2013

Incorporating calculus concepts into a middle school classroom

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INCORPORATING CALCULUS CONCEPTS INTO A MIDDLE SCHOOL CLASSROOM

A Thesis

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Masters of Natural Sciences

in

The Interdepartmental Program in Natural Sciences

by
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B.A., Southeastern Louisiana University, 2001
August 2013

Acknowledgements

I would like to first of all thank my husband for having so much patience with me over the course of this three year process. Your understanding and encouragement has been my biggest push to complete this process. Thank you for never letting me give up!

Thanks to my mom for supporting me in this process. You always made sure I was doing everything I needed to do, as always!

My family has always encouraged me to be the best me that I can be and always making sure I reach my full potential. I know my dad thanks all of you for that. I will be eternally grateful.

To Dr. Neubrander, you NEVER let me stop working, even when I wanted to so badly! You have encouraged me to become a better teacher (and to not lie to my students-what is a line, anyway?).

To Dr. P. Sundar and Dr. Ameziane Harhad, thank you for being on my committee and for all of your help in Algebra and Calculus. I could not have done it without you.

To Leslie Blanchard, thanks for all of the work behind the scenes – we do know how much you do! To Kimberly D'Souza, I'm still not sure if I would have ever made it through that first summer without you! To Austin Scirratt, thanks for all of your help through Algebra and Calculus. To Robyn Carlin, thanks for being our inclusion teacher in geometry.

Thanks to my awesome Ory family, Teri, Amy, and Rachel. You guys have helped me more than you know. Your encouragement helped me see the light at the end of the tunnel.

Finally, thanks to all of my fellow cohorts. Suk and Kathleen, you have always made the ride up I-10 just a bit more bearable and we learned how to make it into a great study session. Danielle and Lauren, you have been my support system, my shoulder to cry on and my rock. We will always have LaMSTI-extra, extra!

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Abstract

The Common Core State Standards for Mathematics were released in June of 2010 and full implementation into our schools is being required in the fall of 2013. The standards require teachers and students to be more focused and rigorous in what they teach and learn. This shift in learning towards focus, coherence, fluency, deep understanding, and applications was to improve math education in the United States. This is so that our students become more competitive in the global economy of the 21st century and to help more students to be college and career ready by grade twelve. It is now necessary for teachers and students to be more prepared for both the Standards for Mathematical Content and the Standards for Mathematical Practice. With teachers giving students more opportunities to experience the usefulness of the mathematical language as outlined in the Standards for Mathematical Practice, students will be entering post-secondary education with a stronger mathematical background. Introducing students to more rigorous, more challenging, and more interesting mathematics in earlier grades will allow them to be more prepared to take on the challenges of a college-preparatory fourth math class in high school, whether it be honors, dual enrollment or advanced placement. This thesis shows how integrating basic calculus concepts into a middle school classroom through linear functions will allow students to be more aware of what a higher math class involves without having to be afraid of the unknown. Implementation of the Common Core Standards is requiring a big chunk of Advanced Mathematics, Geometry, and Algebra II to be moved down into lower grades, and eighth grade math is no exception. The thesis intends to show that eighth grade students are able to understand and apply elementary calculus concepts as long as they are taught with grade-appropriate language using what they already know. In fact, this thesis claims that students in middle school, especially in eighth grade, do have the mental capability and mathematical

maturity to do higher level mathematics and comprehend more advanced mathematical concepts and ideas if presented in the language of linear functions and basic triangle geometry.

Chapter 1: Introduction

Most students (and some adults) believe that math is difficult and hard to understand. Students who believe this usually have had negative mathematical experiences with either teachers they did not get along well with or topics that they did not truly understand. If students are introduced to more challenging mathematics earlier in school by mathematically better trained teachers, they may not be as apprehensive about taking higher level math classes, either later in high school or when in college. If students can see that the mathematics can be understood and is doable with less frustration, they may be more inclined to be led towards a career in science, technology, engineering, or math.

This thesis takes the concept of linear functions that will now be taught in eighth grade mathematics and injects some basic calculus concepts into it. Students are able to see that some calculus topics are easily done using the algebra and geometry they already know. Students will explore properties and representations of linear functions and their inverses, the concepts of slopes and rates of change, areas, arc-lengths, optimization, and arithmetic sequences and series. In my school district, these are topics that were traditionally not touched upon until at least a fourth high school math course, but with a little manipulation and stretching of the Common Core eighth grade linear functions standards, they can be taught in a grade-appropriate way that students can understand.

By the 2013-2014 school-year, all Louisiana public high schools must offer at least one Advanced Placement course in three out of four core subject areas (math, ELA, science and social studies) and by 2014-2015, schools must offer AP courses in all four core content areas (www.louisianabelieves.com). For students to be ready for these Advanced Placement courses,

students must have the rigor from the start of their learning, especially when most of the fundamentals of algebra and geometry are taught – eighth grade. While doing this, teachers can introduce, ever so carefully, some key calculus concepts into their eighth grade curriculum.

This thesis is about incorporating calculus concepts into middle school. It started by taking a table of contents straight from a calculus textbook and proceeded by picking apart the eighth grade mathematics Common Core State Standards to search for the earliest references to functions and, in particular, linear functions. As it turned out, the Common Core State Standards for Mathematics are written in such a way that it is quite easy and quite natural to infuse and use basic calculus concepts in the eighth grade curriculum. With a bit of creativity, any middle school teacher can use these calculus concepts when teaching linear functions throughout the eighth grade curriculum while giving students a grade- and age-appropriate preview of what they will be learning in calculus without scaring the students into a higher-level mathematics black hole.

According to www.commoncore.org, eighth graders are asked to complete more difficult math tasks that will eventually lead to what students will learn in higher math classes. In the Engage New York curriculum, as found on www.engageny.org in the summary of the eighth grade year, the students are expected to do the following during the eighth grade year:

> "Eighth grade mathematics is about (1) formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, and solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem."

According to the same curriculum in the "Grade 8 Sequence of Modules Aligned with the Standards", while in eighth grade, the key area of focus is linear algebra (i.e., studying linear

functions from the real number system into the real number system). Now let us expand on that idea, if students are expected to be able to "grasp the concept of a function and use functions to describe quantitative relationships," should students not be expected to know how a function can and cannot be represented? This is one part of the curriculum that can be expanded upon by expecting students to be able to know that common temperature conversion formulas are in fact linear functions, that they know that the graph and qualitative properties of linear functions depend heavily on the given domains and range spaces, be able to switch freely and comfortably between different representations of linear functions, change one linear equation in two variables into the different algebraic forms of a line, find the area that is enclosed by a graph of a linear function and one of the coordinate axis, approximate the length of a graph of a non-linear function using the Pythagorean theorem and approximating line segments, solve basic optimization problems, and be able to do basic problems involving arithmetic sequences and series.

Students are now expected to have more conceptual and deeper understanding of a less broad but mathematically more significant range of topics under the Common Core Standards. Students are being pushed to a new level of rigor and higher level of content. With this being an expectation and a "new norm," eighth grade students should understand that linear functions are not just another math topic that will soon fade away, but that they are a significant part of the foundation of our quest to come to an understanding of the world around us.

The goal of this thesis is to demonstrate that it is well within reach and well within the goals of the Common Core State Standards for all eighth grade students to be exposed to a grade-appropriate introduction to basic calculus concepts. My hope and belief is that this will help lead more of my students into STEM careers and that this will help to significantly reduce their fear

of upper level mathematics. Students are afraid of the math unknown and they think it is difficult. If students are at least introduced to high school mathematics earlier and in a simpler way, we may see students starting to learn to believe in their math abilities; there are far too many people who believe they are "mathematically challenged." This type of thinking should change; students need to be taught in a different way-a way they can understand that mathematics is not just a boring heap of facts and tricks invented to torture students, but that mathematics provides the basic skills and understanding to have an educated dialogue about daily ideas and information, be it numerical or not. I personally did not take a calculus class until 19 years after high school. I had always thought of myself as a pretty good math student but I never did have the opportunity to take anything above Pre-Calculus. When I entered into college, my elementary education major required college algebra and only two other math classes that required more creativity rather than math ability. I believe that if I had had the opportunity to take a higher math, I may have been swayed into a different career path.

As teachers, we want to be able to give students any advantages possible. Studies have shown that the need for students with science and math degrees will be up about 35% ("Top STEM Jobs") by 2020. Maybe, introducing calculus concepts early on in eighth grade will help them getting there. When it comes to math classes, by starting with a topic they are familiar with and explaining how and why future mathematical concepts will build on it, students can better understand what is being taught and it helps them to have more confidence in their abilities,

The purpose of studying or at the very least introducing calculus concepts in middle school stems from our ever changing world. Technology is becoming a natural part in how we live our daily lives. Students must improve their understanding of math and science to be able to compete in the coming job market. This thesis is designed to engage students in what they are

going to have to understand to keep up with the rest of the world being the ideas of calculus lie at the basis of the mathematics needed to succeed in STEM fields.

Chapter 2: Middle School Common Core

2.1 Overview

The Common Core State Standards define what students should understand and be able to do in each grade level. What is mathematical understanding and how do teacher really know the students "got it"? One way for a teacher to know this, is for the student to have the ability to justify, in a grade-level appropriate way, why a particular mathematical statement is true or where a mathematical rule comes from. It is one thing for a student to be able to spit back out word-for-word (memorize) what the teacher said or did during the lesson instruction but it is another if the student can explain the meaning and expand on the thought process. Mathematical understanding and step-by-step know-how are both equally important, and both are assessable using mathematical tasks of quality and rigor.

With this new rigor come additional "teacher problems." Mainly, the problem of addressing students who do not "get-it" or students on the other end of the spectrum that may be more advanced than most students. The teachers need to figure out how to address typically lower performing students such as English as Second Language Learners and Special Education/504 students. These students may need more time and resources to fully meet the rigor required by the Standards. We expect the same achievement from all of our students, no matter if they aim for college or a professional career. In many classes, teachers that set and expect higher standards from their students will get that back from their students. Students will need to be ready, either for college, or for a career where, by definition, a career is a profession that requires at least two years of post-secondary education. In particular, according to this definition, working in the oil fields as a roustabout or, in general, as an unskilled laborer is a job, not a career. Students of today need to be able to have the understanding of the academic,

technical, and social skills necessary to be able to learn the professional skills related to a career. The students aiming for a professional career need to be held to the same standards that the college-bound students are in order for them to be ready. With the introduction of the new national tests (PARCC), teachers can no longer "dummy-down" the curriculum for those who do not "get-it." With the new tests around the corner, all students in the classroom have to be held to a higher standard than where they are currently. Fully Common Core aligned curricula such as the Engage New York mathematics curriculum will offer all students, regardless of leaning ability, a pathway that allows them to reach a level of understanding sufficient for post-secondary education. The Standards do provide guidelines and four alternative pathways to get all students ready for college and careers. The state of New York created a group of network teams consisting of 3-15 teachers from across the state (along with hundreds of others from across the country). They became the group leading Engage New York (www.engageny.org) to assist teachers in having all of the necessary materials to have successful implementation of the Common Core State Standards across their state. According to the Engage New York website,

> the main design principles in the New York State Common Core Learning Standards (CCLS) for Mathematics standards are focus, coherence, and rigor. These principles require that students and teachers focus their time and energy on fewer topics, in order to form deeper understandings, gain greater skill and fluency, and apply what is learned. Focus in the curriculum is meant to give students an opportunity to understand fewer concepts and have more practice with them in order to reach a better understanding. (www.engageny.org)

In order for students to better understand, they need to learn to rely less on processes and procedures. They need to be able to become fluent in simple tasks and be able to explain why something comes out the way that it does, rather than just know the steps involved in finding the answer.

According to www.corestandards.org,

> The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

When students are fluent in a task, they are able to better understand how things progress from one topic to the next. By having students be introduced to calculus concepts at an earlier age, they are able to see the slow progression of what they are learning to where they will be going next and they form “deeper understandings, gain greater skill and fluency, and apply what is learned.” For example, by finding the areas of more complex 2-D figures, they will apply what they learned about areas of simple geometric figures, by understanding slope in a variety of contexts, they will be better prepared for finding the velocity, acceleration, and equation of a tangent line. By constantly reviewing and reinforcing what students (should have) learned prior to eighth grade mathematics, students will be able to become better math students; the fluency needs to be there to be able to understand and move on to the next level of learning. The ultimate goal for any mathematics teacher is for students to be ready for their next steps, including those after high school towards college and careers. By incorporating calculus concepts into a middle school classroom, they are then on track to becoming better and

more confident math student by developing some understanding of where everything leads to.

2.2 Sixth Grade

The sixth grade math Common Core State Standards are divided into five different domains of what students should understand after completion of the course: (1) Ratios and Proportional Relationships, (2) the Number System, (3) Expressions and Equations, (4) Geometry, and (5) Statistics and Probability (www.corestandards.org).

The main focus of the sixth grade is: students are expected to be able to connect ratio and rate to multiplication and division. They should be able to understand division of fractions, including negative numbers. Students should also be able to write, interpret and use expressions and equations. Finally, sixth grade students should be able to have an introductory understanding of statistical thinking.

The connection of ratio and rate to multiplication and division is necessary for students to be able to better work functions, especially linear functions. This directly leads into the understanding of functions. In eighth grade, the CCSS no longer has a “Ratios and Rates” domain. It then becomes the domain of functions, in particular linear functions

In ratio and proportional understanding, students need to have an understanding of the concept of ratios, using language that describes ratio relationships. Students should be able to understand that the concept of a unit rate being out of 1 - the denominator is never 0. It is either 1 or an understood 1 when written either as a fraction or in words. Ratio and rates can be used to solve real-world or mathematical problems by either using tables, finding missing values, or by plotting the points on a coordinate plane. Students can solve unit rate problems, find a percent of a number and use ratio reasoning to find measurement conversions. According to the Engage

New York Website (www.engageny.org), teachers should spend approximately 35 days teaching the concept of ratios and unit rates to sixth grade students.

Sixth graders build upon their previous knowledge of ratios and proportions to expand their understanding of how ratios and proportions can be solved and what they mean. This is the important skill of ratios and proportions that students will take into seventh grade and use to build on their reasoning and problem solving skills and into eighths grade and use to discuss similar triangles and slopes of lines.

In the number system domain, students compute quotients of fractions and solve word problems involving fractions. Students also easily divide multi-digit numbers and add, subtract, multiply and divide decimal numbers. Students also need to be able to find the greatest common factor and least common multiple of two numbers. Students need to understand what positive and negative numbers are, where they are located on a number line, know and recognize their opposites and understand that absolute value is the distance a number is from zero and how to order all types of numbers (positive, negative, absolute value). They need to be able to plot numbers on a coordinate plane and understand the meaning of negative numbers on the grid, also solve word problems by graphing. Finally, students need to be able to interpret inequality statements and how to write, interpret and explain statements of order using inequality symbols. The suggested time for teaching this unit, is a total of about 50 days. 25 days should be spent on arithmetic operations, including dividing by fractions and the other 25 should be spent studying rational numbers.

This is one of the key points in greater understanding – fluency. The time suggested spending on fluency shows that this is where the focus of sixth grade math needs to be. The majority of the time spent in sixth grade should be used to ensure fluency of simple tasks and

word problems. Students MUST become fluent in the simple tasks so they can be better prepared for the next steps, including deeper understanding. For teachers in eighth through twelfth grade to be able to dig into the higher mathematical concepts, the students need to be fluent in the necessary background knowledge.

In the expressions and equations domain, students need to know how to write and evaluate expressions involving whole-number exponents, and to write, read, and evaluate expressions involving variables (with and without operation signs). Students need to use mathematical terms appropriately, translating from a number sentence to words and vice-versa. They need to be able to evaluate expressions and apply properties of operations to generate equivalent expressions, and be able to identify when two expressions are equivalent. Students need to have an understanding of solving one-variable equations and inequalities in terms of a process of answering a question while using variables to represent numbers in real-world situations. Finally, students need to be able to use variables to represent quantities in real-world problems using dependent and independent variables (www.corestandards.org). It is suggested by Engage New York that 45 days be spent on expressions and equations (www.engageny.org).

Expressions and equations are the foundation for anything beyond sixth grade math. The simple one-step equation turns into a two-step equation, which turns into an equation with variables on both sides, which turns into quadratic equations, absolute value equations, or compound inequalities. Without the foundation being set with simple equations and expressions, students will not be able to be successful with functions.

In geometry, students have to find the area of triangles, special quadrilaterals, and polygons, find volume of rectangular prisms and draw polygons on a coordinate plane if given vertices then use the vertices to find the length of a side or sides. Students must also be able to

represent three-dimensional figures using nets made up of rectangles and triangles (www.corestandards.org). Suggested time spent working with area, volume, and surface area is about 25 days in sixth grade (www.engageny.org).

By introducing calculus concepts in eighth grade, students will then build upon the simple perimeter equations and are able to find the length of a curve. Area formulas can be built upon to find areas under the graphs of linear functions. Moreover, by introducing calculus concepts in the eighth grade, students can take their perimeter or surface area knowledge to help them with some optimization problems concerning areas and volumes.

In the statistics and probability domain, students need to know what a statistical question is and how the answers to those questions can be found knowing how to describe the center, spread and shape of the data, knowing the difference between measures of variation and central tendency. Students need to know how to display the data in different ways and how to summarize the data (www.corestandards.org). Engage New York is allotting 25 days of sixth grade mathematical instructional time (www.engageny.org).

Statistics and probability are so intertwined in all of math. Students will use statistics to find averages of different things, like their grades for the ones that want to know "why do I need this", and later on, like in calculus, the averages of the values $y = f(x)$ for $a \leq x \leq b$.. Statistics can allow students to be able to better see the data they are deciphering.

2.3 Seventh Grade

The seventh grade Common Core State Standards for Mathematics are divided into the same five domains that already defined the sixth grade Standards: (1) Ratio and Proportional

Relationships, (2) The Number System, (3) Expressions and Equations, (4) Geometry, and (5) Statistics and Probability (www.corestandards.org).

The main focus of the seventh grade mathematics course is the following: "developing a deeper understanding and application of proportional relationships, developing understanding of operations with rational numbers while working with linear equations and more expressions, solving problems using scale factor and geometric constructions and finally, drawing inferences based on samples." (www.corestandards.org)

Seventh grade is an extension of what is learned in sixth grade with a few more things added – leading into eighth grade. Students started in sixth grade with developing their understanding of ratios and proportions and now it is being expanded upon in seventh grade. The introduction of linear equations is something new as a term and students will come to understand that it is a particularly important class of equations. Students also expand on their knowledge of perimeter, area, volume, and surface area using geometry while introducing geometric constructions. Their study of statistics and probability is added to with understanding how to draw inferences and sampling methods.

In ratios and proportional relationships, students need to be able to compute unit rate measures in like or different units. Also, students need to be able to recognize and represent proportional relationships between quantities, in which they need to decide whether two quantities are proportional and be able to identify the proportional constant in tables, graphs, or equations. Students need to be able to write proportional relationships using equations and explain the steepness of a line in terms of a proportion. Finally, students need to be able to use proportions to solve multistep ratio and percent problems (www.corestandards.org). The Engage New York website is suggesting that teachers spend approximately 55 days on this domain. It is

split into two different units, one being 30 days on ratio and proportional relationships and another 25 days on percent and proportional relationships (www.engageny.org).

Students are being introduced to the beginning stages of a rate of change, understanding that the ratio of any two sets of points on a line is a constant and leads them into more discussion of linear equations. Also here, students will be better prepared for what lies ahead of them in terms of Algebra and solving multi-step equations. The bulk of suggested curriculum time is spent in this area because is sets the foundations for much of math after seventh grade.

In the number system domain, students need to be able to apply and extend their understanding of addition and subtraction to add and subtract rational numbers, describe situations to make zero (opposite quantities), understand distance between two points, understand the additive inverse property, and to apply properties of numbers to add and subtract rational numbers. Students need to also take what they know about multiplying and dividing fractions and apply it to multiplication and division of positive and negative rational numbers. With this previous understanding, students need to understand how to extend multiplication of fractions to rational numbers by requiring the operations follow the properties and follow the rules of signed numbers. Students need to understand division of integers can occur so long as the divisor is not zero. Students need to understand that fractions become smaller and smaller if the divisors become larger and larger (beginning of limits). Student can apply strategies of operations to rational numbers, do long division, knowing the decimal form will always end in zero or eventually repeat for all rational numbers and finally, use the four operations to solve real-world and mathematical problems (www.corestandards.org). Suggested teaching time of the number system is 30 days (www.engageny.org).

These are again all foundational fluency skills that need to be mastered before eighth grade math. Students will get to the point where they need to have the understanding of knowing the differences of the properties of real, rational, and irrational numbers, along with integers and whole numbers. They need to be able to use this understanding to know when certain numbers are being used appropriately and in context of the situation. In eighth grade, students need to know that depending on the context of a problem, only integers or whole number answers are correct and that the properties of a linear function depend heavily on the number system used to describe the domain and range of the function.

In the expressions and equations domain, students will need to be able to apply properties of operations to be able to add, subtract, factor and expand expressions with rational coefficients. Students also need to be able to know how to rewrite an expression in a different form and solve multi-step real-life and mathematical problems with positive and negative rational numbers. Students need to understand how to use variables to represent quantities and use the variables to write equations and inequalities and then be able to solve for the variable (www.corestandards.org). Engage New York allots 35 days for the expressions and equations units (www.engageny.org).

Seventh grade expressions and equations are leading closer and closer to what students need to have understood by the time they reach eighth grade. The seventh grade standards lend themselves to the expansion of equations. Students are now being able to have a grasp on what makes an equation behave the way it does and be able to manipulate its form to be able to solve for different parts (like finding the height in a triangle area problem). In eighth grade, students will use their knowledge of the behavior of equations to be able to find domain and range of

functions. In turn, the understanding of functions will lead to an understanding of how functions can be linear, nonlinear, piecewise, or any other type of function.

In the geometry domain, students are asked to be able to solve problems using scale drawings, draw geometric figures (given certain conditions), describe the two-dimensional figure resulted in a slice from a three-dimensional figure. Students need to explain facts about angles, supplementary, complementary, vertical and adjacent. Finally, students must be able to explain the formulas and be able to apply them for circumference and area of circles, area, volume, and surface area of two-dimensional and three-dimensional figures made up of triangles, quadrilaterals, polygons, cubes, and prisms (www.corestandards.org). Geometry is given 35 days in seventh grade on www.engageny.org.

By having students be able to describe the 2-D figure from a 3-D figure slice will be helpful later on in calculus when students have to slice a figure rotating on the coordinate plane and find the area of the three-dimensional figure.

In statistics and probability, students need to be able to understand that using statistics can assist with gaining information about a population by studying a sample population and use samples to draw inferences about a population. Students also need to be able to assess the visual overlap and use measures of center and variability to compare two populations. Students will also need to know that probability explains the chance of an event occurring, the difference between experimental and theoretical probabilities, be able to find probabilities and compare the results, and finally to find probabilities of compound events and knowing the probabilities found are fractions of the outcomes possible, find frequencies of compound events and be able to identify the sample space of the events (www.corestandards.org). Engage New York is suggesting 25 days for probability and statistics (www.engageny.org).

2.4 Eighth Grade

Now comes the fun part of math: eighth grade. According to the Common Core State Standards, in eighth grade mathematics students need to master work in five domains: (1) The Number System, (2) Expressions and Equations, (3) Functions, (4) Geometry, and (5) Statistics and Probability (www.corestandards.org).

In particular, compared to sixth and seventh grade, the only change in the domains is that that the domain on ratios and proportional relationships is replaced by the new "function domain." Thus, one of the main focal points of eighth grade mathematics should be formulating and reasoning about expressions and equations, grasping the concept of a function, and getting a firm grasp on the concept of linear functions.

By eighth grade, students should be fluent in math facts and the processes involved in solving different types of problems, transposing equations to solve for different parts, how to plot points on a coordinate plane, knowing properties of and being able to describe two- and three-dimensional figures (and how to find their areas and volumes), basic statistics and probability, and how missing parts of ratios and proportions are found. Eighth grade students will be able to better understand why things are happening the way they are, rather than learning the how-to's. In the number system domain, students need to know that some numbers are not rational and need to know how to approximate them by rational numbers, know where they are graphed on a number line (www.corestandards.org), and know that there is no "closest" rational number for a given irrational number like the square root of 2. In eighth grade, Engage New York is suggesting a time of 35 days to introduce students to irrational numbers using geometry (www.engageny.org) and, in particular, the Pythagorean Theorem.

Students, before this point, have learned only about rational numbers in detail. In eighth grade, teachers will expand on irrational numbers. Students will be able to uncover that between any two rational numbers there is an irrational number, that there are more irrational numbers than rational numbers (using a proof by contradiction and a diagonal argument), and approximate square roots of positive real numbers. In this standard, students have a plethora of calculus topics hiding behind the title of the number system (limits, series and sequences, etc). Students, at this point, can explore the concept of the intermediate value theorem by approximating the point where a curve, like $f(x) = x^2 – 2$, crosses over the x-axis and discovering that it may not be a nice rational number. In the expressions and equations domain, students need to be able to work with radicals and integer exponents, know how to use the square root and cube root symbols, and know how to write numbers in scientific notation and how to perform operations with it (which naturally leads to a discussion about limits to “infinity” and “zero”). Students must also be able to understand how proportions, lines, and linear equations are connected, understand the concept of the slope of a line, and use similar triangles to explain why the slope is the same between any two particular points on a line. With respect to linear equations, students need to be able to solve in one variable, give examples of linear equations with one solution, infinite many solutions or no solutions, solve systems of equations, understand that the solution of a system of two linear equations represents the point of intersection of the two lines associated with the linear equations, and finally, to solve systems of linear equations in two variables by graphing (www.corestandards.org). The bulk of eighth grade is spent on expressions and equations, a total of 60 days, split between two units – 20 days on integer exponents and scientific notation and 40 days on linear equations on the Engage New York curriculum map (www.engageny.org).

In the domain on functions, students in eighth grade need to be able to understand that a function is a rule that gives each input exactly one output, be able to compare properties of differently represented functions, and be able to interpret the equation $y = mx + b$ as defining a linear function and resulting in a straight line (or as a subset of a straight line) when graphed. Students need to be able to construct a function to model a linear relationship between two quantities, determining rate of change and starting point and be able to describe the function relationship between two quantities by analyzing a graph (www.corestandards.org). Students and teachers should spend about 35 days on functions, according to www.engageny.org, 15 days on examples of functions from geometry and 20 days on linear functions.

This domain is wrapped in the domain of expressions and equations. The concept of linear equations leads to linear functions. This thesis aims to show that there are many ways to expand a students learning in these crucial domains in eighth grade math by introducing some basic calculus concepts in the context of linear functions without losing focus or artificially expanding the material covered.

By the time teachers come to the unit on linear functions, the students should have already understood the concepts on linear equations, giving the teacher more opportunity to allow the students to dig deeper into the calculus concepts as explained later on in this thesis.

In the geometry domain, students need to be able to know the properties of rotations, translations, and reflections, describe a congruency statement, and be able to explain the effects of dilations, rotations, translations, and reflections of figures on a coordinate plane. Also, students must understand a similarity statement and finally, understand the effects of a transversal intersecting a pair of parallel lines (www.corestandards.org). The geometry domain

is split between two different units, 25 days on the concept of congruence and another 25 days on similarity (www.engageny.org).

In statistics and probability, eighth grade students need to be able to construct and interpret scatter plots, know that straight lines can model relationships between two quantities, use equations to solve problems involving bivariate data and finally understand patterns seen in data while also using relative frequencies (www.corestandards.org). According to the Engage New York website, eighth grade teachers should spread this domain out among the other domains and teach it in context of all others (www.engageny.org).

Chapter 3: Curriculum

3.1 Functions

What is a function? Usually, textbooks and publishing companies define a function differently, ever so slightly. In regards to what this piece of work is trying to convey, a function is defined as follows:

> If given a set D, called domain, and a set B, called range space, a function is a rule that assigns to each element x in D exactly one element, called f(x), in the set B (Rogawski 22)

The important feature of this definition is that a function is properly defined by:

(i) specifying a domain, D,

(ii) specifying a range space, B,

(iii) a rule f: D $\rightarrow$ B defined by x $\rightarrow$ f(x) for each x in the domain D.

Before we go any further, we must understand what the domain, range, and range space are. McGraw-Hill Companies (Glencoe Course 2 177 and Glencoe Course 3 518) defines domain "as the set of input values of a function." Pearson Education (Algebra 1 24) defines first a relation and then the domain "as the set of all first coordinates in the ordered pairs of a relation". On the website purplemath.com, domain is said to be "all values that x is allowed to be." For us, the domain, D, of a function, *f*, is the set on which the function (or rule) is defined.

Just like with domain, range has different definitions from different publishers and websites. McGraw Hill Companies (Glencoe Course 2 177 and Glencoe Course 3 518), in their

middle school mathematics textbooks, defines the range "as the set of output values of a function," Pearson Education (Algebra 1 241) defines range "as the set of all second coordinates in the ordered pairs of a relation," and the website purplemath.com defines it "as all values that y is allowed to be." For us, the range space B is the set into which a function maps, whereas the range of a function *f* is the set $f(D) \subset B$. For example, for the function $f: (-1, 1) \rightarrow \mathbb{R}$ defined by $f(x) = 1 + x + x^2 + \ldots.$, the range space is $\mathbb{R}$ but the range is $(\frac{1}{2}, \infty)$ since $f(x) = 1 + x + x + \ldots = \frac{1}{1-x}$ for $x \in (-1, 1)$ and $\frac{1}{1-x}$ takes on all values between $\frac{1}{2}$ and infinity for $-1 < x < 1$.

Many publishers do not emphasize the importance of clearly defining the domain and range space. For example, one publisher defines a function as "a correspondence between the elements of one set (domain) and the elements of another set (range), where exactly one element in the range corresponds to each element in the domain" (Stewart 12). Another publisher says "a relation that assigns exactly one value in the range to each value of the domain" (Pearson Education 242). In a very widely used middle school mathematics textbook, a function is defined "as a relation in which each element of the input is paired with exactly one element of the output according to a specified rule"(Glencoe Course 3 517). Online, one definition says that a function "relates an input to its output" (www.mathisfun.com). Also found online, a function is defined as "a rule which relates the values of one variable quantity to the values of another variable quantity, and does so in such a way that the value of the second variable quantity is uniquely determined by (i.e. is a function of) the value of the first variable quantity" (http://www.wmueller.com). Lastly, one more states, "a rule that takes an input, does something to it, and gives a unique corresponding output" (http://www.onemathematicalcat.org). All of the

examples show the inconsistencies and cloudiness of the definitions that vary from publisher to publisher, from on-line site to online site.

3.2 Linear Functions

Linear functions are the first class of functions that are discussed in detail in the K-12 Common Core State Standards (CCSS) for Mathematics. Unfortunately, the CCSS do not define what a linear function is and, once again, the textbooks are all over the place, often starting with the definition of a line. A line is defined as “a series of points that extends in two directions without end” by Prentice Hall, Mathematics, Course 3. This same textbook from Prentice Hall defines linear function as “a function whose points lie on a line.” Stewart’s calculus textbook defines linear function as those when “the graph of a function is a line, which means it can be written in slope-intercept form.” We define a line as follows,

> Let A, B, C be given real numbers. A line is defined as the set of points (x, y) in the coordinate plane $\mathbb{R} \times \mathbb{R}$ that solve the equation

$$Ax + By = C.$$

If B = 0, then the line is called vertical; if A = 0 then the line is called horizontal. If $B \neq 0$, then the points (x, y) in $\mathbb{R}\ x\ \mathbb{R}$ that solve Ax +By = C or

$$y = mx + b$$

(where $m = \frac{-A}{B}$, and $b = \frac{C}{B}$) are called the graph of a linear function, $f\colon \mathbb{R} \rightarrow \mathbb{R}$ defined by the rule $x \rightarrow f(x) = mx + b$. If the domain D and the range B are subsets of, $\mathbb{R}$, then the function $f\colon D \rightarrow B$ defined by $x \rightarrow f(x) = mx + b$ is called linear. In this case, however, the graph $\{(x, f(x)), x \in D\}$ is not a line but only a subset of a line. All lines, where $B \neq 0$, can be transformed into y = mx + b

(slope-intercept form) which is said to be an explicit formula of a line (whereas the form Ax + By = C is called an implicit form for a line). An example of a function written in explicit algebraic form is $g(x) = \frac{1}{2}x - 4$.

Functions with domain and range in the real numbers and, in particular, linear functions may be represented in four different ways:

(i) numerically (e.g. input-output tables),

(ii) verbally (e.g. word problems),

(iii) visually (e.g. coordinate plane graphs),

(iv) Algebraically (e.g. equation).

In introducing students to the concept of functions (linear functions) it is important to make students comfortable in switching freely between these representations.

Example 1: We start with a function *f* whose domain is $D = \{0, 1, 2, 3, 4, 5\}$, whose range space is the set of whole numbers, and whose rule is $f(x) = 2x + 1$. This line is considered a discrete line, meaning that it is a discrete set of points that lie on the line $\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = 2x + 1\}$.

This function can be represented by an input-output table. To represent functions numerically, the most common representation used in middle school is by using function tables. This example of a function table has a set of input values (the x-values), the function rule, and a set of corresponding output values (the y-values).

Table 1

Input *x* Domain {0, 1, 2, 3, 4, 5}	Function rule *f(x)* *f(x)* = 2x + 1	Output *y* Range Space {0, 1, 2, 3, 4, …}
0	2(0) + 1	1
1	2(1) + 1	3
2	2(2) + 1	5
3	2(3) + 1	7
4	2(4) + 1	9
5	2(5) + 1	11

In the table, we notice that only the output values {1, 3, 5, 7, 9, 11} are reached, that is the range f(D) = {1, 3, 5, 7, 9, 11} is a true subset of the range space.

To describe this function verbally a student should be able to find an example like the following: Jackie sells hand-made key chains online. The price at which she sells each key chain is $2 plus $1 shipping (regardless of the number being shipped). All her shipments contain her key chain catalogue which she also ships for a $1 shipping fee to customers that only want to have her catalogue but no key chain. We can write the function rule to represent her charges to the customer as f(x) = 2x + 1. In this situation, the number of key chains she sells will always be a whole number. She cannot sell negative key chains, nor can she sell a part of a key chain. That is, the domain D is given by the set {0, 1, 2, 3, ……..}.

Visually, we can take the same function, f(x) = 2x + 1, and use the inputs and outputs to graph points on a coordinate plane.

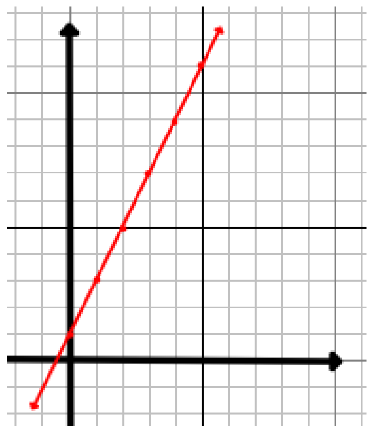


Figure 1

Since this is the first time when students are asked to represent lines of the form y = mx + b in terms of the graphs, special attention must be given to “scaling” and to the fact that not all visual graphs are good examples of “what you see is what you get.” Look at the examples below. All of them are linear functions but the scales have to be changed to be able to see what is happening with the line.

Example 2: $f(x) = 100x + 1$

X	f(x)
1	101
2	102
3	103
4	104
5	105

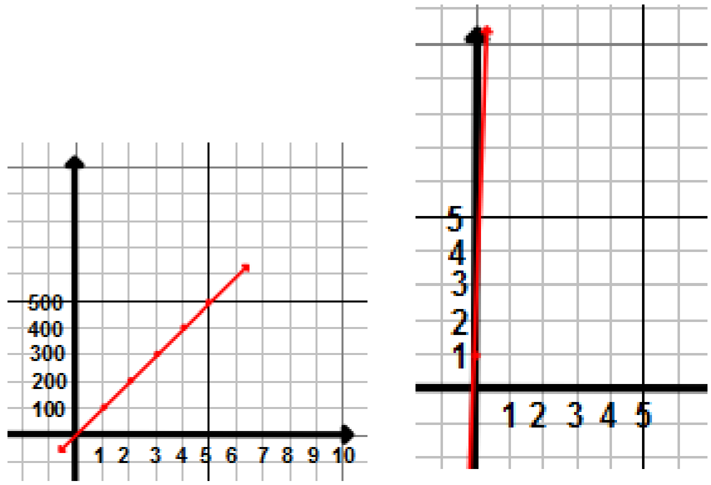


Figure 2

Notice that only the graph with the x-values labeled in 0.01 increments contains enough information to reconstruct the function rule. That is, in strict mathematical terms, this is the only "reasonable" graph for the function f.

Example 3: $f(x) = \frac{x}{10000} + 1$

x	f(x)
1	1.0001
2	1.0002
3	1.0003
4	1.0004
5	1.0005

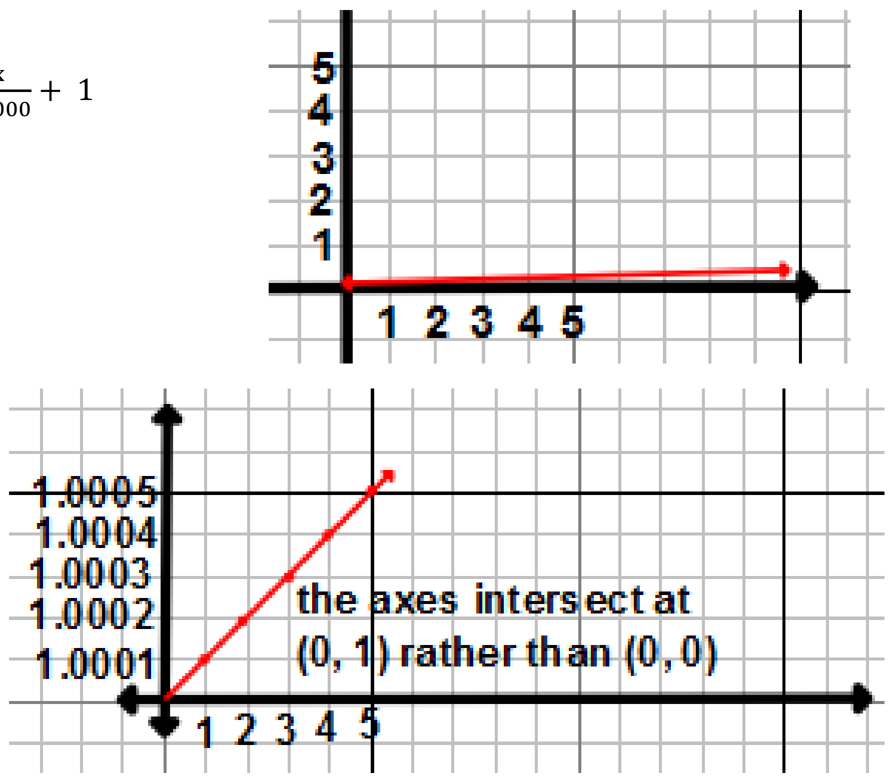


Figure 3

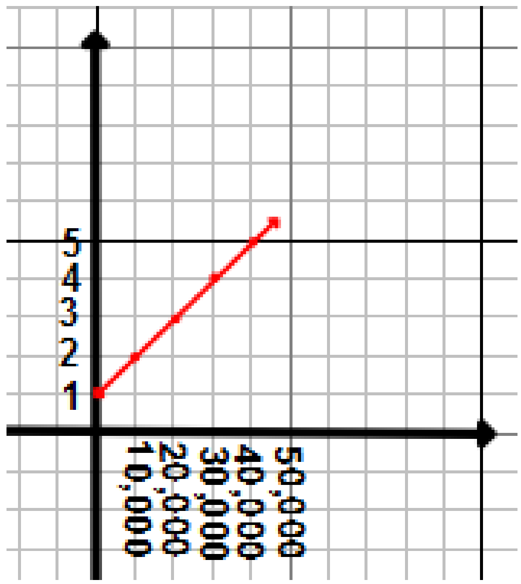


Figure 3 (con't)

The second graph shows that the line is very close to a horizontal line for moderate values of x. However, only the first and the third graph are good representations in the sense that they contain enough detail to be able to reconstruct the algebraic form of the line.

<u>Example 4:</u> For every thousand feet above sea level, the temperature drops 3.6°F (Cook). If the temperature at sea level is 80°F, answer the following questions.

(a) Write a function rule to describe the situation.

(b) Create a function table, using integers, with the rule to show the temperatures at five other heights above sea level (the change in temperature is different for below sea level).

(c) Graph the function on a coordinate plane.

(d) What is the height above sea level at freezing?

(e) What is the temperature at 33,000 feet above sea level (the average altitude of a traveling plane)?

Solutions*:*

(a) $f(x) = 80 - 3.6x$ D = [0, 40] and R = [-64*,* 80].

The beginning of the domain is 0 (in thousand feet) because the temperature change when you go below sea level is a different function compared to above sea level We assume that an appropriate domain value may go up to is 40,000 feet (most transatlantic flights travel between 30,000 and 40,000 feet above sea level at their maximum elevation). If the domain is at its maximum value, the range would be at -64°F while if the domain is at its minimum value, the range would be 80°F. Above 40,000 feet, the temperature function may not behave the same anymore and the rule may differ from the one above that holds when one is close to the Earth's sea level. In this context one may discuss with the students that the rule is certainly not true for values of x above 150,000 feet because then, according to the rule 80-3.6x, the temperature would be below 460° Fahrenheit. This, however, is an impossible temperature since it is below absolute zero which is at 0 Kelvin, -459.67° Fahrenheit, and -273.15° Celsius.

Table 2

(b)

Height (in thousands of feet)	$f(x) = 80 - 3.6x$	Temperature (in °F)
0	80 – 3.6(0)	80
0.25	80 – 3.6(0.25)	79.1
1.5	80 – 3.6(1.5)	74.6
6	80 – 3.6(6)	58.4
10	80 – 3.6(10)	44

(c)

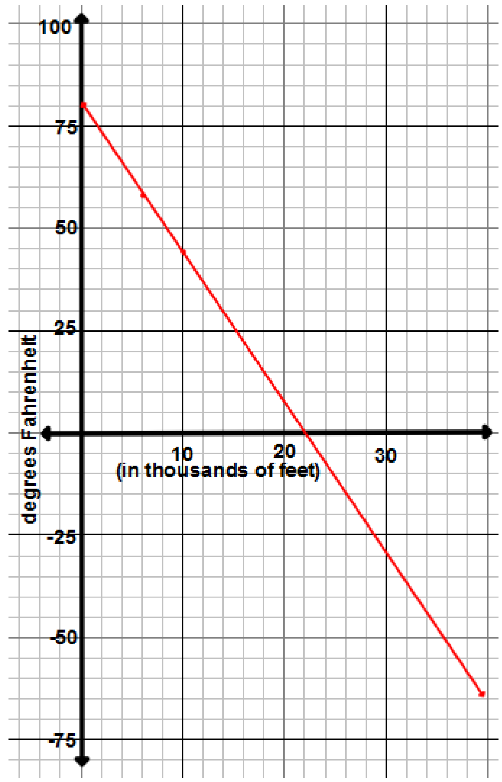


Figure 4

(d) The temperature at freezing is 32° Fahrenheit. In order to find the corresponding height, we have to solve the linear equation 32 = 80 – 3.6x. This gives us that it is at

$$x = 13\tfrac{1}{3} \text{ thousand feet}$$

where we will find freezing temperatures. That is, if it is 80°F at sea level, then it will be freezing when the height above sea level is above $13\frac{1}{3}$ thousand feet.

(e) f(33) = 80 – 3.6(33)= -38.8°F

Thus, according to the rule, if the temperature at sea level is 80°F, then at 33,000 feet above sea level, the temperature is a negative 38.8 degrees Fahrenheit. This is the height about which a plane on a transatlantic flight travels.

Exercise 1

Directions*:* Give a numerical, verbal, and visual description for each of the following functions using the domain or domain subset values of {-2, -1, 0, 1, 2}.

1. $f(x) = 3x – 2$
2. $f(x) = ½ x + 4$
3. $f(x) = -2x + 1$

Directions*:* Translate each linear function written into the explicit form $y = mx +b$ form and give a numerical, verbal, and visual description. Describe the domain and the range space for each function.

4. $3x − 2y = −16$
5. $x − 3y = 6$
6. $6x + 5y = 15$

Students should be able to determine if certain points satisfy the equation defining a line given by $Ax + By = C$ or $y = mx +b$. This is known as verifying the solution.

Exercise 2

Directions: Find at least two pairs of integers (x, y) that satisfy $2x + 4y = 10$.

Exercise 3

Directions*:* (a) Verify if the point (-1, 5) is on the line $-4x + y = 1$.

(b) Find two additional points that do lie on the line and one point that does not.

(c) Graph the line on a coordinate plane.

Exercise 4

You are looking for integer solutions of the equations (i) $3x - y = 0$ and (ii) $3x - y = 1$. Find the domains and range spaces of the resulting functions.

Since a linear function $f(x) = mx + b$ is defined uniquely by the two unknowns m and b, one needs two equations to determine these two unknowns. These two equations can be found if two points are given that lie on the line defined by the linear function f.

Exercise 5

If you are given the points (1, 2) and (3, 5) then the line L that connects these two points is given by $f(x) = mx + b$. Find m and b.

Solution*:* Since (1, 2) and (3, 5) are points on the line, it follows that $2 = m + b$ and $5 = 3m + b$. Subtracting the first equation from the second yields $3 = 2m$ or $m = \frac{3}{2}$. Substituting $m = \frac{3}{2}$ into the first (or second) equation, yields $b = \frac{1}{2}$. Thus the line that connects the two points is given by the function $f(x) = 1.5\,x + 0.5$.

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be points on a line, $y = mx + b$. This leads to the system of linear equations with unknowns of m and b.

$$y_1 = mx_1 + b$$

$$y_2 = mx_2 + b$$

Subtracting the second equation from the first gives

$$y_2 - y_1 = mx_2 - mx_1 = m(x_2 - x_1),$$

$$or \;\; m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Substituting this into the first (or second equation), the results are

$$y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1 + b,$$

$$or \;\; b = y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1.$$

This gives the formula of the line as

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) + y_1,$$

$$or \;\; y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1), \; or \;\; \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \;\; \text{(two-point form).}$$

3.3 Rates of Change

The rate of change of a linear function is the slope, or tilt, of the line – a ratio of a line's rise to its run. More precisely, let (x_1, y_1) and (x_2, y_2) be two points on a line $y = mx + b$. Then the quantity $y_2 - y_1$ is called the rise of the line, the quantity $x_2 - x_1$ is called the run of the line, and the slope of the line is defined as

$$\begin{aligned} \text{slope} &= \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} \\ &= \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} \\ &= m. \end{aligned}$$

The fact that the ratio $\frac{rise}{run}$ is always the same for any two points on a line L can also be seen from the facts we know about similar triangles. Triangles can be drawn between any points on the graph and they are similar. As students can recall, similar triangles are proportional, meaning their corresponding sides all have the same ratio.

<u>Example 5:</u> If you are given the points (1, 2) and (3, 5), we have to find the slope of the line that connects the two points.

$$\text{slope} = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

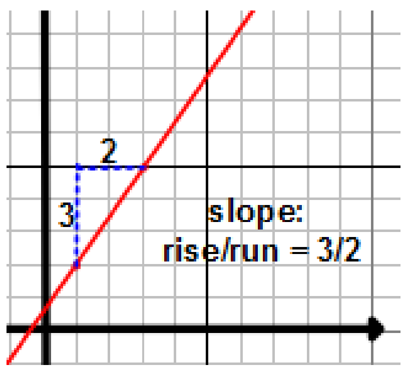


Figure 5

First, we determine the coordinates (x_1, y_1) and (x_2, y_2) of the given points. The first point is given as (1, 2). Thus, we have that $x_1 = 1$ and $y_1 = 2$. For the second point, (3, 5), $x_2 = 3$ and $y_2 = 5$.

Next, using the slope formula, and inputting the numbers from the coordinates gives,

$$\text{slope} = \frac{rise}{run} = \frac{5-2}{3-1} = \frac{3}{2}$$

A slope of $\frac{3}{2}$ means the line that passes through the two points, (1, 2) and (3, 5), rises 3 units for every 2 units it runs.

To see it with similar triangles:

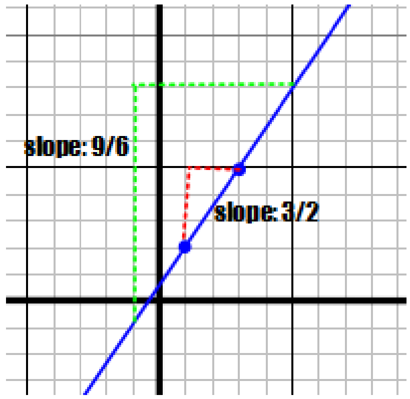
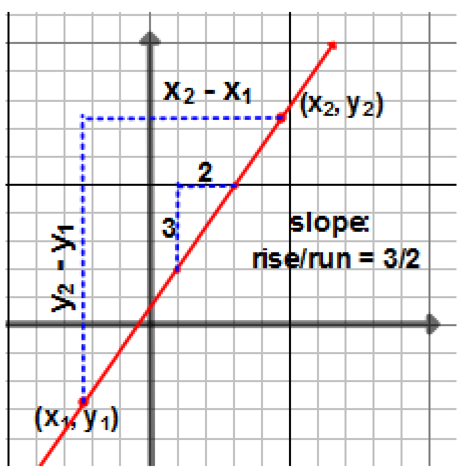


Figure 6

If one triangle, with hypotenuse endpoints of (1, 2) and (3, 5) has a slope of $\frac{3}{2}$, and another, with hypotenuse endpoints (x_1, y_1) and (x_2, y_2) has a slope of $\frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$, then this ratio will reduce to $\frac{3}{2}$, making the two slopes equivalent.



$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Figure 7

Now that we know $y = \frac{3}{2}x + b$, we can find b. Since (1, 2) lies on the line, we can replace x with 1 and y with 2 to solve for b. Thus, the equation for the line $y = \frac{3}{2}x + b$ leads to the equation $2 = \frac{3}{2}(1) + b$ or $\frac{1}{2} = b$. We now have the slope, $m = \frac{3}{2}$, and $b = \frac{1}{2}$, to write the equation of the line as

$$y = \frac{3}{2}x + \frac{1}{2}.$$

As we have seen in the example above, given a line in the explicit form,

$$y = mx + b,$$

the meaning of the constant, m, is the "slope." That is $m = \frac{y_2 - y_1}{x_2 - x_1}$, for any two points (x_1, y_1), (x_2, y_2) on the line $y = mx + b$. But what is the meaning of b? Clearly, if $x = 0$, then $y = m(0) + b = b$. That is, the point (0, b) is on the line $y = mx + b$ and defines the point where the line crosses the y-axis. For this reason, the constant b is called the y-intercept and the equation $y = mx + b$ is called the slope-intercept form of a line.

In general, given two points (x_1, y_1) and (x_2, y_2) that lie on the line defined by $y = mx + b$, then

$$m = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1},$$

$$\textit{or } y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + b.$$

Then, in order to find b, we plug the coordinates (x_1, y_1) into $y = mx + b$ to find b. This yields

$$y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1 + b,$$

$$or\ b = y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1.$$

By substituting the above formula into $y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + b$, we get $y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + y_1 - \frac{y_2 - y_1}{x_2 - x_1}$, *or* $y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$, *or*

$$y = m(x - x_1) + y_1$$

Thus, given a point (x_1, y_1) that is on a line and the slope m of a line, then the "point-slope" form of a line is given as $y = m(x - x_1) + y_1$.

A line can have a positive, negative, zero, or undefined slope. The slope between points on a linear function will always be the same, no matter which two points you choose, it should always reduce to the same slope.

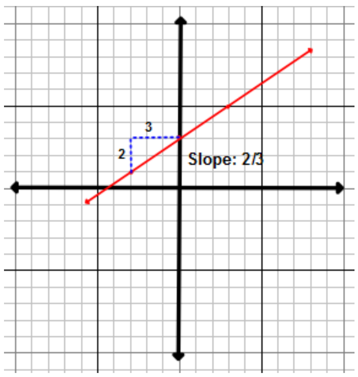
<u>Example 6:</u>

positive slope

$P_1 = (-3, 1)$

$P_2 = (3, 5)$

$m = \frac{2}{3}$, $b = 3$



slope-intercept form:

$$y = \frac{2}{3}x + 3$$

point-slope form:

$$y = \frac{2}{3}(x + 3) - 1$$

two-point form:

$$\frac{y-1}{x-(-3)} = \frac{5-1}{3-(-3)}$$

standard form:

$$2x - 3y = -9$$

Figure 8

negative slope

$P_1 = (-1, 5)$

$P_2 = (0, 2)$

m = -3, b = 2

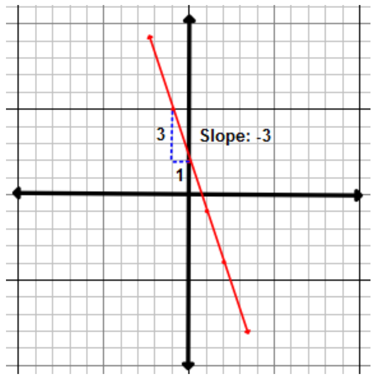


Figure 9

zero slope (horizontal line)

$P_1 = (-5, 3)$

$P_2 = (-2, 3)$

m = 0, b = 3

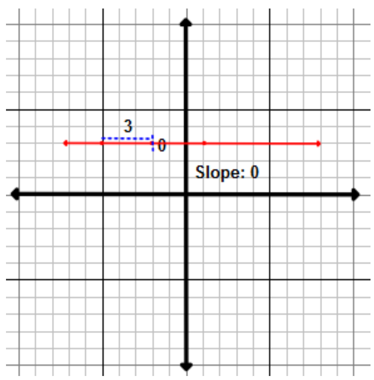


Figure 10

two-point form:

$$\frac{y-3}{x-(-5)} = \frac{3-3}{-2-(-5)}$$

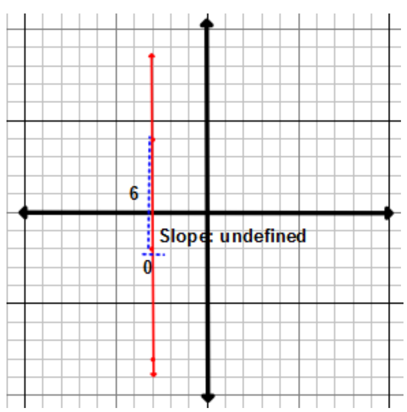
standard form:

y = 3

undefined slope (vertical line)

$P_1 = (-3, -2)$

$P_2 = (-3, 4)$



standard form:

x = -6

Figure 11

Example 7: What is the slope of the line defined by the equation 2x + 3y = 15?

There are two ways to find an answer. The first one is to find two points on the line, like the y-intercept, P_1 = (0, 5) and P_2 = (3, 3). Then m = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-2}{3}$, and therefore y =$-\frac{2}{3}x - 5$. The second way is to solve for y; that is

$$3y = -2x + 15, \qquad or \qquad y = \frac{-2}{3}x + 5.$$

This also yields that m = $\frac{-2}{3}$ and b = 5.

In the equation, 2x + 3y = 15, it takes solving the equation for y to be able to find the slope.

$$2x + 3y = 15$$

Example 8: If we study the graph below, we can see that it has a negative slope and it appears that the points (19, −16), (−7, −10) are on the line. We can use the two points to write the two-point form of a line:

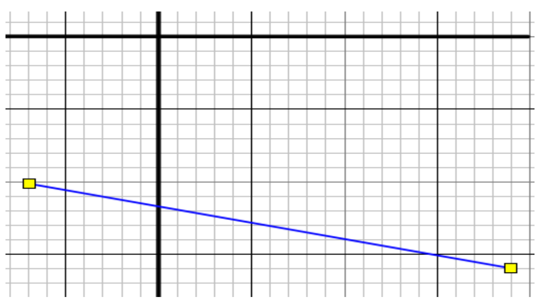


Figure 12

To determine the slope, we can find the difference in the y-values (rise) and divide it by the difference in the x-values (run) and obtain m= $\frac{-10-(-16)}{-7-19} = -\frac{6}{26} = -\frac{3}{13}$.

Using the point-slope form of a line, we get y = $-\frac{3}{13}$ (x – 19) – 16 or y = $-\frac{3}{13}$x + $\frac{57}{13}$ – 16 = $-\frac{3}{13}$x - $\frac{151}{13}$ = $-\frac{3}{13}$x - 11$\frac{8}{13}$. Thus, the y-intercept is at (0, -11$\frac{8}{13}$).

Exercise 6

Give examples in which it is more convenient to use the following line types:

a. two-point form of a line

b. point-slope form of a line

c. slope-intercept form of a line

d. standard form of a line

<u>Example 9:</u> Students can plot the y-intercept of a line then use the slope to find the next point to be able to create a line.

Draw a line that has a y-intercept of 3 and a slope of -2.

Start with the y-intercept of 3. Plot it on the y-axis. From (0, 3), move down two units on the y-axis and right 1 unit. Place another point on the graph. From this point, you can continue moving down two units and one unit right to place more points. Connect the points to create a line.

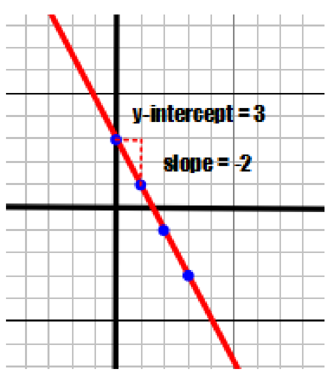


Figure 13

Example 10: Graph a line that is written as x + 5y = -15.

One way to do this is to find two points on the line, like the intercept (0, -3) and the point (5, -4) and to connect them with a ruler. A quicker way is to translate the standard form into slope-intercept form:

$$x + 5y = -15$$

$$5y = -x - 15$$

$$y = -\frac{1}{5}x - 3$$

y-intercept = -3, slope = $-\frac{1}{5}$

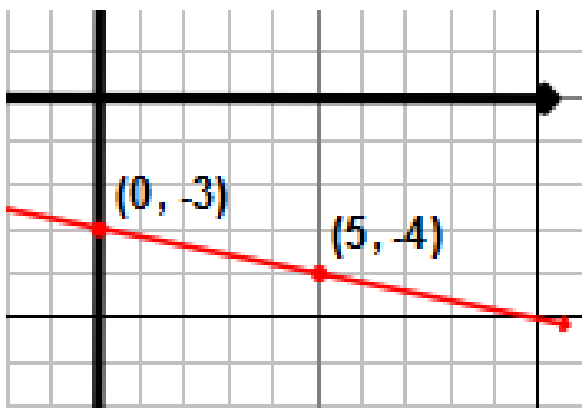


Figure 14

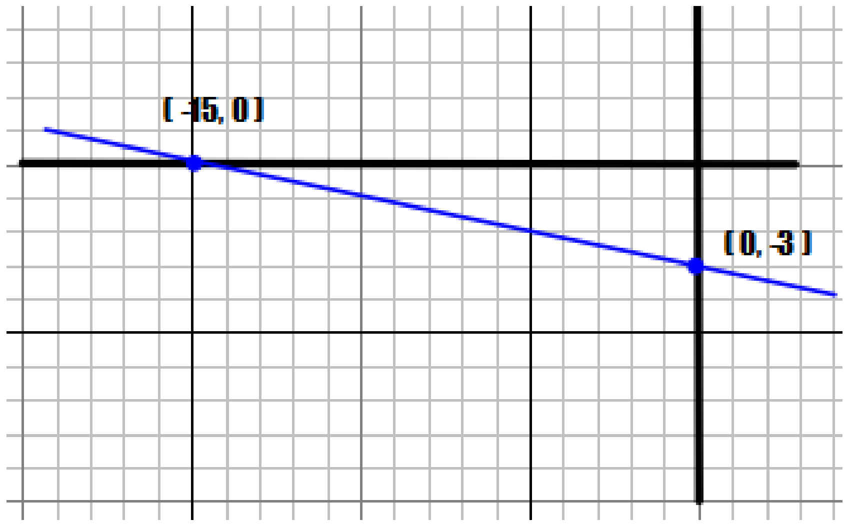
This tells how to graph the line. Start at (0, -3) and now go five to the right and down one. Clearly we could have chosen the first point to be (1, $-\frac{16}{5}$) and the second point would be (7, $-\frac{22}{5}$), but then plotting these points (and the line) would have been not the easiest task being the y-values are not integers.

The easiest way to plot a line, if given in standard form, is to find the x- and y- intercepts. For example, if x + 5y = -15 then,

P_1 = (-15, 0) is on the line (x-intercept) and

P_2 = (0, -3) is on the line (y-intercept).

Now this line can be obtained by connecting the two points with a ruler.



x + y = -1 Figure 15

When lines have the same slope, they are considered to be parallel. For example, the lines y = 3x + 4 and 6x -2y = -8 are parallel because the slope of both lines is 3. Also, $\frac{1}{2}x - \frac{1}{2}y = \frac{3}{2}$ and 3x + 3y = 9 are parallel because they both have the slope of -1.

3.4 Models, Technology, and Graphing Calculators

Below are some online graphing calculators for use on classroom interactive white boards. Some are better than others; some allow for changing of the scales of the window, some allow for plotting only points while others allow linear or trigonometric equations.

http://itools.subhashbose.com/grapher/

http://www.shodor.org/interactivate/activities/Graphit/

http://www.fooplot.com/#W3sidHlwZSI6MCwiZXEiOiJ4XjIiLCJjb2xvciI6IiMwMDAwMDAifSx7InR5cGUiOjEwMDB9XQ--

http://go.hrw.com/math/midma/gradecontent/manipulatives/GraphCalc/graphCalc.html

http://illuminations.nctm.org/ActivityDetail.aspx?ID=82

Classroom teachers should be able to make their own decision on calculator use in their classroom. Students should begin the graphing of linear functions by hand on a coordinate plane as to learn the how-to and why things look the way they do. After students grasp the concepts involved in graphing, they can then move on to learning and using a graphing calculator. Also, learning how to manipulate the graph of a line on a calculator, students know how to adjust the scales on the axes to see a proper graph, zoom-in/out, trace the graph to find out exact coordinates, and many other features. Eighth grade students should be comfortable enough with a scientific calculator to use it properly but a graphing calculator will take some teaching and practice to become proficient with it and feel comfortable in completing assigned tasks on their own.

Each online graphing calculator works differently, allowing for many different final representations. The graph below is what one online graphing calculator shows as the graph of y $= \frac{1}{x-3}$ if the x-window is set to be [-6,6] and the y-window as [-4,4].

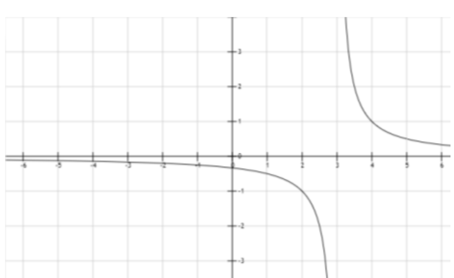
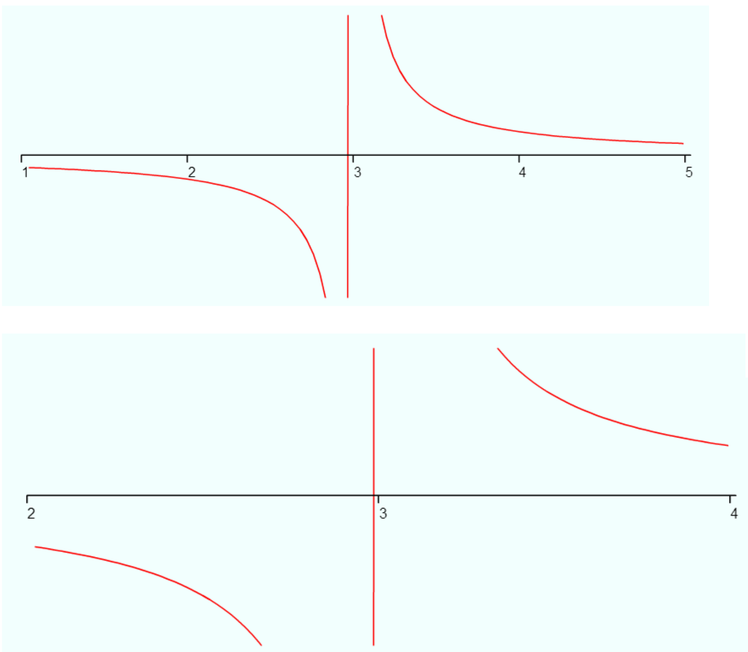


Figure 16

We need to remember that these scales are not the domain and the range. The domain of this graph cannot include 3. At the same time, the range cannot include 0.

In fact, with the restriction that the x-values are between -6 and 6 and the y-values are between -4 and 4 (i.e., the range space is [-4, 4]), the domain of the function is the union of the interval [-6, 2.75] and the interval [3.25, 6]. By simply adjusting the setting for the scale of the x- and y-axis, it can look like you are seeing different graph.



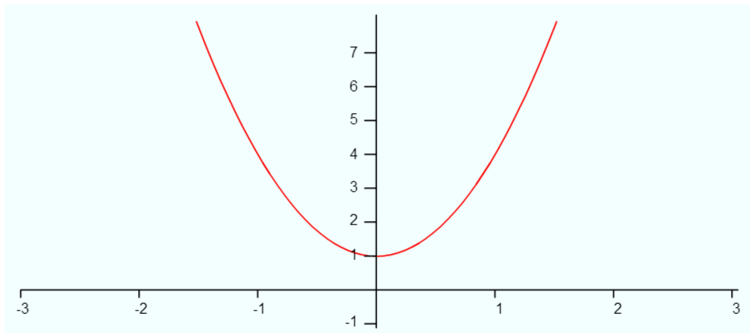
This shows that calculators are great tools for discussion and to deepen students understanding of the concept of domain and range by just adjusting the scales of the axes.

Figure 17

3.5 Nonlinear Functions

Nonlinear functions are functions whose graphs are not a subset of a straight line. Equally, a function y = *f*(x) is not linear if it cannot be written in the form Ax + By = C. Below are examples of some nonlinear functions.   We graph these function using tables and either a graphing calculator or online graphing calculator. Draw a sketch of the graph.

Example 11: $y = 3x^2 + 1$



$y = \frac{1}{4} x^3$

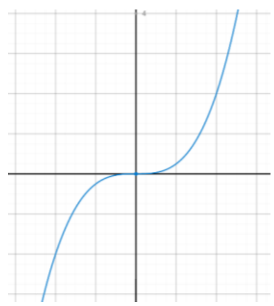


Figure 18

Exercise 7

Directions: Look at the table and use what you know to determine if it is a linear function. Be sure to explain your answer.

Table 3

X	y
1	3
2	5
3	9

Example 12**:** Look at the function y = 2x. Use the input of 2 first. For the second input, take the output from the input of 2. Continue this pattern. Is the graph a linear function? Draw a function table to complete the first five inputs and outputs.

x	Y
2	4
4	8
8	16
16	32
32	64

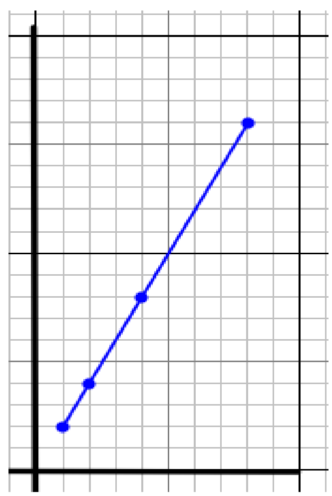


Figure 19

Solution*:* The graph is a linear function although the numbers 2, 4, 8, 16, 32, 64… grow exponentially. In contrast, let us do the same for the nonlinear function $y = 2^x$. Again we use the input of 2 first. For the second input, we take the output from the input of 2 and continue this pattern. Then we get the numbers 2,4,16, 2^{16}= 65,536, and then $2^{65,536} > 10^{19,728}$ which is a

pretty large number with more than 19,729 decimal digits (see Figure 1.5.c below). As one can see, linear functions can grow fast, but exponential functions can grow much faster.

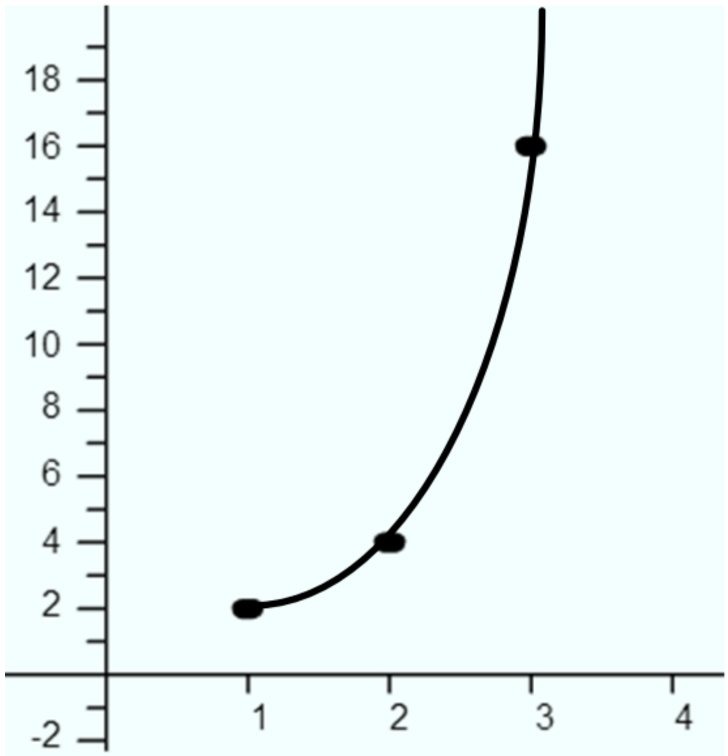
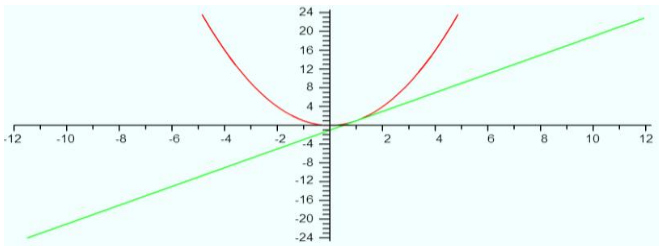


Figure 20

The Tangent Line - The word tangent is derived from a Latin word meaning “touching.” That is exactly what a tangent line does; it touches the graph of a smooth curve at a point on the curve. But is a parabola, really a curve and not a line? This seems to be a silly question, but if we take a closer look, locally, functions like $y=x^2$ appear to be straight although this is a nonlinear function (because x has a power higher than one). At any point on the parabola, you can draw a straight line touching the graph at that point - the tangent line. The curved parabola, when looked at close enough, will begin to resemble this straight line and will look just like the tangent line. The graphs are zoomed-in on x=1 to show the gradual straightening of the graph of the curve, along with the tangent line, passing though (1, 1).

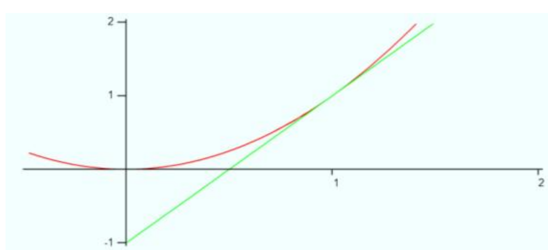
Example 15: Below is an example of how the parabola can almost appear (and at one point is) to be a straight line, similar to the linear function of the tangent line passing through (1, 1). Shown are the settings changed on the x- and y-axis to be able to see how the line can appear to be a straight line, like the tangent line. $y = x^2$ and $y = 2x - 1$



Settings: x-axis: -12 to 12, y-axis:-24 to 24

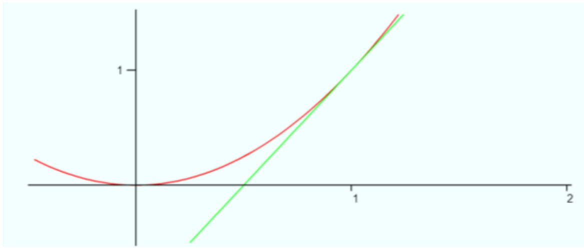
Settings: x-axis: -3 to 3, y-axis:-2 to 6

Settings: x-axis: -2 to 2, y-axis:-2 to 3

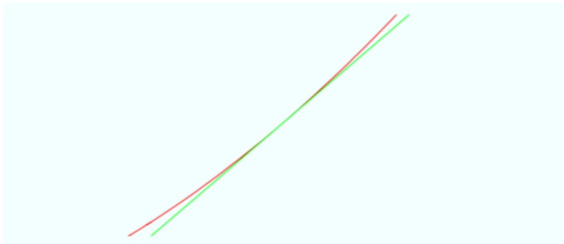


Settings: x-axis: -0.5 to 2, y-axis: -1 to 2

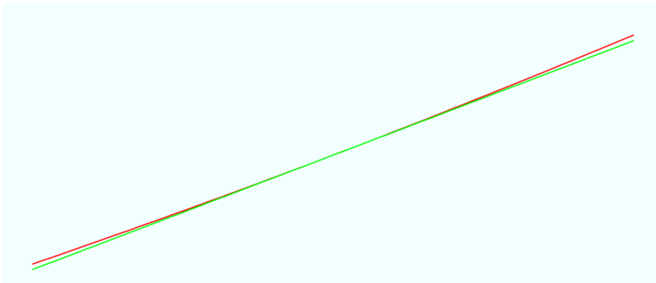
Figure 21



Settings: x-axis: -0.5 to 2, y-axis: -0.5 to 1.5



Settings: x-axis: 0.5 to 1.5, y-axis: 0.5 to 1.5



Settings: x-axis: 0.9 to 1.1, y-axis: 0.8 to 1.25

Figure 21(con't)

3.6 Arc Length

An arc is a part of the curve of a circle or, more general, any piece of a graph of smooth nonlinear functions such as higher order parabolas. As in the picture below, we can use the lengths of the line segments close to the curve to approximate the value of pi, which is the ratio of a circles circumference to its diameter or the arc length of a semicircle of radius 1.

In the first figure, we bisected the two right angles formed to create four 45-45-90 triangles. Each leg of the new triangles is denoted by “a.” The bisector line, denoted by the dashed line, has a length of one. To find the length of “a”, we use the Pythagorean Theorem to solve.

$$a^2 + b^2 = c^2$$

$$a^2 + a^2 = 1^2$$

$$2a^2 = 1$$

$$a^2 = \frac{1}{2}$$

$$a = \sqrt{\frac{1}{2}} \text{ or } \frac{\sqrt{2}}{2}$$

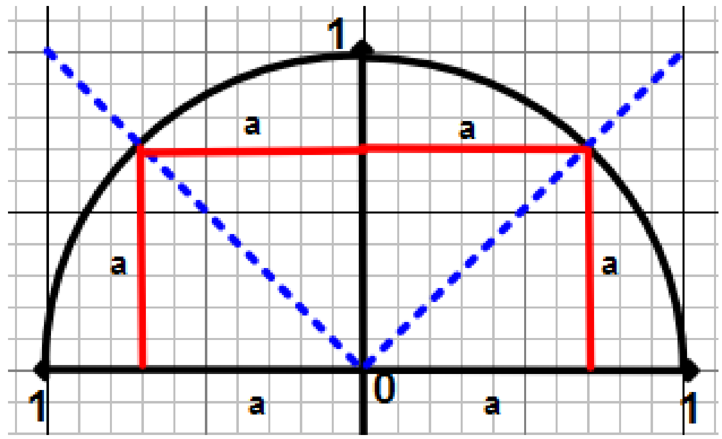


Figure 22

We can use the length of “a” to approximate the length of the semicircle, or arc. The process of approximating the arc length using lines is called linear approximation. If we take the length of “a”, multiply if by 4, we get an approximation of $2\sqrt{2}$ or about 2.83. It is clear from the picture that this is not a good approximation of pi, but, at least we know now that pi is between 2.83 and 4 (that is, the arc length is clearly less than 4 which is the distance traveled going from (1,0) to (1,1) to (-1,1) to (-1,0)).

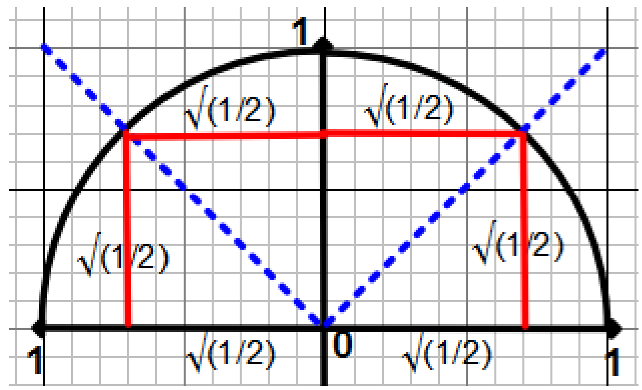


Figure 23

We can approximate it even closer by making more and more lines that are closer and closer to the actual curve. If we create a line connecting the midpoint of the part of the semicircle in located in just quadrant I to (1, 0), we can find the length of the line created, or the new hypotenuse, "b".

$$a^2 + b^2 = c^2$$

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(1 - \frac{\sqrt{2}}{2}\right)^2 = b^2$$

$$\frac{2}{4} + \frac{6-4\sqrt{2}}{4} = b^2$$

$$\sqrt{\frac{2}{4} + \frac{6-4\sqrt{2}}{4}} = b$$

$$0.765367….. = b$$

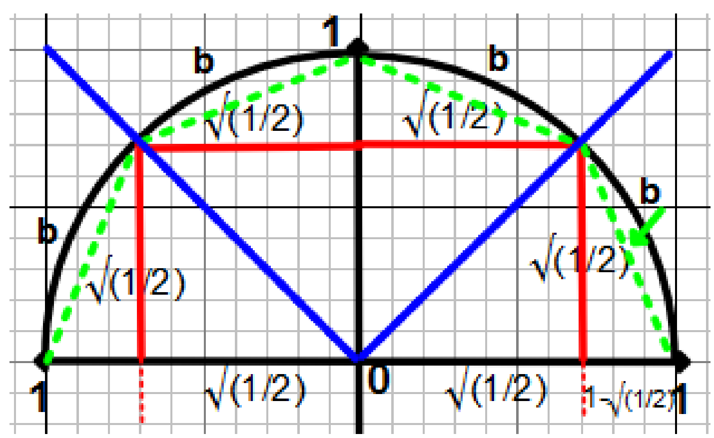


Figure 24

Thus, the approximate length of the semicircle arc is 4b = 3.06147…, which is already much closer to what we know as the standard approximation of pi, namely 3.14159…….

To get an even closer linear approximation of the arc length of a unit semicircle, we can divide the original two 90° angles into each three 30°-60°-90° triangles (a total of 6 triangles).

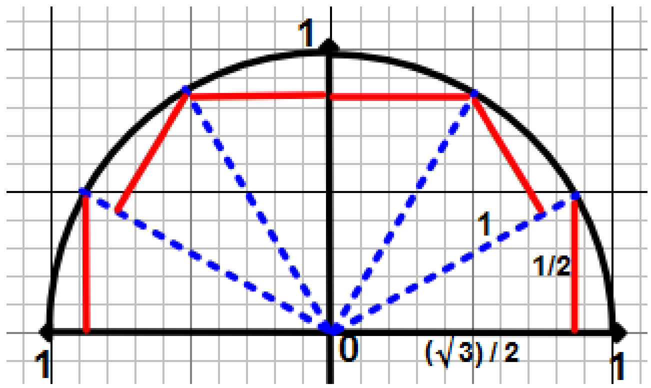


Figure 25

In this 30°-60°-90° triangle, the length of the hypotenuse is 1 which makes the leg across the 30° angle to be $\frac{1}{2}$. To approximate the length of the arc, multiply $\frac{1}{2}$ by 6 to get a length of 3. This is a smaller approximation as before but we can create an even smaller triangle to be able to help with approximating the length of the arc.

In the figure below, we create a smaller triangle. By using the Pythagorean Theorem, we find the length of "d" as $\sqrt{2-\sqrt{3}}$ or about 0.517638….. When we use this estimate to approximate the length of the semicircle arc, we get a value of about 3.10583…. .

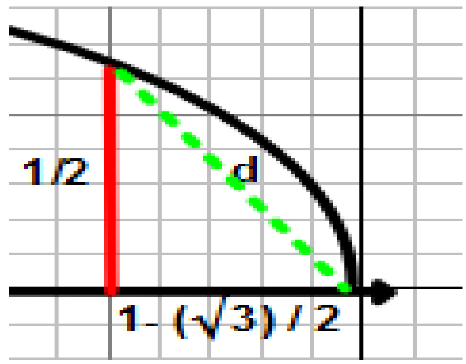


Figure 26

At the middle school level, estimating the value of pi more accurately using angles is challenging because students do not know yet how to find the sin and cos of arbitrary angles. However, students know already the Pythagorean theorem and, therefore, they know that all points (x, y) on the unit circle satisfy the equation

$$x^2 + y^2 = 1.$$

To get a better estimate we consider the following points on the upper quarter circle

$$(0,1), \left(\frac{1}{5}, \frac{1}{5}\sqrt{24}\right), \left(\frac{2}{5}, \frac{1}{5}\sqrt{21}\right), \left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{4}{5}, \frac{3}{5}\right), and\ (1,0).$$

Computing the length of the line segment between (0,1) and $\left(\frac{1}{5}, \frac{1}{5}\sqrt{24}\right)$ gives

$$\sqrt{\frac{1}{25} + \left(1 - \frac{1}{5}\sqrt{24}\right)^2} = 0.201018\ldots\ldots .$$

Computing the length of the line segment between $\left(\frac{1}{5}, \frac{1}{5}\sqrt{24}\right)$ *and* $\left(\frac{2}{5}, \frac{1}{5}\sqrt{21}\right)$ gives

$$\sqrt{\frac{1}{25} + \left(\frac{1}{5}\sqrt{24} - \frac{1}{5}\sqrt{2}\right)^2} = .209772\ldots .$$

Computing the length of the line segment between $\left(\frac{2}{5}, \frac{1}{5}\sqrt{21}\right)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$ gives

$$\sqrt{\frac{1}{25} + \left(\frac{1}{5}\sqrt{21} - \frac{4}{5}\right)^2} = .231464\ldots .$$

Computing the length of the line segment between $\left(\frac{3}{5}, \frac{4}{5}\right)$ and $\left(\frac{4}{5}, \frac{3}{5}\right)$ gives

$$\frac{1}{5}\sqrt{2} = 0.282843 \ldots ,$$

And finally, computing the length of the line segment between $\left(\frac{4}{5}, \frac{3}{5}\right)$ and (1, 0) gives

$$\frac{1}{5}\sqrt{10} = 0.632456 \ldots .$$

Thus, the arc length of the upper right quarter circle can be approximated by the sum s of the lengths of these line segments which is s = 1.55755….. Thus pi, the arc length of the semicircle, can be approximated by 2s = 3.11511….. .

More generally, a nice project for advanced students is to derive the following. If one divides the interval [0, 1] in n pieces of equal length then one obtains the points i/n where I ranges from 0 to n. This yields the n+1 points

$$(\frac{i}{n}, \sqrt{1 - \frac{i^2}{n^2}})$$

on the upper right quarter circle for i between 0 and n. Computing the sum s of the distances between these points leads to the formula that s is the sum of the expressions

$$\sqrt{\frac{1}{n^2} + [\sqrt{1 - \frac{i^2}{n^2}} - \sqrt{1 - \frac{1}{n^2}(1+i)^2}]^2},$$

where i ranges between 0 and n-1. Thus 2s approximates pi – and the following are the results for different vales of n:

Table 4

n	4	5	6	7	8	9	10	11	12
2s	3.1045	3.11511	3.12147	3.12564	3.12854	3.13066	3.13226	3.13351	3.1345

Table 5

n	33	50	100	120	150	200	300	500	1000
2s	3.14004	3.14076	3.1413	3.14137	3.14143	3.14149	3.14157	3.14157	3.14158

Similarly, if we take $y = x^2$, we can find the approximate length of the arc of the parabola from x = 0 to x = 4.

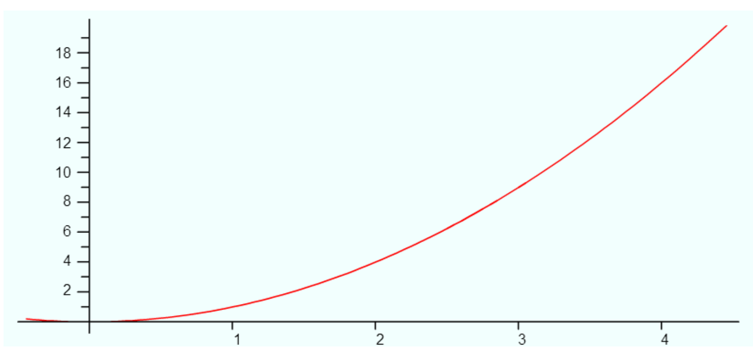


Figure 27

We can start by taking the distance from the starting point (0, 0) and the ending point (4, 16). The distance is $\sqrt{16^2 + 4^2}$ *or about* 16.5. But, if we break it down smaller, using the integer points we know, we can get a more accurate measure.

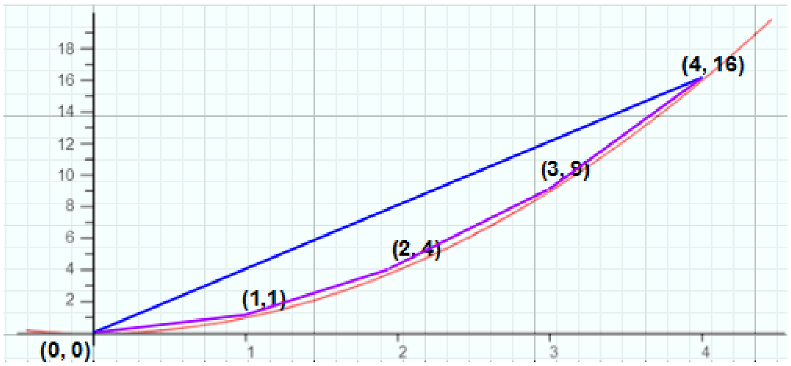


Figure 28

Starting with (0, 0) and (1, 1), we can find the distance to be $\sqrt{2}$ or about 1.4. From (1, 1) to (2, 4), we find the distance to be $\sqrt{10}$ or about 3.2. From (2, 4) to (3, 9) we can find the distance to be $\sqrt{26}$ or about 5.1 and the distance from (3, 9) to (4, 16) to be $\sqrt{50}$ or about 7.1. The approximate length of the arc of the parabola from (0, 0) to (4, 16) is then approximately $\sqrt{2} + \sqrt{10} + \sqrt{26} + \sqrt{50}$, which is about 16.7466…. .

As above, the estimate can be improved if we divide the interval [0,4] in n pieces of length 1/n and consider the n+1 points

$$(\frac{4i}{n}, \frac{16i^2}{n^2})$$

that lie on the parabola for i between 0 and n. Then the distance between one point and the next is

$$\sqrt{(\frac{16(1+i)^2}{n^2} - \frac{16i^2}{n^2})^2 + \frac{16}{n^2}}$$

If one computes the sum s of these distances for i between 0 and n-1 one obtains the following table approximating the arc length of the parabola for x between 0 and 4:

Table 6

n	4	5	8	16	32	64	128	256	512
s	16.7466	16.7691	16.7981	16.8135	16.8173	16.8183	16.8186	16.8186	16.8186

3.7 Inverse Functions

In calculus an inverse function is defined as "Let *f* be a one-to-one function with domain *A* and range *B*. Then its inverse function f^{-1} has domain *B* and range *A* and is defined by $f^{-1}(y) = x \leftrightarrow f(x) = y$." (Stewart 65) For a middle school student, this is a pretty confusing definition of what an inverse function is. So to simplify it for middle school, we define the inverse of a linear function f as the function g that is obtained by reading a table for f from right to left.

Table 7

0	3
1	4
2	5
3	8
4	7

As an example consider the function f(x) = x+3 and its function table.

Then the inverse rule g going from right to left is g(x) = x-3. That is, if we take a number x and apply the rule f then we get a new number, namely y = x+3. Now if we take this number and plug into the rule g then we must subtract 3 from y, and thus we end up with x, the number we started with. That is, if we start with a number and first apply f and then g then we end up where we started.

In the graph below (Figure 1.6.a), what do you notice about the two graphs of the lines y = x + 3 and its inverse y = x - 3, in relation to y = x (the dashed line)? The students will notice that the

lines, y = x + 3 and y = x – 3, are parallel lines that are equidistant from y = x, meaning that y = x is a line of reflection for y = x + 3 and y = x – 3.

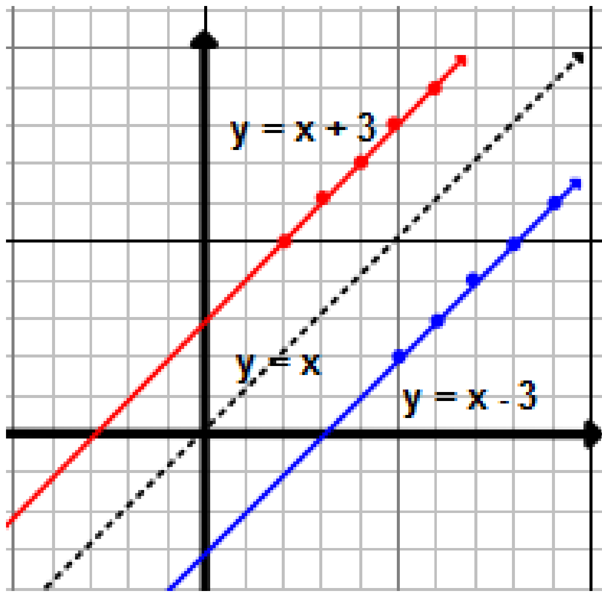


Figure 29

Exercise 8

Directions: (a) Using $f(x) = \frac{1}{2}x + 2$, create a function table. (b) Create a second function table, switching the x- and y-values. (c) Write a function rule for the second table. (d) Graph the two function tables on a coordinate plane. (e) Graph the line y = x and explain its significance.

After completing the above tables and graphs, there is another way to find an inverse function that may be a bit simpler. This time, we are going to take the same function used in the above example, y = x + 3. In this function, switch the x and y in the function and rewrite it as x = y + 3. When you solve the equation for y, you are left with y = x – 3. In this instance, you get the same result without making the chart, drawing the graphs, and writing the function rule. If we take the same function rule as in the exercise above, $y = \frac{1}{2}x + 2$, and switch the x- and y-

values, we are left with x = $\frac{1}{2}$y + 2. If we are to solve for y, we get y = 2x – 4. If we create a graph to represent this inverse function, it looks like this:

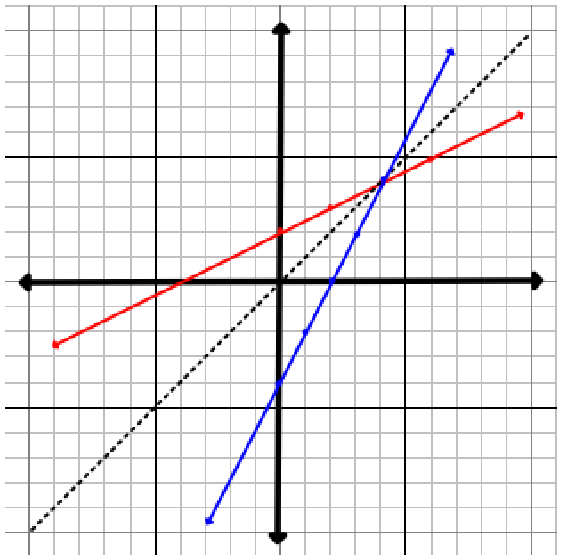


Figure 30

<u>Example 13:</u> The original function is y = 3x + 4. To find its inverse function, we rewrite the function as x = 3y + 4. We need to be able to graph this function, so it needs to be written in a form that makes it able to be easily graphed, y = $\frac{(x-4)}{3}$ or y = $\frac{1}{3}$ x – $\frac{4}{3}$, with a y-intercept of -$\frac{4}{3}$ and a slope of $\frac{1}{3}$. Both the original and inverse functions are graphed on the same graph below. Notice that the point of intersection of both graphs is (-2, -2), which is on the line y = x. If you follow the line of y = x, each of the graphed functions are the same distance away from it, meaning that the line of y = x is a line of reflection for the function and its inverse.

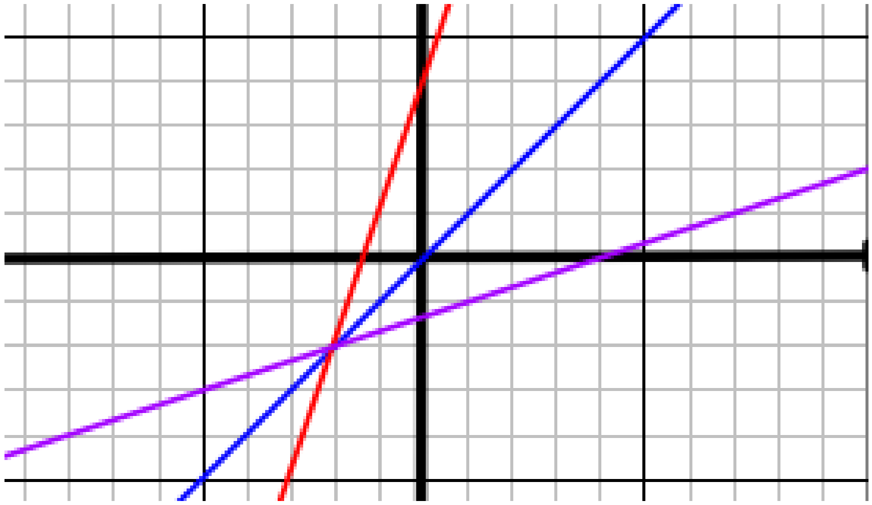


Figure 31

3.8 Optimization

In order to do optimization in eighth grade, the terms maximum and minimum must be discussed first. One way to do this is to create a parabola by graphing a quadratic equation, like $y = 2x^2 + 8$. Students can graph this equation by making a table of points and graphing to get the visual representation of the graph.

x	Y
-2	16
-1	10
0	8
1	10
2	16

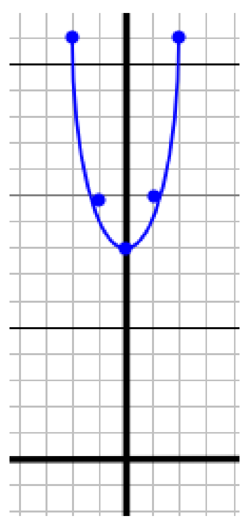
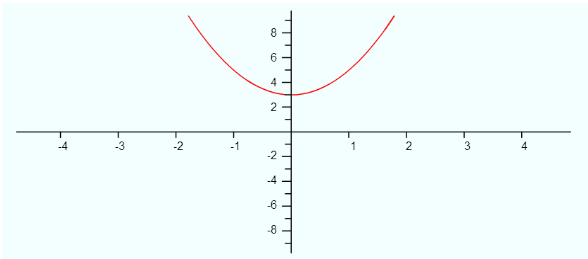


Figure 32

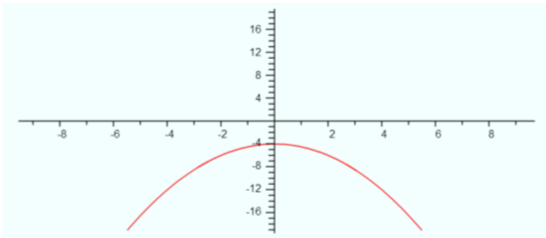
Looking at the graph above, we say that the minimum of the above graph is at the point (0, 8) and that the minimum y-value is 8. In the example of $y = 2x^2 + 3$, the parabola has a minimum y-value of 3 at x = 0 because the y-value will never be less than 3 since the term $2x^2$ is always positive. The range values of this equation are $y \geq 3$, meaning the y-values of this particular equation will never be less than 3.



$y = 2x^2 + 3$

Figure 33

In the example of $y = -1/2\ x^2 – 4$, the parabola opens facing down and has a maximum y-value of -4 at x = 0 because the y-value will never be greater than -4 since the term $-1/2\ x^2$ is always negative. The range values of this equation are $y \leq -4$ because the y-value will never be greater than -4.



$y = -1/2\ x^2 - 4$

Figure 34

Exercise 9

Directions: Determine if each equation has a maximum or minimum value then state what the maximum or minimum value is. Also, state the range of the functions.

1. $y = 3x^2$
2. $y = -2x^2 + 4$
3. $y = 3x+2$ with domain [-2, 6].
4. $y=3x+2$ with domain $x >= -2$

Optimization involves finding the optimal values to achieve certain tasks, be it the best way to maximize profits or minimize costs, or to maximizing area with a given perimeter or minimizing perimeter with a given area. In calculus, the process of optimization is done by finding maxima or minima using the first (or second) derivative test and/or looking at graphs on graphing calculators. In middle school math, students can solve optimization problems by utilizing the skills they know. Students should be familiar enough with being able to create a table to solve these problems. In optimization, student can easily take a problem and get started by creating a table of values to get a sense what the optimal value might be.

<u>Example 14:</u> A farmer has 1,200 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river. What are the dimensions of the field that has the largest area, assuming that length and width are measured in whole feet only?

For students to successfully solve this problem, they first need to be sure they know exactly what is being asked by the problem. The best way to get a clear picture of this is to draw

a picture of what the problem is saying (it doesn't have to be perfect, just a sketch). Label what you know from the problem then create a table to fill in with values.

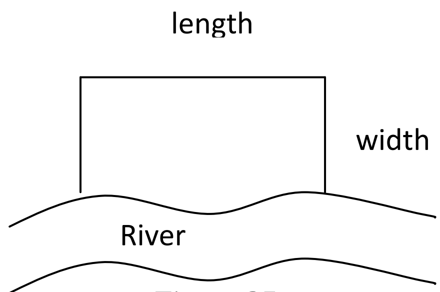


Figure 35

In this problem, start with some number that could represent the length. Subtract that number from the perimeter. Divide the resulting number by two to represent the two sides that are considered the width. Fill in the chart for the length and width. Find the perimeter (it should be 1,200 ft – we got that value from the problem). Find the area. Continue finding values that could represent the length and width. You will find that the areas will go up, and then go back down. You are looking for the largest area possible.

Table 8

Length	Width	Perimeter (length + 2(width))	Area (length ● width)
1000 ft	100 ft	1,200 ft	100,000 ft^2
900 ft	150 ft	1,200 ft	135,000 ft^2
800 ft	200 ft	1,200 ft	160,000 ft^2
700 ft	250 ft	1,200 ft	175,000 ft^2
650 ft	275 ft	1,200 ft	178,750 ft^2
625 ft	287.5 ft	1,200 ft	179,687.5 ft^2

Table 8 (con't)

Length	Width	Perimeter (length + 2(width))	Area (length ● width)
600 ft	300 ft	1200 ft	180,000 ft^2
575 ft	315 ft	1200 ft	179,687.5 ft^2
550 ft	325 ft	1200 ft	178,750 ft^2
500 ft	350 ft	1200 ft	175,000 ft^2
400 ft	400 ft	1200 ft	160,000 ft^2

In this problem, it seems that the largest area is 180,000 ft^2 and it seems that the optimal rectangle has a length of 600 ft and a width of 300 ft. Now one can explain students how to translate the problem into algebra: we have to find the maximum of A = lw given that l + 2w = 1200 and that l and w are positive numbers. Since l = 1200 - 2w we find that (a) w can be at most 600 since l cannot be negative and (b) that A = (1200 - 2w)w = 2(600 - w)w for values of w between 0 and 600. By graphing the parabola A for w between 0 and 600, students see that A has a maximum when w = 300 and l = 1200 - 2(300) = 600.

At this point, in order to adhere to the Mathematical Practices, students should be guided through a discussion about the problem. One way to get started is to ask the students to look at the numbers: for the given perimeter p = 1200, the optimal length was l = 600 and w = 300. Is it an accident that the optimal length is half of the perimeter? To find out, students are then asked to

repeat the problem with a new perimeter, let us say p = 100. Again, after doing the work as above, students will find that in this case the optimal length is l = 50 (and then w = 25). At this point students can then be asked to discuss the general case. That is, they have to find the maximum of A = lw given that l + 2w = p, where p is a given perimeter and where the length l and the width w are positive numbers. Since l = p - 2w we find that (a) w can be at most p/2 since l cannot be negative and (b) that

$$A = (p - 2w)w = -2w^2 + pw = -2\left(w - \frac{P}{4}\right)^2 + p^2/8.$$

Since the term $-2\left(w - \frac{P}{4}\right)^2$ is always negative, the maximum area is A = $p^2/8$ and is attained when the with is w = p/4 and the length is l = p – 2(p/4) = p/2. Here the important learning objective is for students to see that calculators are helpful to get a feeling for a problem, but that at the end one has to do algebra in order to get the final result.

Again, following the Principles of Mathematical Practice, one can ask the students to critically examine the result and ask if there are other possible solutions to the problem. After all, why does it have to be a rectangular field that the farmer wants to fence in? For example, if the farmer would fence in a semicircular field with its base at the river, then the circumference (perimeter) of the field is p = 1200 = π r or r =1200/π. This gives the farmer a fenced area of $A = \pi r^2 = \pi\left(\frac{1200}{\pi}\right)^2 = \frac{1200^2}{\pi} = 458{,}599$ square feet, more than double the area that he can fence in when sticking to the rectangular design! In general, given a perimeter p, the rectangular design leads to an optimal area of $A = p^2/8$ whereas the semicircular design leads to an optimal area of $A = p^2/\pi$ which is $\frac{8}{\pi} = 2.54$ ... times larger than the rectangular one. This can now lead to an open-ended discussion/project on finding the shape with the maximal area attached to a straight

line with a given perimeter p. To get started, students can study triangular fence designs with one side at the river (or one edge away from the river), then rectangular designs (two edges away from the river – this is what we began to cover above), and then other straight lines designs with 3, 4, 5 ,….. edges away from the river.

Other problems to be discussed in this context are optimal area problems with a given perimeter p without any design restrictions (like one side having to be along a straight river) or with even more complicated design restrictions (like the river not being a straight line). These are all open ended problems that are ideal for extended projects (like middle school science fair projects).

Another problem along these lines is to find a rectangle of area $A = lw = 100$ with the smallest perimeter $p = 2l + 2w = 2(l + w)$. If we allow only integer side lengths, this can be restated as follows. Find two positive integers l and w whose product is 100 and whose sum is a minimum.

As allows, a grade appropriate simple approach is via a table like the one below, where a student has to make sure to include all possible cases.

Solution:

Table 9

Factor	Factor	Product	Sum
1	100	100	101
2	50	100	52
4	25	100	29
5	20	100	25
10	10	100	20

The two positive integers whose product is 100 and has a minimum sum is 10 and 10 because it has a sum of only 20.

Now we do the same problem again, this time with the given area to be 90.

Table 10

Factor	Factor	Product	Sum
1	90	90	91
2	45	90	47
3	30	90	33
5	18	90	23
6	15	90	21
9	10	90	19

The two positive integers whose product is 90 and has a minimum sum is l = 9 and w = 10 (or l = 10 and w = 9) because they have a sum of only 19.

A mathematically more inclined eighth grader might think of the following. Given that A = lw is given, it follows that we have to find an integer w > 0 such that

$$p = l + w = \frac{A}{w} + w$$

is at a minimum. For a given value of A this can be done with a graphing calculator by just graphing the function f(x) = A/x + x for x > 0. In doing so for different values of A the guess

would be that the optimal solutions l and w are always factors of A that are “closest to” $\sqrt{A}$. Unfortunately, without calculus or without a clever algebra formula that is equivalent to $\frac{\mathrm{A}}{\mathrm{w}} + \mathrm{w}$ and shows that $\frac{\mathrm{A}}{\mathrm{w}} + \mathrm{w}$ is at a minimum if $= \sqrt{A}$, this problem cannot be solved on an eighth grade level.

3.9 Area

In previous years, formulas for perimeter, area, surface area, and volume were learned and explored. We can take what we learned about perimeter and area and apply it to the coordinate plane. Using the coordinate plane, we can find the area of a figured created by lines or line segments..

For example, we can graph y = m x + b on the coordinate plane. We want to find the area A(x) under the line we graphed from 0 to x.

Graph the equation.

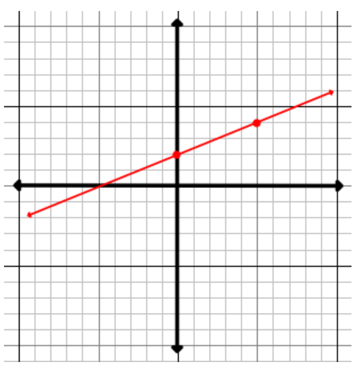


Figure 36

Now look at just quadrant one

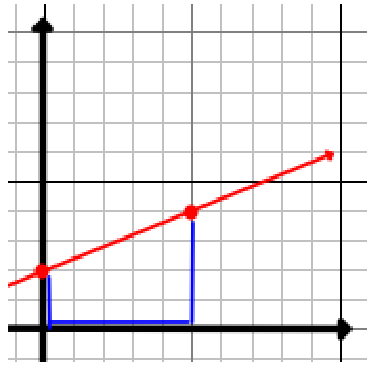


Figure 37

Let's focus on just the graph from 0 to x. Can you divide the area between the line you graphed between 0 and x into two different figures?

Find the area of the triangle.

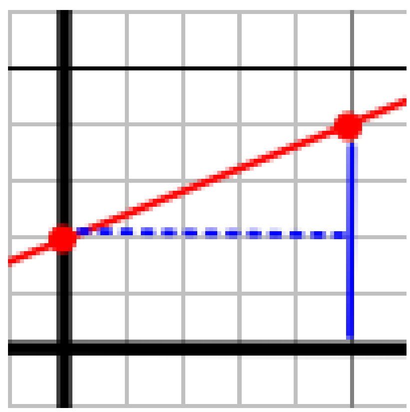


Figure 38

The height of the triangle is mx and the base is x. The area is $\frac{mx^2}{2}$ units^2.

Find the area of the rectangle.

The length is x and the height is m. The area is mx units^2.

What is the area A(x) under the graphed line from 0 to x? The answer is as follows:

Table 11

f(x)	A(x)
$b + mx$	$bx + \frac{m}{2}x^2$

Let's look at another equation, x + 3y = 12. Graph the equation on the coordinate plane.

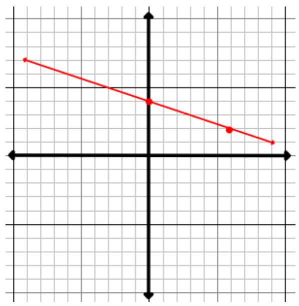


Figure 39

This time, we want to find the area under the line we graphed but between x= 3 and x = 6.

Divide the figure into a triangle and a rectangle and find their areas.

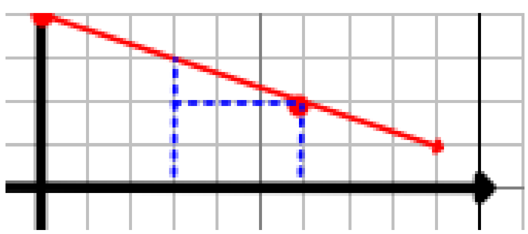


Figure 40

The triangle has a height of 1 and a base of 3. The area is 1.5 units^2. The rectangle has a length of 3 and a height of 2. The area is 6 units^2. The area under the graphed line between x = 3 and x = 6 is 7.5 units^2.

Another way to solve this is to observe that y = 4 - $\frac{x}{3}$. Thus the area function for this line is A(x) = 4x- $\frac{x^2}{6}$. Moreover, the area under the curve between 3 and 6 is the same as the area

under the line from 0 to 6 minus the area under the line from 0 to 3. That is, the area is A = A(6) – A(3) = 18 - (12-3/2) = 7.5.

3.10 Arithmetic Sequences and Series

Beginning in kindergarten, students are taught sequences starting with colored bears (or similar), starting with number patterns and pattern recognition first. Progression of sequences moves from patterns to more concrete numerical data throughout middle school. Students will begin to understand that sequences and series involve carrying out a process (virtually) forever. There are two kinds of sequences that middle school students study, geometric sequences and arithmetic sequences. In a geometric sequence, a common ratio is multiplied to a term to get the next term. In an arithmetic sequence, a common difference is added to a term to get the next term. When using calculus concepts to incorporating sequences into middle school math, we will focus on arithmetic sequences only. Arithmetic sequences form a linear pattern when graphed on the coordinate plane; in fact an arithmetic sequence is nothing but the range of a linear function defined on the natural numbers 0, 1, 2, 3, ……. .

To summarize, if f(x) = dx + b is a linear function, then the chain of numbers $a_n =$ f(n) for n = 0, 1, 2, 3, ….., n, …… is given by

b, d + b, 2d + b, 3d + b, ,,,,,,, , dn +b, ………..

and is called an arithmetic sequence with common difference d and starting value $a_0 =$ b.

Students should be able to easily recognize a common difference in an arithmetic sequence as a slope in a linear relation. We start with a term then add a fixed amount (or common difference) to get the next term. If we take the sequence 1, 4, 7, 10, 13, 16, 19, …, to find the common

difference, we subtract two consecutive terms, 4 – 1 = 3. The number 3 is the common difference d and the number 1 is the starting value. Thus, the explicit formula for this sequence is generated by f(x) = 3x + 1 and we have that

$$a_n = a_0 + dn = 1+3n,$$

where a_0 is the first term of the sequence, d is the common difference, and n is the number of the term of the sequence we have to find. Using the formula, if we know that a sequence begins with 12, it has a common difference of -4, and we are looking for the 10^{th} term in the sequence, we can find it by substituting what we know into the formula:

$$a_n = a_0 + dn = 12 - 4n$$

$$a_{10} = 12 - 4(10) = -28$$

One of the important facts about arithmetic sequences is that their terms can be easily added.

To see this let us assume that we have to find the sum S_n of the first n+1 terms $a_n = b + dn$

$$S_n = a_0 + a_1 + a_2 + \cdots \ldots\ldots\ldots\ldots. a_n = b + (b+d) + (b+2d) + \ldots\ldots (b+nd).$$

After rearranging and collecting like terms we see that

$$S_n = (n+1)b + (1 + 2 + 3 + \cdots + n)d.$$

To find the sum S = 1 + 2 + 3 + …. + n, one uses the wonderful trick that

$$S = 1 + 2 + \cdots + (n-1) + n \text{ and } S = n + (n-1) + \cdots + 2 + 1$$

so that when the two equations are written below each other and added term by term one obtains

$$2S = (n+1) + (n+1) + \cdots + (n+1) + (n+1).$$

This shows that 2S = n(n+1) or

$$S = \frac{n(n+1)}{2}.$$

This shows that the sum S_n of the arithmetic sequence is given by

$$S_n = (n+1)b + (1+2+3+\cdots+n)d = (n+1)b + \frac{n(n+1)}{2}d = \frac{(n+1)}{2}[b + b + nd]$$

$$\text{or} \quad S_n = \frac{(n+1)(a_0 + a_n)}{2}$$

where S_n is the sum of n+1 terms $a_0 + a_1 + \cdots + a_n$. In this summation formula, a_n can be found by using the previous formula, $a_n = a_0 + nd$. To incorporate the two formulas, the series formula could be rewritten as:

$$S_n = \frac{(n+1)\,[a_0 + (a_0 + \text{nd})]}{2}$$

Let's take the formula above to find the sum of the first 10 terms in a sequence that begins with $a_0 = 7$ and has a common difference of 4. That is we have to find $a_0 + a_1 + \cdots + a_9$ = 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 + 43. By the previous formula

$$S_9 = \frac{10\,[\,7+(7+(9)4)]}{2} = \frac{10\,[\,7+43]}{2} = \frac{10\,(50)}{2} = 250.$$

Using this formula is certainly easier than using a calculator, especially if one has to add up the first 100 terms in the arithmetic sequence which is

$$S_{99} = 7 + 11 + \cdots \ldots\ldots + (7 + 99 * 4) = \frac{1000\,[\,7 + (7 + (99)4)]}{2} = 20{,}500,$$

or the 1000 terms in the arithmetic sequence which is

$$S_{999} = 7 + 11 + \cdots \ldots\ldots + (7 + 999 * 4) = \frac{1000\,[\,7+(7+(999)4)]}{2} = 2{,}050{,}000.$$

Here we could ask students if they see a pattern developing. But this is an entirely new story.

Chapter 4: Reflections

Throughout this thesis journey, I have become more aware of what I need to be reinforcing with my students. I have learned the importance of using and understanding correct mathematical vocabulary and the correct definitions – not just taking what the textbook says. I do plan to take this into my classroom and implement what I can in the time allotted. The importance of having my students ready for high school weighs more heavily on me now more than ever. Students need to be prepared for high school. The rigor, focus, and deeper understanding of mathematical concepts that I teach to them have to be there for them to succeed. If it is not I teaching this to them, then who will? I have to be the one that stresses to my students the importance of being the best math students they can be now, while it is still the "easy" math. It only will get more difficult after this. At least, I know that I can help make the transition into calculus a little less painful by introducing some concepts to them using in a way they can understand.

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Vita

Randie Barbera Bailey was born in Houma, Louisiana to Sheila Lapeyrouse and Conrad Barbera. She has two brothers, four nieces, and one nephew. She received her Bachelor of Arts degree in the fall of 2001 from Southeastern Louisiana University in Elementary Education. She currently resides in LaPlace, Louisiana with her husband where she teaches middle school mathematics at John L. Ory Communication Arts Magnet School. She will complete her Masters of Natural Sciences degree at Louisiana State University as a member of the Louisiana Math and Science Teacher Institute in August 2013. She hopes to one day be able to work with teachers to improve their teaching methods and instruction.