11-13-2017

Electromagnetic Resonant Scattering in Layered Media with Fabrication Errors

Emily Anne McHenry
Louisiana State University and Agricultural and Mechanical College, emily.mchenry15@gmail.com

ELECTROMAGNETIC RESONANT SCATTERING IN LAYERED MEDIA WITH FABRICATION ERRORS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Emily Anne McHenry
B.S., Xavier University, 2010
M.S., Louisiana State University, 2011
December 2017

Acknowledgments

I dedicate this dissertation to my parents, Mark and Michelle McHenry, for the many sacrifices they made to support my education; my siblings Colleen, Molly, and Jonathan; and my undergraduate advisor Dr. Bernd Rossa, who first encouraged me to major in mathematics at Xavier University in Cincinnati, Ohio. I would like to extend a special thanks to Molly Ritchie and Michaela Stone for their personal and professional support.

This dissertation was completed under the advisement of Dr. Stephen P. Shipman; and the research contained within was supported by NSF grant DMS-1411393.

Table of Contents

Abstract

In certain layered electromagnetic media, one can construct a waveguide that supports a harmonic electromagnetic field at a frequency that is embedded in the continuous spectrum. When the structure is perturbed, this embedded eigenvalue moves into the complex plane and becomes a complex resonance frequency. The real and imaginary parts of this complex frequency have physical meaning. They lie behind anomalous scattering behaviors known collectively as Fano resonance, and people are interested in tuning them to specific values in optical devices. The mathematics involves spectral theory and analytic perturbation theory and is well understood [16], at least on a theoretical level for deterministic (fixed coefficient) media.

This dissertation is a study of how random fabrication errors in the waveguide structure affect this kind of resonance. The governing equations are the free harmonic Maxwell equations of electromagnetics, which reduce to a system of ODEs in layered media. The material coefficients (dielectric permittivity $\epsilon(z)$ and magnetic permeability $\mu(z)$) are considered to be random variables depending only on the z-variable, with small variance σ, and we are interested in how the real and imaginary parts of the complex resonance $\omega = \hat{\omega} + i\check{\omega}$ behave as random variables depending on these coefficients.

The first main theorem of this thesis states that, if $\epsilon(z)$ and $\mu(z)$ are stationary random variables with respect to the variable z, then the variances of $\hat{\omega}$ and $\check{\omega}$ can be computed by considering only events in which ϵ and μ are random variables that are constant in z.

The significance of this theorem is that it allows one to compute the mean and variance of ω, to leading order in σ, by means of (deterministic) sensitivity analysis of ω as a function of ϵ and μ. One can consider ω to be a complex-analytic function of ϵ (and μ) if $\epsilon = \hat{\epsilon} + i\check{\epsilon}$, where both parts are Hermitian matrices. The Cauchy-Riemann equations, together with known properties of complex resonances allow one to make interesting and useful conclusions about how the frequency of a resonance and the amount of radiation losses depend on the real and imaginary (lossy) parts of the material coefficients..

Chapter 1
Introduction to Maxwell's Equations

Light propagation through an electromagnetic medium is governed by a system of first order partial differential equations (PDEs) known collectively as Maxwell's equations. Absent of sources, the Maxwell equations (in Gaussian units) are given by

$$\begin{aligned}
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0 \\
\nabla \cdot \mathbf{D}(\mathbf{r}, t) &= 0 \\
\nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \\
\nabla \times \mathbf{H}(\mathbf{r}, t) &= \frac{1}{c}\frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t},
\end{aligned}$$

where $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$ and $t \in \mathbb{R}$ denote the spatial and temporal variables, respectively. The full electromagnetic field is composed of solutions

$$(\mathbf{E}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t), \mathbf{D}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)) \quad (\mathbf{r} \in \mathbb{R}^3, t \in \mathbb{R})$$

of the PDEs above, each a vector in 3-space.

In a linear anisotropic medium, time-harmonic fields ($\omega \neq 0$)

$$(\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}), \mathbf{D}(\mathbf{r}), \mathbf{B}(\mathbf{r}))e^{-i\omega t} \quad (\mathbf{r} \in \mathbb{R}^3)$$

that solve the source-free Maxwell PDE system satisfy the matrix equation

$$\begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = -\frac{i\omega}{c} \begin{bmatrix} \mathbf{D} \\ \mathbf{B} \end{bmatrix},$$

where the electric field $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{H} = \mathbf{H}(\mathbf{r}, t)$ are related to the electric displacement $\mathbf{D} = \mathbf{D}(\mathbf{r}, t)$ and magnetic flux density $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ through rank-2

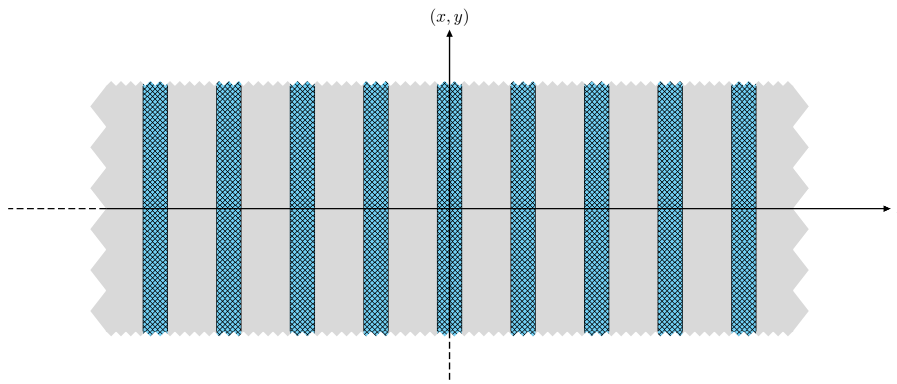

Figure 1.1: Periodic layered medium.

tensors ϵ and μ (called the material coefficients) by the equation

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \epsilon & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}.$$

Hence we need only determine the $\mathbf{E}$ and $\mathbf{H}$ fields to obtain the full electromagnetic field; and for simplicity, we subsequently take the pair $(\mathbf{E}, \mathbf{H})$ to be the electromagnetic field.

The material coefficients ϵ and μ describe the dielectric permittivity and magnetic permeability of the medium, respectively. Encoded in these coefficients are the electromagnetic properties of the material, and in their most general form, they are tensor-valued functions of the spatial variable $\mathbf{r} = (x, y, z)$ and time t. We can study the behavior of electromagnetic fields in various materials by imposing additional conditions on the coefficients $\epsilon(\mathbf{r}, t)$ and $\mu(\mathbf{r}, t)$ and solving the associated Maxwell system.

Let $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$, and consider a one-dimensional (lossless) layered medium as depicted in 1.1; the z-axis is perpendicular to the layers. We assume that each layer is homogeneous, but also anisotropic in x and y, meaning that the material response to an incident wave is directionally dependent, and cannot be defined independently of the polarization.

For such a medium, the time-harmonic Maxwell PDEs reduce to a $6{\times}6$ linear differential-algebraic system. Projecting the electromagnetic field onto the tangential and normal com-

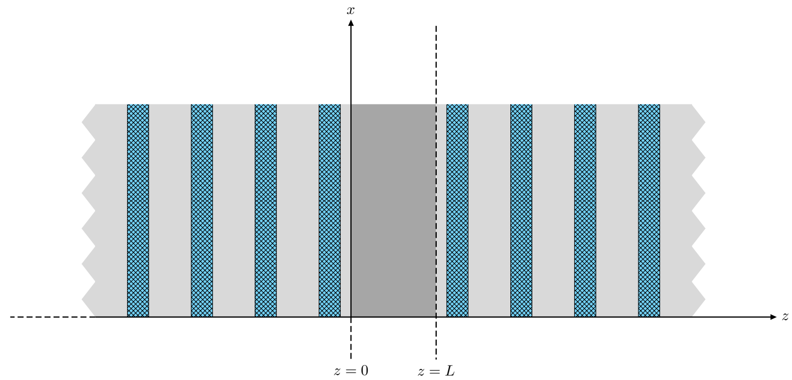

Figure 1.2: Periodic ambient medium interrupted by a defect layer.

ponents (relative to the layers), Maxwell's equations separate into a linear first order ODE system

$$\frac{d\psi}{dz} = \boldsymbol{\mathcal{A}}(\boldsymbol{\kappa}, \omega)\psi \tag{1.1}$$

in the tangential components ψ of the electromagnetic field and an algebraic equation

$$\phi = \Lambda\psi \tag{1.2}$$

for the normal components, where $\boldsymbol{\kappa}$ denotes the tangential wavevector and ω the frequency of an electromagnetic wave supported by the material. We call equation (1.1) the canonical Maxwell ODE for layered media.

1.1 Resonant Scattering Due to Defects in Material Layers

When a layered medium contains a defect layer (slab), as shown in 1.2, it acts as a scatterer of incident waves. In particular, we observe peculiar scattering behavior near the real wavevector-frequency pairs $(\boldsymbol{\kappa}_0, \omega_0)$ that correspond to poles of the scattering matrix, a phenomenon known as Fano resonance. At such pairs, the field $\psi(z)$ is called a *guided mode* of the slab. The field $\psi(z)$ for the pair $(\boldsymbol{\kappa}, \omega)$ is determined by its values on the slab boundary, which are related through the transfer matrix $T = T(\boldsymbol{\kappa}, \omega)$ by the formula $T\psi(0) = \psi(L)$. We obtain the scattering problem by reframing the transferred field in terms of incoming and outgoing fields:

$$S(\boldsymbol{\kappa}, \omega)\Psi^{\text{out}} = \Psi^{\text{in}} \tag{1.3}$$

through the transfer matrix $T = T(\boldsymbol{\kappa}, \omega)$ across the slab, which relates the values of a solution $\psi(z)$ at the interfaces between the slab and ambient medium at $z = 0$ to $z = L$ by the formula

$$T(\boldsymbol{\kappa}, \omega)\psi(0) = \psi(L). \tag{1.4}$$

Not all layered materials will support a guided mode. However, it is possible to construct a material that is guaranteed to have a guided mode by imposing a certain anisotropy in the layers. In the model problem that follows, we explicitly construct such a structure for the simplest case of a layered medium. We use this model to analyze the resonant scattering behavior associated with random perturbations of the material coefficients near these guided mode points.

1.2 The Model Problem

Consider the medium in Figure (1.3), and suppose that the material coefficients, given by,

$$\epsilon(z) = \begin{cases} \epsilon^0, & \text{if } z < 0 \text{ or } z > L \\ \epsilon^1, & \text{if } 0 < z < L \end{cases} \quad ; \quad \mu(z) = \begin{cases} \mu^0, & \text{if } z < 0 \text{ or } z > L \\ \mu^1, & \text{if } 0 < z < L \end{cases}$$

where

$$\epsilon^0 = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mu^0 = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\epsilon^1 = \begin{bmatrix} \epsilon_2 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mu^1 = \begin{bmatrix} \mu_2 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

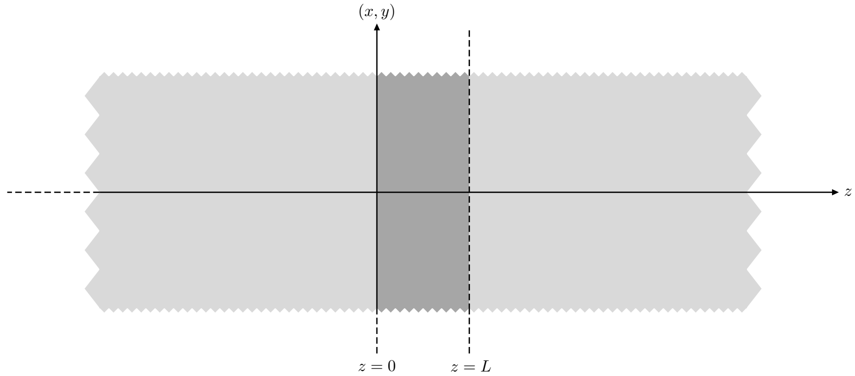

Figure 1.3: A stratified dielectric medium consisting of a contrasting layer of length L suspended in an infinite, anisotropic ambient medium parallel to the xy-plane

satisfy

$$\max\{\epsilon_1, \mu_2\} \leq \min\{\epsilon_2, \mu_1\}$$

and take $k_2 = 0$.

The ω-interval where each medium admits one propgagating mode and one evanescent mode is determined through the spectral analysis, noting that real k_3 corresponds to propagating modes and imaginary k_3 corresponds to evanescent modes. The appropriate ω-interval is the one for which we have the necessary restrictions on the real and imaginary k_3 simultaneously. If we assume $k_1 \neq 0$ and $\max\{\epsilon_1, \mu_2\} \leq \min\{\epsilon_2, \mu_1\}$, the desired ω-interval is computed to be $\left[\frac{k_1}{\sqrt{\mu_1}}, \frac{k_1}{\sqrt{\epsilon_1}}\right]$.

In the ambient medium, the canonical Maxwell ODE gives us

$$k_3 = \pm\left[\epsilon_1\left(\frac{\omega^2}{c^2}\mu_2 - k_1^2\right)\right]^{1/2}, \pm\left[\mu_1\left(\frac{\omega^2}{c^2}\epsilon_2 - k_1^2\right)\right]^{1/2},$$

which correspond to (evanescent and propagating, respectively) eigenvalues:

$$k_3^{0e} = \left[\epsilon_1\left(\frac{\omega^2}{c^2}\mu_2 - k_1^2\right)\right]^{\frac{1}{2}}$$

$$k_3^{0p} = \left[\mu_1\left(\frac{\omega^2}{c^2}\epsilon_2 - k_1^2\right)\right]^{\frac{1}{2}},$$

and the associated eigenspaces:

$$\left\{-\frac{\omega}{c}\epsilon_1 E_1 \pm k_3^{0e} H_2 = 0, E_2 = 0, H_1 = 0\right\} \quad \text{(evanescent)}$$

$$\left\{\frac{\omega}{c}\mu_1 H_1 \pm k_3^{0p} E_2 = 0, H_2 = 0, E_1 = 0\right\} \quad \text{(propagating)}$$

Similarly, we have the eigenvalues inside the slab:

$$k_3^{1p} = \left[\epsilon_2 \left(\frac{\omega^2}{c^2}\mu_1 - k_1^2\right)\right]^{\frac{1}{2}}$$

$$k_3^{1e} = \left[\mu_2 \left(\frac{\omega^2}{c^2}\epsilon_1 - k_1^2\right)\right]^{\frac{1}{2}},$$

with the corresponding eigenspaces:

$$\left\{-\frac{\omega}{c}\epsilon_2 E_1 \pm k_3^{1p} H_2 = 0, E_2 = 0, H_1 = 0\right\} \quad \text{(propagating)}$$

$$\left\{\frac{\omega}{c}\mu_2 H_1 \pm k_3^{1e} E_2 = 0, H_2 = 0, E_1 = 0\right\} \quad \text{(evanescent).}$$

Hence our guided mode solutions have the form

$$\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} = \begin{cases} C_1 \begin{bmatrix} -k_3^{0e} \\ 0 \\ 0 \\ \frac{\omega}{c}\epsilon_1 \end{bmatrix} e^{-ik_3^{0e}z} & z < 0 \\ B_1 \begin{bmatrix} k_3^{1p} \\ 0 \\ 0 \\ \frac{\omega}{c}\epsilon_2 \end{bmatrix} e^{ik_3^{1p}z} + B_2 \begin{bmatrix} -k_3^{1p} \\ 0 \\ 0 \\ \frac{\omega}{c}\epsilon_2 \end{bmatrix} e^{-ik_3^{1p}z} & 0 < z < L \\ C_2 \begin{bmatrix} k_3^{0e} \\ 0 \\ 0 \\ \frac{\omega}{c}\epsilon_1 \end{bmatrix} e^{ik_3^{0e}(z-L)} & z > L. \end{cases}$$

Imposing continuity of solutions at the interfaces $z = 0$ and $z = L$ leads us to the guided mode condition for the model problem:

$$2\cos(k_3^{1p}L) - i\left(\frac{k_3^{0e}}{k_3^{1p}}\frac{\epsilon_2}{\epsilon_1} - \frac{k_3^{1p}}{k_3^{0e}}\frac{\epsilon_1}{\epsilon_2}\right)\sin(k_3^{1p}L) = 0$$

Calculation of guided mode condition:

At $z = 0$:

$$C_1 = \frac{k_3^{1p}}{k_3^{0e}}(B_2 - B_1)$$

$$C_1 = \frac{\epsilon_2}{\epsilon_1}(B_2 + B_1)$$

So

$$\frac{k_3^{1p}}{k_3^{0e}}(B_2 - B_1) = \frac{\epsilon_2}{\epsilon_1}(B_2 + B_1).$$

Hence

$$\frac{k_3^{1p}}{k_3^{0e}}\frac{\epsilon_1}{\epsilon_2} = \frac{B_2 + B_1}{B_2 - B_1},$$

and

$$\frac{k_3^{0e}}{k_3^{1p}}\frac{\epsilon_2}{\epsilon_1} = \frac{B_2 - B_1}{B_2 + B_1}$$

At $z = L$:

$$\begin{aligned}
C_2 &= B_1\frac{k_3^{1p}}{k_3^{0e}}e^{ik_3^{1p}L} - B_2\frac{k_3^{1p}}{k_3^{0e}}e^{-ik_3^{1p}L} \\
&= B_1\frac{k_3^{1p}}{k_3^{0e}}\left[\cos(k_3^{1p}L) + i\sin(k_3^{1p}L)\right] - B_2\frac{k_3^{1p}}{k_3^{0e}}\left[\cos(k_3^{1p}L) - i\sin(k_3^{1p}L)\right] \\
&= -\frac{k_3^{1p}}{k_3^{0e}}(B_2 - B_1)\cos(k_3^{1p}L) + i\frac{k_3^{1p}}{k_3^{0e}}(B_2 + B_1)\sin(k_3^{1p}L)
\end{aligned}$$

Then

$$\frac{C_2}{B_2 - B_1} = -\frac{k_3^{1p}}{k_3^{0e}}\cos(k_3^{1p}L) + i\frac{k_3^{1p}}{k_3^{0e}}\frac{B_2 + B_1}{B_2 - B_1}\sin(k_3^{1p}L) \tag{1.5}$$

And also,

$$\begin{aligned}
C_2 &= B_1\frac{\epsilon_2}{\epsilon_1}e^{ik_3^{1p}L} + B_2\frac{\epsilon_2}{\epsilon_1}e^{-ik_3^{1p}L} \\
&= B_1\frac{\epsilon_2}{\epsilon_1}\left[\cos(k_3^{1p}L) + i\sin(k_3^{1p}L)\right] + B_2\frac{\epsilon_2}{\epsilon_1}\left[\cos(k_3^{1p}L) - i\sin(k_3^{1p}L)\right] \\
&= -\frac{\epsilon_2}{\epsilon_1}(B_2+B_1)\cos(k_3^{1p}L) - i\frac{\epsilon_2}{\epsilon_1}(B_2-B_1)\sin(k_3^{1p}L)
\end{aligned}$$

So

$$\frac{C_2}{B_2-B_1} = \frac{\epsilon_2}{\epsilon_1}\frac{B_2+B_1}{B_2-B_1}\cos(k_3^{1p}L) - i\frac{\epsilon_2}{\epsilon_1}\sin(k_3^{1p}L). \tag{1.6}$$

Making the substitution $\dfrac{B_2+B_1}{B_2-B_1} = \dfrac{k_3^{1p}}{k_3^{0e}}\dfrac{\epsilon_1}{\epsilon_2}$ into Equations (1.5) and (1.6) yields

$$\frac{C_2}{B_2-B_1} = -\frac{k_3^{1p}}{k_3^{0e}}\cos(k_3^{1p}L) + i\left(\frac{k_3^{1p}}{k_3^{0e}}\right)^2\sin(k_3^{1p}L) \tag{1.7}$$

and

$$\frac{C_2}{B_2-B_1} = \frac{\epsilon_2}{\epsilon_1}\left(\frac{k_3^{1p}}{k_3^{0e}}\frac{\epsilon_1}{\epsilon_2}\right)\cos(k_3^{1p}L) - i\frac{\epsilon_2}{\epsilon_1}\sin(k_3^{1p}L). \tag{1.8}$$

Since Equations (1.7) and (1.8) represent the same quantity, we obtain

$$0 = 2\frac{k_3^{1p}}{k_3^{0e}}\cos(k_3^{1p}L) - i\left(\frac{\epsilon_2}{\epsilon_1} - \left(\frac{k_3^{1p}}{k_3^{0e}}\right)^2\frac{\epsilon_1}{\epsilon_2}\right)\sin(k_3^{1p}L),$$

and hence we have the guided mode condition:

$$0 = 2\cos(k_3^{1p}L) - i\left(\frac{k_3^{0e}}{k_3^{1p}}\frac{\epsilon_2}{\epsilon_1} - \frac{k_3^{1p}}{k_3^{0e}}\frac{\epsilon_1}{\epsilon_2}\right)\sin(k_3^{1p}L)$$

To show guided mode exists when $\gamma = 90°$:

$$\epsilon_\gamma^1 = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

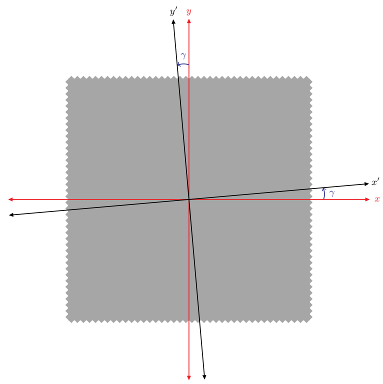

Figure 1.4: xy cross-section of the layered dielectric at a fixed $z \in [0, L]$; The coordinate plane with respect to which the matrix representation of the dielectric permittivity tensor for the ambient medium is diagonal is shown in red, and its rotation by γ shown in black. The rotationally perturbed permittivity matrix inside the slab is a diagonal matrix in the rotated coordinate system (x', y').

vs.

$$\epsilon_{\gamma}^{1} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_2 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Chapter 2
Background

2.1 The Maxwell ODE for Layered Anisotropic Media

For time-harmonic fields propagating in a lossless, non-dispersive layered medium, the z-dependent material coefficients $\epsilon = \epsilon(z)$, $\mu = \mu(z)$ are 3×3 matrix-valued functions in $M_3(\mathbb{C})$. For the purposes of this dissertation, we consider only materials for which ϵ and μ are Hermitian, meaning $\epsilon = \epsilon(z) = \epsilon(z)^*$, $\mu = \mu(z) = \mu(z)^*$. That is, ϵ and μ have the form

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \overline{\epsilon_{12}} & \epsilon_{22} & \epsilon_{23} \\ \overline{\epsilon_{13}} & \overline{\epsilon_{23}} & \epsilon_{33} \end{bmatrix}, \qquad \mu = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \overline{\mu_{12}} & \mu_{22} & \mu_{23} \\ \overline{\mu_{13}} & \overline{\mu_{23}} & \mu_{33} \end{bmatrix}$$

for each fixed value of z, with $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \mu_{11}, \mu_{22}$, and μ_{33} real. We also assume that the functions ϵ and μ are coercive, meaning there exist constants $c_1, c_2 > 0$ for which the inequalities

$$0 < c_1 I \leq \epsilon(z) \leq c_2 I \quad \text{and} \quad 0 < c_1 I \leq \mu(z) \leq c_2 I$$

hold for all $z \in \mathbb{R}$, and bounded.

Such materials support time-harmonic electric and magnetic fields of the form

$$\mathbf{E}(x, y, z; t) = [E_1(z), E_2(z), E_3(z)] e^{i(k_1 x + k_2 y - \omega t)}$$
$$\mathbf{H}(x, y, z; t) = [H_1(z), H_2(z), H_3(z)] e^{i(k_1 x + k_2 y - \omega t)}$$

where $\omega \in \mathbb{C} \setminus \{0\}$ is the angluar frequency, and $\mathbf{k}_\parallel = (k_1, k_2, 0) \in \mathbb{C}^2 \times \{0\} \cong \mathbb{C}^2$ the projection of the two-dimensional wavevector parallel to the layers onto the xy-plane.

In this case, the 6×6 Maxwell PDE system

$$\begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = -\frac{i\omega}{c} \begin{bmatrix} \epsilon & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}.$$

simplifies to a 4×4 system of first order ODEs in the spatial variable perpendicular to the layers, z, for the tangential components of the (time-harmonic) fields $\mathbf{E}$ and $\mathbf{H}$, and a 2×2 algebraic equation (also in z) for the normal components:

$$i^{-1} \frac{d}{dz} G \begin{bmatrix} \mathbf{E}(z) \\ \mathbf{H}(z) \end{bmatrix} = V(z; \boldsymbol{\kappa}, \omega) \begin{bmatrix} \mathbf{E}(z) \\ \mathbf{H}(z) \end{bmatrix},$$

where

$$G = \begin{bmatrix} 0 & i\boldsymbol{e}_3\times \\ -i\boldsymbol{e}_3\times & 0 \end{bmatrix}$$

and

$$V(z; \boldsymbol{\kappa}, \omega) = \frac{\omega}{c} \begin{bmatrix} \epsilon(z) & 0 \\ 0 & \mu(z) \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{k}_\parallel\times \\ -\mathbf{k}_\parallel\times & 0 \end{bmatrix}.$$

Under the projections

$$P_\parallel : \mathbb{C}^6 \to \mathbb{C}^4, \quad P_\perp : \mathbb{C}^6 \to \mathbb{C}^2$$

on to the tangential and normal field components, given by the matrices

$$P_\parallel = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_\perp = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the field $F = [\mathbf{E}, \mathbf{H}]^T$ separates into and ODE for the tangential component $\psi = [E_1, E_2, H_1, H_2]^T$

and an algebraic equation for the normal component $\phi = [E_3, H_3]^T$:

$$i^{-1} J \frac{d\psi}{dz} = A\psi; \quad \psi \in (AC_{loc}(\mathbb{R}))^4$$

$$\phi = \Phi\psi$$

$F = \begin{bmatrix} \psi \\ \phi \end{bmatrix}$ satisfying $P_{\|}F = \psi$, $P_{\perp}F = \phi$ is obtained by setting

$$F = P_{\|}^{*}\psi + P_{\perp}^{*}\psi,$$

where

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and

$$A = \frac{\omega}{c} \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 & 0 \\ 0 & 0 & \mu_{11} & \mu_{12} \\ 0 & 0 & \mu_{21} & \mu_{22} \end{bmatrix} - \frac{c}{\omega} \begin{bmatrix} \epsilon_{13} & \frac{c}{\omega}k_2 \\ \epsilon_{23} & -\frac{c}{\omega}k_1 \\ -\frac{c}{\omega}k_2 & \mu_{13} \\ \frac{c}{\omega}k_1 & \mu_{23} \end{bmatrix} \begin{bmatrix} \frac{1}{\epsilon_{33}} & 0 \\ 0 & \frac{1}{\mu_{33}} \end{bmatrix} \begin{bmatrix} \frac{\omega}{c}\epsilon_{31} & \frac{\omega}{c}\epsilon_{32} & -k_2 & k_1 \\ k_2 & -k_1 & \frac{\omega}{c}\mu_{31} & \frac{\omega}{c}\mu_{32} \end{bmatrix}.$$

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 & 0 \\ 0 & 0 & \mu_{11} & \mu_{12} \\ 0 & 0 & \mu_{21} & \mu_{22} \end{bmatrix}$$

The Maxwell ODE system can also be formulated as an integral equation. By imposing boundary conditions that require continuity across the layers, we arrive at an expression of solutions to the Maxwell ODEs that is an integral equation

$$\psi(z) = \psi(z_0) + \int_{z_0}^{z} iJ^{-1}A(s;\boldsymbol{\kappa},\omega)\psi(s)\,ds\,.$$

For each $\psi_0 \in \mathbb{C}^4$, the integral equation above re-expresses the unique solution $\psi(z)$ to the initial value problem

$$\begin{aligned} \frac{d}{dz}\psi(z) &= iJ^{-1}A(z;\boldsymbol{\kappa},\omega)\psi(z) \\ \psi_0 &= \psi(z_0), \end{aligned}$$

and the solution $\psi(z)$ relates to the initial condition $\psi(z_0)$ through the transfer matrix T by

$$\psi(z) = T(z_0, z)\psi(z_0),$$

which has the following properties $T(z_0, z) = T(z_1, z)T(z_0, z_1)$, $T(z_0, z_1)^{-1} = T(z_1, z_0)$, and $T(z_0, z_0) = I$ for all $z_0, z_1, z \in \mathbb{R}$. It is continuous as a function of z and analytic as a function of wavevector-frequency pair $(\boldsymbol{\kappa}, \omega) \in \mathbb{C}^2 \times \mathbb{C}\setminus\{0\}$. The operator $\boldsymbol{\mathcal{A}}(z;\boldsymbol{\kappa},\omega) := iJ^{-1}A(z;\boldsymbol{\kappa},\omega)$ is the propagator matrix for the Maxwell ODEs, but the perturbations in ϵ that we will impose come only through A since J is simply a matrix of constants.

We can partition the 6×6 matrix A as follows

$$A = \begin{bmatrix} V_{\|\|} & V_{\|\perp} \\ V_{\perp\|} & V_{\perp\perp} \end{bmatrix},$$

where $V_{\|\|}$, $V_{\|\perp}$, $V_{\perp\|}$, and $V_{\perp\perp}$... and the matrix equation for the differential-algebraic system becomes

$$\begin{bmatrix} i^{-1}J\frac{d\psi}{dz} \\ 0 \end{bmatrix} = \begin{bmatrix} V_{\|\|} & V_{\|\perp} \\ V_{\perp\|} & V_{\perp\perp} \end{bmatrix} \begin{bmatrix} \psi \\ \phi \end{bmatrix}.$$

This leaves us with a 4×4 system of ordinary differential equations in the variable z

subject to the condition that $\phi = \Phi\psi$. That is, we have $i^{-1}J\frac{d\psi}{dz} = A\psi$, $\psi \in (AC_{loc}(\mathbb{R}))^4$, and $\phi = \Phi\psi$. So the differential operator is

$$\frac{d\psi}{dz} = iJ^{-1}A\psi$$

2.2 Poynting's Theorem and the Flux-Inner Product

In the ODE system derived for time-harmonic fields in layered media, the matrix A is self-adjoint with respect to the standard inner product in $\mathbb{C}^4$. The flow of electromagnetic energy across the layers is governed by the transfer matrix T. So we expect the appropriate energy conservation law to be given by an inner product with respect to which the matrix T is unitary. This flux inner product, often referred to simply as the energy flux, is an indefinite sesquilinear form derived from the canonical Maxwell ODEs.

The conservation of energy law for electrodynamics is stated in Poynting's theorem, a partial differential equation relating the energy flux and energy density of an electromagnetic field. In a linear, lossless, non-dispersive medium, Poynting's theorem a time-harmonic field with non-zero complex frequency ω is given by

$$-\frac{d}{dt}\int_V U(\mathbf{r},t)d^3\mathbf{r} = \int_V \nabla \cdot \operatorname{Re}\mathbf{S}(\mathbf{r},t)d^3\mathbf{r} = \int_{\partial V} \operatorname{Re}\mathbf{S}(\mathbf{r},t)\cdot \mathbf{n}\,da,$$

where the complex Poynting vector

$$\mathbf{S}(\mathbf{r},t) = \mathbf{S}(\mathbf{r})e^{2\operatorname{Im}\omega t}, \quad \mathbf{S}(\mathbf{r}) = \frac{c}{8\pi}(\mathbf{E}(\mathbf{r}) \times \overline{\mathbf{H}(\mathbf{r})})$$

gives the energy flux

$$\operatorname{Re}\mathbf{S}(\mathbf{r},t),$$

And the energy density satisfies

$$U(\mathbf{r},t) = U(\mathbf{r})e^{2\operatorname{Im}\omega t}, \quad U(\mathbf{r}) = \frac{1}{16\pi}\left((\mathbf{E}(\mathbf{r}),\mathbf{D}(\mathbf{r})) + (\mathbf{B}(\mathbf{r}),\mathbf{H}(\mathbf{r}))\right).$$

We also have the time-averaged Poynting vector

$$\operatorname{Re}\mathbf{S}(\mathbf{r}) = \left\langle \frac{c}{4\pi} \operatorname{Re}\mathbf{E}(\mathbf{r},t) \times \operatorname{Re}\mathbf{H}(\mathbf{r},t) \right\rangle$$

and the total energy density

$$U(\mathbf{r}) = \left\langle \frac{1}{8\pi}(\operatorname{Re}\mathbf{E}(\mathbf{r},t) \cdot \operatorname{Re}\mathbf{E}(\mathbf{r},t) + \operatorname{Re}\mathbf{B}(\mathbf{r},t) \operatorname{Re}\mathbf{H}(\mathbf{r},t)) \right\rangle.$$

On the rectangular volume $V = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$, the z-directional Poynting's theorem simplifies to

$$-2 \operatorname{Im}\omega \int_{z_0}^{z_1} U(z)\, dz = \int_{z_0}^{z_1} \frac{d}{dz}(\operatorname{Re}\mathbf{S}(z) \cdot \mathbf{e}_3)\, dz = [\psi(z_1), \psi(z_1)] - [\psi(z_0), \psi(z_0)],$$

where $[\cdot,\cdot]$ is the energy-flux form (or flux inner product) given by

$$[\psi_1, \psi_2] = \frac{c}{16\pi}(J\psi_1, \psi_2), \qquad \psi_1, \psi_2 \in \mathbb{C}^4$$

In this case we have the z-dependent Poynting vector

$$\mathbf{S}(\mathbf{r}) = \mathbf{S}(z) = \frac{c}{8\pi}(\mathbf{E}(z) \times \overline{\mathbf{H}(z)}),$$

z-dependent energy flux

$$\operatorname{Re}\mathbf{S}(\mathbf{r}) \cdot \boldsymbol{e}_3 = \operatorname{Re}\mathbf{S}(z) \cdot \boldsymbol{e}_3 = [\psi(z), \psi(z)],$$

and z-dependent energy density

$$U(\mathbf{r}) = U(z) = \frac{1}{16\pi}\left((\epsilon(z)\mathbf{E}(z), \mathbf{E}(z)) + (\mu(z)\mathbf{H}(z), \mathbf{H}(z))\right)$$

in the direction perpendicular to the layers.

For each fixed $(z, \boldsymbol{\kappa}, \omega) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \backslash \{0\}$, the matrix A satisfies $A = A^*$ (it is self-adjoint with respect to the standard inner product on $\mathbb{C}^4$) assuming $\epsilon^* = \epsilon$, $\mu^* = \mu$. While $iJ^{-1}A$ is not self-adjoint with respect to the standard inner product, it is self-adjoint with respect to the flux (indefinite) inner product, which is defined through $(\cdot, \cdot)$ and J by:

$$[\psi_1, \psi_2] = \frac{c}{16\pi}(J\psi_1, \psi_2), \quad \psi_1, \psi_2 \in \mathbb{C}^4$$

Proposition 2.1. *The operator $J^{-1}A$ is self-adjoint with respect to the flux inner product $[\cdot, \cdot]$.*

Proof.

$$\begin{aligned}
[J^{-1}A\psi_1, \psi_2] &= \frac{c}{16\pi}(JJ^{-1}A\psi_1), \psi_2) \\
&= \frac{c}{16\pi}(A\psi_1), \psi_2) \\
&= \frac{c}{16\pi}(\psi_1, A\psi_2) \quad \text{since } A \text{ is self-adjoint} \\
&= \frac{c}{16\pi}(\psi_1, JJ^{-1}A\psi_2) \\
&= \frac{c}{16\pi}(J\psi_1, J^{-1}A\psi_2) \\
&= [\psi_1, J^{-1}A\psi_2]
\end{aligned}$$

Hence $J^{-1}A$ is symmetric with respect to $[\cdot, \cdot]$. □

2.3 The Scattering Problem

We can derive the formal scattering problem using the flux-unitary transfer matrix across the slab to decompose solutions at the boundaries $z = 0$ and $z = L$ into incoming and outgoing fields:

$$\psi(0) = \psi_+^{\text{in}}(0) + \psi_-^{\text{out}}(0)$$

$$\psi(L) = \psi_+^{\text{in}}(L) + \psi_-^{\text{out}}(L)$$

. Let P_- and P_+ be the rank-2 complementary projections onto the leftward and rightward subspaces of $\mathbb{C}^4$. Then the block form of the slab transfer matrix is given by

$$\begin{bmatrix} T_{--} & T_{-+} \\ T_{+-} & T_{++} \end{bmatrix} \begin{bmatrix} \psi_-^{\text{out}}(0) \\ \psi_+^{\text{in}}(0) \end{bmatrix} = \begin{bmatrix} \psi_-^{\text{in}}(L) \\ \psi_+^{\text{out}}(L) \end{bmatrix},$$

where

$$T_{--} = P_- T P_-, T_{-+} = P_- T P_+, T_{+-} = P_+ T P_-, T_{++} = P_+ T P_+.$$

Then

$$\begin{bmatrix} -I & T_{-+} \\ 0 & T_{++} \end{bmatrix} \begin{bmatrix} \psi_-^{\text{in}}(L) \\ \psi_+^{\text{in}}(0) \end{bmatrix} + \begin{bmatrix} T_{--} & 0 \\ T_{+-} & -I \end{bmatrix} \begin{bmatrix} \psi_-^{\text{out}}(0) \\ \psi_+^{\text{out}}(L) \end{bmatrix} = 0.$$

And we define the scattering problem as

$$(TP_- - P_+)\Psi^{\text{out}} + (TP_+ - P_-)\Psi^{\text{in}} = 0 \quad \text{(Scattering Problem)}.$$

If $\Psi^{\text{in}} = 0$, then the scattering problem becomes $(TP_- - P_+)\Psi^{\text{out}} = 0$. The matrix $TP_- - P_+$ is a function of $\boldsymbol{\kappa}$ and ω, in particular; so call it as $TP_- - P_+ := S(\boldsymbol{\kappa}, \omega)$. The matrix $S(\boldsymbol{\kappa}, \omega)$ has a nontrivial solution, i.e. a scattered field, if and only if $S(\boldsymbol{\kappa}, \omega)$ is invertible, i.e. $\det S(\boldsymbol{\kappa}, \omega) \neq 0$. So if $S(\boldsymbol{\kappa}, \omega)$ is not invertible, the outgoing field $\Psi^{\text{out}} = 0$, and there is no scattered field. When $S(\boldsymbol{\kappa}, \omega)$ is not invertible, we also know that it has zero as an eigenvalue. Finding the $(\boldsymbol{\kappa}, \omega)$ pairs for which 0 is an eigenvalue of $S(\boldsymbol{\kappa}, \omega)$ is equivalent to determining the $(\boldsymbol{\kappa}, \omega)$ pairs for which $S^{-1}(\boldsymbol{\kappa}, \omega)$ does not exist. If $S^{-1}(\boldsymbol{\kappa}, \omega)$ does not exist, then it has a pole at that $(\boldsymbol{\kappa}, \omega)$. The poles of S^{-1} correspond to the zeros of S. At a true guided mode, $\Psi^{\text{out}} = 0$. But, any nontrivial solutions to $S(\boldsymbol{\kappa}, \omega)\Psi^{\text{out}} = 0$ are called generalized guided modes, and will occur at $(\boldsymbol{\kappa}, \omega)$ pairs near the true guided mode.

Chapter 3
Material Perturbations

This chapter is devoted to the analysis of solutions of the Maxwell ODE system under random perturbations of the dielectric coefficient ϵ. Similar analysis applies to the magnetic coefficient μ.

3.1 Perturbing ϵ and Scattering

We are concerned with the area of scattering theory that analyzes the spectrum of analytic matrix-valued equations under small perturbations.

Starting with the ambient medium from the model problem introduced above with

$$\epsilon^0 = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the slab placed at $90°$ rotation of the same material, i.e.

$$\epsilon^1 = \begin{bmatrix} \epsilon_{22} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

, supports a guided mode, where we have in particular $k_2 = 0$. Then we can look at the affect of a small rotation of ϵ^1 on the resonant characteristics. More generally, the permittivity ϵ would have the form

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

Self-adjointness of JA with respect to the indefinite flux norm ensues if ϵ is Hermitian, which is the case for lossless media. Thus we must have $\epsilon_{21} = \epsilon_{12}^*$. However, since the off-diagonal entries detail the extent to which the electric permittivity of the material is coupled in the x and y directions, such symmetry is expected physically as well. Whether such a material supports a spectrally embedded guided mode in general is still an open question. In particular, there is no known explicit construction for a guided mode in a material for which ϵ_{12} in the slab is nonzero (assuming ϵ is diagonal in the ambient medium).

3.2 Perturbed Transfer Matrix

When presented as an integral equation $\mathcal{A}(\boldsymbol{\kappa}, \omega)\psi = \phi$, the integral operator $\mathcal{A}$ is a Fredholm integral equation of the second kind with index zero (or if the z dependence is retained, which would mean that the upper limit of the integral is variable, then $\mathcal{A}$ is a Volterra integral equation of the second kind). The takeaway here is that either $\mathcal{A}$ has a bounded inverse or a finite-dimensional nullspace and codomain of the same dimension, and the Fredholm alternative applies. This is discussed in [22].

To obtain the integral representation of the transfer matrix, we start by employing the method of variation of parameters. The ODE

$$\frac{d}{dz}\psi(z) = iJ^{-1}A(z; \boldsymbol{\kappa}, \omega)\psi(z)$$

obtained by the reduction of the Maxwell PDEs for time-harmonic waves in layered media has solutions ψ that satisfy the integral equation

$$\psi(z) = \psi(z_0) + \int_{z_0}^{z} iJ^{-1}A(s; \boldsymbol{\kappa}, \omega)\psi(s)\, ds$$

when boundary conditions requiring continuity of the components of ψ across the layers are imposed. The unique solution for the unperturbed problem with $A = A_0$ and initial value $\psi_0 = \psi(z_0)$ is given by the formula $\psi(z) = T_0(z_0, z)\psi(z_0)$ for each initial condition $\psi_0 \in \mathbb{C}^4$. The matrix T_0 is the unperturbed transfer matrix, and like all transfer matrices, it has the properties $T_0(z_0, z) = T_0(z_1, z)T_0(z_0, z_1)$, $T_0(z_0, z_1)^{-1} = T_0(z_1, z_0)$, and $T_0(z_0, z_0) = I$ for all

$z_0, z_1, z \in \mathbb{R}$.

Using the integral equation above, we can derive an integral representation of the transfer matrix as follows by substituting $T(z_0, z)\psi(z_0)$ for $\psi(z)$:

$$\begin{aligned} T(z_0, z)\psi(z_0) &= \psi(z_0) + \int_{z_0}^{z} iJ^{-1}A(s;\boldsymbol{\kappa},\omega)T(z_0,s)\psi(z_0)\,ds \\ \Rightarrow T(z_0, z) &= I + \int_{z_0}^{z} iJ^{-1}A(s;\boldsymbol{\kappa},\omega)T(z_0,s)\,ds \end{aligned}$$

A second-kind Volterra equation for a function ϕ is an integral equation of the form

$$\phi(x) - \lambda\int_0^x K(x,y)\phi(y)dy = f(x)$$

for given (real) functions $f(x)$ and $K(x,y)$. If the function $f(x)$ and the kernel $K(x,y)$ are both in L^2, then there is essentially only one L^2 solution ϕ of the Volterra equaiton, and it is given by the formula

$$\phi(x) = f(x) + \lambda\int_0^x \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,y)f(x)\,dy,$$

where K_{n+1} is defined recursively by

$$\begin{aligned} K_1(x,y) &\equiv K(x,y) \\ K_{n+1}(x,y) &= \int_0^x K(x,z)K_n(z,y)\,dz \end{aligned}$$

This will allow us to write the perturbation of the transfer matrix in integral form in terms of the perturbation A_1 of the propagator matrix A. For convenience we will adopt the notation $T_0(z) := T_0(0,z)$ and $T(z) := T(0,z)$, since $T_0(0,z) = \Phi(z)\Phi^{-1}(0) = \Phi(z)$ because $\Phi(0)$ is the identity.

Proposition 3.1 (Perturbed Transfer Matrix)**.** *The transfer matrix* $T(z) = T(z,\boldsymbol{\kappa},\omega;\sigma)$

for the perturbed problem

$$-iJ^{-1}\frac{d\psi}{dz} = (A_0 + \sigma A_1)\psi, \quad \sigma \ll 1$$

on $[0, L]$ *has the form* $T = T_0 + \sigma T_1 + O(\sigma^2)$ *as* $\sigma \to 0$*, and can be expressed*

$$T(z) = T_0(z) + \sigma\, i \int_0^z T_0(s, z) J A_1(s) T_0(s) ds + O(\sigma^2) \qquad (\sigma \to 0),$$

where $A_1(s) = A_1(s; \boldsymbol{\kappa}, \omega)$*. It can also be expressed in the form* $T = T_0(I + \sigma M) + O(\sigma^2)$*, where*

$$M(z) = i \int_0^z T_0^{-1}(s) J A_1(s) T_0(s) ds.$$

That is,

$$T(z) = T_0(z)\left(I + \sigma\, i \int_0^z T_0^{-1}(s) J A_1(s) T_0(s) ds\right) + O(\sigma^2).$$

Before presenting a proof of the proposition above, we will introduce a notational substitution in order to simplify the ensuing calculations. To that end, let

$$\mathcal{A}_0 = iJA_0 \text{ and } \mathcal{A}_1 = iJA_1.$$

Then the ODE

$$-iJ^{-1}\frac{d\psi}{dz} = (A_0 + \sigma A_1)\psi$$

can be written

$$\frac{d\psi}{dz} = (\mathcal{A}_0 + \sigma \mathcal{A}_1)\psi.$$

Suppose $\Phi(z)$ is a fundamental matrix solution to the unperturbed ODE. Then the unperturbed transfer matrix T_0 from z_0 to z is given by $T_0(z_0, z) = \Phi(z)\Phi^{-1}(z_0)$. Consider the function ϕ defined by $\phi(z) = \Phi^{-1}(z)\psi(z)$, where ψ solves the perturbed ODE with initial condition $\psi(z_0)$. Then we have

$$\psi(z) = \Phi(z)\phi(z)$$
$$\dot{\psi}(z) = \dot{\Phi}(z)\phi(z) + \Phi(z)\dot{\phi(z)}$$

Since $\Phi(z)$ is a fundamental matrix solution of the unperturbed problem, we have $\dot{\Phi}(z) = \mathcal{A}_0(z)\Phi(z)$, So

$$\dot{\psi}(z) = \mathcal{A}_0(z)\Phi(z)\phi(z) + \Phi(z)\dot{\phi}(z) = \mathcal{A}_0(z)\psi(z) + \Phi(z)\dot{\phi}(z)$$

But $\dot{\psi}(z) = \mathcal{A}_0(z)\psi(z) + \sigma\mathcal{A}_1(z)\psi(z)$; so

$$\mathcal{A}_0(z)\psi(z) + \sigma\mathcal{A}_1(z)\psi(z) = \mathcal{A}_0(z)\psi(z) + \Phi(z)\dot{\phi}(z).$$

Then by the uniqueness of the solution, we must have $\Phi(z)\dot{\phi}(z) = \sigma\mathcal{A}_1(z)\psi(z)$, or equivalently, $\dot{\phi}(z) = \sigma\Phi^{-1}(z)\mathcal{A}_1(z)\psi(z)$. Integrating on $[z_0, z]$, we get

$$\phi(z) - \phi(z_0) = \sigma \int_{z_0}^{z} \Phi^{-1}(s)\mathcal{A}_1(s)\psi(s)ds,$$

and so

$$\phi(z) = \phi(z_0) + \sigma \int_{z_0}^{z} \Phi^{-1}(s)\mathcal{A}_1(s)\psi(s)ds\,.$$

Multiplying by $\Phi(z)$ and applying $\phi(z_0) = \Phi^{-1}(z_0)\psi(z_0)$, we obtain

$$\phi(z) = \phi(z_0) + \sigma \int_{z_0}^{z} \Phi^{-1}(s)\mathcal{A}_1(s)\psi(s)ds,$$

$$\Phi(z)\phi(z) = \Phi(z)\Phi^{-1}(z_0)\psi(z_0) + \sigma \int_{z_0}^{z} \Phi(z)\Phi^{-1}(s)\mathcal{A}_1(s)\psi(s)ds.$$

Hence

$$\psi(z) = T_0(z_0, z)\psi(z_0) + \sigma \int_{z_0}^{z} T_0(s, z)\mathcal{A}_1(s)\psi(s)ds.$$

So

$$T(z_0, z)\psi(z_0) = T_0(z_0, z)\psi(z_0) + \sigma \int_{z_0}^{z} T_0(s, z)\mathcal{A}_1(s)T(s, z)\psi(z_0)ds,$$

from which we obtain the integral representation for the transfer matrix

$$T(z_0, z) = T_0(z_0, z) + \sigma \int_{z_0}^{z} T_0(s, z)\mathcal{A}_1(s)T(z_0, s)ds.$$

By setting $z_0 = 0$, we arrive at the transfer matrix through the slab,

$$T(z) = T_0(z) + \sigma \int_{0}^{z} T_0(s, z)\mathcal{A}_1(s)T(s)ds, \tag{3.1}$$

from which will build the scattering problem. This formulation of the perturbed transfer matrix $T(z)$ depends implicitly on itself, but the equivalent expression

$$T(z) - \sigma \int_{0}^{z} T_0(s, z)\mathcal{A}_1(s)T(s)ds = T_0(z) \tag{3.2}$$

is a Volterra integral equation of the second kind for $T(z)$, the solution of which is an asymptotic expansion of $T(z)$ on $[0, L]$ in the parameter σ since the the unperturbed transfer matrix $T_0(z)$ is locally absolutely continuous on $\mathbb{R}$, thus ensuring that the kernel $K(z, s) = T_0(s, z)\mathcal{A}_1(s)$ $K(z, s) = T_0(s, z)\mathcal{A}_1(s)$ and $T_0(z)$ are L^2-functions on $[0, L] \times [0, L]$ and $[0, L]$, respectively.

Thus by the Volterra theorem,

$$T(z) = T_0(z) + \sigma \int_{0}^{z} \sum_{n=0}^{\infty} \sigma^n K_{n+1}(z, s)T_0(s)ds,$$

where the series $\{K_n\}_{n=1}^{\infty}$ is defined recursively by

$$K_1(z,s) = K(z,s)$$
$$K_{n+1}(z,s) = \int_0^z K(z,r)K_n(r,s)dr.$$

The matrix function $T_0(s)$ is bounded, the series $\sum_{n=0}^{\infty} \sigma^n K_{n+1}(z,s)$ converges almost uniformly, and $K_{n+1} = O\left(\frac{1}{n!}\right)$, so $\sigma^n K_{n+1} = O(\sigma^n)$ since $1/n!$ decays faster than σ^n. So, finally, we obtain an asymptotic expansion in σ for the transfer matrix of the perturbed Maxwell ODE system,

$$T(z) = T_0(z) + \sigma \int_0^z T_0(s,z)\mathcal{A}_1(s)T_0(s)ds + O(\sigma^2).$$

Thus to leading order in σ, the perturbed transfer matrix has the form

$$T(z) = T_0(z) + \sigma T_1(z),$$

where

$$T_1(z) = \int_0^z T_0(s,z)\mathcal{A}_1(s)T_0(s)ds.$$

We can also write

$$T_1(z) = T_0(z)\int_0^z T_0^{-1}(s)\mathcal{A}_1(s)T_0(s)ds$$

since $T_0(s,z) = T_0(z)T_0^{-1}(s)$, and express the transfer matrix $T(z)$ as

$$T(z) = T_0(z)\left(I + \sigma \int_0^z T_0^{-1}(s)\mathcal{A}_1(s)T_0(s)ds\right) + O(\sigma^2).$$

The remainder of the section focuses on analyzing the effect of the leading order behavior of the perturbed transfer matrix T across the slab,

$$T = T(L) \approx T_0(L) + \sigma T_0(L) \int_0^L T_0^{-1}(s)\mathcal{A}_1(s)T_0(s)ds,$$

on the resonant characteristics of the scattering problem, which we now formulate for the perturbed system. It is interesting to note that the integrand $T_0^{-1}(s)\mathcal{A}_1(s)T_0(s)$ is the conjugation of the perturbation matrix by the unperturbed transfer matrix. Moreover, we know that T_0 is unitary with respect to the flux inner product, meaning that its flux adjoint is its inverse:

$$T_0^{[*]} = JT_0^*J = T_0^{-1}.$$

This allows us to replace T_0^{-1} in the integrand of $M(z)$ with $JT_0^*J^{-1}$:

$$T_0^{-1}(s)JA_1(s)T_0(s) = JT_0^*(s)J^{-1}JA_1(s)T_0(s) = JT_0^*(s)A_1(s)T_0(s).$$

So

$$M(z) = iJ\int_0^z T_0^*(s)A_1(s)T_0(s)\,ds,$$

and $T_1(z) = T_0(z)M(z)$. Now we are ready to look at how different perturbation matrices A_1 affect the resonance problem through the transfer matrix.

3.3 Variance of the Perturbed Transfer Matrix

We're now ready to move ahead to building up the perturbation theory by introducing progressively more complicated perturbations into A_1, with the goal of modeling random fabrication errors with random variables. We shall decompose A_1 into elementary rank-one components to analyze its effects on T_1.

If $\{\boldsymbol{\phi}_1(z), \boldsymbol{\phi}_2(z), \boldsymbol{\phi}_3(z), \boldsymbol{\phi}_4(z)\}$, where $\boldsymbol{\phi}_i = [\phi_{i1}, \phi_{i2}, \phi_{i3}, \phi_{i4}]^T$ for $i = 1, \ldots, 4$, is a fundamental set of solutions to the homogeneous equation, then the unperturbed transfer matrix computed from the associated fundamental matrix solution $\Phi(z) = [\boldsymbol{\phi}_1 \;\; \boldsymbol{\phi}_2 \;\; \boldsymbol{\phi}_3 \;\; \boldsymbol{\phi}_4]$

has the form

$$T_0(z) = \Phi(z)\Phi^{-1}(0),$$

where $\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3}, \boldsymbol{c_4}$ are the column vectors of the matrix $\Phi^{-1}(0) \in \mathcal{M}_4(\mathbb{C})$. For simplicity, let us take $\Phi^{-1}(0)$ to be the 4×4 identity matrix. If we express the perturbation matrix A_1 as the sum

$$\xi(s) = \sum_{i,j \in [4]} \xi_{ij}(s) \tag{3.3}$$

of rank-1 perturbations (the ξ_{ij} represent the individual entries of ξ), then the matrix

$$M = \int_0^L T_0^{-1}(s) A_1(s) T_0(s)\, ds = \int_0^L T_0^{-1}(s)\xi(s)T_0(s)\, ds,$$

and the entries M_{mn} of M for $m, n = 1, \ldots, 4$ is a sum

$$M_{mn} = \sum_{i,j \in [4]} M_{mn}(i,j)$$

of integrals.

Lemma 3.2. *The effect of the perturbation ξ_{ij} on the mn^{th} entry of M is given by the integral*

$$M_{mn}(i,j) = \int_0^{\mathrm{L}} \xi_{ij}(s)\bar{\phi}_{mj}(s)\phi_{ni}(s)\, ds.$$

The proof uses the flux-unitarity of T_0 and involves only elementary matrix algebra.

When the perturbation $\xi(s)$ is a matrix of random variables, the entries of M will not be deterministic and so we must compute the variance of each M_{mn} in order to quantify the effect of the random perturbation on the perturbed transfer matrix, and subsequently resonant properties of the scattering problem.

Theorem 3.3. *The variance of M_{mn} is given by the equation*

$$\mathrm{Var}(M_{mn}) = \sum_{i,j \in [4]} \int_0^{\mathrm{L}} \left(\bar{\phi}_{mj}(s)\phi_{ni}(s)\right)^2 \mathrm{Var}\left(\xi_{ij}(s)\right)\, ds, \tag{3.4}$$

Proof.

$$\begin{aligned}
\mathrm{Var}(M_{mn}) &= \mathrm{Var}\left(\sum_{i,j\in[4]}\int_0^{\mathrm{L}} \xi_{ij}(s)\bar{\phi}_{mj}(s)\phi_{ni}(s)ds\right) \\
&= \sum_{i,j\in[4]} \mathrm{Var}\left(\int_0^{\mathrm{L}} \xi_{ij}(s)\bar{\phi}_{mj}(s)\phi_{ni}(s)ds\right) \\
&= \sum_{i,j\in[4]} \int_0^{\mathrm{L}} \mathrm{Var}\left(\xi_{ij}(s)\bar{\phi}_{mj}(s)\phi_{ni}(s)\right) ds \\
&= \sum_{i,j\in[4]} \int_0^{\mathrm{L}} \left(\bar{\phi}_{mj}(s)\phi_{ni}(s)\right)^2 \mathrm{Var}\left(\xi_{ij}(s)\right) ds\,.
\end{aligned}$$

□

If the perturbations ξ_{ij} are independent random variables, we have the following corollary.

Corollary 3.4. $\mathrm{Var}(M_{mn}) = \sum_{i,j\in[4]} \mathrm{Var}\left(\xi_{ij}\right) \left\|\bar{\phi}_{mj}\phi_{ni}\right\|_2^2$, *where* $\|\cdot\|_2$ *denotes the usual* L^2*-norm on the interval* $[0, L]$.

Proof. By the theorem above,

$$\begin{aligned}
\mathrm{Var}(M_{mn}) &= \sum_{i,j\in[4]} \int_0^{\mathrm{L}} \left(\bar{\phi}_{mj}(s)\phi_{ni}(s)\right)^2 \mathrm{Var}\left(\xi_{ij}(s)\right)\, ds \\
&= \sum_{i,j\in[4]} \mathrm{Var}\left(\xi_{ij}\right) \int_0^{\mathrm{L}} \left(\bar{\phi}_{mj}(s)\phi_{ni}(s)\right)^2\, ds \\
&= \sum_{i,j\in[4]} \mathrm{Var}\left(\xi_{ij}\right) \left\|\bar{\phi}_{mj}\phi_{ni}\right\|_2^2 \quad \text{if } \xi_{ij}(s) = \xi_{ij} \text{ is constant.}
\end{aligned}$$

□

The proof of this corollary uses the following well-known result from the theory of probability:

Lemma 3.5. *If X, Y are independent random variables, then*

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y).$$

The significance of this result is that, for stationary random perturbations $\xi(z)$, that is, for which $\xi(z_1)$ and $\xi(z_2)$ are independent whenever $z_1 \neq z_2$, one can restrict the analysis to slabs with homogeneous but random perturbations.

Chapter 4
Sensitivity Analysis

Motivated by the corollary of the previous chapter and the related discussion, we turn to an analysis of the sensitivity of the resonant frequency $\omega = \hat{\omega} + i\check{\omega}$ with respect to variations in the tensor ϵ in the slab, when this is taken to be homogeneous, that is, constant in the spatial variable z. This is accomplished through a study of the (deterministic) derivatives of the real and imaginary parts of ω as functions of ϵ. By complexifying ϵ, we accomplish two things: losses are incorporated into the material; and the complex analyticity of ω with respect to ϵ leads to physically meaningful and perhaps surprising relations between different sensitivities (due to the Cauchy-Riemann equations).

4.1 Tuning Resonance Through Material Perturbations

Frequency and wavevector are related through the dispersion relation $\omega(\boldsymbol{\kappa})$. There is a functional relationship between the time-frequency ω and the vector $\boldsymbol{\kappa}$ of spatial frequencies in the x, y and z directions. So this is why it makes sense to treat $\omega = \omega(\boldsymbol{\kappa})$ as a function of $\boldsymbol{\kappa}$ as opposed to $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\omega)$ as a function of ω. This is a complex-analytic dispersion relation for a resonant frequency.

Determining the influence of structural perturbations on the resonance phenomena near a guided mode pair $(\boldsymbol{\kappa}_0, \omega_0)$ is done by a series expansion the complex resonance

$$\omega = \hat{\omega} + i\check{\omega}$$

using analyticity arguments in the material parameters ϵ (and analogously for μ) in a complex neighborhood of the guided mode resonance ω_0. The real $\hat{\omega}$ and imaginary $\check{\omega}$ parts of ω give us information about the central frequency and width of the scattering resonance.

Perturbations in the structural parameters create difficulties in the analysis of resonance that were not seen under perturbations of the wavenumbers of the source field since the conditions for complex analyticity in ϵ and μ are not immediate. The sensitivity analysis presented in this section deals with this problem though a complexification of the dielectric permittivity.

It is our goal to use the Cauchy-Riemann equations to derive some relations between the resonance ω and the dielectric permittivity ϵ. However this requires that the ω be a holomorphic function of the entries of ϵ, which it is not.

To make this more precise, notice, for example, that if ϵ is hermitian, then ω is not a complex-analytic function of the matrix entry ϵ_{12} of ϵ because ω depends formally on both ϵ_{12} and its conjugate, so that $\partial\omega/\partial\bar{\epsilon_{12}} \neq 0$, which means that ω is not analytic in ϵ_{12}. But ω *is* a real-analytic function of the real and imaginary parts of ϵ_{12}, and indeed of the real and imaginary parts of each entry of ϵ. Thus it makes sense to complexify the real and imaginary parts of each component of ϵ.

To do this, we replace ϵ (which we have assumed so far to be hermitian) with a more general matrix

$$\epsilon = \hat{\epsilon} + i\check{\epsilon}$$

which is the sum of a Hermitian matrix $\hat{\epsilon}$ and a anti-Hermitian matrix $i\check{\epsilon}$. This decomposition is canonical, and is determined by

$$\hat{\epsilon} = \frac{1}{2}(\epsilon + \epsilon^*) \qquad \check{\epsilon} = \frac{1}{2i}(\epsilon - \epsilon^*).$$

The component $\hat{\epsilon}$ is the (real) dielectric tensor, and the component $\check{\epsilon}$ incorporates material losses. The latter is required to be a definite matrix, $\check{\epsilon} \geq 0$. More explicitly, ϵ now has the form

$$\epsilon = \begin{bmatrix} \hat{\epsilon}_{11} & \hat{\epsilon}_{12}^{\mathrm{r}} + i\hat{\epsilon}_{12}^{\mathrm{i}} & 0 \\ \hat{\epsilon}_{12}^{\mathrm{r}} - i\hat{\epsilon}_{12}^{\mathrm{i}} & \hat{\epsilon}_{22} & 0 \\ 0 & 0 & \hat{\epsilon}_{33} \end{bmatrix} + i \begin{bmatrix} \check{\epsilon}_{11} & \check{\epsilon}_{12}^{\mathrm{r}} + i\check{\epsilon}_{12}^{\mathrm{i}} & 0 \\ \check{\epsilon}_{12}^{\mathrm{r}} - i\check{\epsilon}_{12}^{\mathrm{i}} & \check{\epsilon}_{22} & 0 \\ 0 & 0 & \check{\epsilon}_{33} \end{bmatrix}$$

with

$$\hat{\epsilon} = \begin{bmatrix} \hat{\epsilon}_{11} & \hat{\epsilon}_{12}^{\mathrm{r}} + i\hat{\epsilon}_{12}^{\mathrm{i}} & 0 \\ \hat{\epsilon}_{12}^{\mathrm{r}} - i\hat{\epsilon}_{12}^{\mathrm{i}} & \hat{\epsilon}_{22} & 0 \\ 0 & 0 & \hat{\epsilon}_{33} \end{bmatrix}$$

and

$$\check{\epsilon} = \begin{bmatrix} \check{\epsilon}_{11} & \check{\epsilon}_{12}^{\mathrm{r}} + i\check{\epsilon}_{12}^{\mathrm{i}} & 0 \\ \check{\epsilon}_{12}^{\mathrm{r}} - i\check{\epsilon}_{12}^{\mathrm{i}} & \check{\epsilon}_{22} & 0 \\ 0 & 0 & \check{\epsilon}_{33} \end{bmatrix}.$$

The 12-entry can be written as

$$\begin{aligned} \epsilon_{12} &= \hat{\epsilon}_{12}^{\mathrm{r}} + i\check{\epsilon}_{12}^{\mathrm{r}} + i(\hat{\epsilon}_{12}^{\mathrm{i}} + i\check{\epsilon}_{12}^{\mathrm{i}}) \\ &= \epsilon_{12}^{\mathrm{r}} + i\epsilon_{12}^{\mathrm{i}} \end{aligned}$$

Radiation losses, on the other hand, refer to loss of energy from the slab due to coupling to radiation states (scattering states of the system). They occur for lossless structures, and the strength of these losses are incorporated in the imaginary part $\check{\omega}$ of the complex resonance. We will sometimes refer to $\hat{\epsilon}$ as the Hermitian part of ϵ, and in a somewhat abuse of notation will refer to $\check{\epsilon}$ as the anti-Hermitian part of ϵ since $i\check{\epsilon}$ is anti-Hermitian. Alternatively, we can call $\hat{\epsilon}$ the lossless part of ϵ and $\check{\epsilon}$ the lossy part of ϵ.

For $\epsilon_{12}^{\mathrm{r}} = \hat{\epsilon}_{12}^{\mathrm{r}} + i\check{\epsilon}_{12}^{\mathrm{r}}$, for example, we have the following

$$\frac{\partial\omega}{\partial\epsilon_{12}^{\mathrm{r}}} = \frac{\partial(\hat{\omega} + i\check{\omega})}{\partial(\hat{\epsilon}_{12}^{\mathrm{r}} + i\check{\epsilon}_{12}^{\mathrm{r}})} = \frac{\partial(\hat{\omega}, \check{\omega})}{\partial(\hat{\epsilon}_{12}^{\mathrm{r}}, \check{\epsilon}_{12}^{\mathrm{r}})} = \begin{bmatrix} \frac{\partial\hat{\omega}}{\partial\hat{\epsilon}_{12}^{\mathrm{r}}} & \frac{\partial\hat{\omega}}{\partial\check{\epsilon}_{12}^{\mathrm{r}}} \\ \frac{\partial\check{\omega}}{\partial\hat{\epsilon}_{12}^{\mathrm{r}}} & \frac{\partial\check{\omega}}{\partial\check{\epsilon}_{12}^{\mathrm{r}}} \end{bmatrix}.$$

The four real derivatives in this matrix have distinct physical significances. For each of the four indices $\alpha \in \{11, 22, 12r, 12i\}$ (allow $\epsilon_{12}^{\mathrm{r}}$ to be denoted also by ϵ_{12r} and $\epsilon_{12}^{\mathrm{i}}$ to be denoted also by ϵ_{12i}),

$\partial\hat{\omega}/\partial\hat{\epsilon}_{\alpha}^{\mathrm{r}} =$ Sensitivity of the frequency of oscillation of the resonance to variations of the (real) dielectric permittivity

$\partial\hat{\omega}/\partial\check{\epsilon}_{\alpha}^{\mathrm{r}} =$ Sensitivity of the frequency of oscillation of the resonance to variations in the loss coming from the electric response of the material

$\partial\hat{\omega}/\partial\hat{\epsilon}_{\alpha}^{\mathrm{r}} =$ Sensitivity of the radiation losses associated with the resonance to variations of the (real) dielectric permittivity

$\partial\check{\omega}/\partial\hat{\epsilon}_{\alpha}^{\mathrm{r}} =$ Sensitivity of the radiation losses associated with the resonance to variations in the loss coming from the electric response of the material

The Cauchy-Riemann equations then imply relations between these quantities, which amount to interesting and useful physical principles. These relations are remarkable in that they assert that the experimental measurement of certain physical quantities determine the values of other physical quantities that are not obviously related. An example in the next chapter will illustrate how measuring radiation losses due to a rotation of the slab determines the amount of detuning that the resonant frequency would sustain if material losses were incurred.

The Cauchy-Riemann equations give us

$$\frac{\partial\hat{\omega}}{\partial\hat{\epsilon}_{12}^{\mathrm{r}}} = \frac{\partial\check{\omega}}{\partial\check{\epsilon}_{12}^{\mathrm{r}}} \quad \text{and} \quad \frac{\partial\hat{\omega}}{\partial\check{\epsilon}_{12}^{\mathrm{r}}} = -\frac{\partial\check{\omega}}{\partial\hat{\epsilon}_{12}^{\mathrm{r}}},$$

from which we get some interesting relations. The first tells us that the detuning of the central frequency due to changes in real dielectric coefficient is equal to the radiation loss generated by material loss. The second tells us that the detuning of the central frequency by the material loss is equal to the negative of the radiation loss generated by changes in the real part of the dielectric coefficient.

A special situation occurs when the complex resonance frequency ω is in fact real. This is the situation in which ω corresponds to a true guided mode of the slab structure. Since ω is a pole of a scattering matrix (which can also be identified with a pole of the analytic continuation through the continuous spectrum of the resolvent of the underlying Maxwell operator), it cannot enter the upper-half plane as long as $\check{\epsilon}$ remains definite (in fact, if ω is real, then $\check{\epsilon} = 0$, that is, the material is lossless). Thus the first derivatives of the $\check{\omega}$ with respect to any of the variables $\hat{\epsilon}_{ij}$ $(i, j \in \{1, 2\})$ must be equal to zero. Together with the Cauchy-Riemann equations, this means that

Theorem 4.1. *If the slab structure considered above admits a true guided mode at real frequency ω for a given Hermitian dielectric tensor ϵ, then for each $\alpha \in \{11, 22, 12r, 12i\}$,*

$$-\frac{\partial \hat{\omega}}{\partial \check{\epsilon}_\alpha} = \frac{\partial \check{\omega}}{\partial \hat{\epsilon}_\alpha} = 0$$

at the guided mode parameters. This implies that, at the guided mode parameters, the sensitiviity of the radiation losses to variations of the (real) dielectric permittivity and the sensitiviity of the frequency of oscillation to variations in the material loss both vanish.

Finer details of the sensitivities when ω is real can be obtained by considering higher derivatives and the relations between them imposed by the Cauchy-Riemann equations. For example,

$$
\frac{\partial^2 \omega}{\partial(\epsilon_{12}^{\mathrm{r}})^2} = \frac{\partial}{\partial \epsilon_{12}^{\mathrm{r}}} \left[\frac{\partial \hat{\omega}}{\partial \hat{\epsilon}_{12}^{\mathrm{r}}} + i \frac{\partial \breve{\omega}}{\partial \hat{\epsilon}_{12}^{\mathrm{r}}} \right] = \begin{bmatrix} \frac{\partial}{\partial \hat{\epsilon}_{12}^{\mathrm{r}}} \left(\frac{\partial \hat{\omega}}{\partial \hat{\epsilon}_{12}^{\mathrm{r}}} \right) & \frac{\partial}{\partial \breve{\epsilon}_{12}^{\mathrm{r}}} \left(\frac{\partial \hat{\omega}}{\partial \hat{\epsilon}_{12}^{\mathrm{r}}} \right) \\ \frac{\partial}{\partial \hat{\epsilon}_{12}^{\mathrm{r}}} \left(\frac{\partial \breve{\omega}}{\partial \hat{\epsilon}_{12}^{\mathrm{r}}} \right) & \frac{\partial}{\partial \breve{\epsilon}_{12}^{\mathrm{r}}} \left(\frac{\partial \breve{\omega}}{\partial \hat{\epsilon}_{12}^{\mathrm{r}}} \right) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \hat{\omega}}{\partial(\hat{\epsilon}_{12}^{\mathrm{r}})^2} & \frac{\partial^2 \breve{\omega}}{\partial(\breve{\epsilon}_{12}^{\mathrm{r}})^2} \\ \frac{\partial^2 \breve{\omega}}{\partial(\hat{\epsilon}_{12}^{\mathrm{r}})^2} & -\frac{\partial^2 \hat{\omega}}{\partial(\breve{\epsilon}_{12}^{\mathrm{r}})^2} \end{bmatrix},
$$

since the second derivative also satisfies the Cauchy-Riemann equations.

Chapter 5
Example Revisited: Perturbation Analysis of the Rotated Slab

The construction of a guided mode in the simplest example of a slab structure is straightforward. Fix $k_2 = 0$ and suppose the material tensors in the ambient medium have diagonal matrix realizations

$$\epsilon^0 = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mu^0 = \begin{bmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} \epsilon_{11} & 0 \\ 0 & \epsilon_{22} \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \epsilon_{22}\sin^2\gamma + \epsilon_{11}\cos^2\gamma & (\epsilon_{11} - \epsilon_{22})\sin\gamma\cos\gamma \\ (\epsilon_{11} - \epsilon_{22})\sin\gamma\cos\gamma & \epsilon_{11}\sin^2\gamma + \epsilon_{22}\cos^2\gamma \end{bmatrix}$$

$$= \begin{bmatrix} (\epsilon_{11} - \epsilon_{22})\cos^2\gamma + \epsilon_{22} & (\epsilon_{11} - \epsilon_{22})\sin\gamma\cos\gamma \\ (\epsilon_{11} - \epsilon_{22})\sin\gamma\cos\gamma & -(\epsilon_{11} - \epsilon_{22})\cos^2\gamma + \epsilon_{11} \end{bmatrix}$$

$$= \begin{bmatrix} \epsilon_{22} & 0 \\ 0 & \epsilon_{11} \end{bmatrix} + (\epsilon_{11} - \epsilon_{22}) \begin{bmatrix} \cos^2\gamma & \sin\gamma\cos\gamma \\ \sin\gamma\cos\gamma & -\cos^2\gamma \end{bmatrix}$$

We can also rewrite R as a perturbation of the original:

$$R_\gamma = \begin{bmatrix} \epsilon_{11} & 0 \\ 0 & \epsilon_{22} \end{bmatrix} + (\epsilon_{11} - \epsilon_{22}) \begin{bmatrix} -\sin^2\gamma & \sin\gamma\cos\gamma \\ \sin\gamma\cos\gamma & \sin^2\gamma \end{bmatrix}$$

Note that $\mathrm{trace}(R) = \epsilon_{11} + \epsilon_{22}$.

The leading order perturbation of R_γ as $\gamma \to 0$ occurs only in the off-diagonal elements:

$$R_\gamma = \begin{bmatrix} \epsilon_{22} & (\epsilon_{22} - \epsilon_{11})\gamma \\ (\epsilon_{22} - \epsilon_{11})\gamma & \epsilon_{11} \end{bmatrix} + O(\gamma^2).$$

It is easily seen that $R \to \epsilon^1$ as $\gamma \to 0$.

The sensitivity of the complex resonance to γ is obtained from $\partial\omega/\partial\epsilon^{\mathrm{r}}_{12}$ and the the chain rule:

$$\frac{\partial\omega}{\partial\gamma} = \frac{\partial\omega}{\partial\epsilon^{\mathrm{r}}_{12}}\frac{\partial\epsilon^{\mathrm{r}}_{12}}{\partial\gamma} = \frac{\partial\omega}{\partial\epsilon^{\mathrm{r}}_{12}}(\epsilon_{22} - \epsilon_{11}),$$

from which we will obtain the relations

$$\frac{\partial\hat\omega}{\partial\gamma} = \frac{\partial\hat\omega}{\partial\check\epsilon^{\mathrm{r}}_{12}}\frac{\partial\check\epsilon^{\mathrm{r}}_{12}}{\partial\gamma} + \frac{\partial\hat\omega}{\partial\hat\epsilon^{\mathrm{r}}_{12}}\frac{\partial\hat\epsilon^{\mathrm{r}}_{12}}{\partial\gamma} = 0 + \frac{\partial\hat\omega}{\partial\hat\epsilon^{\mathrm{r}}_{12}}(\epsilon_{22} - \epsilon_{11}),$$

$$\frac{\partial\check\omega}{\partial\gamma} = \frac{\partial\check\omega}{\partial\check\epsilon^{\mathrm{r}}_{12}}\frac{\partial\check\epsilon^{\mathrm{r}}_{12}}{\partial\gamma} + \frac{\partial\check\omega}{\partial\hat\epsilon^{\mathrm{r}}_{12}}\frac{\partial\hat\epsilon^{\mathrm{r}}_{12}}{\partial\gamma} = 0 + \frac{\partial\check\omega}{\partial\hat\epsilon^{\mathrm{r}}_{12}}(\epsilon_{22} - \epsilon_{11}).$$

Imagine that one could compute the sensitivities $\partial\hat\omega/\partial\hat\epsilon^{\mathrm{r}}_{12}$ and $\partial\check\omega/\partial\hat\epsilon^{\mathrm{r}}_{12}$ experimentally by rotating the slab. The first quantity is a measurement of the frequency detuning, and one deduces, by the Cauchy-Riemann equations, the sensitivity of radiation losses that would result if material losses in the 12-component were incurred. And by measuring the radiation losses resulting from rotation of the slab, one can deduce the amount of frequency detuning that would result due to material losses. In particular, one can say that, if the resonant frequency ω of the slab structure is real (that is, the slab admits a true, spectrally embedded guided mode), then, as we have discussed, $\partial\check\omega/\partial\hat\epsilon^{\mathrm{r}}_{12} = 0$, and thus $\partial\hat\omega/\partial\check\epsilon^{\mathrm{r}}_{12} = 0$, which implies the following physically meaningful result.

Proposition 5.1. *If the the structure in this example is tuned so that the rotation angle of the slab is exactly 90 degrees, then the slab admits a true, spectrally embedded guided mode. In this case, small material losses in the* 12*-component of* ϵ *will not induce significant detuning of the resonant frequency, that is, any detuning will be of quadratic order in the losses.*

References

[1] D. W. Berreman. Optics in stratified and anisotropic media: 4×4-matrix formulation.*J. Opt. Soc. Am.* 62(4), 502510 (1972).

[2] A.-S. Bonnet-Bendhia and F. Starling. Guided waves by electromagnetic gratings and non-uniqueness examples for the diffraction problem. *Math. Methods Appl. Sci.*, 17(5):305-338,1994.

[3] C. Chicone *Ordinary Differential Equations with Applications, 2nd Edition.* Springer, 2000.

[4] L.C. Evans *Partial Differential Equations, 2nd Edition.* American Mathematical Society, 2010.

[5] S. Fan and J.D. Joannopoulos. Analysis of guided resonances in photonic crystal slabs.*Phys. Rev. B,* 65(23)235112-1-8 (2002).

[6] U. Fano. Effects of configuration interaction on intesities and phase shifts. *Physical Review* 124(6):1866-1878 (1961).

[7] P. D. Hislop and I. M. Sigal. *Introduction to Spectral Theory: With Applications to Schrödinger Operators*. Springer, 1996.

[8] J.D. Joannopoulos, S.G. Johnson, J.N. Winn, R.D. Meade *Photonic Crystals: Molding the Flow of Light, 2nd Edition*. Princeton University Press, 2008.

[9] J.D. Joannopoulos and S.G. Johnson *Photonic Crystals: The Road from Theory to Practice*. Kluwer Academic Publishers, 2002.

[10] T. Kato *Perturbation Theory for Linear Operators*. Springer-Verlag, 1980.

[11] N. Ptitsyna and S. P. Shipman. A lattice model for resonance in open periodic waveguides. *Discrete Contin. Dyn. Syst.* 5(5): 9891020 (2012).

[12] M. Reed and B. Simon. *Methods of Mathematical Physics Vol. IV: Analysis of Operators.* Academic Press, 1980.

[13] S. P. Shipman. Resonant scattering by open periodic waveguides. *Progress in Computational Physics, Vol. 1.* Bentham Science Publishers, 2010.

[14] S.P. Shipman and H. Tu. Total resonant transmission and reflection by periodic structures. *SIAM J. Appl. Math.* 72(1), 216239 (2012).

[15] S.P. Shipman and S. Venakides. Resonance and bound states in photonic crystal slabs. *SIAM J. Appl. Math.*, 64(1):322-342 (2003).

[16] S.P. Shipman and S. Venakides. Resonant transmission near non-robust periodic slab modes. *Phys. Rev. E,* 71(1):026611-1-10 (2005).

[17] S.P. Shipman and A. Welters. Resonant electromagnetic scattering in anisotropic layered media. *Journal of Mathematical Physics* 54(10): 103511 (2013).

[18] F.G. Tricomi. *Integral Equations*. Dover, 1985.

[19] P. Yeh. *Optical Waves in Layered Media*. Wiley, 2005.

Appendix A
Volterra Integral Equations

Theorem A.1. *If the kernel $K(x, y)$ and the function $f(x)$ belong to the class L^2, then the Volterra integral equation of the second kind*

$$\phi(x) - \lambda \int_0^x K(x, y)\phi(y)dy = f(x) \quad (0 \le x \le h) \tag{A.1}$$

has one (and essentially only one) L^2 solution on $[0, h]$; this solution is given by the formula

$$\phi(x) = f(x) + \lambda \int_0^x H(x, y; \lambda) f(y)dy, \tag{A.2}$$

where the resolvent kernel $H(x, y; \lambda)$, given by the series

$$H(x, y; \lambda) = -\sum_{v=0}^{\infty} \lambda^v K_{v+1}(x, y). \tag{A.3}$$

of iterated kernels converges almost everywhere. Moreover, the resolvent kernel satisfies

$$\begin{aligned} K(x, y) + H(x, y; \lambda) &= \lambda \int_x^y K(x, z) H(z, y; \lambda) dz \\ &= \lambda \int_x^y H(z, y; \lambda) K(x, z) dz. \end{aligned}$$

$$(I - V)^{-1} = I + V + V^2 + V^3 + \cdots$$

Call $K = V + V^2 + V^3 + \cdots$. Then we want K to be a kernel in the following sense:

$$\begin{aligned} (I - V)\psi &= f \\ \psi &= (I + K) f \end{aligned}$$

If we make f something we can apply an integral kernel to, we can obtain ψ and it doesn't even have to be smooth.

Volterra Integral Equations of the Second Kind:

$$\phi_n(x) = f(x) + \lambda \int_0^x K(x,y)\phi_{n-1}(y)dy$$

$$\begin{aligned}
\lim_{n\to\infty} \phi_n(x) &= \lim_{n\to\infty} \left[f(x) + \lambda \int_0^x K(x,y)\phi_{n-1}(y)dy \right] \\
&= f(x) + \lambda \lim_{n\to\infty} \int_0^x K(x,y)\phi_{n-1}(y)dy \\
&= f(x) + \lambda \int_0^x K(x,y) \lim_{n\to\infty} \phi_{n-1}(y)dy
\end{aligned}$$

assuming bounded or dominated convergence applies. Hence as $n \to \infty$, we have the Volterra equation of the second kind

$$\phi(x) = f(x) + \lambda \int_0^x K(x,y)\phi(y)dy$$

$$\lambda^n \psi_n(x) = \phi_n(x) - \phi_{n-1}(x)$$

$$\lambda^{n-1} \psi_{n-1}(x) = \phi_{n-1}(x) - \phi_{n-2}(x)$$

$$
\begin{aligned}
\psi_n(x) &= \lambda^{-n}(\phi_n(x) - \phi_{n-1}(x)) \\
&= \lambda^{-n}\left[f(x) + \lambda\int_0^x K(x,y)\phi_{n-1}(y)dy - \left(f(x) + \lambda\int_0^x K(x,y)\phi_{n-2}(y)dy\right)\right] \\
&= \lambda^{-n}\left[\lambda\int_0^x K(x,y)\left(\phi_{n-1}(y) - \phi_{n-2}(y)\right)dy\right] \\
&= \lambda^{-n}\left[\lambda\int_0^x K(x,y)\lambda^{n-1}\psi_{n-1}(y)dy\right] \\
&= \lambda^{-n}\left[\lambda^{n}\int_0^x K(x,y)\psi_{n-1}(y)dy\right] \\
&= \int_0^x K(x,y)\psi_{n-1}(y)dy
\end{aligned}
$$

Hence

$$\psi_n(x) = \int_0^x K(x,y)\psi_{n-1}(y)dy.$$

So

$$\psi_1(x) = \int_0^x K(x,y)\psi_0(y)dy = \int_0^x K(x,y)f(y)dy$$

since $\psi_0 = \phi_0 = f$. Now,

$$
\begin{aligned}
\psi_2(x) &= \int_0^x K(x,y)\psi_1(y)dy \\
&= \int_0^x K(x,z)\left(\int_0^z K(z,y)f(y)dy\right)dz \\
&= \int_0^x \int_0^z K(x,z)K(z,y)f(y)dydz,
\end{aligned}
$$

noting that $0 \leq z \leq x$ and $y \leq z$. So when we interchange the limits we define this region by $z \geq y$ and $0 \leq y \leq x$. That is, we integrate y from 0 to x and z from y to x (as opposed to z from 0 to x and y from 0 to z. We get

$$\begin{aligned}\psi_2(x) &= \int_0^x \int_y^x K(x,z)K(z,y)f(y)dzdy \\ &= \int_0^x f(y)dy \int_y^x K(x,z)K(z,y)dz \\ &= \int_0^x \left[\int_y^x K(x,z)K(z,y)dz\right] f(y)dy \\ &= \int_0^x K_2(x,y)f(y)dy,\end{aligned}$$

where

$$K_2(x,y) := \int_y^x K(x,z)K(z,y)dz.$$

In a similar manner we obtain,

$$\psi_n(x) = \int_0^x K_n(x,y)f(y)dy$$

for $n = 1, 2, 3, \ldots$, where K_n is defined recursively by

$$K_1(x,y) \equiv K(x,y),$$

$$K_{n+1}(x,y) = \int_y^x K(x,z)K_n(z,y)dz.$$

So

$$\begin{aligned}\phi_n(x) &= \sum_{v=0}^{n} \lambda^v \psi_v(x) \\ &= f(x) + \sum_{v=1}^{n} \lambda^v \left[\int_0^x K_v(x,y)f(y)dy\right] \\ &= f(x) + \int_0^x \left[\sum_{v=1}^{n} \lambda^v K_v(x,y)\right] f(y)dy\end{aligned}$$

Let

$$H(x, y; \lambda) := -\sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, y).$$

Then we get

$$\begin{aligned}
\lim_{n\to\infty} \phi_n(x) &= f(x) + \int_0^x \left[\lim_{n\to\infty} \sum_{v=1}^{n} \lambda^v K_v(x, y) \right] f(y)dy \\
&= f(x) + \int_0^x \left[\sum_{v=1}^{\infty} \lambda^v K_v(x, y) \right] f(y)dy \\
&= f(x) + \int_0^x \left[\sum_{n=0}^{\infty} \lambda^{n+1} K_{n+1}(x, y) \right] f(y)dy \\
&= f(x) + \int_0^x \left[\lambda \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, y) \right] f(y)dy \\
&= f(x) - \lambda \int_0^x \left[-\sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, y) \right] f(y)dy \\
&= f(x) - \lambda \int_0^x H(x, y; \lambda) f(y)dy.
\end{aligned}$$

Hence

$$\phi(x) = f(x) - \lambda \int_0^x H(x, y; \lambda) f(y)dy.$$

Schwarz inequality:

$$\left[\int_a^b f(x)g(x)dx \right]^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx$$

Assume the kernel $K(x, y)$ is in $L^2([0, h] \times [0, h])$, i.e.

$$\|K\|^2 = \int_0^h \int_0^h K^2(x, y)dxdy \leq N^2$$

exists and is bounded by some constant N^2. Assume also that $f(x) \in L^2[0, h]$, i.e.

$$\|f\|^2 = \int_0^h f^2(x)dx$$

exists and is finite.

Since K is an L^2-kernel,

A. Fubini's theorem implies that

1. The functions

$$A(x) = \left[\int_0^h K^2(x,y)dy\right]^{\frac{1}{2}}$$

,

$$B(y) = \left[\int_0^h K^2(x,y)dx\right]^{\frac{1}{2}}$$

exist almost everywhere for $0 \leq x \leq h$ and $0 \leq y \leq h$,

2. $A(x), B(y)$ are L^2, and

3.

$$\|K\|^2 = \int_0^h A^2(x)dx = \int_0^h B^2(y)dy$$

And B. If $\Phi(x)$ is any L^2-function on $(0, h)$, then the two functions

$$\psi(x) = \int_0^h K(x,y)\Phi(y)dy$$

$$\chi(x) = \int_0^h K(x,y)\Phi(x)dx$$

are also L^2-functions (by the Schwarz inequality). Moreover,

$$\|\psi\| \leq \|K\|\|\Phi\|$$

and

$$\|\chi\| \leq \|K\|\|\Phi\|.$$

Similarly, the composition of L^2-kernels $K(x,y)$, $H(x,y)$:

$$G_1(x,y) = \int_0^h K(x,z)H(z,y)dz$$

and

$$G_2(x,y) = \int_0^h H(x,z)K(z,y)dz$$

are also L^2-kernels such that

$$\|G_1\| \leq \|K\|\|H\|$$

and

$$\|G_2\| \leq \|K\|\|H\|.$$

In particular,

$$\|K_n\| \leq \|K\|^n$$

in the case of iterated kernels.

$$\psi_n(x) = \int_0^x K(x,y)\psi_{n-1}(y)dy$$

$$\psi_1(x) = \int_0^x K(x,y)f(y)dy$$

$$\begin{aligned}
\psi_2(x) &= \int_0^x \int_y^x K(x,z)K(z,y)f(y)dzdy \\
&= \int_0^x f(y)dy \int_y^x K(x,z)K(z,y)dz \\
&= \int_0^x \left[\int_y^x K(x,z)K(z,y)dz\right] f(y)dy \\
&= \int_0^x K_2(x,y)f(y)dy,
\end{aligned}$$

$$
\begin{aligned}
\psi_3(x) &= \int_0^x K(x,y)\psi_2(y)dy \\
&= \int_0^x K(x,z)\psi_2(z)dz \\
&= \int_0^x K(x,z)\int_0^z K(z,s)\psi(s)dsdz \\
&= \int_0^x K(x,z)\int_0^z K(z,s)\int_0^s K(s,y)f(y)dydsdz \\
&= \int_0^x K(x,z)dz\int_0^z K(z,s)ds\int_0^s K(s,y)f(y)dy \\
&= \int_0^x dz\int_0^z ds\int_0^s K(x,z)K(z,s)K(s,y)f(y)dy \\
&= \int_0^x dz\int_0^z dy\int_y^z K(x,z)K(z,s)K(s,y)f(y)ds \\
&= \int_0^x dy\int_y^x dz\int_y^z K(x,z)K(z,s)K(s,y)f(y)ds \\
&= \int_0^x f(y)dy\int_y^x K(x,z)dz\int_y^z K(z,s)K(s,y)ds \\
&= \int_0^x f(y)dy\int_y^x K(x,z)K_2(z,y)dz \\
&= \int_0^x f(y)K_3(x,y)dy \\
&= \int_0^x K_3(x,y)f(y)dy
\end{aligned}
$$

In general,

$$K_{n+1}(x,y) = \int_y^x K(x,z)K_n(z,y)dz$$

For Volterra equations, $K(x,z) \equiv 0$ if $z > x$ and $K_n(z,y) \equiv 0$ if $z < y$, so in this case,

$$\int_y^x K(x,z)K_n(z,y)dz = \int_0^x K(x,z)K_n(z,y)dz,$$

and we can define the recursion as

$$K_{n+1}(x,y) = \int_0^x K(x,z)K_n(z,y)dz.$$

$$
\begin{aligned}
K_3^2(x,y) &= \left[\int_y^x K(x,z)K_2(z,y)dz\right]^2 \\
&\le \int_y^x K^2(x,z)dz \int_y^x K_2^2(z,y)dz \\
&\le \int_0^h K^2(x,z)dz \int_y^x A^2(z)B^2(y)dz \\
&\le A^2(x)B^2(y)\int_y^x A^2(z)dz
\end{aligned}
$$

$$
\begin{aligned}
K_4^2(x,y) &= \left[\int_y^x K(x,z)K_3(z,y)dz\right]^2 \\
&\le \int_y^x K^2(x,z)dz \int_y^x K_3^2(z,y)dz \\
&\le \int_0^h K^2(x,z)dz \int_y^x A^2(z)B^2(y)\int_y^z A^2(s)dsdz \\
&\le A^2(x)B^2(y)\int_y^x A^2(z)dz \int_y^z A^2(s)ds \\
&= A^2(x)B^2(y)F_2(x,y)
\end{aligned}
$$

$$K_{n+2}^2(x,y) = A^2(x)B^2(y)F_n(x,y), \quad n = 1,2,3,\dots$$

$$F_1(x,y) = \int_y^x A^2(z)dz$$

$$F_n(x,y) = \int_y^x A^2(z)F_{n-1}(z,y)dz, \quad n = 1,2,3,\dots$$

Claim:

$$F_n(x,y) = \frac{1}{n!}F_1^n(x,y), \quad n = 1,2,3,\dots$$

Proof: If $n = 1$, expression is trivial $F_1(x, y) = F_1(x, y)$. Assume

$$F_N(x, y) = \frac{1}{N!} F_1^N(x, y), \quad N = 1, 2, 3, \ldots, n-1.$$

$$\begin{aligned}
F_n(x, y) &= \int_y^x A^2(z) F_{n-1}(z, y) dz \\
&= \frac{1}{(n-1)!} \int_y^x A^2(z) F_1^{n-1}(z, y) dz \\
&= \frac{1}{(n-1)!} \int_y^x \frac{\partial F_1}{\partial z} F_1^{n-1}(z, y) dz; \qquad \frac{\partial F_1^n}{\partial z} = n F_1^{n-1} \frac{\partial F_1}{\partial z} \\
&= \frac{1}{(n-1)!} \int_y^x \frac{1}{n} \frac{\partial F_1^n}{\partial z}(z, y) dz \\
&= \frac{1}{n!} \int_y^x \frac{\partial F_1^n}{\partial z}(z, y) dz \\
&= \frac{1}{n!} \left[F_1^n(z, y)\right]_{z=y}^{z=x} \\
&= \frac{1}{n!} [F_1^n(x, y) - F_1^n(y, y)] \\
&= \frac{1}{n!} F_1^n(x, y)
\end{aligned}$$

$$\begin{aligned}
F_1(z, y) &= \int_y^z A^2(t) dt \\
&= \int_0^z A^2(t) dt - \int_0^y A^2(t) dt \\
&= F_1(z, 0) - F_1(y, 0)
\end{aligned}$$

$$\Rightarrow \frac{\partial F_1}{\partial z} = A^2(z)$$

For the theorem, we must prove that

$$\phi(x) = f(x) - \lambda \int_0^x H(x, y; \lambda) f(y) dy$$

1. is in $L^2[0,h]$

2. solves

$$\phi(x) - \lambda \int_0^x K(x,y)\phi(y)dy = f(x)$$

3. $-H(x,y;\lambda) = \sum_{\nu=0}^{\infty} \lambda^\nu K_{\nu+1}(x,y)$ converges almost everywhere

4. H satisfies

$$\begin{aligned} K(x,y) + H(x,y;\lambda) &= \lambda \int_y^x K(x,z)H(z,y;\lambda)dz \\ &= \lambda \int_y^x H(x,z;\lambda)K(z,y)dz \end{aligned}$$

Proposition A.2. *The resolvent kernel*

$$H(x,y;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,y)$$

converges almost everywhere.

Proof.

$$\sum_{n=0}^{\infty} |\lambda^n K_{n+1}(x,y)| = |K_1(x,y)| + |\lambda| \sum_{n=0}^{\infty} |\lambda^n| \cdot |K_{n+2}(x,y)| \leq |\lambda| \sum_{n=0}^{\infty} |\lambda^n| \cdot |K_{n+2}(x,y)|$$

$$F_n(x,y) = \frac{1}{n!} F_1^n(x,y) = \frac{1}{n!} \left(\int_y^x A^2(z)dz \right)^n \leq \frac{1}{n!} \left(\int_0^h A^2(z)dz \right)^n = \frac{\|K\|^{2n}}{n!} \leq \frac{N^{2n}}{n!}$$

$$|K_{n+2}(x,y)| \leq A(x)B(y)F_n(x,y)^{1/2} \leq A(x)B(y)\frac{N^n}{\sqrt{n!}}$$

So

$$|\lambda| \sum_{n=0}^{\infty} |\lambda^n| \cdot |K_{n+2}(x,y)| \leq |\lambda| \sum_{n=0}^{\infty} |\lambda^n| A(x)B(y)\frac{N^n}{\sqrt{n!}} = |\lambda| A(x)B(y) \sum_{n=0}^{\infty} \frac{(|\lambda|N)^n}{\sqrt{n!}}$$

The right hand side is finite (except possible on a set of measure zero) since the series $\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}$ has an infinite radius of convergence. The functions A and B are finite except possibly on a set of measure zero, hence we obtain convergence almost everywhere. $\square$

Appendix B

Theory of First Order ODE Systems

ODE Theory

$\dot{x} = F(t, x), \quad x(t_0) = x_0$ The function $t \mapsto x(t)$ is a solution to the initial value problem if and only if it is a solution of the integral equation

$$x(t) = \int_{t_0}^{t} F(s, x(s))ds.$$

$$\dot{x} = A(t)x + g(t, x), \quad x \in \mathbb{R}^n,$$

where $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a smooth function.

Theorem B.1 (Variation of Parameters Formula)**.** *Consider the initial value problem*

$$\dot{x} = A(t)x + g(t, x), \quad x(t_0) = x_0 \tag{B.1}$$

and let $t \mapsto \Phi(t)$ be a fundamental matrix solution for the homogeneous system $\dot{x} = A(t)x$ that is defined on some interval J_0 containing t_0. If $t_0 \mapsto \phi(t)$ is the solution of the initial value problem defined on some subinterval of J_0, then

$$\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(\phi(s), s)ds. \tag{B.2}$$

$$\frac{\partial \mathbf{F}}{\partial z} = \mathcal{A}\mathbf{F}, \quad \mathbf{F} = [E_1(z), E_2(z), H_1(z), H_2(z)]$$

Here the propagator matrix $\mathcal{A}$ is the operator$A$ from the grant proposal and $iJ^{-1}A$ from Welters' paper. For now we will take $\mathcal{A} = \mathcal{A}_0 + \sigma\mathcal{A}_1 + \mathcal{O}(\sigma^2)$. We now seek to derive the integral representation for the transfer matrix T across the slab from $z = 0$ to $z = L$. Let $T_0(z)$ denote the transfer matrix of the unperturbed problem, i.e. $T_0(z) = e^{\mathcal{A}_0 z}$.

From Chicone's ODE text, for the ODE $\dot{x} = A(t)x$, where $x \in \mathbb{R}^n$, we have the following theory:

$\dot{\Psi}(t) = A(t)\Psi(t) \Rightarrow \Psi$ is matrix solution

$\dot{\Phi}(t) = A(t)\Phi(t)$, $\Phi(t)$ nonsingular $\forall t \in J \Rightarrow \Phi$ is fundamental matrix solution

$\Psi(t) = \Phi(t)\Phi^{-1}(t_0), t_0 \in J \Rightarrow \Psi$ is principal fundamental matrix solution at t_0

Not all matrix solutions are invertible for all t, but such a solution will guarantee the existance of a matrix solution that is invertible for all t, referred to as a fundamental matrix solution. Then given a fundamental matrix solution, we have a principal fundamental matrix solution at each $t_0 \in J$, defined by $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$.

Q: If Φ and Φ' are both FMS, do $\Psi(t) = \Phi(t)\Phi^{-1}(t_0)$ and $\Psi'(t) = \Phi'(t)\Phi'(t_0)^{-1}$ define the same principal FMS? That is, is the PFMS uniquely defined for each $t_0 \in J$?

Since each element of J is associated with a PFMS $t \to \Psi(t)$ as defined above, we can treat J as an index set in the parameter τ to obtain a family of FMS,

$$\left\{t \mapsto \Psi(t, \tau) : \Psi(t, \tau) = \Phi(t)\Phi^{-1}(\tau), \forall \tau \in J\right\},$$

where $\Psi(\tau, \tau) = I$. This family is called the state transition matrix; the elements are the transfer matrices indexed by initial point τ. The state transmission matrix satisfies the following properties:

$$\Psi(\tau, \tau) = I$$

$$\Psi(t, s)\Psi(s, \tau) = \Psi(t, \tau)$$

$$\Psi(t, s)^{-1} = \Psi(s, t)$$

$$\frac{\partial \Psi}{\partial s}(t, s) = -\Psi(t, s)A(s)$$

Consequently, Ψ also satisfies the following equations

$$\Psi(t + s, 0) = \Psi(t, -s) = \Psi(t, 0)\Psi(0, -s) = \Psi(t, 0)\Psi(s, 0).$$

The Maxwell system has a unique solution $\psi(z) = T(z_0, z)\psi(z_0)$ for each initial condition $\psi_0 \in \mathbb{C}^4$.

$$\begin{cases} \frac{d}{dz}\psi = A(z)\psi(z) \\ \psi(z_0) = \psi_0 \end{cases}$$

The transfer matrix $T(z_0, z)$ is a principal fundamental matrix solution at z_0. As with Chicone, most texts present the theory of linear ODEs using time as the independent variable, but this is merely convention. The same results apply immediately to the Maxwell system in the one-dimensional spatial variable z considered here.

For the general inhomogeneous initial value problem

$$\begin{cases} \dot{x} = A(t)x + g(x, t) \\ x(t_0) = x_0 \end{cases}$$

we can express a solution ϕ in terms of the PFMS at the initial time t_0 using the variation of constants formula:

$$\phi(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(\phi(s), s)ds.$$

If $\dot{x} = A(t)x + b(t)$ is T-periodic then the variation of constants formula gives us

$$x(T) = \Phi(T)x(T) + \Phi(T)\int_{0}^{T} \Phi^{-1}(s)b(s)ds.$$

Since $x(T) = x(0)$, it follows that

$$(I - \Phi(T))x(0) = \Phi(T)\int_{0}^{T} \Phi^{-1}(s)b(s)ds.$$

Also useful in the study of periodic systems is Floquet's theorem, which states that if $\Phi(t)$ is a FMS of the T-periodic system $\dot{x} = A(t)x$, i.e. A(t) is continuous with period T, then for all $t \in \mathbb{R}$, $\Phi(t+T) = \Phi(t)\Phi^{-1}(0)\Phi(T)$. In addition, for each matrix B (possibly complex) such that $e^{TB} = \Phi^{-1}(0)\Phi(T)$, there is a (possibly complex) T-periodic matrix function $t \mapsto P(t)$ such that $\Phi(t) = P(t)e^{tB}$ for all $t \in \mathbb{R}$. Also, there is a real matrix R and a real $2T$-periodic matrix function $t \mapsto Q(t)$ such that $\Phi(t) = Q(t)e^{tR}$ for all $t \in \mathbb{R}$.

Appendix C

Spectral Theory of Linear Operators

Defining the adjoint of a linear operator A on a Hilbert space H with inner product $(\cdot,\cdot)$:

$A : H \to H$

Let $g \in H$. If $(Af, g) = (f, g^*) \forall f \in D(A)$ for some g^*, then g is in the domain of A^*, and g^* is said to be the image of g under the adjoint operator A^*, i.e. $g^* = A^*g$. In other words, g^* is the element of H whose inner product with an arbitrary element f is given by (Af, g). But since this requires that f be in the domain of A, when taking the adjoint of an operator in general, we are limited by $(Af, g) = (f, A^*g)$ where $f \in D(A)$. So a symmetric operator could fail to be self-adjoint if the domains don't match. That is, if $D(A) \neq D(A^*)$. Formally, the domain of A^* is defined as

$$D(A^*) = \{g \in H : (Af, g) = (f, g^*) \text{ for all } f \in D(A)\}$$

When considering the standard inner product on $\mathbb{C}$, we have $A^* = \overline{A^T} = \bar{A}^T = \overline{A}^T$. The Hermitian adjoint of a linear operator is a generalization of the conjugate transpose of a complex matrix, and the conjugate transpose of a complex matrix is a generalization of the complex conjugate of a complex number.

Theorem C.1. *If $T : H \to H$ has an inverse T^{-1}, and $D(T)$ and $D(T^{-1})$ are dense in H, then T^* and $(T^{-1})^*$ exist, and in fact $(T^*)^{-1} = (T^{-1})^*$.*

Proof. Let $f \in D(T), g \in D((T^{-1})^*)$ □

Continuation of the eigenvalues (Kato p. 387): If $T(x)$ is a self-adjoint holomorphic family, every isolated eigenvalue λ of $T = T(0)$ with finite multiplicity splits into one or several eigenvalues of $T(x)$ which are holomorphic at $x = 0$ [assuming that $x = 0$ is in the domain of definition D_0 of the family $T(x)$]. Each one $\lambda(x)$ of these holomorphic, together with the associated eigenprojection $P(x)$, can be continued analytically along the real axis, and the resulting pair represents an eigenvalue and an eigenprojection of $T(x)$. This is true even when the graph of $\lambda(x)$ crosses the graph of another such eigenvalue, as long as the eigenvalue is isolated and has finite multiplicity. In this way there is determined a maximal interval I of the real axis in which $\lambda(x)$ and $P(x)$ are holomorphic and represent an eigenvalue and an eigenprojection of $T(x)$. In general this maximal interval I differs from one $\lambda(x)$ to another. At one or the other end of I, $\lambda(x)$ can behave in various ways: it may tend to infinity or be absorbed into the continuous spectrum.

Kato: Analytic perturbation of eigenvalues:
$T(x) = T + xT'$, where x is a small scalar parameter. Can the eigenvalues/vectors of $T(x)$ be expressed as a power series in x? That is, are they holomorphic functions of x in the neighborhood of $x = 0$. If so, then the eigenvalues/vectors change with the same order of magnitude as the perturbation, i.e. x. The eigenvalues of $T(x)$ satisfy

$$\det(T(x) - \zeta) = 0.$$

This is an algebraic equation in ζ of degree $N = \dim X$, with coefficients which are holomorphic in x. The roots of $\det(T(x) - \zeta) = 0$ are (branches of) analytic functions on x with only algebraic singularities. More precisely, the roots for $x \in D_0$ constitute one or several branches of one or several analytic functions that have only algebraic singularities in D_0.

Knopp (4714):
Let z_0 be such that $g_m(z_0) \neq 0$. Then $G(z_0, w) = 0$ has m roots, some of which may be multiple roots. Let w_0 be an α-fold root, $1 \leq \alpha \leq m$. Then we have the following theorem:

Theorem C.2 (Continuity of the roots)**.** *If a circle K_ϵ with sufficiently small radius $\epsilon > 0$*

is described about w_0 as center, then it is possible to draw such a small circle K_δ with radius $\delta = \delta(\epsilon) > 0$ about z_0 as center, that, for every $z_1 \neq z_0$ in K_δ, the equation $G(z_1, w) = 0$ has precisely α distinct roots in K_ϵ.

This theorem gives a deeper interpretation of the multiplicity of a root; for it says that an α-fold root of an equation branches off into α simple roots if the coefficients of the equation are varied a little. If we make the further assumption that $D(z_0) \neq 0$, i.e. $\alpha = 1$ at z_0, then for every $z = z_1$ in K_δ, there is one, and only one root of $G(z, w) = 0$ in K_ϵ. Consequently, this root is a single-valued and continuous function $f_1(z)$ of z, concerning which we have:

Theorem C.3 (Differentiability of the Roots). *$w = f_1(z)$ is a regular function of z in K_δ.*

$T(\beta) = T + \beta T'$ perturbed matrix with eigenvalue $\lambda(\beta)$

Theorem C.4 (Analytic Eigenvalues/Reed and Simon). *Let $F(\beta, \lambda) = \lambda^n + a_1(\beta)\lambda^{n-1} + \cdots + a_n(\beta)$ be a polynomial of degree n in λ whose leading coefficient is one and whose coefficients are all analytic functions of β. Suppose that $\lambda = \lambda_0$ is a simple root of $F(\beta_0, \lambda)$. Then for β near β_0, there is exactly one root $\lambda(\beta)$ of $F(\beta, \lambda)$ near λ_0, and $\lambda(\beta)$ is analytic in β near $\beta = \beta_0$.*

Theorem C.5 (Puiseux Series/ Reed and Simon). *Let $F(\beta, \lambda) = \lambda^n + a_1(\beta)\lambda^{n-1} + \cdots + a_n(\beta)$ be an nth degree polynomial in λ whose leading coefficient is one and whose coefficients are all analytic functions of β. Suppose $\lambda = \lambda_0$ is a root of multiplicity m of $F(\beta_0, \lambda)$. Then for β near β_0, there are exactly m roots (counting multiplicity) of $F(\beta, \lambda)$ near λ_0 and these roots are the branches of one or more multivalued analytic functions with at worst algebraic branch points at $\beta = \beta_0$. Explicitly, there are positive integers $p_1, \ldots, p_k$ with $\sum_{i=1}^k p_i = m$ and multivalued analytic functions $\lambda_1, \ldots, \lambda_k$ (not necessarily distinct) with convergent Puiseux series (Taylor series in $(\beta - \beta_0)^{1/p}$)*

$$\lambda_i(\beta) = \lambda_0 + \sum_{j=1}^{\infty} \alpha_j^{(i)} (\beta - \beta_0)^{j/p_i}$$

so that the m roots near λ_0 are given by the p_1 values of λ_1, the p_2 values of λ_2, etc.

Corollary C.6. *If $T(\beta)$ is a matrix-valued analytic function near β_0 and if λ_0 is an eigenvalue of $T(\beta_0)$ of algebraic multiplicity m, then for β near β_0, $T(\beta)$ has exactly m eigenvalues (counting multiplicity) near λ_0. These eigenvalues are all the branches of one or more multivalued functions analytic near β_0 with at worst algebraic singularities at β_0.*

Suppose $\mathcal{A}(\sigma) = \mathcal{A}_0 + \sigma\mathcal{A}_1$ and λ_0 is a simple eigenvalue of $\mathcal{A}_0$. From Theorem C.4, it follows that for $\sigma \ll 1$, $\mathcal{A}(\sigma)$ has a unique eigenvalue $\lambda(\sigma)$ near λ_0 that is analytic near $\sigma = 0$. So there is a $0 < \delta \ll 1$ such that $\lambda(\sigma)$ is the only eigenvalue of $\mathcal{A}_0 + \sigma\mathcal{A}_1$ in the ball $B(\lambda_0, \delta) = \{\lambda : |\lambda - \lambda_0| < \delta\}$, and when $\mathcal{A}_0$ is self-adjoint, the formula

$$P(\sigma) = -\frac{1}{2\pi i}\oint_{|\lambda-\lambda_0|=\delta} (\mathcal{A}_0 + \sigma\mathcal{A}_1)^{-1}\, d\lambda \tag{C.1}$$

gives the projection onto the eigenspace of the eigenvalue $\lambda(\sigma)$.

Theorem C.7. *Let $H_0+\beta V$ be an analytic family of type (A) in a region R. Then $H_0+\beta V$ is an analytic family in the sense of Kato. In particular, if $0 \in R$ and if E_0 is an isolated nondegenerate eigenvalue of H_0, then there is a unique point $E(\beta)$ of $\sigma(H_0 + \beta V)$ near E_0 when $|\beta|$ is small which is an isolated nondegenerate eigenvalue. Moreover, $E(\beta)$ is analytic near $\beta = 0$.*

Definition 1. Let R be a connected domain in the complex plane and let $T(\beta)$ a closed operator with nonempty resolvent set, be given for each $\beta \in R$. We say that $T(\beta)$ is an analytic family of type (A) if and only if

1. The operator domain of $T(\beta)$ is some set D independent of β.

2. For each $\psi \in D$, $T(\beta)\psi$ is a vector-valued analytic function of β.

Definition 2. A (possibly unbounded) operator-valued function $T(\beta)$ on a complex domain R is called an analytic family (in the sense of Kato) if and only if:

1. For each $\beta \in R$, $T(\beta)$ is closed and has a nonempty resolvent set.

2. For every $\beta_0 \in R$, there is a $\lambda_0 \in \rho(T(\beta_0))$ so that $\lambda_0 \in \rho(T(\beta))$ for β near β_0 and $(T(\beta) - \lambda_0)^{-1}$ is an analytic operator-valued function of β near β_0.

Appendix D

The Weierstrass Preparation Theorem

D.1 The Weierstrass Preparation Theorem

The spectrum of an operator can be studied through analytic functions in the spectral variable. The Weierstrass preparation theorem serves as the bridge between these two theories for our scattering problem.

Theorem D.1 (Weierstrass Preparation Theorem 1)**.** *Let $F(z,w)$ be a function of two complex variables which is analytic in a neighborhood $|z-z_0|<r$, $|w-w_0|<\rho$ of the point (z_0,w_0), and suppose that*

$$F(z_0,w_0)=0, \qquad F(z_0,w)\not\equiv 0.$$

Then there is a neighborhood $|z-z_0|<r'<r$, $|w-w_0|<\rho'<\rho$ in which $F(z,w)$ can be written as

$$F(z,w)=\left[A_0(z)+A_1(z)w+\cdots+A_{k-1}(z)w^{k-1}+w^k\right]G(z,w),$$

where k is such that

$$\frac{\partial F(z_0,w_0)}{\partial w}=\cdots=\frac{\partial^{k-1}F(z_0,w_0)}{\partial w^{k-1}}=0, \qquad \frac{\partial^k F(z_0,w_0)}{\partial w^k}\neq 0,$$

the functions $A_0(z),A_1(z),\ldots,A_{k-1}(z)$ are analytic if $|z-z_0|<r'$, and the function $G(z,w)$ is analytic and nonzero if $|z-z_0|<r'$, $|w-w_0|<\rho'$.

The function $A_0(z)+A_1(z)w+\cdots+A_{k-1}(z)w^{k-1}+w^k$ is known as a *Weierstrass polynomial.* The final statement of the theorem, that the function $G(z,w)$ is analytic and nonzero if $|z-z_0|<r'$, $|w-w_0|<\rho'$, means that there is a neighborhood of the point

(z_0, w_0) where $\left[A_0(z) + A_1(z)w + \cdots + A_{k-1}(z)w^{k-1} + w^k\right] G(z,w) = 0$, but $G(z,w) \neq 0$. Then in that neighborhood, it must be the case that $A_0(z)+A_1(z)w+\cdots+A_{k-1}(z)w^{k-1}+w^k = 0$. In other words, there is a neighborhood of the zero (z_0, w_0) of the function $F(z,w)$ on which $F(z,w) = 0$ can be represented by the Weierstrass polynomial equation $A_0(z) + A_1(z)w + \cdots + A_{k-1}(z)w^{k-1} + w^k = 0$.

The theorem above is presented for an analytic function of two complex variables, but can be easily generalized to analytic functions of $N > 2$ variables.

Theorem D.2 (Weierstrass Preparation Theorem 2)**.** *Suppose that $f(z, \boldsymbol{p})$, $z \in \mathbb{C}$, $\boldsymbol{p} = (p_1, \ldots p_n) \in \mathbb{C}^n$, is an analytic function vanishing at the point $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$, where $z = z_0$ is an m-multiple root of the equation $f(z, \boldsymbol{p}_0) = 0$, i.e.,*

$$f(z_0, \boldsymbol{p}_0) = \frac{\partial f}{\partial z} = \cdots = \frac{\partial^{m-1} f}{\partial z^{m-1}} = 0, \qquad \frac{\partial^m f}{\partial z^m} \neq 0,$$

where the derivatives are taken at the point $z = z_0$, $\boldsymbol{p} = \boldsymbol{p}_0$. Then there exists a neighborhood $U_0 \subset \mathbb{C}^{n+1}$ of the point $(z_0, \boldsymbol{p}_0)$ in which the function $f(z, \boldsymbol{p})$ can be expressed as

$$f(z, \boldsymbol{p}) = ((z - z_0)^m + a_{m-1}(\boldsymbol{p})(z - z_0)^{m-1} + \cdots + a_0(\boldsymbol{p}))b(z, \boldsymbol{p}),$$

where $a_0(\boldsymbol{p}), \ldots, a_{m-1}(\boldsymbol{p}), b(z, \boldsymbol{p})$ are analytic functions uniquely defined by the function $f(z, \boldsymbol{p})$ and $a_i(\boldsymbol{p}_0) = 0, b(z_0, \boldsymbol{p}_0) \neq 0$.

With regard to the Weierstrass Preparation Theorem for analytic $f(z, \boldsymbol{p})$: The function $f(z, \boldsymbol{p})$ can be regarded as a family of functions of a single variable z, where $\boldsymbol{p}$ is the parameter vector. If we are talking about $f(z, z_2, \ldots, z_n)$ with zero at $(0, \ldots, 0)$ and $f(z, z_2, \ldots, z_n) = W(z)h(z_2, \ldots, z_n)$, then the set of zeros of f near $(0, \ldots, 0)$ can be found by fixing any small values of $z_2, \ldots, z_n$ and then solving the equation $W(z) = 0$. The z's such that $W(z) = 0$ form a number of continuously varying branches, in number equal to the degree of W in z. In particular, f cannot have an isolated zero.

Weierstrass preparation allows us to express the reflection $a(\boldsymbol{\kappa}, \omega)$, transmission $b(\boldsymbol{\kappa}, \omega)$, and simple eigenvalue branch $\ell(\boldsymbol{\kappa}, \omega)$, all of which are analytic functions near the true guided mode pair, as the product of a polynomial in one of the variables and a non-vanishing analytic function. This basically means that the leading order behavior of the original functions can be determined by a power series in $\boldsymbol{\kappa}$, the coefficients of which contain information about the resonant characteristics of the scattering problem. To summarize, the WPT allows us to reduce the problem to studying the coefficients of a power series in $\boldsymbol{\kappa}$. This concept is central to studying the resonance near a guided mode pair $(\boldsymbol{\kappa}_0, \omega_0)$.

Vita

Emily Anne McHenry is originally from Springfield, Illinois. She finished her undergraduate studies at Xavier University in Cincinnati, Ohio in May 2010. In August 2010 she came to Louisiana State University to pursue graduate studies, where she earned a Master of Science degree in Mathematics in December 2011. She is currently a candidate for the degree of Doctor of Philosophy in Mathematics, which will be awarded in December 2017.