The Graphs and Matroids Whose Only Odd Circuits Are Small

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THE GRAPHS AND MATROIDS WHOSE ONLY ODD CIRCUITS ARE SMALL

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Abstract

This thesis is motivated by a graph-theoretical result of Maffray, which states that a 2-connected graph with no odd cycles exceeding length 3 is bipartite, is isomorphic to $K_4$, or is a collection of triangles glued together along a common edge. We first prove that a connected simple binary matroid $M$ has no odd circuits other than triangles if and only if $M$ is affine, $M$ is $M(K_4)$ or $F_7$, or $M$ is the cycle matroid of a graph consisting of a collection of triangles glued together along a common edge. This result implies that a 2-connected loopless graph $G$ has no odd bonds of size at least five if and only if $G$ is Eulerian or $G$ is a subdivision of either $K_4$ or the graph that is obtained from a cycle of parallel pairs by deleting a single edge. The main theorem of the dissertation extends Maffray’s theorem to $n$-connected graphs with no odd cycles exceeding size $2n - 1$. To prove this, we first prove the special cases when $n = 3$ and $n = 4$. The proof of the theorem is competed with an argument that treats all $n \geq 5$. 
Chapter 1 Binary Matroids with No Odd Circuits Exceeding Size Three

It is a well known result from graph theory that a graph is bipartite if and only if it has no odd cycles. For each \( n \geq 1 \), let \( K'_{2,n} \) be the graph that is obtained from \( K_{2,n} \) by adding an edge joining the vertices in the two-vertex class (see Figure 1.1). In 1992, Maffray [7, Theorem 2] proved the following result.

Figure 1.1: \( K'_{2,n} \)

**Theorem 1.0.1.** A 2-connected simple graph \( G \) has no odd cycles of length exceeding three if and only if

(i) \( G \) is bipartite;

(ii) \( G \cong K_4 \); or

(iii) \( G \cong K'_{2,n} \) for some \( n \geq 1 \).

There is a long history of generalizing results for graphs to binary matroids (see, for example, [4, 12] or, more recently, [9, Section 15.4]). We shall continue this tradition by proving a generalization of Maffray’s result. A circuit in a matroid is even if it has even cardinality; otherwise, it is odd. A triangle is a 3-element circuit. A binary matroid is affine if all of its circuits are even. Hence the cycle matroid, \( M(G) \), of a graph \( G \) is affine if and only if \( G \) is bipartite. The following is the main theorem of this chapter [10].

**Theorem 1.0.2.** A connected simple binary matroid \( M \) has no odd circuits other than triangles if and only if
(i) $M$ is affine;

(ii) $M \cong M(K_4)$ or $F_7$; or

(iii) $M \cong M(K'_2,n)$ for some $n \geq 1$.

The terminology used here will follow Oxley [9]. Binary affine matroids have several attractive characterizations. Indeed, Welsh [13] proved that the link between bipartite and Eulerian graphs via duality extends to binary matroids. His result is the equivalence of the first two parts of the next theorem (see, for example, [9, Theorem 9.4.1]). The equivalence of the first and third parts was proved independently by Brylawski [2] and Heron [5].

**Theorem 1.0.3.** The following are equivalent for a binary matroid $M$.

(i) $M$ is affine;

(ii) $M$ is loopless and its simplification is isomorphic to a restriction of $AG(r-1,2)$ for some $r \geq 1$;

(iii) $E(M)$ can be partitioned into cocircuits.

Recall that a bond of a graph is a minimal edge cut. The next result follows immediately by applying our Theorem 1.0.2 to the bond matroid of a graph, that is, to the dual of its cycle matroid.

**Corollary 1.0.4.** A 2-connected loopless graph $G$ has no odd bonds of size exceeding three if and only if

(i) $G$ is Eulerian; or

(ii) $G$ is a subdivision of either $K_4$ or the graph that is obtained from an $n$-edge cycle for some $n \geq 2$ by adding an edge in parallel to all but one of the edges.

Another straightforward consequence of Theorems 1.0.2 and 1.0.3 is the following.
Corollary 1.0.5. Let $M$ be a connected cosimple binary matroid of rank at least four. Then $M$ has no odd circuits of size exceeding three if and only if $M$ is affine.

We shall implement the use of the following two lemmas in the proof of Theorem 1.0.2.

Lemma 1.0.6. A simple binary matroid having an even circuit meeting a triangle $T$ in a single element has an odd circuit of size exceeding three.

Proof. From among even circuits that meet $T$ in a single element, choose $C$ to have minimum cardinality. As $M$ is binary, $C \Delta T$ is the disjoint union of $k$ circuits for some $k \geq 1$. As $|C \Delta T| = |C| + 1$, if $k = 1$, then the lemma holds. Thus we may assume that $k \geq 2$. Since each circuit contained in $C \Delta T$ must contain an element of $T - C$, we deduce that $k \leq 2$, so $k = 2$. Thus, as $C \Delta T$ has odd cardinality, it is the disjoint union of an odd circuit and an even circuit, $C_0$, each of which meets $T$ in a single element. As $|C_0| < |C|$, the choice of $C$ is contradicted. \hfill \Box

Our second lemma is more general than we need to prove the theorem. For an integer $n$ exceeding one, let $M_1, M_2, \ldots, M_n$ be matroids such that $E(M_i) \cap E(M_j) = \{p\}$ for all distinct $i$ and $j$ in $\{1, 2, \ldots, n\}$, and $\{p\}$ is not a component of any $M_k$. The parallel connection $P(M_1, M_2, \ldots, M_n)$ is the matroid with ground set $E(M_1) \cup E(M_2) \cup \cdots \cup E(M_n)$ whose set of circuits consists of the union of the sets of circuits of $M_1, M_2, \ldots, M_n$ along with, for all distinct elements $i$ and $j$ of $\{1, 2, \ldots, n\}$, all sets of the form $(C_i - p) \cup (C_j - p)$ where $C_i$ is a circuit of $M_i$ containing $p$, and $C_j$ is a circuit of $M_j$ containing $p$ (see, for example, [9, Proposition 7.1.18]). Thus if $M_k \cong U_{2,3}$ for all $k$, then $P(M_1, M_2, \ldots, M_n) \cong M(K_{2,n}')$. The element $p$ is called the basepoint of the parallel connection.

Lemma 1.0.7. Let $M$ be a simple connected matroid. Then $M$ has an element $p$ such that the only circuits of $M$ that contain $p$ are triangles if and only if $M$ is isomorphic to $U_{1,1}$ or to $U_{2,k}$ for some $k \geq 3$, or $M$ is the parallel connection with basepoint $p$ of some collection of simple rank-2 matroids each of which contains at least three points.
Proof. It is straightforward to check that, for each of the matroids listed, the only circuits containing $p$ are triangles. Now assume that the only circuits of $M$ containing $p$ are triangles. We may assume that $r(M) \geq 3$ otherwise the result certainly holds. As $M$ is connected, each of its elements is in some circuit with $p$. By hypothesis, this circuit must be a triangle. Thus, in $M/p$, every element is in a non-trivial parallel class. If every component of $M/p$ has rank one, then it follows by a result of Brylawski [1] (see also [9, Theorem 7.1.16]) that $M$ is a parallel connection as asserted. Therefore we may assume that $M/p$ has a component of rank exceeding one. Thus $M/p$ has a circuit $D$ of size exceeding two and, as $D \cup p$ is not a circuit of $M$, we deduce that $D$ is a circuit of $M$. Similarly, $(D-d) \cup d'$ is a circuit of $M$ where $d$ is some element of $D$, and $d'$ is parallel to $d$ in $M/p$. Thus $\text{cl}_M(D-d)$ contains $\{d, d'\}$ and so contains $p$. Then $r_{M/p}(D-d) < |D-d|$; a contradiction.

We are now ready to prove Theorem 1.0.2.

Proof of Theorem 1.0.2. It is easily checked that $M(K_4)$, $F_7$, and each $M(K'_2, n)$ are binary having no odd circuits of size greater than three. For the converse, assume that $M$ has no odd circuits of size greater than three. Suppose $M$ is not affine. If $r(M) = 3$, then clearly $M$ is isomorphic to $M(K'_{2,2})$, $M(K_4)$, or $F_7$. Thus we may assume that $r(M) \geq 4$. First we show the following.

1.0.3.1. If $T_0$ is a triangle of $M$ and $C$ is a circuit that meets but is not equal to $T_0$, then $|C| \leq 4$ and $M|(T_0 \cup C) \cong M(K'_{2,2})$.

This is certainly true if $C$ is a triangle, so we assume that $|C| \geq 4$. By Lemma 1.0.6, $|C \cap T_0| = 2$. Then $C \Delta T_0$ is a circuit of $M$ of cardinality $|C| - 1$. Thus $|C| = 4$ and $C \Delta T_0$ is a triangle $T_1$ meeting $T_0$ in a single element. Hence $M|(T_0 \cup C) = M|(T_0 \cup T_1) \cong M(K'_{2,2})$, and (1.0.3.1) holds.

As $M$ is not affine, it contains a triangle $T$. As $M$ is connected, it follows by (1.0.3.1) that $M$ has a triangle $T'$ that meets $T$ in a single element, say $f$. 

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1.0.3.2. For each $g$ not in $\text{cl}(T \cup T')$, there is a triangle that contains $\{g, f\}$.

As $M$ is connected, it has a circuit $D$ that contains $g$ and meets $T \cup T'$. Without loss of generality, we may assume that $D$ meets $T$. By (1.0.3.1), $M|(D \cup T) \cong M(K_{2,2}')$. Thus $M$ has a triangle $T''$ that contains $g$ and meets $T$ in a single element, $h$. We may assume that $h \neq f$ otherwise (1.0.3.2) holds. Then $T''$ meets the 4-element circuit $(T \cup T') - f$ in a single element; a contradiction to Lemma 1.0.6. We deduce that (1.0.3.2) holds.

We may assume that $M$ has a circuit $C'$ that contains $f$ and is not a triangle otherwise the result follows by Lemma 1.0.7. By Lemma 1.0.6, $C'$ meets each triangle containing $f$ in two elements. Moreover, by (1.0.3.1), $|C'| = 4$. Hence $M$ has at most three triangles containing $f$. But, as $r(M) \geq 4$, it follows that $r(M) = 4$, and $M$ has exactly two elements not in $\text{cl}(T \cup T')$, these elements being contained in a common triangle with $f$.

If $T \cup T'$ is a flat of $M$, then $M \cong M(K_{2,3}')$. Thus we may assume that $\text{cl}(T \cup T') - (T \cup T')$ contains an element $h$. Then $M|(T \cup T' \cup h) \cong M(K_4)$, so $T \cup T' \cup h$ contains a 4-circuit $D'$ containing $\{f, h\}$. By (1.0.3.2), $M$ has a triangle that meets $D'$ in $\{f\}$. This contradiction to Lemma 1.0.6 completes the proof of the theorem.
Chapter 2  Graphs with No Odd Cycles Exceeding Size Five

From here, we explored possible extensions of Theorem 1.0.1 and Theorem 1.0.2. Initially we proved a purely graph-theoretical extension of Theorem 1.0.1. Subsequently, we extended this proof to binary matroids. This extension does not appear in this dissertation.

**Theorem 2.0.1.** A 3-connected simple graph $G$ has no odd cycles of length exceeding five if and only if

(i) $G$ is bipartite;

(ii) $G$ is a graph on six or fewer vertices; or

(iii) $G \cong K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 4$ where $K'_{3,n}$, $K''_{3,n}$, and $K'''_{3,n}$ are shown below in Figure 2.1.

Note that $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ can be viewed as $n$ copies of $K_4$ identified at a common triangle with 1, 2 or 3 edges left in respectively.

For the proof of Theorem 2.0.1, we will need the following theorem of Menger [6].

**Theorem 2.0.2.** Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating $A$ from $B$ in $G$ is equal to the maximum of disjoint $A - B$ paths in $G$.

**Proof of Theorem 2.0.1.** It is easily checked that the graphs mentioned in (i), (ii), and (iii) have no odd cycles of length exceeding five. Now assume $G$ is a 3-connected graph with a 5-cycle and $|V(G)| > 6$. Select a 5-cycle, $C$, with vertex set $V(C) = \{v_1, v_2, v_3, v_4, v_5\}$. For all $i$ in $1, 2, 3, 4, 5$, let $e_i$ be the edge $\{v_i, v_{i+1}\}$ where $v_6 = v_1$ as in Figure 2.2

Since $|V(G)| > 6$, there is a vertex in $V(G) - V(C)$; call it $v_0$. Since $G$ is 3-connected, by Theorem 2.0.2, there are three paths from $v_0$ to $V(C)$ where $v_0$ is the only common
vertex of any two of the three paths. Call the paths $p_1$, $p_2$ and $p_3$. By symmetry, we have one of the configurations shown in Figure 2.3.

Let us first consider the case shown in Figure 2.3a. We will use $G_1$ to denote such a graph. All cycles in $G_1$ must have even length or length 3 or 5, since $G_1$ is a subgraph of $G$. Consider the cycles $A = \{p_1, p_2, e_2, e_3, e_4, e_5\}$, $B = \{p_1, p_3, e_3, e_4, e_5\}$ and $C = \{p_2, p_3, e_3, e_4, e_5, e_1\}$ where, for example, $A$ consists of all of the edges of each of $p_1$ and $p_2$ along with the edges $e_2$, $e_3$, $e_4$, and $e_5$. Let $|p_i|$ denote the number of edges in the path $p_i$ and let $|A|$ be the number of edges in cycle $A$. If we sum the lengths of these three cycles, we get $2|p_1| + 2|p_2| + 2|p_3| + 11$. Thus at least one of these cycles has odd length. Thus we have a cycle of length five as each $|p_i|$ is positive. Since $|A| \geq 6$ and $|C| \geq 6,$
we see that the $|B| = 5$ and so $|p_1| = 1 = |p_3|$. As $|A| > 5$, it is even. Thus the cycle \( \{p_1, p_2, e_1\} \) must be odd of length equal to 3 or 5. Thus $|p_2| \in \{1, 3\}$.

We conclude that if we have a graph of the form $G_1$, we are guaranteed one of the substructures in Figure 2.4a or Figure 2.4c in the graph $G$.

Let us consider now the case pictured in Figure 2.3b. Again, $p_1$, $p_2$ and $p_3$ are paths that share $v_0$, but are otherwise disjoint. We will call this $G_2$. Consider the cycles $A = \{p_1, p_2, e_2, e_3, e_4, e_5\}$, $B = \{p_1, p_3, e_3, e_2, e_1\}$, and $C = \{p_2, p_3, e_3, e_2\}$. The sum of their lengths is $2|p_1| + 2|p_2| + 2|p_3| + 9$. By a similar argument as before, exactly one of $|B|$ and $|C|$ has length 5, so either $|p_1|$ and $|p_3|$ have the same cardinality, or $|p_2|$ and $|p_3|$ have the same cardinality.

If $|p_1|$ and $|p_3|$ have the same cardinality, then by the cycles $\{p_1, p_3, e_4, e_5\}$ and $\{p_1, p_3, e_3, e_2, e_1\}$, we deduce that $|p_1| = |p_3| = 1$. Similarly, by considering $\{p_2, p_3, e_4, e_5, e_1\}$ and $\{p_1, p_2, e_2, e_3, e_4, e_5\}$, we see that $|p_2| = 1$.

Now, if $|p_2|$ and $|p_3|$ have different cardinalities, by $\{p_2, p_3, e_3, e_2\}$ and $\{p_2, p_3, e_4, e_5, e_1\}$ one of $p_2$ or $p_3$ has length 1 and the other has length 2. If $|p_2| = 2$, then, by cycles $\{p_1, p_2, e_1\}$ and $\{p_1, p_2, e_3, e_2, e_1\}$, we deduce that $|p_1| = 2$. If $|p_3| = 2$, then by cycles $\{p_1, p_3, e_3, e_2, e_1\}$ and $\{p_1, p_3, e_4, e_5\}$, it follows that $|p_1| = 1$.

We conclude that $G_2$ must be one of the graphs among Figure 2.4b, Figure 2.4d and Figure 2.4e below.
Next we note the following fact.

2.0.2.1. Let \( u \) and \( v \) be vertices of \( G \) such that \( G \) contains an even-lengthed path \( p_e \) and an odd-lengthed path \( p_o \) joining \( u \) and \( v \). If \( |p_e| \geq 6 \) and \( |p_o| \geq 5 \), then \( G \) has no path \( p \) that joins \( u \) and \( v \) and is internally disjoint from both \( p_e \) and \( p_o \).

If such a \( p \) existed, we could examine the cycles \( \{p_e, p\} \) and \( \{p_o, p\} \) These paths have opposite parity and have length greater than five, contradicting our choice of \( G \).

Relabel the graph in Figure 2.4c as \( G_3 \).

2.0.2.2. \( G \) does not have \( G_3 \) as a subgraph.

Assume the contrary. As \( G \) is 3-connected, by Theorem 2.0.2, there is at least one path of \( G \) between \( w \) and \( V(C) = V(G_3) \setminus N(w) = (v_3, v_4, v_5, v_1, v_2) \), where \( N(v) \) is the set of vertices adjacent to \( v \) in \( G_3 \), as shown below in Figure 2.5.

By (2.0.2.1), since \((w, x, v_2, v_1, v_5, v_4, v_3)\) is a path of length 6 and \((w, v_0, v_1, v_5, v_4, v_3)\) is a path of length 5, there is no \( w - v_3 \) path that is internally disjoint from these two paths. By symmetry, there is no \( w - v_1 \) path that is internally disjoint from \( V(C) \cup \{w, x, v_0\} \).
Similarly, using the paths \((w, v_0, v_3, v_2, v_1, v_5, v_4)\) of length 6, and \((w, w, x, v_2, v_1, v_5, v_4)\) of length 5 and \((2.0.2.1)\), there is no \(w - v_4\) path that is internally disjoint from \(V(C) \cup \{w, x, v_0\}\). Again by symmetry, no \(w - v_5\) path that is internally disjoint from \(V(C) \cup \{w, x, v_0\}\) can exist. By the 6-path \((w, v_0, v_1, v_5, v_4, v_3, v_2)\), we see that \(|p|\) must be even.

By the 3-path \((w, v_0, v_1, v_2)\), we see that \(p\) must be of length two. By symmetry between the cycle \(C\) and the cycle with the vertex set \(\{v_0, v_1, v_5, v_4, v_3\}\), we deduce that \(G\) must have a \(x - v_0\) path \(p'\) of length two that is internally disjoint from \(V(C) \cup \{x, w, v_0\}\).

We now know that \(G\) has a \(w - v_2\) path \(p\) that is internally disjoint from \(V(C) \cup \{w, x, v_0\}\).

If the \(x - v_0\) path \(p'\) and the \(w - v_2\) path \(p\) do not intersect, we create a 7-cycle with vertex set \(\{v_1, v_2, y, w, x, z, v_0\}\) where \(y\) is the internal vertex on \(p\) and \(z\) is the internal vertex on \(p'\) as shown below in Figure 2.6. If the \(p\) and the \(p'\) path intersect, we create a 7-cycle with vertex set \(\{v_2, y, v_0, v_3, v_4, v_5, v_1\}\) where \(y\) is the common vertex on \(p\) and

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.5.png}
\caption{One of the dashed paths must exist}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.6.png}
\caption{Paths \(p\) and \(p'\) do not intersect}
\end{figure}
$P'$ as shown in Figure 2.7. We conclude that $G$ cannot have $G_3$ as a subgraph, that is, (2.0.2.2) holds.

![Figure 2.7: Paths $p$ and $p'$ intersect](image)

Relabel the graph shown in Figure 2.4d as $G_4$. By Theorem 2.0.2, $G$ must have a path from $x$ to to $V(G_4) \setminus N(x) = \{v_3, v_4, v_5, w, v_1\}$ that is internally disjoint from $V(C) \cup \{v_0, x, w\}$ as shown in Figure 2.8.

![Figure 2.8: $G_4$](image)

2.0.2.3. $G$ does not have $G_4$ as a subgraph.

Assume the contrary. By (2.0.2.1), using the paths $(x, v_0, w, v_1, v_2, v_3)$ of length 5 and $(x, v_0, w, v_1, v_5, v_4, v_3)$ of length 6, there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{x, w, v_0\}$. By (2.0.2.1), using the paths $(x, v_0, w, v_1, v_5, v_4)$ and $(x, v_0, w, v_1, v_2, v_3, v_4)$ of length 5 and 6 respectively, there is no $x - v_4$ path that is internally disjoint from $V(C) \cup \{x, w, v_0\}$. Similarly, using the paths $(x, v_0, w, v_1, v_2, v_3, v_4, v_5)$ of length 7 and $(x, v_0, v_4, v_3, v_2, v_1, v_5)$ of length 6 and (2.0.2.1), there is no $x - v_5$
path that is internally disjoint from \( V(C) \cup \{x, w, v_0\} \). Likewise, by (2.0.2.1), using the paths \((x, v_0, v_4, v_5, v_1, w)\) and \((x, v_2, v_1, v_5, v_4, v_0, w)\), there is no \( x - w \) path that is internally disjoint from \( V(C) \cup \{x, w, v_0\} \). Finally, using the paths \((x, v_0, v_4, v_3, v_2, v_1)\) and \((x, v_2, v_3, v_4, v_0, w, v_1)\), there is no \( x - v_1 \) path that is internally disjoint from \( V(C) \cup \{x, w, v_0\} \). We conclude that \( G_4 \) cannot be a subgraph of \( G \), that is, (2.0.2.3) holds.

Relabel the graph in Figure 2.4e as \( G_5 \).

2.0.2.4. \( G \) does not have \( G_5 \) as a subgraph.

Assume the contrary. By Theorem 2.0.2, there must be a path from \( x \) to \( V(G_5) \setminus N(x) = (v_1, v_2, v_3, v_5) \) that is internally disjoint from \( V(C) \cup \{x, v_0\} \) as shown in Figure 2.9.

By (2.0.2.1), the paths \((x, v_4, v_3, v_2, v_0, v_1)\) and \((x, v_0, v_2, v_3, v_4, v_5, v_1)\) imply there is no \( x - v_1 \) path that is internally disjoint from \( V(C) \cup \{x, v_0\} \). By symmetry, there is no \( x - v_2 \) path that is internally disjoint from \( V(C) \cup \{x, v_0\} \). Similarly, the paths \((x, v_0, v_1, v_5, v_4, v_3)\) and \((x, v_4, v_5, v_1, v_0, v_2, v_3)\) imply there is no \( x - v_3 \) path that is internally disjoint from \( V(C) \cup \{x, v_0\} \). By symmetry, there is no \( x - v_5 \) path that is internally disjoint from \( V(C) \cup \{x, v_0\} \). We conclude that (2.0.2.4) holds.

This eliminates the cases in Figure 2.4 where \( p_1, p_2, \) or \( p_3 \) has more than one edge. So vertices in \( G \) not on the 5-cycle \( C \) are of the types 2.4a and 2.4b.

We start with the following observation.
2.0.2.5. If a 5-cycle in $G$ has two or more distinct edges that belong to triangles whose third vertices avoid $V(C)$ and are distinct, then $G$ has a 7-cycle.

If we follow the 5-cycle along replacing the edges that are common to the triangles with the other edges of those triangles, we get a cycle of length $5 - 1 + 2 - 1 + 2 = 7$ as shown in Figure 2.10.

Since $|V(G)| > 6$, we have more than one vertex not in $V(C)$. Suppose we have at least one vertex of type 2.4a not on $C$. Since each such vertex creates two triangles off of cycle $C$ and the vertices are distinct, we will always have two edge-disjoint triangles each sharing a single edge with $C$. So by 2.0.2.5, we may not have graphs of type 2.4a as a subgraph.

We now know that all vertices not in $V(C)$ are of type 2.4b. Furthermore, the triangles that meet the 5-cycle must share the same edge; otherwise, we would create disjoint triangles, and thereby a contradiction of 2.0.2.5.

We are left with subgraphs that look like the following (see Figure 2.11).
All extra vertices added not on the 5-cycle meet at the same three points. We may add as many as we like.

Now we must check for possible additional edges within this graph without adding a larger odd cycle. In order to be 3-connected, $v_3$ and $v_5$ in our construction must have additional edges. By our previous argument, none of these edges can be to any of the vertices outside of the 5-cycle. This leaves $v_2$ and $v_3$ as possible neighbors for $v_5$ and $v_1$ and $v_5$ as possible neighbors for $v_3$.

If there is an edge $\{v_3, v_5\}$, we get the 7-cycle $(v_3, v_5, v_1, v_0, v_4, v_2)$. The edges $\{v_3, v_1\}$ and $\{v_2, v_5\}$ create no 7-cycles, so these are the desired necessary edges to complete 3-connectivity.

We look at the remaining possible edges. From our previous argument concerning $v_0$, we know all unknown edges meeting a vertex not in $V(C)$ must join to another vertex not in $V(C)$. Assume $G$ has such an edge $\{v_0, v'_0\}$. This creates a 7-cycle $(v_1, v_0, v'_0, v_2, v_3, v_4, v_5)$ as shown in Figure 2.12.

![Figure 2.12: Forced 7-cycle](image)

Now we need only look at possible edges from the vertices on the 5-cycle to other vertices on the 5-cycle. Remaining edges not in the graph are $\{v_1, v_4\}, \{v_2, v_4\}$, and $\{v_3, v_5\}$. We have already eliminated $\{v_3, v_5\}$. As $\{v_1, v_4\}$ and $\{v_2, v_4\}$ are symmetric, we only need check the cases where one or both are present. Neither causes a larger odd cycle.

This completes the construction of $G$. All vertices meet a 5-cycle at the same three vertices. This creates one side of our partition. The other two vertices of the 5-cycle
connect to the three vertices. The three-vertex side of the bipartition may have one, two, or three edges between them.

Figure 2.13: $K'_3,n$
Chapter 3  4-connected Graphs with No Odd Cycles Exceeding Size Seven

Here we extend the size of the possible odd cycles. The proof of the result is strikingly similar to the previous result in Section 1.2. The infinite class of graphs are built from a bipartite graph with the side of the bipartition that has four vertices having at least one edge.

**Theorem 3.0.1.** A 4-connected simple graph $G$ has no odd cycles of length exceeding seven if and only if

(i) $G$ is bipartite;

(ii) $G$ is a graph on eight or fewer vertices; or

(iii) for some $n \geq 5$, the graph $G$ is isomorphic to a graph that is obtained from $K_{4,n}$ by adding 1, 2, 3, 4, 5, or 6 edges each having both ends in the 4-vertex side of the vertex bipartition as in Figure 3.1.

![Figure 3.1: $K'_{4,n}$](image)

**Proof.** We start with the following observation.

**3.0.1.1.** If a 7-cycle in $G$ has two or more distinct edges that belong to triangles whose third vertices avoid $V(C)$ and are distinct, then $G$ has a 9-cycle.
Follow the 7-cycle along replacing the edges that are common to the triangles with the other edges of those triangles to get a cycle of length $7 - 1 + 2 - 1 + 2 = 9$.

It is easily checked that the graphs mentioned in (i), (ii), and (iii) have no odd cycles of length exceeding seven. Now assume $G$ is a 4-connected graph with a 7-cycle and $|V(G)| > 8$. Select a 7-cycle, $C$, with vertex set $V(C) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. For all $i$ in $\{1, 2, \ldots, 5\}$, let $e_i$ be the edge $\{v_i, v_{i+1}\}$ where $v_8 = v_1$ (see Figure 3.2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3_2}
\caption{7-cycle}
\end{figure}

Since $|V(G)| > 8$, there is a vertex in $V(G) - V(C)$; call it $v_0$. Since $G$ is 4-connected, by Theorem 2.0.2, there are four paths from $v_0$ to $V(C)$ whose only common vertex is $v_0$. By symmetry, we have one of the configurations shown in Figure 3.3 where the wavy lines meeting $v_0$ correspond to paths. These paths are labeled $p_1$, $p_2$, $p_3$, and $p_4$ reading clockwise from the path $p_1$ that joins $v_0$ and $v_1$.

Consider the case shown in Figure 3.3a. We will use $G_1$ to denote such a graph. All cycles in $G_1$ must have even length, or length 3, 5, or 7, as $G_1$ is a subgraph of $G$. Consider the cycle $D_{1,2}$ using $p_1$ and $p_2$ through $(v_1, v_0, v_2, v_3, v_4, v_5, v_6, v_7)$, that is, $D_{1,2}$ uses the path $p_1$ from $v_1$ to $v_0$, the path $p_2$ from $v_0$ to $v_2$, and then the edges $\{v_i, v_{i+1}\}$ for all $i$ in $\{2, 3, \ldots, 7\}$ where $v_8 = v_1$. Similarly, consider the cycles $D_{1,3}$ using $p_1$ and $p_3$ through $(v_1, v_0, v_3, v_4, v_5, v_6, v_7)$, and $D_{2,3}$ using $p_2$ and $p_3$ through $(v_2, v_0, v_3, v_4, v_5, v_6, v_7, v_1)$. Then $D_{1,2}$, $D_{1,3}$, and $D_{2,3}$ have lengths $|p_1| + |p_2| + 6$, $|p_1| + |p_3| + 5$, and $|p_2| + |p_3| + 6$ respectively.

If we sum the lengths of these cycles, we get $2|p_1| + 2|p_2| + 2|p_3| + 17$. Hence at least one of the cycles is odd. Thus we have a cycle of length seven as each $|p_i|$ is positive. Since
$D_{1,2}$ and $D_{2,3}$ have lengths exceeding seven, $D_{1,3}$ must have length seven. So $|p_1| = 1$ and $|p_3| = 1$. By symmetry, we see that $|p_1| = 1$ and $|p_4| = 1$. Thus we have the following.

3.0.1.2. In $G_1$, each $p_i$ has length 1.

Now we consider the configuration shown in Figure 3.3b, which we will call $G_2$. Then $G_2$ has the same cycles $D_{1,2}, D_{1,3},$ and $D_{2,3}$ that were considered in $G_1$. Hence $|p_1| = 1$ and $|p_3| = 1$. Now we look at the cycle $F_{2,3}$ using $p_2$ and $p_3$ through $(v_2, v_0, v_3, v_4, v_5, v_6, v_7, v_1)$ of length $|p_2| + |p_3| + 6 = |p_2| + 7$, the cycle $F_{2,4}$ using $p_2$ and $p_4$ through $(v_2, v_0, v_5, v_6, v_7, v_1)$ of length $|p_2| + |p_4| + 4$, and the cycle $F_{3,4}$ using $p_3$ and $p_4$ through $(v_3, v_0, v_5, v_6, v_7, v_1, v_2)$ of length $|p_3| + |p_4| + 5 = |p_4| + 6$. If we sum the lengths of these cycles, we get $2|p_2| + 2|p_4| + 17$. Thus at least one of the cycles is odd. Since $F_{2,3}$ has length $|p_2| + 7$, we see that this cycle is even, so $|p_2|$ is odd. If $F_{3,4}$ is odd, then $|p_4| = 1$. Since the cycle using $p_2$ and $p_4$ through $(v_2, v_0, v_5, v_6, v_7, v_1)$ is even, the cycle using $p_2$ and $p_4$ through $(v_2, v_0, v_5, v_4, v_3)$ is odd of length $|p_2| + |p_4| + 3 = |p_2| + 4$. So $|p_2| \in \{1, 3\}$. If $F_{2,4}$ is odd, then $|p_2| + |p_4| + 4 = 7$, so $|p_2| + |p_4| = 3$. Since $|p_2|$ is odd, $|p_2| = 1$ and $|p_4| = 2$. We deduce the following.
3.0.1.3. $G_2$ is one of the graphs in Figure 3.4.

Figure 3.4: Path lengths of $G_2$

In the configuration in Figure 3.3c, which we will call $G_3$, we will use the same approach. Consider the cycle $H_{1,2}$ using $p_1$ and $p_2$ through $(v_1, v_0, v_2, v_3, v_4, v_5, v_6, v_7)$, the cycle $H_{1,3}$ using $p_1$ and $p_3$ through $\{v_1, v_0, v_4, v_5, v_6, v_7\}$, and the cycle $H_{2,3}$ using $p_2$ and $p_3$ through $(v_2, v_0, v_4, v_5, v_6, v_7, v_1)$ of lengths $|p_1| + |p_2| + 6$, $|p_1| + |p_3| + 4$, and $|p_2| + |p_3| + 5$ respectively. If we sum the lengths of these cycles, we get $2|p_1| + 2|p_2| + 2|p_3| + 15$. Thus at least one cycle is odd; however, not all cycles are odd, since the first cycle has size larger than 7. Thus, either the second or the third cycle has odd length.

If $H_{1,3}$ is odd, then $|p_1| + |p_3| + 4 = 7$. Thus one of $p_1$ and $p_3$ has length 2 and one has length 1. Suppose $|p_1| = 1$ and $|p_3| = 2$. As $H_{1,2}$ has length $|p_2| + 7$, the path $p_2$ has odd length. As the cycle using $p_3$ and $p_4$ through $(v_4, v_0, v_5, v_6, v_7, v_1, v_2, v_3)$ has length $|p_4| + 8$, the path $p_4$ has even length. The cycle using $p_2$ and $p_4$ through $\{v_2, v_0, v_5, v_6, v_7, v_1\}$ has
the length $|p_2| + |p_4| + 4$, which is odd. Thus $|p_2| = 1$ and $|p_4| = 2$. We deduce that when $|p_1| = 1$ and $|p_3| = 2$, we get $|p_2| = 1$ and $|p_4| = 2$.

Now suppose $|p_1| = 2$ and $|p_3| = 1$. The cycle $H_{1,2}$ has length $|p_2| + 8$. Thus $p_2$ has even length. As the cycle using $p_3$ and $p_4$ through $(v_4, v_0, v_5, v_6, v_7, v_1, v_2, v_3)$ has length $|p_4| + 7$, the path $p_4$ has odd length. Again the cycle using $p_2$ and $p_4$ through $(v_2, v_0, v_5, v_6, v_7, v_1)$ has length $|p_2| + |p_4| + 4$, which is odd. Thus $|p_2| = 2$ and $|p_4| = 1$; that is, when $|p_1| = 2$ and $|p_1| = 1$, we get $|p_2| = 2$ and $|p_4| = 1$. This case is symmetric to the one noted earlier with $|p_1| = 1 = |p_2|$ and $|p_3| = 2 = |p_4|$.

Next suppose $H_{1,3}$ is even. Then $H_{2,3}$ is odd. Thus $|p_2| + |p_3| + 5 = 7$ so $|p_2| = 1$ and $|p_3| = 1$. The even cycle $H_{1,3}$ has length $|p_1| + 5$, so $|p_1|$ is odd. By the cycle using $p_3$ and $p_4$ through $(v_4, v_0, v_5, v_6, v_7, v_1, v_2, v_3)$, which has length $|p_4| + 7$, the path $p_4$ has odd length. The cycle using $p_1$ and $p_4$ through $(v_1, v_0, v_5, v_6, v_7)$ has length $|p_1| + |p_4| + 3 \leq 7$. So, we can have both $p_1$ and $p_4$ of length 1, or one of $p_1$ and $p_4$ is length 3 and the other is length 1. By the symmetry in $G_3$ between $p_1$ and $p_4$, this yields two additional cases. Summarizing the possibilities for $G_3$, we have the following.

**3.0.1.4. $G_3$ is one of the three graphs shown in Figure 3.5.**

Next we consider the configuration in Figure 3.3d, which we shall call $G_4$. Consider the cycle $J_{1,2}$ using $p_1$ and $p_2$ through $(v_1, v_0, v_2, v_3, v_4, v_5, v_6, v_7)$ of length $|p_1| + |p_2| + 6$, the cycle $J_{1,3}$ using $p_1$ and $p_3$ through $(v_1, v_0, v_4, v_5, v_6, v_7)$ of length $|p_1| + |p_3| + 4$, and the cycle $J_{2,3}$ using $p_2$ and $p_3$ through $(v_2, v_0, v_4, v_5, v_6, v_7, v_1)$ of length $|p_2| + |p_3| + 5$. The sum of the lengths of the cycles is $2|p_1| + 2|p_2| + 2|p_3| + 15$. The length of $J_{1,2}$ and the fact that each path is non-empty imply that $J_{1,2}$ has even length and exactly one of the other two cycles is odd.

Suppose the length $|p_2| + |p_3| + 5$ of $J_{2,3}$ is odd. Then $|p_2| = 1$ and $|p_3| = 1$. The cycle $J_{1,3}$ of length $|p_1| + |p_3| + 4$ is even by assumption, so $|p_1|$ is odd. Hence the cycle using $p_1$ and $p_3$ through $(v_1, v_0, v_4, v_3, v_2)$ of length $|p_1| + |p_3| + 3 = |p_1| + 4$ is odd. Thus $|p_1|$ is 1 or 3. From
Figure 3.5: Path lengths of $G_3$

the cycle using $p_2$ and $p_4$ through $(v_2, v_0, v_6, v_5, v_4, v_3)$ of length $|p_2| + |p_4| + 4 = |p_4| + 5$, we deduce that $|p_4| \in \{1, 2\}$. The two cycles using $p_1$ and $p_4$ and the rim of the outer 7-cycle of lengths $|p_1| + |p_4| + 2$ and $|p_1| + |p_4| + 5$ give us cases with $|p_1| = 1$ and $|p_4| = 1$, with $|p_1| = 1$ and $|p_4| = 2$, and with $|p_1| = 3$ and $|p_4| = 2$.

Suppose $J_{1,3}$ is odd. Then $|p_1| + |p_3| + 4 = 7$, so one of $p_1$ and $p_3$ has length 1 and one has length 2. If $p_1$ has length 2 and $p_3$ has length 1, then the size of the cycle using $p_1$ and $p_4$ through $(v_1, v_0, v_6, v_5, v_4, v_3, v_2)$ is $|p_1| + |p_4| + 5 = |p_4| + 7$. Thus $|p_4|$ is odd. From the cycle $J_{1,2}$ of length $|p_1| + |p_2| + 6 = |p_2| + 8$, we deduce that $|p_2|$ is even. Since $|p_2|$ is even and $|p_4|$ is odd, the cycle using $p_2$ and $p_4$ through $(v_2, v_0, v_6, v_5, v_4, v_3)$ has odd length $|p_2| + |p_4| + 4 = 7$. So we get $|p_2| = 2$ and $|p_4| = 1$.

If $p_1$ has length 1 and $p_3$ has length 2, then the size of the cycle using $p_3$ and $p_4$ through $(v_4, v_0, v_6, v_7, v_1, v_2, v_3)$ is $|p_3| + |p_4| + 5 = |p_4| + 7$. Thus the path $p_4$ has odd length. From the cycle using $p_1$ and $p_4$ through $(v_1, v_0, v_6, v_5, v_4, v_3, v_2)$ of length $|p_1| + |p_4| + 5 = |p_4| + 6,
we deduce that $|p_4| = 1$. The cycle $J_{1,2}$ has length $|p_1| + |p_2| + 6 = |p_2| + 7$. Thus the path $p_2$ has odd length. From the cycle using $p_2$ and $p_3$ through $(v_2, v_0, v_4, v_3)$ of length $|p_2| + |p_3| + 2 = |p_2| + 4 \leq 7$, we see that $p_2 \in \{1, 3\}$. Thus $(|p_1|, |p_2|, |p_3|, |p_4|)$ is $(1, 1, 2, 1)$ or $(1, 3, 2, 1)$. These cases are symmetric to those with $(|p_1|, |p_2|, |p_3|, |p_4|)$ equal to $(1, 1, 1, 2)$ or $(3, 1, 1, 2)$, which were identified earlier. We conclude this case by noting that the following holds.

\textbf{3.0.1.5.} $G_4$ is one of the four graphs shown in Figure 3.6.

![Figure 3.6: Path lengths of $G_4$](image)

Summarizing our analysis above, we see that we showed that there is a single possibility for $G_1$, the one in which all of $p_1, p_2, p_3$, and $p_4$ have length one. There are three possibilities for each of $G_2$ and $G_3$, these being shown in Figures 3.4 and 3.5. Finally, there are four possibilities for $G_4$, these being shown in Figure 3.6. We will continue our argument by
looking first at the graphs above in which some \( p_i \) has more than one edge. The following observation plays a key role for much of the rest of the argument.

**3.0.1.6.** Let \( u \) and \( v \) be vertices of \( G \) such that \( G \) contains an even-length path \( p_e \) and an odd-length path \( p_o \) joining \( u \) and \( v \). If \( |p_e| \geq 8 \) and \( |p_o| \geq 7 \), then \( G \) has no path \( p \) that joins \( u \) and \( v \) and is internally disjoint from both \( p_e \) and \( p_o \).

If such a \( p \) existed, we could examine the cycles \( \{p_e, p\} \) and \( \{p_o, p\} \), which have opposite parity and have size greater than seven. This contradicts our choice of \( G \).

Recall the graph in Figure 3.4b is \( G_{2,2} \).

**3.0.1.7.** \( G \) does not have \( G_{2,2} \) as a subgraph.

Assume the contrary. By Theorem 2.0.2, there must be two paths from \( x \) to \( V(G_{2,2}) \setminus N(x) = \{v_0, v_1, v_3, v_4, v_5, v_6, v_7\} \) that have only \( x \) in common and that are internally disjoint from \( V(C) \cup \{v_2, x, w\} \) (see Figure 3.7).

![Figure 3.7: Possible paths from x in G_{2,2}](image)

By (3.0.1.6), using the paths \((x, v_2, v_1, v_7, v_4, v_3)\) of length 7 and \((x, w, v_0, v_1, v_7, v_6, v_5, v_3)\) of length 8, there is no \( x - v_3 \) path that is internally disjoint from \( V(C) \cup \{v_0, x, w\} \). By (3.0.1.6), using the paths \((x, w, v_0, v_1, v_7, v_6, v_5, v_4)\) and \((x, v_2, v_1, v_7, v_6, v_5, v_0, v_3, v_4)\) of lengths 7 and 8 respectively, there is no \( x - v_4 \) path that is internally disjoint \( V(C) \cup \{v_0, x, w\} \). Again, by (3.0.1.6), using the paths \((x, w, v_0, v_3, v_2, v_1, v_7, v_6)\) and \((x, v_2, v_3, v_0, v_1, v_7, v_6, v_5)\) of lengths 8 and 7 respectively, there are no \( x - v_5 \) paths that are internally disjoint from \( V(C) \cup \{v_0, x, w\} \). Similarly, we will find no \( x - v_6 \) paths that are internally disjoint from \( V(C) \cup \{v_0, x, w\} \), by (3.0.1.6) using the paths \((x, v_2, v_3, v_4, v_5, v_0, v_1, v_7, v_6)\) and
(x, w, v_0, v_3, v_2, v_1, v_7, v_6). By the paths (x, v_2, v_3, v_4, v_5, v_0, v_1, v_7) and (x, w, v_0, v_5, v_4, v_3, v_2, v_1, v_7), there are no x−v_7 paths that are internally disjoint from V(C)∪{v_0, x, w}. By the paths (x, v_2, v_3, v_4, v_5, v_6, v_7, v_1) and (x, w, v_0, v_3, v_4, v_5, v_6, v_7, v_1), there are no x−v_1 paths that are internally disjoint from V(C)∪{v_0, x, w}.

We now know that a path from x to {v_0, v_1, v_3, v_4, v_5, v_6, v_7} that is internally disjoint from V(C)∪{v_2, x, w} must end in v_0. As there are at least two such paths that meet only in x, we conclude that G cannot have G_{2,2} as a subgraph, that is, (3.0.1.7) holds.

Recall that the graph in Figure 3.4c is G_{2,3}.

### 3.0.1.8. G does not have G_{2,3} as a subgraph.

Assume the contrary. By Theorem 2.0.2, G has a path from x to V(G_{2,3}) \setminus N(x) = {v_1, v_2, v_3, v_4, v_6, v_7} that is internally disjoint from V(C)∪{v_0, x} as shown in Figure 3.8.

![Figure 3.8: Possible paths from x in G_{2,3}](image)

By (3.0.1.6), the paths (x, v_0, v_3, v_4, v_5, v_6, v_7, v_1) and (x, v_0, v_2, v_3, v_4, v_5, v_6, v_7, v_1) of lengths 7 and 8 imply there is no x−v_1 path internally disjoint from V(C)∪{v_0, x}. Again, the paths (x, v_5, v_6, v_7, v_1, v_0, v_3, v_2) and (x, v_0, v_3, v_4, v_5, v_6, v_7, v_1, v_2) of lengths 7 and 8 imply there is no x−v_2 path that is internally disjoint from V(C)∪{v_0, x}. By paths (x, v_0, v_2, v_1, v_7, v_6, v_5, v_4, v_3) and (x, v_0, v_1, v_7, v_6, v_5, v_4, v_3), there is no x−v_3 path that is internally disjoint from V(C)∪{v_0, x}. If we consider the paths (x, v_5, v_6, v_7, v_1, v_0, v_3, v_4) and (x, v_5, v_6, v_7, v_1, v_0, v_2, v_3, v_4), then we find that there is no x−v_4 path that is internally disjoint from V(C)∪{v_0, x}. Again, the paths (x, v_5, v_4, v_3, v_2, v_1, v_7, v_6) and (x, v_5, v_4, v_3, v_0, v_2, v_1, v_7, v_6) imply that there is no x−v_6 path that is internally disjoint from V(C)∪{v_0, x}. 

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Similarly, by the paths \((x, v_5, v_4, v_3, v_0, v_2, v_1, v_7)\) and \((x, v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)\), there is no \(x - v_7\) path that is internally disjoint from \(V(C) \cup \{v_0\}\). Thus \(x\) has no paths to \(\{v_1, v_2, v_3, v_4, v_6, v_7\}\) internally disjoint from \(V(C) \cup \{v_0, x\}\). Thus (3.0.1.8) holds.

Recall the graph in Figure 3.5a is \(G_{3,1}\).

**3.0.1.9.** \(G\) does not have \(G_{3,1}\) as a subgraph.

Assume the contrary. By Theorem 2.0.2, there must be some path from \(x\) to \(V(G_{2,3}) \setminus N(x) = \{v_1, v_3, v_4, v_5, v_6, v_7, w\}\) that is internally disjoint from \(V(C) \cup \{v_0, x, w\}\) as shown in Figure 3.9.

![Figure 3.9: Possible paths from x in G_{3,1}](image)

By (3.0.1.6), the paths \((x, v_2, v_3, v_4, v_5, v_6, v_7, v_1)\) and \((x, v_2, v_3, v_4, v_0, v_5, v_6, v_7, v_1)\) imply that there is no \(x - v_1\) path that is internally disjoint from \(V(C) \cup \{v_0, x, w\}\). From the paths \((x, v_2, v_1, v_7, v_6, v_5, v_4, v_3)\) and \((x, v_2, v_1, v_7, v_6, v_5, v_0v_4, v_3)\), we see that there is no \(x - v_3\) path that is internally disjoint from \(V(C) \cup \{v_0, x, w\}\). Again, the paths \((x, v_0, w, v_1, v_7, v_6, v_5, v_4)\) and \((x, v_0, v_5, v_6, v_7, v_1, v_2, v_3, v_4)\) of lengths 7 and 8 respectively imply there is no \(x - v_4\) path that is internally disjoint from \(V(C) \cup \{v_0, x, w\}\). Similarly, the paths \((x, v_0, w, v_1, v_2, v_3, v_4, v_5)\) and 

\[(x, v_0, v_4, v_3, v_2, v_1, v_7, v_6, v_5)\]

imply that there is no \(x - v_5\) path that is internally disjoint from \(V(C) \cup \{v_0, x, w\}\). By the paths \((x, v_0, v_4, v_3, v_2v_1, v_7, v_6)\) and \((x, v_0, v_5, v_4, v_3, v_2, v_1, v_7, v_6)\), there is no \(x - v_6\) path that is internally disjoint from \(V(C) \cup \{v_0, x, w\}\). Similarly, the paths \((x, v_2, v_1, w, v_0, v_5, v_6, v_7)\) and \((x, v_2, v_1, w, v_0, v_4, v_5, v_6, v_7)\) imply that there is no \(x - v_7\) path that is internally disjoint from \(V(C) \cup \{v_0, x, w\}\). Finally, the paths \((x, v_0, v_5, v_6, v_7)\) imply that there is no \(x - v_8\) path that is internally disjoint from \(V(C) \cup \{v_0, x, w\}\).
v_4, v_5, v_6, v_7, v_1, w) and (x, v_2, v_3, v_4, v_5, v_6, v_7, v_1, w) imply there is no x - w path that is internally disjoint from \( V(C) \cup \{v_0, x, w\} \). Thus x has no paths to \{v_1, v_3, v_4, v_5, v_6, v_7, w\} internally disjoint from \( V(C) \cup \{v_0, x, w\} \). Thus (3.0.1.9) holds.

Recall the graph in Figure 3.5c is \( G_{3,3} \).

**3.0.1.10.** \( G \) does not have \( G_{3,3} \) as a subgraph.

Assume the contrary. By Theorem 2.0.2, there must be two paths from \( x \) to \( V(G_{3,3}) \setminus N(x) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \) that have only the vertex \( x \) in common and that are internally disjoint from \( V(C) \cup \{v_0, x, w\} \) (see Figure 3.10).

![Figure 3.10: Possible paths from x in G_{3,3}](image)

By (3.0.1.6), the paths \((x, v_0, v_4, v_5, v_6, v_7, v_1, v_2)\) and \((x, v_0, v_4, v_5, v_6, v_7, v_1, v_2)\) of lengths 7 and 8 imply there is no \( x - v_2 \) path internally disjoint from \( V(C) \cup \{v_0, x, w\} \). Similarly, by the paths \((x, w, v_1, v_7, v_6, v_5, v_4, v_3)\) and \((x, v_0, v_4, v_5, v_6, v_7, v_1, v_2, v_3)\) of lengths 7 and 8 respectively, there is no \( x - v_3 \) path that is internally disjoint from \( V(C) \cup \{v_0, x, w\} \).

By paths \((x, v_0, v_2, v_1, v_7, v_6, v_5, v_4)\) and \((x, v_0, v_5, v_6, v_7, v_1, v_2, v_3, v_4)\), there is no \( x - v_4 \) path that is internally disjoint from \( V(C) \cup \{v_0, x, w\} \). If we consider the paths \((x, w, v_1, v_2, v_3, v_4, v_5)\) and \((x, v_0, v_4, v_3, v_2, v_1, v_7, v_6, v_5)\), then we find that there is no \( x - v_5 \) path that is internally disjoint from \( V(C) \cup \{v_0, x, w\} \). Again, the paths \((x, v_0, v_4, v_3, v_2, v_1, v_7, v_6)\) and \((x, v_0, v_5, v_4, v_3, v_2, v_1, v_7, v_6)\) imply that there is no \( x - v_6 \) path that is internally disjoint from \( V(C) \cup \{v_0, x, w\} \). Similarly, by the paths \((x, v_0, v_2, v_3, v_4, v_5, v_6, v_7)\) and \((x, w, v_1, v_2, v_3, v_4, v_5, v_6, v_7)\), there is no \( x - v_7 \) path that is internally disjoint from \( V(C) \cup \{v_0, x, w\} \).
This leaves only $v_1$ in \{$v_1, v_2, v_3, v_4, v_5, v_6, v_7$\} that can be the end of a path from $x$ that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Since there are two such paths that have only the vertex $x$ in common, we conclude that $G$ cannot have $G_{3,3}$ as a subgraph, that is, (3.0.1.10) holds.

Recall the graph in Figure 3.6b is $G_{4,2}$.

**3.0.1.11.** $G$ does not have $G_{4,2}$ as a subgraph.

Assume the contrary. By Theorem 2.0.2, there must be two paths from $x$ to $V(G_{4,2}) \setminus N(x) = \{v_1, v_2, v_3, v_4, v_5, v_7\}$ that have only the vertex $x$ in common and that are internally disjoint from $V(C) \cup \{v_0, x\}$ (see Figure 3.11).

![Figure 3.11: Possible paths from $x$ in $G_{4,2}$](image)

By (3.0.1.6), the paths $(x, v_6, v_5, v_4, v_3, v_2, v_0, v_1)$ and $(x, v_0, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ of lengths 7 and 8 imply there is no $x - v_1$ path internally disjoint from $V(C) \cup \{v_0, x\}$. Similarly, by the paths $(x, v_0, v_4, v_5, v_6, v_7, v_1, v_2)$ and $(x, v_0, v_1, v_7, v_6, v_5, v_4, v_3, v_2)$, there is no $x - v_2$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. By paths $(x, v_0, v_1, v_7, v_6, v_5, v_4, v_3)$ and $(x, v_0, v_2, v_1, v_7, v_6, v_5, v_4, v_3)$, there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$.

If we consider the paths $(x, v_0, v_7, v_1, v_2, v_3, v_4, v_5)$ and $(x, v_0, v_4, v_3, v_2, v_1, v_7, v_6, v_5)$, then we find that there is no $x - v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. Again, the paths $(x, v_0, v_5, v_4, v_3, v_2, v_1, v_7)$ and $(x, v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ imply that there is no $x - v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. The paths $(x, v_0, v_2, v_1, v_7, v_6, v_5, v_4)$ of length 7 and $(x, v_6, v_7, v_1, v_2, v_3, v_4)$ of length 6 imply that any $x - v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$ must have length 1. As the graph $G$ is 4-connected,
if $G_{4,2}$ is a subgraph, there can only be one such path from $x$ to $v_4$. Therefore we do not have the required two paths from $x$ to $V(G_{4,2}) \setminus N(x)$. Thus (3.0.1.11) holds.

Recall the graph in Figure 3.6c is $G_{4,3}$.

**3.0.1.12. $G$ does not have $G_{4,3}$ as a subgraph.**

Assume the contrary. By Theorem 2.0.2, there must be two paths from $x$ to $V(G_{4,3}) \setminus N(x) = \{v_1, v_2, v_3, v_4, v_5, v_7, u, w\}$ that have only the vertex $x$ in common and that are internally disjoint from $V(C) \cup \{v_0, x, u, w\}$ (see Figure 3.12).

![Figure 3.12: Possible paths from $x$ in $G_{4,3}$](image)

By (3.0.1.6), the paths $(x, v_6, v_5, v_4, v_0, w, u, v_1)$ and $(x, v_0, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ of lengths 7 and 8 imply that there is no $x-v_1$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. The paths $(x, v_0, v_4, v_5, v_6, v_7, v_1, v_2)$ and $(x, v_6, v_5, v_4, v_0, w, u, v_1, v_2)$ imply there is no $x-v_2$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$; the paths $(x, v_6, v_7, v_1, v_2, v_0, v_4, v_3)$ and $(x, v_0, v_2, v_1, v_7, v_0, v_4, v_5, v_4, v_3)$ imply there is no $x-v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Similarly, by the paths $(x, v_0, v_6, v_5, v_0, v_4, v_5, v_6, v_7, v_1, v_0, v_4, v_5)$, there is no $x-v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. By the paths $(x, v_0, v_6, v_5, v_0, v_4, v_5, v_6, v_7, v_5)$ and $(x, v_6, v_7, v_1, u, w, v_1, v_7, v_0, v_4, v_5)$, there is no $x-v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Similarly, the paths $(x, v_0, v_6, v_5, v_0, v_4, v_5, v_6, v_7, v_1, v_0, v_4, v_5)$ imply that there is no $x-v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. The paths $(x, v_0, v_6, v_5, v_4, v_3, v_2, v_1, u)$ and $(x, v_6, v_5, v_4, v_3, v_2, v_0, w, u)$ imply there is no $x-u$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Finally, by the paths $(x, v_0, v_6, v_5, v_4, v_3, v_2, v_0, w)$ and $(x, v_6, v_5, v_4, v_3, v_2, v_0, w, u)$,
$v_1, u, w$), there is no $x - w$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Thus $x$ has no paths to $\{v_1, v_2, v_3, v_4, v_5, v_7, u, w\}$ internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Therefore (3.0.1.12) holds.

Recall the graph in Figure 3.6d is $G_{4,4}$.

**3.0.1.13.** $G$ does not have $G_{4,4}$ as a subgraph.

Assume the contrary. By Theorem 2.0.2, there must be two paths from $x$ to $V(G_{4,4}) \setminus N(x) = \{v_2, v_3, v_4, v_5, v_6, v_7, w\}$ that have only the vertex $x$ in common and that are internally disjoint from $V(C) \cup \{v_0, x, w\}$ (see Figure 3.13).

![Figure 3.13: Possible paths from $x$ in $G_{4,4}$](image)

By (3.0.1.6), the paths $(x, v_1, v_7, v_6, v_5, v_4, v_3, v_2)$ and $(x, v_1, v_7, v_6, v_5, v_4, v_0, w, v_2)$ of lengths 7 and 8 imply there is no $x - v_2$ path internally disjoint from $V(C) \cup \{v_0, x, w\}$; the paths $(x, v_1, v_7, v_6, v_0, w, v_2, v_3)$ and $(x, v_1, v_2, w, v_0, v_6, v_5, v_4, v_3)$ imply there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. By paths $(x, v_1, v_2, w, v_0, v_6, v_5, v_4)$ and $(x, v_1, v_7, v_6, v_0, w, v_2, v_3, v_4)$, there is no $x - v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. If we consider the paths $(x, v_1, v_2, v_3, v_4, v_0, v_6, v_5)$ and $(x, v_0, v_4, v_3, v_2, v_1, v_7, v_6, v_5)$, then we find that there is no $x - v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Again, the paths $(x, v_1, v_2, v_3, v_4, v_0, v_7)$ and $(x, v_0, w, v_2, v_3, v_4, v_5, v_6, v_7)$ imply that there is no $x - v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Similarly, by the paths $(x, v_0, v_6, v_5, v_4, v_3, v_2, w)$ and $(x, v_1, v_7, v_6, v_4, v_3, v_2, w)$, there is no $x - w$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Finally, by the paths $(x, v_0, w, v_2, v_3, v_4, v_5, v_6)$ of length 7 and $(x, v_1, v_2, v_3, v_4, v_5, v_6)$ of length 6, any $x - v_6$ path
that is internally disjoint from $V(C) \cup \{v_0, x, w\}$ has length 1. Thus there is only one such $x - v_6$ path; a contradiction. Therefore (3.0.1.13) holds.

This eliminates all cases where $p_1, p_2, p_3,$ or $p_4$ has more than one edge. Thus each vertex in $G$ that is not in the selected 7-cycle is of a type shown in Figure 3.14.

Since $|V(C)| > 8$, we have more than one vertex not in $V(C)$. Suppose we have at least one vertex of type $3.14a, 3.14b,$ or $3.14c$ not on $C$. Since each such vertex creates two triangles sharing a single edge with $C$, we will always have two edge-disjoint triangles sharing a single edge with $C$, as every vertex not on $C$ is in at least one triangle with an edge of $C$. So, by (3.0.1.1), $G$ has no vertices of type $3.14a, 3.14b,$ or $3.14c$. Thus all vertices not on $C$ are of type $3.14d$. Furthermore, if we have two such vertices, they must be adjacent to the same four vertices of $C$ by (3.0.1.1) again. We deduce that $G$ has the graph in Figure 3.15 as a subgraph.
Figure 3.15: A subgraph of $G$

Now we must check for additional possible edges within $G$ that do not create a larger odd cycle. In order for $G$ to be 4-connected, $v_3$, $v_5$, and $v_7$ in our subgraph must be connected to two additional vertices to have degree four. As $|V(C)| > 8$, $v_1$, $v_2$, $v_4$, and $v_6$ already have degree at least four. By our previous argument, each of $v_3$, $v_5$, and $v_7$ can only be adjacent to other vertices of $C$. This leaves $v_1$, $v_5$, $v_6$ and $v_7$ as possible additional neighbors for $v_3$, while $v_1$, $v_2$, $v_3$, and $v_7$ are possible additional neighbors for $v_5$. Finally, $v_2$, $v_3$, $v_4$, and $v_5$ are possible additional neighbors for $v_7$.

If there is an edge $\{v_3, v_5\}$, we get a 9-cycle ($v_5, v_6, v_7, v_1, v_0, v_2, v_4, v_3$). By symmetry, $\{v_3, v_5\}$ is not an edge. If there is an edge $\{v_3, v_7\}$, we get a 9-cycle ($v_3, v_4, v_5, v_6, v'_0, v_2, v_0, v_1, v_7$). For $v_3$, the edges $\{v_1, v_3\}$ and $\{v_6, v_3\}$ remain as possibilities. These edges create no 9-cycles and therefore are the desired necessary edges to complete degree requirements.

For $v_5$, the edges $\{v_1, v_5\}$ and $\{v_2, v_5\}$ remain as possibilities. These edges create no 9-cycles and therefore are the desired necessary edges to complete degree requirements. For $v_5$, the edges $\{v_2, v_7\}$ and $\{v_4, v_7\}$ remain as possibilities. These edges create no 9-cycles and therefore are the desired necessary edges to complete degree requirements. Our subgraph is now 4-connected.

Now we examine remaining possible edges. From our previous argument, we know all edges that meet vertices in $V(C)$ and not in $V(C)$. If any such vertex $v_0$ meets an additional edge, this edge must be $\{v_0, v'_0\}$ for some $v'_0$ not in $V(C)$. Assume such an edge exists. This
would create a 9-cycle \((v_0, v_2, v_3, v_4, v_5, v_6, v_7, v_1, v_0')\). Thus there is no edge between any vertices outside the 7-cycle \(C\).

The remaining possible edges are \(\{v_1, v_4\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_2, v_6\}\) and \(\{v_4, v_6\}\). If we include all such edges, we do not create an odd cycle larger than a 7-cycle. Thus if we include any subset of these edges, we will not create such an odd cycle.

This concludes our construction of \(G\). All vertices not in the 7-cycle \(C\) are adjacent to the same four vertices of \(C\). In our construction, these vertices are \(v_1, v_2, v_4,\) and \(v_6\). These four vertices form one side of a bipartition. The other three vertices of the 7-cycle are adjacent to all four of these vertices (see Figure 3.16). The subgraph induced by the four-vertex side of the bipartition is any subgraph of \(K_4\) having at least one edge. We conclude that Theorem 3.0.1 holds.

\[\square\]

Figure 3.16: \(K'_4,n\)
Chapter 4  \(n\)-connected Graphs with No Odd Cycles Exceeding Size \(2n - 1\)

In this chapter, we will generalize the graph results of the earlier chapters.

**Theorem 4.0.1.** Suppose \(n \geq 2\). Let \(G\) be an \(n\)-connected simple graph having a cycle of length \(2n - 1\). Then \(G\) has no odd cycle of length exceeding \(2n - 1\) if and only if

(i) \(|V(G)| \leq 2n\); or

(ii) for some \(t \geq n + 1\), the graph \(G\) is isomorphic to a graph that is obtained from \(K_{n,t}\) by adding at least one and at most \(n(n-1)\) edges each having both ends in the \(n\)-vertex side of the vertex bipartition.

**Proof.** By Theorem 2.2, Theorem 3.0.1, and Theorem 1.0.1, the theorem holds for graphs with \(n < 5\). Let \(n \geq 5\). If \(G\) satisfies (i) or (ii), it is straightforward to check that \(G\) has no odd cycles of size exceeding \(2n - 1\).

Conversely, suppose \(G\) has no odd cycles of length exceeding \(2n - 1\). If \(|V(G)| < 2n + 1\), there is no larger odd cycle. Assume \(|V(G)| \geq 2n + 1\). Select a \((2n - 1)\)-cycle \(C\) of \(G\) and label its vertices, in order, by \(v_1, v_2, \ldots, v_{2n-1}\). Since \(|V(G)| > 2n - 1\), there is an additional vertex outside of \(V(C)\).

We will now take note of the following observations.

**4.0.1.1.** Suppose \(v_a\) and \(v_{a+1}\) are consecutive vertices on the cycle \(C\) and there are paths \(p_a\) and \(p_{a+1}\) from \(v_a\) and \(v_{a+1}\) to some vertex \(u\) not on \(C\) such that these paths meet only in \(u\). Then \(|p_a|\) and \(|p_{a+1}|\) have the same parity.

Assume not. Then we have a cycle consisting of a path in \(C\) from \(v_{a+1}\) to \(v_a\) having length exceeding one along with the paths \(p_a\) and \(p_{a+1}\). This cycle has length \(2n - 2\) plus the sum of two numbers of opposite parities. So we have an odd cycle of length \(2n + 1\) or greater. Thus \(|p_a|\) and \(|p_{a+1}|\) have the same parity.
Next we show the following.

4.0.1.2. Suppose \(v_a\) and \(v_{a+2}\) are two vertices on the cycle \(C\) that are distance two apart on \(C\), and assume there are paths \(p_a\) and \(p_{a+2}\) from \(v_a\) and \(v_{a+2}\) to some vertex \(u\) not on \(C\) such that these two paths meet only in \(u\). Then \(|p_a|\) and \(|p_{a+2}|\) are either both one or they are have different parities.

In \(G\), we have a cycle \(D\) consisting of a path in \(C\) from \(v_{a+2}\) to \(v_a\) of length \(2n - 1 - 2\) along with the paths \(p_a\) and \(p_{a+2}\). This cycle has length \(2n - 1 - 2 + |p_a| + |p_{a+2}|\). If \(D\) has odd length, then \(|p_a| = 1\) and \(|p_{a+2}| = 1\). If \(D\) has even length, then \(|p_a| + |p_{a+2}|\) is odd, and the paths \(p_a\) and \(p_{a+2}\) have opposite parities.

4.0.1.3. Suppose \(G\) has distinct vertices \(v_0\) and \(v_0'\) not on \(C\). Assume \(C\) has distinct edges \(\{v_a, v_{a+1}\}\) and \(\{v_b, v_{b+1}\}\) such that there are paths \(p_a\) and \(p_{a+1}\) from \(v_a\) and \(v_{a+1}\) to \(v_0\) that meet only in \(v_0\), and there are paths \(p_b\) and \(p_{b+1}\) from \(v_b\) and \(v_{b+1}\) to \(v_0'\) that meet only in \(v_0'\). Assume also that \(p_a\) and \(p_{a+1}\) are vertex disjoint from \(p_b\) and \(p_{b+1}\) except that \(v_a\) may equal \(v_{b+1}\), or \(v_{a+1}\) may equal \(v_b\) but not both. Then \(G\) has an odd cycle of length exceeding \(2n - 1\).

By (4.0.1.1), \(|p_a|\) and \(|p_{a+1}|\) have the same parity, and \(|p_b|\) and \(|p_{b+1}|\) have the same parity. Thus \(|p_a| + |p_{a+1}| = 2j\) and \(|p_b| + |p_{b+1}| = 2k\) for some natural numbers \(j, k\). If we follow the cycle \(C\) replacing the edges \(\{v_a, v_{a+1}\}\) and \(\{v_b, v_{b+1}\}\) with the paths \(p_a\) and \(p_{a+1}\) and \(p_b\) and \(p_{b+1}\), we get a cycle of length \(2n - 1 - 2 + 2j + 2k \geq 2n + 1\).

4.0.1.4. Let \(v_0\) be a vertex not on \(C\). If \(v_a, v_{a+1}\), and \(v_{a+2}\) are three consecutive vertices in order on \(C\) with paths \(p_a, p_{a+1}\) and \(p_{a+2}\) to \(v_0\) that have no other common vertices, then \(|p_a| = 1\), \(|p_{a+2}| = 1\) and \(|p_{a+1}|\) is odd.

By (4.0.1.1) \(|p_a|\) and \(|p_{a+1}|\) have the same parity, and \(|p_{a+1}|\) and \(|p_{a+2}|\) have the same parity. Thus all three paths have the same parity. Hence, by (4.0.1.2), \(|p_a| = 1 = |p_{a+2}|\). Since \(|p_{a+1}|\) has the same parity as \(|p_a|\), we deduce that \(|p_{a+1}|\) is odd.
4.0.1.5. Suppose $v_a, v_{a+1}, \ldots, v_{a+t}$ are consecutive vertices of $C$ with $t \geq 3$ and these vertices are joined to some vertex $v_0$ not on $C$ by paths $p_a, p_{a+1}, \ldots, p_{a+t}$ that meet only in $v_0$. Then all these paths have length one.

By (4.0.1.1), all of $|p_a|, |p_{a+1}|, \ldots, |p_{a+t}|$ have the same parity. By (4.0.1.2), it follows that all of $p_a, p_{a+1}, \ldots, p_{a+t}$ have length one.

Finally, we will need the following observation about path lengths within subgraphs of $G$.

4.0.1.6. Suppose $v_a$ and $v_b$ are vertices of a subgraph $H$ of $G$ and there are two paths in $H$ from $v_a$ to $v_b$ each of length at least $2n - 1$ and of different parities. Then $G$ has no $v_a - v_b$ path disjoint from $H - \{v_a, v_b\}$.

Assume $G$ has a $v_a - v_b$ path $p'$ disjoint from $H - \{v_a, v_b\}$. Let $p_o$ and $p_e$ be $v_a - v_b$ paths in $H$ of odd and even lengths, respectively, each of length at least $2n - 1$. Then $G$ contains cycles of lengths $|p'| + |p_o| \geq 1 + 2n - 1$ and $|p'| + |p_e| \geq 1 + 2n - 1$. Hence we have two cycles whose lengths exceed $2n - 1$ and have opposite parities. Thus there is an odd cycle of size greater than $2n - 1$. We deduce that no such $p'$ exists.

Choose a vertex $v_0$ of $G$ that is not in $C$. By Theorem 2.0.2, there are $n$ paths from $v_0$ to $V(C)$ whose only common vertex is $v_0$. Since we have $2n - 1$ vertices on $C$ and $n$ paths from $v_0$ to distinct vertices of $C$, we will always have at least two consecutive vertices on $C$ that meet distinguished paths from $v_0$.

Now we focus on showing the following.

4.0.1.7. None of the distinguished paths from $v_0$ to $C$ has length exceeding one.

By (4.0.1.5), if all paths from $v_0$ to $C$ meet at consecutive vertices, then all paths have length one. Thus we may assume that the distinguished paths do not all meet $C$ at consecutive vertices.
First, suppose \( G \) has a fan-like subgraph and at least one additional distinguished path to \( C \) from \( v_0 \) as shown in Figure 4.1 where \( f \geq 4 \). The paths from \( v_0 \) to the cycle that meet in the string, \( v_1, v_2, \ldots, v_f \), of consecutive vertices will all have length one by (4.0.1.5). We assume that neither \( v_{2n-1} \) nor \( v_{f+1} \) is the end of one of the distinguished paths from \( v_0 \).

![Figure 4.1: Graph with large fan-like subgraph](image)

Assume at least one distinguished path \( p \) from \( v_0 \) to one of \( v_{f+2}, v_{f+3}, \ldots, v_{2n-2} \) has length greater than one. Label the vertex adjacent to \( v_0 \) on \( p \) as \( v_p \). By Theorem 2.0.2, there are \( n \) paths in \( G \) from \( v_p \) to \( C \) having only the vertex \( v_p \) in common. Consider the paths \((v_p, v_0, v_{i-1}, v_{i-2}, \ldots, v_i)\) and \((v_p, v_0, v_{i-2}, v_{i-3}, \ldots, v_i)\) where \( i \in \{3, 4, \ldots, f+1\} \) and \( i-i = 2n-1 \). These have lengths \( 1+1+((2n-1)-1) = 2n \) and \( 1+1+((2n-1)-2) = 2n-1 \).

By (4.0.1.6), there is no \( v_p - v_i \) path in \( G \) disjoint from \( V(C) \cup \{v_p, v_0\} \). It follows using symmetry that \( v_p \) does not have paths to any of the vertices \( v_{2n-1}, v_1, v_2, \ldots, v_{f+1} \) that have no member of \( V(C) \cup v_0 \) as internal vertices. Since \( f \geq 4 \), there must be \( n-1 \) paths from \( v_p \) to the remaining vertices of \( C \) of which there are at most \( 2n-1-6 \).

Suppose two such paths meet at consecutive vertices on \( C \). The triangle \( \{v_0, v_2, v_3\} \) and these two new paths satisfy the hypotheses of (4.0.1.3) as the paths are internally disjoint from \( C \) and are disjoint from \( v_0, v_2, \) and \( v_3 \). Thus \( G \) has an odd cycle of length exceeding \( 2n-1 \); a contradiction. We deduce that the distinguished paths from \( v_p \) to \( C \) do not end
at consecutive vertices. This is a contradiction, since we have \( n - 1 \) paths but only \( 2n - 7 \) vertices that can be ends of these paths. We deduce that, in this case, (4.0.1.7) holds.

Continuing with the proof of (4.0.1.7), we may now assume the following.

4.0.1.8. If a distinguished path has length greater than one, then the longest sequence of consecutive vertices of \( C \) that are ends of distinguished paths from \( v_0 \) has length at most three.

Now assume that \( C \) has three consecutive vertices \( v_1, v_2, \) and \( v_3 \) that are ends of distinguished paths.

Next we show the following.

4.0.1.9. If the distinguished paths meet three consecutive vertices \( v_1, v_2, \) and \( v_3 \) and each of these paths to the consecutive vertices on \( C \) has length one, then all other distinguished paths have length one.

By (4.0.1.4), we know the distinguished paths from \( v_0 \) to \( v_1 \) and \( v_3 \) have length one while the path from \( v_0 \) to \( v_2 \) has odd length. Suppose the \( v_0 - v_2 \) path has length one. Then one of the other distinguished paths from \( v_0 \) to \( C \) has length greater than one. Let \( v_p \) be the vertex of this path adjacent to \( v_0 \). Then we may use the paths from the previous argument to see that \( G \) does not have a path from \( v_p \) to \( v_{2n-1}, v_1, v_2, v_3, \) or \( v_4 \) that is disjoint from \( V(C) \cup v_0 \). Now, \( G \) contains \( n - 1 \) paths from \( v_p \) to \( V(C) \) avoiding \( v_0 \) and having only \( v_p \) in common. Again by (4.0.1.3) and using \( \{v_0, v_2, v_3\} \), we get an odd cycle of length exceeding \( 2n - 1 \) if two of the paths from \( v_p \) end in consecutive vertices of \( C \). As there are at most \( 2n - 6 \) vertices that are ends of such paths and there are \( n - 1 \) such paths, we obtain a contradiction. We deduce that the \( v_0 - v_2 \) path has odd length exceeding one or (4.0.1.9) holds.
4.0.1.10. If a distinguished path has length greater than one and we have three consecutive vertices meeting distinguished paths at $v_1$, $v_2$, and $v_3$, no two of the distinguished paths from $v_0$ can end in consecutive vertices of $C$ other than those ending in $v_1$, $v_2$, $v_3$.

Since $G$ is at least 5-connected, there are at least two other distinguished paths from $v_0$ apart from those that have $v_1$, $v_2$, and $v_3$ as their ends. Suppose two of these additional paths meet $C$ at adjacent vertices, $v_a$ and $v_{a+1}$, as shown in Figure 4.2.

Consider the vertex on the $v_2 - v_0$ path that is adjacent to $v_0$. Label this vertex $v_p$ and the distinguished $v_0 - v_2$ path $p_2$. Label the distinguished paths from $v_0$ to $v_a$ and $v_{a+1}$ by $p_a$ and $p_{a+1}$. Moreover, label the portion of $p_2$ from $v_p$ to $v_2$ by $p'_2$. By (4.0.1.2), the lengths of $p_a$ and $p_{a+1}$ have the same parity. By Theorem 2.0.2, $v_p$ has $n$ paths to $C$ that meet only in $v_p$.

Consider the path consisting of the union of $(v_p, v_0), p_{a+1},$ and $(v_{a+1}, v_{a+2}, \ldots, v_a)$. Also consider the path consisting of the union of $p'_2$, $(v_2, v_1, v_{2n-1}, \ldots, v_{a+1}), p_{a+1},$ and $(v_0, v_3, v_4, \ldots, v_a)$. These paths have lengths $1 + |p_{a+1}| + ((2n - 1) - 1) = 2n - 1 + |p_{a+1}|$ and $(|p_2| - 1) + ((2n - 1) - 1 - 1) + |p_{a+1}| + 1 = 2n - 3 + |p_{a+1}| + |p_2|$. By (4.0.1.4), $|p_2|$ is odd. By (4.0.1.6), there is no $v_p - v_a$ path internally disjoint from $C$ and $v_0$. By symmetry, there is no $v_p - v_{a+1}$ path internally disjoint from $C$ and $v_0$. By using the paths $(v_p, v_0, v_{a+1}, v_a, v_{a-1}, \ldots, v_{a+2})$ and $(v_p, v_0, v_a, v_{a-1}, \ldots, v_{a+2})$ of lengths $2 + ((2n - 1) - 1) =$
2n and \(2 + ((2n - 1) - 2) = 2n - 1\), we deduce that there is no \(v_p - v_{a+2}\) path internally disjoint from \(C\) and \(v_0\). By symmetry, there is no \(v_p - v_{a-1}\) path internally disjoint from \(C\) and \(v_0\).

Let \(v_q\) be any internal vertex on \(p_a\), provided \(|p_a| > 1\). Label the portion of the path from \(v_q\) to \(p_a\) as \(p'_a\). Consider the path consisting of the union of \((v_p, v_0, p_{a+1}, (v_{a+1}, v_{a+2}, \ldots, v_a)\), and \(p'_a\). Also consider the path consisting of the union of \(p'_2\), \((v_2, v_1, v_{2n-1}, \ldots, v_{a+1})\), \(p_{a+1}\), \((v_0, v_3, v_4, \ldots, v_a)\) and \(p'_a\). These paths have lengths \(1 + |p_{a+1}| + ((2n - 1) - 1) + |p'_a| = 2n-1+|p_{a+1}|+|p'_a|\) and \((|p_2| - 1) + (|p_{a+1}| + |p'_a|) = |p_2| + 2n - 3 + |p_{a+1}| + |p'_a|\).

Since \(|p_2|\) is odd, these lengths have different parities and have size greater than \(2n - 1\). Thus, by (4.0.1.6), there is no \(v_p - v_q\) path disjoint from \(v_0\) and \(C\) for any \(v_q\) in the interior of \(p_a\). By symmetry, there is no \(v_p - v_q\) path disjoint from \(C\) and \(v_0\) for any \(v_q\) in the interior of \(p_{a+1}\). Thus \(v_p\) does not have paths internally disjoint from \(V(C) \cup v_0\) to any of \(v_{a-1}, v_a, v_{a+1}\), or \(v_{a+2}\). By (4.0.1.3), if the paths from \(v_p\) to \(C\) internally disjoint from \(V(C) \cup v_0\) meet \(C\) at two consecutive vertices, we find an odd cycle of length exceeding \(2n - 1\). Thus the \(n - 1\) paths from \(v_p\) to \(C\) that are internally disjoint from \(V(C) \cup v_0\) have their ends in vertices of \(V(C) \setminus \{v_{a-1}, v_a, v_{a+1}, v_{a+2}\}\) that are not consecutive on \(C\). Since there are only \(2n - 6\) such vertices, this is a contradiction. We deduce that if there is a distinguished path with path length greater than one, no two of the distinguished paths can end in consecutive vertices of \(C\) other than those ending in \(v_1, v_2, v_3\), that is, (4.0.1.10) holds.

**4.0.1.11.** If \(G\) has consecutive vertices of \(C\) meeting distinguished paths at \(v_1, v_2, \) and \(v_3\) and if \(p_2\) has length greater than one, then the distinguished paths that do not meet \(C\) at \(v_1, v_2, \) and \(v_3\) may not have distance two on \(C\), that is distinguished paths may not meet at \(v_a\) and \(v_{a+2}\).
To this end suppose there is at least one pair of distinguished paths from $v_0$ to $C$ whose ends are a distance two apart on $C$ aside from those ending in $v_1$, $v_2$ and $v_3$. Label these paths by $p_a$ and $p_{a+2}$, and let their ends on $C$ be $v_a$ and $v_{a+2}$, respectively.

Call a vertex of $C$ that meets a distinguished path a \textit{distinguished vertex}. Let $\mathcal{P}'$ be the set of distinguished paths from $v_0$ to $C$ other than those to $v_1$, $v_2$, and $v_3$.

\textbf{4.0.1.12.} If $G$ has consecutive vertices of $C$ meeting distinguished paths at $v_1$, $v_2$, and $v_3$ and if $p_2$ has length greater than one, then the distinguished paths in $\mathcal{P}'$ do not all have length one.

First suppose all paths in $\mathcal{P}'$ have length one as if Figure 4.3. By (4.0.1.4), we know the distinguished paths from $v_0$ to $v_1$ and $v_3$ have length one while the path from $v_0$ to $v_2$ has odd length. By (4.0.1.10), there are no consecutive vertices in the $2n - 1 - 5$ vertices of $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4\}$ that meet the distinguished paths from $v_0$. Thus there are $n - 3$ paths meeting $2n - 6$ vertices no two of which are consecutive. Therefore in the path $(v_4, v_5, \ldots, v_{2n-1})$, the vertices alternate between undistinguished and being distinguished except in exactly one place where there are two consecutive undistinguished vertices.

Let us examine two arbitrary paths in $\mathcal{P}'$ whose endpoints are distance two on $C$. The path $(v_p, v_0, v_{a+2}, v_{a+3}, \ldots, v_3)$ and the path that consists of the union of $p'_2$ and $(v_2, v_1,$
$v_{2n-1}, \ldots , v_{a+2}, v_0, v_3, v_4, \ldots , v_a$) have lengths $1+1+(2n-1) - 2 = 2n-1$ and $(|p_2|-1) + ((2n-1)-1-2)+1+1 = 2n-3+|p_2|$. Thus there are no $v_p-v_a$ paths that are internally disjoint from $V(C) \cup v_0$. By symmetry, there are no $v_p-v_{a+2}$ paths that are internally disjoint from $V(C) \cup v_0$. Consider the path $(v_p, v_0, v_a, v_{a+1}, \ldots , v_{a-1})$ and the path that consists of the union of $p'_2$ and $(v_2, v_1, \ldots , v_a, v_0, v_3, v_4, \ldots , v_{a-1})$. These paths have lengths $1+1+(2n-1)-1 = 2n$ and $(|p_2|-1)+((2n-1)-1-1)+1+1 = |p_2|+2n-2$. Thus there are no $v_p-v_{a-1}$ paths internally disjoint from $V(C) \cup v_0$. By symmetry, there are no $v_p-v_{a+3}$ paths internally disjoint from $C$ and $v_0$. By using the path $(v_p, v_0, v_a, v_{a-1}, \ldots , v_{a+1})$ and the path that is the union of $p'_2$ and $(v_2, v_1, \ldots , v_{a+2}, v_0, v_3, v_4, \ldots , v_{a+1})$, whose lengths are $1+1+(2n-1)-1 = 2n$ and $(|p_2|-1)+((2n-1)-1-1)+1+1 = |p_2|+2n-2$, we deduce that there are no $v_p-v_{a+1}$ paths disjoint from $C$ and $v_0$. By the paths $(v_p, v_0, v_3, v_4, \ldots , v_1)$ and $p'_2 \cup (v_2, v_3, \ldots , v_1)$, there are no $v_p-v_1$ paths disjoint from $C$ and $v_0$. By symmetry, there are no $v_p-v_3$ paths internally disjoint from $C$ and $v_0$. By the path $(v_p, v_0, v_3, v_2, \ldots , v_4)$ and the path that consists of the union of $p'_2$ and $(v_2, v_1, \ldots , v_4)$, there are no $v_p-v_4$ paths internally disjoint from $C$ and $v_0$. By symmetry, there are no $v_p-v_{2n-1}$ paths internally disjoint from $C$ and $v_0$.

By Theorem 2.0.2, there are $n-1$ paths from $v_p$ to $C$ that avoid $v_0$ and meet $C$ at distinct vertices. We showed above that none of these paths meets $V(C)$ in any member of $\{v_1, v_3, v_4, v_{2n-1}\} \cup \{v_{a-1}, v_a, v_{a+1}, v_{a+2}, v_{a+3}\}$. The union of $\{v_1, v_3, v_4, v_{2n-1}\}$ and the collection of all sets $\{v_{a-1}, v_a, v_{a+1}, v_{a+2}, v_{a+3}\}$ where each of $v_a$ and $v_{a+2}$ meets paths in $\mathcal{P'}$ includes all but at most three vertices of $C$ including $v_2$. This is a contradiction, since it implies there are at most three distinguished $v_p$ paths, so (4.0.1.12) holds. Note that the extreme case occurs when the consecutive non-distinguished vertices of $C$ isolate a distinguished $v_{2n-2}$ or a $v_5$, as otherwise all vertices on $C$ meet paired distinguished paths or the previous collection.
4.0.1.13. If \( G \) has consecutive vertices of \( C \) meeting distinguished paths at \( v_1, v_2, \) and \( v_3 \) and if \( p_2 \) has length greater than one, then there are no pairs of distinguished paths in \( \mathcal{P}' \) that meet at distance two on \( C \) where at least one path has length greater than one.

Suppose there is at least one pair of paths in \( \mathcal{P}' \) from \( v_0 \) to \( C \) that meet \( C \) at vertices that are distance two apart and the one of the path lengths is not one as in Figure 4.4.

![Figure 4.4: Subgraph of \( G \)](image)

By (4.0.1.2) these paths have opposite parities, so one is even. Let \( v_q \) be the vertex on that path adjacent to \( v_0 \). Relabel \( v_p \) to be any vertex on the \( p_2 \) path interior. Let the even path be \( p_a \) with endpoints \( v_0 \) and \( v_a \). By symmetry we may assume the distinguished path \( p_{a+2} \) from \( v_0 \) to \( v_{a+2} \) is odd. Let \( p'_a \) be the path from \( v_q \) to \( v_a \) contained in the larger path \( p_a \). Let \( p'_2 \) be the path from \( v_p \) to \( v_2 \) contained in the larger path \( p_2 \), and let \( p_2 - p'_2 \) be the subpath of \( p_2 \) from \( v_0 \) to \( v_p \). Consider the paths that consist of the union of \( p'_a, (v_a, v_{a-1}, \ldots, v_{a+2}) \), \( p_{a+2} \), and \( p_2 - p'_2 \) and the union of \( p'_a, (v_a, v_{a-1}, \ldots, v_3, v_0) \), \( p_{a+2}, (v_{a+2}, v_{a+3}, \ldots, v_2) \), and \( p'_2 \) of lengths \( (|p_a| - 1) + ((2n - 1) - 2) + |p_{a+2}| + |p_2 - p'_2| = 2n - 4 + |p_a| + |p_{a+2}| + |p_2 - p'_2| \) and \( (|p_a| - 1) + ((2n - 1) - 2 - 1) + 1 + |p_{a+2}| + |p'_2| = 2n - 4 + |p_a| + |p_{a+2}| + |p'_2| \). Since \( p_2 \) has odd length, \( |p'_2| \) and \( |p_2 - p'_2| \) have different parities. Thus, by (4.0.1.6), there is no \( v_q - v_p \) path disjoint from \( C \) and \( v_0 \) where \( v_p \) is on the interior of \( p_2 \). By the path consisting of the union of \( (v_q, v_0), p_2 \), and \( (v_2, v_3, \ldots, v_1) \) of length \( 1 + |p_2| + ((2n - 1) - 1) = 2n - 1 + |p_2| \) and the path \( (v_q, v_0, v_3, v_4, \ldots, v_1) \) of length \( 1 + 1 + ((2n - 1) - 2) = 2n - 1 \), there is no
path from $v_q$ to $v_1$ disjoint from $v_0$ and $C$. By symmetry, there is no $v_q - v_3$ path disjoint from $v_0$ and $C$. The path $(v_q, v_0, v_3, v_4, \ldots, v_2)$ and the path consisting of the union of $p'_a$, $(v_a, v_{a-1}, \ldots, v_3, v_0)$, $p_{a+2}$, and $(v_{a+2}, v_{a+3}, \ldots, v_2)$ have lengths $1 + 1 + (2n - 1) - 1 = 2n$ and $(|p_a| - 1) + ((2n - 1) - 2 - 1) + 1 + |p_{a+2}| = 2n - 4 + |p_a| + |p_{a+2}|$. Since $p_a$ has even length and $p_{a+2}$ has odd length, the second path is odd and has size greater than or equal to $2n - 1$. Thus there are no $v_q - v_2$ paths disjoint from $C$ and $v_0$. By the path $(v_q, v_0, v_3, v_2, \ldots, v_4)$ and the path consisting of the union of $(v_q, v_0)$, $p_2$, and $(v_2, v_1, \ldots, v_4)$ of lengths $1 + 1 + ((2n - 1) - 1) = 2n$ and $1 + |p_2| + ((2n - 1) - 1) = 2n + |p_2|$, there is no $v_q - v_4$ path disjoint from $v_0$ and $C$. By symmetry, there is no $v_q - v_{2n-1}$ path disjoint from $v_0$ and $C$. Thus, there is no path from $v_q$ to $v_{2n-1}$, $v_1$, $v_2$, $v_3$ or $v_4$ not through $v_0$. By Theorem 2.0.2, there are $n - 1$ internally disjoint paths to distinct vertices of $V(C) \setminus \{v_1, v_2, v_3, v_4, v_{2n-1}\}$. However, by (4.0.1.3) and the paths $p_2$ and $(v_0, v_3)$, they may not meet $C$ at consecutive vertices. This requires $2n - 3$ vertices. Thus we deduce that (4.0.1.13) holds.

Since we have that (4.0.1.10), (4.0.1.12) and (4.0.1.13), if there are still distinguished paths that meet at distance two on $C$ and $p_2$ has length greater than one, then we are in exactly the case where the only path in $\mathcal{P}'$ with length greater than one meets an unpaired $v_{2n-2}$ or $v_5$. Without loss of generality, assume the distinguished path meets $v_5$ and call it $p_5$. By (4.0.1.2), $p_5$ and $p_3$ have opposite parities, so $p_5$ is even. By the path consisting of the union of $p_2$, $p_5$, $(v_5, v_6, \ldots, v_1, v_2)$ of length $(2n - 1) - 3 + |p_2| + |p_5|$, there is an odd cycle larger than $2n - 1$ and hence we get the following result.

4.0.1.14. If $G$ has consecutive vertices of $C$ meeting distinguished paths at $v_1$, $v_2$, and $v_3$ and if $p_2$ has length greater than one, then there are no distinguished paths in $\mathcal{P}'$ that meet at distance two or one on $C$.

By (4.0.1.14) and (4.0.1.10), we now know that every two of the $n - 3$ paths in $\mathcal{P}'$ meet $C$ at vertices that are at distance 3 or greater. Thus we need at least $3(n - 3) - 2 = 3n - 11$ vertices remaining in $C$. Since there is no larger string of consecutive vertices that meet
distinguished paths, the paths in \( P' \) do not meet \( v_4 \) or \( v_{2n-1} \). Thus we have \( 2n - 1 - 5 = 2n - 6 \) vertices that can be endpoints of paths in \( C \). So we get \( 2n - 6 \geq 3n - 11 \). Hence \( n \leq 5 \). Since \( n \geq 5 \), this may only occur if \( n = 5 \). If \( n \neq 5 \), then all distinguished paths must have length one.

We now address the case where \( n = 5 \). From before, we do not have the case where distinguished paths from \( v_0 \) to \( C \) meet \( C \) in consecutive vertices other than \( v_1, v_2, \) and \( v_3 \) and paths from \( v_0 \) to \( C \) must meet \( C \) at vertices that are distance three or more. So we get the following configuration in Figure 4.5.

![Figure 4.5: Subgraph of G when n = 5](image)

By (4.0.1.2), the paths \( p_8 \) and \( p_5 \) from \( v_0 \) to \( v_8 \) and from \( v_0 \) to \( v_5 \) are either even or length one, since they are distance two from the edges from \( v_0 \) to \( v_1 \) and \( v_0 \) to \( v_3 \), respectively. The length of \( p_2 \) is odd and not one.

Suppose at least one of \( p_8 \) and \( p_5 \) has even length. Without loss of generality, we may assume it is \( p_8 \). Let \( v_q \) be the vertex on \( p_8 \) adjacent to \( v_0 \). Label the path from \( v_q \) to \( v_8 \) contained in \( p_8 \) as \( p'_8 \). Let \( v_p \) be any vertex on the interior of path \( p_2 \). Label the path from \( v_p \) to \( v_2 \) contained in \( p_2 \) as \( p'_2 \). Consider the path \( (v_q, v_0, v_3, v_4, \ldots, v_9, v_1, v_2) \) and the path consisting of the union of \( p'_2, p'_8, (v_8, v_7, v_6, \ldots, v_3, v_0, v_1, v_2) \), and \( p'_2 \) of lengths

\[
1 + 1 + 8 + |p'_2| = 10 + |p'_2| \quad \text{and} \quad |p_8| + 5 + 1 + 1 + |p_2'| = 8 + |p'_8| + |p'_2| = 8 + |p_8| - 1 + |p'_2| = 7 + |p_8| + |p'_2|,
\]

which have opposite parities since \( p_8 \) has even length. By (4.0.1.6), there
are no $v_q - v_p$ paths internally disjoint from $C$ and $v_0$ for any $v_p$ on the interior of $p_2$.

By the path \{\(v_q, v_0, v_3, v_4, \ldots, v_2\)\} of length 10 and the path consisting of the union of \(p_8\), and \((v_8, v_7, \ldots, v_3, v_0, v_1, v_2)\) of length \(|p'_8| + 8 = |p_8| - 1 + 8 = 7 + |p_8|\), there is no $v_q - v_2$ path internally disjoint from $C$ and $v_0$. By the path \((v_q, v_0, v_3, v_4, \ldots v_1)\) and the path consisting of the union of \((v_q, v_0)\), $p_2$, and \((v_2, v_3, v_4, \ldots v_1)\) of lengths $1 + 1 + 7 = 9$ and $1 + |p_2| + 8 = 9 + |p_2|$ , there is no $v_q - v_1$ path internally disjoint from $C$ and $v_0$. By symmetry, there is no $v_q - v_3$ path internally disjoint from $C$ and $v_0$. By the path \((v_q, v_0, v_2, \ldots, v_4)\) of length 10 and the path consisting of the union of \((v_q, v_0)\), $p_2$, and \((v_2, v_1, \ldots, v_4)\) of length $1 + |p_2| + 7 = |p_2| + 8$, there is no $v_q - v_4$ path internally disjoint from $C$ and $v_0$. By symmetry, there is no $v_q - v_9$ path internally disjoint from $C$ and $v_0$. By the paths consisting of the union of $p'_8$ and \((v_8, v_9, \ldots v_7)\) and the union of $p'_8$, \((v_8, v_9, v_1, v_0)\), $p_2$, and \((v_2, v_3, \ldots v_7)\) of lengths $(|p_8| - 1) + 8 = |p_8| + 7$ and $3 + |p_2| + 5 = 8 + |p_2|$, there is no $v_q - v_7$ path internally disjoint from $C$ and $v_0$, since $|p_2|$ is even and positive. This leaves us with $v_6$, $v_8$, and $v_5$ as vertices where paths from $v_q$ meet $C$ disjoint from $v_0$. As we need $n - 1 = 4$ such vertices, this contradicts 5-connectivity. Thus neither $p_8$ nor $p_5$ is even.

Figure 4.6: Subgraph of $G$ when $n = 5$

Suppose the paths $p_8$ and $p_5$ both have length one. Let $v_p$ be the vertex on $p_2$ that adjacent to $v_0$. Label the path from $v_p$ to $v_2$ contained in $p_2$ as $p'_2$. By the path consist-
ing of the union of $p'_2$ and $(v_2, v_3, \ldots, v_1)$ and the path $(v_p, v_0, v_8, v_7, \ldots, v_1)$, which have lengths $|p'_2| + 1 + 7 = |p_2| - 1 + 8 = |p_2| + 7$ and $9$, there is no $v_p - v_1$ path internally disjoint from $C$ and $v_0$. By symmetry, there is no $v_p - v_3$ path internally disjoint from $C$ and $v_0$. By the path $(v_p, v_0, v_3, v_2, \ldots, v_4)$ and the path consisting of the union of $p'_2$ and $(v_2, v_1, \ldots, v_4)$, there is no $v_p - v_4$ path internally disjoint from $C$ and $v_0$. By symmetry, there is no $v_p - v_9$ path internally disjoint from $C$ and $v_0$. By the paths $(v_p, v_0, v_8, v_9, \ldots, v_7)$ and $(v_p, v_0, v_5, v_4, \ldots, v_7)$, there is no $v_p - v_7$ path internally disjoint from $C$ and $v_0$. By symmetry, there is no $v_p - v_6$ path internally disjoint from $C$ and $v_0$. Thus $v_p$ does not have 5 disjoint paths to $C$ not through $v_0$ as only $v_2, v_8,$ and $v_5$ remain as possible endpoints of such paths. Thus all distinguished paths in this configuration must have length one. So by (4.0.1.9), (4.0.1.14), and the previous case, this completes the proof of the following

4.0.1.15. No three or more of the distinguished paths from $v_0$ meet $C$ at consecutive vertices unless all distinguished paths have length one

Call a vertex distinguished if it meets a distinguished path on $C$. Call a pair of distinguished vertices that are consecutive on $C$ a consecutive distinguished pair. Next we prove the following.

4.0.1.16. Suppose there is a distinguished path of length greater than one. If there are two distinct consecutive distinguished pairs, then these pairs are disjoint. Moreover, each path in $C$ that contains exactly one vertex in each pair must contain two consecutive undistinguished vertices.

Assume that (4.0.1.16) fails. It is immediate from (4.0.1.15) that the two consecutive pairs are disjoint and that each path containing exactly one vertex from each pair has length at least two. Let $q$ be such a path. Then $q$ contains a subpath $q'$ whose endpoints are distinguished vertices, whose vertices are alternately distinguished and undistinguished, and such that the neighbors in $V(C) - V(q')$ of the endpoints of $q'$ are distinguished vertices.

Suppose the sets are separated by a single vertex.
Figure 4.7: Configuration when the pairs of distinguished vertices are distance two on $C$

Label one consecutive distinguished pair of vertices $v_1$ and $v_2$ with distinguished paths $p_1$ and $p_2$ from $v_0$, and the other consecutive distinguished pair of vertices $v_4$ and $v_5$ with paths $p_4$ and $p_5$ from $v_0$. Let $v_3$ be the vertex on $C$ between the pairs. By (4.0.1.1), $|p_1|$ and $|p_2|$ have the same parity, and $|p_4|$ and $|p_5|$ have the same parity. By (4.0.1.2), $|p_2|$ and $|p_4|$ have opposite parities or both have length one. Thus, $|p_1|$ and $|p_4|$ path lengths have opposite parities or both are odd with the inner paths having length one.

4.0.1.17. The configuration in Figure 4.7 may not occur if distinguished paths have length greater than one.

First we will show the following.

4.0.1.18. The configuration in Figure 4.7 may not occur if distinguished consecutive pairs have even length.

Suppose $|p_2|$ and $|p_4|$ have opposite parities. Thus $|p_1|$ and $|p_4|$ have opposite parities and $|p_2|$ and $|p_5|$ have opposite parities. By the cycle through $p_1$, $p_4$ and $V(C) \setminus \{v_2, v_3\}$ of length $((2n - 1) - 3) + |p_1| + |p_4|$, the sum of the lengths of the paths $p_1$ and $p_4$ is three. We deduce that the even path has length two and the odd path has length one. Similarly, by the cycle through $p_2$, $p_5$ and $V(C) \setminus \{v_3, v_4\}$ of length $((2n - 1) - 3) + |p_2| + |p_5|$, the sum of lengths of the paths $p_2$ and $p_5$ is three. Again, we deduce that the even path has
length two and the odd path has length one. Without loss of generality, let $p_1$ and $p_2$ be the even paths as shown in Figure 4.8.

![Figure 4.8: Subgraph when distinguished consecutive pair have distance two on C](image)

Label the vertex on the interior of the $p_2$ path as $v_p$. Label the vertex on the interior of the $p_1$ path as $v_r$. Let $v_a$ be any distinguished vertex in $V(C) \setminus \{v_1, v_2, v_4, v_5\}$ and $p_a$ be distinguished path from $v_0$ to $v_a$. By (2.0.2) $v_p$ has $n - 2$ distinct paths to $C$ not through $v_0$ or $v_2$. By (4.0.1.6), the paths $(v_p, v_2, v_3, v_4, v_{2n-1}, v_1)$ and $(v_p, v_2, v_3, v_4, v_0, v_5, v_6, \ldots, v_{2n-1}, v_1)$ of lengths $((2n-1)-1) + 1 = 2n - 1$ and $((2n-1)-1) + 1 + 2 = 2n$ imply that no $v_p - v_1$ path exists that is internally disjoint from $V(C) \cup \{v_0\}$. Again, by (4.0.1.6), the paths $(v_p, v_2, v_3, v_4, \ldots, v_{2n-1}, v_1, v_r)$ and $(v_p, v_2, v_3, v_4, v_0, v_5, v_6, \ldots, v_{2n-1}, v_1, v_r)$ of lengths $((2n-1)-1) + 1 + 1 = 2n$ and $((2n-1)-1) + 1 + 2 + 1 = 2n + 1$ imply that no $v_p - v_r$ path exists that is internally disjoint from $V(C) \cup \{v_0\}$. The paths $(v_p, v_2, v_1, v_{2n-1}, \ldots, v_5, v_4, v_3)$ of length $1 + ((2n-1)-1) = 2n - 1$ and $(v_p, v_0, v_4, v_5, \ldots, v_{2n-1}v_1, v_2, v_3)$ of length $2 + ((2n-1)-1) = 2n$ imply that there is no $v_p - v_3$ path internally disjoint from $V(C) \cup \{v_0\}$. The paths $(v_p, v_0, v_5, v_6, \ldots, v_{2n-1}, v_1, v_2, v_3, v_4)$ and $(v_p, v_0, v_r, v_1, v_{2n-1}, \ldots, v_5, v_4)$ of lengths $1 + 1 + ((2n-1)-1) = 2n$ and $1 + 2 + ((2n-1)-3) = 2n - 1$ imply that there is no $v_p - v_4$ path that is internally disjoint from $V(C) \cup \{v_0\}$. By (4.0.1.6) and the paths $(v_p, v_0, v_4, v_3, \ldots, v_6, v_5)$ and $(v_p, v_2, v_3, v_4, v_0, v_r, v_1, v_{2n-1}, \ldots, v_6, v_5)$ of lengths $1 + ((2n-1)-1) = 2n$ and $1 + ((2n-1)-1-1) + 1 + 2 = 2n + 1$, no $v_p - v_5$ path exists that is internally disjoint
from \( V(C) \cup \{v_0\} \). Assume there is a vertex \( v_q \) on the interior of \( p_a \). Let \( p'_a \) be the subpath of \( p_a \) from \( v_q \) to \( v_a \) and let \( p_a - p'_a \) be the subpath of \( p_a \) from \( v_0 \) to \( v_q \). The paths that consist of a union of \( (v_p, v_2, v_1, \ldots, v_4, v_0) \) and \( p_a - p'_a \) and a union of \( (v_p, v_2, v_1, \ldots, v_5, v_0) \) and \( p_a - p'_a \) of lengths \( 1 + ((2n - 1) - 2) + 1 + |p_a - p'_a| = 2n - 1 + |p_a - p'_a| \) and \( 1 + ((2n - 1) - 3) + 1 + |p_a - p'_a| = 2n - 2 + |p_a - p'_a| \) imply that there is no \( v_p - v_q \) path that is internally disjoint from \( V(C) \cup \{v_0\} \) for any \( v_q \) on the interior of any \( p_a \).

By (4.0.1.3) and the vertices \( v_0, v_4 \) and \( v_5 \), any path from \( v_p \) to \( v_a \) implies that there is no path from \( v_p \) to \( v_{a-1} \) or \( v_{a+1} \) that is disjoint from \( V(C) \cup v_0 \). Since we have \( n - 2 \) paths from \( v_0 \) to \( V(C) \setminus \{v_1, v_2, v_3, v_4, v_5\} \), a set of size \( 2n - 1 - 5 = 2n - 6 \), without consecutive vertices, this is a contradiction and we may not have the configuration in Figure (4.8), that is (4.0.1.18) holds.

**4.0.1.19.** Suppose there is a distinguished path with length greater than one. The configuration in Figure 4.7 may not occur if distinguished consecutive pairs have odd length.

Suppose \( |p_2| \) and \( |p_4| \) are both one, that is have the same parity from above. By (4.0.1.1), \( |p_1| \) and \( |p_5| \) are both odd. The cycle that is a union of \( p_1 \), \( p_5 \) and \( V(C) \setminus \{v_2, v_3, v_4\} \) of length \( ((2n - 1) - 4) + |p_1| + |p_5| = 2n - 5 + |p_1| + |p_5| \) implies that one of \( p_1 \) and \( p_5 \) has length one and the other has length one or three. Without loss of generality, let \( |p_1| \in \{1, 3\} \).

First, assume \( |p_1| = 1 \). Then there is a distinguished path \( p_a \) to vertex \( v_a \) on \( C \) with length greater than one. Let \( v_q \) be the any vertex on on the interior of the \( p_a \) path. Let \( p'_a \) be the subpath of \( p_a \) from \( v_q \) to \( v_0 \).

By (4.0.1.6) and the cycles consisting of the union of \( p'_a \) and \( (v_0, v_2, v_1, \ldots, v_2, v_3) \) and the union of \( p'_a \) and \( (v_0, v_5, v_6 \ldots, v_2, v_3) \) of lengths \( |p'_a| + 1 + ((2n - 1) - 1) = 2n - 1 + |p'_a| \) and \( |p'_a| + 1 + ((2n - 1) - 2) = 2n - 2 + |p'_a| \), no \( v_q - v_3 \) path internally disjoint from \( V(C) \cup v_0 \) can exist. By (4.0.1.6) and the cycles consisting of the union of \( p'_a \) and \( (v_0, v_4, v_5, \ldots, v_1, v_2) \) and the union of \( p'_a \) and \( (v_0, v_1, v_{2n-1} \ldots, v_3, v_2) \) of lengths \( |p'_a| + 1 + ((2n - 1) - 2) = 2n - 2 + |p'_a| \) and \( |p'_a| + 1 + ((2n - 1) - 1) = 2n - 1 + |p'_a| \), no \( v_q - v_2 \) path disjoint from \( V(C) \cup v_0 \)
can exist. By symmetry, no \( v_q - v_4 \) path internally disjoint from \( V(C) \cup v_0 \) can exist. By (4.0.1.6) and the cycles consisting of the union of \( p'_a \) and \((v_0, v_4, v_3, \ldots, v_6)\) and the union of \( p'_a \) and \((v_0, v_5, v_4 \ldots, v_6)\) of lengths \(|p'_a| + 1 + ((2n - 1) - 2) = 2n - 2 + |p'_a|\) and \(|p'_a| + 1 + ((2n - 1) - 1) = 2n - 1 + |p'_a|\), no \( v_q - v_6 \) path internally disjoint from \( V(C) \cup v_0 \) can exist. By symmetry, there is no \( v_q - v_{2n-1} \) path internally disjoint from \( V(C) \cup v_0 \).

If we include two possible paths from \( v_q \) to \( v_1 \) and \( v_5 \), there are \( n - 3 \) remaining paths from \( v_q \) to \( V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\} \). By the triangle \( \{v_0, v_4, v_5\} \) and (4.0.1.3), \( v_q \) cannot have distinct paths disjoint from \( v_0 \) meeting \( V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\} \) at consecutive vertices. Thus, we have \( n - 3 \) non-consecutive paths in \( 2n - 1 - 7 = 2n - 8 \) vertices, and \( v_q \) is not \( n \)-connected. We deduce that \(|p_1| \neq 1\).

Now assume that \(|p_1| = 3\) as in Figure 4.10. Let \( v_p \) be the vertex on the interior of \( p_1 \) adjacent to \( v_0 \). Let \( p'_2 \) be the subpath of \( p_2 \) from \( v_p \) to \( v_1 \). Let \( v_a \) be any distinguished vertex not in \( \{v_1, v_2, v_4, v_5\} \) and \( p_a \) be the distinguished path from \( v_0 \) to \( v_a \). Let \( v_q \) be any vertex on the interior of \( p_a \). Let \( p'_a \) be the subpath of \( p_a \) from \( v_q \) to \( v_a \). Let \( v_r \) be the additional interior vertex on \( p_1 \).

By Theorem 2.0.2, the graph \( G \setminus \{v_0, v_1\} \) has \( n - 2 \) internally disjoint paths from \( v_p \) to \( C \). By (4.0.1.6) and the paths consisting of the union of \( p'_1 \) and \((v_1, v_{2n-1}, \ldots, v_3, v_2)\) and the union of \( p'_1 \) and \((v_1, v_{2n-1}, \ldots, v_5, v_0, v_4, v_3, v_2)\) of lengths \(|p'_1| + 1 + ((2n - 1) - 1) =

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Figure 4.10: Subgraph when the distinguished consecutive pairs are distance two on $C$

$2n - 1 + |p'_1| = 2n - 2 + 2 = 2n$ and $|p'_1| + ((2n - 1) - 1 - 1) + 1 + 1 = 2n - 1 + |p'_1| = 2n + 1$, no $v_q - v_2$ path internally disjoint from $V(C) \cup v_0$ can exist. By (4.0.1.6) and the path consisting of the union of $p'_1$ and $(v_1, v_{2n-1}, \ldots, v_4, v_3)$ and the path $(v_p, v_0, v_2, v_1, \ldots, v_4, v_3)$ of lengths $|p'_1| + ((2n - 1) - 2) = 2n - 1 + |p'_1| = 2n - 1$ and $1 + ((2n - 1) - 1) = 2n$, no $v_q - v_3$ path internally disjoint from $V(C) \cup v_0$ can exist. The paths $(v_p, v_0, v_2, v_1, \ldots, v_5, v_4)$ and $(v_p, v_5, v_6, \ldots, v_3, v_4)$ of lengths $1 + 1 + ((2n - 1) - 2) = 2n - 1$ and $1 + 1 + ((2n - 1) - 1) = 2n$ imply that there is no $v_p - v_4$ path internally disjoint from $V(C) \cup v_0$. The paths $(v_p, v_0, v_4, v_3, \ldots, v_6)$ and $(v_p, v_0, v_5, v_4, \ldots, v_6)$ of lengths $1 + 1 + ((2n - 1) - 2) = 2n - 1$ and $1 + 1 + ((2n - 1) - 1) = 2n$ imply that there is no $v_p - v_6$ path internally disjoint from $V(C) \cup v_0$. By the paths consisting of the union of $p'_1$, $(v_1, v_{2n-1}, \ldots, v_3, v_2, v_0)$ and $p'_a$ and the union of $p'_1$, $(v_1, v_{2n-1}, \ldots, v_5, v_0)$ and $p'_a$, of lengths $2 + ((2n - 1) - 1) + 1 + |p'_a| = 2n + 1 + |p'_a|$ and $2 + ((2n - 1) - 4) + 1 + |p'_a| = 2n - 2 + |p'_a|$, no $v_p - v_q$ path internally disjoint from $V(C) \cup v_0$ can exist for any $v_q$ on the interior of some $p_a$. The path consisting of the union of $p'_1$ and $(v_1, v_2, \ldots, v_{2n-1})$ and the union $p'_1$ and $(v_1, v_2, v_3, v_4, v_0, v_5, v_6, \ldots, v_{2n-1})$ of lengths $|p'_1| + ((2n - 1) - 1) = 2 + 2n - 2 = 2n$ and $|p'_1| + ((2n - 1) - 1 - 1) + 1 + 1 = 2 + 2n - 1 = 2n + 1$ imply that there is no $v_p - v_{2n-1}$ path internally disjoint from $V(C) \cup v_0$. Allowing for a possible path to $v_5$, there are $n - 3$ remaining paths from $v_p$ to $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\}$. By the triangle $\{v_0, v_4, v_5\}$ and (4.0.1.3), $v_p$ cannot have

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paths disjoint from \( v_0 \) meeting \( V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\} \) at consecutive vertices, otherwise we have a larger odd cycle. Thus, we have \( n - 3 \) non-adjacent paths meeting \( V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\} \) in \( 2n - 1 - 7 = 2n - 8 \) vertices. We deduce \( v_p \) is not \( n \) connected, and (4.0.1.19) holds. By (4.0.1.19) holds and (4.0.1.18), we deduce that if we have longer path length of a distinguished path, then the two consecutive pairs are not distance two on \( C \), that is (4.0.1.17) holds.

Suppose the distance on \( C \) between two consecutive distinguished pairs is greater than two. Label the first adjacent pair of vertices \( v_1 \) and \( v_2 \) with paths \( p_1 \) and \( p_2 \) from \( v_0 \), and the second adjacent pair of vertices \( v_a \) and \( v_{a+1} \) with paths \( p_a \) and \( p_{a+1} \). As noted before, between \( v_2 \) and \( v_a \) on \( C \) is an alternating sequence of distinguished and undistinguished vertices as shown in Figure 4.11. Let \( p_k \) be a distinguished path not in a distinguished consecutive pair from \( v_0 \) to \( v_k \) between \( v_2 \) and \( v_a \).

First we show the following result.

**4.0.1.20. The length of \( p_2 \) is not greater than one.**

Suppose \( p_2 \) has length greater than one. Let \( v_p \) be the vertex on \( p_2 \) adjacent to \( v_0 \). Let \( p'_2 \) as the subpath from \( v_2 \) to \( v_p \).

![Figure 4.11: Subgraph when the sets of paths are distance two](image)

By (4.0.1.1), \( p_1 \) and \( p_2 \) have the same parity and \( p_a \) and \( p_{a+1} \) have the same parity. This is shown in Figure 4.11 with varying path textures. By Theorem 2.0.2, the graph \( G \)
has \( n - 1 \) distinct internally disjoint paths from \( v_p \) to \( C \) not through \( v_0 \). Observe that \(|p_a| + |p_{a+1}|\) is even. Let \( v_q \) be a vertex on the interior of \( p_1 \), and \( p_1' \) be the subpath from \( v_q \) to \( v_1 \) on \( p_1 \). By (4.0.1.6), and the paths consisting of the union of \( p_2' \), \( (v_2, v_3, \ldots, v_{2n-1}, v_1) \), and \( p_1' \) and the union of \( p_2' \), \( (v_2, v_3, \ldots, v_{a-1}, v_a) \), \( p_a, p_{a+1}, (v_{a+1}, v_{a+2}, \ldots, v_{2n-1}, v_1) \) and \( p_1' \) of lengths \( 2n - 2 + |p_1'| + |p_2'| \) and \( 2n - 3 + |p_1'| + |p_2'| + |p_a| + |p_{a+1}| \), no \( v_p - v_q \) path can exist that internally is disjoint from \( V(C) \cup v_0 \). By (4.0.1.6), and the paths consisting of the union of \( p_2' \), and \( (v_2, v_3, \ldots, v_{2n-1}, v_1) \) and the union of \( p_2' \), \( (v_2, v_3, \ldots, v_{a-1}, v_a) \), \( p_a, p_{a+1}, (v_{a+1}, v_{a+2}, \ldots, v_{2n-1}, v_1) \) of lengths \( 2n - 2 + |p_2'| \) and \( 2n - 3 + |p_2'| + |p_a| + |p_{a+1}| \), no \( v_p - v_1 \) path can exist that is internally disjoint from \( V(C) \cup v_0 \). Since \( p_2 \) has a length greater than one, the length of \( p_4 \) has the opposite parity by (4.0.1.2). The paths consisting of the union of \( p_2' \) and \( (v_2, v_1, \ldots, v_4, v_3) \) and the union of \( (v_p, v_0), (v_4, v_5, \ldots, v_3) \) imply there is no \( v_p - v_3 \) path that is internally disjoint from \( V(C) \cup v_0 \). Label an interior vertex of \( p_k \) as \( v_r \), if it exists. Let \( p_k' \) as the subpath from \( v_r \) to \( v_p \) and let \( p_k - p_k' \) be the subpath of \( p_k \) from \( v_r \) to \( v_0 \). When \( p_k \) has length greater than one, \(|p_{k-2}|\) has the opposite parity by (4.0.1.2). Thus \(|p_k'| + |p_{k-2}|\) and \(|p_k| - |p_k'|\) have opposite parities, since \(|p_k'| + |p_{k-2}| + |p_k| - |p_k'| = |p_{k-2}| + |p_k|\). By the paths consisting of the union of \( p_2' \), \( (v_2, v_3, \ldots, v_{2n-1}, v_1) \), \( p_1 \), and \( p_k - p_k' \) and the union of \( p_2' \), \( (v_2, v_3, \ldots, v_{k-2}) \), \( p_{k-2}, p_1 \) \( (v_1, v_{2n-1}, \ldots, v_k) \) and \( p_k' \) of lengths \(|p_2'| + ((2n - 1) - 1) + |p_1| + |p_k - p_k'| = 2n - 2 + |p_2'| + |p_1| + |p_k - p_k'|\) and \(|p_2'| + ((2n - 1) - 2 - 1) + |p_{k-2}| + |p_1| + |p_k'| = 2n - 4 + |p_2'| + |p_1| + |p_{k-2}| + |p_k'|\), no \( v_p - v_r \) path can exist that is internally disjoint from \( V(C) \cup v_0 \) for any interior vertex \( v_r \) on some \( p_k \) with \( k \neq 4 \). If \( k = 4 \), the paths that consist of the union of \( p_2' \), \( (v_2, v_1, \ldots, v_{a+1}) \), \( p_{a+1}, p_a, (v_a, v_{a-1}, \ldots, v_4) \), and \( p_4' \) and the union of \( p_2' \), \( (v_2, v_1, \ldots, v_4) \), and \( p_4' \) suffice to show no \( v_p - v_r \) path can exist that is internally disjoint from \( V(C) \cup v_0 \) for any interior vertex \( v_r \) on \( k = 4 \). Label an interior vertex of \( p_a \) as \( v_s \), if it exists. Let the subpath of \( p_a \) from \( v_s \) to \( v_a \) be \( p_a' \). Since \( p_a \) has a length greater than one, \( p_{a-2} \) has the opposite parity by (4.0.1.2). By (4.0.1.1), \(|p_{a+1}|\) has the same parity as \(|p_a|\). Note that since \(|p_1| + |p_2|\) is even, \(|p_1| + |p_2'|\) is odd. By the paths consisting of the union of \( p_2' \), \( (v_2, v_3, \ldots, v_{a-2}) \), \( p_{a-2}, p_1 \),
and \((v_1, v_{2n-1}, \ldots, v_a)\) of length \(|p'_2| + 2n - 4 + |p_{a-2}| + |p_1| + |p'_a|\) and the union \((v_p, v_0), p_{a-2}, (v_{a-2}, v_{a-3}, \ldots, v_a)\) and \(p'_a\) of length \(1 + 2n - 1 - 2 + |p_{a-2}| + |p'_a|\), no \(v_p - v_s\) path can exist that is internally disjoint from \(V(C) \cup v_0\), if and interior \(v_s\) exists. Let \(p_{a+1}\) have length greater than one. Label an interior vertex of \(p_{a+1}\) as \(v_t\). Let the subpath of \(p_{a+1}\) from \(v_t\) to \(v_a\) be \(p'_{a+1}\). Note that since \(|p_a| + |p'_{a+1}|\) is even, \(|p_a| + |p'_{a+1}|\) and \(|p_{a+1} - p'_{a+1}|\) have the same parity. By the paths consisting of the union of \(p'_2, (v_2, v_3, \ldots, v_1), p_1\), and \(p_{a+1} - p'_{a+1}\) and the union of \(p'_2, (v_2, v_1, \ldots, v_a), p_a, p_1, (v_1, v_{2n-1}, \ldots, v_{a+1})\) and \(p'_{a+1}\) with lengths \(|p'_2| + (2n - 1 - 1) + |p_1| + |p_{a+1} - p'_{a+1}|\) and \(|p'_2| + (2n - 1 - 2) + |p_a| + |p_1| + |p'_{a+1}|\), no \(v_p - v_1\) path can exist that is internally disjoint from \(V(C) \cup v_0\). Let \(p_b\) be an additional distinguished path to \(v_b\); that is, \(b > a + 1\) in our labeling. Label an interior vertex of \(p_b\) as \(v_u\). Let the subpath of \(p_b\) from \(v_u\) to \(v_b\) be \(p'_b\) and the subpath of \(p_b\) from \(v_u\) to \(v_0\) be \(p_b - p'_b\). By the paths consisting of the union of \(p'_2, (v_2, v_3, \ldots, v_1), p_1\) and \(p_b - p'_b\) and the union of \(p'_2, (v_2, v_3, \ldots, v_a), p_a, p_{a+1}, (v_{a+1}, v_{a+2}, \ldots, v_1), p_1\), and \(p_b - p'_b\), no \(v_p - v_u\) path can exist that is internally disjoint from \(V(C) \cup v_0\) for any interior vertex \(v_u\) of \(p_b\). By the paths consisting of the union of \((v_p, v_0), p_{a-2}\), and \((v_{a-2}, v_{a-3}, \ldots, v_a)\) and \(p'_2, (v_2, v_3, \ldots, v_{a+2}), p_{a-2}, p_1,\) and \((v_1, v_{2n-1}, \ldots, v_a)\), no \(v_p - v_a\) path can exist that is internally disjoint from \(V(C) \cup v_0\). By the paths \((v_p, v_0), p_a,\) and \((v_a, v_{a-1}, \ldots, v_{a+2})\) and \((v_p, v_0), p_{a+1},\) and \((v_{a+1}, v_a, \ldots, v_{a+2})\), no \(v_p - v_{a+2}\) path can exist that is internally disjoint from \(V(C) \cup v_0\). By symmetry, no \(v_p - v_{a-1}\) path can exist that is internally disjoint from \(V(C) \cup v_0\). By the paths consisting of the union of \(p'_2\) and \((v_2, v_3, \ldots, v_1)\) and the union of \(p'_2, (v_2, v_3, \ldots, v_a), p_a, p_{a+1},\) and \((v_{a+1}, v_{a+2}, \ldots, v_1)\), no \(v_p - v_1\) path can exist that is disjoint from \(V(C) \cup v_0\). By the paths consisting of the union of \(p'_2\) and \((v_2, v_1, \ldots, v_3)\) and \(p'_2, (v_2, v_1, \ldots, v_{a+1}), p_{a+1}, p_a,\) and \((v_a, v_{a-1}, \ldots, v_3)\), no \(v_p - v_3\) path can exist that is internally disjoint from \(V(C) \cup v_0\). By the paths consisting of the union of \((v_p, v_0), p_1,\) and \((v_1, v_2, \ldots, v_{2n-1})\), no \(v_p - v_{2n-1}\) path can exist that is internally disjoint from \(V(C) \cup v_0\).
Allowing for possible paths to \( v_{a+1} \) and \( v_2 \), there are \( n - 3 \) remaining paths from \( v_p \) to 
\[ V(C) \setminus \{v_{a+2}, v_{a+1}, v_a, v_{a-1}, v_{2n-1}, v_1, v_2, v_3\} \]. By the triangle \( \{v_0, v_a, v_{a+1}\} \) and (4.0.1.3), \( v_p \)
cannot have paths meeting \( C \) at consecutive vertices in 
\[ V(C) \setminus \{v_{a+2}, v_{a+1}, v_a, v_{a-1}, v_{2n-1}, v_1, v_2, v_3\} \].
Thus we have \( n - 3 \) paths meeting non-consecutive vertices in 
\( 2n - 1 - 8 = 2n - 9 \) vertices which is divided in two paths of \( C \). Thus \( p_2 \) does not have length greater than one and 
(4.0.1.20) holds.

The cases where some \( p_k \) path has length greater than one and only \( p_1 \) has length greater
than one are included in the appendix, which completes the proof of (4.0.1.16).

Suppose there is only one consecutive distinguished pair of vertices. Let \( v_1 \) and \( v_2 \) are the adjacent pair meeting distinguished paths \( p_1 \) and \( p_2 \). Observe that the vertices \( v_3, v_4, \ldots v_{2n-1} \) alternate between undistinguished and distinguished vertices the the first
and last being undistinguished as in Figure 4.12.

![Figure 4.12: Subgraph configuration](image)

Suppose some distinguished path \( p_a \) with \( a \notin \{1, 2\} \) has length greater than one as in 
Figure 4.13. Let \( v_a \) be the vertex on \( C \) meeting \( p_a \) and let \( v_p \) be the vertex adjacent to \( v_0 \)
on \( p_a \). Let \( p_p' \) be the subpath from \( v_p \) to \( v_a \).

By Theorem 2.0.2, the graph \( G \) has \( n - 1 \) distinct internally disjoint paths from \( v_p \) to \( C \)
not through \( v_0 \). By (4.0.1.1), \( |p_1| \) and \( |p_2| \) have the same parities, and by (4.0.1.2) \( p_a \) and 
\( p_a \pm 2 \) have opposite parities, since \( p_a \) has length greater than one. Let \( p_2 \) have length greater

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than one. Let $v_t$ be a point on the interior of $p_2$. Let $p'_2$ be the subpath from $v_t$ to $v_2$. By (4.0.1.6) and the paths consisting of the union of $(v_p, v_0), (v_1, v_0, v_{2n-1}, \ldots, v_2)$, and $p'_2$ and the union of $p'_1 (v_a, v_{a-1}, \ldots, v_{a+2}), p_{a+2}$ and $(p_2 - p'_2)$ of lengths $1 + |p_1| + 2n - 1 - 1 + |p'_2|$ and $|p'_a| + 2n - 1 - 2 + |p_{a+2}| + |p_2 - p'_2|$, no $v_p - v_t$ path disjoint from $V(C) \cup v_0$ can exist. By symmetry, no path from $v_p$ to a vertex on the interior of $p_1$ that is disjoint from $V(C) \cup v_0$ can exist. Let the distinguished path with the smallest distance on $C$ from $p_a$ also has length greater than one. Let such a distinguished path be $p_{a+2}$. Let $v_a$ be a point on the interior of $p_{a+2}$. Let $p'_{a+2}$ be the subpath from $v_t$ to $v_{a+2}$. By (4.0.1.6) and the paths consisting of the union of $p'_1, (v_a, v_{a-1}, \ldots, v_{a+2}), p_{a+2}$ and $p'_{a+2}$ of lengths $|p'_a| + 2n - 1 - 2 + |p'_{a+2}|$ and $|p'_a| + 2n - 1 - 2 - 1 + |p_1| + |p_2| + |p_{a-2}|$, no $v_p - v_s$ path disjoint from $V(C) \cup v_0$ can exist. By symmetry, no path from $v_p$ to a vertex on the interior of $p_{a+2}$ that is disjoint from $V(C) \cup v_0$ can exist. Let a distinguished path $p_b$ that does not meet $C$ at $a$ or $a \pm 2$ have length greater than one. Let $v_b$ be the vertex on $C$ that meets the path. Let $v_q$ be an internal point on $p_b$. Let $p_b$ be the subpath from $v_q$ to $v_b$. By (4.0.1.2) $p_b$ and $p_b \pm 2$ have opposite parities, since $p_b$ has length greater than one. By the paths consisting of the union of $p'_1, (v_a, v_{a-1}, \ldots, v_{a+2}), p_{a-2}, p_0 - p'_b$ and the union of $p'_a, (v_a, v_{a+1}, \ldots, v_{b-2}), p_{b-2}, p_{a-2}, (v_{a-2}, v_{a-3}, \ldots, v_b)$ and $p'_b, p'_b$ of lengths $|p'_a| + 2n - 1 - 2 + |p_{a-2}| + |p_b - p'_b|$ and

Figure 4.13: Subgraph configuration
\(|p'_a| + 2n - 1 - 2 - 2 + |p_{b-2}| + |p_{a-2}| + |p'_{b}| \) where \(|p_{b-2}| + |p'_{b}|\) has the opposite parity as \(|p_b - p'_b|\), no \(v_p - v_q\) path that is disjoint from \(V(C) \cup v_0\) can exist. The paths consisting of the union of \((v_p, v_0), p_1, (v_1, v_{2n-1}, \ldots, v_2)\) and the union of \(p'_a, (v_a, v_{a+1}, \ldots, v_1), p_1, p_{a-2}, (v_{a-2}, v_{a-3}, \ldots, v_2)\) imply there is no \(v_p - v_2\) path that is disjoint from \(V(C) \cup v_0\) unless \(a = 4\).

If \(a = 4\), the paths consisting of the union of \(p'_4, (v_4, v_5, \ldots, v_2)\) and the union of \((v_p, v_0), p_1, (v_1, v_{2n-1}, \ldots, v_2)\) suffice. By symmetry, there is no \(v_p - v_1\) path that is disjoint from \(V(C) \cup v_0\). By the paths that consist of a union of \((v_p, v_0), p_1\) and \((v_1, v_{2n-1}, \ldots, v_3)\) and the union of \((v_p, v_0), p_2, (v_2, v_1, \ldots, v_3)\), no \(v_p - v_3\) path may exist that is disjoint from \(V(C) \cup v_0\). By symmetry, no \(v_p - v_{2n-1}\) path may exist that is disjoint from \(V(C) \cup v_0\).

Thus \(v_p\) has \(n - 1\) remaining paths to \(V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3\}\). By (4.0.1.3) and the paths \(p_1\) and \(p_2\) with edge \(v_1 - v_2\), \(v_p\) paths may not meet adjacent vertices on the remainder of \(C\). Thus we have \(n - 1\) non-adjacent vertices into \(2n - 1 - 4 = 2n - 5\) remaining vertices of \(C\). Thus one of our single spoke paths cannot have length greater than one and we find the following

4.0.1.21. All distinguished paths other than the paths that meet \(C\) at adjacent vertices have length one.

Suppose one of the consecutive paths has length greater than one. Let \(p_2\) have length greater than one as in Figure 4.14. Since \(p_4\) has length one, \(p_2\) has an even length by (4.0.1.2).

Let \(v_p\) be the point on \(p_2\) that is adjacent to \(v_0\). Let \(p'_2\) be the subpath of \(p_2\) from \(v_p\) to \(v_2\).

Since \(p_2\) is even, \(p_1\) is also even by (4.0.1.1). Let \(v_q\) be a vertex on the interior of \(p_1\). Let \(p'_1\) be the subpath of \(p_1\) from \(v_p\) to \(v_1\). The path consisting of the union of \(p'_1, (v_2, v_3, \ldots, v_1)\), and \(p'_1\) and the union of \((v_p, v_0, v_4, v_5, \ldots, v_1)\) and \(p'_1\) imply there is no \(v_p - v_q\) path that is internally disjoint from \(V(C) \cup v_0\). By the paths consisting of the union of \(p'_2\), and \((v_2, v_1, \ldots, v_3)\) and \((v_p, v_0), p_1, (v_1, v_{2n-1}, \ldots, v_3)\), no \(v_p - v_3\) path that is disjoint from
Figure 4.14: Subgraph configuration

$V(C) \cup v_0$ can exist. By the paths $(v_p, v_0, v_{2n-2}, v_{2n-3} \ldots v_{2n-1}$ and the union of $(v_p, v_0)$, $p_1, (v_1, v_2, \ldots v_{2n-1})$, there is no $v_p - v_{2n-1}$ path that is internally disjoint from $V(C) \cup v_0$.

Let $a$ be an even number that is not four. By the paths $(v_p, v_0, v_{a-2}, v_{a-3}, \ldots, v_a)$ and the union of $p_2', (v_2, v_3, \ldots, v_{a-2}, v_0)$, $p_1, (v_1, v_{2n-1}, \ldots, v_a)$, there is no $v_p - v_a$ path that is internally disjoint from $V(C) \cup v_0$. Let $a$ be an odd number that is not three. By the paths, $(v_p, v_0, v_{a-1}, v_{a-2}, \ldots, v_a)$ and the union of $p_2', (v_2, v_3, \ldots, v_{a-1}, v_0)$, $p_1, (v_1, v_{2n-1}, \ldots, v_a)$, there is no $v_p - v_a$ path that is internally disjoint from $V(C) \cup v_0$. Thus there are only four possible vertices for $v_p$ paths to meet, and $v_p$ contradicts $n$-connectivity. Thus $p_2$ has path length one and by symmetry $p_1$ has path length one, and we find (4.0.1.7) holds.

Since $|V(C)| > 2n$, we have more than one vertex not in $V(C)$. Label two of these vertices $v_0$ and $v'_0$. Each meets $C$ with $n$ paths of length one. Examine all of the consecutive vertices where $v_0$ paths meet $C$ and the consecutive vertices where $v'_0$ meets $C$. Suppose there is more than one adjacent pair of vertices where both $v_0$ and $v'_0$ meet $C$. There is at least one pair for each $v_0$ and $v'_0$. Label the pair for $v_0$ as $v_1$ and $v_2$ and label the pair for $v'_0$ as $v_a$ and $v_{a+1}$. By the cycle $(v_1, v_0, v_2, \ldots, v_a, v'_0, v_{a+1}, v_{a+2}, \ldots, v_{2n-1})$, there is a cycle of odd length greater than $2n - 1$. Thus there is only one consecutive distinguished pair.

This implies we must have configurations of the type if Figure 4.15 with all paths meeting at the same vertices. We may add as many vertices as we like connecting to the same set.
Now we must check for additional possible edges within $G$ that do not create a larger odd cycle. Let $v_a$ and $v_b$ be any two odd vertices other than one. From above we know that $v_a$ does not connect to any $v_0$. Suppose we have an edge from $v_a$ to $v_b$. Suppose $a > b$. By the $2n+1$-cycle $(v_a, v_{a+1}, \ldots, v_1, v_0, v_2, v_3, \ldots, v_{b-1}, v'_0, v_{a-1}, v_{a-2} \ldots, v_b)$, the odd vertices except $v_1$ are not adjacent. Since they must have degree $n$ this implies they are connected to the even vertices and $v_1$. These edges create no odd cycles. This fulfills the degree requirement of each vertex. By arranging the graph in a bipartite fashion, with one side of the partition $\{v_1, v_2, v_4, \ldots, v_{2n-2}\}$, we see that any edge between the this side of the partition will create no new additional odd cycles as we may only use the a vertex of the other partition exactly once. Thus we may have as many edges as we like between these vertices, and we reach our desired configuration.

$\square$
References


Vita

Kristen Wetzler was born in Arkansas to a long line of educators. She finished her undergraduate degree in mathematics and German at the University of Arkansas, Fayetteville, in 2009. In 2010, Kristen moved to Louisiana to continue her education in mathematics. She has earned her Master of Science degree in mathematics from Louisiana State University and is a degree candidate for Doctor of Philosophy in mathematics, to be awarded in May 2018.