Spaces of Order Arcs in Hyperspaces of Subcontinua.

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by
Mark Lynch
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ABSTRACT

Eberhart, Nadler, and Nowell asked for which Peano continua $X$ is it true that $\Gamma(X)$, the space of order arcs in $C(X)$, is homeomorphic to the Hilbert cube $\mathbb{I}$? They answered the question affirmatively when $X$ contains no free arcs. In Chapter IV, we characterize those 1-dimensional ANR $S$ $X$ for which $\Gamma(X)$ is homeomorphic to $\mathbb{I}$. In Chapters I and II, we develop techniques for analyzing this problem and, in Chapter III, we also apply these techniques in answering some open questions concerning Whitney levels in $C(X)$. 
INTRODUCTION

Preliminaries: The letter $X$ will always denote a metric continuum with metric $d$. For $\epsilon > 0$ and $A \subseteq X$, $N_\epsilon(A) = \{x \in X | d(x, A) < \epsilon\}$. The hyperspace of subcontinua of $X$ is $C(X) = \{K \subseteq X | K$ is a continuum} together with the Hausdorff metric $H_d$ defined by

$$H_d(K, M) = \inf\{\epsilon > 0 | K \subseteq N_\epsilon(M) \text{ and } M \subseteq N_\epsilon(K)\}.$$ Whenever $d$ is understood, we'll write $H(K, M)$. The space $C(C(X))$ is defined similarly and we denote its metric by $H^2$. An order arc $\alpha \in C(C(X))$ is an arc with the property that if $K, M \in \alpha$ then either $K \subseteq M$ or $M \subseteq K$. The space of order arcs in $C(X)$ is $\Gamma(X) = \{\alpha \in C(C(X)) | \alpha$ is an order arc} $\cup \{\{K\} | K \in C(X)\}$ considered as a subspace of $C(C(X))$. A Whitney map (W-map) is a map $\mu : C(X) \rightarrow [0,1]$ satisfying $\mu(\{x\}) = 0$ and if $K \subseteq M$, $K \neq M$, then $\mu(K) < \mu(M)$. The sets $\mu^{-1}(t)$ are called Whitney levels (W-levels) and a topological property $P$ is a Whitney property (W-property) provided whenever $X$ has property $P$, each W-level has property $P$. Kelly [12] was the first to introduce W-maps into the study of hyperspaces and used them as a means of considering order arcs as functions from $[0,1]$ into $C(X)$. For $\alpha \in \Gamma(X)$, let $\cap \alpha = \cap\{M | M \in \alpha\}$ and $\cup \alpha = \cup\{M | M \in \alpha\}$. Denote by $\alpha(t)$ the unique element $K \in \alpha$ such that $\mu(K) = (1 - t) \cdot \mu(\cap \alpha) + t \cdot \mu(\cup \alpha)$. We will think of order arcs in this manner throughout the remainder of this paper.

Some subspaces of $\Gamma(X)$ which we will study are

$$\Lambda_t = \{\alpha \in \Gamma(X) | \alpha(0) \in \mu^{-1}(t) \text{ and } \alpha(1) = X\} \text{ for } t \in [0,1].$$ $\Lambda_0$ is called the space of maximal order arcs. For $M \in C(X)$, let

\[v\]
\[ \Lambda_M = \{ \alpha \in \Gamma(X) | \alpha(0) = M \text{ and } \alpha(1) = X \} \text{ and } \mathcal{C}_M(X) = \{ K \in \mathcal{C}(X) | M \subseteq K \}. \]

The spaces \( \mathcal{C}_M(X) \) are called \textit{intervals of continua}. Each of these spaces will be studied in Chapters I and II.

\( \mathbb{Q} \) will denote the Hilbert cube which is \( \Pi_{i=1}^{\infty} [0,1] \) with the product topology. A space \( Y \) is an \textit{absolute neighborhood retract} (ANR) for metric spaces \( Z \) provided for any closed \( A \subseteq Z \) and any map \( f : A \rightarrow Y \), there is an extension \( \hat{f} : U \rightarrow Y \) of \( f \) to an open set \( U \supseteq A \). If \( U \) can be taken as all of \( Z \), then \( Y \) is called an \textit{absolute retract} (AR).

\textbf{History:} We will restrict ourselves to the history of the study of order arc spaces. For a brief history of hyperspace theory in general, we refer the reader to Nadler's text on the subject [18].

The topic of this thesis had its origin in John L. Kelley's doctoral dissertation, published in the Transactions of the AMS under the title "Hyperspaces of a Continuum" [12]. It was in this paper that Kelley first introduced \( W \)-maps into the study of hyperspaces, and also, studied the spaces \( \Gamma(X) \) for the first time. He showed \( \Gamma(X) \) is compact and used this to show \( \mathcal{C}(X) \) is the continuous image of the cone over the Cantor set.

The next paper on order arc spaces was the joint paper by Eberhart, Nadler, and Nowell entitled "Spaces of order arcs in hyperspaces" [10]. They showed \( \Gamma(X) = \mathbb{Q} \) if \( X \) is locally connected and contains no free arcs. They also showed \( \Gamma(S^1) \neq \mathbb{Q} \) and asked when is \( \Gamma(X) \) homeomorphic to \( \mathbb{Q} \)? We will give a partial answer to this question in Chapter IV by characterizing those 1-dimensional ANR\(^S \) \( X \) for which \( \Gamma(X) = \mathbb{Q} \).
Finally, the only other paper on order arc spaces was written by Doug Curtis "Applications of a selection theorem to hyperspace contractibility" [5]. He studied the spaces of maximal order arcs and gave a characterization of hyperspace contractibility in terms of them. It is interesting to note that Kelley also studied hyperspace contractibility in his paper, although not in terms of order arc spaces. Thus, Kelley and Curtis used order arc spaces as a tool to study the structure of hyperspaces. We will continue their study here and give several applications to hyperspaces.
CHAPTER I
The Spaces \( \Lambda_p \)

We begin the study of \( \Gamma(X) \) by studying its simplest subspace which we call \( \Lambda_p \). Recall that these are the spaces of maximal order arcs starting at a given point \( p \). Although these spaces have never been studied, they have applications to the study of \( W \)-levels in \( C(X) \) where some research in hyperspaces had been focused in the seventies. For example, the main result of this chapter is:

**Theorem 1.1:** Whitney Levels in \( C_p(X) \) are \( AR^S \).

It seems unlikely that a simpler proof of this theorem could be had without the use of \( \Lambda_p \). This result will be used in Chapter III to answer some open questions concerning \( W \)-levels in \( C(X) \). It will also prove useful in Chapter IV where we give a partial solution to a problem raised in [10]. And, Doug Curtis used it in his characterization of stable points in \( C(X) \) [6].

An **equiconnecting function** for a space \( Y \) is a map
\[
\lambda : Y \times Y \times [0,1] \rightarrow Y
\]
such that \( \lambda(a,b,0) = a, \lambda(a,b,1) = b, \lambda(a,a,t) = a \) for all \( a, b \in Y \) and \( t \in [0,1] \). A space which admits an equiconnecting function is called an **equiconnected space**. A set \( C \subseteq Y \) is **convex with respect to \( \lambda \)** provided \( \lambda : C \times C \times [0,1] \rightarrow C \). If \( \lambda \) is understood, we say \( C \) is **convex**. It's easy to see that the spaces \( \Lambda_p \) are equiconnected with \( \lambda : \Lambda_p \times \Lambda_p \times [0,1] \rightarrow \Lambda_p \) defined by \( \lambda(\alpha,\beta,t) = \{\alpha(r) | 0 < r < 1 - t\} \cup \{\alpha(1 - t) \cup \beta(s) | 0 < s < 1\} \) for \( \alpha, \beta \in \Lambda_p \) and \( t \in [0,1] \).

**Lemma 1.2:** Let \( C \subseteq \Lambda_p \) be a closed, convex set. If \( \overline{N}_\varepsilon(C) = \{ \beta \in \Lambda_p | H^2(\beta, C) < \varepsilon \} \), then \( \overline{N}_\varepsilon(C) \) is convex.
Proof: Let $\alpha, \beta \in \overline{N}_\varepsilon(C)$ and $t \in [0, 1]$. Choose $y_1, y_2 \in C$, $\hat{t} \in [0, 1]$ such that $H^2(\alpha, y_1) < \varepsilon$, $H^2(\beta, y_2) < \varepsilon$ and $H(\alpha(1 - t), y_1(1 - \hat{t})) < \varepsilon$.

**Claim:** $H^2(\lambda(\alpha, \beta, t), \lambda(y_1, y_2, \hat{t})) < \varepsilon$

By definition of $H^2$, we must show for all $s \in [0, 1]$, there exists $s' \in [0, 1]$ such that $H(\lambda(\alpha, \beta, t)(s), \lambda(y_1, y_2, \hat{t})(s')) < \varepsilon$ and for all $r \in [0, 1]$, there exists $r' \in [0, 1]$ such that $H(\lambda(\alpha, \beta, t)(r'), \lambda(y_1, y_2, \hat{t})(r)) < \varepsilon$. Since these arguments are similar, we only show the first one. For this, we must show $\lambda(\alpha, \beta, t)(s) \in \overline{N}_\varepsilon(\lambda(y_1, y_2, \hat{t})(s'))$ and $\lambda(y_1, y_2, \hat{t})(s') \in \overline{N}_\varepsilon(\lambda(\alpha, \beta, t)(s))$.

Again, since these arguments are similar, we only show the first containment.

**Case 1:** $s < 1 - t$.

Then, since $H(\alpha(1 - t), y_1(1 - \hat{t})) < \varepsilon$, it's easy to see there exists $s' < 1 - \hat{t}$ with $H(\alpha(s), y_1(s')) < \varepsilon$.

**Case 2:** $s > 1 - t$.

Then, $\lambda(\alpha, \beta, t)(s) = \alpha(1 - t) \cup \beta(r)$ for some $r \in [0, 1]$. Choose $r' \in [0, 1]$ such that $H(\beta(r), y_2(r')) < \varepsilon$. Since union is non-expansive, $H(\alpha(1 - t) \cup \beta(r), y_1(1 - \hat{t}) \cup y_2(r')) < \varepsilon$. Clearly, there exists $s' > 1 - \hat{t}$ such that $\lambda(y_1, y_2, \hat{t})(s') = y_1(1 - \hat{t}) \cup y_2(r')$.

Cases 1 and 2 show $\lambda(\alpha, \beta, t) \in \overline{N}_\varepsilon(\lambda(y_1, y_2, \hat{t}))$. This completes the proof of the claim. Since $C$ is convex, $\lambda(y_1, y_2, \hat{t}) \in C$ and so this proves the lemma. 

The following lemma is a special case of Theorem 3.4 in [7].
Lemma 1.3: If $Y$ is an equiconnected space with each point having a basis of convex sets, then $Y$ is an AR.

Theorem 1.4: $\Lambda_p$ is an AR.

Proof: Follows from Lemmas 1.2 and 1.3. []

A $Q$-manifold is a manifold modelled on the Hilbert cube $Q$.

Chapman [4] showed $Q$ is the only compact, contractible $Q$-manifold.

Several years later, Torunczyk [23] gave a beautiful characterization of $Q$-manifolds and the following theorem is a special case of that result.

Theorem 1.5: Let $Y$ be a compact ANR. If for each $\epsilon > 0$, there exists mappings $f, g : Y \to Y$ such that $d(f(y), y) < \epsilon$, $d(g(y), y) < \epsilon$ for all $y \in Y$, and $f(Y) \cap g(Y) = \phi$, then $Y$ is a $Q$-manifold.

Theorem 1.6: If $\Lambda_p$ is nondegenerate, then $\Lambda_p = Q$.

Proof: Since $\Lambda_p$ is contractible, we need only show it's a $Q$-manifold by Chapman's result [4]. Suppose the following condition holds:

(*) For every $\epsilon > 0$, there exists $\gamma, \beta \in \Lambda_p$, $t < \epsilon$ such that $\gamma(t) \neq \beta(t)$.

Define $f, g : \Lambda_p \to \Lambda_p$ by $f(\eta) = \lambda(\beta, \eta, 1 - t)$, $g(\eta) = \lambda(\gamma, \eta, 1 - t)$.

Then, $H^2(f(\alpha), \alpha) < \epsilon$, $H^2(g(\alpha), \alpha) < \epsilon$ for $\alpha \in \Lambda_p$ and $f(\Lambda_p) \cap g(\Lambda_p) = \phi$ since $f(\alpha)(t) \neq g(\beta)(t)$ for any $\alpha, \beta \in \Lambda_p$. By Theorem 1.5, $\Lambda_p$ is a $Q$-manifold. If (*) doesn't hold, then there exists $0 < \epsilon < 1$ such that for all $\beta, \gamma \in \Lambda_p$, $\beta(t) = \gamma(t)$ for $t < \epsilon$ but for any $\delta > 0$, there exists $\hat{\beta}, \hat{\gamma} \in \Lambda_p$ and $\epsilon < t < \epsilon + \delta$ such that $\hat{\beta}(t) \neq \hat{\gamma}(t)$. $\epsilon$ must be less than one since $\Lambda_p$ is non-
degenerate. We are now in the same situation as in (*) so \( \Lambda_p = \mathbb{Q} \). 

For \( M \in C(X) \), \( \Lambda_M(X) = \{ \alpha \in \Gamma(X) | \alpha(0) = M \text{ and } \alpha(1) = X \} = \Lambda_p(X^*) \), where \( X^* = X/M \) is the quotient space and \( p \in X^* \) is the point corresponding to \( M \). Hence, Theorem 1.6 holds with \( p \) replaced by any \( M \in C(X) \). Let \( e_t : \Lambda_p \rightarrow \mu_p^{-1}(t) \cap C(X) = \mu_p^{-1}(t) \) be the evaluation map defined by \( e_t(\alpha) = \alpha(t) \). For \( K \in \mu_p^{-1}(t) \), \( e_t^{-1}(K) = \{ \alpha \in \Lambda_p | \alpha(t) = K \} \) and it's not hard to see that we can apply the previous arguments to \( e_t^{-1}(K) \) to show these are degenerate or homeomorphic to the Hilbert cube. This gives the following theorem.

**Theorem 1.7**: If the point-inverses of \( e_t : \Lambda_p \rightarrow \mu_p^{-1}(t) \) are non-degenerate, then they are homeomorphic to the Hilbert cube.

Eberhart studied the spaces \( C_p(X) \) \([9]\) and called them intervals of continua. He showed \( C_p(X) \) are AR\(^g \) and gave some sufficient conditions in order that they be homeomorphic to \( \mathbb{Q} \). In the remainder of this chapter, we will show each Whitney level in \( C_p(X) \), denoted \( \mu_p^{-1}(t) \), is an AR. We point out that if \( G \) is a graph, then \( \mu_p^{-1}(t) \) is finite-dimensional and hence, an AR by Theorem 1.7 above and a result of Smale \([22]\). Doug Curtis used this special case in his characterization of stable points in \( C(X) \) for Peano continua \( X \) \([6]\). We will use his result in Chapter IV to characterize those graphs \( G \) for which \( \Gamma(G) = \mathbb{Q} \).

Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda_p \), \( 0 < t_1, t_2, \ldots, t_{n-1} < 1 \). Define \( \beta \in \Lambda_p \) by

\[
\beta = \alpha_1 v_{t_1} \alpha_2 v_{t_2} \cdots v_{t_{n-1}} \alpha_n
\]

\( = \{ \alpha_1(s) | 0 < s < t_1 \} \cup \{ \alpha_1(t_1) \cup \alpha_2(s) | 0 < s < t_2 \} \cup \ldots \)
It is not difficult to see that this construction obtained by attaching elements of \( A \) is continuous in the sense that if \( \alpha^m \rightarrow \alpha \), 
\[ t^m \rightarrow t \] as \( m \rightarrow \infty \) for \( i = 1, 2, \ldots, n \), then
\[ \alpha^m \rightarrow \alpha \] as \( m \rightarrow \infty \). Furthermore, for \( 0 < t_1, \ldots, t_{n-1} < 1 \),

\[ \operatorname{diam} H^2(\{\alpha_1, \ldots, \alpha_n\}) = \operatorname{diam} H^2(\{\alpha_1, \ldots, \alpha_n, \alpha_1 t_1, \alpha_2 t_2, \ldots, \alpha_{n-1} t_{n-1}\}) \].

These results follow from the fact that union is nonexpansive.

**Lemma 1.8:** (Kelly [12]) For \( \epsilon > 0 \), there exists \( \eta > 0 \) such that if \( A, B \in 2^X \) with \( A \subseteq B \) and \( \mu(B) - \mu(A) < \eta \), then \( H(A, B) < \epsilon \).

**Proof:** Suppose this is false. Then, there exists \( \epsilon > 0 \) such that for every \( n \), there exists \( A_n \subseteq B_n \) with \( A_n \subseteq B_n \) and \( \mu(B_n) - \mu(A_n) < \frac{1}{n} \), but \( H(A_n, B_n) > \epsilon \). Since \( 2^X \) is compact, there exist subsequences \( A_{n_i}, B_{n_i} \) of \( A_n, B_n \) respectively, such that \( A_{n_i} \rightarrow \operatorname{C} \) and \( B_{n_i} \rightarrow \operatorname{D} \). Since \( A_{n_i} \subseteq B_{n_i} \), and \( \mu(B_{n_i}) - \mu(A_{n_i}) < \frac{1}{n_i} \), it follows that \( \operatorname{C} \subseteq \operatorname{D} \) and \( \mu(\operatorname{C}) = \mu(\operatorname{D}) \). But then \( \operatorname{C} = \operatorname{D} \) so \( H(A_{n_i}, B_{n_i}) \rightarrow 0 \).

This is a contradiction since \( H(A_{n_i}, B_{n_i}) > \epsilon \) for each \( n \). [1]

We now begin the proof of Theorem 1.1. We will show \( \mu_p^{-1}(t) \) is an absolute extensor for metric spaces. Let \( Z \) be a metric space and \( A \subseteq Z \) be closed. Let \( g : A \rightarrow \mu_p^{-1}(t) \) be a map. We will define an extension \( \hat{g} : Z \rightarrow \mu_p^{-1}(t) \) using a method analogous to Dugundji's construction [8, p 188]. For every \( x \in Z - A \), let

\[ B_x = \{ z \in Z | d(x, z) < \frac{1}{2} d(x, A) \} \] where \( d \) is a metric for \( Z \). Let

\[ U = \{ U_\alpha | \alpha \in A \} \] be a neighborhood finite refinement of \( \{ B_x | x \in Z - A \} \).
and suppose \( A \) is well-ordered. With each \( U_\alpha \in \mathcal{U} \), associate \( a_\alpha \in A \) as follows: Choose \( x_\alpha \in U_\alpha \) and find \( a_\alpha \in A \) with 
\[ d(x_\alpha, a_\alpha) < 2 \cdot d(x_\alpha, A). \]
Let \( \{e_\alpha \mid a \in A\} \) be a partition of unity of \( \mathbb{Z} - A \) subordinated to \( \mathcal{U} \) and for each \( a \in A \), choose \( \beta_a \in \Lambda_p \) such that 
\[ e_t(\beta_a) = g(a). \]
Thus, each element of \( A \) has a corresponding element in \( \Lambda_p \).

Define \( \hat{g} : \mathbb{Z} \rightarrow \mathbb{U}^{-1}(t) \) in the following steps:

1. For \( x \in \mathbb{Z} - A \), let \( U_{a_1}, \ldots, U_{a_n} \) be those elements of \( \mathcal{U} \) containing \( x \), \( a_1 < a_2 < \ldots < a_n \) in \( A \) and define 
\[ e_{\alpha_i}(x) = \frac{\phi_{\alpha_i}(x)}{\phi_{\alpha_1}(x) + \cdots + \phi_{\alpha_n}(x)}, \quad i = 1, 2, \ldots, n. \]

2. Let \( \beta_{\alpha_1}, \ldots, \beta_{\alpha_n} \) be the elements of \( \Lambda_p \) corresponding to the elements \( a_1, \ldots, a_n \) of \( A \) and define 
\[ \beta_x = \beta_{\alpha_1} \cdot e_{\alpha_1}(x) \alpha_2 \cdot e_{\alpha_2}(x) \cdots \cdot e_{\alpha_{n-1}}(x) \alpha_n. \]

3. Define \( \hat{g}(x) = e_t(\beta_x). \)

For every \( a \in A \), set \( \hat{g}(a) = g(a) \). To show \( \hat{g} \) is a map, we need to show \( \hat{g} \) is continuous on \( \mathbb{Z} - A \) and the boundary of \( A \), which we do in the following two claims.

**CLAIM 1:** \( \hat{g} \) is continuous on \( \mathbb{Z} - A \).

Let \( x \in \mathbb{Z} - A \), \( U_{a_1}, \ldots, U_{a_n} \) be those elements of \( \mathcal{U} \) containing \( x \) and let \( x_n \to x \). \( x_n \) eventually belongs to each of these sets and since \( \mathcal{U} \) is locally finite, only finitely many others, say 
\[ U_{a_1}, \ldots, U_{a_n} (r) \] (by passing to a subsequence of \( x_n \) if necessary, we may assume these additional sets are fixed and don't vary with \( n \)). Let 
\( \{W_1, \ldots, W_{m+r}\} \) be the collection \( \{U_{a_1}, \ldots, U_{a_n}, U_{a_1}, \ldots, U_{a_n}\} \) in order according to \( A \). Let \( \beta_1, \ldots, \beta_{m+r} \) be the elements of \( \Lambda_p \).
corresponding to the elements \( a_i \in W_1, \ldots, a_{m+r} \in W_{m+r} \) as in (2) and suppose \( U_i = W_m \). Let \( \hat{t}(x, i) = \frac{\phi_i(x)}{\phi_1(x) + \ldots + \phi_m(x)} \) for \( i = 1, 2, \ldots, m' \) and let \( \beta'_x = \beta_1 V_{\hat{t}(x, 1)} \beta_2 V_{\hat{t}(x, 2)} \ldots V_{\hat{t}(x, m'-1)} \beta_{m'} \). Then, for \( j < m' \) and \( x \not\in W_j \), \( \hat{t}(x, j) = 0 \) so \( \beta'_x = \beta_x \) as defined in (2). Furthermore, since \( \hat{t}(x, m') = 1 \), \( \tau(x_n, m') = 1 \) (in fact, for \( j < m', \tau(x_n, j) \rightarrow \hat{t}(x, j) \)). Hence, if \( \beta'_x = \beta_1 V_{\hat{t}(x_n, 1)} \beta_2 V_{\hat{t}(x_n, 2)} \ldots V_{\hat{t}(x_n, m'-1)} \beta_{m'} \), then there exists \( N \) such that \( n > N \) implies \( e(t(\beta_x)) = \hat{g}(x_n) = e(t(\beta_x)) \). Since the attaching operation is continuous, \( \beta'_x \rightarrow \beta_x \). Thus, \( e(t(\beta_x)) \rightarrow e(t(\beta_x)) \) so \( \hat{g}(x_n) \rightarrow \hat{g}(x) \) and \( \hat{g} \) is continuous on \( Z - A \).

**Claim 2:** \( \hat{g} \) is continuous on the boundary of \( A \).

Let \( a \in A, x_n \in Z - A \), and \( x_n \rightarrow a \). Let \( E_n = \{ \beta \in \Lambda \mid \beta \} \) corresponds to \( x_n \) as in (2)). For each \( n \), \( E_n \) is a finite set and if \( \beta_n \) is any element of \( E_n \), then \( \beta_n(t) \rightarrow g(a) \) (by choice of \( a_u \) for \( U \in U \) and the continuity of \( g \) on \( A \)). Let \( \epsilon > 0 \) be fixed. Choose \( \eta > 0 \) as in Lemma 1.8 for \( \frac{\epsilon}{2} \). Since \( \mu \) is uniformly continuous on \( C(X) \), there exists \( 0 < \delta < \frac{\epsilon}{4} \) such that if \( H(K, M) < \delta \), then \( |\mu(K) - \mu(M)| < \eta \). Choose \( N \) so \( n > N \) implies any \( \beta \in E_n \) satisfies \( H(\mu(a), \beta(t)) < \frac{\delta}{2} \). Then, if \( E_n = \{ \beta_1^n, \ldots, \beta_{m_n}^n \} \) and \( K_n = \bigcup_{i=1}^{m_n} \beta_i^n(t) \), then \( K_n \in C(X) \) and \( H(K_n, \beta_i^n(t)) < \delta \) for \( i = 1, \ldots, m_n \). Hence, \( \mu(K_n) - t < \eta \). Since \( \hat{g}(x_n) \in K_n \) (by definition of \( \hat{g} \)), we have by Lemma 1.8 that \( H(\hat{g}(x_n), K_n) < \frac{\epsilon}{2} \). Thus, for any \( n > N \) and \( j < m_n \),
\[ H(\hat{g}(x_n), \hat{g}(a)) < H(\hat{g}(x_n), K_n) + H(K_n, \beta^n_j(t)) + H(\beta^n_j(t), \hat{g}(a)) \]

\[ < \frac{\varepsilon}{2} + \delta + \frac{\delta}{2} \]

\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} \]

\[ < \varepsilon. \]

Hence, \( \hat{g}(x_n) \rightarrow \hat{g}(a) \) so \( \hat{g} \) is continuous on the boundary of \( A \).

Thus, \( \hat{g} : Z \rightarrow \mu_p^{-1}(t) \) is a continuous extension of \( g \) so \( \mu_p^{-1}(t) \) is an AR. []

Corollary 1.9: \{X\} is unstable in \( C_p(X) \).

Proof: Let \( \varepsilon > 0 \). Choose \( t < 1 \) sufficiently large so that for any \( K \in \mu_p^{-1}([t,1]) = \mu_p^{-1}([t,1]) \cap C_p(X) \), \( H(K,X) < \varepsilon \). Since \( \mu_p^{-1}(t) \) is an AR, there is a retraction \( r : \mu_p^{-1}([t,1]) \rightarrow \mu_p^{-1}(t) \). Define \( R : C_p(X) \rightarrow \mu_p^{-1}([0,t]) \) by

\[ R(K) = \begin{cases} K, & \text{if } K \in \mu_p^{-1}([0,t]), \\ r(K), & \text{otherwise}. \end{cases} \]

Then, \( R(C_p(X)) \subset C_p(X) - \{X\} \) and \( H(R(M),M) < \varepsilon \) for any \( M \in C_p(X) \).

Thus, \( \{X\} \) is unstable in \( C_p(X) \). []

Note that the proof of Corollary 1.9 shows \( \mu_p^{-1}([0,t]) \) is an AR since it's a retract of \( C_p(X) \).

Corollary 1.10: \( C_p(X) - \{X\} \) is an AR.

Proof: Since \( C_p(X) - \{X\} \) is an ANR, it suffices to show it's n-connected for all \( n \in [1,1] \). Let \( f : S^n \rightarrow C_p(X) - \{X\} \) be a map where
$S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$. Choose $t < 1$ large so

$f(S^n) \subseteq \mu^{-1}_p([0,t])$. Since $\mu^{-1}_p([0,t])$ is an AR, $f$ has an extension

$\tilde{f} : B^{n+1} \rightarrow \mu^{-1}_p([0,t]) \subseteq C_p(X) - \{X\}$ where $B^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$. Thus, $\mu^{-1}_p([0,t])$ is n-connected. 

If $X$ has a cut point $p$, then it's easy to see there exists $t_0 < 1$ such that for any $t > t_0$, $\mu^{-1}_p(t) = \mu^{-1}_p(t)$. Hence,

**Theorem 1.11:** If $X$ has a cut point, then:

(a) there exists $t_0 < 1$ such that for $t > t_0$, $\mu^{-1}_p(t)$ is an AR and;

(b) $\{X\}$ is unstable in $C(X)$. 

CHAPTER II
The Spaces $\Lambda_t$

In this chapter, we study the spaces $\Lambda_t$ and investigate the relationships between these spaces and the $W$-levels $\mu^{-1}(t)$. This is motivated by Curtis' paper [5], where a characterization of hyperspace contractibility is given in terms of $\Lambda_0$. The results obtained here are used in Chapter III to study $W$-levels. The main results of this chapter are:

**Theorem 2.1:** If either $\mu^{-1}(t)$ or $\Lambda_t$ has one of the following properties, then so does the other:

(a) Locally connected;
(b) path connected;
(c) hereditarily indecomposable;
(d) pseudo-arc;
(e) ANR;
(f) AR.

**Theorem 2.2:** If $X = \mu^{-1}(0)$ is an ANR, then $\Lambda_0$ is a $Q$-manifold; and in fact, $\Lambda_0 = X \times D$.

We first introduce some further notation and establish some lemmas. Let $f : [0,1] \to C(X)$ be a map and define $h(t) = \cup_{s \leq t} f(s)$. Then, \( \{K \mid K \in h([0,1])\} \) defines an order arc $\alpha_f$ which we call the order arc induced by $f$. $f$ also induces a continuous family of order arcs \( \{\alpha(f,t) \mid t \in [0,1]\} \) as follows: For $t \in [0,1]$, let $h_t(s) = \cup_{r \leq s} f((1-r) \cdot t)$. \( \{K \mid K \in h_t([0,1])\} \) defines an order arc $\alpha(f,t)$. Note that $\alpha(f,t)(0) = f(t)$ and $\alpha(f,t)(1) = \cup_{s \leq t} f(s)$ so...
\(\alpha(f,t)\) is an order arc starting at \(f(t)\) and ending at a set containing \(f(0)\). We will also use the attaching operation as defined in Chapter I.

**Lemma 2.3:** Let \(f : [0,1] \rightarrow \mu^{-1}(t)\) be a map. Then,

\[
\text{diam}_H(f([0,1])) = \text{diam}_2(\{\alpha(f,s) | s \in [0,1]\}).
\]

**Proof:** It is clear that \(\text{diam}_H(f([0,1])) < \text{diam}_2(\{\alpha(f,s) | s \in [0,1]\})\).

Let \(\delta = \text{diam}_H(f([0,1]))\). We must show for all \(r, s \in [0,1]\) with \(r < s\),

\(H^2(\alpha(f,r), \alpha(f,s)) < \delta\). This is done in the following two claims.

**CLAIM 1:** For any \(p_1 \in [0,1]\), there exists \(p_2 \in [0,1]\) with \(H(\alpha(f,r)(p_1), \alpha(f,s)(p_2)) < \delta\).

Recall \(\alpha(f,r)(p_1) = \sum_{q \leq p_1} \alpha(f,s)(p_2) = f((1-q) \cdot r)\). Since \(r < s\), there is a \(p_2 \in [0,1]\) with \((1-p_2) \cdot s = (1-p_1) \cdot r\). Hence,

\(\alpha(f,s)(p_2) = \sum_{q \leq p_2} \alpha(f,s)(p_2) = \alpha(f,r)(p_1)\) so \(H(\alpha(f,r)(p_1), \alpha(f,s)(p_2)) = 0 < \delta\).

**CLAIM 2:** For any \(p_1 \in [0,1]\), there exists \(p_2 \in [0,1]\) with \(H(\alpha(f,s)(p_1), \alpha(f,r)(p_2)) < \delta\).

We have the following two cases:

**CASE 1:** \(r > (1-p_1) \cdot s\)

Then, there exists \(p_2 \in [0,1]\) with \((1-p_2) \cdot r = (1-p_1) \cdot s\) and the proof follows as in claim 1.

**CASE 2:** \(r < (1-p_1) \cdot s\)

Then, \(H(\alpha(f,r)(0), \alpha(f,s)(p_1)) < \delta\) since \(\alpha(f,r)(0) = f(r)\),
\(\alpha(f,s)(p_1) = \bigcup_{q<p_1} f((1 - q) \cdot s)\), and union is nonexpansive. This proves Claim 2 and hence, the lemma.  \[\]

**Lemma 2.4:** Let \(\alpha, \beta \in \Lambda_t\) with \(\alpha(0) = \beta(0)\). Then,
\[
H^2(\alpha, \beta) = \text{diam}_{H^2}(\{\alpha \cup_{1-t} \beta | t \in [0,1]\}).
\]

**Proof:** Clearly, \(H^2(\alpha, \beta) < \text{diam}_{H^2}(\{\alpha \cup_{1-t} \beta | t \in [0,1]\})\). Let \(\delta > H^2(\alpha, \beta)\) and \(N_\delta(\alpha) = \{K \in C(X) | H(K, \alpha) < \delta\}\). We must show for \(s < t\),
\[
H^2(\alpha \cup_{1-s} \beta, \alpha \cup_{1-t} \beta) < \delta
\]
which is equivalent to showing
\[
\alpha \cup_{1-s} \beta \subset N_\delta(\alpha \cup_{1-t} \beta) \quad \text{and} \quad \alpha \cup_{1-t} \beta \subset N_\delta(\alpha \cup_{1-s} \beta).
\]

**CLAIM 1:** \(\alpha \cup_{1-s} \beta \subset N_\delta(\alpha \cup_{1-t} \beta)\).

Recall \(\alpha \cup_{1-s} \beta = \{\alpha(r) | 0 < r < 1 - s\} \cup \{\alpha(1 - s) \cup \beta(r) | 0 < r < 1\}\).

Since \(s < t\), \(1 - t < 1 - s\). We have the following cases:

**CASE 1:** \(0 < r < 1 - t\).

Then, \((\alpha \cup_{1-s} \beta)(r) = \alpha(r) \in N_\delta(\alpha \cup_{1-t} \beta))\).

**CASE 2:** \(1 - t < r < 1 - s\).

\((\alpha \cup_{1-s} \beta)(r) = \alpha(r)\) and there exists \(r'\) with \(H(\alpha(r), \beta(r')) < \delta\). So, \(H(\alpha(r), \alpha(1 - t) \cup \beta(r')) < \delta\) and \(\alpha(1 - t) \cup \beta(r') = (\alpha \cup_{1-t} \beta)(s')\) for some \(s'\). Hence,
\[
(\alpha \cup_{1-s} \beta)(r) \in N_\delta(\alpha \cup_{1-t} \beta).
\]

**CASE 3:** \(1 - s < r\).

\((\alpha \cup_{1-s} \beta)(r) = \alpha(1 - s) \cup \beta(r')\) for some \(r'\). We can find \(r_1\) with \(H(\alpha(1 - s), \beta(r_1)) < \delta\). Let \(r_2 = \max\{r_1, r'\}\). Then,
\[
H(\alpha(1 - s) \cup \beta(r'), \beta(r_1) \cup \beta(r'')) = H(\alpha(1 - s) \cup \beta(r'), \beta(r_2)) < \delta
\]
since union is nonexpansive. So,
\[ H(\alpha(l - s) \cup \beta(r'), \alpha(l - t) \cup \beta(r'_2)) < \delta \]
since union is nonexpansive. Since
\[ \alpha(l - s) \cup \beta(r') = (\alpha \lor_{1-s} \beta)(r) \]
and \[ \alpha(l - t) \cup \beta(r'_2) = (\alpha \lor_{1-t} \beta)(s') \]
for some \( s' \), we have \( (\alpha \lor_{1-s} \beta)(r) \in N_\delta(\alpha \lor_{1-t} \beta) \). This proves
\[ \alpha \lor_{1-s} \beta \in N_\delta(\alpha \lor_{1-t} \beta) \]

**CLAIM 2:** \( \alpha \lor_{1-t} \beta \in N_\delta(\alpha \lor_{1-s} \beta) \).

Similar to Claim 1.

Claims 1 and 2 show \( H^2(\alpha \lor_{1-s} \beta, \alpha \lor_{1-t} \beta) < \delta \) for all \( s < t \). Thus,
\[ \text{diam}_{H^2}(\{ \alpha \lor_{1-t} \beta | t \in [0,1] \}) < \delta. \]
Since \( \delta > H^2(\alpha, \beta) \) is arbitrary,
\[ \text{diam}_{H^2}(\{ \alpha \lor_{1-t} \beta | t \in [0,1] \}) < H^2(\alpha, \beta). \]

**Lemma 2.5:** Let \( \alpha, \beta \in \Lambda_t \), \( f : [0,1] \to \mu^{-1}(t) \) be a path with
\( f(0) = \alpha(0) \) and \( f(1) = \beta(0) \). Then, there is a path \( g : [0,1] \to \Lambda_t \)
with \( g(0) = \alpha \), \( g(1) = \beta \), and \( \text{diam}_{H^2}(\{ g(r) | r \in [0,1] \}) < H^2(\alpha, \beta) + 2 \cdot \text{diam}_{H}(f([0,1])) \).

**Proof:** Let \( \alpha(f, t) \) be the continuous family of order arcs induced by \( f \) as in Lemma 2.3. Then, \( \alpha(f, 0) = \{ \{ f(0) \} \} \) and \( \alpha(f, 1) \) is an order arc starting at \( f(1) = \beta(0) \) and ending at \( u f(t) = \alpha(0) \). Define \( g : [0,1] \to \Lambda_t \)
by
\[ g(r) = \begin{cases} 
\alpha(f, 2r) \lor_{1} \alpha, & \text{if } 0 < r < 1/2, \\
(\alpha(f,1) \lor_{1} \alpha) \lor_{2-2r} \beta, & \text{if } 1/2 < r < 1.
\end{cases} \]

Note that \( g(0) = \alpha(f,0) \lor_{1} \alpha = \alpha \), \( g(1/2) = (\alpha(f,1) \lor_{1} \alpha) \lor_{1} \beta = \alpha(f,1) \lor_{1} \alpha \), and \( g(1) = (\alpha(f,1) \lor_{1} \alpha) \lor_{0} \beta = \beta \) so \( g([0,1]) \) is a path from \( \alpha \) to \( \beta \).
in \( A_t \). We must show for \( s < t \), \( H^2(g(s), g(t)) < H^2(\alpha, \beta) + 2 \cdot \text{diam}_H(f([0,1])) \).

**CASE 1:** \( 0 < s < t < \frac{1}{2} \)

\[ H^2(g(s), g(t)) < \text{diam}_H(f([0,1])) \] by Lemma 2.3 and the fact that \( H^2(\alpha(f,2s)V_1\alpha, \alpha(f,2t)V_1\alpha) = H^2(\alpha(f,2s), \alpha(f,2t)) \).

**CASE 2:** \( \frac{1}{2} < s < t < 1 \)

\[ H^2(g(s), g(t)) < H^2(g(\frac{1}{2}), g(1)) = H^2(\alpha(f,1)V_1\alpha, \beta) \] by Lemma 2.4, and \( H^2(\alpha(f,1)V_1\alpha, \beta) < H^2(\alpha(f,1)V_1\alpha, \alpha) + H^2(\alpha,\beta) < \text{diam}_H(f([0,1])) + H^2(\alpha,\beta) \) by Case 1.

**CASE 3:** \( 0 < s < \frac{1}{2} < t < 1 \)

\[ H^2(g(s), g(t)) < H^2(g(s), g(\frac{1}{2})) + H^2(g(\frac{1}{2}), g(t)) \]

\[ < \text{diam}_H(f([0,1])) + \text{diam}_H(f([0,1])) + H^2(\alpha,\beta) \]

by Cases 1 and 2.

This completes the proof of the lemma. []

We now begin the proof of Theorem 2.1. Suppose \( A_t \) has any one of the properties (a) through (d). Since the evaluation map \( e_0 : A_t \to \mu^{-1}(t) \) defined by \( e_0(\alpha) = \alpha(0) \) is a closed map, it preserves properties (a) and (b). \( e_0 \) is also monotone, and Bing [1] has shown monotone maps preserve properties (c) and (d). Conversely, if \( \mu^{-1}(t) \) has property (a) or (b), then so does \( A_t \) by Lemma 2.5. Suppose \( \mu^{-1}(t) \) has property (c) or (d). We will show \( A_t \) does also by showing that if \( \mu^{-1}(t) \) is hereditarily indecomposable, then
$e_0 : \Lambda_t \longrightarrow \mu^{-1}(t)$ is 1-1 and hence, a homeomorphism. It is known (see [18]) that $X$ is hereditarily indecomposable if and only if for any $K, M \in C(X)$ for which $K \cap M \neq \emptyset$, either $K \subset M$ or $M \subset K$. Let $\alpha, \beta \in \Lambda_t$ and $\alpha(0) = \beta(0)$. We must show $\alpha(r) = \beta(r)$ for all $r \in [0,1]$. Suppose this is false. Then, there is an $r$ such that $\alpha(r) \neq \beta(r)$. Since $\alpha(r)$ and $\beta(r)$ belong to the same $W$-level, $\alpha(r) \notin \beta(r)$ and $\beta(r) \notin \alpha(r)$. Define $\Sigma_\alpha = \{ K \in \mu^{-1}(t) | K \subset \alpha(r) \}$ and $\Sigma_\beta = \{ M \in \mu^{-1}(t) | M \subset \beta(r) \}$. Then, $\Sigma_\alpha$ and $\Sigma_\beta$ belong to $C(\mu^{-1}(t))$ and $\Sigma_\alpha \cap \Sigma_\beta \neq \emptyset$ (both contain $\alpha(0) = \beta(0)$). Hence, since $\mu^{-1}(t)$ is hereditarily indecomposable, $\Sigma_\alpha \subseteq \Sigma_\beta$ or $\Sigma_\beta \subseteq \Sigma_\alpha$. But, this is impossible since $\alpha(r) \notin \beta(r)$ and $\beta(r) \notin \alpha(r)$ implies there exists $x \in \alpha(r)$, $y \in \beta(r)$ with $x \notin \beta(r)$, $y \notin \alpha(r)$. So, there exists $K \in \Sigma_\alpha$, $M \in \Sigma_\beta$ with $x \in K$, $y \in M$ so $K \notin \Sigma_\beta$, $M \notin \Sigma_\alpha$. This contradiction shows $\alpha(r) = \beta(r)$ for all $r$ so $e_0$ is 1-1 and a homeomorphism. This completes the proof of Theorem 2.1, properties (a) through (d).

Before completing the proof of Theorem 2.1 with respect to properties (e) and (f), we establish a few lemmas which have some intrinsic interest.

**Lemma 2.6:** If $\mu^{-1}([t,1])$ is contractible, then $e_0 : \Lambda_t \longrightarrow \mu^{-1}(t)$ is a homotopy equivalence.

**Proof:** It suffices to show there is an embedding $g : \mu^{-1}(t) \longrightarrow \Lambda_t$ such that $e_0 \circ g$ is the identity on $\mu^{-1}(t)$ since it's easy to show (using the attaching operation $\vee$) $g \circ e_0$ is homotopic to the identity on $\Lambda_t$. Let $h : \mu^{-1}([t,1]) \times [0,1] \longrightarrow \mu^{-1}([t,1])$ be a contraction with $h(K,1) = \{X\}$ for all $K \in \mu^{-1}([t,1])$. Let
$K \in \mu^{-1}(t)$ be fixed and note that $h(K,\cdot) : [0,1] \to \mu^{-1}([t,1])$ is a map and so induces an order arc $g(K)$. Since $h(K,1) = \{x\}$, $g(K) \in \Lambda_t$. Hence, $g : \mu^{-1}(t) \to \Lambda_t$ is the desired embedding. \[

\textbf{Corollary 2.7:} \ X \text{ is contractible if and only if } \Lambda_0 \text{ and } C(X) \text{ are contractible.}

\textbf{Proof:} Suppose $\Lambda_0$ and $C(X)$ are contractible. Then $X$ is contractible by Lemma 2.6. If $X$ is contractible, then $C(X)$ is contractible so by Lemma 2.6, $\Lambda_0$ is contractible. \[

\text{We will show later in this chapter that this is the strongest possible result by giving an example of a space } X \text{ which is not contractible but for which } \Lambda_0 \text{ is contractible.}

\textbf{Lemma 2.8:} If $\mu^{-1}(t)$ is contractible, then $\Lambda_t$ is contractible.

\textbf{Proof:} Let $h : \mu^{-1}(t) \times [0,1] \to \mu^{-1}(t)$ be a contraction with $h(K,0) = K$ and $h(K,1) = \hat{A}$ for each $K \in \mu^{-1}(t)$. For $K \in \mu^{-1}(t)$, $\{h(K,s)|0 < s < 1\}$ induces a continuous family of order arcs which we denote by $\alpha(K,s)$ (see Lemma 2.3) where $\alpha(K,s)(0) = h(K,s)$ and $\alpha(K,s)(1) = u \circ h(K,(1-r) \cdot s)$. Define $\hat{A} : \Lambda_t \times [0,1] \to \Lambda_t$ by $\hat{h}(\beta,r) = \alpha(\beta(0),r) \circ \beta$ for each $\beta \in \Lambda_t$. It's easy to see $\hat{A}$ is a map and $\hat{A}(\Lambda_t,1) \subset \hat{A} = \{v \in \Lambda_t | v(0) = \hat{A}\}$ which is contractible. Thus, $\Lambda_t$ is contractible. \[

\textbf{Lemma 2.9:} (See [18], p. 52) Let $Y$ be a continuum. If $\Sigma$ is a locally connected subcontinuum of $C(Y)$ such that $u \circ \sigma \in \Sigma$ for any subcontinuum $\sigma$ of $\Sigma$, then $\Sigma$ is an AR.

\textbf{Lemma 2.10:} If $\mu^{-1}(t)$ is locally connected, then $\mu^{-1}([t,1])$ is an AR.
Proof: Since \( \mu^{-1}(t) \) is locally connected, \( \Lambda_t \) is locally connected by Theorem 2.1(a). Hence, \( \Lambda_t \times [0,1] \) is also. Define

\[ h : \Lambda_t \times [0,1] \to \mu^{-1}([t,1]) \] by \( h(\alpha, r) = \alpha(r) \) for \( \alpha \in \Lambda_t \). Since \( h \) is a closed, surjective map, \( \mu^{-1}([t,1]) \) is locally connected. By Lemma 2.9, \( \mu^{-1}([t,1]) \) is an AR. []

If \( Y \) is a continuum, then let \( F_n(Y) = \{ A \subseteq Y \mid A \text{ has at most } n \text{ points} \} \). Notice that \( F_1(Y) = Y \).

Lemma 2.11: If \( \mu^{-1}(t) \) is locally connected, then

\[ \Gamma(\mu^{-1}([t,1])) = \{ \alpha \in \Gamma(X) \mid \alpha(0) \in \mu^{-1}([t,1]) \} \] is an AR.

Proof: Clearly, \( \Gamma(\mu^{-1}([t,1])) \) is locally connected so \( C(\Gamma(\mu^{-1}([t,1]))) \) is an AR. We can use the same argument as in Lemma 3.3 of [10] to show \( F_1(\Gamma(\mu^{-1}([t,1]))) \) is a retract of \( C(\Gamma(\mu^{-1}([t,1]))) \) and therefore, an AR. []

We may now show that \( \Lambda_t \) is an ANR (AR) if and only if \( \mu^{-1}(t) \) is an ANR (AR). Suppose \( \Lambda_t \) is an ANR. Then it's locally connected so \( \mu^{-1}(t) \) is locally connected by Theorem 2.1(a). By Lemma 2.10, \( \mu^{-1}([t,1]) \) is an AR and hence, contractible. As in the proof of Lemma 2.6, there exists an embedding \( g : \mu^{-1}(t) \to \Lambda_t \) such that \( e_0 \circ g \) is the identity on \( \mu^{-1}(t) \). Let \( Y \) be any metric space, \( A \subseteq Y \) closed, and \( f : A \to \mu^{-1}(t) \) a map. Then, \( g \circ f : A \to \Lambda_t \) has an extension \( (g \circ f)^A : U \to \Lambda_t \) where \( A \subseteq U \) open in \( Y \). It's clear that \( e_0 \circ (g \circ f)^A \) is the desired extension of \( f \) so \( \mu^{-1}(t) \) is an ANR. Now suppose \( \mu^{-1}(t) \) is an ANR. There exists an open set \( U \subseteq \mu^{-1}([t,1]) \) and a retraction \( r : U \to \mu^{-1}(t) \). Define \( g : (U \times \{0,1\}) \cup (\mu^{-1}(t) \times [0,1]) \to \mu^{-1}([t,1]) \) by \( g(K,0) = K, g(K,1) = \)
r(K) for all K \in U and g(K,s) = K for (K,s) \in \mu^{-1}(t) \times [0,1].

Since \mu^{-1}([t,1]) is an AR and \langle U \times \{0,1\} \cup \mu^{-1}(t) \times [0,1] \rangle is closed in U \times [0,1], g has an extension \hat{g} : U \times [0,1] \rightarrow \mu^{-1}([t,1]).

Note that for K \in U, \{\hat{g}(K,s) \mid 0 < s < 1\} is a path in \mu^{-1}([t,1]) from \hat{g}(K,0) = K to \hat{g}(K,1) = r(K) \in \mu^{-1}(t). This induces an order arc

\alpha_t(K) with \alpha_t(K)(0) = r(K) and \alpha_t(K)(1) = K (this is similar to the construction in Lemma 2.3). Let \emptyset = \{\beta \in \Gamma(\mu^{-1}([t,1])) \mid \beta(0) \in U\}. \emptyset is open in \Gamma(\mu^{-1}([t,1])) and so, is an ANR. We'll define a retraction R : \emptyset \rightarrow \Lambda_t. As in the proof of Lemma 2.6, there is an embedding h : \mu^{-1}([t,1]) \rightarrow \{\beta \in \Gamma(\mu^{-1}([t,1])) \mid \beta(1) = \{X\}\} such that e_0 \circ h is the identity on \mu^{-1}([t,1]). R : \emptyset \rightarrow \Lambda_t is defined by

R(\gamma) = \alpha_t(\gamma(0))V_1 \gamma V_1 h(\gamma(1))

Since \alpha_t(\gamma(0))(0) \supseteq \gamma(0), \alpha_t(\gamma(0))V_1 \gamma is well-defined. Also, since \gamma(1) = h(\gamma(1))(0), \gamma V_1 h(\gamma(1)) is well-defined. Since R is a composition of maps, it's continuous. Further, R(\gamma)(0) = \alpha_t(\gamma(0))(0) = r(\gamma(0)) \in \mu^{-1}(t) and R(\gamma)(1) = h(\gamma(1))(1) = \{X\} so R(\gamma) \in \Lambda_t. It is easy to see R is the identity on \Lambda_t so \Lambda_t is a retract of \emptyset and hence, an ANR. This proves Theorem 2.1(e). Theorem 2.1(f) follows from Theorem 2.1(e), Lemma 2.6, and Lemma 2.8. This completes the proof of Theorem 2.1.

If f : X \rightarrow X is a map, then f induces a map

\hat{f} : C(X) \rightarrow C(X) defined by \hat{f}(K) = \{f(x) \mid x \in K\} which induces a map

\hat{\gamma} : \Gamma(X) \rightarrow \Gamma(X) defined by \hat{\gamma}(\alpha) = \{\hat{f}(\alpha(t)) \mid t \in [0,1]\}. Furthermore, if \varepsilon > 0 and d(f(x),x) < \varepsilon for all x \in X, then H(\hat{f}(K),K) < \varepsilon for all K \in C(X) and H^2(\hat{\gamma}(\alpha),\alpha) < \varepsilon for all \alpha \in \Gamma(X). Bing [2] has
shown all Peano continua admit a convex metric. If $X$ is a Peano continuum, then $d$ will always denote a convex metric. For each $M \in C(X)$, define $c(M,t) = \{x|d(x,M) < t\}$ and $\hat{c}(M,t) = \{c(M,s)|0 < s < t\}$. When $d$ is convex, $\hat{c}(M,t)$ is an order arc from $M$ to $c(M,t)$ and is called the order arc induced by convex growth.

We now begin the proof of Theorem 2.2, using Torunczyk's characterization of $\mathbb{Q}$-manifolds (Theorem 1.5). There are two cases: Either $X$ admits an open embedding of $[0,1)$ or it doesn't. However, we find it convenient to first prove the following statement which overlaps these two cases.

**CLAIM:** If $X$ has at least two unstable points, then $\Lambda_0$ is a $\mathbb{Q}$-manifold.

Let $p,q \in X$ be two unstable points and let $\epsilon > 0$. Since $X$ is locally connected, there exists $0 < \delta < \epsilon$ such that if $d(x,y) < \delta$, then there is a path from $x$ to $y$ of diameter less than $\epsilon$. Let $f : X \to X - \{p,q\}$ be the induced map. Let $g_p,g_q : [0,1] \to X$ be maps with $g_p(0),g_q(0) \in f(X)$, $g_p(1) = p$, $g_q(1) = q$ and $\text{diam}(g_p([0,1])) < \epsilon$, $\text{diam}(g_q([0,1])) < \epsilon$. If $\delta$ is chosen sufficiently small, we can be sure that $p \notin g_q([0,1])$ and $q \notin g_p([0,1])$. Hence, if $\alpha_p$ and $\alpha_q$ are the order arcs induced by $g_p$ and $g_q$ respectively, then $\alpha_p(0), \alpha_q(0) \in f(X)$, $p \notin \alpha_q(1)$, and $q \notin \alpha_p(1)$. Define $h_1,h_2 : \Lambda_0 \to \Lambda_0$ by

$$h_1(\alpha) = \hat{f}(\alpha)V_1\alpha_p V_1\hat{c}(f(X) \cup \alpha_p(1),1)$$
$$h_2(\alpha) = \hat{f}(\alpha)V_1\alpha_q V_1\hat{c}(f(X) \cup \alpha_q(1),1)$$

Then, $H^2(h_i(\alpha),\alpha) < \epsilon$ for each $\alpha \in \Lambda_0$ and $i = 1,2$, and
h_1(\Lambda_0) \cap h_2(\Lambda_0) = \emptyset \text{ since } p \notin f(X) \cup q(1) \text{ and } q \notin f(X) \cup q(1)$. By Theorem 1.5, \Lambda_0 is a $\mathcal{Q}$-manifold. This completes the proof of the claim.

**CASE 1:** $X$ does not admit an open embedding of $[0,1]$.

It is routine to show that for every $\epsilon > 0$, there exists a finite family of homotopies $\{h_i | i = 1, 2, \ldots, n\}$ of $X$ such that:

1. $h_i | X \times \{0\} = \text{identity for each } i$;
2. $\text{diam}(\{h_i(x,t) | t \in [0,1]\}) < \epsilon$ for each $i$ and each $x \in X$;
3. Each $x \in X$ is moved by some $h_i$ (i.e. there exists $j$ and $s$ such that $h_j(x,s) \neq x$);
4. Each $h_i$ is supported on a set of diameter less than $2\epsilon$.

Let $\epsilon > 0$ and find homotopies $\{h_i | i = 1, \ldots, N\}$ satisfying (1) through (4) above. For each $x \in X$ and $i = 1, 2, \ldots, N$, $\{h_i(x,s) | 0 < s < 1\}$ is a path of diameter less than $\epsilon$, and for some $j$, is nondegenerate by (3). If $\alpha_{x,i}$ denotes the order arc induced by the $i$th path, then it has diameter less than $\epsilon$, and for some $j$, is nondegenerate. Let $\beta, \gamma \in \Lambda_0$ and suppose $\alpha_{\beta(0),j} = \mathcal{E}(\gamma(0), \epsilon/2)$ for some $j$. Then, $\text{bd}(N_{\beta(0)})$ is degenerate for each $t \in [0,1]$ so that $\pi : [0,1) \to X$ defined by $\pi(t) = \text{bd}(N_{\beta(0)})$ is an open embedding of $[0,1)$ into $X$. This contradicts our assumption and so the following order arcs are distinct, nondegenerate order arcs starting at $\beta(0)$ and having diameter less than $\epsilon$:

$$\alpha_{\beta(0),1}^{V_1 \alpha_{\beta(0),2}} \alpha_{\beta(0),2}^{V_1 \cdots} \alpha_{\beta(0),N}$$
Define $h_1, h_2 : \Lambda_0 \to \Lambda_0$ by

$$h_1(\beta) = \alpha_{\beta(0), 1}V_1\alpha_{\beta(0), 2}V_1 \cdots V_1\alpha_{\beta(0), N}V_1\beta$$

and,

$$h_2(\beta) = \alpha(\beta(0), \frac{\epsilon}{2})V_1\beta.$$
point of \( g(X) \) has an open neighborhood of the form \([0,1)\) in \( X \), and no point of \( g(X) \) is unstable in \( X \). We can now apply a similar argument as in Case 1 to the points of \( g(X) \). Let \( \{h_i | i = 1,2,\ldots,N\} \) be homotopies of \( X \) as in Case 1, \( \alpha_g(x) \) the induced order arcs for each \( i \), and \( \alpha_{\mathcal{C}}(g(x), \frac{\epsilon}{2}) \) the convex order arcs where convex growth takes place in \( X \). As in Case 1, for each \( x, y \in X \), \( \alpha_{\mathcal{C}}(g(x), \frac{\epsilon}{2}) \) and \( \alpha_{g(y)}, 1^\alpha_g(y), 2^\alpha_g(y), \ldots, N^\alpha_g(y), N \) are distinct order arcs of diameter less than \( \epsilon \). Define \( h_1, h_2 : \Lambda_0 \to \Lambda_0 \) by

\[
h_1(\beta) = \alpha_{g(\beta(0))}, 1^\alpha_{g(\beta(0))}, \ldots, 1^\alpha_{g(\beta(0))}, N^\alpha_{g(\beta(0))}, N^\alpha_{g(\beta)}, 1^\alpha_{\mathcal{C}(g(X), 1)}
\]

and

\[
h_2(\beta) = \alpha_{\mathcal{C}(g(\beta(0)), \frac{\epsilon}{2})}, 1^\alpha_{g(\beta)}, N^\alpha_{g(\beta)}, N^\alpha_{g(\beta)}, 1^\alpha_{\mathcal{C}(g(X), 1)}.
\]

Then, \( H^2(h_1(\alpha), \alpha) < \epsilon \) for each \( \alpha \in \Lambda_0 \) and \( i = 1,2 \), and \( h_1(\Lambda_0) \cap h_2(\Lambda_0) = \emptyset \). By Torunczyk's theorem (Theorem 1.5), \( \Lambda_0 \) is a \( \mathbb{Q} \)-manifold. This completes the proof of the first statement in Theorem 2.2. To prove the second statement, note that since \( \Lambda_0 \) is a \( \mathbb{Q} \)-manifold, \( \Lambda_0 = \Lambda_0 \times \mathbb{Q} \) ([4]). \((e_0 \times \text{id}) : \Lambda_0 \times \mathbb{Q} \to X \times \mathbb{Q} \) defined by \((e_0 \times \text{id})(\alpha, q) = (\alpha(0), q)\) is an AR-map and hence, a near-homeomorphism ([4]). Thus, \( \Lambda_0 = \Lambda_0 \times \mathbb{Q} \simeq X \times \mathbb{Q} \). This completes the proof of Theorem 2.2.

[]

In general, we do not know when \( \Lambda_0 = X \times \mathbb{Q} \). If \( X \) is hereditarily indecomposable, then by Theorem 2.1(d), \( \Lambda_0 \) is also, so \( \Lambda_0 \neq X \times \mathbb{Q} \). Another example of such a space is given below. This is an example of a space \( X \) which is not contractible but for which \( \Lambda_0 \) is contractible (see Corollary 2.7).
**Example:** Let $X$ be the following subspace of the plane:

![Diagram of Example](image)

Clearly, $X$ is not contractible. We will show $A_0$ is contractible.

Let $P$ be the positive fan in $X$, let $N$ be the negative fan, and let $[-1,1], [-1,0]$ and $[0,1]$ denote the intervals containing $((-1,0),(1,0)), ((-1,0),(0,0))$ and $((0,0),(1,0))$ respectively. We will contract $A_0$ by deforming it to a set $A_1$ with $e_0(A_1) = [-1,0] \cup P$, then deform $A_1$ to a set $A_2$ with $e_0(A_2) = [-1,1]$, and finally, deform $A_2$ to a set $A_3$ with $e_0(A_3) = (0,0)$. By Theorem 1.4, $A_3$ is contractible so $A_0$ is contractible. We show how to deform $A_0$ to $A_1$ since the other deformations can be defined similarly.

Let $0 < \epsilon < \frac{1}{2}$, $L = \{a \in A_0 | a(0) \in N - [-1,0]\}$, and $O_L$ be an $\epsilon$-neighborhood of $L$. Let $f : A_0 \to [0,1]$ be a Urysohn function with $f^{-1}(0) = A_0 - O_L$ and $f^{-1}(1) = N$, the closure of $L$. Notice that $e_0(O_L) = N \cup [0,\epsilon)$. Define a function $h : X \times [0,1] \to X$ by $h|X \times \{0\} = \text{identity}$, $h(x,t) = x$ for $x \notin e_0(O_L)$, and let $h$ collapse $N$ to $(-1,0)$ taking $[0,\epsilon]$ onto $[-1,\epsilon]$. For $x \in X$,$$
abla_{h(x,s)} (0 < s < 1)$$is a path so induces a continuous family of order arcs as in Lemma 2.3 which we denote by $\alpha(x,t)$. If $x \in N$, then $\alpha(x,1)(0) = (-1,0)$. Define $\hat{h} : A_0 \times [0,1] \to A_0$ by
\[ \hat{h}(\beta, t) = \alpha(\beta(0), t \cdot f(\beta)) \]

If \( \beta \in O_L \), then \( f(\beta) = 0 \) so \( \hat{h}(\beta, t) = \alpha(\beta(0), 0) = \beta \) for all \( t \).
If \( \beta \in N \), then \( f(\beta) = 1 \) so \( \hat{h}(\beta, 1) = \alpha(\beta(0), 1) \) is an element of \( \Lambda_0 \) starting at \((-1, 0)\). Hence, \( e_0(\hat{h}(\Lambda_0, 1)) = [-1, 0] \cup P \).

**CLAIM:** \( \hat{h} \) is a map.

Suppose \( (\beta_n, t_n) \rightarrow (\beta, t) \). We consider three cases:

**CASE 1:** \( \beta \in O_L \)

Then, there exists \( N \) such that \( n > N \) implies \( \beta_n \in O_L \) so \( \beta_n(0) \notin N \cup \{0, \epsilon\} \). Since \( h \) is continuous when restricted to \( N \cup \{0, \epsilon\} \), it follows that \( \hat{h}(\beta_n, t_n) \rightarrow \hat{h}(\beta, t) \).

**CASE 2:** \( \beta \notin O_L \)

Then, there exists \( N \) such that \( n > N \) implies \( \beta_n \notin O_L \) so \( \hat{h}(\beta_n, t_n) = \beta_n \rightarrow \beta = \hat{h}(\beta, t) \).

**CASE 3:** \( \beta \) lies on the boundary of \( O_L \).

Then, \( f(\beta) = 0 \) so \( \hat{h}(\beta, t) = \beta \). Also, \( f(\beta_n) \rightarrow 0 \) so \( \text{diam}_n(\alpha(\beta_n(0), t_n \cdot f(\beta_n))) \rightarrow 0 \) so \( h^2(\hat{h}(\beta_n, t_n), \beta_n) \rightarrow 0 \). Since \( \beta_n \rightarrow \beta \), \( \hat{h}(\beta_n, t_n) \rightarrow \beta = \hat{h}(\beta, t) \). This proves the claim so \( \Lambda_0 \) is contractible.

In the above example, \( \Lambda_0 \) is contractible although \( X \) is not. However, it's easy to see \( X \) has trivial shape [16]. In the remainder of this chapter, we show \( X \) and \( \Lambda_0 \) always have the same shape.

An **inverse sequence** \( \{Y_n, f_n\}_{n=1}^{\infty} \) is a sequence of spaces and mappings \( f_n : Y_{n+1} \rightarrow Y_n \). The spaces \( Y_n \) are called **coordinate spaces**
and the maps $f_n$ are called bonding maps. If $\{Y_n, f_n\}_{n=1}^{\infty}$ is an inverse sequence, then the inverse limit of $\{Y_n, f_n\}_{n=1}^{\infty}$, denoted $\lim_{\leftarrow} (Y_n, f_n)$, is the subspace of $\prod_{n=1}^{\infty} Y_n$ (with the product topology) given by:

$$\{(y_1, y_2, \ldots, y_n, \ldots) \in \prod_{n=1}^{\infty} Y_n | f_n(y_{n+1}) = y_n \text{ for each } n = 1, 2, \ldots\}.$$

We will concern ourselves with inverse sequences whose coordinate spaces are continua. If $f : X \to Y$ is a map of continua, then we have seen that $f$ induces maps $\hat{f} : C(X) \to C(Y)$ and $\check{f} : \Gamma(X) \to \Gamma(Y)$. If $f$ is onto, then $\check{f}|A_0(X) : A_0(X) \to A_0(Y)$, where $A_0(X)(A_0(Y))$ denotes the space of maximal order arcs in $C(X)(C(Y))$ and the vertical line denotes restriction. The next theorem is due to J. Segal but we refer the reader to [18], p. 171, for a proof.

**Theorem 2.12:** Let $X$ be a continuum and assume $X = \lim_{\leftarrow} X_n$ where each $X_n$ is a continuum. Then, $C(X) = \lim_{\leftarrow} C(X_n), \hat{f}_n\}$. The homeomorphism $h : \lim_{\leftarrow} (C(X_n), \hat{f}_n) \to C(X)$ is defined by

$$h((K_1, K_2, \ldots)) = \lim_{\leftarrow} [K_1, f_1 | K_1 + 1].$$

Similarly, we can define a homeomorphism $h^* : \lim_{\leftarrow} (C(C(X_n)), \hat{f}_n) \to C(C(X))$ by

$$h^*((\sigma_1, \sigma_2, \ldots)) = \lim_{\leftarrow} [\sigma_1, \check{f}_1 | \sigma_1 + 1],$$

where $\hat{f}_n : C(C(X_{n+1})) \to C(C(X_n))$ is induced by $\hat{f}_n$. Note that $\check{f}_n = \hat{f}_n|\Gamma(X_{n+1})$. Then, $\{\Gamma(X_n), \check{f}_n\}$ is a well-defined inverse sequence and if we assume the bonding maps $\{f_n\}$ are onto (which we may do without loss of generality), then $\{A_0(X_n), \check{f}_n | A_0(X_{n+1})\}$ is a well-defined inverse sequence also. In fact, $h^*(\lim_{\leftarrow} \Gamma(X_n), \check{f}_n) = \Gamma(X)$ and
Theorem 2.13: If \( X \) is a continuum and \( X = \lim_{n \to \infty} (X_n, f_n) \) with onto bonding maps, then \( \Gamma(X) = \lim_{n \to \infty} (\Gamma(X_n), \hat{f}_n) \) and \( \Lambda_0(X) = \lim_{n \to \infty} (\Lambda_0(X_n), \hat{f}_n|_{\Lambda_0(X_{n+1})}) \).

Theorem 2.14: (See [19], p. 305-306) Any metric continuum is homeomorphic to an inverse limit \( \lim_{n \to \infty} (P_n, f_n) \) where the \( P_n \) are compact, connected polyhedra and the bonding maps are onto.

Theorem 2.15: If \( X \) is a continuum, then \( X \) and \( \Lambda_0 \) have the same shape.

Proof: Write \( X = \lim_{n \to \infty} (P_n, f_n) \) as in Theorem 2.14. By Theorem 2.13, \( \Lambda_0(X) = \lim_{n \to \infty} (\Lambda_0(P_n), \hat{f}_n|_{\Lambda_0(P_{n+1})}) \). By Theorem 2.1(e), each \( \Lambda_0(P_n) \) is an ANR and so determines the shape class of \( \Lambda_0(X) \) [16]. Consider the following diagram:

\[
\begin{array}{c}
\Lambda_0(P_1) \overset{\hat{f}_1}{\rightarrow} \Lambda_0(P_2) \overset{\hat{f}_2}{\rightarrow} \ldots \overset{\hat{f}_\infty}{\rightarrow} \Lambda_0(X) \\
e_1 \downarrow \quad g_1 \downarrow \quad e_2 \downarrow \quad g_2 \downarrow \quad \ldots \quad \downarrow e_\infty \\
P_1 \overset{f_1}{\leftarrow} P_2 \overset{f_2}{\leftarrow} \ldots \overset{f_\infty}{\leftarrow} X
\end{array}
\]

where \( e_i : \Lambda_0(P_i) \to P_i \) are the evaluation maps for each \( i \), the \( g_i : P_i \to \Lambda_0(P_i) \) are their homotopy inverses as defined in Lemma 2.6, and \( e_0 : \Lambda_0(X) \to X \) is the evaluation map and is induced by the \( e_i \).

Since the diagram commutes up to homotopy, \( X \) and \( \Lambda_0(X) \) have the same shape.
A topological property \( P \) is said to be a **Whitney property** (W-property) provided whenever \( X \) has property \( P \), so does each W-level for all W-maps defined on \( C(X) \). W-properties were studied by a number of mathematicians in the seventies (see [18], chapter 14) and many properties were established as being, or not being, W-properties. In particular, Kransinkiewicz and Nadler [14] asked if any of the following are W-properties: ANR, AR, trivial shape, contractible, and acyclic. Rogers [21] showed that being acyclic is a W-property for the class of 1-dimensional continua. However, Ann Petrus [20], one of his students, showed that none of these properties are W-properties for all continua by giving counterexamples in each case. The spaces \( X \) in her examples were 2-dimensional and so the questions remained open for the class of 1-dimensional continua. The main results of this chapter are:

**Theorem 3.1:** The following are W-properties for the class of 1-dimensional continua:

(a) ANR;
(b) AR;
(c) Trivial shape.

**Corollary 3.2:** If \( X \) is 1-dimensional and contractible, then each W-level has trivial shape.

Thus, it is still an open question whether contractibility is a W-property for 1-dimensional continua although we do know the W-levels have trivial shape.
Theorem 3.3: If \( \mu^{-1}(s) \) has one of the following properties, then so does \( \mu^{-1}(t) \) for all \( t > s \):

(a) Locally connected;
(b) Path connected;
(c) Hereditarily indecomposable;
(d) Pseudo-arc.

Proof: We do all the properties at once. If \( \mu^{-1}(s) \) has one of these properties, then so does \( A_s \) by Theorem 2.1. Let \( t > s \) and choose \( r \in (0,1) \) such that \( e_r(\beta) = \beta(r) \in \mu^{-1}(t) \) for all \( \beta \in A_s \). Since \( e_r : A_s \rightarrow \mu^{-1}(t) \) is a proper, monotone map, it preserves each of these properties.

All the results in Theorem 3.3 are known except (b), which answers a question raised by Nadler in [17]. The following result is also known although the proof given here is simpler than that in the literature.

Corollary 3.4: All the properties in Theorem 3.3 are W-properties.

Proof: Follows from Theorem 3.3 with \( s = 0 \).

A continuum \( X \) is said to be locally unicoherent at \( K \in C(X) - F_1(X) \) provided there exists \( \epsilon > 0 \) such that if \( M \in C(X) \) and \( H(K,M) < \epsilon \), then \( \phi \neq K \cap M \in C(X) \). If \( X \) is locally unicoherent at each \( K \in C(X) - F_1(X) \), then \( X \) is said to be locally unicoherent.

Lemma 3.5: If \( G \) is a graph, then \( G \) is locally unicoherent.

Proof: Let \( K \in C(G) - F_1(G) \). If \( K \) is acyclic, then clearly \( G \) is locally unicoherent at \( K \) so assume \( K \) is not acyclic. Let
\( \{ e_i | i = 1, \ldots, n \} \) and \( \{ v_j | j = 1, \ldots, m \} \) be the sets of edges and vertices, respectively, in \( G \) and for any \( M \in C(G) \), let \( M_e = \bigcup (e_i \in M) \cup \{ v_j | v_j \in M \} \). Choose \( \epsilon > 0 \) so that the following conditions are satisfied:

1. If \( H(K, M) < \epsilon \), then \( K \cap M \neq \emptyset \) and;

2. if \( \overline{N}_\epsilon(K) = \{ x \in G | d(x, K) < \epsilon \} \), then \( (\overline{N}_\epsilon(K))_e = K_e \).

**CLAIM:** If \( M \in C(G) \) and \( H(K, M) < \epsilon \), then \( K \cap M \in C(G) \).

\( \overline{N}_\epsilon(K) \neq \emptyset \) by (1) and \( M_e \subset K_e \) by (2). If \( M_e = \emptyset \), then \( M \) is contained in the interior of an edge of \( G \) and \( K \cap M \in C(G) \) by (2).

Assume \( M_e \neq \emptyset \). \( M \) is the union of \( M_e \) together with partial edges meeting \( M_e \) at its vertices. If one of these partial edges of \( M \) meets one of \( K \), then one must be contained in the other. Hence, \( K \cap M \) is \( M_e \) union the intersection of the partial edges of \( M \) with \( K \) and is connected. \[
\]

If \( X \) is a 1-dimensional ANR, then \( X = \lim\{ G_i, r_i \} \) where the \( G_i \) are graphs, \( G_{i+1} \) is obtained from \( G_i \) by the addition of a single "sticker", and \( r_i : G_{i+1} \to G_i \) is a retraction which collapses the sticker to its vertex (this characterization is in the folklore). Note that all the cycles of \( X \) lie in \( G_1 \).

**Proposition 3.6:** If \( X \) is a 1-dimensional ANR, then \( X \) is locally unicoherent.

**Proof:** Write \( X = \lim\{ G_i, r_i \} \) as above and let \( K \in C(X) - F_1(X) \). If \( K \cap G_1 = \phi \) or degenerate, then clearly \( X \) is locally unicoherent at \( K \). If \( K \cap G_1 \in C(G_1) - F_1(G_1) \), then \( X \) will be locally unicoherent at
K provided \( G_1 \) is locally unicoherent at \( K \cap G_1 \) and this follows by Lemma 3.5. Hence, \( X \) is locally unicoherent. 

We now begin the proof of Theorem 3.1 with respect to properties (a) and (b). Suppose \( X \) is a 1-dimensional ANR. Let \( t > 0 \) and \( K \in \mu^{-1}(t) \). We will show \( \mu^{-1}(t) \) is a local ANR. If \( K \) has a cut point \( p \), then there exists \( \varepsilon > 0 \) such that if \( H(K, M) < \varepsilon \), then \( p \in M \). Hence, \( \mu_p^{-1}(t) \), which is an AR by Theorem 1.1, is a neighborhood of \( K \) in \( \mu^{-1}(t) \) so \( K \) has ANR neighborhoods. Suppose \( K \) has no cut points. Then, \( K \in C(G_1) \) (in fact, \( K \) is a subgraph of \( G_1 \)). Let \( p, q \in K \) with \( p \neq q \), \( p, q \) lie on the same edge of \( G_1 \), and neither \( p \) nor \( q \) are vertices of \( G_1 \). Clearly, \( \mu_p^{-1}(t) \cup \mu_q^{-1}(t) \) is a neighborhood of \( K \) in \( \mu^{-1}(t) \) and so we need only show it's an ANR and we do this by showing \( \mu_p^{-1}(t) \cap \mu_q^{-1}(t) \) is an ANR. Let \( M \in \mu_p^{-1}(t) \cap \mu_q^{-1}(t) \) and define \( \Sigma_M = \{ \alpha \in \Gamma(X) | \alpha(0) = \{ p, q \}, \alpha(1) = \{ X \} \} \). If \( \alpha(s) \notin C(X) \), then \( \alpha(s) \in M \). \( \Sigma_M \) is an AR by the same argument showing \( \Lambda_p \) is an AR. Let \( r \in (0, 1) \) be chosen so that \( e_r(\beta) = \beta(r) \in \mu_p^{-1}(t) \cap \mu_q^{-1}(t) \) for all \( \beta \in \Sigma_M \). Again, we can use the same argument as in Theorem 1.1 to show \( e_r(\Sigma_M) \) is an AR \( (e_r: \Sigma_M \to \mu_p^{-1}(t) \cap \mu_q^{-1}(t) \) may not be surjective). \( e_r(\Sigma_M) \) is a neighborhood of \( M \) in \( \mu_p^{-1}(t) \cap \mu_q^{-1}(t) \) for if \( p, q \in N \in \mu^{-1}(t) \) and \( N \) is close to \( M \), then \( M \cap N \in C(X) \) by local unicoherence. Thus, there exists \( \beta \in \Sigma_M \) such that \( \beta(s) = M \cap N \) for some \( s < r \) and so \( N \in e_r(\Sigma_M) \). Hence, \( \mu_p^{-1}(t) \cap \mu_q^{-1}(t) \) is a local ANR so \( \mu^{-1}(t) \) is an ANR. This completes the proof of (a). Petrus [20] has shown the W-levels of a 1-dimensional AR are contractible so (b) follows from this and (a).

In the proof of Theorem 3.1(a), we showed \( \mu_p^{-1}(t) \cap \mu_q^{-1}(t) \) is an
ANR provided $X$ is locally unicoherent. However, there exists a 2-cell $Y$, a $W$-map $\mu : C(Y) \to [0,1]$, $p, q, \epsilon Y$, and $t \in (0,1)$ such that $\mu_p^{-1}(t) \cap \mu_q^{-1}(t)$ is a convergent sequence.

**Corollary 3.7:** If $X$ is a 1-dimensional ANR, then $\mu^{-1}([0,t])$ is an ANR for all $t \in [0,1]$.

**Proof:** Since $\mu^{-1}(t)$ is an ANR, there exists open $U \supset \mu^{-1}(t)$ and a retraction $r : U \to \mu^{-1}(t)$ where $U \subset C(X)$. Note that $\mu^{-1}([0,t]) \cup U$ is an open set containing $\mu^{-1}([0,t])$ and hence, is an ANR since $C(X)$ is an AR. Define a retraction $R : \mu^{-1}([0,t]) \cup U \to \mu^{-1}([0,t])$ by:

$$R(K) = \begin{cases} K, & \text{if } K \in \mu^{-1}([0,t]), \\ r(K), & \text{if } K \notin \mu^{-1}([0,t]). \end{cases}$$

It follows that $\mu^{-1}([0,t])$ is an ANR. []

Notice the proof of corollary 3.7 shows if $X$ is a Peano continuum and $\mu^{-1}(t)$ is an ANR, then $\mu^{-1}([0,t])$ is an ANR.

Before completing the proof of theorem 3.1, we will study the $W$-levels of 1-dimensional ANR $\mathbb{S}$ in more detail. By Theorem 3.1(a), they must be ANR $\mathbb{S}$. However, they need not be contractible. Our next results are an attempt at characterizing the AR $W$-levels of 1-dimensional ANR $\mathbb{S}$.

**Theorem 3.8:** Let $G$ be a graph. The following are equivalent:

(a) $G$ has a cut point;

(b) There exists $t_0 < 1$ such that $\mu^{-1}(t)$ is an AR for all $t > t_0$;
(c) \( \{G\} \) is unstable in \( C(G) \).

Proof: (a) implies (b) by Theorem 1.11. (b) implies (c) by Corollary 1.9. (c) implies (a) by a result of Curtis [6]. []

It follows from Theorem 3.8 that the high \( W \)-levels of a graph \( G \) are \( ARS \) if and only if \( G \) has a cut point. As we will see, cut points play a crucial role in determining which \( W \)-levels are \( ARS \). In fact, we make the following conjecture:

Conjecture: Let \( X \) be a 1-dimensional ANR. Then, \( \mu^{-1}(s) \) is an AR if and only if for each \( t > s \) and for each \( K \in \mu^{-1}(t) \), \( K \) has a cut point.

We begin by reducing the study of \( W \)-levels in 1-dimensional ANRs to \( W \)-levels in graphs.

Theorem 3.9: Let \( X \) be a Peano continuum and \( t \in [0,1] \). \( \mu^{-1}(t) \) is an ANR (AR) if and only if \( \mu^{-1}([0,t]) \) is an ANR (AR).

Proof: The proof of Corollary 3.7 shows if \( \mu^{-1}(t) \) is an ANR, then so is \( \mu^{-1}([0,t]) \). To complete the proof, we will show \( \mu^{-1}(t) \) is a strong deformation retraction of \( \mu^{-1}([0,t]) \). Define \( h : \mu^{-1}([0,t]) \times [0,1] \to \mu^{-1}([0,t]) \) by:

\[
h(K,s) = c(K,s \cdot r_K)
\]

where \( r_K = \min \{ r \mid c(K,r) \in \mu^{-1}(t) \} \) and \( c \) is as in Chapter 2 on page 19. \( h \) is continuous provided the function \( K \to r_K \) is continuous.

Let \( K_n \to K \) and \( r_{K_n} = r_n \). We need to show \( r_n \to r_K \). Suppose false. Then there exists a subsequence \( r_{n_i} \) of \( r_n \) such that \( r_{n_i} \to s \neq r_K \).

Since \( c \) is a map, \( c(K_{n_i},r_{n_i}) \to c(K,s) \) and \( c(K_{n_i},r_{n_i}) \in \mu^{-1}(t) \) for
all \( n \) implies \( c(K,s) \in \mu^{-1}(t) \). This means \( c(K,s) = c(K,r_K) \) so \( s = r_K \) since \( d \) is convex. This is a contradiction. Hence, \( h \) is a map. \[
\]

Now suppose \( X \) is a 1-dimensional ANR and write
\[ X = \varprojlim(G_i)^s. \]
Let \( \pi : X \to G \) denote the projection map. Then, there exists a deformation \( h : X \times [0,1] \to X \) such that \( h(x,0) = x \), \( h(x,1) = \pi(x) \), \( h \) collapses the stickers of \( X \) and for \( r < s \), \( h(X,r) = h(X,s) \). Let \( \hat{\mu} : C(X) \to C(G) \) and \( \hat{\mu} : C(X) \times [0,1] \to C(X) \) be the induced maps. It's easy to see for any \( t \in [0,1] \),
\[
\hat{\mu}^{-1}([0,t]) \times [0,1] : \mu^{-1}([0,t]) \times [0,1] \to \mu^{-1}([0,t]) \text{ and }
\hat{\mu}^{-1}([0,t]),1 = \hat{\mu}^{-1}([0,t])) = \mu^{-1}([0,t]) \cap C(G). \]
This, together with Theorem 3.1(a) and Theorem 3.9, give the following.

**Proposition 3.10:** Let \( X = \varprojlim(G_i)^s \) be a 1-dimensional ANR. Then, \( \mu^{-1}(t) \) is an AR iff \( \mu^{-1}(t) \cap C(G) \) is an AR.

Proposition 3.10 reduces the problem of studying AR W-levels of 1-dimensional ANRGs to a problem of AR W-levels of graphs. Now, suppose \( G \) is a graph and let \( S \) denote the collection of all subgraphs of \( G \) (with respect to a particular triangulation of \( G \)). For each \( s < 1 \), there exists \( \epsilon > 0 \) such that if \( 0 < \delta < \epsilon \), then for each \( K \in \mu^{-1}(s) \), \( \overline{N_\delta(K)} \) does not contain an element of \( S - \mu^{-1}(s) \) since \( \mu^{-1}(s) \) is compact and \( S \) is finite.

**Lemma 3.11:** Let \( G \) be a graph, \( s < 1 \) and \( \epsilon > 0 \) as above. Then, for every \( K \in \mu^{-1}(s) \), \( \mu^{-1}(s) \cap C(\overline{N_\epsilon(K)}) \) is contractible.

**Proof:** By Theorem 3.9, it suffices to show \( \mu^{-1}([0,s]) \cap C(\overline{N_\epsilon(K)}) \) is contractible. Let \( h : \overline{N_\epsilon(K)} \times [0,1] \to \overline{N_\epsilon(K)} \) be a deformation collapsing the stickers of \( \overline{N_\epsilon(K)} \) in such a way that if
\( \hat{h} : C(N_\epsilon(K)) \times [0,1] \rightarrow C(N_\epsilon(K)) \) is the induced deformation, then for each \( M \in C(N_\epsilon(K)) \), \( \mu(h(M,r)) < \mu(M) \) and \( \mu(h(N_\epsilon(K),l)) < s \) (by choice of \( \epsilon \)). Thus, \( \hat{h} : \{ \mu^{-1}([0,s]) \cap C(N_\epsilon(K)) \} \times [0,1] \rightarrow \mu^{-1}([0,s]) \cap \\
C(N_\epsilon(K)) \) and the image of \( \hat{h}(\epsilon,1) \) is contained in \( C(h(N_\epsilon(K),l)) \) which is contractible. Hence, \( \mu^{-1}([0,s]) \cap C(N_\epsilon(K)) \) is contractible. 

Proposition 3.12: Let \( G \) be a graph, \( s < 1 \), and \( \mu^{-1}(s) \) an AR. Then, there exists \( t_0 > s \) such that if \( s < t < t_0 \), then \( \mu^{-1}(t) \) is an AR.

Proof: Choose \( \epsilon > 0 \) as in Lemma 3.11. Choose \( t_0 > s \) so that if \( s < t < t_0 \), then \( H^2(\mu^{-1}(s),\mu^{-1}(t)) < \epsilon \). Since \( \mu^{-1}(s) \) is an AR, so is \( A_s \). Let \( r > 0 \) so \( e_r(\beta) = \beta(r) \in \mu^{-1}(t) \) for each \( \beta \in A_s \). We will show \( e_r : A_s \rightarrow \mu^{-1}(t) \) has contractible point inverses and so is a homotopy equivalence [13]. We know by Lemma 3.11 that \( \{ \alpha(0) \mid \alpha \in e_r^{-1}(K) \} \) is contractible. Hence, \( e_r^{-1}(K) \) contracts as in Lemma 2.8. Hence, \( \mu^{-1}(t) \) is contractible and an AR. 

Proposition 3.13: Let \( G \) be a graph with \( \mu^{-1}(s) \) an AR for all \( s \) in some interval \( (r,t) \). Then, \( \mu^{-1}(r) \) is an AR.

Proof: We need only show \( \mu^{-1}([0,r]) \) is an AR. Since \( \mu^{-1}([0,r]) \) is an ANR, there exists an open set \( U \supset \mu^{-1}([0,r]) \) and a retraction \( R : U \rightarrow \mu^{-1}([0,r]) \). Choose \( r < s < t \) with \( s - r \) small so \( \mu^{-1}([0,s]) \subset U \). Then, \( R : \mu^{-1}([0,s]) \rightarrow \mu^{-1}([0,r]) \) is a retraction and since \( \mu^{-1}([0,s]) \) is an AR, \( \mu^{-1}([0,r]) \) is an AR. Thus, \( \mu^{-1}(r) \) is an AR. 

Notice that Proposition 3.12 shows that if we are given some W-level is an AR, then there are higher W-levels which are ARs. The
next results will tell us when lower W-levels are ARs.

**Lemma 3.14:** Let $G$ be a graph and suppose each $K \in \mu^{-1}(t)$ has a cut point. Then, there exists $r < t$ such that if $s \in [r,t)$, then for each $K \in \mu^{-1}(t)$, $\cap \{M| M \subset C(K) \cap \mu^{-1}(s)\} \neq \emptyset$.

**Proof:** Suppose this is false. Then, for each $r_n < t$, there exists $K_n \in \mu^{-1}(t)$ such that no point of $K_n$ belongs to every member of $C(K_n) \cap \mu^{-1}(r_n)$. Let $r_n \rightarrow t$ and assume without loss of generality $K_n \rightarrow K \in \mu^{-1}(t)$. Let $p$ be a cut point of $K$. Then, there exists $N$ such that $n > N$ implies $p$ is a cut point of $K_n$. Clearly, there exists $r < t$ such that if $s \in [r,t)$, then $p \in \cap \{M| M \subset C(K) \cap \mu^{-1}(s)\}$. Hence, the same must hold true for each $K_n$, $n > N$. This is a contradiction.

**Proposition 3.15:** Let $G$ be a graph, $t < 1$ such that $\mu^{-1}(t)$ is an AR and each $K \in \mu^{-1}(t)$ has a cut point. Then, there exists $r < t$ such that $\mu^{-1}(s)$ is an AR for each $s \in [r,t)$.

**Proof:** Choose $r < t$ satisfying the conditions of Lemma 3.14. We will show $\mu^{-1}(s)$ is an AR by showing $\Lambda_s$ is an AR. Choose $t_0 \in (0,1)$ so that $e_{t_0}(\beta) = \beta(t_0) \in \mu^{-1}(t)$ for each $\beta \in \Lambda_s$. As in Proposition 3.12, we need only show $e_{t_0} : \Lambda_s \rightarrow \mu^{-1}(t)$ has contractible point inverses and hence, is a homotopy equivalence. Let $K \in \mu^{-1}(t)$. Then, $e_{t_0}^{-1}(K) = \{a \in \Lambda_s| a(t_0) = K\}$. But, $\{a(0)| a(t_0) = K\} = \mu^{-1}(s) \cap C(K)$ for some cut point $p \in K$ and this is contractible. Hence, as in Proposition 3.12, we conclude $e_{t_0}^{-1}(K)$ is contractible. Since $\mu^{-1}(t)$ is contractible, $\Lambda_s$ and hence, $\mu^{-1}(s)$, is contractible. Thus, $\mu^{-1}(s)$ is an AR.
Proposition 3.13 shows a W-level can be an AR even though it may contain elements without cut points but Proposition 3.15 suggests no W-levels below this are ARs. We will now show how our previous results reduce our conjecture to the following statement:

\((*)\) If \(G\) has no cut points, then no W-level of \(G\) below the top level can be an AR.

Assume \((*)\) is true and let \(G\) be a graph with cut points. If \(G\) is acyclic, then each W-level is an AR by Theorem 3.1(b) so assume \(G\) is not acyclic. By Theorem 3.8, all sufficiently high W-levels are ARs and by Propositions 3.13 and 3.15, there exists \(t_0\) such that \(\mu^{-1}(t)\) is an AR for \(t > t_0\), each element of \(\mu^{-1}(t)\) has a cut point for \(t > t_0\), and there exists \(G_1 \in \mu^{-1}(t_0)\) without a cut point. Since \(G_1\) has no cut points,

\((**): \mu_{G_1}^{-1}(s) = \mu^{-1}(s) \cap C(G_1)\) is not an AR for \(s < t_0\) by \((*)\).

Since \(G\) has cut points, write \(G = G_1^1 \cup G_2^1, G_1, G_2^1\) subgraphs of \(G\) and \(G_1^1 \cap G_2^1 = \{v_1\}\), a cut point of \(G\). Then, \(G_1 \subseteq G_1^1\) or \(G_1 \subseteq G_2^1\). Assume \(G_1 \subseteq G_1^1\). If \(G_1 \neq G_1^1\), then \(G_1^1\) has a cut point. Continue the process with \(G_1^{i-1} = G_1^i \cup G_2^i, G_1^i, G_2^i\) subgraphs of \(G_1^i, G_1^i \cap G_2^i = \{v_1\}\), a cut point of \(G_1^{i-1}\), and \(G_1 \subseteq G_1^i\). Eventually, there exists \(N\) with \(G_1 = G_1^N\). Let \(r_i: G_1^{i-1} \rightarrow G_1^i\) be a retraction collapsing \(G_1^{i-1}\) to \(v_1\) and let \(\tilde{r}_i: C(G_1^{i-1}) \rightarrow C(G_1^i)\) be the induced map. Then, for each \(s < t_0\), \(\tilde{r}_n \circ \ldots \circ \tilde{r}_1\) is a retraction. By \((*)\), \(\mu_{G_1}^{-1}([0,s])\) is not an AR so \(\mu^{-1}([0,s])\), and hence \(\mu^{-1}(s)\), is not an AR. This proves the conjecture.
Remark: In the above argument, if we don't assume (*), then all we can show is there exists $s_0 < t_0$ such that $\mu^{-1}(s)$ is not an AR for $s_0 < s < t_0$. We need to replace (***) by the following statement, which is a consequence of Theorem 3.8: There exists $s_0 < t_0$ such that $\mu^{-1}_{G_1}(s) = \mu^{-1}(s) \cap G(G_1)$ is not an AR for $s_0 < s < t_0$. However, our results don't allow us to conclude that the W-levels below $\mu^{-1}(s_0)$ are not ARs.

Example: Let $X = \cdots$, the Hawaiian earring, let $S_1$ be the 1th circle and $\mu : C(X) \to [0,1]$ any W-map. Although $X$ is not an ANR, the techniques employed in the proof of Theorem 3.1(a) can be applied to show $\mu^{-1}(t)$ is an ANR for all $t > 0$. Also, if $t_0 = \max \{\mu(S_1) \mid i = 1, 2, \ldots\}$, then $\mu^{-1}(t)$ is an AR if and only if $t > t_0$ by the arguments used to reduce the conjecture to the statement (*).

Example: Let $X = \cdots$, let $S_1$ be the 1th circle, and $\mu : C(X) \to [0,1]$ any W-map. All the results of the previous example hold for this example for the same reasons.

We now turn our attention to completing the proof of Theorem 3.1 and Corollary 3.2. We'll need to introduce some notation and prove a few lemmas. Recall from chapter 2 that if $X$ is a continuum, then we can write $X = \lim\{P_i, f_i\}$ where the $P_i$ are compact, connected polyhedra and $f_i : P_{i+1} \to P_i$ are onto. Also, $h : \lim\{C(P_i), f_i\} \to C(X)$ defined by $h((K_1, K_2, \ldots)) = \lim\{K_i, f_i |_{K_{i+1}}\}$ is a homeomorphism. Notice that $\lim\{C(P_i), f_i\} \subset \prod_{i=1}^{\infty} C(P_i) \subset C(\prod_{i=1}^{\infty} P_i)$ and if $\mu : C(X) \to [0,1]$ is a W-map, then $\mu \circ h : \lim\{C(P_i), f_i\} \to [0,1]$ is
a map satisfying the following conditions: \( \mu \circ h((x_1,x_2,\ldots)) = 0 \) for each \((x_1,x_2,\ldots) \in \lim_{i=1}^{\infty} \Pi_i P_i \) and if \((K_1,K_2,\ldots) \in (M_1,M_2,\ldots) \) and \((K_1,K_2,\ldots) \neq (M_1,M_2,\ldots) \), then \( \mu \circ h((K_1,K_2,\ldots)) < \mu \circ h(M_1,M_2,\ldots) \). Ward [24] has shown \( \mu \circ h \) extends to a \( W \)-map

\[ \hat{\mu} : C(\Pi P_i) \to [0,\infty) \]

Let \( \hat{\mu} = \sum_{i=1}^{\infty} \mu_i \) and let

\[ \pi^m : \Pi C(P_i) \to \Pi C(P_i) \]

be the projections. Let \((p_1,p_2,\ldots) \in \Pi P_i \) be fixed and define

\[ i_m : C(P_m) \to \Pi C(P_i) \]

by

\[ i_m(K) = (i_1^{-1}(K),i_2^{-1}(K),\ldots,i_{m-1}(K),K,p_{m+1},p_{m+2},\ldots) \]

where

\[ i_1 \circ \cdots \circ i_m = i_m \]

Define \( \mu_m : C(P_m) \to [0,\infty) \) by

\[ \mu_m(K) = \min_{i=1}^{\infty} \{ \hat{\mu}(\pi^m(M)) \mid M \in \Pi C(P_i) \} \]

For each \( K \in C(P_m) \), let \( K^* \in \Pi C(P_i) \) denote some element with \( \hat{\mu}(K^*) = \mu_m(K) \) and \( \pi^m(K^*) = \mu_m \circ i_m(K) \), and let \( \sigma_m(K^*) = (\pi_{m+1}(K^*),\pi_{m+2}(K^*),\ldots) \) be the tail of \( K^* \) beyond the \( n^{th} \) coordinate.

Lemma 3.16: \( \mu_m : C(P_m) \to [0,\infty) \) is a \( W \)-map for each \( m \).

Proof: It's easy to see \( \mu_m(\{x\}) = 0 \) for each \( \{x\} \in F_1(P_m) \). Suppose \( M,N \in C(P_m) \) and \( M \subseteq N, M \neq N \). If \( \mu_m(N) < \mu_m(M) \), then

\[ \hat{\mu}(N^*) < \hat{\mu}(M^*) \]

But, \((\pi^m(M^*),\sigma_m(N^*)) \subseteq N^*, (\pi^m(M^*),\sigma_m(N^*)) \neq N^* \) so

\[ \hat{\mu}(\pi^m(M^*),\sigma_m(N^*)) < \hat{\mu}(N^*) < \hat{\mu}(M^*) \]

This contradicts the choice of \( N^* \).

Hence, \( \mu_m(M) < \mu_m(N) \). We now show \( \mu_m \) is a map. Let \( K_n,K \in C(P_m) \) and \( K_n \to K \). Assume without loss of generality that \( K_n \to M \in \Pi C(P_i) \) and note that \( \pi^m(M) = \pi^m(K_n^*) \). Since \( \hat{\mu} \) is a map, \( \mu_m(K_n) = \hat{\mu}(K_n^*) = \hat{\mu}(M) \). We need to show \( \hat{\mu}(M) = \mu_m(K) = \hat{\mu}(K^*) < \hat{\mu}(M) \).

If
\( \hat{\mu}(K^*) < \hat{\mu}(M) \), then for \( K'_n = (\pi^m(K^*_n), \sigma^m(K^*_n)) \), it's clear
\( \hat{\mu}(K'_n) \to \hat{\mu}(K^*) \) since \( K'_n \to K^* \). Hence, there exists \( N \) such that
\( n > N \) implies \( \hat{\mu}(K'_n) < \hat{\mu}(K^*) \) and this contradicts the choice of \( K^*_n \).
Thus, \( \hat{\mu}(K^*) > \hat{\mu}(M) \) so \( \mu_m \) is a map. \[ \]

It follows from the definition of \( \mu_m : C(P_m) \to [0,\infty) \) that \( \hat{\mu}_m \)
maps \( \mu^{-1}_m([0,t]) \) into \( \mu^{-1}_m([0,t]) \) for each \( t \). Thus,
\( \{\mu^{-1}_m([0,t]), \hat{\mu}_m\} \) is a well-defined inverse sequence for each \( t \).

**Theorem 3.17:** \( \mu^{-1}([0,t]) = \lim\{\mu^{-1}_m([0,t]), \hat{\mu}_m\} \) for each \( t \).

**Proof:** We show \( h : \lim\{C(P_1), \hat{\mu}_1\} \to C(X) \) takes \( \lim\{\mu^{-1}_m([0,t]), \hat{\mu}_m\} \)
onto \( \mu^{-1}([0,t]) \). If \( K \in \mu^{-1}([0,t]) \), then \( \mu_m \circ \pi^m(h^{-1}(K)) < t \) for
each \( m \) so \( h \) takes \( \lim\{\pi^m(h^{-1}(K)), \hat{\mu}_m \circ \pi^m(h^{-1}(K)) \} \) to \( K \). Suppose
\( K = (K_1, K_2, \ldots) \in \lim\{\mu^{-1}_m([0,t]), \hat{\mu}_m\} \). We must show \( \mu \circ h(K) = \hat{\mu}(K) < t \). But this follows from the fact that \( \mu_m \circ \pi^m(K) < t \) for
each \( m \). \[ \]

**Corollary 3.18:** If \( X \) is 1-dimensional and has trivial shape, then
\( \mu^{-1}([0,t]) \) has trivial shape for all \( t \).

**Proof:** Case and Chamberlin [3] have shown a 1-dimensional continuum
with trivial shape can be written as an inverse limit of acyclic graphs.
Thus, \( X = \lim\{G_1, f_1\} \). By Theorem 3.17, \( \mu^{-1}([0,t]) = \lim\{\mu^{-1}_m([0,t]), \hat{\mu}_m \circ \mu^{-1}_m([0,t])\} \). Since the \( G_1 \) are acyclic graphs, each
\( \mu^{-1}_m([0,t]) \) is an AR by Theorem 3.1(b) and Theorem 3.9. Hence,
\( \mu^{-1}([0,t]) \) has trivial shape. \[ \]

We now complete the proof of Theorem 3.1 with respect to property
(c). We assume all the previously introduced notation. Write
\[ X = \lim\{G_i, f_i\} \] where the \( G_i \) are acyclic graphs for each \( i \) and let \( h : \lim\{C(G_i), \hat{\omega}_i\} \to C(X) \) be the above homeomorphism. To show \( \mu_i^{-1}(t) \) has trivial shape, we show \( h^{-1}(\mu_i^{-1}(t)) \) has trivial shape. Let \( Y \) be a metric ANR and \( g : h^{-1}(\mu_i^{-1}(t)) \to Y \) a map. It suffices to show \( g \) is null-homotopic. Let \( \hat{\gamma} : 0 \to Y \) be an extension of \( g \) to an open set \( 0 = U_1 \times U_2 \times \ldots \times U_n \times \Pi C(G_i) \). For \( m > n \) large, \( g_m = \hat{\gamma} \circ i_m \circ \tau_m : h^{-1}(\mu_i^{-1}(t)) \to Y \) is homotopic to \( g \). We will show \( g_m \) is null-homotopic for \( m \) sufficiently large.

**CLAIM:** There exists \( m > n \) such that for each \( i < m \),

\[ f_i^{-1}(\mu_i^{-1}(\tau_m h^{-1}(\mu_i^{-1}(t)))) \subseteq U_i. \]

Suppose this is false. Let \( \Sigma_m = \mu_m(\tau_m h^{-1}(\mu_i^{-1}(t)))) \). Then for each \( m > n \), there exists \( K_m \in \mu_m^{-1}(\Sigma_m) \) such that \( f_j^{-1}(K_m) \notin U_j \) for some \( j < m \). Let \( K^*_m \in \Pi C(G_i) \) be as before and assume without loss of generality that \( K^*_m \to K \). We will show \( K \in h^{-1}(\mu_i^{-1}(t)) \). By definition of \( \mu_m, \mu_m \circ \tau_m h^{-1}(C(X)) \) converges uniformly to \( \hat{\chi} h^{-1}(C(X)) \). Hence, \( \mu_m \circ \tau_m h^{-1}(\mu_i^{-1}(t)) \to t \) as \( m \to \infty \). Since \( K_m \in \mu_m^{-1}(\Sigma_m), \mu_m(K_m) \in \Sigma_m \). Thus, \( \mu_m(K_m) = \hat{\chi}(K_m^*) \to t \) so \( \hat{\chi}(K_m^*) = t \) and \( K \in h^{-1}(\mu^{-1}(t)) \). It follows that there exists \( M_0 > n \) such that \( m > M_0 \) implies \( K_m \in \Sigma_m \) since \( \Sigma_m \) is open and contains \( h^{-1}(\mu_i^{-1}(t)) \). By definition of \( K_m^* \), this means \( f_i^{-1}(\tau_m(K_m^*)) = f_i^{-1}(K_m^*) \subseteq U_i \) for all \( m > M_0 \) and for all \( i < m \). This contradiction proves the claim.

To complete the proof, note that \( \mu_m(\tau_m h^{-1}(\mu_i^{-1}(t)))) = \Sigma_m \) is an interval so \( \mu_m^{-1}(\Sigma_m) \) is contractible since \( G_m \) is an acyclic graph.

Let \( m > M_0 \) be fixed and let \( \beta_r : \mu_m^{-1}(\Sigma_m) \to \mu_m^{-1}(\Sigma_m) \) be a contraction with \( \beta_0 \) the identity. Define \( F_r : h^{-1}(\mu_i^{-1}(t)) \to Y \) by:
By the above claim, \( i_m \circ \beta_r \circ \pi_m (h^{-1}(\mu^{-1}(t))) = 0 \) for all \( r \) so \( F_r \) is well-defined. Since \( F_0 = g_m \), this shows \( g_m \) is null-homotopic for \( m > M_0 \). Hence, \( g \) is null-homotopic and \( \mu^{-1}(t) \) has trivial shape. This completes the proof of Theorem 3.1. []

Corollary 3.2 follows from Theorem 3.1(c) for if \( X \) is contractible, then it has trivial shape.
CHAPTER IV
The Spaces $\Gamma(X)$

Eberhart, Nadler, and Nowell [10] studied the spaces $\Gamma(X)$ when $X$ is a Peano continuum and showed $\Gamma(X) = \mathbb{Q}$ when $X$ contains no free arcs. In case $X$ has free arcs, they were able to show $\Gamma(S^1) \neq \mathbb{Q}$ where $S^1$ denotes the unit circle. In this chapter, we will characterize the 1-dimensional ANR $X$ for which $\Gamma(X) = \mathbb{Q}$. It will be convenient to first characterize those graphs $G$ for which $\Gamma(G) = \mathbb{Q}$. We denote by $G \cup_p [0,1]$ the graph obtained from $G$ by attaching an arc to $G$ at $p$ and consider $G \cup_p [0,1]$ to be a graph with $p$ as a vertex. Our main results are:

**Theorem 4.1:** Let $G$ be a graph, $\Sigma = \{G = G_1, G_2, \ldots, G_N\}$ be the collection of all subgraphs of $G$ such that for each $i > 1$, $G_i$ has a neighborhood of the form $G_i \cup_p [0,1]$ in $G$. Then, $\Gamma(G) = \mathbb{Q}$ if and only if each $G_i$ has a cut point for $i > 1$.

**Theorem 4.2:** Let $X = \varprojlim \{G_i, r_i\}$ be a 1-dimensional ANR written as an inverse limit of graphs as in Chapter 3 (i.e. each $G_i$ is a graph and each $r_i : G_{i+1} \to G_i$ is a strong deformation retraction). Let $\Sigma = \{H_1, H_2, \ldots, H_k\}$ be the collection of subgraphs of $G_1$ which have no cut points. Then, $\Gamma(X) = \mathbb{Q}$ if and only if no $H_i$ has a neighborhood of the form $H_i \cup_p [0,1]$ in $X$.

We begin the proof of Theorem 4.1 by showing the conditions of the theorem are necessary.
Theorem 4.3: \{G\} is unstable in C(G) if and only if \{\{G\}\} is unstable in \Gamma(G).

Proof: Suppose \{G\} is unstable in C(G), and let \(\varepsilon > 0\). By Theorem 3.8, there exists a retraction \(r: C(G) \rightarrow \mu^{-1}([0,t])\) with 
\(r(\mu^{-1}([t,1])) = \mu^{-1}(t)\), for some \(t < 1\) such that 
\(\text{diam}_{\mu^{-1}([t,1])} < \varepsilon\). Define \(R: \Gamma(G) \rightarrow \Gamma(G) - \{\{G\}\}\) by:

\[
R(\alpha) = \begin{cases} 
\{\alpha(s)\mid 0 < s < s', \text{ if } \mu(\alpha(0)) < t \text{ and } \alpha(0) \in [0,1] \} & \text{if } \mu(\alpha(0)) < t \text{ and } s' = \max\{s\mid \mu(\alpha(s)) < t\}; \\
\{\{r(\alpha(0))\}\} & \text{otherwise.}
\end{cases}
\]

It's easy to see \(R\) is a map, \(H^2(\alpha, R(\alpha)) < \varepsilon\) for each \(\alpha \in \Gamma(G)\), and \{\{G\}\} \notin R(\Gamma(G)). Conversely, suppose \{\{G\}\} is unstable in \Gamma(G), and let \(\varepsilon > 0\). There exists a map \(f: \Gamma(G) \rightarrow \Gamma(G) - \{\{G\}\}\) with 
\(H^2(f(\alpha), \alpha) < \varepsilon\) for each \(\alpha \in \Gamma(G)\). Let \(e_0: \Gamma(G) \rightarrow C(G)\) and \(i: C(G) \rightarrow \Gamma(G)\) be defined by \(e_0(\alpha) = \alpha(0)\) and \(i(K) = \{\{K\}\}\), respectively. Then, \(e_0 \circ f \circ i: C(G) \rightarrow C(G) - \{G\}\) and \(H(e_0 \circ f \circ i(K), K) < \varepsilon\) for each \(K \in C(G)\). Hence, \{G\} is unstable in C(G). \[
\]

Corollary 4.4: If \(G\) has no cut points, then \(\Gamma(G) \neq \emptyset\).

Proof: If \(G\) has no cut points, then \{G\} is stable in C(G) by a result of Curtis [6]. Theorem 4.3 implies \{\{G\}\} is stable in \Gamma(G) so \(\Gamma(G) \neq \emptyset\). \[
\]

Theorem 4.5: Let \(G = G' \cup_p [0,1]\) be a graph and suppose \(G'\) has no cut points. Then, \(\Gamma(G) \neq \emptyset\).

Proof: (Due to Doug Curtis) By Lemma 2.1 of [15], the complement of each finite-dimensional compactum in \(\emptyset\) has trivial homology. Thus if
\( \Gamma(G) = \emptyset, \Gamma(G) - A \) has trivial homology, where \( A \) is the arc \( \{ \alpha \in \Gamma(G) | \alpha(0) = \{g'\} \} \). Doug Curtis [6] has shown \( \{g'\} \) has cone neighborhoods in \( C(G') \) with base \( B \) having non-trivial homology. Let \( \text{Cone}(B) \subset C(G') \) be such a neighborhood. We will show \( i(B) \) is a retract of \( \Gamma(G) - A \), where \( i : C(G) \to \Gamma(G) \) is the embedding as in Theorem 4.3. This contradiction shows that \( \Gamma(G) \neq \emptyset \). Consider the following diagram:

\[
\begin{array}{ccc}
\Gamma(G) - A & \xrightarrow{r_1} & i(C(G) - \{g'\}) & \xrightarrow{r_2} & i(C(G') - \{g'\}) & \xrightarrow{r_3} & i(B) \\
\end{array}
\]

\( r_1 : \Gamma(G) - A \to i(C(G) - \{g'\}) \) is defined by \( r_1(\alpha) = \{\alpha(0)\} \).

Certainly, \( r_1 \) is a retraction. \( r_2 \) is obtained in the following steps:

1. Retract \( i(C([0,1])) \) to \( i(C_p([0,1])) \);
2. retract \( i(C_p(G) - \{g'\}) \) to \( i(C_p(G') - \{g'\}) \);
3. extend by the identity to the rest of \( C(G) - \{g'\} \).

(2) is possible since \( C_p(G') - \{g'\} \) is an AR by Corollary 1.10 and is closed in \( C_p(G) - \{g'\} \). \( r_3 \) is obtained by radial projection and convex growth. Clearly, \( r_3 \circ r_2 \circ r_1 : \Gamma(G) - A \to i(B) \) is a retraction.

\textbf{Corollary 4.6:} Let \( G = G' \cup_p [0,1] \) be as in Theorem 4.5. Then, \( \Gamma(G' \cup_p [0,1]) \) is not a \( \mathbb{Q} \)-manifold.

\textbf{Proof:} Let \( \Sigma = \{ \alpha \in \Gamma(G) | l \in \alpha(1) \} \). \( \Sigma \) is a Z-set in \( \Gamma(G) \) and \( \Gamma(G) - \Sigma = \Gamma(G' \cup_p [0,1]) \). By a result of Torunczyk [23], if \( \Gamma(G) - \Sigma \) were a \( \mathbb{Q} \)-manifold, then \( \Gamma(G) \) would be a \( \mathbb{Q} \)-manifold. Since \( \Gamma(G) \) is compact and contractible, it would have to be \( \mathbb{Q} \) [4]. This contradicts Theorem 4.5. Hence, \( \Gamma(G' \cup_p [0,1]) \) is not a \( \mathbb{Q} \)-manifold. \( \square \)
Corollary 4.7: Let $G$ be a graph, $G'$ a subgraph of $G$ such that $G'$ has no cut point and $G'$ has neighborhoods of the form $G' \cup_p [0,1]$ in $G$. Then, $\Gamma(G) \neq \mathbb{Q}$.

Proof: If $\Gamma(G) = \mathbb{Q}$, then $\Gamma(G' \cup_p [0,1])$ would be a $\mathbb{Q}$-manifold since it's open in $\Gamma(G)$. But this contradicts Corollary 4.6 so $\Gamma(G) \neq \mathbb{Q}$. []

Corollaries 4.4 and 4.7 prove the necessity part of Theorem 4.1. Corollary 4.4 shows $G$ must have a cut point and Corollary 4.7 shows each $G_i \in \Xi$ for $i > 1$ must have a cut point. We will now complete the proof of Theorem 4.1 by showing the conditions are also sufficient.

Lemma 4.8: Let $G$ be a graph, $G'$ a subgraph of $G$ with a cut point and $G' \cup_p [0,1]$ a neighborhood of $G'$ in $G$. Then, for each $\varepsilon > 0$, there exists a map $f : C(G) \to C(G) - \{K \in C(G) | G' \subset K\}$ with $H(f(M), M) < \varepsilon$ for each $M \in C(G)$.

Proof: Note that $p$ need not be a cut point of $G'$ but must be a cut point of $G$. Let $\varepsilon > 0$, $\mu : C(G) \to [0,1]$ a $W$-map, and $\hat{\mu} = \mu|C(G')$.

Choose $s < \mu(G')$ so $\hat{\mu}^{-1}(s)$ is an AR and $\text{diam}_H(\hat{\mu}^{-1}([s, \mu(G')]))) < \varepsilon$ (by Theorem 3.8). Define a retraction $r : \hat{\mu}^{-1}([s, \mu(G')]))) \to \hat{\mu}^{-1}(s)$ and let

$\hat{r} : \hat{\mu}^{-1}(s) \cup \hat{\mu}^{-1}([s, \mu(G')]))) \to \hat{\mu}^{-1}(s)$ be defined by:

$$\hat{r}(K) = \begin{cases} K, & \text{if } K \in \hat{\mu}^{-1}(s); \\ r(K), & \text{if } K \in \hat{\mu}^{-1}([s, \mu(G')]))). \end{cases}$$

Since $\hat{\mu}^{-1}(s)$ is an AR, $\hat{r}$ has an extension $\hat{r} : \hat{\mu}^{-1}([s, \mu(G')]))) \to \hat{\mu}^{-1}(s)$. Extend $\hat{r}$ to $C(G')$ by the identity and note that $H(\hat{r}(K), K) < \varepsilon$ for each $K \in C(G')$, and if
p \in K \in C(G')$, then $p \in \hat{f}(K)$. Now define $f : C(G) \to C(G) - \{K \in C(G) | G' \subset K\}$ by:

$$f(K) = \begin{cases} K, & \text{if } K \cap G' = \emptyset; \\ \hat{f}(K \cap G') \cup (K - G'), & \text{if } K \cap G' \neq \emptyset. \end{cases}$$

$f$ is a well-defined map since $\hat{f}$ is a map and if $p \in K$, then $p \in \hat{f}(K)$ so $\hat{f}(K \cap G') \cup (K - G')$ is connected. Furthermore, $G' \notin f(K)$ for any $K \in C(G)$ since if $K \cap G' \neq \emptyset$, $G' \notin \hat{f}(K \cap G')$ by definition. It's easy to see $H(f(K), K) < \varepsilon$ for each $K \in C(G)$. 

**Corollary 4.9:** Let $G$ be a graph and let $\Sigma$ be as in Theorem 4.1. If each $G_i \in \Sigma$ has a cut point, then for every $\varepsilon > 0$, there exists a map $f : C(G) \to C(G) - \Sigma$ such that $G_i \notin f(K)$ and $H(f(K), K) < \varepsilon$ for each $i$ and each $K \in C(G)$.

**Proof:** Since $G$ has a cut point, $\{G\}$ is unstable in $C(G)$. Hence, $f$ may be obtained as a finite composition of maps as defined in Corollary 1.9 and Lemma 4.8 above.

We now prove the sufficiency of Theorem 4.1 by using Torunczyk's characterization of the Hilbert cube (Theorem 1.5). Let $\varepsilon > 0$, and choose $f : C(G) \to C(G) - \Sigma$ as in Corollary 4.9. If $G$ has end points, then we may assume no $f(K)$ contains an end point of $G$ for each $K \in C(G)$. Notice if $K \in C(G)$, then $f(K)$ does not have a neighborhood of the form $f(K) \cup_p [0, 1]$ in $G$. Define

$$\hat{f} : \Gamma(G) \to \Gamma(G) \text{ by } \hat{f}(\alpha) = \{\hat{f}(\alpha(s)) | 0 < s < 1\}. \text{ Since union is non-expansive, } H^2(\hat{f}(\alpha), \alpha) < \varepsilon \text{ for each } \alpha \in \Gamma(G). \text{ Further, if }$$

$$e_0 : \Gamma(G) \to C(G) \text{ is defined by } e_0(\alpha) = \alpha(0), \text{ then } e_0 \circ \hat{f}(\alpha) = \hat{f}(\alpha(0)) \text{ does not have a neighborhood of the form } e_0 \circ \hat{f}(\alpha) \cup_p [0, 1] \text{ in }$$
G and $e_0 \circ \hat{f}(\alpha) \neq \{G\}$ for each $\alpha \in \Gamma(G)$. The remainder of the proof is similar to the argument used in Theorem 2.2. Let

$A = \{p_1, \ldots, p_m, p_{m+1}, \ldots, p_n\}$ be an $\varepsilon$-$\mathcal{F}$-net in $G$ with $\{p_1, p_2, \ldots, p_m\}$ being the set of all vertices of $G$. For each $i = 1, 2, \ldots, n$, there exists $\varepsilon$-homotopies $h_i : G_i \times [0,1] \rightarrow G_i$ satisfying the following conditions:

1. $h_i(x,0) = x$, for each $x \in G$;
2. $h_i(x,t) = x$, for each $(x,t) \in (G - \mathcal{N}_\varepsilon(p_i)) \times [0,1]$;
3. For $1 < i < m$ (corresponding to the vertices of $G$), then $h_i$ collapses each segment of $\mathcal{N}_\varepsilon(p_i)$ to $p_i$ alternately, and moves each point of $\mathcal{N}_\varepsilon(p_i)$ except $p_i$. For $m < i < n$, let $C_i$ denote the component of $p_i$ in

$\mathcal{N}_\varepsilon(p_i) - \{p_1, \ldots, p_m\}$. $C_i$ is an open arc contained in a segment of $G$. $h_i$ moves only points of $C_i$ and collapses half of each arc on either side of $p_i$ in $C_i$ alternately.

In addition to the above homotopies (which don't move the vertices), we define $\varepsilon$-homotopies $\hat{h}_i : G \times [0,1] \rightarrow G$ for $m < i < n$ such that $\hat{h}_i$ satisfies (1) and (2) above but instead of (3) satisfies:

(3') $\hat{h}_i(x,1) = p_i$, for each $x \in \mathcal{N}_\varepsilon(p_i)$ and moves each point of $\mathcal{N}_\varepsilon(p_i)$ including any vertices it may contain.

For convenience, we re-name these homotopies $\{g_k | k = 1, 2, \ldots, 2n - m\}$ with $g_k = h_k$ for $1 < k < n$ and $g_k = \hat{h}_i$ for $k = n + 1, \ldots, 2n - m$ and $i = m + 1, \ldots, n$. Each $g_k$ induces $\hat{f}_k : C(G) \times [0,1] \rightarrow C(G)$ by

$\hat{f}_k(K,t) = \cup \{g_k(x,t) \}$. Define $G_k : C(G) \rightarrow \Gamma(G)$ by $G_k(K)(t) = \cup \{\hat{f}_k(K,s) \}$. Finally, define $\hat{G} : C(G) \rightarrow \Gamma(G)$ by
\[ \hat{\Gamma}(K) = G_1(K) V_1 G_2(K) V_1 \cdots V_1 G_{2n-m}(K) \]

\( \hat{\Gamma} \) is well-defined since \( G_k(K)(0) = K \) for each \( K \in C(G) \) and each \( k = 1,2,\ldots,2n - m \), and \( H^2(\hat{\Gamma}(K),\{\{K\}\}) < \varepsilon \) for each \( K \in C(G) \). Let \( \hat{c} : C(G) \to \Gamma(G) \) be defined by \( \hat{c}(K) = \) convex arc from \( K \) to \( \{x \in G | d(x,K) < \frac{\varepsilon_1}{2}\} \). Note that \( \hat{c} \circ f, \hat{\Gamma} \circ f : C(G) \to \Gamma(G) \) have disjoint images since \( \hat{c} \circ f(K) \) grows simultaneously in at least two directions in \( G \) since \( f(K) \) hasn't a neighborhood of the form \( f(K) \cup [0,1] \) in \( G \) and \( \hat{\Gamma} \circ f(K) \) can move in only one direction at a time. To see this, we have two cases:

**CASE 1:** \( f(K) \) is not a subgraph of \( G \).

Then, it's easy to see \( f(K) \) is changed in only one direction at a time by some \( G_k \) for \( 1 < k < n \).

**CASE 2:** \( f(K) \) is a subgraph of \( G \).

Then, \( f(K) \) is unchanged by the \( G_k \) for \( 1 < k < n \). However, since it's a subgraph, it is changed in only one direction by some \( G_k \) for \( n + 1 < k < 2n - m \).

Hence, the functions \( F_1,F_2 : \Gamma(G) \to \Gamma(G) \), defined as follows, have disjoint images: For each \( \alpha \in \Gamma(G) \), let

\[ F_1(\alpha) = \hat{\Gamma}(e_0 \circ \hat{f}(\alpha)) V_1 \hat{f}(\alpha) \]

and

\[ F_2(\alpha) = \hat{c}(e_0 \circ \hat{f}(\alpha)) V_1 \hat{f}(\alpha). \]

It's easy to see \( H^2(F_i(\alpha),\alpha) < 2\varepsilon \) for \( i = 1,2 \), and for each \( \alpha \in \Gamma(G) \). Hence, by Torunczyk's theorem, \( \Gamma(G) = \mathbb{Q} \). This completes the proof of Theorem 4.1. \[ \square \]
We now begin the proof of Theorem 4.2. The necessity of the conditions follows from the proof of Corollary 4.7. To prove sufficiency, no \( H_j \) has a neighborhood of the form \( H_j \cup_{p_i} [0,1] \) in \( X \) so the convex arc from \( H_j \) in \( X \) doesn't grow along an arc. Let \( \epsilon > 0 \), and choose \( n \) large so projection \( \pi_n : X \to G_n \) satisfies 
\[
d(\pi_n(x), x) < \epsilon \text{ for each } x \in X.
\]
Let \( \pi_n : \Gamma(X) \to \Gamma(G_n) \) be the induced map and let \( \hat{\pi}_n : \Gamma(G_n) \to \Gamma(G_n) \) be as in Theorem 4.1. Let \( F_1 : \Gamma(G_n) \to \Gamma(G_n) \) be as in Theorem 4.1 and let \( F_2 : \Gamma(G_n) \to \Gamma(X) \) be as in Theorem 4.1 except use convex growth in \( X \) instead of \( G_n \). Then, \( F_1 \circ \hat{\pi}_n : \Gamma(X) \to \Gamma(X) \) satisfies \( \mathcal{H}^2(F_1(a), a) < 3\epsilon \) for \( a \in \Gamma(X) \) and \( i = 1, 2 \). Also, \( F_1 \circ \hat{\pi}_n(\Gamma(X)) \cap F_2 \circ \hat{\pi}_n(\Gamma(X)) = \emptyset \) as in Theorem 4.1. By Torunczyk's theorem, \( \Gamma(X) = \emptyset \). This completes the proof of Theorem 4.2. 

Example: Let \( X = \bigcirc \cdots \bigcirc | \bigcirc | \bigcirc | \bigcirc | \bigcirc | \bigcirc | n \)-stickers

Then \( \Gamma(X) = \emptyset \) by Theorem 4.2 but \( \Gamma(G_n) \neq \emptyset \) for any \( n \) by Theorem 4.1.

Theorem 4.10: Let \( X \subset \mathbb{R} \) be a Peano continuum, \( X = \bigcup_{n=1}^{\infty} X_n \) where \( X_n \subset X_{n+1} \), \( \Gamma(X_n) = \emptyset \) for all \( n \) sufficiently large, \( r_n : X \to X_n \) a retraction for each \( n \), and \( r_n \to \text{identity} \) as \( n \to \infty \). Then, \( \Gamma(X) = \emptyset \).

Proof: Since \( X \) is Peano, \( \Gamma(X) \) is an AR. If \( \hat{r}_n : \Gamma(X) \to \Gamma(X_n) \) are induced by the \( r_n \), then \( \hat{r}_n \to \text{identity} \) as \( n \to \infty \). Since \( \Gamma(X_n) = \emptyset \) for large \( n \), \( \Gamma(X) = \emptyset \) by Torunczyk's theorem.
Example: Let $X = \cdots$, $X_n = \cdots \cdot \cdot \cdot$...

By Theorem 4.1, $\Gamma(X_n) \approx \mathbb{Q}$ if and only if $n > 2$. By Theorem 4.10, $\Gamma(X) \approx \mathbb{Q}$ since there exists retractions $r_n : X \rightarrow X_n$ such that $r_n \rightarrow$ identity.

Example: Let $X = \cdots$, the Hawaiian ear ring, $X_n = \cdots \cdot \cdot \cdot$...

By Theorem 4.1, $\Gamma(X_n) \approx \mathbb{Q}$ if and only if $n > 2$ so $\Gamma(X) \approx \mathbb{Q}$ as in the above example.


VITA

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