacent to internal vertices of T, but a is adjacent to at most u_1 from T, and possibly u_2, then $G - \{u_1, u_2\}$ is a tree. Thus, by Theorem 2.36, G is a member of $\mathcal{S}$, which is a contradiction.

This concludes our proof of Case 4.

References

[1] R. Bixby, A simple theorem on 3-connectivity, *Linear Algebra Appl.* **45** (1982), 123-126.

[2] T. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer Math. Soc.* **171** (1972), 235-282.

[3] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Third edition, Chapman & Hall, London, 1996.

[4] W. Cunningham, On matroid connectivity, *J. Combin. Theory Ser. B* **30** (1981), 94-99.

[5] W. Cunningham and J. Edmonds, A combinatorial decomposition theory, *Canad. J. Math.* **32** (1980), 734-765.

[6] R. Dawes, Minimally 3-connected graphs *J. Combin. Theory Ser. B* **40** (1986), 159-168.

[7] R. Diestel, *Graph Theory*, Fourth edition, corrected printing, Springer, Berlin, 2012.

[8] G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27** (1952), 85-92.

[9] G. A. Dirac, Some results concerning the structure of graphs, *Canad. Math. Bull.* **6** (1963), 183-210.

[10] D. Eppstein, Parallel recognition of series-parallel graphs, *Inform. and Comput.* **98** (1992), 41-55.

[11] J. Geelen, B. Gerards, and G. Whittle, The highly connected matroids in minor-closed classes, *Ann. Comb.* **19** (2015), 107-123.

[12] A. J. Goodall, A. de Mier, S. D. Noble, and M. Noy, The Tutte polynomial characterizes simple outerplanar graphs, *Combin. Probab Comput.* **20** (2011), 609-616.

[13] R. Halin, Untersuchungen über minimale n-fach zusammenhängende Graphen, *Math. Ann.* **182** (1969), 175-188.

[14] R. Halin and H. A. Jung, Über Minimalstrukturen von Graphen, insbesondere von n-fach zusammenhängenden Graphen, *Math. Ann.* **152** (1963), 75-94.

[15] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 96-115.

[16] B. Oporowski, Graph theory, Course lecture notes, Louisiana State University, https://www.math.lsu.edu/ bogdan/7410/handout.pdf, Accessed May 1, 2013.

[17] J. Oxley, *Matroid Theory*, Second edition, Oxford University Press, Oxford, 2011.

[18] J. Oxley and T. Reid, The smallest rounded sets of binary matroids, *European J. Combin.* **11** (1990), 47-56.

[19] J. Oxley and J. Taylor, On two classes of nearly binary matroids, *European J. Combin.* **36** (2014), 251-260.

[20] J. Oxley and H. Wu, On the structure of 3-connected matroids and graphs, *European J. Combin.* **21** (2000), 667-688.

[21] P. D. Seymour, A note on the production of matroid minors, *J. Combin. Theory Ser. B* **22** (1977), 289-295.

[22] P. D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305-359.

[23] P. D. Seymour, Minors of 3-connected matroids, *European J. Combin.* **6** (1985), 375-382.

[24] F. T. Tseng and K. Truemper, A decomposition of the matroids with the max-flow min-cut property, *Discrete Appl. Math.* **15** (1986), 329-364.

[25] W. Tutte, A theory of 3-connected graphs, *Indag. Math.* **23** (1961), 441-455.

[26] W. Tutte, Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301-1324.

[27] K. Wagner, Über eine Erweiterung eines Satzes von Kuratowski, *Deut. Math.* **2** (1937), 280-285.

[28] L. Warshauer, *Some Classes of Graphs That Are Nearly Cycle-Free*, Ph.D. thesis, Louisiana State University, 2011.

[29] H. Whitney, Non-separable and planar graphs, *Trans. Amer. Math. Soc.* **34** (1932), 339-362.

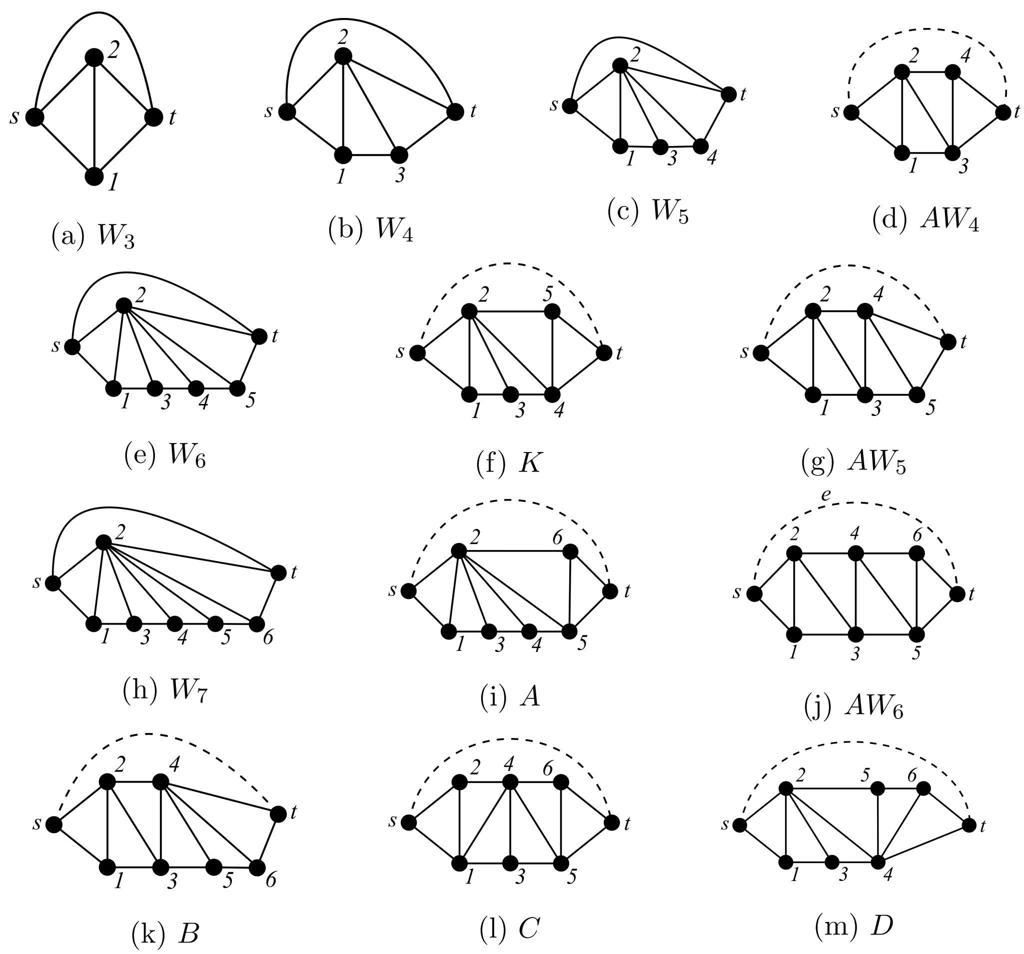
[30] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932), 150-168.

[31] H. Wu, On contractible and vertically contractible elements in 3-connected matroids and graphs, *Discrete Math.* **179** (1998), 185-203.

Appendix A: Daisy-Chain Menagerie

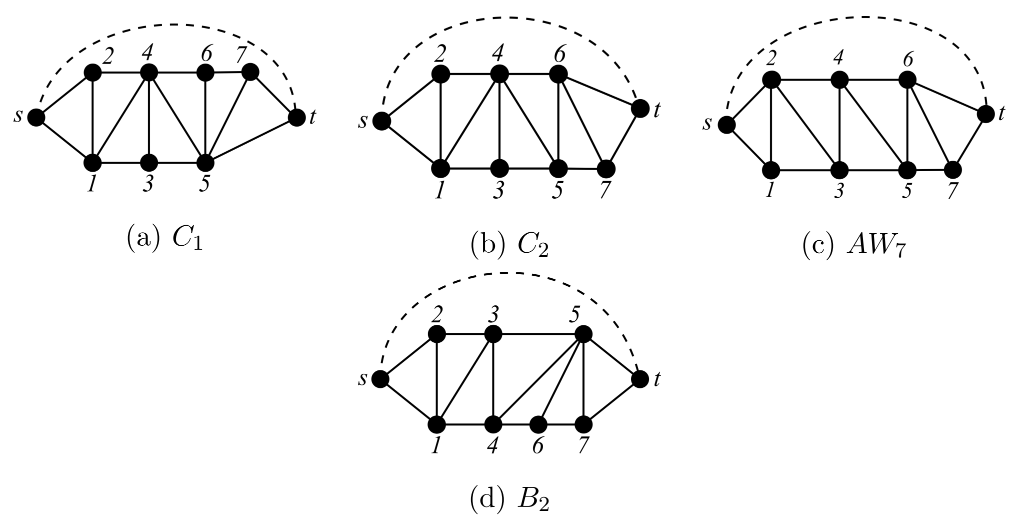
This appendix contains all the daisy chains that appear in the main body of the dissertation. The label of each graph is that of the unidentified daisy chain associated with the illustration; however, a dotted edge between the distinguished vertices s and t indicates the identified daisy chain also exists.

Figure 2.105 is an exhaustive summary all daisy chains having $n \leq 6$ inside vertices and no inside 4-cycles. We also include selected daisy chains for $n = 7$ with no inside 4-cycles in Figure 2.106, followed by selected daisy chains with one, two, or three inside 4-cycles in Figures 2.107, 2.108, and 2.109.



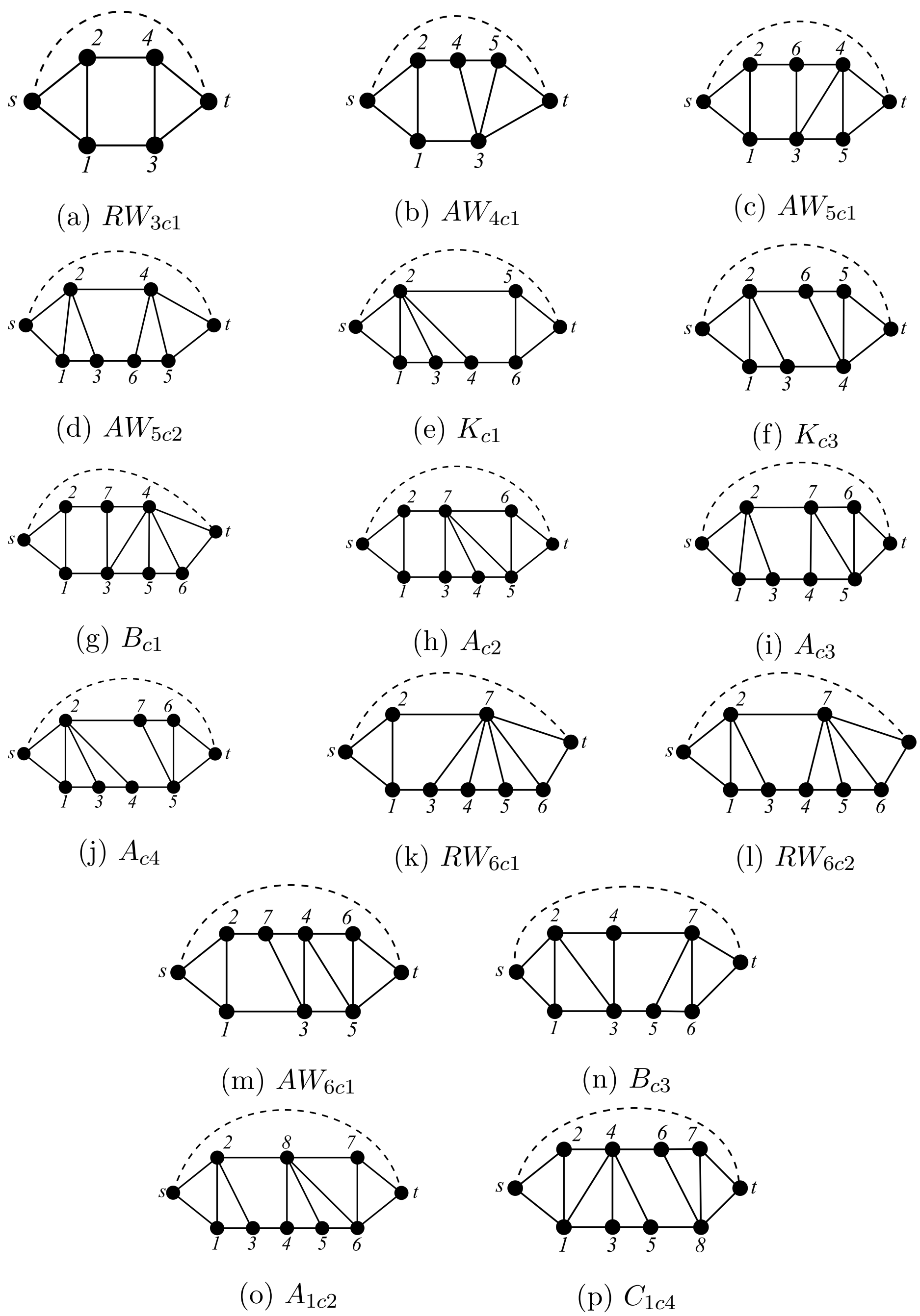
(a) W_3 (b) W_4 (c) W_5 (d) AW_4 (e) W_6 (f) K (g) AW_5 (h) W_7 (i) A (j) AW_6 (k) B (l) C (m) D

FIGURE 2.105: The daisy chains having $n \leq 6$ and only inside 3-cycles.



(a) C_1 (b) C_2 (c) AW_7 (d) B_2

FIGURE 2.106: Selected daisy chains having $n = 7$ and only inside 3-cycles.



(a) RW_{3c1} (b) AW_{4c1} (c) AW_{5c1}

(d) AW_{5c2} (e) K_{c1} (f) K_{c3}

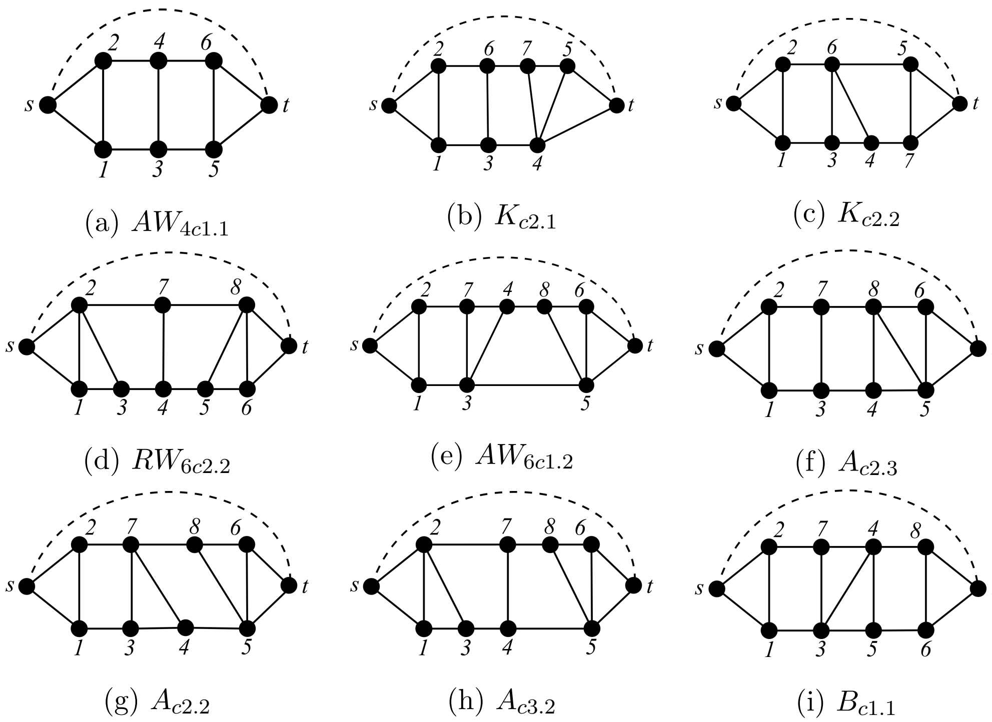
(g) B_{c1} (h) A_{c2} (i) A_{c3}

(j) A_{c4} (k) RW_{6c1} (l) RW_{6c2}

(m) AW_{6c1} (n) B_{c3}

(o) A_{1c2} (p) C_{1c4}

FIGURE 2.107: Selected daisy chains having exactly one inside 4-cycle.

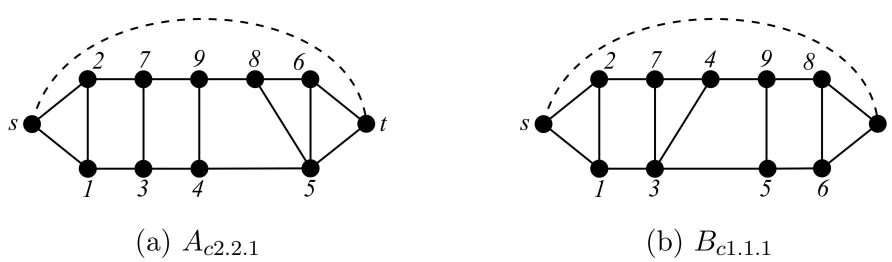


(a) $AW_{4c1.1}$ (b) $K_{c2.1}$ (c) $K_{c2.2}$

(d) $RW_{6c2.2}$ (e) $AW_{6c1.2}$ (f) $A_{c2.3}$

(g) $A_{c2.2}$ (h) $A_{c3.2}$ (i) $B_{c1.1}$

FIGURE 2.108: Selected daisy chains having exactly two inside 4-cycles.



(a) $A_{c2.2.1}$ (b) $B_{c1.1.1}$

FIGURE 2.109: Selected daisy chains having exactly three inside 4-cycles.

Appendix B: Sage Code

This appendix contains the SageMath (now CoCalc) code written for checking if a graph or any of its minors are members of $\mathcal{S}$ or $\mathcal{S}^*$.

The first function, `check_s(G)`, takes a graph G and checks to see if deleting any pair of its vertices results in a tree. If it is true that deleting some pair of vertices is a tree, then G is a member of $\mathcal{S}$, by Theorem 2.36. The function `check_s_delv(G,k)` deletes vertex k of graph G, and then checks to see if deleting any pair of vertices from $G\backslash k$ is a tree. The function `check_s_dele(G,u,v)` checks to see if deleting any pair of vertices from $G\backslash(u,v)$ is a tree, where (u,v) is an edge of G; the function `check_s_cone(G,u,v)` does the same for $G/(u,v)$.

```
def check_s(G):
    V_G=G.vertices()
    for i in range(len(V_G)):
        if i < (len(V_G)-1):
            for j in range(i+1,len(V_G)):
                H=copy(G)
                H.delete_vertices([V_G[i],V_G[j]])
                print H.is_tree()

def check_s_delv(G,k):
    if k in G.vertices():
        K=copy(G)
        K.delete_vertex(k)
        V_G=K.vertices()
        for i in range(len(V_G)):
                if i < (len(V_G)-1):
                    for j in range(i+1,len(V_G)):
                        H=copy(K)
                        H.delete_vertices([V_G[i],V_G[j]])
                        print H.is_tree()

def check_s_dele(G,u,v):
    if (u,v) in G.edges(labels=False):
        print (u,v)
        K=copy(G)
        K.delete_edge(u,v)
        V_G=K.vertices()
        for i in range(len(V_G)):
```

```
                if i < (len(V_G)-1):
                    for j in range(i+1,len(V_G)):
                        H=copy(K)
                        H.delete_vertices([V_G[i],V_G[j]])
                        print H.is_tree()

def check_s_cone(G,u,v):
    if (u,v) in G.edges(labels=False):
        print (u,v)
        K=copy(G)
        K.merge_vertices([u,v])
        V_G=K.vertices()
        for i in range(len(V_G)):
                if i < (len(V_G)-1):
                    for j in range(i+1,len(V_G)):
                        H=copy(K)
                        H.delete_vertices([V_G[i],V_G[j]])
                        print H.is_tree()
```

The function `dualcheck_s(G)` takes a graph G, and checks to see if any of its vertices can be split so that the resulting graph has treewidth less than three. By Theorems 1.7 and 1.8, such a graph is series-parallel, and, therefore, G is a member of $\mathcal{S}^*$. The function returns a list of series-parallel graphs that can be obtained by splitting vertices of G, if G is a member of $\mathcal{S}^*$. The function `dualcheck_s_delv(G,k)` deletes vertex of k of graph G, and then runs `dualcheck_s(G)` on $G - k$. The functions `dualcheck_s_dele(G,u,v)` and `dualcheck_s_cone(G,u,v)` respectively delete or contract the edge (u, v) from G, and then run `dualcheck_s(G)` on $G\backslash(u, v)$ or $G/(u, v)$.

```
def dualcheck_s(G):
    VG=G.vertices()
    Is_SP=[]
    Isnt_SP=[]
    for i in range(len(VG)):
        H=G.copy()
        v=VG[i]
        n=G.degree(v)
        N=G.neighbors(v)
        Nset=Set(N)
        d=len(G.vertices())
        H.delete_vertex(v)
```

```
    L=[]
    if is_odd(n):
        nf=floor(n/2)
        for j in range(1,(nf+1)):
            Lj=list(Subsets(N,j))
            L.extend(Lj)
    elif is_even(n):
        n2=(n/2)
        for j in range(1,n2+1):
            if j < n2:
                Lj=list(Subsets(N,j))
                L.extend(Lj)
            elif j == n2:
                Lj=list(Subsets(N,j))
                for k in range((len(Lj)-1)):
                    if k < len(Lj)-2:
                        for l in range(k+1,len(Lj)):
                            if Lj[k] == Nset.difference(Lj[l]):
                                break
                            if l == len(Lj)-1:
                                L.append(Lj[k])
                    if k== len(Lj)-2:
                        if Lj[k] == Nset.difference(Lj[k+1]):
                            L.append(Lj[k])
                        if Lj[k] != Nset.difference(Lj[k+1]):
                            L.extend([Lj[k],Lj[k+1]])
    H.add_vertices([d,d+1])
    for i in range(len(L)):
        Li=list(L[i])
        Li_comp=list(Nset.difference(L[i]))
        J=H.copy()
        for j in range(len(Li)):
            J.add_edge((d,Li[j]))
        for k in range(len(Li_comp)):
            J.add_edge((d+1,Li_comp[k]))
        t=J.treewidth()
        if t < 3:
            Is_SP.append(J)
        if t >= 3:
            Isnt_SP.append(J)
if len(Is_SP) == 0:
    print 'G is not a member of S*.'
else:
    print 'Congratulations, G is a member of S*.'
```

```
        return Is_SP

def dualcheck_s_delv(G,k):
    if k in G.vertices():
        K=copy(G)
        K.delete_vertex(k)
        ord_K=order(K)
        K.relabel([0..ord_K])
        dualcheck_s(K)

def dualcheck_s_dele(G,u,v):
    if (u,v) in G.edges(labels=False):
        print (u,v)
        K=copy(G)
        K.delete_edge(u,v)
        dualcheck_s(K)
    else:
        print 'That was not an edge.'

def dualcheck_s_cone(G,u,v):
    if (u,v) in G.edges(labels=False):
        print (u,v)
        K=copy(G)
        K.merge_vertices([u,v])
        ord_K=order(K)
        K.relabel([0..ord_K])
        dualcheck_s(K)
    else:
        print 'That was not an edge.'
```

Vita

Victoria "Iah" Fontaine received a B.S. in Mathematics from Seton Hall University, summa cum laude and with departmental honors, as Salutatorian of the Class of 2012. She was awarded a Board of Regents Fellowship at Louisiana State University, where she received an M.S. in Mathematics in 2014, followed by an M.M. in Organ Performance in 2016. Iah is currently a graduate teaching assistant and a candidate for a Ph.D. in Mathematics at Louisiana State University.