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Mechanistic analysis and reduced order modeling of forced film cooling flows

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MECHANISTIC ANALYSIS AND REDUCED ORDER MODELING OF FORCED FILM COOLING FLOWS

A Dissertation

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Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
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in
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by
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Abstract

Unforced and forced film cooling jets are investigated in view to develop a reduced order model of the velocity and temperature fields. First, a vertical jet in cross-flow, a configuration well documented at high blowing ratios, is investigated at low blowing ratios using experimental visualizations and large eddy simulations. The unforced study reveals that dominant structures at low blowing ratio can be significantly different from the ones formed at high blowing ratio and describes their evolution and transition as the blowing ratio is changed. The forced jet investigations extend the results of past numerical studies in terms of starting vortex classification to partly modulated jets, and evidence a quantitative mismatch in the transition blowing ratio threshold between experiments and simulations. Film cooling performance estimations show that, at fixed mass flow rate, unforced jets perform better than the forced jets while at fixed pressure supply, forced jets can bring some improvements over their unforced counterpart. A survey of a more application relevant inclined jet is carried out using comparable methods. The unforced study shows the attached inclined and vertical jet vortical structures have strong similarities yet one of them is absent from the former configuration therefore leading to smoother regime transitions. The forced inclined jet study reveals some common starting vortex regimes with the ones of the vertical jet, but also exhibits unique sets of structures not observed before. Film cooling performance of inclined jets is also assessed and compared to relevant vertical jet results. Derivations of reduced order models of two-dimensional systems for both velocity and temperature fields using the Proper Orthogonal Decomposition (POD) - Galerkin method are used to establish the numerical methods and potential caveats of the method. Then, POD of unforced and forced inclined jet is carried out as a statistical analysis tool to evidence the energetically dominant flow structures. Finally, reduced order models of the unforced jet are obtained in the attached and transitional regimes and reduced order models for the forced jet are derived in both instantaneous and phase averaged fields. Most of the derived models show reasonable agreement with the projected empirical data once stabilized using appropriate linear damping methods. The flow analysis, models and methods presented herein constitute an essential step towards the development of close-loop controlled film cooling system.
Chapter 1
Introduction

1.1 Motivations

While modern cooling techniques, including film cooling, have permitted significant increase in gas turbine engine firing temperatures, they also impose a serious toll on the overall engine performance as they divert potentially working fluid from the work producing path, therefore decreasing the output of the device. In modern engines, as much as 25 percent of the fluid can be diverted for cooling purposes which represent a comparable loss in net work output (see Nikitopoulos et al. 2006). Film cooling also has the potential to penalize the turbine’s output by decreasing the working fluid temperature because of dilution with coolant, as well as affecting the turbine isentropic effectiveness due to aerodynamic and turbulent losses. Optimized film cooling systems capable of decreasing even further turbine components temperature could lead to an increase in turbine inlet temperature and associated net work output and efficiency, or a decrease in the amount of coolant necessary to maintain a certain part temperature.

Individual film cooling jets belong to the family of jets in cross-flow and involve large scale vortical structures potentially promoting mixing. Most of the optimization efforts throughout the years have been oriented towards geometrical shaping of the jet orifice to improve film cooling performance by altering the dominant vortical structures. However, these systems remain passive in nature and although they can withstand global changes in operating conditions such as the one imposed by a typical flight plan, they can do so at rather high coolant expenses. They also remain highly sensitive to more localized energetic fluctuations generated by the wake of upstream turbine stages which causes passive jet forcing and has a dramatic impact on the instantaneous coverage. Actively controlled film cooling jets could have the advantage of mitigating passive jet forcing while providing increased adaptability to changes in operating conditions. Forced film cooling systems have been studied rather sporadically in the past in different setups and provided mixed results. However, none of the past studies has been able to provide physical insights on the reasons of the enhancement or decrease in film cooling performance.

This study investigates the potential of forced film cooling jets from a fluidic perspective, providing description of the flow field and involved vortical structures and their impact on film cooling performance. Furthermore, it lays the foundations to the development of closed-loop controlled film cooling systems by studying reduced order modeling of unforced and forced jets using the POD-Galerkin method, necessary for the design of such controller.

1.2 Historical Perspective

Mechanical devices using expanding gases to generate motion can be found as early as 130 B.C. with Hero’s Aeolipile. Intended as a toy, the Aeolipile consisted of a hollow sphere which could rotate on a pair of hollow tubes mounted on its poles. Those tubes were connected to a cauldron under which a fire was burning to produce steam. The steam coming from the cauldron entered the sphere and exited it through a pair of nozzles placed on its equator and directing the steam flow in the azimuthal direction and causing the sphere to rotate. This is the first known device to convert expanding gas energy into motion. However, it is not until the early 16th century with Leonardo da Vinci’s “chimney jack” that the concept of recuperating this energy through a turbine-like device is documented. Using hot gases from a fire flowing up a chimney, da Vinci entrained a fan-shaped wheel and transmitted its rotation through a series of shafts and gears to turn a roasting skewer. The history of gas turbine engine though, is generally admitted to start in 1791 when John Barber is awarded a patent to power a horseless carriage using a chain driven reciprocating gas compressor, an “exploder” (combustion chamber) and a paddle wheel (turbine). While it is usually considered to be the first real gas turbine, involving the essential parts of modern engines, the resulting device was incapable of providing any net work output. The first successful applications of Barber’s concept appear in fact almost a century later: In 1884, Sir Charles Parsons patented a reaction steam turbine and gas turbine followed in 1894 by a steam turbine propelled ship, the Turbinia, which at the time demonstrated
to be 7 knots faster than the fastest Royal Navy ship. In 1890 several of Parsons’ steam turbines were implemented in Newcastle Forth Banks Power Station to provide single-phase electric power at 1,000 volts and 80 Hz. This power station was the first one relying only on steam turbines for electricity generation and ran for 17 years, undergoing several upgrades throughout its operating life, before being replaced by a more modern adjacent plant in 1907. In 1903, Édouard Bibert and his team successfully designed and implemented a turbine for water cooling system with a series of channels passing through the turbine disk and the turbine blades themselves. Current modern cooling designs replaced for obvious reasons water by air, but still rely on the same principles. In 1926 A. A. Griffith published *An Aerodynamic Theory of Turbine Design* using an approach based on gas flow past airfoils rather than flow through passages, and completed a demonstrator consisting of an axial compressor and a two-stage turbine in 1928. In 1930 the Whittle patent his design of a centrifugal gas turbine, leading to the successful static test in 1937 of the first jet engine under the newly created Power Jets Ltd. In 1939, a 4 Megawatt gas-turbine-driven electrical power system is assembled by Brown, Boveri & Cie and used in an emergency power station in Switzerland. That same year, the company Ernst Heinkel flies the first gas turbine jet (HE178). Three years later, Messerschmitt successfully tests the Me 262, first jet powered fighter aircraft. For more details, the reader can consult to the references used to compile this short history (Weston (1992), Gunston (1998), Giampaolo (2003), Boyce (2011)).

Over the past half century, gas turbine engines have significantly changed air travel. Bringing large increases in thrust and operating at higher altitude than their piston counterpart, turbojets and turbofans have allowed larger airplanes to develop, thus bringing flying costs down. Today, more than 5 billion passengers\(^1\) and 15 millions tons of freight\(^2\), respectively representing approximately 5 trillion RPK (Revenue Passenger-Kilometers)\(^3\) and 200 billion RTK (Revenue Tons-Kilometer)\(^4\) rely on jet engines, their performance and reliability. Gas turbines are not exclusively restricted to aircraft propulsion but are also used in electric power generation. In the latter configuration, the usable output is no longer the engine thrust (exhaust gases velocity), but the engine shaft rotation speed and torque, which are converted into electric power through an electric generator. Modern land-based gas turbines have thermal efficiencies approaching 40 percent, competing with other conventional land-based power generators. However, the overall electric efficiency can be increased to 60 percent when used in co-generation (Combined Cycle Gas Turbine - CC/GT) where the hot exhaust gases from the gas turbine are recuperated and used in a second power producing cycle through a steam turbine. This combination constitutes one of the most efficient land-based power generation sources\(^5\). Another advantage of land-based gas turbines resides in their capacity to be fully operational in only a few minutes from a cold start making them ideal to absorb peak hours electricity demand. This is the reason why in the United States, their use has gone up from 15 percent in 1998 to 25 percent in 2010 and is expected to reach as much as 40 percent in 2020 according the U.S. DOE\(^6\).

Based on the last decade averaged yearly increase for passenger and freight traffic of 5-to-6 percent as well as the average 5 percent yearly increase in electric power consumption combined with even more stringent environmental regulations, improving gas turbine engine performance, reliability and efficiency appear as a major objective to achieve in the near future.

### 1.3 Gas Turbine Engine Principle and Simplified Brayton Cycle

Typical single cycle gas turbine engines are composed of three key elements: a compressor, a combustor and a turbine. The compressor is connected to the turbine by a rotating shaft which is used to transmit

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\(^1\)2010 - Airports Council International - Annual World Airport Traffic Report
\(^2\)2010 - Research and Innovative Technology Administration, Bureau of Transportation Statistics - Freight Transportation: Global Highlights
\(^3\)2011-2030 - Boeing - Current Market Outlook
\(^4\)2010 - Boeing - World Air Cargo Forecast
\(^5\)2003 - EURELECTRIC / VGB - Efficiency in Electricity Generation
\(^6\)2010 - U.S Energy Information Administration - Annual Electric Generator Report - Form EIA-860
the necessary power to drive the compressor back from the turbine. In the case of a power generation unit, the shaft is also used as an output to the electric generator, while for turbofans, the shaft also powers the engine fan. The cycle starts by increasing inlet gas flow pressure (usually air) through the compressor. Heat is provided to the gases through the combustor which are finally expanded through the turbine to produce power. In turbojets and turbofans, the energy drawn by the turbine from the working fluid is only enough to power the compressor and fan so that the primary output is the exhaust gases velocity to generate thrust. In power generating gas turbines however, most of the working fluid energy is converted through the turbine to power the compressor but also transmitted to an electric generator. In both cases, this cycle corresponds to an open version of the Brayton Cycle presented in figure 1.1.

In the ideal Brayton cycle, the compression is isentropic from state 1 to state 2s, the heat input between state 2s and 3 is done at constant pressure, while the expansion to state 4s is also isentropic and the closure is an isobaric process back to state 1. In this cycle, the isentropic compression gives \( p_2 = r p_1 \) where \( p_1 \) and \( p_2 \) are the pressure respectively in state 1 and 2s and \( r \) is called the pressure ratio. The temperature increases consequently between state 1 and 2s so that \( T_{2s} = T_1 r^{(\gamma - 1)/\gamma} \) with \( \gamma \) the specific heat ratio of the working fluid and the work provided to the fluid by the compressor is given by \( w_c = C_p(T_1 - T_{2s}) \) and the associate power \( P_c = \dot{m}_a w_c \) with \( \dot{m}_a \) the working fluid mass flow rate. In the combustion chamber, the heat added isobarically to the working fluid using a mass fuel-to-air ratio of \( f \) can be expressed by:

\[
Q_a = \dot{m}_a (1 + f) C_p(T_3 - T_{2s})
\]  (1.1)

Finally, in the turbine, the isentropic process leads to \( T_{4s} = T_3(p_4/p_{3p})^{(\gamma_s - 1)/\gamma_s} \) with \( \gamma_s \) the specific heat ratio of the reacted working fluid and fuel mixture. The work extracted by the turbine is such that \( w_t = C_p, g(T_3 - T_{4s}) \), and the associated power is \( P_t = \dot{m}_a (1 + f) w_t \). From this set of relationships, the effective work output of the gas turbine engine is \( w_n = (1 + f) w_t + w_c \) and the net power \( P_n = \dot{m}_a [(1 + f) w_t + w_c] \), while the thermal efficiency is given by \( \eta_{th} = P_n/Q_a \).

Unfortunately, no real gas turbine engine can achieve the performance predicted by the ideal Brayton cycle. In particular, the isentropic compressions and expansions and the isobaric heat addition between states 2 and 3 are unachievable due to inevitable losses. A somewhat realistic deviation from the ideal cycle is presented in figure 1.1(b). To account for the losses through the compressor and turbines, the isentropic efficiencies of both stages (resp. \( \eta_c \) and \( \eta_t \)) can be used in the expressions of the respective work so that \( w_c = C_p(T_1 - T_2) = C_p(T_1 - T_{2s})/\eta_c \) and \( w_t = C_p, g(T_3 - T_{4s}) = \eta_t C_p(T_3 - T_{4s})/\eta_c \), therefore yielding a net produced work of \( w_n = (1 + f) \eta_t C_p(T_3 - T_{4s}) + C_p(T_1 - T_{2s})/\eta_c \). Neglecting the fuel-to-air ratio, and assuming \( \gamma_g \approx \gamma \), a non-dimensional net work output \( w_{ns}^* \), normalized by \( C_p T_1 \) can be expressed as a function of the pressure ratio and the temperature ratio \( T_3/T_1 \), along with a corresponding optimum pressure ratio \( r^* \) at which the net work output is maximized:

---

**Figure 1.1:** Brayton cycle: (a) Flow diagram; (b) T-s diagram: ideal cycle (solid line) and realistic cycle (red dashed line).
Figure 1.2: Realistic single stage Brayton cycle: (a) Thermal efficiency $\eta_{th}$ as a function of pressure ratio ($r$) for several turbine inlet temperatures $T_3$; (b) Net normalized work $w^*_n$ as a function of pressure ratio ($r$) for several turbine inlet temperatures $T_3$ along with maximum output at corresponding conditions (symbol); (c) Net normalized work $w^*_n$ as a function thermal efficiency $\eta_{th}$.

\[
\begin{align*}
  w^*_n &= w_n/C_pT_1 \approx \eta_t(T_3/T_1)(C_{p,g}/C_p)(1 - 1/r^{(\gamma-1)/\gamma}) + (1 - r^{(\gamma-1)/\gamma})/\eta_c \\
  r^* &= (\eta_t\eta_h(T_3/T_1))^{2/(\gamma-1)}
\end{align*}
\]  

(1.2)

Figure 1.2 shows the evolution of the thermal efficiency $\eta_{th}$ and the normalized net work output $w^*_n$ as a function of the pressure ratio for several turbine inlet temperatures $T_3$ computed for $\eta_t = 0.8$, $T_1 = 300K$ and $C_p = C_{p,g} = 1.004kJ/kgK$. Figure 1.2(a) demonstrates that higher turbine inlet temperatures lead to overall higher efficiency. Figure 1.2(b) shows increasing maximum net work output for higher turbine inlet temperatures and resulting increase in optimum pressure ratio. Based on the results obtained for the simple cycle gas turbine, increase in gas turbine efficiency and work output can be achieved by increasing $T_3$, which itself leads to an increase in the operating pressure ratio in order to obtain maximum work output.

1.4 Modern Gas Turbines Designs

While temperature and pressure ratio increases can lead to higher performance for the simple Brayton cycle, they are limited, at least to the first order, by the technology available to achieve and/or sustain higher temperatures and pressure ratios. Other first order design modifications to the Brayton cycle have been made to improve both net work output and efficiency.

One of the first improvements is the implementation of heat regenerator through heat exchangers to preheat the compressed gases before entering the combustion chamber. Increase in combustor inlet temperature leads to higher combustor outlet temperature for a fixed heat input and therefore to higher efficiencies. Another advantage of this method is the decrease in exhaust gases temperature. However, the addition of a heat exchanger results in pressure losses across the device therefore effectively reducing the pressure ratio across the turbine, thus the resulting amount work. The use of regenerators can lead to increases of up to 10 percent in thermal efficiency over the simple Brayton cycle without significantly affecting the net work output. The added complexity and weight though make them more suitable for land-based and marine power generation primarily. A second modification to the Brayton cycle consists in adding multiple stage compression with intercooling. Indeed, as seen in the previous paragraph, the work input necessary for the compressor is directly proportional to the temperature difference across the compressor. Ideally, to limit the work required, the compression should be isothermal. In multistage compression with intercooling, the working fluid is compressed through a first stage then cooled down before being compressed through a second stage. The resulting point on the T-s diagram for stage 2 ends up being to the left and on the same isobar as the one of the simple Brayton cycle of figure 1.1(b), therefore resulting in lower work input from the compressor. The disadvantages of intercooling include lower combustor inlet temperature which directly affect the turbine inlet temperature, therefore the work output from the turbine, as well as increased weight and complexity. Similarly, a multiple stage expansion with reheat can be implemented in the work producing
part of the cycle, effectively increasing the temperature after an expansion through the turbine’s first stage. This is usually achieved through post-combustion or even secondary combustion. While the work output through the turbine is increased, the thermal efficiency is decreased due to an increase input in heat to the system. Although independently inter-cooling and reheat can lead to undesired losses, combined with heat regeneration, they can lead to higher efficiency and work output. However, the increase in manufacturing and maintenance costs associated with these systems have most of the time kept manufacturers away from them. For more details on cycle improvements see Weston (1992).

In propulsion applications, major advances were made when transitioning from turbojets to turbofans. While turbojets are pure gas turbines using the turbine exhaust gases for output, in turbofans, the incoming air splits after passing through the fan and the flow bypassing the core is accelerated through a nozzle and expelled without any heat input. The rest of the fluid passes through the gas turbine core to produce work which is used to power the fan. The ratio of mass flow-rate bypassing the core to the mass flow-rate of air entering it is called the bypass ratio and modern aircraft engines can have bypass ratio of the order of 10. Similarly to the reheat introduced in the previous paragraph, an additional post-combustion stage or afterburner can be added at the turbine outlet to increase the temperature and therefore exhaust gases velocity. Although afterburners provide tremendous increase in thrust, they also significantly affect fuel consumption and are generally used only in military application during critical maneuvers.

Although modern gas turbine engines differ significantly from the idealized Brayton cycle device, the principal optimization parameters still remain the turbine inlet temperature and pressure ratio. This is evidenced by 50 years of jet engine and power generating gas turbines engineering. In figure 1.3(a) actual data from existing power generating gas turbine engines show the increase in efficiency associated with machines with higher pressure ratios for all four manufacturers considered. Figure 1.3(b) evidences the increase in power associated to higher turbine inlet temperature from the Mitsubishi Heavy Industries (MHI) lineup. Recently, with its J series MHI has reached turbine inlet temperatures of $1600^\circ C$ ($\sim 1900K$). Both of these trends highlights the need for higher turbine inlet temperatures and associated higher optimum pressure ratio for increased performance in real life devices, in agreement with the simple Brayton cycle findings. Similarly in figure 1.4, the evolution of jet engines pressure ratio through the years show a clear trend and current overall pressure ratios are above 50, more than three times larger than those of engines produced fifty years before. The direct consequence of this is shown in figure 1.4(b) showing the associated increase in thrust as a function of the pressure ratio. It should be noticed that similarly to the gas turbine trends, the increase in overall jet engines pressure ratio is associated with an increase in gas turbine inlet temperature, if the engine is to be operated at its optimal working conditions.

Based on the simple Brayton cycle results and the global trends of past and actual devices, it appears obvious that performance enhancement of real life gas turbine engines, both in terms of efficiency and net
power output, involves an increase in turbine inlet temperature and associated increase in overall pressure ratio. Modern combustors can reach temperatures well beyond the ones used in current gas turbines and therefore do not constitute a serious roadblock when it comes to increasing the turbine inlet temperature. The principal challenge resides in the turbine components, and in particular the ones of the first stages, which cannot withstand high temperature while operating at high speeds without being deteriorated. As seen previously, this problem was already faced in 1903 by Armengaud and Lemale in their own design. Consequently, to increase the turbine inlet temperature, one must first investigate ways to protect and/or cool the turbine components.

1.5 Turbine Parts Thermal Management

As mentioned previously, the problem of overheating and degrading turbine components is as ancient as the first functional devices. For the past fifty years, turbine parts have been improved and engineered to increase their tolerance to hot exhaust gases. A short history summary of these advances is presented in
Indeed, turbine blades and vanes are subjected to extreme conditions including mechanical and thermal stresses due to high temperatures, high centrifugal forces and thermal cycling, all accelerating the propagation of defects naturally present in the material. The development of parts made from wrought alloys at first, then cast alloys, directionally solidified alloys, and finally single-crystal Nickel-based super-alloys, has permitted to minimize defects growth while increasing the material resistance to corrosion and high temperatures. More recently, thin ceramic thermal barrier coatings (TBCs) applied on the exposed surfaces of the turbine components, have allowed to push even further the parts melting point. Modern Yttria Partially Stabilized Zirconia (YPSZ) TBCs with high temperature resistance and low thermal conductivity have allowed increases of the turbine inlet temperature by as much as $100\,^\circ C$.

On top of increasing material tolerances, turbine parts can also be cooled down using a source of coolant fluid. This is in fact one of the first thermal management solution used in 1903 by Armengaud an Lemale as they circulated water through internal channels inside the turbine blades. Current internal cooling schemes, while still based on this concept, replaced water by air drawn from one of the compressor stages. Intricate serpentine channels lined with ribs and chevrons are used to enhance heat transfer inside the blade as depicted in figure 1.5(b). In regions such as in the shower head, where very localized cooling is required, internal walls or plates pierced with a series of holes generate impingement cooling jets hitting the blade’s internal wall to locally increase the heat transfer coefficient. The last cooling scheme which is also the one of interest in this study is called film cooling, and aims at insulating the outer surface of the turbine parts. Film cooling consists in arrays of holes passing through the blade, vane or end-wall shell, to reach inside internal cooling passages. Air supplied inside the blade exits through the film cooling holes to form an insulating layer of coolant fluid over the external surface. The efficiency of film cooling systems is judged on their ability to cool the blade, and limit heat exchange between the blade and the hot free stream gases. Both of these metrics are usually measured by the wall adiabatic effectiveness $\eta_{aw}$ and the Stanton number $St$ or the Frossling number $Fr$ defined by:

$$\eta_{aw} = \frac{T_{aw} - T_\infty}{T_c - T_\infty}$$

$$St = \frac{Nu}{RePr}$$

$$Fr = \frac{Nu}{\sqrt{Re}}$$

Where $T_{aw}$, $T_\infty$ and $T_c$ are respectively the adiabatic wall, free-stream and coolant temperatures, $Nu$ the Nusselt number associated with the ratio of convective to conductive heat transfer across a boundary, $Re$ the Reynolds number associated with the ratio of inertial to viscous forces within a fluid and $Pr$ the Prandtl number associated with the ratio of viscous to thermal diffusion rates. Considering these definitions, high adiabatic effectiveness values (approaching 1) and low Stanton and Frossling numbers (approaching 0) are desirable for film cooling application. Effectively, both of these values are obtained through tests involving different boundary conditions and can be synthesized in a more global figure of merit, the heat flux ratio:

$$\frac{q_f}{q_0} = St_f \left( 1 - \frac{\eta_{aw}}{\phi} \right)$$

Where $St_f$ and $St_0$ are respectively the Stanton numbers obtained with and without film cooling and $\phi$ is a dimensionless temperature parameter. Heat flux ratios close to zero are desirable for film cooling applications.

### 1.6 Literature Survey

#### 1.6.1 Jets in Cross-flow

Locally, individual film cooling jets belong to a flow category called jets in cross-flow or transverse jets. These jets are present in many natural or industrial configurations at various scales and intensities. Smoke stacks, VSTOL aircraft, fuel injectors, are few of the many engineering systems involving jets in cross-flow and triggered numerous flow studies over the past 70 years. Among them, Fric & Roshko (1994) documented
Figure 1.6: Jets in cross-flow dominant vortical structures. *Slv*: Shear layer vortices; *CRVP*: Counter-Rotating Vortex Pair; *Hv*: Horseshoe vortex; *Wv*: Wake vortices.

The four fundamental vortical structures involved in this type of flow configuration: the shear layer vortices, Counter-Rotating Vortex Pair (CRVP), horseshoe vortex and wake vortices which are presented in figure 1.6. The shear layer vortices are near-field structures, typically taking the form of consecutive vortex rings convected in the free stream and generally considered to be the result of Kelvin-Helmholtz type instability of the jet cylindrical shear layer (see Fric & Roshko 1994, Kelso *et al.* 1996, Yuan *et al.* 1999). The CRVP, first identified by Kamotani & Greber (1972), is considered as the predominant mixing structure and has been the topic of many studies concerned with mixing enhancement or prevention. Formed downstream of the jet exit and dominating the far field, the CRVP consists of a pair of counter-rotating quasi-streamwise vortices. They are also often referred to as “kidney vortices” because of the shape of the cross-section obtained when observing them using smoke or dye imaging techniques. Several initiation mechanisms have been proposed among which Yuan *et al.* (1999), who identified a pair of hanging vortices at the base of the jet column as the origin of the CRVP vorticity. Other research groups suggested that the CRVP vorticity was directly issued from the jet shear layer being reoriented under the influence of the cross-flow (Andreopoulos & Rodi 1984, Haven & Kurosaka 1997). In particular, studies of the shear layer vortical structures by Sykes *et al.* (1986) and Kelso *et al.* (1996), showed that vertical vorticity was provided to the CRVP by the folding of the shear layer vortices through their side arms, while the upstream and downstream rollups canceled each other. This mechanism was later supported by the transient numerical simulations of Cortelezzi & Karagozian (2001) and later of Marzouk & Ghoniem (2007) using vortex simulations methods which showed the initiation of the CRVP through the folding of the starting structures. Other theories such as the one formulated by Blanchard *et al.* (1999), proposed that the shear layer vortices were generated due to a loss of stability of the jet caused by the elliptical nature of CRVP cross-section. The horseshoe vortex constitutes the third characteristic vortical structure of transverse jets, and is the result of cross-flow boundary layer separation upstream of the jet exit, comparable to the flow separation upstream of wall-mounted cylinder. Krothapalli & Lourenco (1990) and Kelso & Smits (1995) have conducted in depth studies of the horseshoe vortex system, evidencing a dynamic coupling with the jet shear layer structures causing the leading horseshoe vortex to oscillate in the streamwise direction upstream of the jet exit. Finally, the wake vortices were described by Fric & Roshko (1994) as tornado like quasi-vertical vortices, located downstream of the jet exit and below the jet core, originating from cross-flow boundary layer separation and due to the adverse pressure distribution past the jet column. According to this mechanism, initially spanwise oriented rollers
are reoriented and stretched vertically by interaction with the jet core (Rivero et al. 2001). Numerous studies like the one of Smith & Mungal (1998) have dealt with the characterizations of general quantities such as jet penetration, spread and mixing rate for instance, and have identified the jet-to-cross-flow mass-flux ratio (defined as the blowing ratio – \( BR = (\rho_j U_j) / (\rho_\infty U_\infty) \)) and the jet-to-cross-flow momentum ratio (defined as the momentum ratio) as two of the principal scaling parameters for transverse jets. Though a considerable number of studies have been documenting jets in cross-flow, unanimous agreement has yet to be reached on the formation mechanisms of the different vortical structures, because of the high complexity associated with these highly three-dimensional flows. Moreover, since mixing enhancement for combustion application and VSTOL propulsion have been the main concern in the past, most of these studies have focused on rather high blowing ratio jets whereas only few of them have considered blowing ratios less than 1.0, which were usually treated as marginal configurations. Among them, Bergeles et al. (1976) and Gopalan et al. (2004) showed that the vortical structures at low blowing ratios (\( BR < 0.5 \)) could be fundamentally different from the ones at higher blowing ratios. While high blowing ratio jets are fully lift off, these low blowing ratio ones exhibit reattachment leading to the formation of a recirculation region downstream of the jet exit, enclosed by a semi-cylindrical vortical shear layer, as well as jet exit velocity profiles significantly skewed towards the downstream direction.

1.6.2 Unforced Film Cooling

Film cooling studies branched off the classic transverse jets studies influenced by VSTOL research as soon as the effects of injection angle on film cooling performances were evidenced. In the early years, most of the work on film cooling has been focused on thermal measurements. During the late 1960s optimum injection angles of 35° with respect to the mainstream direction and blowing ratios of \( BR = 0.5 \) were rapidly identified for film cooling jets with circular feeding tubes (see Golkstein 1971). Lower blowing ratios were found to have lower performance due to their lack of coolant mass-flow, while higher blowing ratio jets had the tendency to lift off the bottom surface therefore bringing only a marginal amount of coolant close to the wall. Second order parameters such as the effects of incoming boundary layer characteristics (Goldstein & Yoshida, 1982), jet spacing (Brown & Saluja, 1979), jet density ratio (Sinha et al., 1991), compound angle (Schmidt et al. 1996, Goldstein & Jin 2001), length-to-diameter ratio (Burd et al. 1998, Lutum & Johnson 1999) and blade internal geometry and flow direction (Hale et al. 2000, Wilfert & Wolff 2000), were later investigated to fine tune film cooling metrics. Overall, it was found that incoming boundary layer characteristics had little effect on film cooling metrics being either the adiabatic effectiveness or the heat transfer coefficient. Optimum jet spacing over a flat plate was found to be approximately 3 jet diameters, a parameter widely used in current designs. Higher density ratios were found to have mixed impact on film cooling performance, as they tend to prevent film separation but sensibly decrease the film spread at the wall. Compound angles at either 45° or 90° to the cross-flow direction were found to significantly increase film cooling spread in the hole area over their straightly injected counterparts. Investigations on length-to-diameter ratio showed that longer feeding tubes lead to higher film cooling metrics, as they tend to decouple blade internal flow from the external one therefore demoting passive pulsation and coolant lift-off. Although values of length-to-diameter ratio of 7 and beyond would be necessary to achieve decoupling, effectively, values of 3 to 4 are usually found in modern designs due to weight and space constraints. Finally, the influence of internal flow direction and internal blade geometry was found to have a significant impact on film cooling performance, in particular for short injection holes. Potentially, changes in film cooling effectiveness ranging from 5 to 65 percent can be observed depending on the internal blade geometry. The internal flow direction was also found to significantly affect the adiabatic effectiveness in configurations with inclined and normal holes. The greatest advances in film cooling engineering were accomplished by implementing complex holes geometries. In an early study, Goldstein et al. (1974) already reported significant improvement of film cooling metrics corresponding to jets with expanding holes. In their study, Gritsch et al. (1998) investigated multiple fan shaped hole film cooling geometries, evidencing a laid back fan-shaped geometry as optimum for spread. Today, various geometries have been proven to improve film cooling performances over a wider range of blowing ratios compared to cylindrical jets because of the decrease in coolant momentum their expanding geometry implies. An extensive review of shaped film cooling holes has been conducted by Bunker (2005). Other features and configurations such as the full coverage film cooling of Harrington et al. (2001a), the upstream mounted triangular tabs of Nasir et al. (2003), the sister holes configuration of Ely & Jubran (2008) or the anti vortex film cooling...
Forced film cooling jets were investigated in an attempt at mitigating the adverse effects of cross-flow unsteadiness, and decrease the amount of coolant required to insulate turbine blades, following the findings of Bons et al. (2001, 2002) who achieved turbine blade separation suppression using pulsed vortex generator jets while reducing by almost 50 percent the amount of fluid required over comparable unforced apparatus. Ekkad et al. (2006) studied forced film cooling jets on a shower-head model with a flow/no-flow type of excitation without cross-flow unsteadiness. They concluded that over the range of forcing parameters considered, forced jets could provide overall higher adiabatic effectiveness at matching heat transfer coefficients over comparable unforced jets. Shortly after, Coulthard et al. (2007b, a) investigated full film cooling modulation of a row of 5 jets over a flat plate. Their findings were that overall jet forcing did not provide absolute improvement over the best performing unforced jet but that some forced cases at the highest forcing frequencies showed improvements over lift off unforced jets at equivalent blowing ratio. The degradation of film cooling performance in forced cases was attributed to jet lift off at the impulse, resulting in a decrease in adiabatic effectiveness with increasing forcing frequency. At the highest forcing frequency, the jet was shown to never reach an equilibrium so that even during the “off part” of the cycle, low momentum residual flow exited the film cooling hole, therefore increasing the time averaged film cooling effectiveness. In a later study using a similar experimental apparatus equipped with a spoked wheel, the potential of forced film cooling in unsteady cross-flow environment was investigated by Womack et al. (2008a). Forced experiments at blowing ratios equivalent to the optimum unforced jet, showed systematic decrease in film cooling performance.
performance over the steady state results with or without incoming wakes. The decrease in performance in forced conditions was reported to be even greater when wakes were generated than in steady free-stream conditions, showing that forced jets had an effect opposite to the one desired. At higher blowing ratios, forced jet in unsteady cross-flow showed some improvement over equivalent experiments in steady conditions but still a net performance degradation over comparable unforced jets. The phase difference between the wake passage and forcing signal was found to have a significant impact on the instantaneous coverage at low forcing frequencies but almost none on time averaged quantities.

The limited amount of research available on forced film cooling systems and the even more restricted information on the flow physics associated with forced film cooling jets makes the interpretation of the discrepancies between results obtained by Ekkad et al. (2006) and Coulthard et al. (2007b) difficult to explain other than by the obvious difference in nature of the experimental apparatus involved. Unfortunately, the complexity associated with the measurements and understanding of the underlying fluidic phenomena responsible for thermal performance often relegate those considerations to the background, as can attest the almost 30 years separating the first steady state film cooling thermal studies in the late 1960s and the first comprehensive explanations of the physics associated to them in the early 2000s.

1.6.4 Forced Jets in Cross-flow

Forced jets in cross-flow were primarily investigated for their potential in mixing and penetration enhancement. Gogineni et al. (1998) obtained up to 30 percent increased penetration by simply exciting the jet shear layer using piezoelectric actuators, while Johari et al. (1999) reached mixing rate enhancements of the order of 50 percent in a fully modulated jet. In the latter study, two forced jet regimes were observed; one for long injection times that consisted of “successive puffs resembling the steady state jet”, the other for shorter injection times associated with the formation of a starting vortex ring at the jet impulse and providing significant penetration and mixing enhancement. Both studies agreed on the fact that both duty cycle and forcing frequency could greatly affect the jet behavior. Eroglu & Breidenthal (2001) conducted detailed water tunnel experiments, examining both steady state and fully modulated jets in cross-flow. Their observations confirmed the formation of series of starting vortex rings in the forced jet configuration, penetrating deeper in the cross-flow compared to the unforced jet. At the same time, M'Closkey et al. (2002) attempted penetration amplification using acoustically excited jets in cross-flow coupled with a digital compensator and obtained significant improvement for a specific optimal pulse width, found to be in agreement with previous work from Gharib et al. (1998). In their study, Gharib et al. (1998) had used impulsively started jets and isolated a maximum stroke ratio beyond which the leading vortex ring could not gain additional circulation and would depart from the wall. When the injection was maintained, series of smaller trailing ring vortices would form and follow the leading structure. Johari (2006) provided a mapping of the different types of starting vortices for strong forced jets in cross-flow ($BR > 3$) as a function of the duty cycle and the stroke ratio. They highlighted the existence of 4 different types of starting vortices: “distinct vortex rings”, “vortex rings/puff + tail”, “turbulent puffs” and “elongated steady jet-like structures”. More recently, Sau & Mahesh (2008) investigated the formation and behavior of vortex rings in cross-flow emanating from a fully modulated forced jet using large eddy simulations (LES), also including rather low average blowing ratios below 2.0. They found that for blowing ratios below 2.0, large hairpin vortices instead of ring vortices would be generated at the pulse. They also provided a mapping of the starting vortex regimes as a function of the blowing ratio and the stroke ratio, evidencing three distinct structures: “discrete vortex ring”, “vortex ring with trailing column” and “hairpin-like vortex structures”. These results constitute a generalization of the findings of Gharib et al. (1998) on the existence of a formation number (maximum stroke ratio) beyond which a trailing column is formed for forced jets in a quiescent environment, though for forced jets in cross-flow, the formation number was found to vary with the value of the blowing ratio.

The previously mentioned studies highlight the fact that forced jets in cross-flow are complex in nature and their physics respond to other parameters such as the stroke ratio in addition to the conventional forcing signal parameters: forcing frequency, duty cycle and blowing ratio. Changes in the forcing parameters can lead to significantly different flow configurations which in film cooling application could have serious impact on the performance. Unfortunately, no study has been conducted to investigate the fluid dynamics associated with forced inclined jets in cross-flow.
1.6.5 Reduced Order Modeling for Control of Fluid Flows

Low-dimensional models attempting to describe flow instabilities and predict their stability margins can be traced back to the late nineteenth century. In the 1960s, low-dimensional models have been developed for weather prediction purposes, leading to the derivation of the famous Lorenz strange attractor (Lorenz 1963). In the 1970s, the low-dimensional character of the chaos associated with Taylor-Couette flows was evidenced by Swinney (1978), showing discrete frequencies for flows within the linear instability regime. More recently, a growing number of studies have been using reduced order models in view to investigate closed-loop control of fluid flows. Sophisticated controllers usually require a set of governing equations describing the system dynamics. Unfortunately, the Navier-Stokes and energy equations describing fluid flows and their associated heat transfer are far too complex and the models developed for computational fluid dynamics purposes remain impractical to use for controller design. For such application, less computationally expensive sets of equations must be derived. Such equations should be more relevant to the dynamic system at stake rather than possessing the same level of abstraction and universality as the Navier-Stokes equations. Reduced order modeling techniques such as the Proper Orthogonal Decomposition (POD - also known as the Karhunen-Loève decomposition) with Galerkin projection have been used to obtain reduced order models of simple fluid flows. The POD, since it provides an orthogonal set of basis functions with optimum energy distribution, has primarily been used as a statistical analysis tool in the description of turbulent flows (see Lumley 1970, Sirovich 1987, Aubry et al. 1988, Meyer et al. 2007 or Bagheri et al. 2009). It was also used in conjunction with the Galerkin projection to obtain reduced order models of various two-dimensional flows. Turbulent and transitional boundary layers were amongst the first flows to be investigated using this method by Holmes et al. (1998) who successfully captured the sweep/burst mechanisms using only 5 modes. Additional systems have been studied since then with and ROMs for flows past a circular (Bergmann et al. 2007) or square cylinder (Couplet et al. 2005), laterally heated cavities (Megerian et al. 2007), lid driven cavities (Kasnakoglu 2010), or supersonic jets (Moreno et al. 2004) have been extracted relatively successfully using the POD-Galerkin model reduction. Using this method, Graham et al. (1999a,b) investigated vortex shedding control behind a cylinder and evidenced the difficulties associated with obtaining a representative and robust reduced order model for active control purposes. Based on this work, Bergmann et al. (2007) successfully minimized the wake behind an actively controlled cylinder using a self adapting reduced order model through trust-region POD. Although the method is well established and successful in two-dimensional low Reynolds number flows, it has been found to often lead to models with low accuracy or even fail in complex three-dimensional systems. Indeed, such systems often involve multiple length and energy scales which can create some difficulties when the turbulent interactions are not always captured by the POD-Galerkin method. In such cases, calibration methods (see Couplet et al. 2005) or alternative orthogonalization methods such as the balanced POD (see Rowley 2005) and or other reduction techniques such as the eigensystem realization algorithm (see Ma et al. 2011) have been used, yet none of these has been found to be a universal solution.

1.7 Objectives and Plan of the Current Study

In view of the past studies on forced film cooling systems, the identification of ideal forcing parameters to obtain increased film cooling performance using results from a predetermined test matrix appears as a challenging if not impossible task, particularly when considering the rather poor knowledge of the fluidics associated with forced film cooling flows. Active control of the film cooling jets subjected to a closed loop controller appears as a viable solution to converge towards optimum forcing conditions in steady but particularly unsteady cross-flow environment. It was established previously that integration of the complete Navier-Stokes equations is neither desirable, nor feasible, when considering the design of a controller-based system. Thus, one of the key steps in the implementation of a close-loop controlled system resides in the derivation of low order equations representative of the flow of interest, here in particular film cooling flows. The current study intends at establishing the framework for developing a controller based system by providing a reduced order model of unforced and forced film cooling systems. The specific objectives for this work are:

1. To perform a fundamental mechanistic analysis of unforced and forced low blowing ratio jets in cross-flow. To this end, vertical low blowing ratios forced and unforced jets in cross-flow will be investigated in an effort to extend the otherwise extensive literature concerning those flows at high blowing ratios. Investigations will include experimental measurements and visualizations, as well as Large Eddy
Simulations. Inclined jets in cross-flow, more relevant to film cooling application, yet much less documented, will then be investigated using similar methods and the results put in perspective with the ones obtained for the vertical jet.

2. To develop a reduced order model of unforced and forced film cooling velocity and temperature fields. To this end, reduced order model equations of canonical two-dimensional flows will be obtained using the Proper Orthogonal Decomposition - Galerkin Projection method. The velocity and temperature fields from the numerical simulations of unforced and forced film cooling jets will be statistically analyzed using POD to identify energetically dominant vortical features and potential shortcomings associated with POD for reduced order modeling purposes. Finally, reduced order models for the velocity and temperature fields of realistic three-dimensional unforced and forced film cooling jets will be derived using the POD-Galerkin method and their performance assessed.
Chapter 2
Vertical Jet in Cross-flow

In order to gain physical insight on forced film cooling jets, the vortical structures generated in low blowing ratio partly modulated vertical jets in cross-flow are investigated first. Though not strictly application-relevant when it comes to film cooling, these jets offer the advantage of having been well documented in the past, though at blowing ratio values usually relatively high, and constitute a good starting point for a mechanistic analysis and understanding of forced film cooling flows.

In the first part of this chapter, a detailed unforced survey is presented to provide a description of the vortical structures found in low blowing ratio transverse jets as well as their evolution as the blowing ratio is increased to reach a more commonly studied detached configuration. In a second part, partially modulated transverse jets are described with a particular focus on the transient regimes introduced by jet forcing. This study is based on experimental results using hot wire anemometry and Mie scattering visualizations while monitoring the jet flow rate in a time-resolved manner. Large Eddy Simulation (LES) as implemented in Ansys Fluent is also used to provide additional insights on the behavior of the vortical structures. Film cooling performance is evaluated from the numerical simulations for the forced jets and compared to the unforced results.

2.1 Experimental and Numerical Setup

2.1.1 Wind Tunnel Experiments

Experiments are conducted in an open loop aerodynamic wind tunnel with a 9m long test section and a $0.9 \times 0.6m$ cross-section, schematically presented in figure 2.1. A set of four conditioning screens followed by a contraction with an area ratio of 16 : 1 are located directly upstream of the test section to provide stable inlet conditions during the experiments. Optical access is available through a set of transparent acrylic walls constituting the top and one of the side walls of the test section, respectively allowing visualizations in planes parallel to the bottom wall ($X-Y$) and parallel to the jet symmetry plane ($X-Z$). The coordinates $x$, $y$, and
are respectively associated with the stream-wise, span-wise and vertical directions of the flow as shown in figure 2.1, the origin is taken at the center of the jet exit and the quantities $X_j$, $Y_j$, and $Z_j$ correspond to the normalized coordinates with respect to the jet diameter $D_j$ (resp. $x/D_j$, $y/D_j$ and $z/D_j$). The free stream flow in the test section has turbulence intensity levels of less than 0.5%, with a boundary layer Reynolds number at the jet level of $Re_{\infty} = U_{\infty} \delta/\nu = 1,700$ (with $\delta$ the 99% boundary layer thickness and $\nu$ the air kinematic viscosity). In this study aimed at the observation of the vortical structures generated under pulsed conditions, the choice of a laminar boundary layer is made in order to limit the number of factors affecting the jet behavior. The jet exits from a 25.4mm ($D_j$) round exit located 762mm downstream of the test section inlet and is mounted flush to the bottom wall. The air supply for the jet is provided by an industrial compressor, dried using an inline desiccant dryer regulated at 1.4bar. At the jet inlet, the setup is composed of two branches, a principal and a bypass, each comporting a metering needle valve to control the flow in the branch. The bypass also comports a computer-controlled solenoid valve which is used to pulse the flow during forced flow experiments (see figure 2.1). This system provides the ability to independently and accurately set the low ($BR_l$) and high ($BR_h$) jet blowing ratios in forced experiments, recorded in a time resolved manner by an inline flow-meter. The jet and cross-flow are at ambient temperature therefore the density ratio is approximately one. The experimental setup is studied using laser sheet Mie scattering visualizations, hot wire anemometry and time resolved flow rate records. An in-house seeding system injects in two locations water and Titanium Tetrachloride ($TiCl_4$) vapors inside the jet to instantaneously generate sub-micron Titanium Dioxide ($TiO_2$) particles used as a tracer in Mie scattering visualizations. Turning off the jet water seed allows for reactive Mie scattering visualizations since the reaction only occurs in the test section as $TiCl_4$ vapors reacts with moist cross-flow in the mixing layers. To control the solenoid valve and acquire data from the different sensors, a set of analogue-to-digital acquisition modules from National Instruments™ coupled with a fully automated 3-axis traverse system placed on top of the test section are used to conduct anemometric measurements and visualizations in a synchronized manner, particularly under forced conditions. Flow rate records are acquired with a $1kHz$ sampling frequency, through long enough periods of time to extract statistically significant averaged and phase averaged quantities, with more than 30 cycles per record during pulsed-flow experiments. Hot wire measurements, used for spectral characterization, are performed at a sampling frequency of $10kHz$ using appropriate Nyquist filtering at $5kHz$ by means of a single wire probe (TSI model 1212-T1.5), with a sensing length of $1.27mm$ ($D_j/20$). Records used for statistical purposes are acquired with a sampling interval longer than the local integral time scale including enough points to ensure less than 5% uncertainty on the computed average quantities. Wavelet analysis is performed on high sampling records to provide better insight on deterministic signatures observed in unsteady flow conditions, and often masked in Fourier analysis. The wavelet computations are made using a modified version of the algorithm presented in Torrence & Compo (1998) with a Morlet wavelet of characteristic parameter 6 giving satisfactory temporal and spectral resolutions between 0 and 50Hz. Contrast Limited Adaptive Histogram Enhancement (CLAHE) from Matlab™ is used in Mie scattering visualizations to locally improve the contrast.

### 2.1.2 Numerical Simulations

Numerical simulations are carried out in parallel to the experiments in order to provide more detailed information on the vortical structures and their formation mechanisms under forced and unforced conditions. A commercial solver (Ansys Fluent™) is used to simulate the unsteady, turbulent flow through incompressible Large Eddy Simulation (LES) using a dynamic Smagorinsky sub-grid scale model with second order accuracy for both spatial and temporal discretizations. Figure 2.2 shows the simulated domain consisting of a rectangular box representing a part of the wind-tunnel test section and of the jet pipe as well as the applied boundary conditions. The main computational domain is $18D_j$ long (stream-wise x-direction), $8D_j$ wide (span-wise y-direction) and $6D_j$ tall (vertical z-direction) and accounts for $174 \times 92 \times 76$ cells along the respective dimensions. The jet exit center is located $6D_j$ downstream from the domain inlet and the jet pipe domain is $8D_j$ long to allow development of the artificial inlet boundary conditions, the parameters of which are adjusted to approximate the natural flow induced during experiments. The jet tube mesh consists of an O-grid type mesh with 3060 cells in the cross-section and 150 cells along the pipe axis. The full mesh is structured and contains overall 1.8 million cells. The first cells in contact with solid walls are on average $0.03D_j$ tall in the normal direction for $y^+$ values below or close to unity. For all cases, the
Figure 2.2: Vertical jet numerical grid details and boundary conditions. (a) X-Y view, (b) X-Y view detail, (c) X-Z view, (d) Y-Z view, (e) global view and imposed boundary conditions.
simulation process starts with an initial RANS (Reynolds Averaged Navier Stokes) simulation with a $k-\epsilon$ model as implemented in Fluent, to initialize the flow domain and obtain a maximum viscous dissipation rate ($\epsilon$) estimate in the domain. While the pressure, velocity and temperature fields are used to initialize the LES simulations, the viscous dissipation rate is used to compute the time step used in LES simulations according to the Kolmogorov time scale ($\tau_K = \sqrt{(\nu/\epsilon)}$). Therefore in the current simulations, the time step varies from $5.10^{-4}s$ at $BR = 0.150$, to $5.10^{-5}s$ at $BR = 4.3$. To obtain statistically significant data, the flow is integrated over several thousands iterations for each case on high performance computing platforms at Louisiana State University (IBM Power5+ at 1.9GHz), using an average of 24 processors per run. Velocity characteristics and boundary layer profiles for the inlet of the computational domain are obtained from hot wire measurements performed in the wind tunnel as summarized in table 2.1, where $\delta^*, \theta$ and $H$ correspond respectively to the boundary layer displacement, momentum thickness and shape factor.

At the inlet of the jet pipe, uniform velocity is set so as to equal the volumetric flow rate of the experiment. In the pulsed jet simulations, the velocity is modulated by using the signal of the unsteady volumetric flow rate measurement from the flow-meters during experiments. Spatial profile perturbations are provided using the spectral synthesizer method implemented in Fluent with a characteristic length scale equal to $D_j$ and a cross-flow characteristic boundary layer length scale equal to $0.4\delta$. Based on the experimental measurements presented in table 2.1, a 0.5% perturbation level is imposed at the cross-flow inlet. Because the experimental setup does not allow measurements $8D_j$ inside the jet pipe, the perturbation levels at the jet inlet are adjusted so that simulated and experimental perturbation levels at the jet exit would match. The jet and test section walls are set as adiabatic. Periodic boundary conditions are applied to the side boundaries of the test section box and a standard convective outflow boundary condition is used on the outlet plane. Finally, conversely to the experiment, the jet and cross flow fluids are maintained at constant temperatures of respectively 300 and 330K similarly to what was done in the experiments of Ekkad et al. (2006). The density difference associated with those temperature gradients is sufficiently low as to not affect the velocity field.

A grid independence study is carried out at a blowing ratio of $BR = 0.75$ for a geometry comparable to the one presented here in terms of dimensions and features, using grids of 0.3, 0.7 and 1.6 million cells.
Figure 2.4: Experimental (symbols) and simulated (solid line) velocity magnitude profiles in the symmetry plane at $X_j = 1$ and $X_j = 3.5$ for (a) $BR = 0.15$, (b) $BR = 0.25$, (c) $BR = 0.465$.

Table 2.2: Relative velocity magnitude error with respect to the 1.6$M$ grid values on multiple streamwise profiles.

<table>
<thead>
<tr>
<th>$X_j$</th>
<th>$X_j = 0$</th>
<th>$X_j = 2.5$</th>
<th>$X_j = 5$</th>
<th>$X_j = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max. avg.</td>
<td>max. avg.</td>
<td>max. avg.</td>
<td>max. avg.</td>
</tr>
<tr>
<td>0.3$M$</td>
<td>6.1% 1.0%</td>
<td>25.2% 2.0%</td>
<td>9.3% 2.4%</td>
<td>9.3% 2.9%</td>
</tr>
<tr>
<td>0.7$M$</td>
<td>5.9% 0.9%</td>
<td>9.5% 2.0%</td>
<td>7.1% 2.9%</td>
<td>6.8% 3.0%</td>
</tr>
</tbody>
</table>

which results are presented in figure 2.3 in terms of normalized mean velocity magnitude. The overall shape of the velocity profiles is consistent for all three simulations and at all streamwise locations. The relative error on the velocity magnitude are calculated with respect to the most refined grid and are presented in table 2.2. The maximum local error levels are below 10% while the average error remains below 3%. These results provide good confidence in the consistency of the results obtained with the highest resolution grid.

Figure 2.4 shows a comparison between experimentally measured and predicted velocity magnitude profiles at two stream-wise locations, $X_j = 1.0$ and $X_j = 3.5$ in the symmetry plane $Y_j = 0$ for three values of the blowing ratio. Because experimental profiles are obtained using single component hot wire anemometry, no distinction can be made between stream-wise and vertical velocity components, thus velocity magnitudes are chosen as a comparison basis. In addition, recirculatory flow present near the wall at $X_j = 1.0$ cannot be measured accurately because of hot wire signal rectification, thus no experimental data is reported in this region. The velocity profiles in figure 2.4 compare reasonably well at $BR = 0.15$ and $BR = 0.465$ for both stream-wise locations, while at $BR = 0.25$, the simulations appear to under predict the velocity magnitude in the jet wake. This result is consistent with a mismatch in the transition behavior between experimental and numerical data which will be described later. Overall, the agreement of the rudimentary mean flow comparisons between simulation and experimental results is quite good and re-enforces the qualitative and mechanistic comparisons between the simulation results and the visualizations presented in the following sections. Thus, both physical and simulated flows are deemed comparable enough to be used concurrently in interpreting and understanding fundamental processes.

2.2 Unforced Jet

A rather extensive study of the unforced jet over a wide range of blowing ratios from $BR = 0.150$ to $BR = 4.3$ is conducted as a baseline for the subsequent pulsed jet study. Based on the characteristic vortical
structures of the jet at various blowing ratios, attached and detached jet regimes are identified at blowing ratios below 0.275 and above 0.6 respectively. The latter exhibits classic transverse jet vortical structures (horseshoe vortex, ring shear layer vortices, wake vortices). At intermediate values of the blowing ratio \(0.275 \leq BR < 0.600\) the transition cases exhibit features from both types of flows.

### 2.2.1 Attached Jet

Only few studies have addressed low blowing ratio transverse jets and to the authors’ knowledge, none has been particularly oriented towards describing and understanding their vortical structures or their evolution as \(BR\) is increased. Similarly to the detached configuration, the attached jet exhibits a set of characteristic vortical structures resulting from the interaction between jet and cross-flow. In the reactive Mie scattering visualizations of figure 2.5(a) at \(BR = 0.188\), a recirculation region located directly downstream of the jet exit is revealed by reacted seed, also evidenced in the LES results of figure 2.6 by a low pressure region (\(P^-\)). In the time-averaged streamlines of figure 2.6, the recirculation region is mainly supplied by jet fluid issuing from the sides of the jet tube, although a certain amount of cross-flow fluid (moist air in the experiment) is entrained and reacts to generate the seed particles in the visualization of figure 2.5(a) in that region. In the experimental visualizations, the mixing layers, appearing as bright lines of reacted seed both at the leading and trailing edges, depict the existence of the respective shear layers near the jet exit. This assumption is only valid in the vicinity of the jet exit as the mass diffusivity of titanium dioxide particles is typically two to three orders of magnitude lower than the momentum diffusivity of air. At such low blowing ratios, the jet trajectory is highly diverted by the cross-flow with significantly higher momentum, and both upper and lower shear layers merge within the first one to two jet diameters downstream from the jet exit. Figures 2.5(a) and 2.7 show the typical shear layer structures found in the attached configuration consisting of interlocked hairpin vortices (\(Hp\)) similar to the structures described by Perry & Lim (1978) in co-flowing jets. Comparable vortical structures were previously observed in detached transverse jets by Camussi et al. (2002), at blowing
ratios of 2.0 and above though at extremely low jet Reynolds numbers of \( Re_j = (U_\infty D_j)/\nu = 100 \) relative to the current value of 2,700. These vortices are formed periodically in the lower shear layer with increasing formation frequency as the blowing ratio is increased. Multiple self-induction and mirrored induction effects dictate the behavior of the hairpin vortices as they are convected downstream by the cross-flow momentum. The schematics of figures 2.8(a) and 2.8(b) show how the mirrored induction with respect to the bottom wall (figure 2.8a and 2.8b - hollow arrows) causes the legs of the hairpin vortices to converge toward the symmetry plane \( (Y_j = 0) \) and tends to reduce the convection velocity of the head, while the convective effect from the cross-flow carries the structures downstream. Simultaneously, the mutual induction from the legs (figure 2.8b - solid arrows) generates a vertical velocity component which is reinforced by the converging effect of the mirrored induction as the legs approach each other. Finally, the hairpin vortices are also subject to line vortex induction due to curvature, (see Batchelor (2000)), which in the present case corresponds to vertical and a counter-streamwise component (figure 2.8a - dashed arrow).
Figure 2.9: X-Y view of Laplacian of the pressure iso-surfaces at $\Delta P = 1.2 \text{kPa.m}^{-2}$ extracted from LES simulations at $BR = 0.188$, colored using stream-wise vorticity from black (negative) to white (positive). Successive time instants are such as (a) $t_c = t_0$; (b) $t_c = t_0 + 2.2 (35 \text{ms})$; (c) $t_c = t_0 + 5.4 (85 \text{ms})$; (d) $t_c = t_0 + 8.5 (135 \text{ms})$.

A time series of instantaneous simulated flow fields at $BR = 0.188$ is presented in figure 2.9. Iso-surfaces of the Laplacian of the pressure, colored to reflect stream-wise vorticity levels, are generated to identify vortical structures based on the principle that vortex cores are associated with local pressure minima. This choice was made based on the proportionality between the second invariant of the velocity gradient tensor ($Q$) and the Laplacian of the pressure in incompressible flows and on the equivalence between $Q$ and the second eigenvalue ($\lambda_2$) of the tensor $S^2 + \Omega^2$, as described in Jinhee & Fazle (1995), where $S$ corresponds to the stress tensor and $\Omega$ the rotation rate tensor. The reported time scale is the convective time scale such that $t_c = t.U_\infty/D_j$, with $t$ the time in seconds. The hairpin vortices observed in the experiments are also present in the simulations and denoted as $Hp1$, $Hp2$, and $Hp3$ in figure 2.9. Upstream from the hairpin vortices $Hp1$ and $Hp2$, the spatial separation between upper and lower jet shear layers is extremely low, therefore yielding sharp velocity gradients and resulting in the formation of an inverted hairpin vortex denoted as $IHp$. Compared to regular hairpin vortices, inverted ones have identical y-vorticity in the ‘head’ but opposite x-vorticity in the legs which instead of converging towards the symmetry plane, tend to spread away from it due to opposite induction effect from their respective mirrored images. Because inverted hairpins are formed near the wall as well as in between and below regular hairpins, their presence in the Mie scattering visualizations can only be hinted at in figure 2.5(a) by a rollup located directly beneath the most downstream hairpin vortex, and in figure 2.7 where V-shaped structures precede the regular hairpin vortices. Overall, the dynamics of the hairpin vortices observed in the simulations were found to be consistent with the one described in the experiments.

In figures 2.5 and 2.7, a stable horseshoe vortex ($H$) is visible above and on the leading edge of the jet exit. This vortical system is the one described by Fric & Roshko (1994) and Kelso & Smits (1995), resulting from the cross-flow boundary layer separation due to the jet blockage effect. Relatively small at the lowest blowing ratios, (approximately $0.19D_j$ in radius at $BR = 0.15$), the primary vortex becomes larger in diameter with increasing values of the blowing ratio (up to $0.26D_j$ at $BR = 0.275$), while being pushed further upstream from the leading edge of the jet. Although only a single rollup is clearly visible in figures
2.5 and 2.7, at least one other small scale rollup was intermittently observed upstream of the first one. The large eddy simulations in figure 2.9 confirm the presence and location of the horseshoe vortex at $BR = 0.188$. In figure 2.5, the horseshoe vortex is seeded with jet tracer particles entrained from the upper lip of the jet into the horseshoe vortex system, suggesting transport of jet fluid by the structure. This is in agreement with the time averaged LES results of figure 2.6, showing streamlines exiting from the jet into the horseshoe vortex system. Additional stream-wise vortices were also found in both Mie scattering visualizations and numerical simulations. In figure 2.7, these side vortices ($S$), are located on the outside of the horseshoe vortex and appear to interact downstream with the shear layer vortices. In figure 2.9, the side vortices have opposite stream-wise vorticity compared to the horseshoe vortex and are denoted by $CH$ near the jet exit, as counter-horseshoe vortices. The origins of these secondary vortices are explained in figure 2.10(a) by a strong positive spanwise (or locally radial outwards) velocity gradient in the vertical (wall-normal) direction due to jet fluid injected around the jet hole, as well as cross-flow deflection imposed by the jet blockage. Conversely to higher blowing ratio studies (e.g. Kelso & Smits 1995) describing complex horseshoe vortex systems with multiple positive and negative horseshoe vortices ahead of the jet exit, no evidence of negative span-wise vorticity was found upstream of the jet exit. This suggests that the current counter-horseshoe vortex is not the result of cross-flow boundary layer separation due to adverse pressure gradient at the jet leading edge, but rather the result of cross-flow deflection and induced vertical velocity gradients as presented in figure 2.10(a). Further downstream, as shown in figure 2.9, both horseshoe and counter-horseshoe vortices interact with the shear layer structures, to respectively form positive and negative side-vortices $S1$ and $S2$ (named after their stream-wise vorticity sign in the first space quadrant). The relatively stronger opposite-sign vorticity from the hairpin vortices legs tends to weaken the negative side-vortices, allowing the positive ones to expand in these regions thus resulting in apparently disrupted stream-wise vortices of varying strength and size along the jet core. Due to their closeness to the wall and their opposite vorticity, both types of side-vortices behave differently as they are convected. Figure 2.8(c) illustrates first order mutual induction effects as well as mirrored induction effects. On one hand the negative side-vortex ($S1$) has a mirrored induction motion toward the symmetry plane, whereas its interaction with the positive side vortex ($S2$) tends to entrain it away from the wall. On the other hand, the positive side-vortex induced velocity, entrains it away from the symmetry plane, while its interactions with both the negative side-vortex and the hairpin vortex have an opposite yet weaker influence, such that the positive side-vortex tends to have a downward motion. Thus, as they are convected downstream, the negative side-vortices will have a tendency to ride on top of the positive ones while converging toward the symmetry plane, whereas the latter will diverge from the symmetry plane. The different relative motions of the side vortices result in “X-patterned” structures lying on each side of the jet core, which can be observed both in the Mie scattering visualizations of figure 2.7 and the LES visualizations in figure 2.9, and are illustrated in the schematic of figure 2.8(d).

Amongst the few low-blowing-ratio studies of the past, Gopalan et al. (2004) suggested that at low blowing ratios, the span-wise vorticity from the boundary layer, resulting in the formation of the horseshoe vortex, interacted with the jet upper shear layer vorticity and ultimately canceled it so that, conversely to
the higher blowing ratio cases, the upstream shear layer was not rolling and no vortical structure was formed at the leading edge of the jet. However, in the current study, the side visualizations of figure 2.5(b), realized in fully reacted Mie scattering experiments, show the presence of a stable rollup of reacted seed, or inner vortex \((Iv)\), resulting from jet flow separation inside the pipe and located at the leading edge. The shedding of this vortex is prevented by the blockage exerted by the cross-flow on the jet as suggested by the relatively advanced position of the horseshoe vortex. This flow separation inside the jet pipe was also reported in the large eddy simulations from Guo \textit{et al.} (2006) where, at low blowing ratios of 0.1, the rollup occupied almost half the jet pipe. Accordingly, in figure 2.6, a high pressure region is located upstream and above the jet exit, creating an adverse pressure gradient for part of the jet fluid exiting from the pipe, thus forcing it to roll up at the jet leading edge. The time averaged stream lines show that jet fluid from the upstream region of the jet exits on the sides of the jet pipe as well as upstream, underneath the primary horseshoe vortex. The closed vortex line in figure 2.10(a) (gray line) also shows that the inner vortex and the consecutive hairpin vortices form a series of interlocked rings illustrated schematically in figure 2.10(b).

In the time averaged stream-wise vorticity field of figure 2.11, a pair of stream-wise counter-rotating vortices is located on each side of the symmetry plane (filled arrows). This result is a common feature in transverse jets and is usually identified as the CRVP. According to the instantaneous observations, the vorticity of this vortex pair is generated by the hairpin vortices. Initially observed in the average flow field of the detached jet, the CRVP has also been observed in the instantaneous flow field in Rivero \textit{et al.} (2001). It could be tempting to directly interpret the vortex pair found in figure 2.11 as the usual CRVP, although a few considerations lead us to do so with caution. First, the perceptible continuity of the vortex pair in the average vorticity field of figure 2.11, is in fact induced by the downstream convection of the hairpin shear layer vortices. Instantaneously though, this vortex pair is discontinuous, as can be seen in figure 2.9(d) near \(X_j = 9\), where no particular vortex pair is found. This constitutes a first difference with the usual CRVP structure which, although it can be asymmetric in strength with respect to the symmetry plane (Smith & Mungal 1998; Rivero \textit{et al.} 2001), has never been documented as discontinuous. Furthermore in figure 2.11 at \(X_j = 12\), the vorticity in the vortex pair appears twice as low as the one in the average side vortices found near the wall (hollow arrows), which is somewhat in contradiction with the fundamental notion that the CRVP is the dominant structure in the far field. This last point is supported by the distribution of the passive scalar concentration (in our case the normalized temperature field \((\eta = (T - T_\infty)/(T_j - T_\infty))\) contours, which is highly distorted near the wall, in an inverted T shape, and does not assume the well-known kidney shape in the neighborhood of the jet exit or circular distribution further downstream. This shows that the side vortices are dominantly participating in the scalar quantity transport. Therefore, although it appears similar to the familiar structure found in most high blowing ratio studies, the stream-wise vortex pair present in the average vorticity field of figure 2.11 exhibits significant differences when compared to the “traditional” CRVP.
Figure 2.12: Mie scattering visualizations in the plane $Y_j = 0$ and $BR = 0.25$ at (a) $t_c = t_0$, (b) $t_c = t_0 + 6 (95\, ms)$, (c) $t_c = t_0 + 7.9 (125\, ms)$, (d) $t_c = t_0 + 9.5 (150\, ms)$. $Hn$: Horseshoe vortex $\neq n$, $Iv$: Inner vortex.

Figure 2.13: Instantaneous spanwise vorticity $X-Z$ slices and streamlines at $BR = 0.25$. Time sequence (a–d) corresponding to figure 2.12. $Hn$: Hairpin vortex $\neq n$, $Iv$: Inner vortex.

For blowing ratios in the range $0.225 \leq BR \leq 0.275$, the vortical structures are overall identical to the one found in the lower blowing ratio cases described previously. In addition, these cases exhibit irregular transport of the horseshoe vortex, which is convected downstream on top of the jet upper shear layer as evidenced in figure 2.12. Originally located upstream of the jet exit in figure 2.12(a), the horseshoe vortex is shown to ride on top of the jet/cross-flow interface in figure 2.12(b). This destabilization is likely provoked by an interaction between the horseshoe vortex and the inner vortex, which is partially outside the jet pipe in figure 2.12(b). Once “detached”, the horseshoe vortex is convected downstream, where it eventually merges with hairpin vortices of identical span-wise vorticity sign. Although not immediately seeded and entirely visible, a second horseshoe vortex very quickly replaces the convected one in the cross-flow boundary layer upstream of the jet as seen in figure 2.12(d). The transport is also occurring in the simulation results of figure 2.13 at identical blowing ratio, where a second horseshoe vortex $H2$ is instantaneously formed while the first one, $H1$, is transported downstream. This shedding affects the inner vortex which can be seen oscillating inside the jet pipe in both experimental and numerical visualizations. In the numerical results of the sequence shown in figure 2.13, the transported horseshoe vortex (figure 2.13a) loses its coherence as it is convected and diffused in the jet upper shear layer. In the experimental Mie scattering visualizations however, the shed vortices were found to stay more coherent and visibly transported over the jet.

2.2.2 Transitional Jet

Although the horseshoe vortex transport described previously constitutes an early sign of the jet transitioning from the low blowing ratio regime to the high blowing ratio one, the majority of the vortical structures are still analogous to those of the attached jet configuration. However, for blowing ratios between 0.275 and 0.6, a major difference appears as the previously stable inner vortex starts to shed. A typical convection sequence of this vortical structure is presented in figure 2.14. The inner vortex ($Iv$), still partly inside the jet
Figure 2.14: Fully Reacted Mie scattering visualizations at $Y_j = 0$ and $BR = 0.365$ (a) $t_c = t_0$; (b) $t_c = t_0 + 6$ ($95\text{ms}$); (c) $t_c = t_0 + 7.9$ ($125\text{ms}$); (d) $t_c = t_0 + 9.8$ ($155\text{ms}$). $Hn$ - Horseshoe vortex #n, $Iv$ - Inner vortex, $Hp$ - Hairpin vortices.

Figure 2.15: Laplacian of the pressure iso-surfaces at (a-d) $\Delta P = 10kPa.m^{-2}$; (e) $\Delta P = 2.5kPa.m^{-2}$ from LES at $BR = 0.465$, colored using streamwise vorticity from black (negative) to white (positive). Time stamps (a) $t_c = t_0$; (b) $t_c = t_0 + 1.6$ ($25\text{ms}$); (c-e) $t_c = t_0 + 2.2$ ($35\text{ms}$); (d) $t_c = t_0 + 3.5$ ($55\text{ms}$). $Hn$ - Hairpin vortex #n, $Rv$ - Ring vortex #n, $Iv$ - Inner vortex #n.

tube in figure 2.14(a), is convected in figure 2.14(b), which is followed by an ingestion of cross-flow fluid inside the jet pipe at the leading edge. The perturbation of the jet/cross-flow interface also triggers the transport of the horseshoe vortex ($H1$) in figure 2.14(c), followed by the formation of a large hairpin vortex ($Hp$) in figure 2.14(d). A second horseshoe vortex ($H2$), shown in the reversed grayscale window in figure 2.14(a, b, c), replaces the convected one ahead of the jet exit almost instantaneously. At values of the blowing ratio approaching 0.6, the size of the transported inner vortex becomes comparable to the downstream shear layer vortices and the ingestion of cross-flow fluid that follows the inner vortex transport, is gradually decreased, until it completely disappears beyond $BR = 0.6$.

The evolution of vortical structures observed in the LES results is consistent with experimental observations. Figure 2.15(a-d) presents the pressure Laplacian iso-surfaces describing an inner vortex transport sequence comparable to the one in figure 2.14. In addition, the simulations show a connection between the inner vortex and the downstream hairpin vortices, thus forming a ring vortex ($RvI$), which is a characteristic vortical structure encountered in higher blowing ratio regimes. Once shed, the upstream rollup is folded and quickly dissipated in the vicinity of the jet exit by the overall positive span-wise vorticity of the shear layers. After the inner vortex (or ring vortex $RvI$) is transported, the horseshoe vortex ($H1$) is subsequently destabilized and convected (e.g. figure 2.15c and 2.15e). A second inner vortex ($Iv2$) and horseshoe vortex ($H2$) are formed respectively inside the jet pipe and ahead of the jet exit. Hairpin vortices, typical of lower blowing ratio jets, are also found in figure 2.15(e) ($Hp$), downstream of the ring vortices.

The average stream-wise vorticity field in figure 2.16 shows a decrease in the strength of the side vortices (hollow arrows) near the wall, compared to the principal vortex pair (solid arrows). Consequently, the
Figure 2.16: Time averaged stream-wise vorticity (filled) and normalized temperature contours by increments of 15% (line) from LES at $BR = 0.465$.

Figure 2.17: Fully reacted Mie scattering visualization in the plane $Y_j = 0$ at $BR = 1.5$. $Rv$ - Ring vortices, $Wv$ - Wake vortices.

Normalized temperature (scalar) concentration distribution is impacted by the vortex pair, to become more comparable with the typical distribution of transverse jets at higher blowing ratios (see Smith & Mungal 1998). It should be noted that the inner vortex ($Iv$) in the experiments was destabilized and convected at blowing ratios above 0.275, while similar sequences were clearly observed in the large eddy simulations for $BR \geq 0.415$. This discrepancy was attributed to inconsistencies in the spectra of the jet and cross-flow inlet fluctuations, between experiment and simulations, to which the interaction between horseshoe vortex and inner vortex is expected to be highly sensitive. Such sensitivity to “initial” conditions is well known in shear flows with quasi-deterministic vortical structures.

### 2.2.3 Fully Detached Jet

In the current study, cases with blowing ratios above 0.6 exhibited a completely detached jet as shown in figure 2.17 for $BR = 1.5$, a configuration which has been widely studied and described by Fric & Roshko.
Figure 2.18: Laplacian of the pressure iso-surfaces at $\Delta P = 20kPa.m^{-2}$ and streamlines extracted from LES simulations at $BR = 1.0$, colored using span-wise vorticity from black (negative) to white (positive). Contour slices show streamwise vorticity from blue (negative) to red (positive). Successive time instants such as (a) $t_c = t_0$; (b) $t_c = t_0 + 0.63$ (10ms); (c) $t_c = t_0 + 1.25$ (20ms); (d) $t_c = t_0 + 2.9$ (30ms); $H$ - Hairpin vortex, $Rv$ - Ring vortex $#n$, $Wv$ - Wake vortex.

(1994) and Kelso et al. (1996) amongst others. This threshold value was found to be consistent with previous studies such as Harrington et al. (2001b). In figure 2.17, shear layer vortices ($Rv$), appearing as well defined rollups on both the upper and lower jet/cross-flow interface, shed periodically, while wake vortices ($Wv$) located under the jet core are evidenced by seed particles, thus suggesting these structures transport jet fluid.

In figure 2.18, the large eddy simulations at $BR = 1.00$ show that on the downstream side of the jet, the shear layer rolls up outside the jet pipe, whereas the upstream part rolls inside it. The different pressure conditions at the jet exit on the windward (adverse pressure gradient) and leeward (favorable pressure gradient) sides are in part responsible for this. The dynamics of the ring vortices observed in figure 2.18 are comparable to the one described by Kelso et al. (1996). After shedding, the vortex rings are folded in such a way that the lower and upper rollups approach each other, thus increasing their induced upward velocity, while the connecting side arms are stretched in the downward direction. The horseshoe vortex, invisible because unseeded in the Mie scattering visualizations, is located upstream of the jet exit and oscillates back and forth due to the fluctuating pressure field associated with the shedding of shear layer vortices, in agreement with the results from Kelso & Smits (1995). The stream-lines in figure 2.18 show the formation of wall and wake vortices from the cross-flow boundary layer separation around the jet exit. A pair of vortex tubes is also identified in the instantaneous Laplacian of the pressure iso-surfaces consistent with the CRVP, joining the individual vortex rings. An initialization mechanism for this structure is described in the following sections. In figure 2.19, the average stream-wise vorticity field has the typical kidney shaped distribution in the near field of the jet exit, while in the far field the counter rotating vortex pair appears to be the predominant feature both in terms of size and strength, and accordingly the normalized temperature ($\eta$) distribution contours are in good agreement with the previous results in detached jet experiments mentioned earlier.

### 2.2.4 Characteristic Frequency Modes

Single component hot wire measurements were performed at select locations, chosen based on the visualizations in order to identify frequency modes associated with the previously described structures. The
Figure 2.19: Time averaged stream-wise vorticity (filled) and \( \eta \) contours by increments of 15\% (line) from LES at \( BR = 1.0 \).

Figure 2.20: Fundamental frequencies scaled using (a) time averaged jet velocity; (b) cross-flow velocity. In the experiment these were identified from hot wire records using wavelet analysis. The symbol \( \bigcirc \) corresponds to the jet exit \( (X_j = 0, Z_j = 0) \); \( \triangledown \) is from location \( (X_j = 0, Z_j = 0.25) \) and \( \nabla \) from location \( (X_j = 0, Z_j = 0.5) \), both inside the upper shear layer; \( \blacksquare \) corresponds to the fundamental and \( \blacklozenge \) to its subharmonic from downstream location \( (X_j = 3.5, Z_j = 0.75) \); \( \square \) is from location \( (X_j = 3.5, Z_j = 1.25) \). The symbol \( \bullet \) indicates characteristic frequencies from the LES at the center of the jet exit \( (X_j = 0, Z_j = 0) \).
measurement locations were at the jet exit, in the upper shear layer above the jet exit, and at a downstream location \((X_j = 3.5)\). The velocity signals were subjected to both Fourier and wavelet spectral analysis, the latter between 0 and 500 Hz (corresponding to Strouhal numbers \(St_\infty\) between 0 and 0.794), guided by the results from the former. The results are presented in figure 2.20, summarizing the fundamental frequencies found at each location scaled by jet diameter and mean jet velocity \((St_j = f.D_j/U_j)\) in figure 2.20(a) and scaled by jet diameter and cross-flow velocity \((St_\infty = f.D_j/U_\infty)\) in figure 2.20(b). In figure 2.21, wavelet analysis mappings show that measurements performed at the jet exit (and above) revealed no particular frequencies for \(BR \leq 0.2\), due to the weakness of the shear layer instability at this location at low blowing ratios, while the one performed at \(X_j = 3.5\) exhibited no clear signature for \(BR \geq 0.3\). Results from the attached jet regime show an increase in Strouhal number with increasing blowing ratio, corresponding to the rate of formation of hairpin vortices. Similarly, results from the detached jet at \(BR = 0.6\) show a nearly linear rate of increase in Strouhal number with blowing ratio at both locations, corresponding to the formation rate of ring vortices. The simulation results show an identical trend but with overall lower frequencies and a slightly lower rate of increase. This is in agreement with previous results reported by Megerian et al. (2007) for \(BR < 3.5\), even though their study showed Strouhal numbers increasing at a rate of 1/5, while in the current study the rate of increase is approximately 2/5. However, it should be noted that in the previously cited study, stronger jets with top hat exit profiles were involved, whereas in the current investigation, the jets were weaker, with larger shear-layer momentum thicknesses. Fundamental frequencies obtained for the attached jet appear to be significantly higher than equivalent signatures obtained at higher blowing ratios. This is justified by the fact that data for the first regime were obtained at \(X_j = 3.5\), near the wall, where measurements were influenced by the passage of inverted hairpin vortices formed in between the primary hairpin vortices, thus doubling the apparent passage frequency of the primary structures. In order to provide a more relevant comparison base, the corresponding sub-harmonics for those measurements (also found in the wavelet records but with lower energy content) are also provided in figure 2.20 (black diamonds) and show better agreement with characteristic frequencies at higher blowing ratios. The usefulness of the wavelet analysis in unforced conditions was found in the transition regime, where events (or series of events) have a tendency to occur rather intermittently, as seen in figure 2.21(c), making the identification of characteristic frequencies difficult with Fourier spectrum analysis. Overall, the value of \(St_\infty\) appears relatively constant around approximately 0.2 over the range 0.225 < \(BR < 0.425\), while \(St_j\) decreases continuously from 0.780 to 0.32 at the jet exit. This indicates that the dominant feature, in this case the transport of the horseshoe vortex and the inner vortex, are better scaled by the cross-flow velocity and thus suggest that the transport mechanism is initiated by the horseshoe vortex. On the other hand, for 0.425 < \(BR < 0.6\), \(St_j\) values tend to reach saturation, while \(St_\infty\) is clearly increasing, indicating a shift in the transport initiating structure from the horseshoe vortex to the inner vortex.

### 2.2.5 Film Cooling Performance

From a film cooling perspective, it is interesting to identify the vortical structures potentially influencing the wall temperature field. Figures 2.22(a) and 2.23(a) show that the area of highest adiabatic effectiveness in the attached configuration is located in the recirculation region, which is essentially seeded by jet fluid. The fluid forced to exit on the sides of the jet pipe by the high pressure region is also providing additional coverage as seen in figure 2.24. This figure also shows the effect of the horseshoe vortex on wall adiabatic effectiveness as it entrains significant amounts of coolant upstream of the jet due to its position above the jet exit. The counter-horseshoe and horseshoe vortices have an opposite impact on the jet spread at the wall because of their respective velocity fields. Indeed, near the wall the horseshoe vortex has a velocity field pointing away from the symmetry plane and tend to entrain cooler fluid from the jet core region toward the wall, whereas the counter-horseshoe vortices have an associated opposite velocity field carrying free stream fluid near the bottom surface. Consequently, the formation of the X-patterned structures directly downstream of the hairpin vortices results in dramatic decrease in local wall adiabatic effectiveness. Along the symmetry plane, the legs of the hairpin vortices are responsible for the transport of most of the coolant. These results show the significant role of the horseshoe vortex and the inner vortex in the jet coverage near the wall.

As observed in the experiments for \(BR = 0.25\), the horseshoe vortex is convected, while the inner vortex oscillates inside the jet pipe. After the transport of the horseshoe vortex is initiated (figure 2.22(b) - H1),
Figure 2.21: Wavelet analysis mapping of hot wire time records in the plane $Y_j = 0$ for (a) $BR = 0.188$, $X_j = 0$, $Z_j = 0$; (b) $BR = 0.188$, $X_j = 3.5$, $Z_j = 0.75$; (c) $BR = 0.3$, $X_j = 0$, $Z_j = 0$; (d) $BR = 0.3$, $X_j = 3.5$, $Z_j = 1.0$; (e) $BR = 0.6$, $X_j = 0$, $Z_j = 0$; (f) $BR = 0.6$, $X_j = 3.5$, $Z_j = 1.5$. 
Figure 2.22: $\Delta P$ iso-surfaces at two superposed time instants separated by $\Delta t_c$, first iso-surface colored by temperature: (a) $BR = 0.188$, $\Delta P = 1500$, $\Delta t_c = 1.25$; (b) $BR = 0.25$, $\Delta P = 2000$, $\Delta t_c = 2.2$; (c) $BR = 0.465$, $\Delta P = 5000$, $\Delta t_c = 0.6$; (d) $BR = 0.9$, $\Delta P = 1500$, $\Delta t_c = 0.6$.

Figure 2.23: Time averaged wall adiabatic effectiveness at: (a) $BR = 0.188$; (b) $BR = 0.25$; (c) $BR = 0.465$; (d) $BR = 0.9$. 
Figure 2.24: Instantaneous streamlines and wall temperature contours from 300K (black) to 330K (white) after LES at $BR = 0.188$. Streamlines colored with flow temperature from 300K (blue) to 330K (red).

A second one is instantaneously formed in the cross-flow boundary layer (figure 2.22(b) - H2) so that the coolant supply upstream of the jet exit is never interrupted and coverage is continuously provided to this area. As $BR$ increases, the horseshoe vortex is strengthened and pushed further upstream from the jet exit, thus augmenting the jet coverage at the wall as seen in figure 2.23(b). Finally, the hairpin vortices at $BR = 0.25$ are stronger when compared to the lower blowing ratio cases, and stay coherent further downstream, carrying more coolant along the symmetry plane. Figure 2.22(c) and 2.23(b) present results corresponding to a transitional jet at $BR = 0.465$, showing the immediate lift off of the hairpin vortices’ legs after the jet exit because of increased mutual induction, as well as the shedding of the inner vortex. Each inner vortex is directly connected to a downstream rollup and forms a ring vortex, typical of higher blowing ratio regimes and shown in figure 2.22(c) as R1, rapidly dissipated as it is convected in the free stream. The drop in coverage observed in the contours of adiabatic effectiveness in figure 2.23(c) is explained by the consecutive transports of the inner and horseshoe vortices, interrupting the coolant flow upstream and on the sides of the jet exit. Hence the transitional regime marks the beginning of coverage degradation occurring well before the detached regime settles in. According to figure 2.22(d), at $BR = 0.9$, the shear layer ring vortices (R2) are formed on a periodic basis and are directly convected away from the wall, therefore not providing coverage. Conversely to the transitional regime, the horseshoe vortex formed ahead of the jet is not convected, though it oscillates as the shear layer ring vortices are shed, and its impact on the wall temperature is only minor in 2.22(d) and 2.23(d). Marginal amounts of coolant reach the jet wake region transported by the wake vortices as evidenced by the $\eta$ contours as well as the presence of seed particles in experimental Mie scattering visualizations of figure 2.17. Figure 3.12(a) shows spanwise averaged adiabatic effectiveness from the numerical results. As expected from the contour plots in figure 2.23, $\eta_{span}$ increases with increasing values of the blowing ratio, until the transitional threshold is reached and then drops precipitously. As $BR$ continues to increase, the effectiveness decreases asymptotically toward zero. A local minimum in $\eta_{span}$ profiles for the transitional jet is observed downstream of the jet exit, corresponding to the location where the cross-flow recovers behind the jet and tends to move upstream toward the jet exit as $BR$ increases. Maximum coverage is reached at $BR = 0.415$.

The area averaged adiabatic effectiveness ($\eta_{area}$) presented in figure 3.12(b) was computed from the simulations, assessing the overall film cooling performance at a given blowing ratio. Accordingly to $\eta_{span}$ trends, film cooling is improved for increasing values of $BR$ below the transitional threshold, and decreases
dramatically beyond this point. A coverage coefficient ($C_c$) was defined and computed from the simulations based on total wall area over which the adiabatic effectiveness maintains a value above a set threshold normalized by the jet exit area according to the following definition:

$$C_c(\eta_x) = \frac{1}{A_j} \int_{\eta > \eta_x} dA_{wall}$$

As opposed to $\eta_{area}$, the coverage coefficient gives a global performance index in terms of covered area. Estimating performance based on $\eta_{area}$ only, does not distinguish a configuration with a localized high $\eta$ peak from one with lower peak but greater and more homogeneous coverage. Figure 3.12(b) presents three trends of $C_c$ for adiabatic effectiveness thresholds of 0.2, 0.3 and 0.5. From these results, all trends show a drastic drop of each metric beyond $BR = 0.415$, after an increase with $BR$ up to this threshold for most. The combination of the reduction in both $\eta_{area}$ and $C_c$ shows that both the overall value of adiabatic effectiveness and the covered area are decreased. The coverage degradation is associated with the passage of the jet from the attached to the transitional regime and the disruption of coolant supply to the upstream and side regions resulting from the shedding of the inner vortex. The abruptness of the performance degradation beyond the transitional blowing ratio suggests that operating film cooling systems near this point would be highly hazardous since a slight change in cross-flow velocity could have dramatic consequences on the coverage.

Investigations of the unforced jet have shown that the transitional threshold constitutes a limitation to the improvement of film cooling performance because of the transport of the inner vortex and the associated interruption of coolant supply to the upstream region. Based on the evolution of the film cooling metrics in the attached jet regime, delaying the transport of the inner vortex to higher values of $BR$ could lead to improved overall coverage.

### 2.3 Forced Jet

#### 2.3.1 System Characterization

Forced jet experiments are performed using a nominal square wave actuation signal for the solenoid valve shown in figure 2.1. A wide range of forcing parameters’ values for $BR_m$, $BR_l$, $BR_h$, $BR_{pp}$, (respectively mean, low, high, peak to peak blowing ratios over a cycle) and $DC$ (duty cycle) is covered, a part of which is presented in more detail herein and summarized in table 2.3. Each individual case is observed at 4 distinct forcing frequencies ($f_J$) of 0.5, 1.0, 5.0 and 10.0 Hz, respectively corresponding to Strouhal numbers of $St_{\infty} = 0.008, 0.016, 0.079$ and 0.159. These were selected to stay below the shear layer natural frequencies of the unforced jet (see figure 2.21b) so as to minimize the potential of direct resonant amplification (see Nikitopoulos et al. 2006), although resonance with sub-harmonic and other lower harmonics giving rise to
Table 2.3: Forced cases parameters †: No flow-meter record available, ‡: No experiments at these conditions, *: Simulations carried at these conditions.

<table>
<thead>
<tr>
<th>Case#</th>
<th>$BR_m$</th>
<th>$BR_l$</th>
<th>$BR_h$</th>
<th>$BR_{pp}$</th>
<th>DC(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1†</td>
<td>0.250</td>
<td>0.188</td>
<td>0.438</td>
<td>0.250</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
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<td>0.075</td>
<td>0.325</td>
<td>0.250</td>
<td>70</td>
</tr>
<tr>
<td>3*</td>
<td>0.350</td>
<td>0.188</td>
<td>0.836</td>
<td>0.648</td>
<td>25</td>
</tr>
<tr>
<td>4*</td>
<td>0.350</td>
<td>0.188</td>
<td>0.513</td>
<td>0.325</td>
<td>50</td>
</tr>
<tr>
<td>5</td>
<td>0.350</td>
<td>0.225</td>
<td>0.475</td>
<td>0.250</td>
<td>50</td>
</tr>
<tr>
<td>6</td>
<td>0.350</td>
<td>0.175</td>
<td>0.425</td>
<td>0.250</td>
<td>70</td>
</tr>
<tr>
<td>7‡</td>
<td>0.750</td>
<td>0.200</td>
<td>2.20</td>
<td>2.00</td>
<td>25</td>
</tr>
<tr>
<td>8‡</td>
<td>1.25</td>
<td>0.200</td>
<td>4.20</td>
<td>4.00</td>
<td>25</td>
</tr>
<tr>
<td>9‡*</td>
<td>1.25</td>
<td>0.200</td>
<td>2.20</td>
<td>2.00</td>
<td>50</td>
</tr>
<tr>
<td>10‡*</td>
<td>0.365</td>
<td>0.200</td>
<td>2.20</td>
<td>2.00</td>
<td>8</td>
</tr>
<tr>
<td>11‡*</td>
<td>1.28</td>
<td>0.700</td>
<td>3.00</td>
<td>2.30</td>
<td>25</td>
</tr>
<tr>
<td>12*</td>
<td>0.465</td>
<td>0.188</td>
<td>0.712</td>
<td>0.524</td>
<td>50</td>
</tr>
</tbody>
</table>

Figure 2.26: Left: Typical flow rate (gray line), vertical velocity at the jet exit (black line), valve signal (dashed line) records in forced experiments corresponding to Case 4 at forcing frequencies of (a) $St_{\infty} = 0.016$; (b) $St_{\infty} = 0.079$. Right: (c) Overshoot scaling.

The use of time resolved flow-meter measurements allows for precise verification of the forcing conditions. Typical flow rate profiles from Case 4, and corresponding jet exit velocity magnitudes are presented in figure 2.26. These records attest to the good conservation of the square wave features such as the duty cycle and the peak to peak values, if one excludes the initial transients from the low to the high part of the cycle and vice versa. Calibrated duty cycle values are used to account for valve response delays at higher frequencies, so that effective duty cycles of 50% are obtained with actuation $DC$ of 40% and 30% at respectively $St_{\infty} = 0.079$ and 0.159. An oscillatory signature, due to a characteristic of the actuation system, can be observed in the flow rate at the valve opening and closing, with a constant frequency of $St_{\infty} = 0.8$ (47.5Hz) and an exponential decay rate regardless of the forcing parameters. This frequency was well above the jet shear layer characteristic frequencies for $BR \leq 1$ so that no significant direct interaction was expected with the natural jet modes. The phase averaged value of the first blowing ratio overshoot ($BR_{os}$) scales relatively well using the bi-linear dependence on $BR_l$, $BR_h$, $BR_{pp}$ such as $BR_{os} = 1.5.BR_h - 0.5.BR_l = BR_h + 0.5.BR_{pp}$, as evidenced in figure 2.26(c).
The oscillating time in the high blowing ratio part is almost constant over the considered range of blowing ratios, with a slight sensitivity to the $BR_h$ value. The average $e$-folding time is $24 \pm 3\,\text{ms}$ and the average 99% attenuation time is $110 \pm 15\,\text{ms}$, shorter times being achieved at higher values of $BR_h$. Oscillations in the low part of the cycle after the valve closing have an average $e$-folding time of $41.3 \pm 7\,\text{ms}$ and a 99% attenuation time of $192 \pm 32\,\text{ms}$. Because the forcing cycles periods in this study range from 2s down to $100\,\text{ms}$, cases at $St_\infty = 0.008$ and 0.016 have cycle periods much longer than the transient time scales, whereas cases at $St_\infty = 0.079$ and 0.159 have comparable or shorter cycle periods. From these basic preliminary considerations, the jet is expected to behave differently in cases of low forcing frequencies compared to higher forcing frequencies where the transient dynamics introduced by the forcing would dominate the jet behavior. These observations are supported by the jet exit velocity records of figure 2.26(a) at $St_\infty = 0.016$, showing distinct behavior after the transition period of the high part of the cycle, which are not found in figure 2.26(b) at $St_\infty = 0.0159$. Finally, in view of the large number of parameters attached to partly modulated transverse jets, the use of the cross-flow Strouhal number $St_\infty$, defined in the previous sections, is preferred over the parameter $St_j$, necessitating the identification of characteristic velocity scales for each forced case. In the rest of the chapter, the dimensionless time $t^* = t \times f_j$ is used to report instants inside a cycle. Using this parameter, the transition from low to high blowing ratio occurs at $t^* = 0$, while the transition from high to low blowing ratio occurs at $t^* = DC$.

The study of the Mie scattering visualizations leads to a qualitative identification of the different vortical structures generated during a forced cycle. As expected, the jet behavior is found to be different at forcing Strouhal numbers of $St_\infty = 0.008$ and 0.016, compared to higher forcing Strouhal numbers ($St_\infty = 0.079, 0.159$). Figure 2.27 presents visualizations from Case 3 at forcing frequencies of $St_\infty = 0.016$ (top) and $St_\infty = 0.159$ (bottom) for identical phase locked positions. In figure 2.27(a-d), the forced case at $St_\infty = 0.016$ exhibits four distinct phases: a transition from low to high blowing ratio with starting vortices (figure 2.27a), a quasi-unforced plateau in the high blowing ratio part of the cycle with typical structures encountered in the unforced jet at $BR = BR_h$ (figure 2.27b), a transition from high to low blowing ratio with cross-flow ingestion inside the jet pipe (figure 2.27c) and finally a quasi-unforced plateau in the low part of the cycle with typical structures encountered in the unforced jet at $BR = BR_l$ (figure 2.27d). In figure 2.27(e-h) however, only two overlapping phases are found: a transition from low to high blowing ratio with the formation of a starting structure (figure 2.27e, f) and a transition from high to low blowing ratio with cross-flow ingestion inside the jet pipe (figure 2.27g, h). These observations are in agreement with the findings of Ligrani et al. (1996a) who observed quasi-unforced behavior in free stream bulk flow pulsation experiments for $St_\infty < 2.4$, although jet forcing was only passive and imposed through sinusoidal cross-flow pressure fluctuations, which could explain the discrepancy in the limiting Strouhal number values. Similarly, Johari et al. (1999) mention that in their experiment the jet behaves as a “portion of the unforced jet in the far field” for injection times greater than $300\,\text{ms}$.

Among the four possible phases encountered in a typical pulsed case, the quasi-unforced behaviors in both high and low blowing ratios are very similar to the ones described in the previous section for unforced
jets and therefore will not be further discussed. The next part will then focus primarily on the transient phases from low to high blowing ratio and vice-versa.

2.3.2 Low-to-high Blowing Ratio Transition: Starting Vortices

At the high blowing ratio onset, a starting vortex is formed at the edge of the jet exit due to the higher shear generated across the jet/cross-flow interface by the bulk flow rate increase. These starting structures have been observed and described in many pulsed jets in cross-flow experiments and simulations for their potential in mixing and penetration improvement, though generally at higher average blowing ratios and forcing frequencies than in the current work.

Over the range of forcing parameters covered in the present study, two principal types of starting vortices can be observed: vortex rings formed at rather high values of $BR_h$ and hairpin vortices very similar to the one observed under unforced conditions, yet considerably larger as shown in figure 2.28(c), at lower values of $BR_h$. As in Johari (2006) and Sau & Mahesh (2008), the types of starting ring vortices regimes can be divided into two sub-domains depending on the pulse width $\tau$ ($\tau = DC/f$) corresponding to the duration of the high part of the cycle. At low $\tau$ values only a single starting vortex ($S_v$), evidenced in figure 2.28(a, c), is generated while for longer pulse widths, a trailing column composed of smaller vortices ($T_v$) is formed in the wake of the principal starting vortex ($S_v$) as shown in figure 2.28(b, d). These observations are in good agreement with the one presented in previously mentioned work, although trailing vortices in the case of starting hairpin vortices were not clearly documented before. The dynamics of the different starting vortices are described and compared to the DNS results from Sau & Mahesh (2008) in the following sections.

Starting Hairpin Vortex

As mentioned previously, the starting hairpin vortices are in essence identical to those extensively described in unforced conditions. In several cases, such as Case 1 presented in figure 2.29, starting hairpin vortices are generated even though the $BR_h$ value ($BR_h = 0.438$ in Case 1) is significantly higher than the unforced jet transitional blowing ratio threshold ($BR_{tr} = 0.275$). In figure 2.28(c), the inner vortex can be seen partially out of the jet pipe yet without being convected, and eventually pushed back inside for the rest of the cycle. The mechanisms responsible for this behavior will be described later using LES results. Figures 2.29(a) and 2.29(b), respectively before and after the transition from $BR_l$ to $BR_h$, show that starting hairpin vortices are considerably larger than the ones observed in unforced or quasi-unforced conditions and penetrate deeper into the cross-flow. The side and horseshoe vortices are also present in the quasi-unforced regime of figure 2.29(c), and in figure 2.29(d) are entrained in the wake of the leading starting vortex.
hairpin vortex and reoriented in the vertical direction in a similar manner to the formation mechanism of the tornado-like wake vortices described by Fric & Roshko (1994).

A case with higher $BR_h$ value corresponding to Case 4 is investigated both experimentally and numerically. Qualitative comparison between LES results and experimental visualizations can be found in Bidan & Nikitopoulos (2011). The instantaneous stream lines in figure 2.30(a) issuing from both the jet and the cross-flow show significant flow entrainment near the starting hairpin vortex legs, in a pattern comparable to the one observed in figure 2.29(d). A quasi-streamwise vortex ($Wv$) is formed in the wake of the starting vortex and is reoriented in the vertical direction as the starting hairpin vortex is convected, which is also consistent with the experimental observations.

As pointed out earlier, hairpin vortices are formed for a certain period of time after the high blowing ratio onset, although the value of $BR_h$ is higher than the transitional blowing ratio found in unforced conditions. The absence of transport of the inner vortex, leading to the formation of a starting hairpin vortex rather than a complete ring vortex, is explained in the 2D streamline visualizations from the simulations of Case 4 at $St_\infty = 0.016$ in figure 2.31. Upon increase in blowing ratio, the inner vortex is pushed outside and upstream from the jet pipe as seen in figure 2.31(b), repelling the horseshoe vortex further upstream from the jet exit compared to its previous location (also supported by experimental observations in figure 2.29(d)). In figure 2.31(c), as the initial overshoot decays, the inner vortex that had been pushed out, weakens due to
Figure 2.31: Instantaneous pressure difference contours $P - P_{\text{origin}}$ and 2D streamlines for Case 4 at $St_\infty = 0.016$ at time instants (a) $t^* = 0$; (b) $t^* = 0.017$; (c) $t^* = 0.065$; (d) $t^* = 0.265$; (e) $t^* = 0.33$; (f) $t^* = 0.370$ in the plane $Y_j = 0$.

its interaction with the horseshoe vortex of opposite spanwise vorticity and is pushed back inside the pipe by the cross-flow momentum and induction effects from both the horseshoe vortex and its image vortex. At the same time, a second horseshoe vortex is formed upstream of the first one, while it relocates downstream, yet further above the wall, which causes entrainment of jet fluid in the upstream region of the jet exit. In figure 2.31(d, e) as the inner vortex strengthens again, the induction between both structures increases and the horseshoe vortex is pulled closer to the wall, interrupting the flow of jet fluid to the upstream region of the jet exit. Finally, as quasi-unforced regime is reached in figure 2.31(f), the inner vortex is convected in agreement with corresponding unforced cases at $BR = BR_h$, entraining the horseshoe vortex in its wake, while the upstream part of the shear layer rolls up inside the pipe to form the next inner vortex.

**Starting Single Vortex Ring**

In the rest of this chapter, the tilting directions of vortex rings will be described with respect to rotation around the y-axis of the vertical unit normal vector to the vortex ring plane. Using this convention, negative rotation around the y-axis will be described as upstream rotation, and positive rotation described as downstream rotation. In figure 2.27(e-h) and figure 2.28(a), both corresponding to Case 3, a single starting vortex ring is formed. The starting vortex tilts towards the upstream direction as it is convected downstream and away from the wall. This is in good agreement with observations from both Chang & Vakili (1995) and Sau & Mahesh (2008) in which a tilting angle of $3^\circ$ to $4^\circ$ (compared to approximately $20^\circ$ here for Case 3) was observed and attributed to a Kutta-Joukowski lift effect exerted on the ring by the cross-flow. This explanation was however refuted by experimental and numerical studies from respectively Lim et al. (2008) and Cheng et al. (2009) showing that the initial tilt was more likely to be attributed to an interaction between the cross-flow boundary layer and the upstream part of the vortex ring, and that the extent of the tilt was to be related to the spanwise vorticity magnitude ratio between boundary layer and vortex ring. Sau & Mahesh (2008) also reported a size imbalance between the upstream and downstream rollups attributed to the shearing effect of the cross-flow on the structure. In the current study, size imbalance appears as well in the early stages of the vortex ring formation as observed in figure 2.27(f), while the vortex ring is still near the wall, but is not significantly amplified after the starting vortex departs from it.

Among the experiments monitored using the time-resolved flow rate measurement, only Case 3 at $St_\infty = 0.159$ exhibits a single vortex ring. This case is also simulated using the experimental flow meter records. The starting vortex formed in the simulations is shown in figure 2.30(c, d). Although in figure 2.30(c) a vortex ring is formed, the upstream rollup is not convected as fast as the downstream rollup thus
resulting in the elongated starting vortex in figure 2.30(d). This is attributed to the same causes responsible for the mismatch in the value of the transitional blowing ratio under unforced conditions. To observe the dynamics of single starting vortex rings, a forcing signal using an idealized square wave at $St_\infty = 0.159$ is used such that $BR_l = 0.200$, $BR_h = 2.20$, $DC = 8\%$ (identified as Case 10 in table 2.3). Iso-surfaces of the pressure Laplacian corresponding to Case 10 are presented in figure 2.32 at the moment of the transition from low to high blowing ratio. The observed dynamics agree fairly well with the one found in the experimental visualizations of figure 2.27 and provide additional information on the three-dimensional evolution of the structure. As the starting vortex departs from the bottom wall (figure 2.32b), the structure acquires an initial tilt angle toward the upstream direction which increases progressively as shown in the sequence of frames in figure 2.32. Under the action of the cross-flow, the vortex ring is deformed and progressively folded, the upstream and downstream parts approaching each other, while the side arms relatively appears to drop, in a manner similar to the one of the shear layer ring vortices in the unforced detached jet shown in figure 2.18. According to the previously mentioned studies, the rollup size imbalance and the upstream tilting of the starting vortex are to be attributed to an initial weakening of the leading edge part of the structure because of interaction with the boundary layer flow with opposite vorticity. However, the velocity profiles in figure 2.32(f) show a continuous momentum and vorticity deficit at the leading edge, compared to the trailing edge, throughout the starting vortex formation process. Figure 2.32(f) suggests that the circulation and initial convection velocity imbalances also originate from the characteristic skewness of the jet in cross-flow velocity profiles (maximum velocity shifted downstream). Both these effects combined cause the unevenness in circulation and speed between upstream and downstream rollups, and therefore influence the dynamics of the resulting asymmetric vortex ring.
Figure 2.34: Interactions between leading starting vortex and first trailing vortex based on inviscid considerations. (b) corresponding to figure 2.33(c); (c) corresponding to figure 2.33(e). Mutual induction effect are represented (matching gray arrows), as well as overall resulting vortex velocity (black arrows).

Figure 2.35: Reactive Mie scattering visualizations in the plane $Y_j = 0$ for Case 7 $St_\infty = 0.008$. Images enhanced using CLAHE algorithm show the passage of the jet from transient structures (trailing column) to transverse jet structures (upper and lower shear layer rollups and CRVP roots).

Starting Vortex Ring and Trailing Column

For longer injection times, a significantly different behavior can be observed. The leading ring vortex is followed by a series of smaller vortex rings constituting a trailing column, a typical formation sequence of which is shown in figure 2.33 from the experiments of Case 8. In figure 2.33(a), the leading vortex ring initially tilts in the upstream direction due to interaction with the cross-flow boundary layer and velocity distribution skewness. As it departs from the wall and the first trailing vortex ring is formed in figure 2.33(b), the leading vortex ring tilting is reversed towards the downstream direction, “swallowing” the downstream part of the first trailing vortex as shown in figure 2.33(c). This “partial leapfrogging” of the first trailing vortex as well as the downstream tilting of the leading vortex ring can easily be explained by 2D inviscid considerations. Figure 2.34 presents qualitative inviscid vortex interactions based on the positions of the vortices in figure 2.33 albeit with arbitrary vortex strengths.

Indeed, the convective effect of the cross-flow on the leading starting vortex shifts its axis downstream such that the downstream rollup of the first trailing vortex is placed in its central velocity field, as shown in figure 2.34(a). Consequently, the vertical velocity component of the starting vortex ($S_v$) downstream rollup is decreased and its streamwise velocity increased. Simultaneously, the downstream part of the trailing vortex ($T_v1$) gains in vertical and streamwise velocity, resulting in a the velocity vector pointing toward the upstream rollup of $S_v$, in a manner comparable to leapfrogging of concentric vortex rings (see Green 1995). The interaction between the upstream part of $S_v$ and downstream part of $T_v1$ results in higher vertical velocities for both of them. Conversely, the upstream part of $T_v1$ interacts with the upstream part of $S_v$ providing it with additional upward velocity while convected downstream. The relative difference in velocities experienced by the upstream and downstream rollups of $S_v$ results in an overall rotation of the
structure towards the downstream direction, despite the size imbalance of the rollups, while \( Tv1 \) tilts towards the upstream direction. In figure 2.34(b, c) the downstream part of \( Tv1 \) approaches the upstream part of the starting vortex such that the vertical velocity component of these vortices increases while the upstream rollups of \( Tv1 \) and downstream rollups of \( Sv \) have relatively lower velocities, thus accentuating the tilting mechanisms. The following trailing vortices \( (Tv2, Tv3 \text{ and } Tv4) \) experience a velocity field comparable to the one encountered by \( Tv1 \) in figure 2.34(a), such that they successively tilt in the upstream direction. In addition to the induction mechanism, the upstream parts of the vortex rings are exposed to the high pressure region created by the cross-flow deflection, while their downstream counterparts are facing a lower pressure in the jet wake, thus promoting tilting in the upstream direction as they exit from the jet tube.

The reorientation of the trailing vortices is an important feature since it generates vertical vorticity from the side-arms of the trailing column ring vortices and therefore provides a source for the initialization of the CRVP. The reactive Mie scattering visualizations in figure 2.35(a-d) show the passage from transient to quasi-unforced behavior of the jet in \textit{Case 7}. Overall the starting vertical structure assembly is very much the same as the one described previously. However, the transparency offered by this type of visualization technique, where only mixing regions are visualized, permits the identification in the background of a quasi-vertical structure joining the leading vortex ring to the base of the jet. These hanging vortices, as described in the LES results from Yuan \textit{et al.} (1999), have been observed in many jets in cross-flow experiments such as Kelso \textit{et al.} (1996) and have been associated with the initialization of the CRVP. In figure 2.35(b, c) quasi vertical vortices are visible in the jet wake and appear connected to the downstream part of the trailing ring vortices. Structures consistent with the well-known wake vortices formed in the jet wake are shown figure 2.35(d). These visualizations suggest that the wake vortices originate from an interaction between the downstream shear layer vortices and the boundary layer as mentioned by Fric & Roshko (1994). A similar interaction was observed previously in the dynamics of the hairpin starting vortices of figure 2.30(a).

The simulated dynamics of the leading vortex ring and the trailing vortices for \textit{Case 9} are presented in figure 2.36 through the rendering of iso-surfaces of the pressure Laplacian. These are in good qualitative agreement with the experimental observations, in spite of the fact that idealized square waves are used for the excitation signal in the numerical simulations because of the lack of time-resolved flow rate records from the corresponding experiment. The sequence of snapshots in figure 2.36 confirms the partial leapfrogging of the trailing vortices as well as the tilting dynamics, in this higher blowing ratio case as well. Additional insight is gained on the behavior of the side arms of the trailing vortices as they are convected in the free stream and away from the wall. The first trailing vortex is distorted due to the rapid ingestion of its downstream
with $x=pp$ or $x=h$. 

Figure 2.37: Classification of the starting vortices observed in forced experiments with respect to characteristic stroke ratios and blowing ratios (a) $SR_h$, $BR_h$; (b) $SR_{pp}$, $BR_{pp}$ ○ Leading vortex ring and trailing column; ● Single vortex ring; ▲ Limiting case vortex ring with/without trailing column; □ Leading hairpin vortex and trailing hairpins; ■ Single hairpin vortex; ◊ Limiting case hairpin/ring vortex; ▽ Limiting case hairpin vortex with/without trailing vortices. $F_0 = 3.6$ - Asymptotic formation number from Sau & Mahesh (2008). Transition stroke number fit $(SR_x)_{tr} = F_0 - A_1 \exp(-A_2 BR_x)$ (a) $(SR_h)_{tr}$: $A_1 = 3.6$, $A_2 = 1.51$; (b) $(SR_{pp})_{tr}$: $A_1 = 3.6$, $A_2 = 1.26$. 

Starting Vortices Classification

In view of the good qualitative agreement with the numerical results from Sau & Mahesh (2008), a similar classification of the starting vortices observed in the current experiments is conducted. At this point, a set of characteristic forcing parameters must be identified in order to provide a proper classification. In the results from Gharib et al. (1998), the piston stroke ratio $SR = L/D$ was equated to the ratio of length to diameter of the fluid column ejected such that $L/D = <U_p> \tau/D_j$, where $<U_p>$ was the average piston velocity and $\tau$ the duration of the velocity program (using Gharib's terminology). Similarly, in the fully modulated jet numerical simulations of Sau & Mahesh (2008), the characteristic stroke ratio was extrapolated to $L/D = <U_{exit}^j> \tau/D_j$, where $<U_{exit}^j>$ was the average jet nozzle exit velocity, consistent with the velocity $U_p$ in incompressible flow approximation in a cylinder of constant diameter, $\tau$ being the pulse width. While both of these studies have been dealing with flow/no flow type of jet forcing, the current study involving non-zero values of $BR_j$ pushes us to consider at least three characteristic jet velocities $U_h$, $U_{pp}$ and $U_m$ corresponding to $BR_h$, $BR_{pp}$, and $BR_m$ respectively to define the stroke ratio. While the values of $U_h$, and potentially $U_l$ (thus $U_{pp}$), are expected to influence the starting vortex formation mechanism, the duration of the low part of the cycle ($\tau(1-DC)/DC$) is unlikely to affect it. Consequently, the average value $U_m$, which changes as the low part of the cycle duration is extended when all other parameters are held constant, is discarded. The pulse width $\tau$ is logically taken as the duration of the high blowing ratio regime so that $\tau = DC/f_j$. Finally, two sets of parameters are used to classify the vortical structures: $(SR_h, BR_h)$ and $(SR_{pp}, BR_{pp})$ such as:

$$SR_x = U_x \tau/D_j = BR_x U_\infty DC/(f_j D_j) = BR_x DC/St_\infty$$  \hspace{1cm} (2.2) 

with $x=pp$ or $x=h$. 

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The parameters used in the classification of the starting vortices are directly computed from the values measured by the flow-meter and are presented in figure 2.37(a, b). Limiting cases are identified when multiple types of starting structures can be observed in consecutive cycles. Overall, both sets of parameters exhibit a similar distribution to the one provided by Sau & Mahesh (2008), with a differentiation of the four different starting vortices. A better separation between the regimes is achieved when using the set \((SR_h, BR_h)\) with less overlapping of the zones compared to the one obtained with \((SR_{pp}, BR_{pp})\). The distinction between the two principal starting structures regimes observed in figure 2.37(a) shows that hairpin vortices are formed in cases with \(BR_h < 0.584 \pm 3\%\), while this separation occurs for cases with \(BR_{pp} < 0.294 \pm 12\%\) in figure 2.37(b). Based on these observations, the set \((SR_h, BR_h)\) appears to provide a more precise classification map of the starting vortices.

To confirm the better suitability of the scaling using \(U_h\) over \(U_{pp}\), a set of three distinct cases, with idealized square wave models of different amplitudes and pulse widths, are simulated using LES. Based on a two dimensional assumption, the total circulation is computed at the jet exit by integrating the spanwise vorticity on the domains \(\{\Omega_1: -2 \leq X_j < 0, Y_j = 0, 0 < Z_j \leq 2.5\}\) and \(\{\Omega_2: 0 < X_j \leq 2, Y_j = 0, 0 < Z_j \leq 2.5\}\), respectively at the leading and trailing edges of the jet exit. Figure 2.38 presents the scaled results using the two scaling parameters used in figure 2.37: \(U_c = U_h\) and \(U_c = U_{pp}\). The initial time chosen in figure 2.38 corresponds to the transition from low to high blowing ratio, while the final time corresponds to the moment when either one of the downstream or upstream parts of the vortex ring leaves \(\Omega_1\) or \(\Omega_2\). According to the results from Rosenfeld et al. (1998) and Krueger et al. (2006), using the proper velocity scale \(U_c\) results in a fixed proportional relationship between the scaled total circulation at the jet exit \((\Gamma^* = \Gamma/(U_cD_j))\) and the formation time \((t^* = tU_c/D_j)\) for all forcing conditions. Based on this criterion, the higher blowing ratio velocity \(U_h\) clearly provides an overall better scaling of the rate of creation of circulation during the formation of the starting vortex (figure 2.38a), compared to the peak to peak velocity \(U_{pp}\) (figure 2.38b), since the total circulation trends collapse better using \(U_h\) as the velocity scale. This result confirms the experimental observations of figure 2.37(a, b) and suggests that the jet average high velocity \(U_h\) should be used as the scaling parameter instead of the peak to peak velocity.

The limiting stroke number at fixed high blowing ratio value, between single starting vortex and starting vortex with trailing vortices, is presented in figure 2.37 as the transition stroke number \((SR_x)_{tr}\), using to the nomenclature introduced by Sau & Mahesh (2008). An exponential fit is used to obtain a continuous limiting value, again according to the previously mentioned study. The decrease in the value of the transition stroke number \((SR_x)_{tr}\) for decreasing values of \(BR_h\) (or \(BR_{pp}\)) is consistent with the numerical results from Sau & Mahesh (2008) and can be explained by previous research from Rosenfeld et al. (1998) and Krueger et al. (2006). Indeed, in the first study the value of the formation number was directly affected by the jet velocity profile and decreased formation numbers were observed for parabolic profiles (weak jet) compared to more uniform profiles (strong jet). Such variation in the jet velocity profile is expected in the current experiments, as the jet velocity profile becomes weaker when the blowing ratio decreases. In the study by
Figure 2.39: Mie scattering visualizations in the plane $Y_j = 0$ evidencing (a) leading edge ingestion and horseshoe vortex transport in Case 1 at $St_\infty = 0.079$ and (b) peripheral ingestion in Case 9 at $St_\infty = 0.016$.

Figure 2.40: Instantaneous temperature contours and 2D streamlines for Case 3 at $St_\infty = 0.016$ at time instants (a) $t^* = DC+0.018$; (b) $t^* = DC+0.022$; (c) $t^* = DC+0.13$; (d) $t^* = DC+0.19$; (e) $t^* = DC+0.37$ in the plane $Y_j = 0$.

Krueger et al. (2006) on vortex ring formation in co-owing flows, a decrease of the formation number was observed as the co-owing velocity ratio (equivalent to $1/BR$ here) was increased. Similarly, in the current work, the co-owing component of the free stream with the jet ow increases as the blowing ratio decreases and the jet deflection in the streamwise direction is more important.

Although the current results agree qualitatively well with the observations from Sau & Mahesh (2008) in terms of the type of starting vortices generated and their dynamics, a discrepancy is found in the threshold value of $BR_h$ between the formation of starting hairpin vortices and starting ring vortices. Indeed in Sau & Mahesh (2008) a threshold value of $BR_h = 2.0$ was established, whereas a value of $BR_h = 0.584$ is found here in the current experimental results. However, it should be noted that the current numerical simulations also evidence a threshold significantly higher (around $BR_h = 2.20$) than the experimental threshold based on the available observations. As for the discrepancies in transitional thresholds in unforced conditions, this disparity is attributed to mismatching perturbation levels and initial length scales between simulations and experiments as well as potential errors introduced by numerical modeling.

### 2.3.3 High-to-low Blowing Ratio Transition: Cross-flow Ingestion

At the jet shutdown, most of the cases with $BR_{pp} > 0.150$ exhibit clear ingestion of cross-flow uid inside the jet feeding tube. As shown in figure 2.39, two types of ingestion can be observed. In figure 2.39(a), at low $BR_h$ values, the ingestion principally occurs at the leading edge of the jet exit and triggers the convection of the horseshoe vortex in the downstream direction. However, in figure 2.39(b), for higher values of $BR_h$, the cross-flow ingestion occurs on the whole circumference of the jet exit. The lack of seed in the cross-flow and the absence of visibility in the jet pipe prevents a more detailed characterization of this transition using Mie scattering visualizations. Numerical simulations are then used to gain additional physical insight on these transient regimes.
Leading Edge Ingestion

Figure 2.40, corresponding to Case 3 at $St_{\infty} = 0.016$, shows that at the jet shutdown, the horseshoe vortex is partially ingested inside the jet pipe (figure 2.40(b)), while the inner vortex is pulled further inside. The stream traces show flow separation at the jet leading and trailing edges around $Z_j = -1.5$. The higher shear introduced by the backward flow entrains the rollup of the jet shear layer which forms a complex vortical system, evidenced in figure 2.40(c) and 2.41(a, b), composed of three vortices of negative spanwise vorticity and two of opposite vorticity near the wall. In the three-dimensional representation of figure 2.41(b) corresponding to figure 2.40(c), the streamlines exit on the side of the jet and merge to form the legs of the hairpin vortex seen in figure 2.41(b). Over time, the multiple inner vortices eventually pair in figure 2.40(d, e), so that only one stable rollup persists in the jet pipe, in the same fashion as the corresponding unforced regime. The time stamps of figure 2.40 show that the formation time scale of the inner vortex system inside the jet pipe at the conditions of Case 3 are of the order of 100ms ($t^* = 0.1$), whereas the jet recovers from the transient regime into a quasi-unforced regime after approximately 400ms ($t^* = 0.4$). These observations explain the lesser extent of ingestion observed in simulations at higher forcing frequencies of $St_{\infty} = 0.159$, since the vortical system does not have enough time to fully develop until the next pulse. Ingestion of cross-flow fluid can also be observed in the snapshot of figure 2.30(b), corresponding to Case 4, with a $BR_{pp}$ almost half of the of the one in Case 3. Although significant ingestion is also occurring, the extent of it is found to be somewhat lessened, with no horseshoe vortex ingestion. These observations suggest that the $BR_{pp}$ parameter, directly related to the mass deficit between high and low parts of the cycle, plays a significant role in the scaling of the ingestion extent. This is also supported by the experimental observations that no ingestion was observed for $BR_{pp} < 0.15$.

Peripheral Ingestion

Peripheral ingestion, as the one evidenced in figure 2.39(b), is also witnessed in the numerical simulations of Case 10 for instance, as evidenced by the jet exit vertical velocity profile of figure 2.32(f), showing negative vertical velocity values at both the leading and trailing edges at $t^* = DC + 0.005$. In figure 2.41(c), corresponding to Case 9 at $t^* = DC + 0.02$ after the transition from high to low flow rate, streamlines from both the upstream and downstream regions, converge toward the jet exit and enter the jet pipe. The flow inside the jet pipe separates over the total circumference of the jet, therefore reducing the effective flow area and increasing the effective blowing ratio. The jet shear layer is clearly disrupted with the formation of regions of vorticity opposite to the natural shear layer vorticity at the jet exit. The negative values of the vertical velocity all around the jet exit and the streamline trajectories in figure 2.41(d) show the peripheral character of the ingestion. Only two regions at the jet exit, evidenced in figure 2.41(d) by two tailless hollow arrows, maintain positive velocity values and correspond to the location of the roots of the CRVP.

The extent of the separation regions in both cases and the evacuation of transient structures inside the feeding pipe explain the rather “chaotic” character of the flow during the early moments of the low part of the cycle.

2.3.4 Characteristic Frequencies

The time-dependent capabilities of the wavelet analysis are particularly useful for forced experiment records where several regimes and characteristic frequencies may be encountered throughout a single cycle. Figure 2.42 shows hot wire records taken at several locations under different forced conditions and analyzed using wavelet decomposition. As for the unforced records, the hot wire measurements are obtained at the center of the jet exit ($X_j = 0, Y_j = 0, Z_j = 0$), in the upper shear layer directly above the jet exit ($X_j = 0, Y_j = 0, Z_j = 0.5$) and at a downstream location ($X_j = 3.5, Y_j = 0, Z_j = 1.25$). Records taken at the jet exit (figure 2.42a, d), show that the shape of the excitation signal is fairly well preserved throughout the jet tube in most of the cases. The characteristic acoustic frequency of the system is clearly identified at the jet exit ($St_{\infty} = 0.75$), but does not appear in any other records at the other locations, thus supporting the fact that this volumetric resonance has little effect on the global jet behavior. In agreement with previous observations, low forcing frequency cases ($St_{\infty} = 0.008, St_{\infty} = 0.016$) exhibit distinct characteristic signatures associated with quasi-unforced regimes, which are absent from higher forcing frequency records ($St_{\infty} = 0.079, St_{\infty} = 0.159$). In figure 2.42(a-c), corresponding to Cases 1 and 2, signatures at respectively...
Figure 2.41: Ingestion: Instantaneous 3D streamlines for Case 3 at $St_{\infty} = 0.016$ at $t^* = DC + 0.13$ (a) X-Z side view and spanwise vorticity contours; (b) Y-Z front view, streamlines colored by spanwise vorticity from negative (black) to positive (gray). Case 9 at $St_{\infty} = 0.016$ at $t^* = DC + 0.13$ (c) Instantaneous 2D streamlines and velocity profiles at $Z_j = 0$ and $Z_j = -1$ from negative (black) to positive (white); (d) 3D streamlines and velocity profiles at $Z_j = 0$ and $Z_j = -1$ from negative (black) to positive (white) values.

$St_{\infty} = 0.155$, $St_{\infty} = 0.28$ and $St_{\infty} = 0.23$ are found during either the high or low quasi-unforced regimes of the cycle, which are comparable to the ones found in unforced cases at corresponding blowing ratios; for example $St_{\infty} = 0.145$ for $BR = 0.438$, $St_{\infty} = 0.302$ for $BR = 0.188$ and $St_{\infty} = 0.216$ for $BR = 0.325$. The first harmonic and sub-harmonic are also present in figure 2.42(a). At higher forcing frequencies of $St_{\infty} = 0.079$ and $St_{\infty} = 0.159$, only the structures associated with the transient regimes are observed in the visualizations, which is consistent with the wavelet decomposition. Figure 2.42(d-f) are overwhelmed by their respective forcing frequency signatures at $St_{\infty} = 0.079$ and $St_{\infty} = 0.159$, as well as their first harmonics corresponding to the passage of the starting vortices at the probe location and generated only once per cycle.

### 2.3.5 Film Cooling Performance

As observed previously in 2.3.1, during the early stages of the $BR_t$ to $BR_h$ transient regime, jet fluid continuously flows from the upstream edge of the jet, weakening the inner vortex while slightly elevating and enhancing the principal horseshoe vortex, therefore temporarily preventing shedding of the inner vortex. Figures 2.45(a-c) show an increase of the cooled area around the jet during the low-to-high transient part of the cycle, relative to the previous low blowing ratio part. This is confirmed in figure 2.44(a) by the trends of the time dependent $\eta_{area}$ and $C_c$ values over a cycle. An increase in $\eta_{area}$ and $C_c(\eta < 0.5)$ is observed immediately after the transition from low to high blowing ratio over one quarter of the cycle period ($t^* < 0.25$), compared to the values during the low part of the cycle. The supply of jet fluid to the upstream region is slowly disrupted as the pressure overshoot from the low-to-high blowing ratio transition settles and the difference in pressure increases across the jet/cross-flow interface. When the inner vortex is free to be transported, the jet behaves in a quasi-unforced manner at $BR = 0.512$ and the coverage is dramatically decreased in figure 2.45(d) as the jet becomes transitional. This is also evidenced in figure 2.44(a) as the values of $\eta_{area}$ and $C_c$ decrease continuously during the remaining high part of the cycle, therefore partly counter-balancing the positive effects of the first part of the cycle. These trends could provide some guidance towards tailoring the forcing signal in order to improve film-cooling metrics. Truncating the high part of
Figure 2.42: Top: Wavelet analysis mapping of hot wire time records in the plane $Y_j = 0$ for (a) Case 1 at $X_j = 0, Z_j = 0 - St_\infty = 0.016$; (b) Case 1 at $X_j = 3.5, Z_j = 1.25 - St_\infty = 0.008$; (c) Case 2 $X_j = 0, Z_j = 0.5 - St_\infty = 0.016$; (d) Case 4 at $X_j = 0, Z_j = 0 - St_\infty = 0.079$; (e) Case 6 at $X_j = 0, Z_j = 0.5 - St_\infty = 0.159$; (f) Case 5 at $X_j = 3.5, Z_j = 1.25 - St_\infty = 0.159$. Bottom: Corresponding normalized velocity record and instantaneous flow rate time-record.
Figure 2.43: Instantaneous wall adiabatic effectiveness contours from LES. Top $BR_m = 0.35$, $DC = 50\%$ (Case 4) at $St_\infty = 0.016$, time stamps: (a) $t^* = -0.03$; (b) $t^* = 0.08$; (c) $t^* = 0.20$; (d) $t^* = 0.43$; (e) $t^* = DC + 0.10$.

Figure 2.44: Top: Instantaneous coverage coefficient (solid) and area averaged adiabatic effectiveness (dashed); Bottom: Typical blowing ratio profile over a cycle for: (a) $BR_m = 0.35$, $DC = 50\%$ (Case 4) at $St_\infty = 0.016$ vertical bars represent snapshot instants for figure 2.43; (b) $BR_m = 0.35$, $DC = 50\%$ (Case 4) at $St_\infty = 0.0159$ vertical bars represent snapshot instants for figure 2.45.

the cycle, before the quasi-unforced high blowing ratio state settles (e.g. for Case 4 before $t^* = 0.25$), could prevent the performance degradation after the transient associated with the arrival of the pulse has washed out, while retaining the initial improvement associated with the arrival of the pulse.

At the high-to-low blowing ratio transition, ingestion of cross-flow fluid at the jet upstream lip occurs, and hot cross-flow fluid is entrained inside the jet pipe by the inner vortex as seen in figure 2.40 and an instantaneous decrease in film coverage around the jet exit can be observed in figure 2.45(e). As the jet settles in the low blowing ratio quasi-unforced regime, this pocket of hot fluid is slowly evacuated. This explains in part the lowest performance of the pulsed jet during the $BR_h$ to $BR_l$ transient part of the cycle relative to the unforced jet observed in 2.44(a). Eventually the jets slowly settles in quasi-unforced regime, although the cooling performance is not stabilized until the next pulse as all three coverage coefficients keep changing into the next cycle. Given the time stamps of figure 2.44(a), it appears that under the forcing conditions corresponding to Case 4 at $St_\infty = 0.016$, the high-to-low transient regime time scale is only slightly less than $500\,ms$ ($t^* = 0.50$), therefore explaining the strong negative impact on the overall cooling.
performance. This indicates that a smoother transition from $BR_0$ to $BR_l$, for instance via a ramp signal, could suppress part of the ingestion by decreasing the mass deficit induced by the rapid jet drawdown.

Simulations corresponding to the same flow conditions (Case 4), at higher forcing frequency of $St_\infty = 0.159$, show in 2.45 that injection of jet fluid in the upstream region occurs briefly at the jet onset but is not as prominent as the one observed at lower forcing frequency and is rapidly stopped as the jet draws down. The instantaneous values of $\eta_{span}$ and $C_c$ presented in 2.44(b) show almost no fluctuations throughout the cycle. Time averaged values for $\eta_{span}$, $\eta_{area}$ and $C_c$ are extracted from simulated adiabatic effectiveness and presented in figure 2.46 and table 2.4. Results corresponding to Case 4 show that forced jets at $St_\infty = 0.016$ and $St_\infty = 0.159$ have very comparable performances in terms of spanwise averaged effectiveness and coverage, despite the significant fluctuations of instantaneous values of $\eta_{span}$ and $C_c$ observed in figure 2.44 at $St_\infty = 0.016$, compared to quasi-unforced values at $St_\infty = 0.159$. Overall, these two cases have spanwise averaged adiabatic effectiveness values approximately 30% lower than the corresponding unforced jet at equivalent mean blowing ratio ($BR = BR_m$). Accordingly, the coverage coefficients in table 2.4 are on average 50% to 70% lower than corresponding unforced jet values at constant mass flow. The area averaged adiabatic effectiveness paints a similar picture. Simulations for Case 3 showed significantly deteriorated performance compared to equivalent unforced results with 90% and 65% decrease in $C_c$ and $\eta_{area}$ respectively, attributed to the high value of $BR_{pp}$, yielding strong starting vortices carrying coolant away from the wall, while generating large mass deficit at the jet drawdown and considerable cross-flow ingestion. The last two simulations at $BR_m = 0.45$, corresponding to Case 12, show some improvement over the corresponding unforced jet results directly downstream of the jet exit for $X_j < 4$. The coverage coefficients of the forced cases in table 2.4 match or exceed the values of the unforced jet when $St_\infty = 0.159$, while the values of $\eta_{area}$ are comparable to the unforced one. Overall, the forcing frequency appears to have only a limited influence on film cooling performance of the jet for Case 4 and Case 12. However, flow visualizations clearly show that the degradation of the film cooling performance at $St_\infty = 0.016$ can be attributed to the settling of the jet in the high part of the cycle with transitional features such as the transport of the inner vortex, while at $St_\infty = 0.159$ the high entrainment of cross-flow fluid generated by the starting vortex is responsible for such performance decrease. Effectively the same outcome in terms of thermal performance is perpetrated by different flow dynamic mechanisms. Overall, jet forcing around an average value below the unforced transitional threshold has a greater influence (yet detrimental) on the film cooling metrics, than forcing around an average value beyond the transitional threshold. This shows that the beneficial effect of decreasing the blowing ratio below the transitional value during the low part of the cycle is less significant than the detrimental action of increasing it above this threshold.

### 2.4 Conclusion

This chapter has been dealing with the description and understanding of vortical structure dynamics involved in low blowing ratio unforced and forced vertical jets. The investigations were carried out using experimental visualizations and hot wire anemometry along with numerical LES methods.
Table 2.4: Coverage coefficient and area averaged adiabatic effectiveness from LES under forced conditions compared to equivalent mass flow rate unforced cases.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>( C_c )</th>
<th>( \eta = 0.2 )</th>
<th>( \eta = 0.3 )</th>
<th>( \eta = 0.5 )</th>
<th>( \eta_{area} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( BR = BR_m = 0.350 - St_{\infty} = 0 )</td>
<td>12.82</td>
<td>7.02</td>
<td>2.15</td>
<td>0.226</td>
<td></td>
</tr>
<tr>
<td>( BR = BR_h = 0.513 - St_{\infty} = 0 )</td>
<td>7.05</td>
<td>1.47</td>
<td>0.21</td>
<td>0.123</td>
<td></td>
</tr>
<tr>
<td>Case 4 - ( St_{\infty} = 0.016 )</td>
<td>( 7.22 )</td>
<td>( 3.23 )</td>
<td>0.81</td>
<td>0.162</td>
<td></td>
</tr>
<tr>
<td>Case 4 - ( St_{\infty} = 0.159 )</td>
<td>( 7.26 )</td>
<td>( 2.63 )</td>
<td>0.66</td>
<td>0.145</td>
<td></td>
</tr>
<tr>
<td>( BR = BR_h = 0.836 - St_{\infty} = 0 )</td>
<td>1.91</td>
<td>0.21</td>
<td>0.05</td>
<td>0.071</td>
<td></td>
</tr>
<tr>
<td>Case 3 - ( St_{\infty} = 0.159 )</td>
<td>( 1.99 )</td>
<td>( 0.75 )</td>
<td>0.30</td>
<td>0.079</td>
<td></td>
</tr>
<tr>
<td>( BR = BR_m = 0.450 - St_{\infty} = 0 )</td>
<td>7.25</td>
<td>1.68</td>
<td>0.30</td>
<td>0.122</td>
<td></td>
</tr>
<tr>
<td>( BR = BR_h = 0.720 - St_{\infty} = 0 )</td>
<td>3.96</td>
<td>0.47</td>
<td>0.08</td>
<td>0.093</td>
<td></td>
</tr>
<tr>
<td>Case 12 - ( St_{\infty} = 0.016 )</td>
<td>( 5.62 )</td>
<td>( 1.91 )</td>
<td>0.20</td>
<td>0.121</td>
<td></td>
</tr>
<tr>
<td>Case 12 - ( St_{\infty} = 0.159 )</td>
<td>( 4.29 )</td>
<td>( 1.39 )</td>
<td>0.34</td>
<td>0.113</td>
<td></td>
</tr>
<tr>
<td>( BR = 0.415 - St_{\infty} = 0 )</td>
<td>14.42</td>
<td>7.87</td>
<td>2.17</td>
<td>0.247</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.46: Spanwise averaged adiabatic effectiveness for forced jet in cross-flow and associated constant mass flow rate unforced jet results after LES.

The unforced transverse jet study provided identification and a detailed description of the evolution of the vortical structures dynamics when jet blowing is increased. Although numerous studies have extensively described the vortical structures in high blowing ratio detached configuration, only few had been dealing with the attached one, let alone treated the transition from one regime to the other in details. In attached jets with \( BR \leq 0.275 \), inverted hairpin vortices were formed in the lower jet shear layer, along with the previously documented shear layer hairpin structures. The horseshoe vortex system was found to entrain jet fluid through an interaction with a stable inner vortex, resulting from upper shear layer rollup inside the jet pipe at the leading edge. Side vortices were born from shear caused by the injection of jet fluid along the jet exit perimeter, itself created by pressure buildup at the core of the jet exit from the jet interaction with the cross-flow. Induction mechanisms were responsible for the formation of X-patterned structures located on each side of the jet core, which carried significant strength in the far field. Although a counter-rotating vortex pair was identifiable in the mean velocity field, the relative strength and the spatial discontinuity of the associated instantaneous hairpin vortex structures made the resemblance with the well known CRVP of the detached jet only tenuous. A clear interaction between the horseshoe vortex and the inner vortex led to the convection of the former for \( 0.225 \leq BR < 0.275 \) and of both for \( 0.275 \leq BR < 0.6 \). The convection of the inner vortex marked the transition from the attached to the detached jet regime in which it evolved into the well-known upper shear layer rollups observed in many detached jet studies. The side vortices eventually evolved into wake vortices at the higher blowing ratios. Beyond \( BR = 0.6 \), the jet column and its associated pressure gradient became strong enough to prevent the convection of the horseshoe vortex which remained near the jet exit leading edge as the the detached jet regime was established.

Although the origins of the detached jet vortical structures could be traced back to ones observed in the attached jet, the nature of their interactions (i.e. horseshoe vortex/inner vortex and side vortices/shear layer...
vortices) and their relative strength changed significantly as the jet lifted off, leading to considerably modified dynamics. As such, the evolution of the jet from one regime to the other can be seen as a "competition" between different vortical structures. On one hand, the horseshoe vortex / inner vortex interaction is such that, at low $BR$ values, the inner vortex is rather weak and maintained inside the jet pipe by a relatively stronger horseshoe vortex. But, as the blowing ratio is increased, the inner vortex gains in strength and pushes the horseshoe vortex further away from the jet exit leading edge and is free to be convected and to form the well-known upper shear layer rollups of the detached jet configuration. On the other hand, the shear layer / side vortices interaction is such that, at low $BR$ values, the side vortices are relatively stronger than the hairpin vortices and play a significant role in the far-field, therefore challenging the notion of CRVP due to their physical proximity. However, at higher blowing ratios, the shear layer structures gain in strength as well as cross-flow penetration, thus changing the nature of their interaction with the side vortices which evolve into the well-known wake vortices and let the CRVP assume its expected detached configuration shape. This transition was also manifest in the characteristic formation frequency signatures, isolated using wavelet analysis, with a tipping point around $BR = 0.425$.

The forced jet exhibited two distinct transient phases during a forcing cycle, corresponding to the transition from low to high blowing ratio and the reverse. Quasi-unforced phases were also present between both transients for cases with lower forcing frequencies. At the transition from low to high blowing ratio, starting hairpin or ring vortices were generated, on the low and high ends of the $BR_h$ range respectively, due to sudden increase in shear at the jet exit. Trailing structures were formed in the wake of the leading starting vortices, be they hairpin or ring, when the stroke ratio exceeded the transition stroke number value at the corresponding $BR_h$. Starting hairpin vortices were formed at low $BR_h$ values and their dynamics were found to be conform to those observed in the smaller regular hairpin vortices of the unforced jet. A strong interaction between starting vortex and horseshoe vortex led to the reorientation of the latter in a similar way as wake vortices were reoriented in the unforced detached jet. It was observed that starting hairpin vortices could be generated at $BR_h$ values well above the unforced transitional blowing ratio due to a modified interaction between horseshoe and inner vortices. Single starting vortex ring dynamics were found to be in agreement with previous studies indicating upstream tilting of the structure due to interaction of its upstream part with the cross-flow boundary layer. The inherent skewness of the jet exit velocity profile was also identified as a source of the strength imbalance between upstream and downstream rollups responsible for the tilt. When a trailing column was formed, the interaction and dynamics of the starting structures could be explained by inviscid induction mechanisms including the partial leapfrogging of the first trailing ring vortices. The reorientation of the structures trailing the starting vortex led to the formation of a coherent counter-rotating vortex pair on the lee-side of the jet responsible for the initialization of the CRVP.

Estimations of the rate of generation of circulation under multiple forcing conditions led to the identification of $SR_h$ and $BR_h$ as a suitable pair of scaling parameters for the classification of the starting vortices in partly modulated forced jets. Using those parameters a limiting value of $BR_h = 0.584$ was found below which hairpin starting vortices were generated and above which starting ring vortices were formed. This study constitutes an experimental verification of past numerical studies both in terms of the dynamics of the starting vortical structures as well as the classification proposed by Sau & Mahesh (2008). It also further extends those results to partly modulated jets with non-zero low blowing ratios.

At the transition from high to low blowing ratio, cross-flow ingestion inside the jet pipe occurred for $BR_{pp}$ values above 0.15 from the mass deficit created as the blowing ratio was suddenly dropped. Two ingestion mechanisms were observed, both during experiments and in numerical simulations results. A leading edge ingestion occurred at rather low values of $BR_h$, when the jet was attached during the high part of the cycle. Conversely, a peripheral ingestion occurred at rather high values of $BR_h$, when the jet was detached during the high part of the cycle. In both cases, the mass deficit led to a jet flow separation and backflow inside the feeding pipe. In the case of leading edge ingestion, the separation triggered the formation of a complex vortical system which was slowly dissipated as the jet settled in the low part of the cycle. The parameter $BR_{pp}$ was observed to have direct impact on the extent of the cross-flow ingestion.

Evaluation of the film cooling metrics showed that in the attached regime, the horseshoe vortex had an overall beneficial impact on film spread at the wall, by entraining jet coolant upstream of the jet exit and away from the symmetry plane. Above the transitional blowing ratio threshold, the transport of the inner vortex triggered intermittent disruptions of the horseshoe vortex system, allowing early cross-flow penetration directly downstream of the jet exit and interrupting coolant flow to the upstream regions which significantly
impacted film cooling performance. Under certain forcing conditions, the transient dynamics introduced at
the pulse onset caused a significant increase in coverage by temporarily suppressing the convection of the
inner vortex, while operating the jet at blowing ratios above the unforced transitional regime threshold.
However this improvement was counterbalanced by subsequent degradation of the performance associated
with the quasi-unforced regime in the second half of the high part of the cycle. The abrupt decrease in
jet flow rate at the transition from high to low blowing ratio, created a significant mass deficit resulting
in internal jet flow separation and ingestion of cross-flow fluid inside the jet pipe, strongly affecting film
cooling performance throughout most of the low part of the cycle. Film cooling performance of forced
jets was compared to unforced cases at fixed mass flow rate and found to be consistently lower in terms
of spanwise average adiabatic effectiveness, coverage coefficients and area averaged adiabatic effectiveness
with respect to the best performing unforced jet. Considering a transitional regime baseline at $BR = 0.45$,
forcing the jet showed improvement in adiabatic effectiveness and coverage coefficient in the vicinity of the jet
exit, while area averaged adiabatic effectiveness was unaffected. Although the duty cycle effect at constant
$BR_m$ was significant, forcing frequencies appeared to have only minor influence on the jet performance,
even though they greatly affected the jet flow behavior. By observing and understanding the dynamics of
the characteristic vortical structures formed under pulsed conditions and their impact on the wall cooling
performance, some basis was provided to engineer an optimum forcing signal. In particular, exploiting the
improvement observed in the transient part (low to high blowing ratio transition) after the arrival of the
pulse, but shortening the high part of the cycle to suppress the quasi-unforced high blowing ratio state,
could alleviate performance degradation associated with jets in the transitional regime. In addition, a more
progressive transition from high to low blowing ratio could lessen the extent of cross-flow ingestion and the
associated film cooling performance degradation.
Chapter 3
Inclined Jet in Cross-flow

This chapter carries over the work presented in the previous one using a more application-relevant geometry of a 35° inclined modulated jet over a flat plate using reactive Mie scattering visualizations, hot wire anemometry and PIV measurements, complemented by Large Eddy Simulations (LES) to provide details on the flow and temperature fields in the near-field of the jet exit. As for the vertical jet, the first part focuses on the characterization of the unforced jet baseline, while the second section investigates the specifics of modulated inclined jets.

3.1 Experimental and Numerical Setup

3.1.1 Wind Tunnel Experiments

The experiments for the 35° jet are carried out in the same wind tunnel as the one used in the 90° jet study. Notations and coordinate systems for the inclined jet are also identical to the previous chapter, taking the origin at the jet exit center. The only major difference between both configurations is the implementation of a PIV (Particle Image Velocimetry) section in the jet pipe consisting in a square acrylic tube of $6D_j$ long presented in figure 3.1. This section will be used in a close future for PIV (Particle Image Velocimetry) measurements in view of obtaining realistic inlet conditions for LES. The cross-flow conditions are also identical to the 90° jet study and are provided in table 2.1. The jet exits from a 25.4mm round tube mounted flush to the bottom wall at an angle of 35° with respect to the cross-flow direction and zero compound angle. The tube length, including the PIV section is approximately $12D_j$. A seeding system injecting $TiCl_4$ and water inside the jet is also used during reactive and fully reacted Mie scattering visualizations. The jet supply system is identical to the one of the vertical jet with two branches one of which includes a computer-controlled solenoid valve used to pulse the flow (figure 3.1). The experiment was designed so that the jet natural frequencies would be relatively low ($< 100Hz \sim St_\infty = 1.5$) and forcing is applied at frequencies lower than these natural frequencies consistently with the scaled-down (relative to this work) experiments of Elkad et al. (2006) and theoretical assessments of Nikitopoulos et al. (2006).

![Figure 3.1: Inclined jet experimental apparatus.](image-url)
In the forced jet experiment, an additional visualization plane (Y-Z) is acquired at four distinct stream-wise locations of $X_j = 0, 0.5, 1$ and 1.5. For those visualizations, the camera is set downstream of the visualization plane at an angle of approximately 30° with the streamwise direction and the focus plane is adjusted using a Scheimpflug mount offsetting the lens axis from the camera sensor axis to satisfy the Scheimpflug condition (see Raffel et al. (1998)). The obtained images are then dewarped based on a linear transformation algorithm, using a previously acquired calibration image. These images (and the one in the $Y_j = 0$ plane for the forced jet) are obtained in a time resolved manner at 1000 frames per second (fps) using a Dantec Dynamics Nanosense MkIII camera with a 1280 $\times$ 1024 sensor. Finally, the forced jet is also documented using 2D-PIV measurements in the $Y_j = 0$ plane. These measurements are obtained using the same camera operated at 500fps with an average time delay between both laser shots of $\Delta t = 250ms$. An adaptive correlation algorithm is used with an initial interrogation window of 64 pixels, a final interrogation window of 16 pixels and a 50% overlap. At least 30 cycles are acquired for each forcing conditions to obtain accurate phase averages presented in the following sections. Seeding for PIV measurements is achieved using $2\mu m$ porous silica particles with extremely low specific gravity and high reflectivity, following the flow very accurately.

### 3.1.2 Numerical Simulations

As for the previous configuration, numerical simulations are carried out in parallel to the experiments using Ansys Fluent™. Large Eddy Simulations (LES) with dynamic Smagorinsky sub-grid scale models are used. A set of three simulation grids is used in this part of the study. All three of them are built on the same principles as the vertical jet grid, representing a part of the jet pipe and of the wind-tunnel test section.

**Mechanistic Analysis Grid**

The first grid corresponds to the one used in Vezier (2009). This grid was primarily used for flow understanding and a preliminary statistical POD analysis. The domain consists of a rectangular box representing a part of the wind-tunnel test section and part of the jet feeding tube. The computational domain is $18D_j$ long (x-direction), $8D_j$ wide (y-direction) and $8D_j$ tall (z-direction), respectively discretized into $160 \times 100 \times 90$ hexahedral cells, while the jet pipe feeding is $8D_j$ long, discretized by 2000 nodes in the cross-section and 120 nodes in the axial direction. The grid is structured and counts approximately 1.6 million hexahedral cells. The jet exit center is located $6D_j$ downstream from the domain inlet. The first cells in contact with the walls are approximately $0.03D_j$ tall, providing average $y^+$ values of the order of 1, thus reaching the viscous sub-layer and allowing direct solving of the wall shear stress from laminar stress-strain relationship, without using wall functions.

Velocity characteristics and boundary layer profiles for the inlet of the computational domain are obtained from hot wire measurements performed in the wind tunnel as summarized in table 2.1. The numerical solver used is pressure based with second order accuracy in time and space. The integration time steps used range from $5.10^{-5}s$ at $BR = 1.2$ to $5.10^{-4}s$ at $BR = 0.15$ so as to remain below the minimum Kolmogorov time scale in the domain. At the inlet of the jet pipe, uniform velocity profile is set so as to equal the volumetric flow rate of the experiment. The jet and cross flow fluids are maintained at constant temperature of respectively $300$ and $330K$, not affecting the velocity field. A grid independence study was carried out using this particular grid and is available in Vezier (2009).

**Reduced Order Modeling Grid**

The second grid is presented in figure 3.2 and corresponds to a refined and extended version of the first grid. It is primarily used to obtain flow and temperature fields snapshots for reduced order modeling purposes. The domain is extended of $3D_j$ in the downstream direction and the jet exit relocated $4D_j$ downstream of the domain inlet to give a $19D_j$ long grid. The spanwise dimension is left unchanged while the height is brought down to $6D_j$. The pipe geometry is slightly modified to account for an updated, longer jet pipe PIV section. The main domain comport $240 \times 148 \times 75$ hexahedral cells in respectively the streamwise, spanwise and vertical directions. The jet pipe is meshed using an O-grid of 6500 cells in the cross-section and 175 cells in the pipe axis direction. This accounts for a total of over 4 million cells. The first cell in contact with the wall is on average $0.005D_j$ tall resulting in much less than unity $y^+$ values. The
Figure 3.2: Inclined jet numerical grid details and boundary conditions. (a) X-Y view, (b) X-Y view detail, (c) X-Z view, (d) Y-Z view, (e) global view and boundary conditions.

Table 3.1: Relative velocity magnitude maximum and averaged error with respect to the 4M grid values on multiple streamwise profiles up to \( Z_j = 1.0 \) for \( BR = 0.15 \) and \( Z_j = 2.5 \) for \( BR = 1.0 \).

<table>
<thead>
<tr>
<th>( X_j )</th>
<th>( BR = 0.15 )</th>
<th>( BR = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 1M )</td>
<td>( 2M )</td>
</tr>
<tr>
<td></td>
<td>max.</td>
<td>avg.</td>
</tr>
<tr>
<td>( X_j = 0 )</td>
<td>7.3%</td>
<td>0.9%</td>
</tr>
<tr>
<td>( X_j = 2.5 )</td>
<td>3.4%</td>
<td>0.4%</td>
</tr>
<tr>
<td>( X_j = 5 )</td>
<td>1.3%</td>
<td>0.2%</td>
</tr>
<tr>
<td>( X_j = 10 )</td>
<td>1.3%</td>
<td>0.2%</td>
</tr>
</tbody>
</table>

55
The pipe inlet velocity profile is obtained from experimental measurements as summarized in table 2.1. The corresponding time steps are summarized in table 3.2. High perturbation levels had to be implemented to recreate the fluctuation levels introduced by the injection jets of the seeding system (see inlet condition grid below). The integration time step for each individual blowing ratio case is evaluated based on the Kolmogorov time-scale determined from initial $k - \epsilon$ RANS simulations. The corresponding time steps are summarized in table 3.2.

A grid independence study is carried out using three smaller grids of 1, 2 and 3 million cells. The results are presented in figure 3.3 at the two extreme blowing ratios of $BR = 0.15$ and $BR = 1.0$ based on the average velocity magnitude profiles. Overall good convergence is obtained with increasing number of cells towards the most refined grid. Quantitative error levels are also extracted based on the highest resolution velocity profiles and are provided in table 3.1. They show overall decreasing average and maximum relative error levels with increasing cell count. Average errors of the order of 1 to 2% are reached between the 3 and 4 million cells grids at both investigated blowing ratios. The qualitative profiles in figure 3.3 and the quantitative error levels in table 3.2 provide good confidence on the grid independence of the following numerical results.

A series of constant temperature anemometry measurements are performed at blowing ratios of $BR = 0.15$, $BR = 0.5$ and $BR = 1.0$, at 4 different streamwise locations ($X_j = 0, 2.5, 5, 10$) and presented in figure 3.4 along with extracted corresponding simulation results. Overall the LES results compare extremely well with the experimental measurements at all streamwise locations and at all three blowing ratio. Only a minor discrepancy is found directly at the jet exit at $BR = 0.15$ which can be attributed to rectification of the hot wire probe signal. Indeed, at such low blowing ratio, the seeding flow rate accounts for a large amount of the overall jet flow rate (more than 60%) and the fluctuation levels introduced by the seeding jets along the hot wire direction are likely non-negligible and can locally increase the sensed velocity magnitude. These validation measurements provide good confidence in the following numerical results.

### Inlet Conditions Grid

A third and last grid is used to obtain the jet inlet velocity profile. This grid is based on the previous one, with an extended pipe geometry, including the complete jet PIV section, seeding injection block and jets and an approximation of the flow-meter geometry. Figure 3.5 shows the extended pipe geometry. The complete mesh comports 5.7 million cells. The simulations are run with a $k - \omega$ model with fully developed inlet velocity profiles for the main pipe inlet as well as the seed injection jets. Average velocity magnitude profiles are extracted at the location of the ROM grid inlet plane and used as inlet conditions for these simulations.

Finally, figure 3.6 shows the geometries of all the different grids for comparison purposes.

### 3.2 Unforced Jet

In unforced conditions, the jet is studied using reactive and fully reacted Mie scattering visualizations. Selected cases at $BR = 0.15, 0.3, 0.4, 0.75, 1.0$ and $1.2$ are simulated with LES using the mechanistic analysis grid. Similarly to the vertical jet in cross-flow described in the previous chapter, several unforced jet regimes are observed. At low blowing ratios under $BR = 0.4$, the jet is fully attached to the wall while at blowing ratios above $BR = 1.0$, the jet is found to be completely detached from it. At intermediate blowing ratios, the jet exhibits vortical structures from both regimes.

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<table>
<thead>
<tr>
<th>Fluctuation levels</th>
<th>$BR = 0.15$</th>
<th>$BR = 0.5$</th>
<th>$BR = 1.0$</th>
<th>Case IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of vortices</td>
<td>30%</td>
<td>50%</td>
<td>60%</td>
<td>50%</td>
</tr>
<tr>
<td>Time step</td>
<td>$1 \times 10^{-4}$</td>
<td>$5 \times 10^{-5}$</td>
<td>$1 \times 10^{-5}$</td>
<td>$1 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

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Table 3.2: Simulation parameters for the ROM grid.
Figure 3.3: Normalized velocity magnitude for 4 different grids: 1M (·· dot), 2M (−· dash-dot), 3M (−− dashed) and 4M (− solid) at multiple streamwise locations.
Figure 3.4: Experimental (*symbols*) and LES (*solid line*) time averaged velocity magnitude profiles at (a) $BR = 0.15$; (b) $BR = 0.5$; (c) $BR = 1.0$ at $X_j = 0, 2.5, 5$ and 10.
Figure 3.5: Extended inlet conditions grid.

Figure 3.6: Geometries of the different grids.
3.2.1 Attached Jet

The attached jet is the most relevant configuration to film cooling and shows strong similarities with the vertical jet configuration. The dominant shear layer vortical structures consist in interlocked hairpin vortices observed in the experimental side-views of figure 3.7(a) as well as the LES results of figure 3.10(a-b) at $BR = 0.15$. These vortices are developing as a result from Helmholtz or Landman-Saffman type of instability in the jet shear layers and have been observed in previous studies such as Tyagi & Acharya (2003). In the experimental case at $BR = 0.15$, the instability develops sooner in the upper shear layer where the initial rollup occurs first. Eventually both upper and lower shear layers merge a few diameters downstream of the jet exit and the hairpin vortex is fully formed. The dynamics of the hairpin vortices are comparable to those observed for the vertical jet and are dictated by cross-flow convection as well as mutual, self, and mirrored inductions with respect to the bottom wall. While the convection carries the structures downstream, the mutual induction of one leg onto the other creates a positive vertical velocity component entraining the hairpins legs away from the wall. In the meantime, the induction from the mirrored image of the legs with respect to the bottom wall generates a spanwise motion toward the symmetry plane ($Y_j = 0$) pushing the legs toward each other and consequently reinforcing the induced upward motion as the distance between the vortex cores is decreased. Similarly, the mirrored induction on the head of the hairpin vortices as well as the self-induction of the curved vortex line result in a backward/upward motion for the hairpins heads. Upstream of the jet exit, the horseshoe vortex system is absent from the experimental side visualizations at the lowest blowing ratios of $BR = 0.15$ and $BR = 0.3$, as well as from the LES results of figure 3.10(a). However, a pair of streamwise vortices containing vorticity coherent with the horseshoe vortex develops on the sides of the jet without connecting upstream in the usual U-shape as evidenced in figure 3.8. These streamwise vortices are also observed in the experimental visualizations of figure 3.9 on each side of the jet since they carry reacted particles. The absence of a full horseshoe vortex has been documented in previous studies such as Tyagi & Acharya (2003) or Guo et al. (2006) in cases where the cross-flow blockage due to the presence of the jet was weak, which is clearly the case here at such low blowing ratios. Yet, the presence of the jet creates a cross-flow deflection and generates a spanwise velocity gradient in the vertical direction at the origins of the side vortices near the jet exit as evidenced in the velocity profiles of figure 3.8(b). This is identical to the mechanism responsible for the formation of the counter-horseshoe vortex and the side vortices in the 90° jet configuration. At $BR = 0.15$, in the far-field of figure 3.10(a) X-patterned structures are formed around $X_j = 10$, again very much like what was observed in the attached vertical jet setup. These become stronger and start forming earlier as the jet strength increases and the associated cross-flow deflection as well.

As the blowing ratio is increased, a coupling between upper and lower shear layers fluctuations occurs. At $BR = 0.3$ (figure 3.7b), instability in the lower shear layer starts to develop and eventually at $BR = 0.4$ (figure 3.7c) both shear layers exhibit distinct rollups, each corresponding to a hairpin vortex, merging as
they are convected downstream. The hairpin vortices legs gain in strength and tend to detach earlier from the wall and entrain more cross-flow fluid near the bottom surface. This is observed in the reactive Mie scattering visualizations of figure 3.9 where at $BR = 0.15$ (figure 3.9a) the legs of the hairpin vortices lift-off between $X_j = 3$ and $X_j = 4$, whereas at $BR = 0.75$ (figure 3.9b) the legs are completely detached beyond $X_j = 1$. Directly downstream of the jet exit in figure 3.7(a) as well as figure 3.9(a) at $X_j = 2$, an area of reacted seed evidences the characteristic recirculation region present in the attached jet configuration. This region, mainly supplied by jet fluid, is also observed in the LES results and corresponds in figure 3.10(a) and 3.10(c) to an area of lower temperatures encompassed by the hairpin vortices legs. At $BR = 0.4$, the jet blockage is important enough to cause the cross-flow boundary layer to separate upstream of the jet exit and form a complete horseshoe vortex in figure 3.7(c) (tail-less arrow) and 3.10(c). Further downstream, the side vortices appear much stronger than the one formed at lower blowing ratios and start developing around $X_j = 3$, and X-patterned structures around $X_j = 6$. Conversely to the vertical jet, there is no stable inner vortex inside the jet pipe of the 35° jet. This is because in the latter configuration, the adverse pressure gradient for the flow exiting the jet pipe is much lower than in the vertical configuration due to the lower jet deflection. It should be noted that conversely to the vertical jet, the horseshoe vortex was never found to be transported over the jet. This is also explained by the absence of destabilizing interaction with the inner vortex in this setup.

### 3.2.2 Transitional Jet

The transitional regime for the inclined jet was defined based on the vertical jet observations when negative spanwise vorticity, effectively from the destabilized inner vortex, started to shed in the upper shear layer. This is observed in the inclined jet configuration at blowing ratios beyond $BR = 0.4$ as evidenced in the experimental visualizations in figure 3.7(d) as well as in the LES results of figure 3.10(f) for $BR = 0.75$. Overall though, downstream of the jet exit, rollups with positive spanwise vorticity still develop in the upper shear layer, so that the dominant vortical structures remain hairpin vortices. The increased vorticity in the

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**Figure 3.8:** (a) Laplacian of the pressure iso-surfaces $\Delta P = 1.6kPa.m^{-2}$ and instantaneous wall temperature; (b) Laplacian of the pressure iso-surfaces $\Delta P = 0.8kPa.m^{-2}$ and local velocity profiles with spanwise and vertical component only after LES at $BR = 0.300$.

**Figure 3.9:** Experimental Mie scattering visualizations on a plane inclined at -30° with respect to the Y-Z plane for (a) $BR = 0.15$; (b) $BR = 0.75$. 
legs and initial vertical momentum at the jet exit triggers significant lift-off. Organized motion taking the form of two counter-rotating vortices underneath the jet core appears to develop as suggested by large rollups in figure 3.9(b). At $BR = 0.75$, where the fluctuation levels are significantly higher, multiple scales of shear layer structures develop at the jet/cross-flow interface, as seen in figure 3.9(b), and smaller kidney/anti-kidney vortices are formed along with the larger scale counter-rotating vortex pair. Such multiplicity of the shear layer structures has been previously documented by Haven & Kurosaka (1997) and New et al. (2003) and attributed to convex/concave warping of the vortex sheet. Side vortices and X-patterned structures remain present, yet with less regularity. It should be noted that overall the flow symmetry, fairly well conserved at lower blowing ratios, is no longer observed in the simulations at $BR = 0.75$.

The transitional regime in the inclined configuration does not exhibit the dramatic changes in flow behavior that were observed in the vertical jet with the destabilization and transport of the inner vortex, making the transition from attached to detached jet much smoother. In fact, only the presence negative vorticity in the upper shear layer (figure 3.7d) allows us to define a transitional inclined jet regime.

### 3.2.2.3 Detached Jet

For $BR \geq 1$, the jet exhibits the usual jet in cross-flow vortical structures extensively documented at high $BR$ values. Rollups of negative spanwise vorticity are consistently formed in the upper shear layer as evidenced in figure 3.7(e-f) and 3.10(g-h), as well as tornado-like wake vortices located below the jet core which are seeded by particles and therefore transport jet fluid. A complete, yet rather short, horseshoe vortex is observed in the LES results of figure 3.10(g), although not in the experimental visualizations due to the absence of seed, indicating that the horseshoe vortex does not entrain significant amount of jet fluid at high blowing ratios.
3.2.4 Film Cooling Performance

The impact of the different vortical structures on the wall temperature and the film cooling performance are investigated using the results from the simulations. As seen in figure 3.8 and figure 3.10(c-d), in the attached jet configuration, the shear layer vortical structures have a strong influence on the wall temperature field since they carry most of the coolant fluid from the jet while generating a velocity field responsible for entraining cross-flow fluid into the wall region. In figure 3.8(a) the streamlines show that jet fluid exits from the sides of the jet exit, very close to the wall to provide significant wall coverage as evidenced by the temperature contours of figure 3.10(a) and 3.10(c). Quantitative information is extracted from the numerical simulations to provide a performance benchmark to the forced results and is also compared to the vertical jet results. Figure 3.11 presents time-averaged wall adiabatic effectiveness for the inclined jet. At $BR = 0.15$, (figure 3.11a), the area of high adiabatic effectiveness directly downstream of the jet exit corresponds to the recirculation region enclosed by the legs of the successive hairpin vortices. As the blowing ratio is increased up to $BR = 0.4$, the hairpin vortices legs with higher circulation tend to lift off the wall sooner due to self induction and therefore carry coolant away from the wall. This results in film coverage breakup and entrainment of hot cross-flow fluid near the wall. Figure 3.10(a) through 3.10(d) show the impact of the hairpin vortices lift-off and consequent cross-flow entrainment on the adiabatic wall temperature. On the other hand, from $BR = 0.15$ to $BR = 0.4$, a relative increase in spread is observed further away from the jet exit $(X_j > 7)$ due to the beneficial effect of the side vortices and their favorable velocity field, as well as near the jet exit due to the strengthening and formation of the horseshoe vortex. However, due to the absence of inner vortex in the inclined jet configuration, the interaction between the inner vortex and the horseshoe vortex responsible for significant coolant entrainment in the upstream region of the vertical jet exit is absent in the 35° jet configuration and the upstream region is almost not cooled at all.

As the jet enters the transitional regime $(0.4 < BR \leq 0.9)$, the wall coverage downstream of the hole is degraded due to complete lift off of the shear layer structures. The coverage finally becomes marginal in the detached configurations where the jet wake is the only region showing some form of cooling effect.

Figure 3.12 presents downstream spanwise averaged adiabatic effectiveness $\eta_{span}$ as well as center-line adiabatic effectiveness $\eta_{centerline}$ and comparable results from the vertical jet. In figure 3.12(a), directly downstream of the jet exit $(X_j < 6)$, the case at $BR = 0.15$ shows improved performance due to the presence of the steady recirculation region fed by jet fluid. However further downstream $(X_j > 6)$ the case at $BR = 0.3$ provides increased adiabatic effectiveness due to the increased coolant flow rate and spread over the case at $BR = 0.15$. As the blowing ratio is increased beyond 0.3, the spanwise averaged performance of the inclined jet decreases consistently. The center-line results for the inclined jet in figure 3.12(b) show...
Figure 3.12: Film cooling performance: (a) Spanwise averaged adiabatic effectiveness $\eta_{\text{span}}$; (b) center-line adiabatic effectiveness $\eta_{\text{centerline}}$ after LES for the inclined jet (solid line, filled symbols) and the vertical jet (open symbols).

Figure 3.13: Unforced film cooling performance (a) Area averaged adiabatic effectiveness $\eta_{\text{area}}$ for the inclined jet (solid line) and the vertical jet (dashed line); (b) coverage coefficient $C_c$ for thresholds $\eta = 0.1$, 0.2, 0.3, 0.5.

Similar trends. The local decrease in $\eta_{\text{centerline}}$ observed for $BR \geq 0.3$ directly downstream of the jet exit is due to early lift-off of the shear layer vortices, allowing cross-flow penetration in the vicinity of the jet exit. For $BR \leq 0.3$, the inclined jet shows greater spanwise averaged and center line adiabatic effectiveness, compared to the vertical jet. However, the inclined jet performance is degraded beyond $BR = 0.3$, which was not the case for the vertical jet where performance continued to increase up to $BR = 0.415$. Consequently, at $BR = 0.4$, the vertical jet exhibits greater $\eta_{\text{span}}$ values than those of the inclined configuration. At higher blowing ratios ($BR \geq 0.75$), the inclined jet performs better than the vertical jet, as it is well known to do so. The $\eta_{\text{centerline}}$ trends show that the inclined jet always provides greater center-line effectiveness over the vertical jet. This confirms that the overall better performance of the vertical jet at $BR = 0.4$ comes essentially from a greater spanwise spread of the coverage (see the previous chapter).

Figure 3.13 presents area-averaged adiabatic effectiveness and coverage coefficient for both inclined and vertical jets, giving a global performance index at every blowing ratio. The $\eta_{\text{area}}$ trends of figure 3.13(a) for the inclined jet exhibit a constant decrease in performance past $BR = 0.3$ with a slight improvement from $BR = 0.15$ to $BR = 0.3$. This result is consistent with the previous observations based on $\eta_{\text{span}}$ values. According to figure 3.11(a), the improvement from $BR = 0.15$ to $BR = 0.3$ corresponds to an increase in $\eta$ values beyond $X_j = 5$. A local maximum is expected to exist between $BR = 0.15$ and $BR = 0.3$ which is rather low for an inclined jet configuration and can be justified by the use of single jet in laminar cross-flow conditions. The comparisons with the vertical jet results show that the expected performance degradation as the blowing ratio increases is less abrupt in the inclined configuration than in the vertical setup. Interestingly, the vertical jet performs better at $BR = 0.415$ with an area averaged adiabatic effectiveness of 0.225 within the considered field of view ($-1.2 < X_j < 12$). The coverage coefficient trends in figure 3.13(b) confirm the above mentioned results with a consistent decrease in performance beyond $BR = 0.3$ for the inclined jet. Once again when compared to the vertical jet, the inclined jet provides more coverage at high effectiveness ($\eta \geq 0.5$) values. The vertical jet though provides significantly greater coverage at lower effectiveness levels (almost double in some cases), sign of a greater spread at higher injection angle. Higher spread and performance in the neighborhood of the jet exit ($X_j < 12$) for high injection angles were previously reported in studies such as Yuen & Martinez-Botas (2003).
Table 3.3: Forced inclined jet cases. ∗: Simulations carried at these conditions using mechanistic analysis grid, †: Simulations carried out at these conditions using ROM grid.

<table>
<thead>
<tr>
<th>Case#</th>
<th>BRm</th>
<th>Br</th>
<th>Brh</th>
<th>BRpp</th>
<th>DC(%)</th>
</tr>
</thead>
<tbody>
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<td>0.150</td>
<td>0.250</td>
<td>0.100</td>
<td>0.7</td>
</tr>
<tr>
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<td>0.100</td>
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<tr>
<td>III</td>
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<td>0.150</td>
<td>0.500</td>
<td>0.350</td>
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</tr>
<tr>
<td>IV†</td>
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<td>0.150</td>
<td>0.500</td>
<td>0.350</td>
<td>50</td>
</tr>
<tr>
<td>V</td>
<td>0.156</td>
<td>0.150</td>
<td>1.000</td>
<td>0.850</td>
<td>0.7</td>
</tr>
<tr>
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<td>0.150</td>
<td>1.000</td>
<td>0.850</td>
<td>50</td>
</tr>
<tr>
<td>VII</td>
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<td>0.150</td>
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</tr>
<tr>
<td>VIII</td>
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<td>1.85</td>
<td>50</td>
</tr>
<tr>
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<td>3.000</td>
<td>1.85</td>
<td>0.6</td>
</tr>
<tr>
<td>X*</td>
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<td>0.150</td>
<td>0.450</td>
<td>0.300</td>
<td>50</td>
</tr>
<tr>
<td>XI*</td>
<td>0.450</td>
<td>0.150</td>
<td>0.750</td>
<td>0.600</td>
<td>50</td>
</tr>
<tr>
<td>XII</td>
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<td>0.750</td>
<td>1.000</td>
<td>0.250</td>
<td>50</td>
</tr>
<tr>
<td>XIII</td>
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<td>0.650</td>
<td>1.500</td>
<td>0.850</td>
<td>50</td>
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<td>1.75</td>
<td>2.000</td>
<td>0.250</td>
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</tr>
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</table>

The overall greater performance of the vertical jet in this study should be put in perspective with respect of two considerations. First, the domain of the current study was limited to the near field \((X_j < 12)\), which in Yuen & Martínez-Botas (2003) was the approximate streamwise location beyond which the inclined jet started to evidence better performance than the vertical one. This is obviously not accounted for in the results of figure 3.13. Also, it was explained in the previous chapter that the sudden decrease in performance in the 90° injection configuration was attributed to a destabilization of the inner vortex formed inside the jet tube, extremely sensitive to both cross-flow and jet inlet conditions. Hence, it is expected that outside the laminar cross-flow boundary layer conditions used in this study, the transition for the vertical jet could happen significantly sooner while not affecting the inclined jet as much, therefore leading to better performance for the latter configuration.

3.3 Forced Jet

Forced jet experiments are carried out using a nominal square wave excitation according to the forcing parameters summarized in Table 3.3. Most of these cases are observed at 4 forcing frequencies of 0.5, 1.0, 5.0 and 10.0 Hz, respectively corresponding to Strouhal numbers of \(St_\infty = 0.008, 0.016, 0.079\) and 0.159.

3.3.1 System Characterization

The use of time-resolved flow-meter measurements allowed for precise verification of the forcing conditions. Typical phase-averaged flow meter records are provided in figure 3.15(a, b) respectively corresponding to Case X at \(St_\infty = 0.016\) and 0.159. In the time records, flow rate oscillations are observed at the transition from the low part to the high part of the cycle and inversely. These correspond to a characteristic of the actuation valve and occurred at a constant frequency of \(St_\infty = 0.8\) throughout the tests, regardless of forcing parameters. Such oscillations are commonly found in forced systems and present in numerous studies such as Ou & Rivir (2006) or Johari et al. (1999).

As for the forced vertical jet configuration, a total of four phases can be identified during forced cycles. The first one evidenced in figure 3.14(a-c) corresponds to the transient introduced at the transition from \(BR_l\) to \(BR_h\) and consists in the formation of a single or series of starting vortices. The second phase corresponds to a quasi-unforced regime established as the transient associated with the starting vortices is washed away and while the jet remains at high flow rate. In this phase, the jet behaves identically to an unforced jet at equivalent blowing ratio of \(BR_h\) as seen in figure 3.14(d). The third phase corresponds to a transient triggered by the sudden decrease in mass-flow at the moment the solenoid valve is closed and the
Figure 3.14: Instantaneous reactive Mie scattering visualizations in the plane \( Y_j = 0 \) (left) and temperature field from LES (right) at: (a) \( t^* = 0.04 \); (b) \( t^* = 0.09 \); (c) \( t^* = 0.13 \); (d) \( t^* = 0.49 \); (e) \( t^* = DC + 0.07 \); (f) \( t^* = DC + 0.19 \); (g) \( t^* = DC + 0.47 \) for Case X at \( St_{\infty} = 0.016 \).

blowing ratio transitions from \( BR_h \) to \( BR_l \). During this phase shown in figure 3.14(e, f), cross-flow fluid is ingested inside the jet pipe and the jet flow is disrupted. Finally, the fourth regime is established as the transient dynamics from the ingestion are washed away while the jet remains in low flow rate. In this part of the cycle, the jet behaves similarly to an unforced jet at equivalent blowing ratio of \( BR_l \) as seen in figure 3.14(g). As for the vertical jet, the two quasi-unforced regimes are not always observed depending on the duty cycle and forcing frequency which determine the duration of the low and high parts of the cycle and therefore the available time for the transients to be evacuated before the next transient is introduced. In the current study, a majority of the cases at \( St_{\infty} = 0.008 \) and \( St_{\infty} = 0.016 \) (respectively corresponding to 0.5 and 1Hz) exhibits all four regimes while cases at \( St_{\infty} = 0.08 \) and \( St_{\infty} = 0.159 \) (respectively 5 and 10Hz) only exhibits the transients.

3.3.2 Low-to-High Blowing Ratio Transition: Starting Vortices

In the forced inclined jet configuration, starting vortices are also generated as the blowing ratio transitions from \( BR_l \) to \( BR_h \) due to the associated sudden increase in shear at the jet/cross-flow. While for the vertical jet only two principal regimes of starting structures/systems were identified, the inclined jet exhibits three dominant types of starting vortices, which are presented in figure 3.16. At the lower end of the investigated \( BR_h \) range, the starting vortices consist of rather large hairpin vortices as evidenced in figure 3.16(e). Cases with low values of the stroke ratio (see equation 2.2) exhibit only one large starting hairpin vortex whereas cases with longer injection times trigger the formation of additional trailing structures, visible in figure 3.16(e). On the higher end of the \( BR_h \) range, the starting vortices consist of ring vortices. Cases with short injection times generate a single starting ring vortex (figure 3.16a) while cases with longer injection times trigger the formation of a leading starting vortex ring, followed by a series of smaller trailing ring vortices (figure 3.16b). While the dynamics of the vortical structures in those regimes are different from one configuration to the other and will be described in the coming sections, it can be argued that both the
Figure 3.15: Phase Averaged $\eta_{\text{area}}$ (top), relative coverage coefficient fluctuation for $\eta = 0.1, 0.2, 0.3$ and $0.5$ (center), blowing ratio (bottom) for Case X at (a) $St_\infty = 0.016$; (b) $St_\infty = 0.159$.

Figure 3.16: Mie scattering visualizations in the plane $Y_j = 0$ for (a) Case V at $St_\infty = 0.008$ - Single starting vortex ring; (b) Case VIII at $St_\infty = 0.008$ - Leading starting vortex ring and trailing vortices; (c) Case III at $St_\infty = 0.008$ - Single compound starting vortex; (d) Case IV at $St_\infty = 0.008$ - Compound starting vortex and trailing vortices; (e) Case II at $St_\infty = 0.008$ - Starting hairpin vortex and trailing vortex. ($S_v$) Starting vortex, ($T_v$) Trailing vortices.
inclined and vertical jets have those regimes in common. However, the inclined jet also exhibits a third kind of starting vortex at intermediate values of $BR_h$. This type of complex starting vortex involves in fact multiple structures and for this reason is referred to as “compound starting vortex system”. As for the other two types, long injection times lead to the formation of multiple starting structures (figure 3.16d) while short injection times only trigger the formation of a single compound starting vortex system (figure 3.16c).

The dynamics of the different kind of starting vortices are described in the following sections based on experimental data, and numerical simulations when available. Closer attention is brought to the modified dynamics of the starting vortex ring regimes, and to the description of the new compound starting vortices after touching very briefly on the starting hairpin vortices.

**Starting Hairpin Vortex**

As for the vertical jet, the dynamics of the starting hairpin vortices are found to be essentially identical to their unforced equivalent and therefore will not be extensively discussed. The Mie scattering visualizations in figure 3.17 show a formation sequence of a starting hairpin vortex. As the jet transitions from $BR_l$ to $BR_h$, a mass surplus is created inside the jet tube which is balanced by a pressure wave traveling down the jet pipe and perpendicularly to its axis from the valve towards the jet exit. Because of the inclined geometry of the jet, the wave reaches the leading edge of the jet exit first and triggers the formation of a
shear layer rollup with negative vorticity at the leading edge in figure 3.17(b). Due to the relatively low magnitude of the pulse in Case II, the vorticity generated at the leading edge remains low and the rollup is rapidly dissipated by the incoming boundary layer with opposite vorticity before it can accumulate enough circulation to be transported. Hence, the transition from low to high blowing ratio does not carry enough energy to force the jet shear layer to roll into a coherent starting vortex. However, the propagation of the expansion wave into the free stream acts as a perturbation for the natural jet shear layer instability and triggers the formation of a large hairpin vortex in figure 3.17(d). A second structure follows the first one and constitutes a trailing hairpin vortex. In figure 3.17(e-h), a small rollup appears to be formed in the lower shear layer between \( t^* = 0.039 \) and \( t^* = 0.054 \) and to be convected. It is however rather unclear whereas this rollup is a primary consequence of the jet pulse or an instability of the lower shear layer triggered by the transport of the starting hairpin vortex.

**Starting Vortex Ring and Trailing Column**

At high enough values of \( BR_h \), the starting vortices are of the ring kind, a regime which was previously observed in the vertical jet configuration as well. However, the inclined jet starting system exhibits a few differences. In figure 3.18(b), the starting vortex starts to form at the leading edge where the upper shear layer is rolling up first. At this particular instant (\( t^* = 0.003 \)), no rollup is observed at the downstream edge of the jet yet. The combination of the \( Y_j = 0 \) and the \( X_j = cst \) slices in figure 3.19 shows the progressive rollup of the shear layer from the upstream edge towards the downstream edge between approximately \( t^* = 0.003 \) to \( t^* = 0.008 \), moment at which a clear rollup is observed at the trailing edge. This is a consequence of the expansion wave reaching the leading edge first due to geometrical considerations. This is quantitatively shown in the PIV measurements of figure 3.20, where the jet exit velocity profiles show a peak localized at the jet leading edge in 3.20(b) which then progressively propagates towards the downstream edge from figure 3.20(c) to (f). Evidence of negative vorticity is found at the leading edge at \( t^* = 0.003 \), while none is found at the trailing edge until \( t^* = 0.008 \). The complete set of velocity profiles extracted directly above the jet exit from the PIV measurement is presented in figure 3.21 showing the vertical velocity magnitude across the jet exit as a function of \( t^* \). Figure 3.21(a), corresponding to Case VIII, shows the downstream shift of the velocity profile through time as well as a decrease in the vertical momentum at the leading edge between \( t^* = 0.005 \) and 0.015, attributed to the transport of the starting vortex and its associated velocity field (lower vertical velocity on the upstream side, higher ones on the downstream side). The slope of the lower values region provides an indication on the starting vortex upstream rollup convection velocity. Shallow angles correspond to fast convection speeds, while large angles correspond to slow convection speeds. Because of this delay, the formation process time-lines for the upstream and downstream part of the starting vortex ring are offset in time and the upstream part of the starting vortex departs the leading edge first between \( t^* = 0.003 \) and \( t^* = 0.008 \) after having accumulated sufficient amount of circulation, while the downstream part departs the wall later between \( t^* = 0.008 \) and \( t^* = 0.013 \). Consequently, the starting vortex axis rotates continuously in the positive direction from 0° to approximately 50° (figure 3.18b-d) at the moment of the departure of the downstream rollup and reaches an equilibrium at approximately 60° while it is convected in the free stream (figure 3.18d-g). It is also interesting to notice that while the upstream rollup of the starting vortex ring appears to be significantly larger than its downstream counterpart in the early moments of the cycle (up to \( t^* = 0.015 \)), the size imbalance appears to be resorbed once the starting vortex is convected in the free stream. This is observed in the vorticity levels of figure 3.20 showing comparable circulation in both upstream and downstream rollups. Had the strength of the upstream and downstream rollups been significantly different, their mutual induction would have caused the structure to keep rotating while convected in the free stream, which is not the case here. The \( X_j = cst \) slices show cuts of the starting ring vortex at multiple time instants as it is convected downstream. While the initial moments (therefore at locations close to the origin) show a well organized ring structure, the slices obtained at \( X_j = 1.5 \) reveal the development of typical three-dimensional azimuthal instabilities of smaller scale “inside” the starting vortex ring leading to its turbulent burst (see Shariff & Leonard 1992, Sbrizzai et al. 2004).

Once the starting vortex departs from the wall, the jet shear layer rolls up into additional trailing vortices. Although smaller in size, they are in nature very similar to the starting vortex ring and follow comparable trajectories. This is in agreement with the vertical jet dynamics, however due to the time-line delay between the upstream and downstream edges, the trailing vortices start to rollup first on the jet upper shear layer,
Figure 3.18: Time resolved Mie scattering visualizations in the plane $Y_j = 0$ for Case VIII at $St = 0.008$ at multiple time instants. Dashed lines corresponds to $X_j = cst$ visualizations planes for figure 3.19.
Figure 3.19: Time resolved Mie scattering visualizations in the planes $X_j = 0, 0.5, 1$ and $1.5$ for Case VIII at $St = 0.008$ at multiple time instants.
Figure 3.20: Phase averaged PIV measurements in the plane $Y_j = 0$ for Case VIII at $St = 0.008$ at multiple time instants. Spanwise vorticity field (left); jet exit vertical velocity (center); velocity magnitude and vector field (right). Time instants corresponding to figure 3.18.
Figure 3.21: Vertical velocity profiles across the jet exit in the plane $Y_j = 0$ as a function of time extracted from PIV for (a) Case VIII, (b) Case VII, (c) Case VI, (d) Case V, (e) Case IV, (f) Case III, (g) Case II, (h) Case I. Dashed lines representing the slope of the low velocity region.
as seen in figure 3.18(c), even before the downstream rollup of the starting vortex has finished to form. The lower shear layer counterparts to the trailing vortices are in fact formed much later between \( t^* = 0.015 \) and \( t^* = 0.023 \) for Case VIII. Conversely to the vertical jet, pairing of consecutive trailing vortices is clearly observed on both the upper and lower shear layers and in particular between \( t^* = 0.023 \) and \( t^* = 0.033 \). Pairings can also be found in the phase averaged PIV measurement in figure 3.20 for the first pair of trailing vortices, attesting of the repeatability of the event. The \( X_j = \text{cst} \) slices in figure 3.19 evidence multiple trailing vortices, in particular at the first three streamwise locations, often capturing two distinct structures at once. They also reveal the presence of smaller scale anti-kidney vortices on the jet upper shear layer, in particular at \( t^* = 0.013 \) for \( X_j = 0.5 \) and \( X_j = 1 \), which are once more coherent with the instabilities described in previous unforced studies by Haven & Kurosaka (1997), New et al. (2003). Finally, the slices at \( X_j = 1.5 \) show the vortical system developing in the wake of the starting vortex after \( t^* = 0.023 \) taking the form of a mushroom-like shape. A pair of strong counter-rotating vortices develops on the lower part of the jet body, coherent with the CRVP structure of high blowing ratio unforced detached jets. Under the draft generated by the starting vortex, the jet fluid exhausted during the previous cycle is pushed toward the symmetry plane (\( Y_j = 0 \)) and creates a column of seeded fluid below the counter rotating vortex pair. This column is also visible in the \( Y_j = 0 \) slices in figure 3.18(f, g).

**Starting Single Vortex Ring**

At high values of \( BR_h \) but for short injection times, only a single starting ring vortex is generated as depicted in figure 3.22 for Case V. As for the previous case, the upper shear layer starts rolling up first due to the way the pressure wave is traveling down the jet pipe, thus creating a local increase in velocity at the jet leading edge (see figure 3.24(b) and 3.21(d)). The upper rollup of the starting vortex deports the wall between \( t^* = 0 \) and \( t^* = 0.005 \) while the lower shear layer rollup is just starting to form as evidenced by the vorticity contours from the PIV measurements. However, as the upstream part of the starting vortex is convected above the jet exit, the pulse is interrupted and the actuation valve is closed. The mass deficit created inside the jet pipe at that moment generates a pressure compression wave traveling down the jet pipe, perpendicularly to the jet axis. The compression wave reaches the leading edge first and generates negative vertical velocities in that region, similarly to the dynamics of the expansion wave causing the upper shear layer to rollup first. This is evidenced in the PIV measurements of figure 3.24 showing first the local velocity peak at the leading edge at \( t^* = 0.005 \) and the negative velocities at the same location when \( t^* = DC + 0.003 \).

At this moment, the upstream part of the starting vortex ring, still located above the jet exit, is exposed to strong opposite counter-flow and vorticity at the jet leading edge and is partly ingested inside the jet pipe. The jet exit vertical velocity maps in figure 3.21(e, d) show a comparison between Case V and Case VI with equivalent forcing conditions with exception of the duty cycle. Although a local decrease in velocity at the leading edge is observed in Case VI as the starting vortex upstream part is convected above the jet exit, the velocity values remain positive. However, in Case V, a clear region of negative vertical velocity, consequence of the jet drawdown, develops at the leading edge and propagates downstream. The backward flow has a strong impact on the vorticity levels of the upstream part of the starting vortex as seen in figure 3.24. Consequently, the rotation of the starting vortex ring, initially positive, is interrupted and even reverted and in figure 3.22(f-h) the starting vortex appears to have a neutral angle. As it is convected, the weakened starting vortex starts to dissipate and loses its coherence rapidly. Following the convection of the starting vortex as the compression wave travels down the jet exit, a new localized velocity peak is formed at the leading edge causing the formation of a smaller, local quasi-vertical jet in cross-flow system. This is picked up by both the Mie scattering visualizations in figure 3.22(e, f) and the corresponding PIV measurements in figure 3.24(e, f) and figure 3.21(d). This is due to the characteristic response of the actuator used in the experiments, which oscillates for a short amount of time after the valve is shutdown. The \( X_j = \text{cst} \) visualizations evidence the “side arms” of the starting vortex ring and the complete absence of jet flow in the wake of the starting vortex. Two pairs of counter-rotating streamwise vortices can be observed near the wall on each side of the jet exit at \( t^* = DC + 0.013 \) and correspond to the horseshoe vortex system. Usually located much further away from the jet exit, it is entrained in its vicinity by both the cross-flow ingestion mechanism and the draft from the starting structure. In the \( X_j = 1.5 \) slice at \( t^* = DC + 0.023 \), the same pair of counter rotating vortices is found significantly away from the wall as it is picked up by the starting vortex structure. This entrainment is even more obvious when looking at the full, animated time sequence.
Figure 3.22: Time resolved Mie scattering visualizations in the plane $Y_j = 0$ for Case V at $St = 0.008$ at multiple time instants. Dashed lines correspond to $X_j = cst$ visualizations planes for figure 3.23.
Figure 3.23: Time resolved Mie scattering visualizations in the plane $X_j = 0, 0.5, 1$ and $1.5$ for Case V at $St = 0.008$ at multiple time instants.
Figure 3.24: Phase averaged PIV measurements in the plane $Y_j = 0$ for Case V at $St = 0.008$ at multiple time instants. Spanwise vorticity field (left); jet exit vertical velocity (center); velocity magnitude and vector field (right). Time instants corresponding to figure 3.22.
and constitutes an additional example of how streamwise, wall vorticity can be reoriented by jet shear layer vortices to generate tornado-like wake vortices. The localized jet in cross-flow system developing after the ingestion is also very well captured in the $X_j = \text{cst}$ visualizations. Indeed, at all streamwise locations a pair of counter-rotating vortices corresponding to a starting vortex ring is visible. Part of the dynamics of the ingestion occurring at the jet shutdown has been described here for Case V due to their strong interaction with the starting vortex itself. This is a feature unique to the inclined jet due to its low vertical momentum and the delayed formation process of the starting vortex. Because of this, the starting structure is convected close to the jet exit which makes it prone to interactions with the ingestion dynamics. As for the vertical jets, a section will later be dedicated to the transition from high to low blowing ratios.

As for the vertical jet, ingestion of the first trailing vortex by the leading starting vortex, or “partial leapfrogging”, is not uncommon (see figure 3.18d) in the inclined setup. However, due to the departure delay between the upstream and downstream parts of the starting structure, it is also not unusual to observe trailing vortices ingestion at the upstream edge even before the downstream part of the starting structure is fully formed and departs the wall itself. Such case is presented in figure 3.25. In this high $BR_h$ case, the upstream rollup departs from the wall very early (much before $t^* = 0.005$). However, the high shear introduced by the high value of $BR_h$ triggers the shear layer instability responsible for the formation of the trailing vortices very early, so that at $t^* = 0.005$, the shear layer inside the starting vortex ring becomes unstable and the first trailing vortices start to form “inside” the leading starting vortex. In this particular case, the jet flow is dropped down between $t^* = 0.005$ and $t^* = 0.006$, while the downstream part of the starting vortex is still forming. Although somewhat extreme in terms of the physics involved (the destabilization of the jet shear layer inside the starting vortex structure is only observed in those extreme $BR_h$ cases), this case constitutes a valid example of ingestion of trailing vortices at the upper shear layer and poses a dilemma in terms of definition. Indeed, it is rather difficult to decide whether or not trailing structures were generated in this particular case. On one hand, in fact on one edge, a fully formed starting vortex ring followed by smaller trailing structures are generated, and on the other hand, rather the other edge, an undeveloped downstream rollup corresponding to the leading starting structure has yet to depart the wall when the jet is transitioned to a lower blowing ratio value. In the following classification, such structures were qualified as starting vortex rings with trailing structures.

**Compound Starting Vortices and Trailing Vortices**

It was mentioned previously that for intermediate values of $BR_h$ (how intermediate will be established in the coming sections), the forced inclined jet could generate starting structures of a third kind evidenced in 3.14 and 3.16(e, d). As for the previous starting structures, at the transition from $BR_l$ to $BR_h$, the sudden rise in flow rate results in the propagation of an expansion wave towards the jet exit, reaching the leading edge first due to the inclined jet geometrical configuration. The increased shear at the jet/cross-flow upstream interface triggers the rolling of the upper shear layer to form a starting vortex, visible in figure 3.14(a) at $t^* = 0.004$ and figure 3.26(b) at $t^* = 0.008$. A downstream rollup appears to form in figure 3.26 at $t^* = 0.023$ and to be convected at $t^* = 0.030$, yet no evidence is found in reactive visualizations of figure 3.14. The PIV measurements however confirm the presence of a region of very strong positive vorticity at the trailing edge, being convected between $t^* = 0.023$ and 0.030. At this moderate $BR_h$ value, the upstream rollup only has relatively low vertical momentum and as it is convected away from the wall, it remains very close to the jet exit. This is visible in the PIV mappings of figure 3.21(e), where a region of negative

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Figure 3.25: Time resolved Mie scattering visualizations in the plane $Y_j = 0$ for Case IX at $St = 0.008$ at multiple time instants.
Figure 3.26: Time resolved Mie scattering visualizations in the plane $Y_j = 0$ for Case IV at $St = 0.008$ at multiple time instants. Dashed lines corresponds to $X_j = cst$ visualizations planes for figure 3.27.
Figure 3.27: Time resolved Mie scattering visualizations in the plane $X_j = 0, 0.5, 1$ and $1.5$ for Case IV at $St = 0.008$ at multiple time instants.
Figure 3.28: Phase averaged PIV measurements in the plane $Y_j = 0$ for Case IV at $St = 0.008$ at multiple time instants. Spanwise vorticity field (left); jet exit vertical velocity (center); velocity magnitude and vector field (right). Time instants corresponding to figure 3.26.
vertical velocity develops at the leading edge and propagates in the downstream direction which corresponds to the draft associated with the upstream rollup of the starting vortex. After the shedding of the starting vortex, a region of positive velocity redevelops at the leading edge and triggers the formation of a trailing structure which is visible in the Mie scattering visualizations in figure 3.26 at $t^* = 0.023$. This pattern is repeated, though with less amplitude, until two trailing structures are formed in the wake of the leading starting vortex ring. A comparable set of starting structures is also visible in the reactive Mie scattering visualizations of figure 3.14. As the starting vortex and its trailing vortices penetrate the free stream in figure 3.26(e-g), secondary positive rollups develop downstream of the convected upper shear layer rollups. The PIV measurements show that these secondary rollups, later on evolving into large hairpin vortices, originate from the jet upper shear layer natural vorticity. Eventually, the primary starting vortex ring and trailing vortices (with negative vorticity) are diffused by the overall positive vorticity and only the secondary starting vortices subsist in figure 3.14(c) and figure 3.26(g). Almost no trace of negative vorticity can be found in the corresponding PIV measurements. The $X_j = cst$ visualizations in figure 3.27 provide additional information on the three-dimensional development of this compound starting vortex system. At the beginning of the pulse, the upstream part of the starting vortex develops close to the jet exit with a streamwise vorticity sign consistent with the formation of a ring-like vortex. However, while it is convected and as the secondary vortex forms (around $t^* = 0.03$), two distinct large scale counter rotating vortical structures of opposite streamwise vorticity are visible at the $X_j = 1.5$ location, one located “on top” of the jet core, the other one at the “bottom”. The top vortex pair in fact corresponds to the “side arms” of the upstream part of the primary starting vortex while, the bottom one is associated to the sidearms of the downstream part of the primary vortex. This shows that while it is convected, the primary vortex ring is folded in a manner similar to the schematic depiction of figure 3.29. It should be noted in figure 3.21(a, e, c) the progressive increase in the low velocity region slope, associated with decreased convection velocities of the starting structures for diminishing values of $BR_h$.

More details on the starting structures are gathered using tomographic phase-locked reactive Mie scattering visualization and numerical simulations. Figure 3.30 shows a comparison between those results for Case X at $St_{\infty} = 0.016$. Overall, the experimental and numerical results agree well and confirm the dynamics described previously. The $\Delta P$ iso-surfaces in figure 3.30(b) in the simulations evidence the formation of the downstream rollup to the primary starting vortex (magenta arrow) which, conversely to the upstream rollup, retains its coherence throughout the transient, to ultimately form a hairpin vortex in figure 3.30(f'). The experimental tomographic visualizations emphasize the secondary positive rollup occurring in figure 3.30(e, f) (cyan arrow) although the Laplacian of the pressure iso-surfaces show this structure earlier in 3.30(d'). The mutual induction between the downstream rollup of the leading starting vortex and the secondary shear layer vortex results in increased vertical velocity for the former and inversely decreased vertical velocity for the latter. Therefore in figure 3.30(d'), the starting vortex rollup is well below the secondary rollup while in figure 3.30(f'), they are approximately at the same height. Unfortunately, in the simulations as well, the upstream part of the primary ring vortex (orange arrow) is rapidly dissipated which prevents any observation of the folding mechanism. The starting vortex formation sequence for Case XI with a higher value for $BR_h$, 

Figure 3.29: Author's representation of the primary vortex folding mechanism.
presented in figure 3.31, evidences a more coherent primary starting vortex ring 3.31(a) (magenta arrow), being folded as it is convected. However, the folding mechanism demonstrated in the simulations does not appear to fit well the experimental observations of figure 3.27, since the upstream part of the primary vortex ring does not gain enough upward momentum to make the vortex ring rotate significantly towards the downstream direction. Such discrepancy is attributed to the quality and density of the grid above the jet exit, which can have a strong effect on the dynamics of the upstream part of the primary vortex. Nevertheless, the simulations provide additional insights on the development of wall structures directly downstream of the starting system (red arrow in figure 3.31e). Such pair of vortex tubes is comparable to the wake vortices formed in the forced vertical jet configuration when a starting hairpin vortex was generated as in figure 2.30(a). Here, the wall structures eventually evolve into large scale X-patterned structures on each side of the jet, once again in agreement with the dynamics of starting hairpin vortices in the vertical configuration.

**Single Compound Starting Vortex System.**

Similarly to the starting vortex regime, shorter injection times lead to the development of a single set of compound starting structures. For example, the $Y_h = 0$ visualizations in figure 3.32 for Case III show somewhat modified dynamics for the primary starting vortex when compared to figure 3.26 from Case IV at identical forcing condition with exception of the duty cycle (i.e. injection time). In Case III, as the upstream rollup of the primary starting vortex departs the jet exit leading edge, the actuation valve is closed. This generates an instantaneous mass deficit and cross-flow ingestion inside the jet pipe. This is evidenced in the PIV mappings of figure 3.21(f), which show a significantly larger and stronger region of negative vertical velocity compared to figure 3.21(e). Consequently, the vortex ring follows a trajectory closer to the jet exit, and its circulation is decreased as shown in the PIV results of figure 3.34(c), compared to equivalent results for Case IV in figure 3.28. The PIV results for Case III also show strong vorticity at the trailing edge, corresponding to the downstream rollup of the primary starting vortex ring, though weaker than the equivalent one in Case IV. This is explained in the vertical velocity mappings showing much lower velocity amplitudes at the trailing edge in Case III than in Case IV, since the valve is closed before the flow has time to develop. This in turns results in lower generated shear and circulation. Although weaker, the primary starting vortex exhibits dynamics which are comparable to the ones observed in Case IV and a secondary hairpin vortex develops downstream of the primary structure from natural shear layer vorticity (figure 3.34d, e). The two pairs of counter rotating vortices in the $X_j = c_{st}$ slices of figure 3.33 evidence an identical folding mechanism for the primary vortex ring to the one described in the previous section represented in figure 3.29.

**Starting Vortices Classification**

The different types of starting structures are mapped against two sets of potential scaling parameters $(SR_h, BR_h)$ and $(SR_{pp}, BR_{pp})$ and the results presented in figure 3.35. As for the vertical jet, both sets of scaling parameters provide satisfying and very similar maps. Using the $(SR_h, BR_h)$ scaling in figure 3.35(a), starting hairpin vortices are formed for $BR_h$ values below 0.33, starting ring vortices are formed for $BR_h > 0.68$ and compound vortices are generated at intermediate values. The transition stroke ratio $(SR_h)_{tr}$ is found to decrease consistently with decreasing values of $BR_h$. A formation number (asymptotic value of $(SR_h)_{tr}$) of $F_0 = 2.0$ is identified, which is significantly lower than for the vertical jet. The $(SR_{pp}, BR_{pp})$ classification in figure 3.35(b) shows identical shaped regions for the different starting structures. Overall the threshold values are so that starting hairpin vortices are formed for $BR_{pp} < 0.18$, starting ring vortices for $BR_{pp} > 0.49$, and compound starting vortices are formed at intermediate values. The transition stroke number $(SR_{pp})_{tr}$, is also found to decrease with decreasing values of $BR_{pp}$ and the formation number is found at approximately $F_0 = 1.75$, which is also lower than the vertical jet value. This value is however very consistent with the one found in the $(SR_h, BR_h)$ mappings. Such decrease in the formation number is explained by the geometrical configuration of the inclined jet which leads to higher coflowing components of the velocity between the jet and the cross-flow compared to the vertical jet. Indeed it was shown in Krueger et al. (2006) that coflowing flows led to decreased values of the formation number compared to jets in quiescent environments.

As for the vertical jet, rates of generation of the circulation at both the leading and trailing edges are evaluated, this time based on PIV measurements. The leading edge and trailing edge domains are taken
Figure 3.30: Comparative tomographic visualizations of the starting structure for Case X at $St_\infty = 0.016$ from (a-f) Phase locked reactive Mie scattering and (a’-f’) Corresponding LES simulations with $\Delta P$ isosurfaces materializing the starting structures. Time stamps are going from $t^* = DC + 0.995$ to $t^* = 0.095$ by increments of $\Delta t^* = 0.01$. 

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to be the areas of the PIV domain, respectively \( \Omega_1 \) above and \( \Omega_2 \) below the dashed line in figure 3.20 \((Y_j = \frac{2.9}{6}X_j + 0.2)\) which was found to accommodate all the considered cases. The results are scaled using the experimentally determined values of \( BR_h \) and \( BR_{pp} \) and presented in figure 3.36. It appears clearly that the scaled results using \( U_h \) causes the circulation trends on both domain to collapse nicely onto a single line. On the other hand, the trends scaled using \( U_{pp} \) only seem to partly regroup cases with equivalent \( BR_{pp} \) values such as Cases VI, XIII, XVI and Cases XII, XV. Based on these considerations, \((SR_h, BR_h)\) are found to be a better set of scaling parameters.

### 3.3.3 High-to-Low Blowing Ratio Transition: Cross-flow Ingestion

At the transition from \( BR_h \) to \( BR_l \), the rapid decrease in flow rate is accompanied by an ingestion of cross-flow fluid at the jet inlet leading edge, observed in both experiments (figure 3.14e) and simulations (figure 3.14e'). In figure 3.38, the temperature fields for both Case X and XI at \( St_{\infty} = 0.016 \) are shown at \( t^* = DC + 0.03 \) (30 ms after the transition from \( BR_h \) to \( BR_l \)). In both cases, the flow separates inside the jet pipe and cross-flow fluid is ingested at the upstream edge of the jet and the horseshoe vortex is convected downstream (figure 3.37f and 3.37m) as the adverse pressure gradient caused by the presence of the jet vanishes. The ingestion is more consequent in Case XI with a greater \( BR_h/BR_{pp} \) value since the mass deficit at the closing of the solenoid valve is more significant. In figure 3.39(e) corresponding to a higher forcing frequency, ingestion is also observed at the leading edge of the jet exit (arrow head) and the disturbances caused by the flow separation inside the jet pipe appear to persist throughout the low part of the cycle, as evidenced by cross-flow ingestion in 3.39(d), shortly before the next cycle starts. Conversely to the vertical jet, no particular quasi-stationary vortical structures appear to develop after the flow separation and all transients are directly evacuated. This can be explained by the difference in pressure gradient across the jet exit between both configurations in the same way the inner vortex was not formed in unforced conditions. Only a single ingestion mode corresponding to the leading edge ingestion is observed, even in high \( BR_h \) cases.

The velocity mappings of figure 3.21 show the strong effect of ingestion on the jet exit velocity profiles. This is also supported by the low seed particles content in the visualizations following the jet transition from \( BR_h \) to \( BR_l \) in figure 3.33 and 3.23, showing a strong film disruption. The redeveloping jet flow following the ingestion generates a multitude of streamwise wall structures seen between \( t^* = DC + 0.023 \).
Figure 3.32: Time resolved Mie scattering visualizations in the plane $Y_j = 0$ for Case III at $St = 0.008$ at multiple time instants. Dashed lines corresponds to $X_j = cst$ visualizations planes for figure 3.33.
and $t^* = DC + 0.048$, visible in figure 3.33 for Case III as well as figure 3.23 for Case V. It appears when comparing the $X_j = cst$ slices from those two cases, that the one with the lowest value of $BR_{pp}$ recovers faster, judging from the film homogeneity at $t^* = DC + 0.048$.

### 3.3.4 Film Cooling Performance

The impact of the starting structures on the temperature field is investigated using LES results for Case X and Case XI at $St_\infty = 0.016$. In figure 3.14 instantaneous wall adiabatic effectiveness and Laplacian of the pressure iso-surfaces are presented. For Case X in figure 3.14(b-d), an increase in spread around the jet exit is observed at the onset of the high blowing ratio, due to jet fluid exiting on the side of the jet exit. The coverage is also affected by draft induced by the secondary starting vortices and the associated cross-flow fluid entrainment. A ‘pinch’ in the coverage develops near the secondary starting structures legs as they are convected downstream. In figure 3.14(e), the jet behaves in a quasi-unforced manner. For Case XI corresponding to a higher value of $BR_{h}$, although a relatively low increase in the spread can be found in figure 3.14(j-l), the cooling performance of the jet is significantly degraded as the jet lifts partially off the wall shortly after the transition to $BR_{h}$.

Figure 3.40 shows similar views as those presented in figure 3.37 from simulations at a higher forcing frequency of $St_\infty = 0.159$. Instantaneous coverage is brought to the wall essentially by the secondary starting vortices (tailless arrows) as they are convected downstream. In figure 3.40(a-d) for Case X, the coverage beyond $X_j = 6$ is marginal as the starting structures consistently lift off the wall at this level. For Case XI in figure 3.40(e-h) with a higher value of $BR_{h}$, the average coverage breakup point is even closer to the jet.
Figure 3.34: Phase averaged PIV measurements in the plane $Y_j = 0$ for Case III at $St = 0.008$ at multiple time instants. Spanwise vorticity field (left); jet exit vertical velocity (center); velocity magnitude and vector field (right). Time instants corresponding to figure 3.32.
Figure 3.35: Classification of the starting vortices observed in forced experiments against characteristic stroke and blowing ratios (a) $SR_h$, $BR_h$; (b) $SR_{pp}$, $BR_{pp}$. ◯ Starting vortex ring and trailing column; ● Single vortex ring; ▽ Limiting case single/multiple starting structures; □ Leading hairpin vortex and trailing hairpins; ■ Single hairpin vortex; ▼ Limiting case hairpin vortex with/without trailing vortices. Transition stroke number fit $(SR_x)_{tr} = F_0 - A_1 \exp(-A_2.BR_x)$ (a) $(SR_h)_{tr}$: $F_0 = 2.0$, $A_1 = 1.8$, $A_2 = 1.82$; (b) $(SR_{pp})_{tr}$: $F_0 = 1.75$, $A_1 = 1.75$, $A_2 = 1.7$.

Figure 3.36: Total 2D circulation from PIV measurements, scaled using (a) $U_h$ and (b) $U_{pp}$ for Case VI (●), Case VIII (◇), Case XII (□), Case XIII (●), Case XIV (○), Case XV (●).
Figure 3.37: Instantaneous wall adiabatic effectiveness and Laplacian of the pressure iso-surfaces from LES for (a-h) Case X; (i-p) Case XI (bottom) at $St_\infty = 0.016$ at $t^* = DC + 0.49, 0.06, 0.12, 0.21, 0.34, DC + 0.03, DC + 0.014$ and $DC + 0.34$.

Figure 3.38: Temperature field and 2D U-W streamlines for (a) Case X; (b) Case XI at $t^* = DC + 0.03$.

Figure 3.39: Reactive Mie scattering visualizations for Case X at $St_\infty = 0.08$ at (a) $t^* = 0.13$; (b) $t^* = 0.43$; (c) $t^* = DC + 0.07$; (d) $t^* = DC + 0.37$. 
exit, about \( X_j = 4 \), as the starting structures possess more strength and vertical momentum compared to the previous case, thus lifting off earlier. In both cases, large X-patterned structures from the combination of side vortices and wake vortices associated with the leading starting vortex velocity field, are formed downstream of the starting structures, and while the upper legs of the ‘X’ appear to affect negatively the coverage as the velocity field they generate near the wall is oriented toward the symmetry plane, the lower legs of the ‘X’ with opposite vorticity tend to favor the spread, hence the “rosary-like” pattern observed at the wall from the succession of pinch/spread.

Although the ingestion causes a disturbance and a disruption of coolant supply for a short amount of time, it significantly affects the coverage during most of the low part of the cycle. Indeed, in figure 3.14(f) and 3.14(f’), even 140\( ms \left( t^* = DC + 0.16 \right) \) after the transition from \( BR_h \) to \( BR_h \), the jet has not recovered from the ingestion and a clear coverage breakup is visible around \( X_j = 4 \). In figure 3.37(f, g) for Case X, the effect of the ingestion is visible mainly in the near-field of the jet exit, while in figure 3.37(h), 435\( ms \left( t^* = DC + 0.45 \right) \) after the transition, the values of adiabatic effectiveness and spread beyond \( X_j = 7 \) are still low compared to the one immediately before the jet onset (figure 3.37a) or the corresponding unforced jet at \( BR = 0.15 \). Figure 3.37(m-p) shows results for Case XI after the valve closing, where coverage breakup is even more significant when comparing with Case X at identical phase positions. While the wall coverage is still redeveloping when the next pulse is triggered, vortical structures formed in the jet shear layers are of the hairpin type and consistent with the corresponding unforced jet at \( BR = 0.15 \).

As for the unforced jet, quantitative information was extracted from the LES simulations to assess the instantaneous as well as average performance of the forced jet. Figure 3.41 presents time averaged \( \eta_{span} \) and \( \eta_{centerline} \) values for Case X and XI, along with relevant unforced jet trends at constant low, high and average blowing ratios. For Case X, the span-wise average effectiveness trends (figure 3.41a) show that the forced cases at \( St_{\infty} = 0.016 \) and \( St_{\infty} = 0.079 \) have performance comparable to the case at \( BR = BR_h = 0.45 \) with yet an average mass-flow rate 1.5 times lower. These observations are consistent with the findings of Ekkad et al. (2006). On the other hand, the center-line adiabatic effectiveness (figure 3.41b) appears to be greatly affected by jet forcing and decreases consistently with increasing forcing frequency.

The case at \( St_{\infty} = 0.016 \) still shows reasonable center-line performance, though inferior to the compared unforced jet values while the two trends at \( St_{\infty} = 0.079 \) and \( St_{\infty} = 0.159 \) exhibit significantly degraded \( \eta_{centerline} \) values. Directly downstream of the jet exit, the case at \( St_{\infty} = 0.159 \) shows improvement over the other forced cases and the unforced jet at \( BR = BR_h = 0.45 \). For \( \eta_{span} \) as for \( \eta_{centerline} \), increasing the forcing frequency has a negative impact on the jet performance. The combination of both \( \eta_{centerline} \) and \( \eta_{span} \) suggests an increase in spread for the forced cases with overall more homogeneous wall adiabatic effectiveness values over the unforced jet. In figure 3.41(c) for Case XI, although no improvement in \( \eta_{span} \) is found in forced cases over the \( BR = BR_h \) and \( BR = BR_m \) unforced jets, forcing the jet at \( St_{\infty} = 0.016 \) provided overall higher \( \eta_{span} \) values than the one at \( BR = BR_h \) particularly for \( X_j < 6 \). At \( St_{\infty} = 0.159 \) higher \( \eta_{span} \) values were achieved for \( X_j < 4 \) over the case at \( BR = BR_h \) with lower values downstream of this point. Concerning the center-line adiabatic effectiveness in figure 3.41(d), no improvements are observed.
in forced conditions over the unforced cases at \( BR = BR_t \) and \( BR = BR_{m} \) but higher \( \eta_{\text{centerline}} \) values are achieved over the case \( BR = BR_0 \) for \( St_{\infty} = 0.016 \) and \( St_{\infty} = 0.159 \) for respectively \( X_j < 4 \) and \( X_j < 2.5 \). The phase averaged temporal evolutions of the coverage were established for Case X at \( St_{\infty} = 0.016 \) and \( St_{\infty} = 0.159 \) and are shown in figure 3.15(a) and 3.15(b) respectively. In these figures, a certain delay between the values of \( \eta_{\text{area}} \) with respect to the instantaneous blowing ratio is expected since the wall values do not respond immediately to events occurring at the jet exit. The lower forcing frequency case clearly shows the negative impact of the transition from \( BR_h \) to \( BR_l \) on the value of \( \eta_{\text{area}} \), due to the ingestion of cross-flow fluid at the jet exit and the momentary disruption of coolant. As mentioned previously, while commenting figure 3.37, even though this event occurs at \( t^* = 0.50 \), the coverage is degraded all along the low part of the cycle and increases again only after the onset of the high part of the following cycle. A stall in the progression of the coverage during the high part of the cycle is observed around \( t^* = 0.45 \), as the jet enters the quasi-unforced regime mentioned earlier. The amplitude of the variations during a cycle is significant when compared to the unforced jet values (see figure 3.13a). The averaged maximum \( \eta_{\text{area}} \) value over a cycle reaches 0.193 (with instantaneous peaks at above 0.20), which is virtually identical to the maximum value found in unforced conditions at \( BR = 0.3 \) (\( \eta_{\text{area}} = 0.196 \)). The relative variations of the coverage coefficients \( (\tilde{C}_c = (C_c - \bar{C}_c)/\bar{C}_c) \) are provided in figure 3.15(a). The evolution of the coverage coefficient during the cycle is similar to the evolution of \( \eta_{\text{area}} \) with decrease in the low part of the cycle and increase in the high part. The trends of the coverage coefficient for the higher values of \( \eta \) respond faster to the changes in blowing ratio since the regions of higher effectiveness are usually closer to the jet exit. This can also be observed in the coverage coefficient values of figure 2.44(a) with coverage peaks shifted in time as the threshold is decreased. Jet forcing introduces greater relative variation of the coverage coefficient related to higher values of the adiabatic effectiveness. Indeed, while \( C_c(\eta = 0.1) \) has a relative standard deviation of 9% over a cycle, the same quantity increases to 20% for \( C_c(\eta = 0.5) \) and 28% for \( C_c(\eta = 0.75) \). Although the impact of each individual regime can be observed for the forced case at \( St_{\infty} = 0.016 \), the shorter time scales involved at \( St_{\infty} = 0.159 \) make this impossible since multiple cycles effectively affect the instantaneous wall temperature field. This explains the phase shifts in the variations of \( C_c \) at different \( \eta \) thresholds. Overall the coverage appears relatively constant over a cycle and the relative standard deviation for \( \eta_{\text{area}} \) is only 0.9% (compared to 9% at \( St_{\infty} = 0.016 \)) and the relative standard deviation for \( C_c(\eta = 0.1) \) and \( C_c(\eta = 0.5) \) are respectively 0.6% and 4.3%.

Global performance for the forced jet is summarized in table 3.4 along with relevant unforced jet performance. In all cases, forcing leads to decreased performance over the equivalent unforced jet at constant mean blowing ratio \( (BR = BR_0) \). However, the forced cases at \( St_{\infty} = 0.016 \) performed systematically better than the equivalent unforced jets at fixed high blowing ratio \( (BR = BR_{h}) \). At fixed \( BR_{h}, BR_l, DC \), the jets forced at the highest forcing frequency \( (St_{\infty} = 0.159) \) had systematically the lowest performance.
Table 3.4: Coverage coefficient and area averaged adiabatic effectiveness from LES under forced conditions compared to equivalent mass flow rate unforced cases. Values for the unforced jet are interpolated from figure 3.13.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>$\eta = 0.2$</th>
<th>$\eta = 0.3$</th>
<th>$\eta = 0.5$</th>
<th>$\eta_{\text{area}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$BR = BR_{m} = 0.3 - St_{\infty} = 0$</td>
<td>26.0</td>
<td>21.4</td>
<td>11.9</td>
<td>0.197</td>
</tr>
<tr>
<td>$BR = BR_{h} = 0.45 - St_{\infty} = 0$</td>
<td>22.2</td>
<td>16.8</td>
<td>7.29</td>
<td>0.152</td>
</tr>
<tr>
<td>Case XI - $St_{\infty} = 0.016$</td>
<td>25.5</td>
<td>17.9</td>
<td>8.96</td>
<td>0.168</td>
</tr>
<tr>
<td>Case XI - $St_{\infty} = 0.159$</td>
<td>19.9</td>
<td>10.9</td>
<td>5.31</td>
<td>0.131</td>
</tr>
<tr>
<td>$BR = BR_{h} = 0.75 - St_{\infty} = 0$</td>
<td>18.3</td>
<td>13.0</td>
<td>4.43</td>
<td>0.109</td>
</tr>
<tr>
<td>Case XI - $St_{\infty} = 0.016$</td>
<td>20.3</td>
<td>13.0</td>
<td>4.42</td>
<td>0.129</td>
</tr>
<tr>
<td>Case XI - $St_{\infty} = 0.159$</td>
<td>10.4</td>
<td>6.47</td>
<td>3.07</td>
<td>0.091</td>
</tr>
</tbody>
</table>

3.4 Conclusion

An unforced inclined jet study was conducted with blowing ratios ranging from 0.150 to 1.2 to obtain a baseline both in terms of performance and vortical structures dynamics for the forced jet results. Experimental visualizations and Large Eddy Simulations were carried out in parallel and compared to gain more information on the flow and temperature fields. Particular attention was brought to the attached jet configuration ($BR < 0.4$), more relevant to film cooling, and the principal vortical structures as well as their impact on the wall temperature field. Overall, the dominant vortical structures found in the 35° jet were found to have strong similarities with the 90° jet configuration. One exception was found with the absence of the inner vortex from the the inclined system due to the lower pressure gradient across the jet/cross-flow interface at the leading edge. For the same reasons, the horseshoe vortex was formed at higher blowing ratios in the inclined jet and was relatively smaller when compared to its vertical jet counterpart at equivalent $BR$ values. Due to the absence of the inner vortex, no destabilization of the horseshoe vortex was observed for the inclined jet and the structure remained attached to the wall throughout the investigated range. The dynamics of hairpin vortices and side vortices were found to be almost identical to their equivalent in the vertical jet configuration. The transitional regime for the inclined jet was defined by the coexistence of positive and negative spanwise vorticity in the upper shear layer, and did not exhibit changes in flow dynamics as dramatic as the ones observed in the vertical jet. Once again because of the absence of inner vortex. The transition from attached to detached jet regime occurred therefore more smoothly with a progressive lift off the wall of the different vortical structures as the blowing ratio was increased. The detached inclined jet exhibited vortical structures comparable to the well documented vertical detached jet configuration, including wake vortices and shear layer rollups of negative vorticity in the upper shear layer and of opposite sign in the lower one. On one hand, the velocity field associated with the hairpin vortices was found to have a rather negative impact on wall temperature, as they entrain warm cross-flow fluid near the wall. On the other hand, the side vortices were found to have a rather positive impact on the film spread, particularly beyond $X_{j} = 7$. Film cooling performance for the unforced jet were evaluated based on LES results and the best performing inclined jet was found to be at a blowing ratio of 0.3. Interestingly, the comparison with the vertical jet performance showed that the overall best performing jet was a vertical jet at $BR = 0.415$ due to significantly greater spread. This result was however found to be in agreement with previous studies and explained by the relatively restricted domain in the streamwise direction considered for this study.

A range of actuated jets using a square wave excitation was also investigated. In forced conditions, a maximum of four distinct phases were observed over a forcing cycle, comparable to the one described in the vertical jet configuration. Two transient regimes initiated by the low-to-high and high-to-low blowing ratio transitions and two quasi-unforced regimes established during the high and low part of the cycle as the transients are evacuated. The transition from low to high blowing ratio triggered the formation of starting vortical structures due to increased shear at the jet/cross-flow interface. Three distinct starting structures were found in the inclined jet configuration, compared to only two for the vertical setup. At the lower end of the $BR_{h}$ range, the starting vortices were of the hairpin type and their dynamics were quite similar to those of comparable structures in unforced conditions. At the higher end of the $BR_{h}$ range, the starting...
vortices were of the ring type with modified dynamics compared to the corresponding vertical jet ones due to the inclined nature of the jet and the time delay it causes in the formation process of the different parts of the starting structures. Consequently, the leading ring vortex was found to rotate continuously in the positive direction in the early part of the cycle and reach a stable inclination at approximately 60°, as it was carried away by the free stream. Cases with long injection times evidenced the formation of trailing ring vortices with dynamics comparable to the one of the leading vortex ring. Shorter injection times led to the formation of a single starting vortex ring, which was strongly affected at the jet shutdown by the counter flow or ingestion it generated. Hence, the starting vortex rotation, initially towards the downstream direction, was eventually reverted towards the upstream direction and the convected structure lost its coherence very rapidly. The third type of starting vortices, exclusive to the inclined jet setup, was found to develop at intermediate values of BR. Composed of two distinct structures: a primary vortex ring generated at the jet onset and a secondary hairpin vortex formed later on, this type of starting vortex system was referred to as compound starting vortex system. Additionally, a folding mechanism for the primary vortex ring was observed in both experimental visualizations and numerical simulations. The upstream rollup of the vortex ring with negative vorticity was rapidly dissipated by the overall positive vorticity present in the surrounding shear layers, so that ultimately the compound starting vortex system evolved into large scale hairpin vortices and associated X-patterned structures. As for the other two types of starting vortices, longer injection times led to the generation of multiple starting structures, while shorter ones led to the formation of a single compound starting vortex system. For the single starting ring vortex, the upstream part of the primary starting vortex ring was found to be strongly affected by the cross-flow ingestion induced at the valve closing, leading to a trajectory closer to the jet exit. Eventually though, the single compound starting vortex system was still subjected to a folding mechanism comparable to the one of compound starting vortices and trailing vortices and was found to also evolve into a large hairpin vortex further downstream. Evaluations from PIV measurements of the rate of generation of the circulation at both leading and trailing edges showed that the set (SRb, BRh) provided better scaling then the set (SRpp, BRpp), which is in agreement with the vertical jet results. Maps of the vortical structures against values of SR and BR showed that hairpin vortices were formed for values of BR below 0.33, while starting ring vortices were formed for BR values above 0.68. The asymptotic formation number was found to be approximately 2.0, which is twice as low as the corresponding vertical jet parameter. At the transition from high to low blowing ratio, an ingestion of cross-flow fluid inside the jet feeding pipe was observed in most of the investigated cases due to the mass-flow deficit introduced by the sharp change in blowing ratio. This also triggered flow separation inside the jet pipe though no particular quasi-stationary vortical system was found to develop, conversely to the vertical jet configuration. In addition, only a single, leading edge, ingestion mode was observed in the inclined configuration, which constitutes another difference with the vertical one.

In terms of film cooling performance, the starting structures had a tendency to increase the spread in the jet exit region while also promoting mixing directly downstream of themselves due to strong cross-flow entrainment. At the jet pulse shutdown, the cross-flow ingestion significantly impacted the film cooling performance during the low part of the cycle. Time averaged measurements of coverage and adiabatic effectiveness show that a pulsed jet can provide increased coverage over some of the comparable unforced jet counter-parts, such as the one at constant pressure supply (BR = BRh), but not over the one at constant mass flow (BR = BR). The phase averaged coverage and area averaged adiabatic effectiveness show that the transition from BRh to BRl is mainly to blame for the poor performance. Overall, higher forcing frequencies led to poorer performance due to the domination of the transient features during the shorter cycles.

It appears now rather clear how previous studies from Ekkad et al. (2006) and Coulthard et al. (2007b) (or Womack et al. 2008a which has operating conditions identical to those of Coulthard et al. 2007b) could have provided such disparities in their results. First of all, while they were carried out at comparable blowing ratios and forcing frequencies, the corresponding scaled parameters, using the scaling defined in the current study, show that both studies actually did not investigate the jet in the same conditions. Moreover, when using the current study delimitation of the (SRh, BRh) map, it is rather clear that the sampling of the different starting vortex regimes in Ekkad et al. (2006) is strongly biased towards the starting ring vortex and trailing column regime, and only one value of BRh is found in the compound starting vortices and trailing structures region and only a single data point in the starting hairpin vortex and trailing structures regime. None of these covers any of the single starting structure regimes, therefore only three out of the six.
Figure 3.42: Scaled operating points for previous forced inclined jet studies from Ekkad et al. (2006), Coulthard et al. (2007b), and Womack et al. (2008a) based on the scaling parameters identified in the current work.

Possible jet regimes are investigated. In Coulthard et al. (2007b), the sampling bias is a little less pronounced and the data sets are covering four of the six regimes. However, only one data point is found in the single starting vortex ring and compound starting vortex system regions, which is expected to be a more favorable regime. In addition, only one value of $BR_h$ is investigated in each region, which makes the sampling of those particular regimes very localized. None of the data sets of Coulthard et al. (2007b) or Womack et al. (2008a) are covering the starting hairpin vortex region.

Finally, because of the differences in setup used in both studies, it is very likely the physics associated with these jets were significantly different. While Coulthard et al. (2007b) used a setup similar to the one of the current study, Ekkad et al. (2006) used an inclined jet with a shallower angle of 20° with respect to the streamwise direction, but also a compound angle of 90°. Given the considerable impact the introduction of compound angles can have on unforced jets (see McGovern & Leylek 2000), it is strongly believed it will also affect the formation mechanisms and dynamics of the starting structures.
Chapter 4
POD-Galerkin Reduced Order Models of Canonical Flows

In this chapter, two-dimensional canonical flows, a two-sided lid driven cavity and a cylinder in cross-flow, are investigated. This is used to develop the necessary numerical tools and understanding of the POD-Galerkin model reduction method.

4.1 POD-Galerkin Method

The POD-Galerkin method is a combination of two operations providing a set of ordinary differential equations approximating the flow and temperature field empirical solutions. The following section provides a short description of both the Proper Orthogonal Decomposition (POD) and the Galerkin projection method and sets the definitions and notations used in the remaining of this chapter.

4.1.1 Proper Orthogonal Decomposition

Formulation

The principle of the Proper Orthogonal Decomposition (POD) is rather simple and can be described following the definitions in Holmes et al. (1998). Given a set of scalar fields \( \{ u_k \} \) each being a function of the type \( u_k = u_k(x) \), \( x \in \Omega_x \) where \( u \) belongs to the space of linear infinite-dimensional Hilbert space \( L^2(\Omega_x) \) of square integrable complex functions with inner product \( (.,.) \):

\[
(f, g) = \int_{\Omega_x} f(x)g(x^*)dx
\] (4.1)

We would like to identify a set of orthonormal basis functions \( \{ \varphi_j(x) \}_{j=1}^{\infty} \) in \( L^2(\Omega_x) \) which provides the best finite representation of

\[
u_N(x) = \sum_{j=1}^{N} a_j \varphi_j(x)
\] (4.2)

in the \( L^2 \) norm sense compared to any other set of functions. This can be expressed for one function \( \varphi \), by maximizing the average projection of \( u \) onto \( \varphi \):

\[
\max_{\varphi \in L^2} \frac{\langle |(u, \varphi)|^2 \rangle}{\|\varphi\|^2}
\] (4.3)

where \( \langle . \rangle \) denotes the temporal average, \( |.| \) denotes the modulus and \( \| . \| \) the norm on \( L^2(\Omega_x) \) so that \( \| f \| = (f, f)^{\frac{1}{2}} \). The problem in equation 4.3 is a problem in the calculus of variations, which consists in maximizing \( \langle |(u, \varphi)|^2 \rangle \) while imposing \( \| \varphi \| = 1 \) which is described by the functional \( I [\varphi] \):

\[
\left\{ \begin{array}{l}
I [\varphi] = \langle |(u, \varphi)|^2 \rangle - \lambda \left( \| \varphi \|^2 - 1 \right) \\
\frac{d}{d\delta} (I [\varphi + \delta \psi]) |_{\delta=0} = 0, \delta \in \mathbb{R}
\end{array} \right.
\] (4.4)

The second expression being for extrema condition. Equation 4.4 yields then

\[
\int_{\Omega_x} \left[ \int_{\Omega_x} \langle u(x)u^*(x') \rangle \varphi(x')dx' - \lambda \varphi(x) \right] \psi^*(x)dx = 0
\]
where \( \psi \) is an arbitrary function, which can be reduced to the eigenvalue problem:

\[
\int_{\Omega_x} R(x,x') \varphi(x') dx' = \lambda \varphi(x) \tag{4.5}
\]

Where the kernel of equation 4.5 is the autocorrelation function \( R(x,x') = \langle u(x)u^*(x') \rangle \) and provides to the basis functions \( \{ \varphi_j \} \) their empirical character. Because of the linear character of the decomposition, the POD modes inherits all linear properties of the field such as boundary conditions, and particularly for velocity fields, incompressibility (\( \nabla \cdot \vec{u} = 0 \)). The reader is referred to Holmes et al. (1998) for more details on the theory behind proper orthogonal decomposition.

**Discretization of the Eigenvalue Problem**

In finite dimensional space, where the \( N_t \) observations \( \{ u_k \}_{k=1}^{N_t} \) are \( N_x \)-dimensional vectors, the autocorrelation function \( R(x,x') \) is equivalent to a \( N_x \times N_x \) tensor and the time average can be estimated by the finite arithmetic average:

\[
\langle u \rangle_T = 1/T \int_0^T u(t) dt \sim \frac{1}{N_t} \sum_{k=1}^{N_t} u_k
\]

so that equation 4.5 can be approximated by:

\[
\frac{1}{N_t} \sum_{k=1}^{N_t} u_k(x) \int_{\Omega_x} u_k^*(x') \varphi(x') dx' = \lambda \varphi(x) \tag{4.6}
\]

Similarly, approximating the integral using the trapez or Simpson’s rule gives:

\[
\int_{\Omega_x} u_k^*(x') \varphi(x') dx' \sim \sum_{i=1}^{N_x} u_k^*(x_i) \varphi(x_i) w_i
\]

Equation 4.6 becomes then:

\[
\frac{1}{N_t} \sum_{k=1}^{N_t} \left[ u_k(x) \sum_{i=1}^{N_x} u_k^*(x_i) \varphi(x_i) w_i \right] = \lambda \varphi(x) \tag{4.7}
\]

Multiplying equation 4.7 left and right by \( \sqrt{w_i} \) and defining \( \hat{u} \) and \( \hat{\varphi} \), by \( \hat{u} = \left[ \frac{u(x_1) \sqrt{w_1}}{u(x_2) \sqrt{w_2}} \ldots \frac{u(x_{N_x}) \sqrt{w_{N_x}}}{N_x} \right] \) and

\[
\hat{\varphi} = \left[ \begin{array}{c} \varphi(x_1) \sqrt{w_1} \\ \varphi(x_2) \sqrt{w_2} \\ \vdots \\ \varphi(x_{N_x}) \sqrt{w_{N_x}} \end{array} \right]_{N_x}
\]

equation 4.7 becomes

\[
\frac{1}{N_t} \sum_{k=1}^{N_t} \hat{u}_k \hat{\varphi} = \lambda \hat{\varphi} \quad \text{or, defining } C = \left[ \frac{1}{N_t} \sum_{k=1}^{N_t} \hat{u}_k \hat{\varphi} \right]_{N_x \times N_x}
\]

\[
C \hat{\varphi} = \lambda \hat{\varphi} \tag{4.8}
\]

This method usually referred to as the direct method, was initially introduced by Lumley (1970) and involves the computation of a \( N_x \times N_x \) eigenvalue problem. This can be acceptable for two-dimensional vector fields (\( N_x \approx 10^4 \)), in which case the problem size is \( 2N_x \times 2N_x \approx 10^8 \), but can be extremely problematic for three-dimensional flows in which \( N_x \) is usually of the order of \( 10^6 \) and the size of the eigenproblem becomes \( 3N_x \times 3N_x \approx 10^{13} \). Such large scale problems are common in computational fluid dynamics and require the reformulation of the eigenvalue problem to a lower order equation. Sirovich (1987) proposed an alternative formulation called the snapshot method where the problem is reduced to a \( N_t \times N_x \) size problem by branching off of equation 4.6 and defining \( d_k = \int_{\Omega_x} u_k^*(x') \varphi(x') dx' \) so that equation 4.6 can be expressed as:
\[
\frac{1}{N_t} \sum_{k=1}^{N_t} d_k u_k(x) = \lambda \varphi(x) \tag{4.9}
\]

Multiplying both sides of equation 4.9 by \( u_i^* (x) \) and integrating over \( \Omega_x \) gives:

\[
\frac{1}{N_t} \sum_{k=1}^{N_t} d_k \int_{\Omega_x} u_i^* (x) u_k(x) dx = \lambda \int_{\Omega_x} u_i^* (x) \varphi(x) dx \tag{4.10}
\]

Defining \( e_{ij} = \int_{\Omega_x} u_i^* (x) u_j(x) dx \), equation 4.10 can be written as \( E d = \lambda d \), where \( E = [e_{ij}]_{N_t \times N_t} \), and \( d = [d_k]_{N_t} \) are respectively \( N_t \times N_t \) tensor, and \( N_t \)-dimensional vector. Once the eigenvalues \( \lambda_n \) and eigenvectors \( d^{(n)} = [d_1^{(n)}, d_2^{(n)}, \ldots, d_{N_t}^{(n)}]^T \) have been identified, the original POD modes and coefficients can easily be retrieved using equation 4.9, by projecting the empirical field \( \{ u_k \} \) onto the eigenvector basis \( \{ d^{(n)} \} \):

\[
\varphi^{(n)} (x) = \frac{1}{\lambda_n N_t} \sum_{k=1}^{N_t} u_k(x) d_k^{(n)} \tag{4.11}
\]

Or using matrix notations:

\[
\Phi = \frac{1}{N_t} U D \tag{4.12}
\]

\[
\Phi = \begin{bmatrix} \varphi^{(1)}(x) & \varphi^{(2)}(x) & \cdots & \varphi^{(N_t)}(x) \\ \varphi_1^{(1)} & \varphi_1^{(2)} & \cdots & \varphi_1^{(N_t)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \cdots & \varphi_2^{(N_t)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{N_t}^{(1)} & \varphi_{N_t}^{(2)} & \cdots & \varphi_{N_t}^{(N_t)} \end{bmatrix}_{N_t \times N_t} \tag{4.13}
\]

\[
D = \begin{bmatrix} d_1^{(1)}/\lambda_1 & d_1^{(2)}/\lambda_2 & \cdots & d_1^{(N_t)}/\lambda_{N_t} \\ d_2^{(1)}/\lambda_1 & d_2^{(2)}/\lambda_2 & \cdots & d_2^{(N_t)}/\lambda_{N_t} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N_t}^{(1)}/\lambda_1 & d_{N_t}^{(2)}/\lambda_2 & \cdots & d_{N_t}^{(N_t)}/\lambda_{N_t} \end{bmatrix}_{N_t \times N_t} \tag{4.14}
\]

\[
U = [u_1, u_2, \ldots, u_{N_t}] = \begin{bmatrix} u_1^1 & u_1^2 & \cdots & u_1^{N_t} \\ u_2^1 & u_2^2 & \cdots & u_2^{N_t} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N_t}^1 & u_{N_t}^2 & \cdots & u_{N_t}^{N_t} \end{bmatrix}_{N_t \times N_t} \tag{4.15}
\]

If the POD basis \( \{ \Phi \} \) is not orthonormal, it is normalized into:

\[
\Phi = \begin{bmatrix} \varphi^{(1)}(x) & \varphi^{(2)}(x) & \cdots & \varphi^{(N_t)}(x) \\ \| \varphi^{(1)} \| & \| \varphi^{(2)} \| & \cdots & \| \varphi^{(N_t)} \| \end{bmatrix}_{N_t \times N_t} \tag{4.16}
\]

By definition, the \( a_i(t) \) are the coefficients of the projection of \( \tilde{u} \) on the POD basis \( \{ \varphi^{(n)} \} \) and can be obtained from the projection of the empirical field onto the POD basis:

\[
a_i(t_k) = \left( \tilde{u}_k^i(x), \varphi_i^j(x) \right) = \int_{\Omega_x} \tilde{u}_k^i(x') \varphi_i^j(x') dx' = \Phi^T U \tag{4.17}
\]

In addition from equation 4.7 and the orthonormality of \( \{ \varphi_i^j \} \), the \( a_i \) obey:
\[ \langle a_i a_j^* \rangle_T = \delta_{ij} \lambda_i, \quad \langle a_i \rangle_T = 0 \tag{4.18} \]

The resulting POD modes and eigenvalues from the snapshot method provide an approximation of the first \( N_t \) POD modes obtained using the direct method. In general, converged averages and POD modes for turbulent flows are obtained for \( N_T \sim 10^3 \), therefore using the snapshot method reduces considerably the computational expenses to obtain a POD basis. Considering the extent of the numerical domains investigated in this study (\( N_x = \mathcal{O}(10^5) \) to \( \mathcal{O}(10^6) \)), the snapshot method was preferred to obtain proper orthogonal decompositions.

4.1.2 Galerkin Projection

The Galerkin projection method applied to fluid flows consists in reducing the incompressible Navier-Stokes and energy equations (equations 4.22, 4.23 and 4.24) to a finite set of ordinary differential equations (ODE) by projecting them onto a finite-dimensional subspace such as the basis function obtained through proper orthogonal decomposition, although any other basis function can be used. The Navier-Stokes and energy equations are:

\[
\begin{align*}
\nabla \bar{u} &= 0 \tag{4.19} \\
\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} &= -\frac{1}{\rho} \nabla P + \nu \Delta \bar{u} \tag{4.20} \\
\frac{\partial T}{\partial t} + (\bar{u} \cdot \nabla) T &= \alpha \Delta T \tag{4.21}
\end{align*}
\]

These equations are often presented in their scaled form using:

\[
\bar{u}^* = \frac{\bar{u}}{\bar{U}}, \quad T^* = \frac{T - T_1}{T_2 - T_1}, \quad \rho^* = \frac{P}{\rho \bar{U}^2}, \quad (x^*, y^*) = \left( \frac{x, y}{L} \right), \quad t^* = \frac{t \bar{U}}{L}, \quad \text{Re} = \frac{UL}{\nu}, \quad \text{Pr} = \frac{\nu}{\alpha}
\]

\[
\nabla \bar{u}^* = 0 \tag{4.22} \\
\frac{\partial \bar{u}^*}{\partial t^*} + (\bar{u}^* \cdot \nabla) \bar{u}^* &= -\nabla P^* + \frac{1}{\text{Re}} \Delta \bar{u}^* \tag{4.23} \\
\frac{\partial T^*}{\partial t^*} + (\bar{u}^* \cdot \nabla) T^* &= \frac{1}{\text{Pr}} \frac{1}{\text{Re}} \Delta T^* \tag{4.24}
\]

where \( U, T_1, T_2, L \) are respectively characteristic velocity, temperature and length scales of the problem, and \( \text{Re} \) and \( \text{Pr} \), the Reynolds and Prandtl numbers. The \((. \)^* notation will be dropped, although the considered quantities remain the non-dimensional ones in the following.

For reasons that will be exposed in the coming sections, instead of considering the entire field, we will decompose the velocity and temperature fields into a time averaged part and a fluctuation part according to Reynolds notations:

\[
\begin{align*}
\bar{u} &= \bar{\bar{u}} + \bar{u}' \\
\bar{T} &= \bar{\bar{T}} + T' \\
\langle \bar{u}' \rangle &= 0 \\
\langle T' \rangle &= 0
\end{align*}
\]

The Navier-Stokes equations can then be expanded into:
\begin{align*}
\nabla \left( \bar{\vec{u}} + \vec{u}' \right) &= 0 \\
\frac{\partial \vec{u}'}{\partial t} + \left( \left( \bar{\vec{u}} + \vec{u}' \right) \cdot \nabla \right) \left( \bar{\vec{u}} + \vec{u}' \right) &= -\nabla P + \frac{1}{Re} \triangle \left( \bar{\vec{u}} + \vec{u}' \right) \\
\frac{\partial T'}{\partial t} + \left( \left( \bar{\vec{u}} + \vec{u}' \right) \cdot \nabla \right) \left( \bar{T} + T' \right) &= \frac{1}{PrRe} \triangle \left( \bar{T} + T' \right)
\end{align*}

(4.25) \quad (4.26) \quad (4.27)

It can easily be shown that by taking the time average of the continuity equation 4.25, the fluctuation field \( \vec{u}' \) is also divergence free.

To simplify the description of the projection the momentum and energy equations 4.26 and 4.27 will be expressed as:

\begin{align*}
\frac{\partial \vec{u}'}{\partial t} &= M \left( \vec{u}', \bar{\vec{u}} \right) \\
\frac{\partial T'}{\partial t} &= E \left( \vec{u}', \bar{\vec{u}}, \bar{T}, T' \right)
\end{align*}

(4.28) \quad (4.29)

where

\[ M \left( \vec{u}', \bar{\vec{u}} \right) = - \left( \left( \bar{\vec{u}} + \vec{u}' \right) \cdot \nabla \right) \left( \bar{\vec{u}} + \vec{u}' \right) - \nabla P + \frac{1}{Re} \triangle \left( \bar{\vec{u}} + \vec{u}' \right) \]

and

\[ E \left( \vec{u}', \bar{\vec{u}}, \bar{T}, T' \right) = - \left( \left( \bar{\vec{u}} + \vec{u}' \right) \cdot \nabla \right) \left( \bar{T} + T' \right) + \frac{1}{PrRe} \triangle \left( \bar{T} + T' \right) \]

The fluctuation fields \( \vec{u}' \) and \( T' \) are expanded using the POD notations, \( \vec{u}'(x,t) = \sum_{n=1}^{N_v} a_n(t) \vec{\varphi}_n(x) \) (\( \lambda_n \) are the associated eigenvalues) and \( T'(x,t) = \sum_{m=1}^{N_T} b_m(t) \psi_m(x) \) (\( \sigma_m \) are the associated eigenvalues) and can be replaced in equations 4.28 and 4.29 to give:

\begin{align*}
\frac{\partial}{\partial t} \left( \sum_{n=1}^{N_v} a_n(t) \vec{\varphi}_n(x) \right) &= M \left( \sum_{n=1}^{N_v} a_n(t) \vec{\varphi}_n(x), \bar{\vec{u}} \right) \\
\frac{\partial}{\partial t} \left( \sum_{m=1}^{N_T} b_m(t) \psi_m(x) \right) &= E \left( \sum_{n=1}^{N_v} a_n(t) \vec{\varphi}_n(x), \bar{\vec{u}}, \sum_{m=1}^{N_T} b_m(t) \psi_m(x), \bar{T} \right)
\end{align*}

(4.30) \quad (4.31)

The next step consists in projecting equations 4.30 and 4.31 on their respective POD basis, using the inner product defined on \( L^2(\Omega_x) \) in equation 4.1 with \( \vec{\varphi}_r \) and \( \psi_n \), and using the orthonormality of \( \{ \vec{\varphi}_r \} \) and \( \{ \psi_i \} \). The projection steps being quite fastidious, they have been left out of this section and relegated to the Appendix. The resulting reduced order model equations are then for the velocity field:

\[ a_r(t) = \sum_{i=1}^{N_v} \left( C_{i,0} + \frac{1}{Re} D_i \right) a_i(t) + \sum_{i=1}^{N_v} \sum_{j=1}^{i} C_{i,j} a_i(t) a_j(t) + \left( C_0^r + \frac{1}{Re} D_0^r \right) + P_r(t) \]

(4.32)

where
\[ \ddot{u} = \ddot{\bar{u}} + u'' = \bar{u} + \sum_{n=1}^{N_v} a_n(t) \varphi_n(x) \]

\[ C_0^r = - ((\bar{u} \nabla) \bar{u}, \varphi_r) = - \int_{\Omega_x} \bar{u}_l \frac{\partial}{\partial x_l} (\bar{u}_m) \varphi_{r,l} dx \]

\[ D_0^r = (\Delta \bar{u}, \varphi_r) = \int_{\Omega_x} \frac{\partial}{\partial x_m \partial x_m} (\bar{u}_l) \varphi_{r,l} dx \]

\[ C_{i,0}^r = - ((\bar{u} \nabla) \varphi_i, \varphi_r) - ((\varphi_i \nabla) \bar{u}, \varphi_r) = - \int_{\Omega_x} (\bar{u}_m \varphi_{i,l} + \varphi_{i,m} \frac{\partial}{\partial x_m} \bar{u}_l) \varphi_{r,l} dx \]

\[ D_i^r = (\Delta \varphi_i, \varphi_r) = \int_{\Omega_x} \frac{\partial}{\partial x_m \partial x_m} (\varphi_{i,l}) \varphi_{r,l} dx \]

\[ C_{i,j}^r = - ((\varphi_i \nabla) \varphi_j, \varphi_r) = - \int_{\Omega_x} (\varphi_{i,m} \varphi_{j,l} + \varphi_{j,m} \varphi_{i,l}) \varphi_{r,l} dx \]

\[ P_r(t) = - (\nabla P(t), \varphi_r) \]

and for the temperature field:

\[ \dot{b}_n(t) = \sum_{i=1}^{N_v} E_{s,i}^0 a_i(t) + \sum_{j=1}^{N_T} \left( E_{s,j}^0 + \frac{1}{Pr \text{Re}} F_{s,j} \right) b_j(t) + \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} E_{s,j} a_i(t) b_j(t) + \left( E_{s,0}^0 + \frac{1}{Pr \text{Re}} F_{s,0} \right) \]

where

\[ T = T + T' = T + \sum_{m=1}^{N_T} b_m(t) \psi_m(x) \]

\[ E_{s,0}^0 = - ((\bar{u} \nabla) T, \psi_s) = - \int_{\Omega_x} \bar{u}_l \frac{\partial}{\partial x_l} (T) \psi_s dx \]

\[ F_{s,0} = (\Delta T, \psi_s) = \int_{\Omega_x} \frac{\partial}{\partial x_m \partial x_m} (T) \psi_s dx \]

\[ E_{s,0}^i = - ((\varphi_i \nabla) T, \psi_s) = - \int_{\Omega_x} \varphi_{i,m} \frac{\partial}{\partial x_m} (T) \psi_s dx \]

\[ E_{s,j}^0 = - ((\bar{u} \nabla) \psi_j, \psi_s) = - \int_{\Omega_x} \bar{u}_l \frac{\partial}{\partial x_l} (\psi_j) \psi_s dx \]

\[ F_{s,j} = (\Delta \psi_j, \psi_s) = \int_{\Omega_x} \frac{\partial}{\partial x_m \partial x_m} (\psi_j) \psi_s dx \]

\[ E_{s,j}^i = - ((\varphi_i \nabla) \psi_j, \psi_s) = - \int_{\Omega_x} \varphi_{i,m} \frac{\partial}{\partial x_m} (\psi_j) \psi_s dx \]

In both equations, \( \bar{u}_m \) corresponds to the \( m \)-th component of the vector \( \bar{u} \) and \( \varphi_{r,l} \) corresponds to the \( l \)-th component of the vector \( \varphi_r \). Redundant indexes imply summation on those indexes according to Einstein notation.

Provided with initial values for \( A(t_0) = [a_1(t_0), a_2(t_0), \ldots, a_{N_v}(t_0)]^T \) and \( B(t_0) = [b_1(t_0), b_2(t_0), \ldots, b_{N_T}(t_0)]^T \) the set of ODEs can be integrated to obtain a prediction of the velocity and temperature field.

Both projected momentum and energy equations RHS are quadratic functions in terms of the coefficients \( a_i \) and \( b_j \). Similarly to the full incompressible Navier Stokes and energy equations, the equations are only one-way coupled. The velocity equations are self-sufficient, while the temperature solution depends on the velocity field. A few terms in equations 4.32 and 4.33 can be simplified if the velocity and temperature fields
are homogeneous on the domain boundaries \((u'|_{\partial \Omega} = 0, T'|_{\partial \Omega} = 0)\). Indeed, in most of the terms involving the Laplacian \((\Delta)\) operator, the coefficients can be reduced to first order derivatives by using integration by part:

\[
D_0^i = (\Delta \vec{\pi}, \varphi_r) = \int_{\Omega} \Delta (\vec{\pi}) \varphi_{r,i} dx = \left[ \nabla \vec{\pi}, \varphi_r \right]_{\partial \Omega} - \int_{\Omega} \nabla \vec{\pi} \cdot \nabla \varphi_{r,i} dx = 0
\]

\[
D^t_i = (\Delta \vec{\varphi}_{i}, \varphi_r) = \int_{\Omega} \Delta (\vec{\varphi}_{i,t}) \varphi_{r,i} dx = \left[ \nabla \vec{\varphi}_{i,t}, \varphi_r \right]_{\partial \Omega} - \int_{\Omega} \nabla \vec{\varphi}_{i,t} \cdot \nabla \varphi_{r,i} dx = 0
\]

\[
F_{s0} = (\Delta \vec{T}, \psi_s) = \int_{\Omega} \Delta (\vec{T}) \psi_s dx = \left[ \nabla \vec{T}, \psi_s \right]_{\partial \Omega} - \int_{\Omega} \nabla \vec{T} \cdot \nabla \psi_s dx = 0
\]

\[
F_{s0} = (\Delta \psi_j, \psi_s) = \int_{\Omega} \Delta (\psi_j) \psi_s dx = \left[ \nabla \psi_j, \psi_s \right]_{\partial \Omega} - \int_{\Omega} \nabla \psi_j \cdot \nabla \psi_s dx = 0
\]

In addition, the term \(P_r(t)\) in equation 4.32 is a pressure term which can be quite problematic since it requires additional computation. However this term vanishes also in cases where the velocity field fluctuation is homogeneous at the boundaries of the domain:

\[
(\nabla P(t), \varphi_r) = \int_{\Omega} \nabla P \cdot \varphi_r dx = \left[ P \varphi_r \right]_{\partial \Omega} - \int_{\Omega} P \nabla \varphi_r dx = 0 \tag{4.34}
\]

This is the reason why it is preferable to use the Reynolds decomposition of the velocity rather than the complete field when decomposing using POD. In the eventuality the velocity field is not homogeneous at the boundaries, the pressure term can either be modeled with additional computations implying to solve the pressure-Poisson equation or approximated by an empirical field projection (see Noack et al. 2005), or simply neglected. The latter solution leads to an inevitably less accurate model, although neglecting the pressure terms has often less effect on the model accuracy than the truncation of the higher order POD modes itself.

In the rest of this document, we will use indifferently \(a_i, a_{i, vel}\) or \(a_{i,t}\) for the velocity POD temporal coefficients, and \(b_i, a_{i, temp}\) or \(a_{i,t}\) for the temperature POD temporal coefficients.

### 4.2 Two-dimensional Two-sided Lid Driven Cavity

In order to develop the methodology, the algorithms and the understanding of the POD-Galerkin reduced order modeling method, simple problems were first investigated. The first investigated problem is a simple rectangular uniform cartesian grid, corresponding to a closed domain with homogeneous boundary conditions, low energy fluctuations and rather uniform length scales. In addition, to simplify and speed up the numerical simulations, the problem is chosen to be two-dimensional. These considerations lead the choice of a two-sided lid driven cavity flow (2D-2SLDC or 2SLDC).

#### 4.2.1 Numerical Setup

The numerical setup for this system is presented in figure 4.1. The domain is a rectangular domain of dimensions \(1 \times 5m\) with a uniform mesh of \(40 \times 200\) cells for a total of 8000 interior cells. The two "long" walls at \(X = 0\) and \(X = 1\) are driven at the same velocity of \(V_0 = 0.05m.s^{-1}\) in the same direction but are maintained at two different temperatures of \(T_1 = 300K\) and \(T_2 = 350K\). The corresponding Reynolds number based on the cavity height \(Re = V_0 h/\nu\), was approximately 1600. The two remaining "short" boundaries are adiabatic wall. This is a closed system without any mass going in or out the domain. The numerical model used was a two-dimensional approximation of Large Eddy Simulation model as implemented in Ansys Fluent™. The solver was second order in space and time. The time step used was of \(\Delta t = 0.25s\) corresponding to a corrective time step of \(\Delta t_c = 2.5 \times 10^{-3} (t_c = tV_0/h)\) and resulting in maximum Courant’s number \((u\Delta t/\Delta x)\) of less than 0.5. In the rest of this section, the reported quantities will be normalized velocities \(u = u^*/V_0, v = v^*/V_0\) and temperature \(\theta = (T - T_1)/(T_2 - T_1)\).
Figure 4.1: (a) Numerical domain and applied boundary conditions for the 2SLDC problem. (b) Time averaged flow field with V-velocity contours and U-V streamlines; (c) Instantaneous normalized flow field with V-velocity contours and U-V streamlines; (d) Time averaged normalized temperature field; (e) Instantaneous temperature field.

Figure 4.2: Velocity magnitude signal from probe located inside the cavity at $X = 0.5, Y = 2.5$ as a function of time.
4.2.2 Base Flow

Although not of particular interest in the current study, the base flow is rapidly described to put the results in the next sections in perspective. At low Reynolds numbers ($V_0 \approx 5 \times 10^{-3}$, $Re \approx 160$), the flow was found to be stable and stationary. In this regime, the flow field consisted of two large scale recirculation regions on each side of the problem symmetry line at $X = 0.5$. This configuration did not serve our purposes as some unsteadiness was required to obtain a meaningful reduced order model. As the value of the Reynolds number was increased to values of $V_0 = 0.05 \text{m.s}^{-1}$ instability started to develop and the recirculation cells started to be convected from the top domain to the bottom in a quasi-periodic manner. Figure 4.1(b) shows the time averaged velocity field with three distinct pairs of symmetric recirculation cells with respect to the center-line. An instantaneous snapshot in figure 4.1(c) shows three recirculation cells and highlights the fact that the instantaneous flow field is not symmetric. The associated time averaged and instantaneous temperature fields are also provided in figure 4.1(d, e). This flow configuration was judged suitable for experimenting with reduced order modeling methods. A probe was placed inside the flow at the center of the domain ($X = 0.5, Y = 2.5$) to record the velocity magnitude throughout the simulations. The signal is presented in figure 4.2. This signal shows that the flow is quasi-periodic with a dominant frequency of $9.25 \times 10^{-3} \text{Hz}$ corresponding to a scaled frequency of $St = St_0 = 0.925$ where $St$ is the Strouhal number $St = fh/V_0$. This signal is also modulated at subharmonic frequency of $St_0/2 = 0.463$.

4.2.3 Proper Orthogonal Decomposition

Proper Orthogonal Decomposition is carried out using Sirovich Snapshot method on the fluctuation signal. Several decompositions including successively 10, 25, 50, 75, 100, 150 and 200 snapshots are investigated to observe the impact of the number of snapshots on the decomposition metrics. The snapshots are separated by 50 time steps for an equivalent convective time step of $\Delta t_c = 0.125$. This sampling is materialized by the symbols in figure 4.2 against the probe signal. Figure 4.3 shows the energy repartition for the different computations. The energy distribution converges rather rapidly to a unique distribution for the lower order modes (highest energy content) for $N > 50$ for the velocity field, but settles only for $N > 150$ for the temperature decomposition. This is a result of the non-optimality of the POD for the temperature signal since the thermal energy does not scale with the square of the temperature. In the rest of this section, the computations are carried out using the $N = 200$ POD modes for both temperature and velocity fields, in order to prevent any bias on the reduced order models introduced by an inaccurate temperature POD. For $N = 200$, the two first velocity POD modes carry as much as 94% of the total turbulent kinetic energy and a total of 5 and 9 modes are necessary to reconstruct respectively 99% and 99.9% of the total turbulent kinetic energy. On the other hand, the first two temperature modes carry “only” 73% of the total temperature signal fluctuation energy (pseudo-thermal energy) and a total of 11 and 15 POD modes are necessary to reconstruct respectively 99% and 99.9% of the total pseudo-thermal energy. These numbers are comparable to previous two-dimensional systems with simple flows.
Figure 4.4: Velocity POD modes: (a) Mode1; (b) Mode2; (c) Mode3; (d) Mode4; (e) Mode5; (f) Mode6; (g) Mode20 for $N = 200$. U-V streamlines and V contours.

Figure 4.5: Temperature POD modes: (a) Mode1; (b) Mode2; (c) Mode3; (d) Mode4; (e) Mode5; (f) Mode6; (g) Mode20 for $N = 200$. 
The POD modes for the velocity and temperature are presented in figure 4.4 and figure 4.5 respectively. The first two modes are antisymmetric with respect to the flow center-line and exhibit three large recirculation regions. According to figure 4.3(a), they carry comparable amounts of energy, respectively 48% and 47% of the total energy. Peaks of identical sign between modes 1 and 2 are however shifted in the Y-direction of approximately a quarter of a wavelength (mainly in the core of the flow). In addition, the temporal coefficients $a_1$ and $a_2$ in figure 4.6 appear identical with a phase shift of a quarter wavelength as well. These combined characteristics (identical energy levels, shape function and temporal coefficients shifted of a quarter of a wavelength) are typical of POD modes describing the convection of flow structures in a particular direction. Indeed, the sum of the two partial velocity fields $a_1 \phi_1$ and $a_2 \phi_2$ in the reconstructed signal results in the convection of the recirculation cells observed in the shape functions. Modes 3 and 4 on the other hand are symmetric and do not carry the same amount of energy. They describe the convection of smaller scale recirculation cells on each side of the symmetry line (2 cells per domain width). Modes 5 and 6 are again antisymmetric and of comparable energy content, and represent convection of again smaller scale recirculation cells (3 cells per domain width). As for modes 1 and 2, the temporal coefficients for modes 3 and 4 as well as for modes 5 and 6 have respectively similar frequencies but a quarter wavelength phase shift. Finally, mode 20 captures the dynamics of rather small scales and no real coherence can be found. The decrease in length scale captured by the successive modes is a classic and essential feature of the POD which by nature constructs a set of basis function with decreasing energy content, thus captures ever smaller fluctuations and length scales in higher order modes. This feature is in fact a key element in the process of reduced order modeling using truncated POD series to a finite number of POD modes therefore retaining only the lower order modes with the highest energy content susceptible to govern and represent the best the flow dynamics. Overall, the velocity shape functions appear to be organized in pairs of symmetric and antisymmetric modes with comparable energy content and temporal coefficient dominant frequency. The POD modes for the temperature decomposition show some similarities with the velocity decomposition. The first two POD modes capture rather large scale fluctuation areas, yet the decrease in length scale with increasing mode order is not as obvious as it is in the velocity decomposition. Similarly, the energy decay rate of energy distribution in figure 4.3(b) is not as significant as the velocity one. POD modes still appear to be associated in pairs of alternating symmetric and antisymmetric, yet the energy repartition between the individual modes of every pair is not as even as the one in the velocity decomposition. Once again, these results are attributed to the non-optimality of the POD to the decomposition of temperature signals.

The temporal coefficients $a_1^{vel}$ and $a_1^{temp}$ are presented in figure 4.6 and show that not only the temporal coefficients corresponding to a same pair of modes have similar amplitude (i.e. energy) and frequency, but that the frequency captured by the different pairs is increased with increasing modes orders. Modes 1 and 2 capture dynamics with a fundamental frequency of $St = 0.47 \approx St_0/2$, modes 3 and 4 with a frequency 1.5 times higher of $St = 0.705$, and modes 5 and 6 with a frequency twice as high $St = 0.94 \approx St_0$ where $St_0$ is the fundamental frequency directly evidenced in the probe signal of figure 4.2.
Figure 4.7: First four ROM temporal coefficients for the velocity and temperature fields obtained for several values of $N_V$ and $N_T$ (solid lines) along with corresponding POD temporal coefficients (symbols). $a_1$ (red), $a_2$ (green), $a_3$ (blue), $a_4$ (magenta).
4.2.4 Reduced Order Model

Reduced order models are obtained for several combinations of the pair of values \((N_V, N_T)\), respectively the number of velocity and temperature POD modes used in the reconstruction of the fields. The Galerkin projection is carried out on the POD results including 200 velocity fields and 200 temperature fields. The spatial derivatives of the respective POD modes are calculated using Tecplot™, utilizing a centered second order accurate scheme. Although the derivation scheme used by Tecplot to obtain the derivatives is most likely not identical to the one implemented in the Fluent solver, the error introduced by the change in scheme is deemed negligible with respect to the dropped POD terms in the velocity and temperature fields reconstructions. Moreover, the relatively refined mesh used in this simulation ensured local errors of the order, or inferior to, \(\Delta x^2 \sim 6.10^{-4}\). The Galerkin coefficients of equations 4.32 and 4.33 are evaluated using the trapezoidal integration approximation. In this particular case, the velocity field and temperature fields are homogeneous at the boundary so that the pressure term \(P_t(t)\) in equation 4.32 vanishes and does not require any additional calculations. The set of ODEs are integrated using both a 5th order Runge-Kutta solver (Dormand-Prince pair 4/5 Dormand & Prince 1980 - DP45) and a variable order Adams-Bashforth-Moulton PECE solver (see Shampine & Gordon 1975) with a relative tolerance between \(10^{-8}\) and \(10^{-2}\). The sensitivity of both solvers on the tolerance parameter was investigated and the solution did not evidence any significant differences beyond a tolerance of \(10^{-4}\). The presented following results are the one obtained using the DP45 algorithm with a relative tolerance of \(10^{-4}\). Both projected momentum and energy equations are solved simultaneously in a system of \(N_V + N_T\) coupled ODEs.

The results from the integration of the reduced order model obtained from the Galerkin projection method are presented in figure 4.7 for several values of \(N_V\) and \(N_T\), along with the corresponding POD coefficients corresponding to the projected LES data onto the POD basis. The results for the first four temporal coefficients show relatively good tracking for the velocity over the first periods corresponding to the dominant frequency as soon as \(N_V = 4\). While the fidelity of the model appears to be improved by increasing the number of velocity POD modes to \(N_V = 10\), no obvious improvement seems to occur between \(N_V = 10\) and \(N_V = 24\). This shows that the model is fully converged with 10 modes and that the remaining error is to be attributed to the numerous approximations made to obtain the set of ODEs including but not limited to, errors introduced by the solver numerical integration scheme, inaccuracy of the solution due to domain discretization, errors introduced by the mismatch in derivation scheme between the solver and the post-processing software, integrals approximations with the trapezoidal rule, etc. As expected from the fact that the velocity solution is decoupled from the temperature one, the changes in the value of \(N_T\) at fixed values of \(N_V\) do not visually affect the fidelity of the velocity ROM.

The impact of \(N_V\) and \(N_T\) on the temperature ROM fidelity is also evidenced in figure 4.7. With \(N_T = 4\), the accuracy of the temperature ROM is poor and the tracking mediocre. However, the quality of the ROM is greatly improved for \(N_T = 10\). When including 24 temperature modes, no significant improvement is observed on the first three coefficients and a certain degradation of the profile of the fourth coefficient occurs. Conversely to the velocity ROM, the changes in values of \(N_V\) at fixed \(N_T\) have a moderate yet visible influence on the temperature ROM tracking, in particular the “long term” accuracy of the model appears to be improved for increased number of velocity POD modes included.

The accuracy of the different ROMs is evaluated with an estimate of the error between the ROM and the POD coefficients using the instantaneous error \(n_{\varepsilon}^V\):

\[
n_{\varepsilon}^V(N_V, N_T, n, t) = \frac{\left\| a_n^{ROM}(N_V, N_T, t)\varphi_n - a_n^{POD}(t)\varphi_n \right\|^2}{\left\langle \left\| a_n^{POD}(t)\varphi_n \right\|^2 \right\rangle} = \frac{1}{\lambda_n} \int_{\Omega} \left( a_n^{ROM}(N_V, N_T, t)\varphi_n - a_n^{POD}(t)\varphi_n \right)^2 dx = \frac{1}{\lambda_n} \left( a_n^{ROM}(N_V, N_T, t) - a_n^{POD}(t) \right)^2
\]

Similarly, we have for the temperature:
Figure 4.8: Normalized error $nE^V_1$ for the velocity ROM (red) and $nE^T_1$ temperature ROM (blue) as a function of the mode index for several values of $N_V$ and $N_T$.

$$nE^T_1 = \frac{\|b^\text{ROM}_n(N_V,N_T,t)\psi_n - b^\text{POD}_n(t)\psi_n\|^2}{\langle \|\psi_n^\text{POD}(t)\psi_n\|^2 \rangle} = \frac{1}{\sigma_n^2} (b^\text{ROM}_n(N_V,N_T,t) - b^\text{POD}_n(t))^2$$

These quantities are estimating the instantaneous error on mode $n$, normalized by the total kinetic energy (or temperature signal energy) contained in the mode during the POD decomposition. Integrating $e_n$ over a period of time $T_p$ we also define $nE_1$:

$$nE^V_1(N_V,N_T,n,T_p) = \frac{1}{T_p} \int_0^{T_p} n e^V_n(N_V,N_T,n,t) \, dt \quad (4.35)$$
$$nE^T_1(N_V,N_T,n,T_p) = \frac{1}{T_p} \int_0^{T_p} n e^T_n(N_V,N_T,n,t) \, dt \quad (4.36)$$

which is estimating the average error on mode $n$ over a period $T_p$.

Finally, a global figure of merit $E_1$ can be built based on $nE_1$:
\begin{align*}
E_V^1(N_V, N_T, T_p) &= \sum_{n=1}^{N_V} E_n^V(N_V, N_T, n, T_p) \\
&= \frac{\left\langle \left\| \sum_{n=1}^{N_V} a_n^{ROM}(N_V, N_T, t) \phi_n - \sum_{n=1}^{N_V} a_n^{POD}(t) \phi_n \right\|^2 \right\rangle_{T_p}}{\left\langle \sum_{n=1}^{N_V} \| a_n^{POD}(t) \phi_n \|^2 \right\rangle_{T_{POD}}} \\
E_T^1(N_V, N_T, T_p) &= \sum_{n=1}^{N_T} E_n^T(N_V, N_T, n, T_p) \\
&= \frac{\left\langle \left\| \sum_{n=1}^{N_T} b_n^{ROM}(N_V, N_T, t) \psi_n - \sum_{n=1}^{N_T} b_n^{POD}(t) \psi_n \right\|^2 \right\rangle_{T_p}}{\left\langle \sum_{n=1}^{N_T} \| b_n^{POD}(t) \psi_n \|^2 \right\rangle_{T_{POD}}}
\end{align*}

which evaluates the total error made on the modeled part of the flow only over a period of time \( T_p \) when including \( N_V \) velocity POD modes and \( N_T \) temperature POD modes.

This figure, while providing information on how good the model is when only considering the modeled part of the flow (the first \( N_V \) and \( N_T \) modes) does not represent the total error made on the reconstructed flow. Indeed, if only the first \( N_V \) modes are modeled by the ROM, then the energy contained in the remaining modes should also count as a loss. We define now \( E_V^2 \):

\[ E_V^2(N_V, N_T, t) = \frac{\left\| \sum_{n=1}^{N_V} a_n^{POD}(t) \phi_n - \sum_{n=1}^{N_V} a_n^{ROM}(N_V, N_T, t) \phi_n \right\|^2}{\left\langle \sum_{n=1}^{N_V} \| a_n^{POD}(t) \phi_n \|^2 \right\rangle_{T_{POD}}} \]

Because of the orthogonality of the modes, this reduces down to:

\[ E_V^2(N_V, N_T, t) = \sum_{n=1}^{N_V} (a_n^{POD}(t) - a_n^{ROM}(N_V, N_T, t))^2 + \sum_{n=N_V+1}^{N} (a_n^{POD}(t))^2 \]

Similarly for the temperature ROM:

\[ E_T^2(N_V, N_T, t) = \sum_{n=1}^{N_T} (b_n^{POD}(t) - b_n^{ROM}(N_V, N_T, t))^2 + \sum_{n=N_T+1}^{N} (b_n^{POD}(t))^2 \]

Finally, we define a global figure of merit \( \overline{E}_2^V \) based on \( E_V^2 \):

\[ \overline{E}_2^V(N_V, N_T, T_p) = \left\langle E_V^2(N_V, N_T, t) \right\rangle_{T_p} \]

in the same way for the temperature we have:
Figure 4.9: Estimates of \( \overline{E}_T^1 \) and \( \overline{E}_T^2 \) over one fundamental period \( 1/St_0 \) (left), two fundamental periods (center) and three fundamental periods (right) for multiple values of \( N_V \) and \( N_T \).

\[
\overline{E}_T^2(N_V, N_T, T_P) = \langle E_T^2(N_V, N_T, t) \rangle_{T_P} \tag{4.42}
\]

These values evaluate the global or total error of the reduced order model over a period of time \( T_P \), when including \( N_V \) velocity modes and \( N_T \) temperature modes on the complete flow.

The normalized individual errors \( nE_V^1 \) and \( nE_T^1 \), for the velocity and temperature ROMs evaluated for \( T_P = 1/St_0 \), are presented in figure 4.8. Overall, for both ROMs, the normalized error increases with increasing mode order at fixed values of \( N_V \) and/or \( N_T \), which shows that the modes carrying the most energy are the one with lowest error. Specifically, the first two velocity and temperature modes have normalized errors of the order of 0.01%. The insensitivity of the velocity ROM to the value of \( N_T \) is clearly evidenced here. The increase of the number of velocity POD modes from \( N_V = 4 \) to 10 shows a threefold decrease of the error levels for the first two modes, and twofold decrease for the next two modes. The decrease in error from \( N_V = 10 \) to \( N_V = 24 \) is only minor for the 10 first POD modes, less than 20% on average. For \( N_V = 10 \), the first 6 modes, which are carrying more than 99% of the total turbulent kinetic energy according to figure 4.3(a), exhibit normalized errors less or equal to 1%, with an average relative error on the first two dominant modes of 0.01%.

The estimates of the normalized error for the temperature ROM show that the error on the first four modes decreases globally from \( N_T = 4 \) to \( N_T = 10 \), but not homogeneously across the modes. Indeed, the increase of the number of temperature POD modes from 10 to 24 generates an increase in the error associated with the fourth mode, but an overall decrease for all the other modes. The decrease in error at fixed values of \( N_T \) with increasing values of \( N_V \) is not systematic for all modes as initially expected. On this last point, the definition of the normalized error is probably influencing this trend, since it covers only one period \( T_P \), while the qualitative improvement for increasing values of \( N_V \) was noted in the relative long term in figure 4.7. Overall for \( N_T = 24 \), the relative error for the first 11 temperature modes carrying more than 99% of the total pseudo-thermal energy is inferior or equal to 6%, with an average of 0.02%, for the first two dominant modes.
Figure 4.10: Estimates of $\overline{E_T^1}$ and $\overline{E_T^2}$ over one fundamental period $1/St_0$ (left), two fundamental periods (center) and three fundamental periods (right) for multiple values of $N_V$ and $N_T$.

The evaluations of $\overline{E_V^1}$ and $\overline{E_V^2}$ over one (short term), two (medium term), and three (long term) fundamental periods $1/St_0$, presented in figure 4.9, confirm the insensitivity of the velocity field to the temperature field, since the surfaces are essentially two-dimensional. In the short term (figure 4.9a and d), the error on the modeled part $\overline{E_V^1}$ reaches a minimum early around $N_V = 10$, increases sharply to reach a maximum at $N_V = 16$, then decreases again up to $N_V = 22$ and reaches another local maximum around $N_V = 34$, to finally flatten out as $N_V$ approached 40. This shows that to obtain an accurate model, the strongly correlated POD modes with comparable energy content (see figure 4.3a and figure 4.6a), should all be included together. In the medium and long term, $\overline{E_V^1}$ decreases consistently with increasing values of $N_V$, reaching a minimum around $N_V = 26$. Both cases exhibit a couple of “ripples” corresponding to the local peaks in the short term estimate. Similarly, the total error $\overline{E_V^2}$ decreases consistently as $N_V$ is increased, with a minimum around $N_V = 24$ beyond which the error (being $\overline{E_V^1}$ or $\overline{E_V^2}$) no longer decreases and reaches a steady value. All of the $\overline{E_V^2}$ estimates also exhibit local ripples at $N_V = 16$ and $N_V = 34$.

Similarly, evaluations of $\overline{E_T^1}$ and $\overline{E_T^2}$ over one, two and three fundamental periods are presented in figure 4.10. This time, the surfaces are fully three-dimensional, evidencing sensitivity of the temperature ROM to both $N_V$ and $N_T$. The impact of $N_V$ on the error of the modeled temperature field becomes very small beyond $N_V = 16$ where a local increase in $\overline{E_T^1}$ is found, probably associated to the corresponding increased error in the velocity field. The temperature ROM shows a very strong dependency to the value of $N_T$ and reaches a minimum around $N_T = 6$, before increasing to reach a steady value for $N_T > 18$. This trend is also observed on the total error $\overline{E_T^2}$ for all three estimates. This evidences the fact that when using POD based reduced order models, more is not always synonym of better, and that adding lower order modes to a ROM can lead to overall higher error levels. This is in part due to the inaccurate modeling of the small scale vortical structures (and their associated temperature field) responsible for most of the energy dissipation. There are multiple reasons to this, starting with the numerical model (LES), which by definition does not resolve those smaller scales, but also the spatial sampling which does not allow to capture scales smaller than or of the order of the local grid cell dimensions. To these should be added the temporal sampling which tend to filter the dynamics of the vortical structures with fundamental frequencies below the sampling frequency.
Figure 4.11: (a) Numerical domain and local cell surface area from $10^{-5}D^2$ (blue) to $10^{-3}D^2$ (red). (b) Time averaged flow field with $U$-velocity contours and $U-V$ streamlines; (c) Instantaneous normal vorticity; (d) Time averaged normalized temperature field; (e) Instantaneous temperature field.

As a result, the physics of the small scale structures, usually captured by the higher order modes, is not always properly represented by the POD and therefore the ROM.

Overall, very satisfying results are obtained using only 22 velocity modes and 6 temperature modes.

4.3 Two-dimensional Cylinder in a Cross-flow

While 2D-2SLDC problem constituted a good starting point, this flow was very well behaved and corresponded to a closed-flow configuration. To investigate open flow systems, a two-dimensional cylinder in a cross-flow is selected to assess the impact of neglecting the pressure terms in equation 4.32. In addition, the extensive documentation on this type of flow makes it an excellent candidate.

4.3.1 Numerical Setup

The numerical setup for the cylinder in cross-flow consists of a cylinder inside of diameter $D = 1m$ and a rectangular domain $16D$ wide (in the $X$ direction) and $33D$ long (in the $Y$ direction). A close-up view of the domain is provided in figure 4.11(a). The origin of the domain is located at the center of the cylinder and the inlet of the domain is at $Y = -8D$, therefore the outlet at $Y = 33D$. The grid is multi-block with an O-grid of $3D$ diameter surrounding the cylinder. The cell height at the cylinder wall is $2 \times 10^{-4}D$ for corresponding $y^+$ values of less than 0.25, well inside the boundary layer viscous sub-layer. The mesh counts a total of 250,000 cells. Because of the high density of the mesh, the surface area of the individual cells is presented in figure 4.11(a), instead of the mesh itself, and still provides a good overview of the mesh distribution. At the inlet of the domain, normal velocity is imposed at $V_0 = 2.235 \times 10^{-3}m.s^{-1}$ for a corresponding Reynolds number ($Re_D = V_0D/\nu$) of 150, within the unstable two-dimensional laminar regime. At the outlet of the domain, an outflow boundary condition is imposed. The sides of the domains are taken far away from the core of the flow and periodic boundary conditions are applied to them. A constant temperature of $T = T_1 = 300K$ is imposed at the domain inlet and a constant temperature of $T = T_2 = 350K$ is imposed at the cylinder wall. The numerical model used is a 2D approximation of the Large Eddy Simulation model as implemented in Ansys Fluent™ with a dynamic Smagorinsky-Lilly sub-grid scale model. The discretization scheme for the pressure, momentum and energy equation are respectively second order upwind, second order bounded central-differences and QUICK (Quadratic Upstream Interpolation for Convective Kinematics). The time discretization scheme is also second order. The integration time step used during the simulations is $\Delta t = 1s$.
corresponding to a convective time step of $\Delta t_c = 2.2 \times 10^{-2}$ with $t_c = t V_0 / D$. The maximum Courant’s number ($u \Delta t / \Delta x$) is less than 0.9.

### 4.3.2 Base Flow

The base flow for this problem is well documented (see Williamson 1996 for instance) and will only be described to put in perspective the following results. A recirculation region is located directly downstream of the cylinder (see figure 4.11b) and contains a pair of vortical regions of opposite vorticity (see figure 4.11c). At Reynolds numbers above the Hopf Bifurcation stability margin ($Re_D \approx 40$), the wake becomes unstable and the vortices composing the wake starts shedding in an alternating periodic manner to form the well known Kármán vortex street observed in figure 4.11(c). The wake is characterized by the Strouhal number ($St = fD/V_0$) associated with the shedding frequency of the vortices, which is a function of the Reynolds number. In the current simulations at $Re_D = 150$, the fundamental shedding frequency is found to be $St_0 = 0.198$ in the lift coefficient $C_L$ history of figure 4.12 while the drag coefficient $C_D$ signature exhibits a fundamental twice as high. This is in agreement with previous findings from Thompson et al. (1996). For additional information on the velocity field of flows past circular cylinder, the reader is referred to Williamson (1996). The temperature field is also presented in figure 4.11(d-e). Since the cylinder wall is the only surface at a prescribed temperature above cross-flow temperature, the average field shows that the only areas with elevated temperatures are the surroundings of the cylinder and the wake. The instantaneous temperature field in figure 4.11(e) is highly correlated to the vorticity field in figure 4.11(c) highlighting the thermal convection from the shedding wake vortices.

### 4.3.3 Proper Orthogonal Decomposition

As for the previous case, the proper orthogonal decomposition is carried out using the snapshot method. A sampling of $\Delta t_c = 5.5 \times 10^{-2}$, corresponding to 50 time steps, is used which is equivalent to 45 snapshots per period of the fundamental frequency and evidenced by the symbols in figure 4.12. Four decompositions including 45, 68, 90 and 135 snapshots are made for both the velocity and temperature fields in order to evaluate the influence of the number of snapshots on the convergence of the decomposition. Figure 4.13 presents the energy distribution from the decomposition of the velocity and temperature fields. For the velocity, the first 10 POD modes are identical for the decompositions including integer numbers of periods whereas the decomposition including a non-integer number of periods appears slightly different beyond the second mode. This is a well known result concerning POD decomposition of periodic flows, which in order to be accurate and optimum, requires an integer number of periods during the sampling (see Rowley 2005).
For the rest of this section, the decomposition including 135 snapshots is used for reduced order modeling purposes, covering 3 shedding cycles, therefore allowing us to estimate the long term behavior of the ROMs. The first two velocity POD modes carry approximately 47% of the total turbulent kinetic energy each. A total of 5 and 8 velocity POD modes are required to respectively capture 99% and 99.9% of the total turbulent kinetic energy.

The difference between integer and non-integer sampling of the period is less obvious in the temperature decomposition yet remain visible. The first 18 modes for the temperature decomposition including an integer number of periods appear to be identical when including 1, 2 or 3 periods. Indeed, the energy distribution of the temperature decomposition is less optimal than the one in the velocity decomposition and the first two modes carry “only” 30% of the total pseudo-thermal energy. A total of 8 and 14 temperature POD modes are required to capture respectively 99% and 99.9% of the total pseudo-thermal energy which is significantly higher than for the velocity field decomposition. Once again, these observations evidence that while the POD is optimal in the sense of the kinetic energy, it is not in the sense of the thermal energy, therefore providing modes with lower energy content for the temperature decomposition.

The normal vorticity extracted from the shape functions of the velocity field decomposition is presented in figure 4.14. As for the 2SLDC results, the decomposition yields pairs of modes which are similar in terms of the size and shape of the vortical features they capture. The first mode in figure 4.14(a) is symmetric with respect to the flow symmetry line (X = 0) and captures vortical structures with scales of the order of the cylinder diameter, with alternating vorticity sign. The second mode is almost identical to the first one, with a shift in the pattern of a quarter wavelength towards the downstream direction. This shift, coupled with the \( \pi/4 \) shift in the phase of the temporal coefficient of mode 2 with respect to the one of mode 1 in figure 4.15,
results in the downstream convection of vortical cells in the reconstructed signal $a_1(t)\phi_1(x) + a_2(t)\phi_2(x)$. This is comparable to the results obtained in the 2SLDC configuration. Modes 3 and 4 are the first antisymmetric modes and contain vortical cells of opposite sign on each side of the symmetry line with alternating sign in the downstream direction. The vorticity levels are rather high close to the recirculation region and decrease in intensity beyond $Y = 16$, with in addition, a bifurcation in the vortical structures trajectories beyond this point. It is rather unclear if this is to be related to the mesh density change beyond $Y = 16$ (visible in figure 4.11a) or to the natural decay of the vorticity due to diffusion. As for modes 1 and 2, the temporal coefficients for modes 3 and 4 are comparable in amplitude and frequency with a $\pi/4$ phase shift. Modes 5 and 6 exhibit a symmetric pattern similar to the one of modes 1 and 2, with yet twice as many changes in vorticity sign in the wake region. The vortical cells are rather elongated in the X-direction. The shape functions contain single cells with high vorticity up to $Y = 16$ and double cells on each side of the symmetry line with lower vorticity levels beyond that point. The temporal coefficients of modes 5 and 6 also exhibit a quarter period shift. Finally, mode 20 exhibits small scale fluctuations in the wake of the cylinder but, conversely to the 2SLDC results, conserves symmetry and coherence. It is worth noting that this latter mode also captures rather large scale structures (at least when compared to the ones directly downstream of the cylinder) near the domain outlet. This is a good example of the potential issues faced when using POD on flows involving various length and energy scales. Indeed, it is observed in figure 4.11(c) that the vorticity levels, and therefore the energy content, of the structures located directly downstream of the cylinder are significantly higher than the ones of the previously convected vortices on which diffusion has had an effect. Consequently, the POD will have a tendency to capture across the first POD modes the vortical structures in the near-field of the cylinder (as evidenced in figure 4.14a-f) and across the higher order modes the less energetic far-field structures. However, there is a certain overlap such that in mode 20, the far-field structures are captured along with smaller features of the near-field structures carrying less energy.

The POD modes for the temperature decomposition presented in figure 4.16 show strong similarities with the velocity ones. In particular modes 1 and 2 are symmetric with respect to the flow center-line, shifted in the downstream direction of a quarter wavelength with respect to each other and show strong similarities with the vorticity field of modes 5 and 6 from the vorticity field decomposition. Modes 3 and 4 as well as 5 and 6 are antisymmetric. Once again, the higher order modes such as mode 20, capture simultaneously increasingly smaller scale features in the near-field but also larger, less energetic ones in the far-field.

The temporal coefficients for the velocity and temperature decompositions are presented in figure 4.15(a) and 4.15(b) respectively. While both decompositions result in series of pairs of coefficients with identical fundamental frequency and a $\pi/4$ phase shift, the fundamental frequency of the dominant velocity and temperature modes are different. While modes 1 and 2 from the velocity field decomposition have a fundamental frequency equal to $St_0 = 0.198$ identical to the one found in figure 4.12 for the lift coefficient $C_L$, the first two temperature POD modes have a fundamental frequency of $2St_0$ corresponding to the one of the drag coefficient $C_D$ in the same figure. Velocity modes 3 (and 4), modes 5 (and 6) and mode 20 have fundamental frequencies of respectively $2St_0$, $3St_0$ and $8St_0$ while temperature modes 3 (and 4), modes 5 (and 6) and mode 20 have respective fundamental frequencies $St_0$ and $4St_0$ and $5St_0$. Conversely
Figure 4.16: Temperature POD modes: (a) Mode 1; (b) Mode 2; (c) Mode 3; (d) Mode 4; (e) Mode 5; (f) Mode 6; (g) Mode 20 for $N = 135$.

to the 2SLDC decompositions, the fundamental frequencies captured by the “corresponding” velocity and temperature POD modes do not match. This is a reminder that although the velocity and temperature POD series are calculated based on the same data set of corresponding velocity and temperature fields, the obtained POD modes are decoupled from one another and there is no one-on-one equivalent between the velocity and temperature decompositions.

### 4.3.4 Reduced Order Model

Reduced order models are obtained for several combinations of the values $N_V$ and $N_T$, respectively the number of POD modes used in the velocity and the temperature reconstructions, based on the decomposition including 135 snapshots described previously. As for the 2SLDC problem, the derivatives of the POD shape functions are obtained using Tecplot. A trapezoidal integration approximation is used to evaluate the Galerkin coefficients in equations 4.32 and 4.33 while the pressure term $P_r(t)$ in equation 4.32 is neglected. The ODEs are integrated using both the DP45 and ABM-PECE algorithms, with tolerance parameters of $10^{-8}$ to $10^{-2}$. Once again, no significant differences are observed between both solvers and the integration results are sensibly identical beyond a relative tolerance of $10^{-4}$. In the following, the equations for the velocity and the temperature are solved simultaneously using the DP45 algorithm with a tolerance of $10^{-4}$.

The results from the integration of reduced order models are presented in figure 4.17 as a function of the parameters $N_V$ and $N_T$ for the first four temporal coefficients along with the corresponding POD coefficients (projected LES data). The results for the velocity ROM show excellent tracking over two periods for the first two temporal coefficients as early as $N_V = 4$. The accuracy of the ROM remains excellent for the following two modes with $N_V = 8$ and $N_V = 18$. As for the 2SLDC velocity ROM, no visible influence of $N_T$ on the velocity solution is observed at fixed values of $N_V$. For $N_T = 4$, the temperature ROM is somewhat less accurate than its velocity counterpart on modes 1 and 2, but surprisingly good on mode 3 and 4. The ROM starts to diverge in the third period and the influence of $N_V$ on the temperature ROM appears to be weak. For $N_T = 8$, the accuracy of the temperature ROM is greatly improved for the first two temporal coefficients over the $N_T = 4$ solution. The influence of $N_V$ is however visible between $N_V = 4$ and $N_V = 8$, in particular for the third and fourth temporal coefficients. Finally, the results for $N_T = 18$ show reasonable accuracy at $N_V = 4$ and excellent qualitative tracking for $N_V = 8$ and $N_V = 18$. It is however worth noting that the accuracy in the third period appears to be slightly worse for the case with higher $N_V$, especially on the third and fourth modes. Overall good qualitative tracking of both velocity and temperature ROMs is obtained for $N_V = 8$ and $N_T = 8$.

The accuracy of the ROM is evaluated using the normalized error parameter $n_E^V$ and $n_E^T$ defined in equation 4.35 and 4.36 over a the period $T_p = 1/St_0$ corresponding to the fundamental frequency of the first velocity POD mode. The error estimates on the velocity signal confirm the qualitative observations with a very low normalized error of less than 0.001% for the first two velocity modes at $N_V = 4$, but above 1% for the next two modes. As modes 5 and 6 are introduced in the ROM, the error on modes 3 and 4 drops
Figure 4.17: First four ROM temporal coefficients for the velocity and temperature fields obtained for several values of $N_V$ and $N_T$ (solid lines) along with corresponding POD temporal coefficients (symbols). $a_1$ (red), $a_2$ (green), $a_3$ (blue), $a_4$ (magenta).
Figure 4.18: Normalized error $\eta E_V^1$ for the velocity ROM (red) and $\eta E_T^1$ temperature ROM (blue) as a function of the mode index for several values of $N_V$ and $N_T$.

significantly around 0.002%. This is to be attributed to the strong correlation between the modes pairs 3-4 and 5-6 (see figure 4.15a) and therefore, obtaining an accurate reconstruction of modes 3-4 requires the incorporation of modes 5-6. Adding higher order modes has a positive effect on modes 1-2, 5-6 and 7-8 which $\eta E_V^1$ levels approach respectively 0.0002%, 0.004% and 0.04% but does increase the error on modes 3 and 4 back to 0.007%. The normalized error value for the velocity is insensitive to the changes in $N_T$ at fixed values of $N_V$. As for the 2SLDC problem, the relative error consistently increases for increasing velocity mode order and can reach almost 100% for high order modes (e.g mode 18). The global error of the first 8 modes carrying 99.9% of the total turbulent kinetic energy is less than 1% for $N_V = 8$, and less than 0.1% for $N_V = 18$.

The normalized error estimates for the temperature ROMs show the impact of $N_V$ on the error levels. Overall, adding more velocity modes at fixed $N_T$ value decreases the error levels. Similarly, adding more temperature modes tend to have a positive impact on the error levels. The maximum individual error level on the first 14 temperature modes carrying more than 99.9% of the temperature field energy is less than or close to 0.1%.

The evaluations of $\overline{E_V^1}$ and $\overline{E_T^1}$ over one (short term), two (medium term), and three (long term) fundamental periods $1/St_0$ are presented in figure 4.19. They confirm that the velocity ROM is insensitive to the number of temperature modes introduced since the surfaces are essentially two-dimensional. In all three time spans, (figure 4.19a-c) the error on the modeled part $\overline{E_V^1}$ reaches a minimum around $N_V = 8$, beyond which any additional mode actually decreases the overall ROM accuracy. A plateau is reached around $N_V = 32$ so that the error levels remain stable beyond that point. This evidences once more that increasing the number of POD modes in the model is not always beneficial in terms of cumulative error. An identical trend is observed on the error estimates $\overline{E_T^1}$ over the total energy content of the flow with a minimum around $N_V = 8$.

Similarly, evaluations of $\overline{E_V^2}$ and $\overline{E_T^2}$ over one, two and three fundamental periods are presented in figure 4.20. As for the 2SLDC case, the surfaces are fully three-dimensional, showing that the temperature ROMs are sensitive to both $N_V$ and $N_T$. The quality of the velocity ROM has a clear impact on the temperature error levels as evidenced by the local minimum obtained around $N_V = 8$. As for the velocity ROM, the
Figure 4.19: Estimates of $\hat{E}_1^V$ and $\hat{E}_2^V$ over one fundamental period $1/St_0$ (left), two fundamental periods (center) and three fundamental periods (right) for multiple values of $N_V$ and $N_T$.

Figure 4.20: Estimates of $\hat{E}_1^T$ and $\hat{E}_2^T$ over one fundamental period $1/St_0$ (left), two fundamental periods (center) and three fundamental periods (right) for multiple values of $N_V$ and $N_T$. 
values of both $E_T^1$ and $E_T^2$ exhibit a minimum around $N_T = 18$. Beyond this value, introducing additional temperature modes results in overall increase of the error levels. Once again, the poor accuracy of the ROM when considering the smaller dissipative scales is to blame.

Overall, good qualitative and quantitative agreement with the projected LES data is obtained using $N_V = 8, N_T = 18$, even though the pressure terms are being neglected.

4.4 Conclusion

POD-Galerkin reduced order modeling method was applied to two simple two-dimensional flows: a two-sided lid driven cavity and a cylinder in cross-flow. The first problem showed that the POD decomposition for the velocity and temperature converged beyond a number of snapshots of respectively 50 and 150. The velocity POD captured 99% of the total turbulent kinetic energy with 5 modes and the temperature decomposition captured 99% of the pseudo-thermal energy of the signal fluctuation within 11 modes. For both the velocity and temperature fields, the shape functions were associated by pairs with shifted patterns and associated temporal coefficient with phase shifts of $\pi/4$. These shifts resulted in the reconstruction of the convective motion of the individual structures captured by the particular mode pairs. The integration of the ordinary differential equations for several numbers of velocity and temperature modes included has shown that the velocity reduced order model is insensitive to the number of temperature POD modes, while the temperature ROM is not insensitive to the number of velocity modes. The tracking capabilities of the ROM were found to be satisfying for 10 velocity modes and 24 temperature modes with normalized errors for the first modes carrying 99% of the energy below respectively 1% and 6%.

The POD for the two-dimensional cylinder revealed the necessity to use even numbers of periods in the decomposition in order to obtain a maximum energy decay rate across the POD modes in strongly periodic flows. The first 8 modes of the velocity POD and 18 modes of the temperature POD were found to be captured accurately with only one period sampling. The velocity POD captured 99% of the turbulent kinetic energy with 5 modes, while 9 modes were necessary to reach similar levels in the temperature decomposition. Similarly to the 2SLDC problem, the modes were found to be associated in pairs with shape functions patterned shifted in the downstream direction and temporal coefficients shifted in time, resulting in the convection of individual flow structures. The reduced order models were integrated for different numbers of velocity and temperature POD modes included in the reconstructions. The qualitative and quantitative observation of the reduced order models showed that the velocity ROMs had excellent tracking abilities for as few as 8 POD modes and that the overall minimum error levels on the total flow were obtained for $N_V = 8$ and below 0.6% over the long term estimates. The temperature ROM was also very accurate when as little as 8 modes were included. The minimum normalized total error was obtained with $N_V = 8$ and $N_T = 18$ and in the long term were below 0.8%. In this particular case, although the pressure terms arising from the projection of the momentum equation onto the POD modes were theoretically non-zero (particularly at the outlet), neglecting those terms did not seem to affect the reduced order models significantly, at least over the investigated time-span over which they were integrated.

It was found that each ROM had optimum values for $N_V$ and $N_T$ beyond which adding any more modes to the system only resulted in comparable or worse performance. This was explained by the fact that on a mode-to-mode basis, the relative error levels between the projected LES data and the ROM integration results was increasing with increasing mode order. Therefore, even though adding more POD modes to the decomposition reduced the amount of non-modeled energy, the poor quality of the model at these scales made the overall ROM performance worse than other systems integrated with less modes.
Chapter 5
POD-Galerkin Reduced Order Models of Film Cooling Flows

5.1 Preliminary Statistical POD Analysis of Film Cooling Jets

Before using the POD for reduced order modeling purposes, the decomposition is used for its flow analysis capabilities in order to isolate the dominant flow vortical structures in a quantitative way and identify any potential shortcomings of the method. POD is applied to both unforced and forced inclined jets described in the previous chapter. In this section, the Proper Orthogonal Decomposition is carried out on the complete velocity and temperature signals, without beforehand separating them into time averaged and fluctuation components.

5.1.1 Unforced Jets

The three-dimensional flow field and temperature field from LES at $BR = 0.15$ are analyzed using the snapshot method described in Chapter 4. A set of 200 statistically independent flow and temperature fields with a spatial density of 10 points per jet diameter is used. It is found that at the considered low turbulence levels, 300 snapshots and 20 vectors per jet diameters lead to sensibly identical results with yet considerably longer computation times. The analyzed sub-domain is presented in figure 5.1 and is such as $\{-1 \leq X_j \leq 12, -2 \leq Y_j \leq 2, 0 < Z_j \leq 2\}$ yielding 106,600 spatial points at the lowest resolution.

Figures 5.2(a-c), 5.3(a, b) present slices of the first four velocity POD modes (including the mean flow field) as well as the corresponding $\lambda_2$ (2nd invariant of the velocity divergence tensor) iso-surfaces. The time averaged flow field exhibits the classical features of inclined jet in cross-flow and compares qualitatively well with previously reported results of Bernsdorf et al. (2006). In particular the presence of a pair of counter-rotating vortices (CRVP) is visible in the $X_j = 6$ and 10.6 slices (solid arrows) as well as in the $Z_j = 0.25$

![Figure 5.1: Proper Orthogonal Decomposition domain.](image)
Figure 5.2: Mean flow (0th POD Mode) and first 3 velocity POD modes at \( BR = 0.15 \) \((a-c)\) Mode0; \((d-f)\) Mode1; \((g-i)\) Mode2; \((j-l)\) Mode3. Slices at \( X_j = 6 \) (left), \( X_j = 10.6 \) (center) with \( U \) velocity contours and \( V-W \) streamlines. \( \lambda_2 \) iso-surfaces (right) computed from the corresponding POD modes colored by \( U \) velocity and mean wall temperature contours (gray scale).

Figure 5.3: Mean flow and first 3 velocity POD modes at \( BR = 0.15 \) \((a, b)\) Mode0; \((c, d)\) Mode1; \((e, f)\) Mode2; \((g, h)\) Mode3. Slices at \( Y_j = 0 \) with \( V \) velocity contours and \( U-W \) streamlines (left) and slices at \( Z_j = 0.25 \) with \( W \) velocity contours and \( U-V \) streamlines (left).

The CRVP is generated in the average field by the downstream convection of the shear layer hairpin vortices legs as for the vertical jet configuration. In addition at \( X_j = 10.6 \) a secondary pair of counter-rotating streamwise vortices (hollow arrows) of opposite vorticity is formed near the wall on each side of the jet, corresponding to the side vortices observed in figure 3.10. The side vortices are observed in figure 5.2(c) for \( X_j > 9 \) and materialized in the average flow field by vortex tubes along the jet core as well as in figure 5.3(b) through the formation of another ‘stripe’ of positive vertical motion at \( Y_j = \pm 0.9 \).

The remaining POD modes provide an orthogonal decomposition of the fluctuation part of the velocity. According to figure 5.4(a), the first two POD modes virtually capture the same amount of energy while being also very similar in shape when comparing figure 5.2(f) and 5.2(i), as well as figure 5.3(e, d) and 5.3(e, f) with a phase shift in the streamwise direction. Both modes exhibit alternating changes in the sign of the velocity components in the downstream direction. Hence, the cumulative effects of Mode1 and 2 alternating positive and negative velocity regions combined with the phase shifted variations in the signs of \( a_1^{Ve} \) and \( a_2^{Ve} \) in figure 5.4(b), generates a downstream motion. In the present case, this behavior is associated with the convection of the shear layer hairpin vortices which are the principal structures observed in the attached
jet configuration, supported by $\lambda_2$ iso-surfaces in figures 5.3(f) and 5.3(i) exhibiting hairpin shapes similar to the one of hairpin vortices observed in figure 3.8(a) and 3.10(a). The velocity vector field associated with the first two POD modes also exhibits in figures 5.3(c) and 5.3(e) focus points corresponding to the hairpin legs. It is also observed that the first modes are strong rather far away from the jet exit but not particularly significant near the jet exit. This is explained by the fact that proper orthogonal decomposition ‘sorts’ the modes with respect of the amount of kinetic energy they contain, thus sorting the scales of the structures as well, such that the most energetic modes (first modes) will usually represent the largest structures. The third mode presented in figure 5.2(k-m) and 5.3(g, h) exhibits similarities with the first two modes with alternating positive and negative vertical velocity in the plane $Z_j = 0.25$, though the pattern appears less regular. In the plane $Y_j = 0$ of figure 5.3(g), the streamlines clearly show the formation of rollups in the jet upper shear layer. A strong negative streamwise velocity region is located near the symmetry plane ($Y_j = 0$) between the hairpin legs which is visible in the plane $X_j = 10.6$ as well. While streamwise vorticity from the fluctuations of the CRVP and side vortices is captured in the first two modes, no significant x-vorticity appears to be contained in the third mode at $X_j = 6$ or 10.6. Although according to figure 5.4(a) several other modes appear to carry a significant amount of energy, they will not be presented in the current document for the sake of brevity. However, it should be noted that the fourth mode resembles closely the third one yet with a ‘phase shift’ in the same way the first and second modes were related. The additional modes ($N \geq 5$) describe principally the dynamics of the near-wall region, particularly the side vortices and their interaction with the hairpin vortices.

The temperature field is analyzed as well using POD. Although the norm used in the standard proper orthogonal decomposition does not maximize the thermal energy, this method has been used in previous studies to provide a set of orthonormal functions that can be used to obtain a reduced order model of the temperature field (see previous sections). However, it can be argued that the decomposition may not be as ‘optimal’ as the one obtained for the velocity field in terms of energy. The $0^{th}$ mode (average temperature field) is presented in figure 5.5(a-c), and 5.6(a, b). The temperature field at the wall (figure 5.6b) is consistent with the one presented in figure 3.11(a). The contours in figure 5.5(a, b) show the averaged impact of the CRVP and the side-vortices on the temperature field. While the CRVP entrains hot cross-flow fluid near the wall, the side-vortices tend to increase the spread by varying jet fluid away from the jet core as the two lumps of cooler fluid at $Y_j = \pm 1$ suggest. In figure 5.4(a), the two first modes carry equivalent pseudo-thermal energy while figures 5.5(f) and 5.5(i) show that both modes, related to the hairpin shear layer vortices according to the shape of their iso-surfaces, are overall identical with a phase shift in the streamwise direction. Based on these considerations and given the distribution of the phase diagram in figure 5.4(c)
Figure 5.5: Mean temperature field ($0^{th}$ POD mode) and first significant temperature POD modes at $BR = 0.15$ (a-c) Mode0; (d-f) Mode1; (g-i) Mode2; (j-k) Mode5. Slices $X_j = 6$ (left), $X_j = 10.6$ (center) with temperature contours. Iso-temperature surfaces (right) computed from corresponding POD modes (red: positive blue: negative) and mean wall temperature contours (gray scale).

Figure 5.6: Mean temperature field and first significant temperature POD modes at $BR = 0.15$ (a, b) Mode0; (c, d) Mode1; (e, f) Mode2; (g, h) Mode5. Slices at $Y_j = 0$ with temperature contours (left) and slices at $Z_j = 0$ with temperature contours (right).

showing alternating signs for $a_{1}^{Temp}$ and $a_{2}^{Temp}$, it appears clearly that the first two modes describe the convective effect of the hairpin vortices on the temperature field. Near the wall at $Z_j \approx 0$ (figure 5.6d, f), the two modes exhibit similar shifted patterns yet a noticeable difference exists near $X_j = 10$ with the presence in the second mode of a pair of patches with a negative value (dark) on each side of the jet (around $Y_j = \pm 1$) which counterpart is not clearly found in the first mode. A comparable discrepancy between the two first modes is found in the same plane around $X_j = 8$, where in Mode1 a pair of patches of rather high positive values (white) near $Y_j = \pm 1$ are present yet no equivalent counterpart is found in Mode2. These differences can be attributed to ‘quasi-stationary’ events occurring near the wall where the velocity is low and the convective effects are not as strong. Thus, while the positive patches in Mode1 appear to correspond to the effect of the hairpin vortices on the pinching of the jet coverage observed in the mean temperature field and attributed to cross-flow entrainment from the legs of the structures, the negative patches in Mode2 are likely to be associated to the spreading of the jet coverage due to the effects of the side vortices. For the sake of brevity, the $3^{rd}$ and $4^{th}$ modes are not presented here as they appeared very similar to the two first modes although at smaller length scales.
Figure 5.7: Reconstructed velocity field for multiple values of \( N_r \). Stream traces correspond to in-plane velocity, contours to streamwise velocity.

The 5th mode, however associated with a lower eigenvalue (figure 5.4a), captures a different behavior as shown in figure 5.5(k-m) and figures 5.6(g, h). At \( X_j = 6 \) and \( X_j = 10.6 \), Mode5 exhibits two continuous zones in the streamwise direction with positive and negative values. This mode appears to capture a different type of mixing since conversely to Mode1 and 2 the spatial distribution of the mode is not alternating signs in the downstream direction and no other POD mode was found to assume a similar distribution. These observations suggest Mode5 is associated with the mixing behavior induced by the CRVP rather than the downstream convective effect of the hairpin vortices. Hence, the stacking of an area with negative values on top of one with positive values can be interpreted by the CRVP bringing cooler fluid to the upper part of the domain while entraining hotter fluid near the wall and toward the symmetry plane. Near the wall in figure 5.6(h), a large area of negative values is visible around \( X_j = 8 \), \( Y_j = \pm 0.5 \) where the jet coverage starts to degrade due to cross-flow entrainment. A similar distribution is found near the wall in Mode4 although the rest of the distribution is more consistent with the Mode3. From these observations, it can be argued that the temperature POD modes are more correlated than the velocity modes and part of the information associated with specific phenomena can be captured across several modes.

Figure 5.4(a) provides an estimation of the total amount of kinetic and pseudo-thermal energies captured by the POD modes as well as the cumulative energy captured by the \( N \) first modes. According to the latter, 61 velocity POD modes and 53 temperature POD modes are required to gather at least 90% of the total energy and respectively 156 and 163 modes to gather 99% while modes beyond respectively \( N_V = 17 \) and \( N_T = 15 \) carry less than 1% of the total energy.

In figure 5.7 an instantaneous velocity field is reconstructed from truncated POD series to assess the number of modes necessary to rebuild the dominant flow features. In the plane \( Y_j = 0 \), the first reconstruction using only 2 modes does not render the formation of the individual hairpin vortex present in the original snapshot at \( X_j = 10.6 \). This was to be expected due to the absence of information carried by higher order modes with respect to the hairpin vortices as seen in 5.2(l), thus justifying the lack of accuracy of the reconstructed field. The individual hairpin vortex is observed for \( N_r \geq 8 \). In the near field of the jet \((X_j < 6)\) the reconstruction appears relatively close to the original field with as few modes as 8, due to the low fluctuation levels and limited length scales of the structures in this region. In the plane \( X_j = 6 \) however, the features of the original flow such as the two counter rotating vortices located at \( Z_j = 0.3 \), \( Y_j = \pm 1 \) are only consistently reconstructed for \( N_r \geq 20 \). Similarly in the plane \( X_j = 10.6 \), the location and scale of the multiple structures present in the original snapshot are only captured in reconstructions including 20 modes or more. Overall, although closer from the actual snapshot, the reconstruction using 60 modes does not appear to constitute a significant improvement over the one with 20 modes in terms of the dominant features.
Figure 5.8: Reconstructed temperature field for multiple values of $N_r$.

Figure 5.8 shows reconstructed and original temperature fields corresponding to figure 5.7. Overall in the plane $Y_j = 0$, the reconstructed temperature field appears to render individual structures beyond $N_r = 8$, with a relatively satisfying definition for $N_r \geq 20$. Similarly, in the planes $X_j = 6$ and $X_j = 10.6$, the reconstructed fields show very little differences. Beyond the qualitative examination of the reconstructed fields, time averaged error distributions with respect to the original field were also calculated to assess the quantitative accuracy obtained with a given number of POD modes. For the velocity field, the error in local kinetic energy. This choice has the advantage of combining all three velocity components into a single figure of merit in addition of being directly related to the norm used for the orthogonal decomposition. However, as the 0th mode corresponds to the time averaged velocity field, not carrying any intrinsic error, it appears more adequate to monitor the error on the turbulent kinetic energy (TKE) rather than the total kinetic energy. The errors distributions on the turbulent kinetic energy are presented in figure 5.9. In the plane $Y_j = 0$, two zones can be identified. Beyond $X_j = 6$ the error on the decreases significantly to be almost zero for $N_r = 60$. However for $X_j < 6$, the error, although decreasing, stays significant in the jet shear layers even with as many as 100 POD modes. It should also be noted that the extent of this zone of relatively high error decreases with increasing number of POD modes included in the decomposition. This can be explained by the nature of the proper orthogonal decomposition and the norm associated with it, imposing a hierarchy of the modes based on the amount of kinetic energy they capture so that the first modes will capture the large scale structures of the domain. In jets in cross-flow, the principal instability responsible for the formation of the shear layer hairpin vortices is of the convective type and exhibits multiple length scales as it develops in the downstream direction to reach maximum intensity in the far-field. The POD modes thus capture first the downstream motion carrying the largest amount of energy and then progressively capture the lower shear layer fluctuations by going in a reverse direction to the development of the instability justifying the relatively poor quality of the reconstruction in the near field of the jet. This constitutes one of the limitations of the standard POD which may lead to dynamical inaccuracy when trying to implement a reduced order model of the flow as pointed out by Ilak & Rowley (2008).

Since no perturbation on the temperature field is imposed at the inlets of the domain and by nature the energy equation is free of an explicit non-linear term in $T$ capable of producing or sustaining fluctuations, in a large part of the domain the true temperature fluctuation is extremely low, even null, and thus induced extremely high errors on the fluctuation value when monitoring the reconstructed temperature field. Based on these considerations, it is decided to monitor the absolute relative error on the total temperature field rather than the fluctuation part, reported in figure 5.10. The error on the temperature field decreases consistently with increasing number of POD modes included in the reconstruction. Most of the error is concentrated beyond $X_j = 5$ where most of mixing and temperature fluctuations occur. It should be noted that similarly to the turbulent kinetic energy, the error in the shear layer of the near-field, appears to decrease.
Figure 5.9: Average TKE error of the reconstructed fields for various values of \( N_r \). Maximum contour value 1 (white).

Figure 5.10: Average temperature error of the reconstructed field at various \( N_r \) values. Maximum contour value: \( 7 \times 10^{-3} \) (white).

at a slower rate than in the far-field. Overall with \( N_r = 20 \), the maximum error was found to be inferior to 2%, decreasing to less than 1% with \( N_r = 60 \).

5.1.2 Forced Jets

As for the steady state, LES flow fields and temperature fields for Case X at \( St_{\infty} = 0.016 \) are analyzed using 3D-POD. The domain and spatial sampling used for the forced cases are identical to the one described in the previous section. Based on the work of Vernet et al. (2009) on 2D-POD of a pulsed detached jet in cross-flow, an initial temporal sampling of 25 phase locked positions over 10 periods is considered. However, given that the forcing signal used in Vernet et al. (2009) was a sinusoidal, thus significantly different from the square wave used in the current study, a finer sampling of 50 phase locked positions over 10 cycles is preferred. The POD is computed on both the full time sequence and the phase averaged signal. As seen in figure 5.13, both methods provide sensibly similar results for the first 20 modes and starts to diverge for higher order modes. This result is expected and consistent with the findings of Vernet et al. (2009). Indeed, when considering the complete time sequence, the fluctuation part due to more fine-grained turbulence is included in the signal and requires a large number of modes to be fully resolved. On the other hand, when analyzing the phase averaged signal, the turbulent fluctuation is removed from the signal thus requiring fewer modes to capture the bulk flow fluctuations (phase averaged). While in the steady state POD analysis, the turbulent fluctuations were considered important to model the flow behavior, in the pulsed system, the
Figure 5.11: Mean flow ($0^{th}$ POD Mode) and first significant velocity POD modes for Case X at $St_{\infty} = 0.016$.

(a-c) Mode0; (d-f) Mode1; (g-i) Mode2; (j-l) Mode6. Slices at $X_j = 6$ (left), $X_j = 10.6$ (right) with $U$-velocity contours and $V-W$ streamlines. $\lambda_2$ iso-surfaces (right) from corresponding POD modes colored by $U$-velocity and mean wall temperature contours (gray scale).

Figure 5.12: Mean flow and first significant velocity POD modes for Case X at $St_{\infty} = 0.016$; (a, b) Mode0; (c, d) Mode1; (e, f) Mode2; (g, h) Mode6. Slices at $Y_j = 0$ with $V$-velocity contours and $U-W$ streamlines (left) and slices at $Z_j = 0.25$ with $W$-velocity contours and $U-V$ streamlines (right).

Significant part was considered to be the phase averaged variation thus the results presented in this paper are based on the POD analysis of the phase averaged signal.

For the sake of brevity, figures 5.11 and 5.12 only present modes 0, 1, 2 and 6 issued from the decomposition of Case X at $St_{\infty} = 0.016$. As for the steady state, the $0^{th}$ mode corresponds to the average flow field. Based on the interpretations of the velocity shape functions it is possible to qualitatively identify which features of the forced jet are being captured by individual modes. The $1^{st}$ mode corresponds to the bulk flow fluctuation associated with the change in blowing ratio and captures the global jet expansion and shrinking occurring during a cycle. No evidence of a vortical structure is observed in the upper shear layer in figure 5.12(c), although a counter rotating vortex pair is visible in the constant $X_j$ slices of figure 5.11(d) and 5.11(e). The $2^{nd}$ mode exhibits large scale structures in the jet upper shear layer in figure 5.12(e), converging velocity field toward the jet exit as well as strong vertical vorticity in the plane $Z_j = 0.25$ of figure 5.12(f), and is predominantly significant in the near field of the jet in figure 5.11(i). These considerations suggest that Mode2 is correlated to the large scale structures introduced at the transition from $BR_l$ to $BR_h$ and...
Figure 5.13: POD decomposition metrics for Case X at \(St_\infty = 0.016\) (a) Temperature and velocity POD modes eigenvalues and cumulative energy; (b) Velocity POD coefficients; (c) Temperature POD coefficients. Open symbols correspond to full time sequence POD. Arrow points toward \(t^* = 0\) in the time sequence.

from \(BR_h\) to \(BR_l\), being starting vortices and/or ingestion as described in the previous chapter. It should be noted that in this forced case, the cumulative captured kinetic energy of modes 1 and 2 is equivalent to 53\% of the total kinetic energy which is to be put in perspective with the 30\% in the steady state case at \(BR = 0.15\). Modes 3 to 5 were very similar to the 2\(^{nd}\) mode with finer scales were considered as to capture smaller features associated with the introduction of the transient regimes and thus were not presented here for brevity. However, Mode6 in figure 5.12(g) evidences the presence of shear layer vortices consistent with the natural hairpin vortices encountered in unforced conditions, and the \(\lambda_2\) iso-surfaces of figure 5.11(m) are very comparable to the one found in dominants modes at \(BR = 0.15\) (see previous section). This implies that the 6\(^{th}\) mode is likely related to the quasi-unforced behavior during either the low or high part of the cycle. This last point will be developed later and lead to the observation of the POD mode segregation. Conversely to the steady state, the circular shape of the phase diagram in figure 5.23(b) should not be interpreted as a sign of correlation between the 1\(^{st}\) and 2\(^{nd}\) modes since the signal is by nature periodic. However, the circular shape shows that both modes operate at the same frequency and with almost equal influence. Interestingly, clusters of points can be observed before the beginning (red arrow) of the cycle and before the transition from high to low blowing ratio (diametrically opposed to the red arrow) where both 1\(^{st}\) and 2\(^{nd}\) mode values stagnate, the former at a maximum (or minimum) and the latter at a zero value. This observation confirms the qualitative interpretation previously made where Mode1 would correspond to the bulk modulation of the jet envelope thus would be predominant away from the transition points, and Mode2 would correspond to the large scale structures of the transition and would have high values at the transitions.

The temperature field is analyzed as well using POD, the results of which are presented in figure 5.14 and 5.15. Similarly to the velocity field decomposition, the 0\(^{th}\) mode in figure 5.14(a-c), 5.15(a, b) corresponds to the average temperature field, while Modes 1 and higher correspond to the fluctuations around this time averaged field. As for the velocity decomposition, the 1\(^{st}\) mode, which is the most energetic, appears to describe the global temperature fluctuations due to the change in penetration associated with jet forcing. The 2\(^{nd}\) mode shows a more localized distribution with positive values in the vicinity of the jet exit and negative values in the far field. The phase diagram in figure 5.13(c) shows strong similarities with the one of the velocity field and suggests that while Modes1 and 2 have similar overall impact in terms of amplitude and frequency, the moments at which they are acting on the flow are different. Hence Mode1 has a stronger influence away from the transitions while Mode2 affects the flow mainly at the transitions moments. Modes 3 to 9 exhibited distributions similar to the one of Mode2 with yet smaller length scales and were considered to describe the smaller scale perturbations introduced by the transients thus not presented here. The 10\(^{th}\) mode
though shows fluctuations in the jet shear layer further away from the jet exit which are consistent with the one observed in unforced conditions at $BR = 0.15$. This suggests that this mode captures the quasi-unforced nature of the jet away from the transitions. The energy distribution in figure 5.13(a) shows that a total of 41 velocity modes and 33 temperature modes are required to gather 99% of the total fluctuation energy.

To confirm the qualitative interpretations of the POD modes the coefficients $a_1$, $a_2$, $a_6$ for the velocity field and $a_1$, $a_2$, $a_{10}$ for the temperature field were plotted in figure 5.16(a) versus $t^*$ along with the phase averaged blowing ratio profile. In both decompositions, $a_1$ has broad periods of maxima in absolute value beyond $t^* = 0.20$ up to $t^* = 0.55$ and from $t^* = 0.80$ to $t^* = 1$, both corresponding to the respective established quasi-unforced regimes while $a_2$ exhibits more localized maxima, directly after the transitions, from one part of the cycle to the other. Modes 6 for velocity and 10 for temperature have significant values during the quasi-unforced regime in the high part of the cycle also confirming the qualitative analysis. The plots of intermediate coefficients ($a_3$ to $a_5$ for velocity and $a_3$ to $a_9$ for temperature) showed that their support of action was also localized within the transient regions of the cycle and the amplitude of their respective maxima was decreasing along with the width of the peak, exposing a more localized and finer
Figure 5.16: Temporal evolution of the POD modes coefficients $a_i^{Vel}$ for the velocity (top) and $a_i^{Temp}$ for temperature (middle) decompositions along with forcing blowing ratio profile (bottom) at (a) $St_\infty = 0.016$; (b) $St_\infty = 0.0159$. The values of $a_{41}$ in (a) are multiplied by a factor 10 for increased visibility.

scale influence. Although the transient and the high quasi-steady regimes are captured by modes 1 to 6, none of the first significant modes, except for the first one describing the bulk flow modulations, seem to have significant non-zero values during the low quasi-steady part of the cycle. Only modes beyond the 40$^{th}$ one have their support of action localized during this part of the cycle as seen in figure 5.16(a) for $a_{41}$ (multiplied by 10 to increase visibility). This can be explained by the fact that vortical structures formed during the low quasi-unforced part of the cycle are relatively energetically weak, compared to the transient vortical structures or even the one formed during the quasi-steady high part of the cycle, thus are relegated to the end of the POD spectrum as weak perturbations.

Figure 5.17 shows the phase locations of the maximum and minimum values of the modal coefficients for each mode revealing the moment in the cycle where they have maximum influence. For both velocity and temperature, a clear pattern appears in the distribution of the maxima and minima. Indeed most of the 40 first modes have an influence on the high part of the cycle with some of them, similarly to Mode2 described previously, having effects at the transition from $BR_h$ to $BR_l$ as well. However, the support for modes 40 and above is almost exclusively located in the low quasi-unforced part of the cycle. Such segregation of the POD modes could have a negative impact on the reconstructed flow field as well as on a reduced order model resulting from truncation of the POD series as it would obliterate a significant part of the cycle.

Although Modes 1 and 2 are the most energetic, the complexity of the flow field generated at the transitions from one part of the cycle to the other prevents us from drawing definitive conclusions based only on the observation of these modes. They however provide a first order estimate of the impact of jet forcing on the temperature field, and particularly at the wall from a film cooling point of view. The first order effect of jet forcing on the wall temperature (corresponding to Mode1 in figure 5.14d-f, and 5.15c, d) appears to be located directly around the jet exit and corresponds to the increase in spread observed in figure 3.40(a) at the jet onset as well as directly downstream of the jet exit due to increase in coolant mass flow. Because $a_1$ changes sign throughout the cycle (positive over the high part and negative over the low part), the first order effect of jet forcing is a decrease in wall temperature during the high part and an increase during the low part. Overall, the highest values for Mode1 are located away from the wall suggesting a considerable waste of coolant in the free-stream. The second order effect represented by Mode2 shows that the transitional regime (during which Mode2 is dominant) affects more strongly the wall temperature than the bulk effect of forcing. The effect of the jet onset ($a_2 < 0$) over the average temperature field is to decrease the wall temperature locally around the jet exit due to local increase in spread, but increase it further downstream probably due to the increased entrainment associated with the starting structures lifting off of the wall. At the jet shutdown ($a_2 > 0$), the temperature around the jet increases due to the shrinking of the jet coverage associated with the decrease in coolant mass flow and the ingestion of cross-flow fluid. The downstream effect on the wall temperature is overall positive as the weaker vortical structures generated during the low part of the cycle do not entrain as much cross-flow and tend to remain attached to the wall.
Figure 5.17: Phase distribution of the minima (green diamond) and maxima (red squares) associated with the POD modes coefficients $a_i^{Vel}$ for the velocity (top) and $a_i^{Temp}$ for the temperature (middle) along with phase averaged forcing signal (bottom).

Figure 5.18: Reconstructed temperature field for multiple values of $N_r$ at four different phase locations: $t^* = 0.06, 0.26, 0.56, 0.76$. 
Figure 5.19: Error on the reconstructed velocity field for different $N_r$ values estimated with phase averaged fluctuation of kinetic energy at $t^* = 0.06, 0.26, 0.56, 0.76$. Maximum value (white) is 9%.

Figure 5.20: Error on the reconstructed temperature field for different values of $N_r$ estimated using the total value of the temperature at $t^* = 0.06, 0.26, 0.56, 0.76$. Maximum value (white) is 9%.

Figure 5.18 presents the reconstructed temperature fields with $N_r = 2, 6, 15$ modes along with the original temperature field at four phase locked positions. Overall the temperature field appears relatively well reconstructed with only 15 POD modes. However, while the increase from 2 to 15 modes brings significant improvement in the first 3 phase positions, the reconstruction at $t^* = 0.76$ does not show the same details as the one at $t^* = 0.06$ for $N_r = 15$. This is an illustration of the effect of the absence of the higher order modes capturing the behavior in the quasi-steady low part of the cycle. It should be noted that the reconstructed temperature field at the wall does exhibit the dominant features of the original field with only 6 modes.

Although the reconstructed velocity field is not presented, an estimate of the error on the reconstruction is shown in figure 5.19. Because the POD analysis was performed on the phase averaged signal, it is impossible to base the estimate of the error on the turbulent kinetic energy. Instead, the error ($e_r$) is estimated on the kinetic energy of the phase average fluctuation normalized by the total kinetic energy. Using $U_r = \bar{U} + \tilde{U}_r$ where $U_r$ is the total reconstructed phase averaged field, $\bar{U}$ is the time averaged signal (equal to the true LES time averaged field) and $\tilde{U}_r$ the reconstructed phase averaged fluctuation using $N_r$ POD modes. Similarly, the true phase averaged velocity field can be decomposed into $U = \bar{U} + \tilde{U}$. The expression for the error on the phase averaged fluctuation is then:

$$e_r = \frac{\tilde{U}_r^2 + \tilde{V}_r^2 + \tilde{W}_r^2 - (\bar{U}^2 + \bar{V}^2 + \bar{W}^2)}{\bar{U}^2 + \bar{V}^2 + \bar{W}^2}$$

Figure 5.19 shows that the error decreases consistently with increasing numbers of modes although not equally across all phase positions. Indeed while the error decreases significantly from $N_r = 2$ to $N_r = 25$ at $t^* = 0.06$, it still stays relatively high for the other phase locations until $N_r = 40$. Even with $N_r = 40$, the reconstructions at $t^* = 0.26$ and 0.76 still show some error due to the truncation although in absolute value below 4% of the total instantaneous kinetic energy.
As for the steady state, the error on the reconstructed temperature field is estimated based on the relative error on total temperature (mean and fluctuation) and is presented in figure 5.20. Similarly to the velocity field, the error decreases consistently with increasing number of POD modes used in the reconstruction, although not homogeneously across all the phase positions. While the maximum error at $t^* = 0.06$ decreases from 9.2% at $N_r = 2$ to 0.2% at $N_r = 25$, corresponding to a factor of 46, it decreases at $t^* = 0.26$, 0.56 and 0.76 only by factors of respectively 2, 5, and 1.75. Nevertheless, the relative error does not exceed a maximum of 4% across the cycle with $N_r = 10$ and 2% with $N_r = 25$ and above. On both velocity and temperature fields, it is verified that the error asymptotically converges towards 0 when using all the modes for the reconstruction so that no loss of information is introduced by the proper orthogonal decomposition.

Simulation results from Case X at $St_\infty = 0.159$ are analyzed using 3D-POD as well. Identical domain and spatial samplings are used. The temporal sampling consists here of 25 phase locked positions over 10 cycles accounting for a total of 250 snapshots. As for the previous forced case, the decomposition is carried out on both the complete time sequence and the phase averaged signal. The complete energy distribution

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**Figure 5.21:** Mean flow ($0^{th}$ POD Mode) and first significant velocity POD modes for Case X at $St_\infty = 0.159$ (a-c) Mode0; (d-f) Mode1; (g-i) Mode3; (j-l) Mode5. Slices at $X_j = 6$ (left), $X_j = 10.6$ (center) with $U$-velocity contours and $V$-$W$ streamlines. $\lambda_2$ iso-surfaces (right) computed from corresponding POD modes and correlated mode (white) colored by the corresponding $U$-velocity and mean wall temperature contours (gray scale).

**Figure 5.22:** Mean flow and first significant velocity POD modes for Case X at $St_\infty = 0.159$ (a, b) Mode0; (c, d) Mode1; (e, f) Mode3; (g, h) Mode5. Slices at $Y_j = 0$ with $V$-velocity contours and $U$-$W$ streamlines (left) and slices at $Z_j = 0.25$ with $W$-velocity contours and $U$-$V$ streamlines (right).
Figure 5.23: POD decomposition metrics for Case X at $St_{\infty} = 0.159$ from LES (a) POD modes eigenvalues and cumulative energy for temperature and velocity; (b) Velocity POD coefficients; (c) Temperature POD coefficients. Open symbols correspond to full time sequence POD. Arrow points toward $t^* = 0$ in the time sequence.

and the phase diagrams for the first 6 pairs of modes are presented in figure 5.23. Once again, decompositions of instantaneous and phase averaged flows are identical up to the $10^{th}$ POD mode and diverge for higher order modes due to the presence of the turbulent fluctuation in the full time sequence series. As for the previous case at $St_{\infty} = 0.016$, the decomposition on the phase averaged signal is preferred. The first noticeable difference between the two forced cases is found in the shape of the captured energy distribution. While in figure 5.13(a) at $St_{\infty} = 0.016$, the energy distribution does not exhibit a particular shape, the one at $St_{\infty} = 0.159$ assumes a clear stair-like shape. It should also be noted that the cumulative kinetic energy captured by the first two POD modes correspond to more than 65% of the total energy. Although in the presence of a forced case where the POD modes are automatically correlated to the forcing signal, the quasi-prefect circular distribution of the successive pairs of modes presented in the phase diagrams of figure 5.23(b) and 5.23(c) is a consequence of the stream-wise homogeneity of the flow making POD modes converge toward Fourier modes (see Holmes et al. 1998). This result is a consequence of the fact that at $St_{\infty} = 0.159$, multiple forcing cycles affect the flow field and that the dominant events are the generation and convection of the starting vortices. Hence the problem involves less length and time scales compared to the lower frequency case where the jet exhibited four distinct regimes each with distinct time and length scales.

POD modes issued from the decomposition of the velocity fields are presented in figures 5.21 and 5.22. While the $0^{th}$ POD mode corresponds to the average flow field, the $1^{st}$ and $2^{nd}$ are virtually identical with the exception of a phase shift in the downstream direction as seen in the superposition of $\lambda_2$ iso-surfaces in figure 5.21(f). Both modes assume the shape of multiple large scale hairpin vortices penetrating deeply in the free stream. A set of side vortices is visible in the $\lambda_2$ iso-surfaces as well as in the $X_j = 6$ slice of the flow field in figure 5.21(c). These correspond to the large scale side vortices formed near the wall close to the starting vortices and observed in instantaneous snapshots of figure 3.40. The third and fourth modes, also largely identical with a shift in phase, correspond to the first harmonic of the $1^{st}$ and $2^{nd}$ modes respectively. The phase diagram corresponding to these modes in 5.23(b) shows that while $a_1^{Vel}$ and $a_2^{Vel}$ complete a single revolution during a cycle, $a_3^{Vel}$ and $a_4^{Vel}$ complete two. Similarly, Modes 5 and 6 are the second harmonics of Modes 1 and 2 respectively. The energy distribution in figure 5.23(a) shows that a total of 12 velocity modes and 15 temperature modes are required to reconstruct 99% of the total fluctuation energy.

As for the previous pulsed case, the modal coefficients $a_1^{Vel}$ through $a_6^{Vel}$ are plotted versus time along with the forcing signal in figure 5.16(b). The phase shift between the correlated pairs of modes is easily quantifiable and corresponds to a quarter of the period for the first two, an eighth for modes 3 and 4 and a
Figure 5.24: Mean temperature field (0th POD mode) and first significant temperature POD modes for Case X at $St_\infty = 0.159$ (a-c) Mode0; (d-f) Mode1; (g-i) Mode3; (j-l) Mode5. Slices at $X_j = 6$ (left), $X_j = 10.6$ (center) with temperature contours. Iso-T surfaces (right) computed from corresponding pairs of POD modes (transparent) and mean wall temperature contours (gray scale).

Figure 5.25: Mean temperature field and first significant temperature POD modes for Case X at $St_\infty = 0.159$ (a, b) Mode0; (c, d) Mode1; (e, f) Mode3; (g, h) Mode5. Slices at $Y_j = 0$ with temperature contours (left) and slices at $Z_j = 0$ with temperature contours (right).

sixteenth for modes 5 and 6. Conversely to the $St_\infty = 0.016$ case, none of the modes appears to dominate over one particular part of the forcing cycle and the coefficient converge towards cosine and/or sine functions.

The POD modes corresponding to the temperature field analysis are presented in figures 5.24 and 5.25. As in the velocity POD, the first two modes are quasi-identical with a shift in the stream-wise direction. The iso-surfaces in figure 5.24(f) have a hairpin-like shape, alternating positive and negative values in the stream-wise direction as in the unforced case at $BR = 0.15$. Higher order modes are paired similarly to the velocity field decomposition and correspond the successive harmonics of the first two modes. The evolution of the modal coefficients $a_{1\text{Temp}}^T$ through $a_{6\text{Temp}}^T$ with respect to time in figure 5.16(b) appears almost identical to the one obtained for the velocity field.

In opposition to the lower forcing frequency case, the impact of the different modes on the wall temperature decreases in scale and amplitude as mode order increases. Almost no impact on the wall temperature is observed beyond $X_j = 6$. In the vicinity of the jet exit, the 1st temperature POD mode shows alternating positive and negative values due to the formation and convection of the starting structures. A band of high
positive values covering part of the upstream edge of the jet exit is associated with the successive cross-flow ingestions and coverage increases occurring periodically at the jet shutdown and onset.

The reconstructed temperature field using 2, 4 and 6 POD modes is presented in figure 5.26. The reconstruction quality is homogeneous with time and although the finest details are not represented, the fields including 4 and more modes provide qualitatively a reasonably good reconstruction. The error on the reconstructed velocity field and temperature fields are also investigated using different values of $N_r$. In figure 5.27 the error on the velocity field is estimated in the same way as for the lower forcing frequency case. The bulk of the error resides in the region where the core of the starting vortex is located which also corresponds to the regions of higher velocity fluctuations. In this higher forcing frequency case, the relative error decreases consistently with increasing number of included POD modes at all time steps. For $N_r = 15$, the overall maximum error does not exceed 4%. Finally, the error on the temperature field is also evaluated and presented in figure 5.28. As for the velocity reconstruction, the error decreases at all phase locations consistently with increasing number of POD modes included in the reconstruction. The effect of the individual modes is clearly visible on the reduction of the error as the scale of ‘error patches’ decreases with increasing values of $N_r$ involving higher harmonics thus lower length scales. Six modes are required to obtain a maximum error inferior to 4% while 10 modes ensure a maximum error on the temperature field of the order of 2%.

These results evidence the sorting mechanism associated with proper orthogonal decomposition in both unforced and forced jets. In particular, in forced jet conditions, it is found that the flows forced at low frequencies have decompositions which are much different from the ones at higher forcing frequencies when all other parameters are maintained fixed. In addition, due to the discrete length and energy scales introduced in forced jet conditions, the energy distribution of velocity and temperature decompositions are much more...
organized and exhibits a much more pronounced hierarchy compared to the unforced jet, even at the lowest blowing ratio.

5.2 Reduced Order Models

Simulation results from the ROM grid described in Chapter 3 are used to obtain reduced order models of both the unforced and forced inclined jets. The POD-Galerkin method described in Chapter 4 is essentially applied to the film cooling jet simulations, although the three-dimensional, turbulent character of realistic film cooling flows can greatly affect the reduced order model accuracy and stability. Additional considerations are therefore required to stabilize the equations.

5.2.1 Stabilization Methods

As mentioned previously, many studies involving two-dimensional low Reynolds number flows have provided very accurate reduced order models using the POD-Galerkin method. Two examples of quite successful model reductions using this method are in fact presented in Chapter 4 for a tall double driven cavity and a cylinder in cross-flow. However, many other studies have shown the complexity of the task when dealing with more realistic three-dimensional flows, or flows involving turbulence. Obviously, trying to obtain reduced order models for three-dimensional turbulent flows constitutes an even more challenging task. Several reasons can be provided to explain the increased difficulty associated with reduced order modeling of realistic flows. One of them resides in the nature of the proper orthogonal decomposition sorting mechanisms. As evidenced in the preliminary POD analysis of forced film cooling flows, while the decomposition prioritizes the modes in term of their energy content, this does not necessarily mean it sorts them according to their “dynamic relevance”. In simple flows involving a unique length scale, such as the ones presented in Chapter 4, energy content and dynamic relevance often go hand in hand so that the most energetic modes are also the most relevant ones. However when the flow involves many different types of vertical structures, some dynamically relevant ones can be overwhelmed by others and be later on neglected when truncating the POD series. This was evidenced in the previous section by the mode segregation at low forcing frequencies. Other examples of dynamically relevant low order modes can be found in Rowley et al. (2004), Rowley (2005), Ilak & Rowley (2008). Another source of error can arise from the truncated modes and their associated dissipative mechanism. Most of the energy transfer is usually considered to go from the larger scales towards the lower ones through the “energy cascade”, to be finally dissipated through viscous forces into thermal energy at the Kolmogorov scale level. The effect of the POD series truncation is to effectively interrupt the energy cascade and prevent the natural energy dissipation scheme from taking place. This usually causes accumulation of energy into the POD based reduced order models which very often leads to system model inaccuracy and instability.

To balance the underestimated diffusivity of the system, many solutions have been developed, most of which consist in adding a turbulent viscosity $\nu_T$ term to the already existing viscosity $\nu = \frac{1}{Re}$ to account for additional diffusivity from the neglected modes so that the momentum diffusivity effectively becomes
\( (\nu + \nu_T) = \frac{1}{Re} + \nu_T \). This follows one of the most common way to model turbulence using Boussinesq eddy viscosity assumption (see Boussinesq 1878). Early models such as the one from Aubry et al. (1988) or Holmes et al. (1998) modeled the turbulent viscosity using a constant based on mixing length theory so that \( \nu_T = \alpha U_T L_T \) where \( U_T \) and \( L_T \) are unresolved velocity and length scales estimates and \( \alpha \) a factor of order one. Later on, Cazemier et al. (1998) proposed to model the truncated modes diffusivity using a first order linear term \( S_r^V \) added to the ROM ordinary differential equations 4.32 (the pressure term is neglected here) so that:

\[
\dot{a}_r(t) = \sum_{i=1}^{N_r} \left( C_{r0}^r + \frac{1}{Re_j} D_i^r \right) a_i(t) \\
+ \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} C_{ij}^r a_i(t) a_j(t) \\
+ \left( C_{r0}^r + \frac{1}{Re_j} D_0^r \right) a_r(t) S_r^V (5.1)
\]

In the rest of the chapter, the characteristic velocity, temperature and length scales mentioned in equations 4.22, 4.23 and 4.24 will be taken as \( U = BR \times U_\infty \), \( T_1 = T_{jet} \), \( T_2 = T_\infty \), \( L = D_j \) so that \( Re = Re_j = \frac{BR \times U_\infty \times D_j}{\nu} \) is the jet Reynolds number.

Effectively, this provides a turbulent viscosity term that is no longer a simple constant as in Aubry et al. (1988), but is mode dependent. The coefficient \( S_r^V \) is then calculated based on energy conservation considerations for each individual mode. The kinetic energy contained in mode \( r \) corresponds to \( a_r^2(t) \), therefore, the associated rate of change of the energy is:

\[
\frac{d}{dt}(a_r^2) = 2a_r \dot{a}_r
\]

This can be expressed using 5.1 so that:

\[
\frac{1}{2} \frac{d}{dt}(a_r^2) = \sum_{i=1}^{N_r} \left( C_{r0}^r + \frac{1}{Re_j} D_i^r \right) a_i(t) a_r(t) \\
+ \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} C_{ij}^r a_i(t) a_j(t) a_r(t) \\
+ \left( C_{r0}^r + \frac{1}{Re_j} D_0^r \right) a_r(t) + a_r^2(t) S_r^V (5.2)
\]

The parameter \( S_r^V \) is then calculated using the fact that the rate of change of the energy in mode \( r \) must on average be zero. Taking the average \( \langle . \rangle \) through time of equation 5.2 and using the properties from the POD temporal coefficients \( \langle a_i a_j \rangle = \delta_{ij} \lambda_i \) (see equation 4.18), and \( \langle a_i \rangle = 0 \) we obtain:

\[
\langle \frac{1}{2} \frac{d}{dt}(a_r^2) \rangle = \left( C_{r0}^r + \frac{1}{Re_j} D_r^r \right) \lambda_r + \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} C_{ij}^r \langle a_i(t) a_j(t) a_r(t) \rangle + \lambda_r S_r^V = 0
\]

Therefore:

\[
S_r^V = -\frac{1}{\lambda_r} \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} C_{ij}^r \langle a_i(t) a_j(t) a_r(t) \rangle - \left( C_{r0}^r + \frac{1}{Re_j} D_r^r \right) (5.3)
\]
The choice of $U = BR \times U_\infty$ over simply $U_\infty$ was made so that the blowing ratio would explicitly appear in the reduced order model equations 4.32 and 4.33 as part of the $\frac{1}{Re}F_{s_0}$ term. Based on the considerations from Smith et al. (2005), this can provide a ROM approximation at any operating blowing ratio. Effectively, we believe this will be true for values of $BR$ in the (very) close neighborhood of the value used to derive the reduced order model. Using considerations similar to the one leading to the linear damping of the velocity ROM equations, we propose a linear diffusion term $S_T$ for the energy reduced order model so that equation 4.33 is reformulated into:

$$
\dot{b}_s(t) = \sum_{i=1}^{N_V} E_{s_0}^i a_i(t) \\
+ \sum_{j=1}^{N_T} \left( E_{s_j}^0 + \frac{1}{Pr} \frac{1}{Re_j} F_{s_j} \right) b_j(t) \\
+ \sum_{i=1,j=1}^{N_V,N_T} E_{s_j}^i a_i(t) b_j(t) \\
+ \left( E_{s_0}^0 + \frac{1}{Pr} \frac{1}{Re_j} F_{s_0} \right) b_s + b_s S_T 
$$

(5.4)

The pseudo-thermal energy (or the temperature signal energy) in mode $s$ is $b_s^2(t)$ and its rate of change:

$$
\frac{1}{2} \frac{d}{dt} (b_s^2) = 2b_s \dot{b}_s
$$

This can be expressed using equation (5.4) so that:

$$
\frac{1}{2} \frac{d}{dt} (b_s^2) = \sum_{i=1}^{N_V} E_{s_0}^i a_i(t) b_s(t) \\
+ \sum_{j=1}^{N_T} \left( E_{s_j}^0 + \frac{1}{Pr} \frac{1}{Re_j} F_{s_j} \right) b_j(t) b_s(t) \\
+ \sum_{i=1,j=1}^{N_V,N_T} E_{s_j}^i a_i(t) b_j(t) b_s(t) \\
+ \left( E_{s_0}^0 + \frac{1}{Pr} \frac{1}{Re_j} F_{s_0} \right) b_s(t) + b_s^2(t) K_s 
$$

(5.5)

Taking the average of equation (5.5) through time and using the properties from the POD temporal coefficients, we obtain:

$$
\langle \frac{1}{2} \frac{d}{dt} (b_s^2) \rangle = \sum_{i=1}^{N_V} E_{s_0}^i \langle a_i(t) b_s(t) \rangle + \left( E_{ss}^0 + \frac{1}{Pr} \frac{1}{Re_j} F_{ss} \right) \sigma_s + \sum_{i=1,j=1}^{N_V,N_T} E_{s_j}^i \langle a_i(t) b_j(t) b_s(t) \rangle + \sigma_s S_T \\
= 0
$$

Therefore:

$$
S_T = -\frac{1}{\sigma_s} \left( \sum_{i=1}^{N_V} E_{s_0}^i \langle a_i(t) b_s(t) \rangle + \sum_{i=1,j=1}^{N_V,N_T} E_{s_j}^i \langle a_i(t) b_j(t) b_s(t) \rangle \right) - \left( E_{ss}^0 + \frac{1}{Pr} \frac{1}{Re_j} F_{ss} \right) 
$$

(5.6)
Building on Cazemier et al. (1998) derivations, Vigo (1998) proposed a cubic formulation of the stabilizing terms for incompressible flows which although exhibiting very good performance, were however computationally very expensive according to the author himself. Other closure models for POD-based reduced order models have been formulated, mimicking the efforts in turbulence modeling for numerical integration. Among others, models using spectral vanishing viscosity in Sirisup & Karniadakis (2004), Smagorinsky in Borggaard et al. (2011), Wang et al. (2011) and dynamic Smagorinsky sub-grid scale models in Wang (2012) can be mentioned.

Additional studies using optimization algorithms have been attempting to obtain optimal turbulent viscosity values. Bergman (2004) identified mode dependent and both mode and time dependent turbulent viscosity values providing a good fit of the DNS projected values while Couplet et al. (2005) went to the extent of calibrating the complete set of Galerkin coefficients to obtain a proper fit of the experimental data. One of the major issues however with “fitting methods” resides in the accuracy of the model beyond the calibration region and while the method from Bergman (2004) retains some physical meaning, the one presented by Couplet et al. (2005) is purely mathematical and has a tendency to make the model very unstable outside of the calibrated range.

We propose here a variant of the model from Cazemier et al. (1998) by using the turbulent viscosity notation in the governing equations and attempting to balance the energy budget using this term rather than a purely linear term on the rth equation.

We begin by writing the modified velocity model based on 4.32, introducing the mode dependent turbulent viscosity term \( \nu^r_T \):

\[
\dot{a}_r(t) = \sum_{i=1}^{N_u} \left( C^r_{i0} + \left( \frac{1}{Re_j} + \nu^r_T \right) D^r_i \right) a_i(t) \\
+ \sum_{i=1}^{N_u} \sum_{j=1}^{i} C^r_{ij} a_i(t) a_j(t) \\
+ \left( C^r_0 + \left( \frac{1}{Re_j} + \nu^r_T \right) D^r_0 \right) a_r(t) 
\]  

(5.7)

Right away we can see that the modified equations 5.7 are extremely close to the one from Cazemier et al. (1998) in equation 5.1, but instead of letting the rth linear term bear alone the stabilizing action, it is spread out onto all the linear terms. The energy content of mode r is equal to \( a^2_r(t) \), and its rate of change through time is:

\[
\frac{1}{2} \frac{d}{dt} (a^2_r) = \sum_{i=1}^{N_u} \left( C^r_{i0} + \left( \frac{1}{Re_j} + \nu^r_T \right) D^r_i \right) a_i(t) a_r(t) \\
+ \sum_{i=1}^{N_u} \sum_{j=1}^{i} C^r_{ij} a_i(t) a_j(t) a_r(t) \\
+ \left( C^r_0 + \left( \frac{1}{Re_j} + \nu^r_T \right) D^r_0 \right) a_r(t) 
\]  

(5.8)

Taking the time average of equation 5.8 and using the properties of the POD temporal coefficients we get:

\[
\left\langle \frac{1}{2} \frac{d}{dt} (a^2_r) \right\rangle = \left( C^r_{i0} + \left( \frac{1}{Re_j} + \nu^r_T \right) D^r_i \right) \lambda_r + \sum_{i=1}^{N_u} \sum_{j=1}^{i} C^r_{ij} \left\langle a_i(t) a_i(t) a_r(t) \right\rangle \\
= 0
\]

This provides an expression for \( \nu^r_T \):
\[ \nu_T = -\frac{1}{\lambda_r D_r} \left( \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} C_{ij}^T \langle a_i(t) a_j(t) a_r(t) \rangle + \lambda_r \left( C_{rr}^T + \frac{1}{Re_j} D_r^T \right) \right) \] (5.9)

Similarly, we also propose a stabilization for the temperature reduced order model, introducing the turbulent thermal diffusivity \( \alpha_T^s \) for mode \( s \) in the temperature ROM equations:

\[
\dot{b}_s(t) = \sum_{i=1}^{N_v} E_{s0}^i a_i(t) \\
+ \sum_{j=1}^{N_T} \left( E_{s}^{0} + \left( \frac{1}{Pr} + \alpha_T^s \right) \frac{1}{Re_j} F_{sj} \right) b_j(t) \\
+ \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} E_{s}^{i} a_i(t) b_j(t) \\
+ \left( E_{s0}^0 + \left( \frac{1}{Pr} + \alpha_T^s \right) \frac{1}{Re_j} F_{s0} \right) b_s(t) \tag{5.10}
\]

Once again, the impact of the stabilization term is now spread across all the linear terms rather than only on the \( s^{th} \) term. The rate of change of the energy content in the \( s^{th} \) mode is:

\[
\frac{1}{2} \frac{d}{dt} \langle b_s^2 \rangle = \sum_{i=1}^{N_v} E_{s0}^i a_i(t) b_s(t) \\
+ \sum_{j=1}^{N_T} \left( E_{s}^{0} + \left( \frac{1}{Pr} + \alpha_T^s \right) \frac{1}{Re_j} F_{sj} \right) b_j(t) b_s(t) \\
+ \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} E_{s}^{i} a_i(t) b_j(t) b_s(t) \\
+ \left( E_{s0}^0 + \left( \frac{1}{Pr} + \alpha_T^s \right) \frac{1}{Re_j} F_{s0} \right) b_s(t) \tag{5.11}
\]

Taking the time average and using the properties of the POD temporal coefficients we obtain:

\[
\langle \frac{1}{2} \frac{d}{dt} \langle b_s^2 \rangle \rangle = \sum_{i=1}^{N_v} E_{s0}^i \langle a_i(t) b_s(t) \rangle + \left( E_{s}^{0} + \left( \frac{1}{Pr} + \alpha_T^s \right) \frac{1}{Re_j} F_{sj} \right) \sigma_s + \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} E_{s}^{i} \langle a_i(t) b_j(t) b_s(t) \rangle \\
= 0
\]

Therefore providing an expression for \( \alpha_T^s \):

\[
\alpha_T^s = -\frac{1}{Re_j F_{sj}} \left( \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} E_{s}^{i} \langle a_i(t) b_j(t) b_s(t) \rangle + \sum_{i=1}^{N_v} E_{s0}^i \langle a_i(t) b_s(t) \rangle + \left( E_{s}^{0} + \frac{1}{Pr} \frac{1}{Re_j} F_{sj} \right) \sigma_s \right) \tag{5.12}
\]

As mentioned previously, both stabilization methods are very close, so close in fact that the values of \( S_r^V \) and \( S_r^T \) can be directly related to the values of \( \nu_T^* \) and \( \alpha_T^s \) such that:
5.2.2 Unforced Jets

The first reduced order models obtained are based on unforced jets results. Two regimes, attached and transitional, are investigated. Due to the unlikelihood of the detached regime to benefit any film-cooling application, reduced order modeling of these flows is not pursued. In the perspective of obtaining a reduced order model for flow control, the complete domain has to be considered, in particular the jet pipe and its inlet plane which corresponds to the input plane for the control action.

Attached Regime

The $BR = 0.15$ unforced jet is the first one investigated. The LES simulations are carried out using the parameters described in Chapter 3. A POD analysis is performed using a set of 1,000 POD modes following Sirovich snapshot method. This represents a significant increase with respect to the preliminary study presented in the previous section which is justified by the extended domain in the downstream direction, the inclusion of the jet pipe in the decomposition, and the increased grid resolution. Snapshot sampling is done with an equivalent sampling frequency of $300Hz$ ($St_s = 4.8$). The POD metrics are presented in figure 5.29 for both the velocity and temperature fields decompositions. The first 6 to 8 POD velocity and temperature modes are organized in pairs carrying equivalent energy. The first two velocity modes carry each a little more than 5% of the total kinetic energy and the first two temperature modes carry each a little less than 10% of the total pseudo-thermal energy. To capture 99% of the kinetic energy, a total of 395 modes is required, while 99% of the thermal energy is captured by the first 313 temperature modes. These constitute rather large numbers of modes and could pose an issue when integrating the reduced order models ordinary differential equations. The first 14 velocity POD modes $\lambda_2$ iso-surfaces are presented in figure 5.30 and confirm that the first POD modes are associated in pairs, shifted by a quarter of wave length in the
downstream direction. Conversely to the preliminary decomposition, the first mode pair does not correspond to the hairpin vortices, but rather the near-wall side vortices. This is due to the ROM grid being extended in the downstream direction, therefore capturing more of the developing wall structures, which highlights the impact of the POD domain selection onto the hierarchy of the captured vortical structures. The first two modes also exhibit vortical structures in the jet tube developing due to the perturbations imposed at the jet inlet during the simulations. Hairpin vortices start appearing in the second pair of velocity modes, in particular in the mid-field. Higher order modes show structures similar to the ones of modes 1, 2 and 3, 4 or a combination of them at either higher spatial frequencies (modes 5, 6) or different locations (modes 13, 14). The first temperature modes in figure 5.31 are also associated in pairs of similar modes shifted in the downstream direction by a quarter wavelength, which is consistent with the preliminary statistical POD analysis. The first two modes capture the convective effect of the hairpin vortices on the temperature field, while modes 3 and 4 seem to capture temperature fluctuations closer to the wall and may be associated to the effects of the side vortices. Although the first pair of temperature modes captures fluctuations primarily in the mid-field, higher order modes capture features in both the near-field and far-field. Interestingly, modes 7 and 8 seem to represent fluctuations in the transverse planes, likely attributed to the draft of the hairpin vortices legs towards the symmetry plane.

Reduced order models are obtained using the Galerkin projection method described in the previous chapter. For all the following ROMs, the pressure terms arising in equation 4.32 at the boundaries have been neglected. Given the extent of the domain, not all Galerkin coefficients corresponding to all 1,000 POD modes are computed. In addition, the integration cost of more than 1,000 non-linear coupled ODEs seem prohibitive and a maximum limit of 500 equations for the velocity and 500 temperature ROMs is set. Given the large domain and the important number of modes to be included, a parallel MPI code in Fortran 90 was developed to perform the projection across multiple platforms.

To evaluate the sensitivity of the ROMs to the values of $N_V$ and $N_T$, error surfaces similar to the ones obtained for the canonical flows were generated for $\bar{E}_1^V$, $\bar{E}_2^V$, $\bar{E}_1^T$ and $\bar{E}_2^T$ (see equations 4.37, 4.41, 4.38 and 4.42) and are presented in figure 5.44 and 5.45 for ($N_V$, $N_T$) values in the range $[30, 500] \times [30, 500]$. Because of the large number of integrations required, another parallel code using Fortran 90 was developed to simultaneously perform integrations on multiple platforms. Three estimates were made over two, four and six convective time scale units and will be referred as short, medium and long term estimates. The velocity error surfaces show overall rather high error levels regardless of the considered time frame. The similarity between surfaces of $\bar{E}_1^V$ and $\bar{E}_2^V$ shows that the error on the modeled part is so large that the intrinsic error on the non-modeled part of the flow is negligible compared to it. Nevertheless, the error levels tend to decrease with increasing number of POD modes and an optimum number of modes is found for all three time frames at approximately $N_V = 250, 400$ and 500 for respectively the short, medium and long term error estimates. The temperature error surfaces also show significant error levels on both the modeled part of the flow and the total flow. Conversely to the velocity ROM, the short term behavior of the temperature ROM evidences increasing error levels with increasing number of temperature modes though a local minimum is found at $N_V = 250$, $N_T = 120$. The error levels of the medium and long term estimates decrease monotonously with increasing number of velocity POD modes, and increases monotonously with increasing number of temperature modes. This is in agreement with the results obtained for the canonical flows where higher error levels were generally obtained when integrating higher order modes. Including these modes can have a serious impact on the overall accuracy and stability of the the ROM. Given the amplitude of the error, it seems appropriate to primarily investigate the short term behavior of the best performing ROM.

Figure 5.46 shows the integrated temporal coefficients $a^V_i$ for the velocity ROM and $a^T_i$ for the temperature ROM at $N_V = 250$ and $N_T = 120$, along with the corresponding projected LES results (POD temporal coefficients). Both velocity and temperature ROMs show reasonable agreement with the projected LES data up to $t_c = 1$ but start diverging from them beyond that point. The divergence occurs both in amplitude and phase and is particularly obvious in the first temperature coefficients. Several explanations can be thought of to justify the diverging behavior of the velocity and temperature modes. The first one comes from the truncation of the POD series and the presence of unresolved modes, potentially leading to less accurate models. While this may be true, the error estimates in figure 5.44 and 5.45 show that increasing the number of POD modes does not necessarily lead to more accurate models. Another potential source of error could arise from neglecting the pressure term initially present in the equations. To verify the
influence of the pressure term, an empirical pressure correction method as described in Noack et al., 2005 is implemented and the complete ROM equations are integrated over identical time frames. The results are presented in figure 5.34 and show only very little difference with the non-corrected ROM. This provides an a posteriori confirmation of the original assumption of neglecting the pressure term. Another source of error can also come from the chaotic character of turbulent Navier Stokes equations which can lead to very different solutions. Unfortunately, such chaotic behavior is intrinsic to turbulent fluid flows and can only be kept in mind when assessing the accuracy of reduced order models. Finally, a more fundamental source of error can originate from the initial set of snapshots considered to generate the POD basis. First, the sampling frequency used to collect the velocity and temperature fields snapshots acts as a temporal filter and cannot realistically provide a good representation of the vortical structures with a characteristic time-scale of the order of twice the sampling period according to Nyquist criterion. Moreover, the snapshots used in the current study being extracted from LES simulations, suffer from the inherent spatial and temporal filtering introduced by the LES model at the sub-grid scale level. Both temporal, and spatial filters introduced by the sampling but also the chosen turbulence model affect the representation of the smaller dissipative scales and can, combined with the POD series truncation, lead to significant underestimates of the effective momentum and thermal diffusivities.

To try to alleviate the underestimated diffusion, both local and broad linear stabilization methods are used to integrate the velocity and temperature ROMs at multiple values of \( N_V \) and \( N_T \) and the corresponding error surfaces are generated and presented in figure 5.35, 5.36, 5.37, 5.38. Overall, the error on both the velocity and temperature fields drops significantly when using either one of the stabilization methods. The error levels on both \( E_V^N \) and \( E_T^N \) are divided by approximately a factor of 30 using the local stabilization, and approximately a factor of 25 for the broad method over all three considered time frames. Similarly, the temperature error levels drop by a factor 30 for the short term estimate, 150 for the medium term estimate, and almost 300 for the long term estimate when using the local stabilization, and 30, 130 and 200 respectively using the broad method. The short term error surfaces evidence an optimum set of \( (N_V, N_T) = (275, 100) \) at which minimum error levels are reached for both ROMs. Although they are comparable in magnitude, the error levels for the local stabilization are consistently slightly inferior to the corresponding ROMs integrated using the broad stabilization.

The best performing stabilized ROM, with \( N_V = 275 \), and \( N_T = 100 \), is presented in figure 5.39 for the velocity and temperature models integrated using both the local and broad damping methods. For the velocity, the agreement with the LES data is clearly improved over the non-stabilized ROM of figure 5.34 and reasonably good tracking is now obtained up to 3 convective time scale units. The temperature ROM accuracy is also enhanced, and in particular the amplitude divergence observed in the non-stabilized model is no longer visible. Once again, the local and broad stabilization methods provide very comparable ROMs although the former seems to outperform the latter by a small margin.

Finally, the normalized turbulent momentum \( (\nu_T^N/\nu) \) and thermal \( (\alpha_T^N/\alpha) \) diffusivities for the stabilized ROM calculated using equations 5.9 and 5.12 for \( N_V = 275 \) and \( N_T = 100 \) are presented in figure 5.40. The turbulent viscosity is always positive and increases consistently with the mode order. This confirms the fact that low order modes suffer from poorer modeling of the effective flow diffusivity and require more viscous damping. Indeed, while from the lower order mode point of view (larger scales), the energy cascade appears relatively accurately modeled, from the point of view of the higher order modes (smaller scales), closer to the truncation point, the energy cascade becomes less and less realistic and the absence of smaller dissipative scales or inaccuracies are more and more significant. The turbulent thermal diffusivity is for most of the modes positive but decreases in amplitude with increasing mode rank. Higher order modes even evidence negative turbulent thermal diffusivity values, effectively acting as a source term in the energy equation. This is to be expected since the truncated temperature POD series and the LES model prevent the modeling of heat generation through viscous dissipation.

**Transitional Regime**

Simulations at \( BR = 0.5 \) are now used to obtain a reduced order model of the transitional jet. Simulations are obtained using the ROM grid and parameters described in Chapter 3. The POD analysis is performed using the same parameters as for the attached jet decomposition, including 1,000 snapshots and an equivalent sampling frequency of \( 300Hz \) \( (St_a = 4.8) \) based on Sirovich snapshot method. The POD metrics are presented in figure 5.41, 5.42, 5.43, 5.44. The agreement with the LES data is clearly improved over the non-stabilized ROM of figure 5.34 and reasonably good tracking is now obtained up to 3 convective time scale units. The temperature ROM accuracy is also enhanced, and in particular the amplitude divergence observed in the non-stabilized model is no longer visible. Once again, the local and broad stabilization methods provide very comparable ROMs although the former seems to outperform the latter by a small margin.

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Figure 5.30: First 14 velocity POD modes $\lambda_2$ iso-surfaces for POD decomposition at $BR = 0.15$. Mode energy provided in percent of the total energy.
Figure 5.31: First 14 temperature POD modes iso-surfaces for POD decomposition at $BR = 0.15$. Pink and cyan contours are for negative values, red and blue for positive ones. Mode energy provided in percent of the total energy.
Figure 5.32: Estimates of $\log(E_V^1)$ and $\log(E_V^2)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.15$ ROMs integrated at multiple values of $N_V$ and $N_T$.

Figure 5.33: Estimates of $\log(E_T^1)$ and $\log(E_T^2)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.15$ ROMs integrated at multiple values of $N_V$ and $N_T$. 
Figure 5.34: $BR = 0.15$ ROM temporal coefficients for the velocity and temperature fields obtained at $N_V = 250$, $N_T = 120$ without pressure correction (solid lines) and with pressure correction (dashed lines) along with corresponding POD temporal coefficients (symbols).
Figure 5.35: Estimates of $\log(E_{V1}^2)$ and $\log(E_{V2}^2)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.15$ ROMs integrated at multiple values of $N_V$ and $N_T$ using local linear stabilization.

Figure 5.36: Estimates of $\log(E_{T1}^2)$ and $\log(E_{T2}^2)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.15$ ROMs integrated at multiple values of $N_V$ and $N_T$ using local linear stabilization.
Figure 5.37: Estimates of $\log(E_{V1}^T)$ and $\log(E_{V2}^T)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.15$ ROMs integrated at multiple values of $N_V$ and $N_T$ using broad linear stabilization.

Figure 5.38: Estimates of $\log(E_{T1}^T)$ and $\log(E_{T2}^T)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.15$ ROMs integrated at multiple values of $N_V$ and $N_T$ using broad linear stabilization.
Figure 5.39: $BR = 0.15$ ROM temporal coefficients for the velocity and temperature fields obtained at $N_V = 275$, $N_T = 100$ with local linear stabilization (solid lines) and broad linear stabilization (dashed lines) along with corresponding POD temporal coefficients (symbols).
Figure 5.40: $BR = 0.15$ ROM (a) normalized turbulent momentum diffusivity $\nu_T^*/\nu$ and (b) normalized turbulent thermal diffusivity $\alpha_T^*/\alpha$ calculated for $N_V = 275$ and $N_T = 100$.

Figure 5.41: POD decomposition metrics for velocity (red) and temperature (blue) fields with $N = 1,000$ at $BR = 0.5$. 
presented in figure 5.41 for both velocity and temperature decompositions. Compared to the previous case at $BR = 0.15$, both energy distributions are fuller which is a sign of a more turbulent flow involving a wider range of length scales. The first two POD modes have comparable energy content at approximately 1.75% each, while the first two temperature modes capture approximately 2.5% of the total pseudo-thermal energy. This is to be put in perspective with the respective 5% and 10% obtained at $BR = 0.15$. The 99% energy threshold is reached using 592 velocity modes and 521 temperature modes. This is beyond the limit which was set at 500 modes for each individual decomposition, though not by a large extent, so that effectively 97.8% of the total kinetic energy and 98.8% of the total pseudo-thermal energy can still be captured using 500 velocity and temperature modes. These numbers are assumed to be high enough to capture the significant dynamics of the flow. The velocity POD modes $\lambda_2$ isosurfaces are presented in figure 5.42. As for the lower blowing ratio case, the first 12 modes are mostly associated in pairs of two modes with equivalent energy content, with features shifted in the downstream direction by a quarter of a wavelength from one mode to the other. The first pair captures large scale structures such as hairpin and wall vortices predominantly in the far-field. On the other hand, the second and third pairs represent comparable features but mostly in the near-field. Higher order modes are grasping ever smaller structures throughout the domain, including the near-field. When comparing the POD modes obtained at $BR = 0.15$ and those obtained at $BR = 0.5$, it is interesting to notice how much “busier” the latter look and how all the modes at $BR = 0.5$ possess features across the complete flow domain. This is a result of the increased range of length and energy scales present in the higher blowing ratio case and is once more characteristic of more turbulent flows. The temperature POD modes are shown in figure 5.43 and present some interesting characteristics of their own. While in the $BR = 0.15$ case, the predominant modes were capturing fluctuations in the streamwise direction, the first two pairs of modes at $BR = 0.5$ evidence simultaneously streamwise fluctuations in the far field and transverse ones in the near field close to the jet exit. This shows an evolution of the dominant convective mechanisms which at $BR = 0.15$ were primarily present in the far-field and associated to the hairpin vortices heads and side-vortices, but are now at $BR = 0.5$ also very strong in the near-field and associated with the counter-rotating vortex pair (through the hairpins legs). The temperature POD modes can also generally be matched in pairs of equivalent shape and energetic content, though some pairs having comparable energy levels can be mixed. For example, it appears rather clearly that modes 5 and 8 form a pair while modes 6 and 7 form another one which is “inserted” in between modes 5 and 8.

Reduced order models of the transitional unforced jet are obtained using the POD-Galerkin method. The ROM sensitivity to $N_V$ and $N_T$ is evaluated using estimates of the error surfaces for the velocity and temperature models which are presented in figure 5.44 and 5.45 for $N_V$ and $N_T$ values in the range $[30, 500] \times [30, 500]$. All surfaces exhibit missing areas corresponding to pairs of $(N_V, N_T)$ values at which the ROM integration diverged within the time frame considered for the error evaluation. The primary reason for divergence resides in the velocity ROM instability, and only a few integrations with $N_V \leq 100$ were successful all the way to the $t_c = 6$ limit. The error levels on the velocity ROM, even in the short term, are considerable, conversely to the one on the temperature which are quite reasonable when compared to the non-stabilized results at $BR = 0.15$. Once again, considering the extent of the error levels, looking at medium and long term results would be rather hasty, and focus will be brought to the short term performance. The velocity ROM has a minimum error for $N_V = 350$, while the temperature ROM reaches minimum error on the modeled flow for $(N_V, N_T) = (350, 30)$ and minimum total error for $(N_V, N_T) = (30, 350)$. Figure 5.46 shows the integrated optimum ROMs. The model clearly diverges beyond $t_c = 2.5$ and the discrepancy appears to originate from the higher order velocity modes which amplitudes are growing significantly beyond $t_c = 1.5$. As the error propagates in the system, the lower order velocity and temperature modes are eventually diverging as well. It should be noted that since the integrated systems presented in figure 5.46 are not corresponding to each other, the diverging point do not clearly match. Nevertheless, both the velocity and temperature ROMs evidence rather good agreement with the projected LES data before the system starts to diverge ($t_c \leq 1$), which is an encouraging feature.

Following what was done in the attached jet model reduction, stabilizing terms are calculated and added to the velocity and temperature ROMs which are integrated using both local and broad stabilization formulations. The error surfaces for both methods are presented in figures 5.47 and 5.48 for the local method and figures 5.49 and 5.50 for the broad stabilization. Based on the error surfaces, or rather their missing parts, the ROM integrated with the local stabilization method appear to diverge more often than the ones integrated at the same values of $(N_V, N_T)$ using the broad stabilization. This is especially the case for large
Figure 5.42: First 14 velocity POD modes $\lambda_2$ iso-surfaces for POD decomposition at $BR = 0.5$. Mode energy provided in percent of the total energy.
Figure 5.43: First 14 temperature POD modes iso-surfaces for POD decomposition at $BR = 0.5$. Pink and cyan contours are for negative values, red and blue for positive ones. Mode energy provided in percent of the total energy.
Figure 5.44: Estimates of $\log(E_{1}^{V})$ and $\log(E_{2}^{V})$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.5$ ROMs integrated at multiple values of $N_{V}$ and $N_{T}$.

Figure 5.45: Estimates of $\log(E_{1}^{T})$ and $\log(E_{2}^{T})$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.5$ ROMs integrated at multiple values of $N_{V}$ and $N_{T}$. 

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Figure 5.46: $BR = 0.5$ ROM temporal coefficients for the velocity at $N_v = 350$ and temperature fields obtained at $(N_v, N_T) = (30, 350)$ (solid lines) along with corresponding POD temporal coefficients (symbols).
Figure 5.47: Estimates of $\log(E^1_V)$ and $\log(E^2_V)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.5$ ROMs integrated at multiple values of $N_V$ and $N_T$ using local linear stabilization.

Figure 5.48: Estimates of $\log(E^1_T)$ and $\log(E^2_T)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.5$ ROMs integrated at multiple values of $N_V$ and $N_T$ using local linear stabilization.
Figure 5.49: Estimates of $\log(E_1^V)$ and $\log(E_2^V)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.5$ ROMs integrated at multiple values of $N_V$ and $N_T$ using broad linear stabilization.

Figure 5.50: Estimates of $\log(E_1^T)$ and $\log(E_2^T)$ over two (left), four (center) and six (right) convective time scale units for $BR = 0.5$ ROMs integrated at multiple values of $N_V$ and $N_T$ using broad linear stabilization.
Figure 5.51: $BR = 0.5$ ROM temporal coefficients for the velocity and temperature fields obtained at $N_V = 350$, $N_T = 350$ with local linear stabilization (solid lines) and $N_V = 500$, $N_T = 300$ with broad linear stabilization (dashed lines) along with corresponding POD temporal coefficients (symbols).
values of $N_V$. In fact, none of the locally stabilized ROMs including more than 350 velocity modes can be integrated up to $t_c = 2$ (short term time frame) whereas it is possible to include all 500 velocity modes using the broad stabilization and integrate up to at least $t_c = 4$ (medium term time frame). Looking back at the $BR = 0.15$ stabilized ROMs, it appeared also impossible to obtain a converged solution for $N_V > 400$ when using the local stabilization, whereas $N_V = 500$ ROMs could be successfully integrated using the broad formulation stabilization. Overall, the minimum error levels attained for the velocity ROM are found to be on average one to two orders of magnitude lower when using broad stabilization compared to local stabilization. While the error on the modeled part of the velocity $E^V_1$ increases with increasing number of velocity modes before reaching a maximum around $N_V = 100$ and decreases again beyond that point, the total error $E^V_2$ consistently decreases with increasing number of velocity modes. Thus, and because of the improved stability of the broadly stabilized ROMs, lower values of $E^V_2$ can be reached for $N_V = 500$ which are unattainable by ROMs using the local stabilization method. The temperature ROM appears to be much less sensitive to the stabilization method used and the error levels using both methods are comparable in the short term, yet with a slight advantage to the broad stabilization method. This is due to the fact that the temperature ROMs for the transitional jet are almost insensitive to the value of $N_V$ and can reach the lowest error levels with only a few velocity modes. In the medium and long term, the broad stabilization method also shows improved performance over the local one. Overall in the short term, the locally stabilized ROM reach minimum values of $E^V_2$ and $E^T_2$ value at $(N_V, N_T) = (350, 350)$ while the broadly stabilized ROM reach a minimum error values for $(N_V, N_T) = (500, 300)$. Similarly, the medium term error estimates show a minimum at $(N_V, N_T) = (350, 140)$ for the former and $(N_V, N_T) = (500, 200)$ for the latter.

Figure 5.51 presents the integrated ROMs using both local stabilization at $(N_V, N_T) = (350, 350)$ and broad stabilization for $(N_V, N_T) = (500, 300)$. The results show a clear improvement over the non-stabilized

Figure 5.52: $BR = 0.5$ ROM (a, c) normalized turbulent momentum diffusivity $\nu_T^e/\nu$ and (b, d) normalized turbulent thermal diffusivity $\alpha_T^e/\alpha$ calculated for (a, b) $N_V = 350$ and $N_T = 350$ and (c, d) $N_V = 500$, $N_T = 300$. 
No divergence of the extent of the one observed in the non-stabilized equations is found in the stabilized velocity ROMs though the amplitudes of the higher order modes of the locally stabilized equations start to increase significantly beyond \( t_c = 3 \). This is clearly not the case for the broadly stabilized model which seems to follow the projected LES data all the way up to \( t_c = 4 \) (medium term). Less obvious differences are visible in the temperature ROMs which seem qualitatively very well behaved and follow the projected data nicely. A growing discrepancy is observed for the locally stabilized ROM towards the end of the considered time span \( t_c \sim 4 \), and is most likely a result of the growing amplitude of the corresponding high order velocity modes. Overall the broad stabilization method provides more stable and more accurate models of the transitional jet, in particular when including large numbers of POD modes compared to the locally stabilized method. The turbulent viscosity and thermal diffusivity for the optimum locally and broadly stabilized ROM are shown in figure 5.52. The trends for the turbulent viscosity \( \nu_T \) are consistent with the one obtained for the attached jet, with increasing positive turbulent diffusivity for increasing mode order. The thermal diffusivity trends also follow the ones of the \( BR = 0.15 \) case, with positive values for lower order modes decreasing with mode rank. The higher order temperature modes evidence negative thermal diffusivity, accounting for heat generation due to viscous dissipation through the smaller scales.

### 5.2.3 Forced Jets

Although reasonably good reduced order models were obtained for the unforced jet, it is unlikely those models will be able to capture and reproduce the dynamics of forced jets in cross-flow, and in particular the starting structures, which are generated in actuated conditions. To capture those dynamics, forced jet simulations were used to generate reduced order models by means of the POD-Galerkin method.

#### Instantaneous Flow

Instantaneous velocity and temperature fields were the first one to be investigated in view to obtain a reduced order model of the forced flow.

#### Modified Formulation

At this point it is necessary to introduce a new decomposition of the velocity field for forced jet simulations. Indeed, while it was deemed acceptable in the unforced jet (and confirmed a posteriori) to neglect the non-homogeneous velocity terms at the jet inlet surface arising from the perturbation introduced in the simulations (see equation 4.34), it is not possible to do so for the forced jet. Throughout the cycle the jet inlet velocity fluctuates from values corresponding to \( BR_h \) to values corresponding to \( BR_l \), which cannot be homogenized by simply subtracting the time averaged flow field from the instantaneous one according to Reynolds decomposition. Therefore, instead of Reynolds decomposition, the velocity decomposition proposed by Graham et al. (1999a) and Ravindran (2000) is used instead:

\[
\tilde{u} = \bar{u} + \gamma(t)\bar{u}_{ref} + \tilde{u} \tag{5.13}
\]

where \( \gamma(t) \) is the control function, \( \bar{u}_{ref} \) a reference velocity field, \( \bar{u} \) the average field and \( \tilde{u} \) the fluctuation. In this study, \( \gamma(t) \) is a square wave of amplitude 1. The reference velocity field is a time independent field chosen to represent the global impact of the control function on the flow. Effectively this term is used to homogenize the velocity at the control surface (jet inlet). For this particular study, the reference field \( \bar{u}_{ref} \) is defined as:

\[
\bar{u}_{ref} = \bar{u}(BR = BR_h) - \bar{u}(BR = BR_l) \tag{5.14}
\]

The average field is then obtained by taking the time average of the velocity field from which the reference field has been subtracted:

\[
\bar{u} = \bar{u} - \gamma(t)\bar{u}_{ref} \tag{5.15}
\]

Finally, the fluctuation part of the velocity is calculated by subtracting both the average and reference fields from the initial velocity field:

\[
\tilde{u} = \bar{u} - \gamma(t)\bar{u}_{ref} - \bar{u} \tag{5.16}
\]
The fluctuation part is then decomposed using POD so that:

\[ \vec{u}' = \sum_{m=1}^{N_V} a_m(t) \vec{\varphi}_m(x) \]  
\[ \vec{u} = \vec{u} + \gamma(t) \vec{u}_{ref} + \sum_{m=1}^{N_V} a_m(t) \vec{\varphi}_m(x) \]  

Because in this study the temperature field is always homogeneous at the jet inlet, there is no need to modify the decomposition for the temperature which can remain:

\[ T = T + T' = T + \sum_{n=1}^{N_T} b_n(t) \psi_n(x) \]

Effectively, using this velocity decomposition will affect the general form of the reduced order model equations 4.32 and 4.33.

Rewriting the Navier-Stokes equations using the previously introduced velocity decomposition:

\[ \nabla \left( \vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}' \right) \]  
\[ \frac{\partial}{\partial t} \left( (\vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}') \cdot \nabla \right) \left( \vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}' \right) = -\nabla P + \frac{1}{Re_j} \Delta \left( \vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}' \right) \]

\[ \frac{\partial T'}{\partial t} + \left( \left( \vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}' \right) \cdot \nabla \right) \left( T + T' \right) = \frac{1}{Pr} \frac{1}{Re_j} \Delta \left( T + T' \right) \]  

Since \( \vec{u}_{ref} \) was taken to be divergence free, \( \vec{u} \) is consequently divergence free, and so is \( \vec{u}' \).

Using similar notation as the one used in the previous chapter, we introduce:

\[ \frac{\partial \vec{u}'}{\partial t} = \dot{M} \left( \vec{u}', \vec{u}, \vec{u}_{ref}, \gamma \right) \]  
\[ \frac{\partial T'}{\partial t} = \dot{E} \left( \vec{u}', \vec{u}, \vec{u}_{ref}, \gamma, T, T' \right) \]

where

\[ \dot{M}(\vec{u}', \vec{u}, \vec{u}_{ref}, \gamma,) = - \left( \left( \vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}' \right) \cdot \nabla \right) \left( \vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}' \right) \]

\[ -\nabla P + \frac{1}{Re_j} \Delta \left( \vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}' \right) - \frac{\partial}{\partial t} \left( \gamma(t) \vec{u}_{ref} \right) \]

and

\[ \dot{E}(\vec{u}', \vec{u}, \vec{u}_{ref}, \gamma, T, T') = - \left( \left( \vec{u} + \gamma(t) \vec{u}_{ref} + \vec{u}' \right) \cdot \nabla \right) \left( T + T' \right) + \frac{1}{Pr} \frac{1}{Re_j} \Delta \left( T + T' \right) \]

As for the unforced flows, the next step consists in projecting the momentum and energy equations onto their respective POD basis in order to obtain the reduced order model equations. This task being quite fastidious, it has been left out of this section, and only the final reduced order equations are presented here. The details of the derivation are available in the Appendix.

The reduced order model equations for the controlled velocity field are then:
\[
\dot{a}_r(t) = \sum_{i=1}^{N_v} \left( \tilde{C}_{i,0}^r + \frac{1}{Re_j} \tilde{D}_s^r \right) a_i(t) + \sum_{i=1}^{N_v} \sum_{j=1}^N \tilde{C}_{i,j}^r a_i(t)a_j(t) + \left( \tilde{C}_o^r + \frac{1}{Re_j} \tilde{D}_o^r \right) + P_r(t) \\
+ \gamma(t) \left[ \tilde{H}_o^r + \frac{1}{Re_j} \tilde{L}_o^r + \sum_{i=1}^{N_v} \tilde{H}_{i,0}^r a_i(t) \right] + \gamma^2(t) \tilde{K}_o^r + \frac{d\gamma(t)}{dt} \tilde{G}_o^r \tag{5.22}
\]

where

\[
\bar{u} = \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' = \bar{u} + \gamma(t) \bar{u}_{ref} + \sum_{n=1}^{N_v} a_n(t) \bar{\varphi}_n(x) \\
\tilde{C}_o^r = -((\bar{u} \cdot \nabla) \bar{\varphi}_r, \bar{\varphi}_r) = -\int_{\Omega_x} \bar{\varphi}_r \frac{\partial}{\partial x_1} \left( \bar{u}_l \right) \varphi_{r,l} dx \\
\tilde{D}_o^r = (\Delta \bar{u}, \bar{\varphi}_r) = \int_{\Omega_x} \frac{\partial}{\partial x_1} \left( \bar{u}_l \right) \varphi_{r,l} dx \\
\tilde{C}_{i,0}^r = -((\bar{\varphi}_i \cdot \nabla) \bar{\varphi}_r, \bar{\varphi}_r) - ((\bar{\varphi}_i \cdot \nabla) \bar{\varphi}_r, \bar{\varphi}_r) = -\int_{\Omega_x} \left( \bar{\varphi}_i \frac{\partial}{\partial x_1} \varphi_{i,l} + \varphi_{i,m} \frac{\partial}{\partial x_1} \bar{u}_m \right) \varphi_{r,l} dx \\
\tilde{D}_o^r = (\Delta \bar{\varphi}_i, \bar{\varphi}_r) = \int_{\Omega_x} \frac{\partial}{\partial x_1} \left( \varphi_{i,l} \right) \varphi_{r,l} dx \\
\tilde{C}_{i,j}^r = -((\bar{\varphi}_i \cdot \nabla) \bar{\varphi}_j, \bar{\varphi}_r) - ((\bar{\varphi}_j \cdot \nabla) \bar{\varphi}_i, \bar{\varphi}_r) = -\int_{\Omega_x} \left( \bar{\varphi}_i \frac{\partial}{\partial x_1} \varphi_{j,l} + \varphi_{j,m} \frac{\partial}{\partial x_1} \bar{u}_m \right) \varphi_{r,l} dx \\
P_r(t) = -((\nabla P(t), \bar{\varphi}_r) \\
\tilde{H}_o^r = -((\bar{u} \cdot \nabla) \bar{u}_{ref}, \bar{\varphi}_r) - ((\bar{u} \cdot \nabla) \bar{\varphi}_r, \bar{\varphi}_r) = -\int_{\Omega_x} \left( \bar{u}_{ref,l} \frac{\partial}{\partial x_1} \bar{u}_{ref,l} + \bar{u}_{ref,m} \frac{\partial}{\partial x_1} \bar{u}_m \right) \varphi_{r,l} dx \\
\tilde{L}_o^r = (\Delta \bar{u}_{ref}, \bar{\varphi}_r) = \int_{\Omega_x} \frac{\partial}{\partial x_1} \left( \bar{u}_{ref,l} \right) \varphi_{r,l} dx \\
\tilde{H}_{i,0}^r = -((\bar{u}_{ref} \cdot \nabla) \bar{\varphi}_i, \bar{\varphi}_r) - ((\bar{\varphi}_i \cdot \nabla) \bar{u}_{ref}, \bar{\varphi}_r) = -\int_{\Omega_x} \left( \bar{u}_{ref,l} \frac{\partial}{\partial x_1} \varphi_{i,l} + \varphi_{i,m} \frac{\partial}{\partial x_1} \bar{u}_{ref,l} \right) \varphi_{r,l} dx \\
\tilde{K}_o^r = -((\bar{u}_{ref} \cdot \nabla) \bar{u}_{ref}, \bar{\varphi}_r) = -\int_{\Omega_x} \bar{u}_{ref,l} \left( \bar{u}_{ref,m} \right) \varphi_{r,l} dx \\
\tilde{G}_o^r = -((\bar{u}_{ref}, \bar{\varphi}_r) = -\int_{\Omega_x} \bar{u}_{ref,l} \varphi_{r,l} dx \\
\]
\[
T = T + T' = T + \sum_{m=1}^{N_T} b_m(t) \psi_m(x)
\]
\[
\tilde{E}_{s,0} = - ((\tilde{\pi}\cdot\nabla) T, \psi_s) = - \int_{\Omega_x} \varpi_m \frac{\partial}{\partial x} (T) \psi_s dx
\]
\[
\tilde{F}_{s,0} = (\Delta T, \psi_s) = \int_{\Omega_x} \frac{\partial}{\partial x} (T) \psi_s dx
\]
\[
\tilde{E}_{s,0} = - ((\tilde{\varphi}_i \cdot \nabla) T, \psi_s) = - \int_{\Omega_x} \varphi_{i,m} \frac{\partial}{\partial x} (T) \psi_s dx
\]
\[
\tilde{F}_{s,j} = (\Delta \psi_j, \psi_s) = \int_{\Omega_x} \frac{\partial}{\partial x} (\psi_j) \psi_s dx
\]
\[
\hat{E}_{s,j} = - ((\tilde{\varphi}_i \cdot \nabla) \psi_j, \psi_s) = - \int_{\Omega_x} \varphi_{i,m} \frac{\partial}{\partial x} (\psi_j) \psi_s dx
\]
\[
\hat{N}_{s,j} = - ((\tilde{u}_{ref} \cdot \nabla) T, \psi_s) = - \int_{\Omega_x} u_{ref,m} \frac{\partial}{\partial x} (T) \psi_s dx
\]
\[
\hat{N}_{s,j} = - ((\tilde{u}_{ref} \cdot \nabla) \psi_j, \psi_s) = - \int_{\Omega_x} u_{ref,m} \frac{\partial}{\partial x} (\psi_j) \psi_s dx
\]

It is clearly visible that the velocity decomposition from Graham et al. (1999a) has the great advantage to make the control variable \( \gamma(t) \) explicitly appear in the reduced order model equations, therefore making it very practical for controller design and implementation purposes.

**Stabilization Methods**

As for the unforced flows, the truncation of the POD series and the temporal and spatial filtering, resulting from both the numerical model and snapshot sampling, can lead to significant underestimations of the overall effective diffusivity of the flow. To alleviate this problem, stabilizing terms including turbulent momentum and thermal diffusivity are added to the reduced order model equations and their values are computed based on average energy conservation considerations for each individual mode. However, in the case of controlled flows, the conservation of energy at the mode level is not a valid assumption since large deterministic energy fluctuations are introduced through the control signal \( \gamma(t) \tilde{u}_{ref} \). Therefore, although it is technically possible to follow the same method to stabilize the ROM equations 5.22 by assuming the time average energy content of each individual mode must be conserved on average on an integer number of cycle periods, this leads to significant overestimation of the diffusivity coefficient on an instantaneous basis.

In forced conditions, the energy conservation constraint must effectively take into account all the modes so that \( \frac{d}{dt} \left( \tilde{\pi} + \gamma(t) \tilde{u}_{ref} + \sum_{i=1}^{N_V} \left(a_i \tilde{\varphi}_i \right) \right)^2 = 0 \), which prevents from identifying mode specific stabilization coefficients.

Instead, a partial stabilization method, consisting in damping the uncontrolled system \( \gamma = 0 \) only, is proposed. Using the locally and broadly damped formulations of the stabilizing terms and setting \( \gamma(t) = 0, \frac{d}{dt} \gamma(t) = 0 \) in equations 5.22 and 5.23 provides the following stabilized ROM equations for the local stabilization:

\[
a_r(t) = \sum_{i=1}^{N_V} \left( \tilde{C}_{r,0}^i + \frac{1}{Re_e} \tilde{D}_r^i \right) a_i(t) + \sum_{i=1}^{N_V} \sum_{j=1}^{i} \tilde{C}_{r,j}^i a_i(t)a_j(t) + \left( \tilde{C}_0^r + \frac{1}{Re_e} \tilde{D}_0^r \right) P_r(t) + \gamma(t) \left[ \tilde{H}_0^r + \frac{1}{Re_e} \tilde{L}_0^r + \sum_{i=1}^{N_V} \tilde{H}_{i,0}^r a_i(t) \right] + \gamma^2(t) \tilde{K}_0^r + \frac{d\gamma(t)}{dt} \tilde{G}_0^r + a_r(t) \tilde{S}_r^V
\]
\[ \dot{b}_s(t) = \sum_{i=1}^{N_v} \tilde{E}_{s,0}^{i} a_i(t) + \sum_{j=1}^{N_T} \left( \tilde{E}_{s,j}^0 + \frac{1}{Pr} \frac{1}{Re_j} \tilde{F}_{s,j} \right) b_j(t) + \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} \tilde{E}_{s,j}^{i} a_i(t) b_j(t) + \left( \tilde{E}_{s,0}^0 + \frac{1}{Pr} \frac{1}{Re_j} \tilde{F}_{s,0} \right) \]

\[ + \gamma(t) \left[ \tilde{N}_{s,0}^0 + \sum_{j=1}^{N_T} b_j \tilde{N}_{s,j}^0 \right] + b_s(t) \tilde{S}_s^T \]

(5.25)

with

\[ \tilde{S}_s^V = -\frac{1}{\lambda_s} \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} \tilde{C}_{ij}^r (a_i(t) a_j(t) a_r(t)) - \left( \tilde{C}_{r0}^s + \frac{1}{Re_j} \tilde{D}_r^s \right) \]

(5.26)

and

\[ \tilde{S}_s^T = -\frac{1}{\sigma_s} \sum_{i=1}^{N_v} \tilde{E}_{s0}^i (a_i(t) b_s(t)) + \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} \tilde{E}_{s,j}^i (a_i(t) b_j(t) b_s(t)) - \left( \tilde{E}_{ss}^0 + \frac{1}{Pr} \frac{1}{Re_j} \tilde{F}_{ss} \right) \]

(5.27)

The broad stabilization formulation for the controlled jet is:

\[ \dot{a}_r(t) = \sum_{i=1}^{N_v} \left( \tilde{C}_{i,0}^r + \frac{1}{Re_j} \tilde{D}_i^r \right) a_i(t) + \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} \tilde{C}_{ij}^r a_i(t) a_j(t) + \left( \tilde{C}_{r0}^r + \frac{1}{Re_j} \tilde{D}_0^r \right) + P_r(t) \]

\[ + \gamma(t) \left[ \tilde{H}_0^r + \frac{1}{Re_j} \tilde{L}_0^r + \sum_{i=1}^{N_v} \tilde{H}_{i,0}^r a_i(t) \right] + \gamma^2(t) \tilde{K}_0^r + \frac{d\gamma(t)}{dt} \tilde{G}_0^r \]

(5.28)

\[ \dot{b}_s(t) = \sum_{i=1}^{N_v} \tilde{E}_{s,0}^i a_i(t) + \sum_{j=1}^{N_T} \left( \tilde{E}_{s,j}^0 + \frac{1}{Pr} + \tilde{\alpha}_T^s \right) \frac{1}{Re_j} \tilde{F}_{s,j} b_j(t) + \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} \tilde{E}_{s,j}^i a_i(t) b_j(t) \]

\[ \left( \tilde{E}_{s,0}^0 + \frac{1}{Pr} + \tilde{\alpha}_T^s \right) \frac{1}{Re_j} \tilde{F}_{s,0} b_s(t) + \gamma(t) \left[ \tilde{N}_{s,0}^0 + \sum_{j=1}^{N_T} b_j \tilde{N}_{s,j}^0 \right] \]

(5.29)

with

\[ \tilde{\nu}_T = -\frac{1}{\lambda_r} \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} \tilde{C}_{ij}^r (a_i(t) a_j(t) a_r(t)) + \lambda_r \left( \tilde{C}_{r0}^s + \frac{1}{Re_j} \tilde{D}_r^s \right) \]

(5.30)

and

\[ \tilde{\alpha}_T^s = -\frac{1}{\sigma_s} \frac{E_{s,j}}{F_{s,j}} \left( \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} \tilde{E}_{s,j}^i (a_i(t) b_j(t) b_s(t)) + \sum_{i=1}^{N_v} \tilde{E}_{s,0}^i (a_i(t) b_s(t)) + \left( \tilde{E}_{s,j}^0 + \frac{1}{Pr} \frac{1}{Re_j} \tilde{F}_{s,j} \right) \sigma_s \right) \]

(5.31)

The relationships between \( \tilde{S}_s^V, \tilde{S}_s^T \) and \( \tilde{\nu}_T, \tilde{\alpha}_T^s \) still hold:

\[ \tilde{S}_s^V = \frac{\tilde{\nu}_T}{D_r}, \quad \tilde{S}_s^T = Re_j \frac{\tilde{\alpha}_T^s}{F_{ss}} \]
In an attempt to provide a more complete formulation of a stabilization method, we also propose a third method which builds on the the partial local linear stabilization by adding a $\gamma$ dependent term to $\tilde{S}_r^V$ and $\tilde{S}_s^T$ so that the stabilized equations become:

$$\dot{a}_r(t) = \sum_{i=1}^{N_V} \left[ \tilde{C}_{i,0}^r + \frac{1}{Re_j} \tilde{D}_r^i \right] a_i(t) + \sum_{i=1}^{N_V} \sum_{j=1}^{N_V} \tilde{C}_{i,j}^r a_i(t) a_j(t) + \left( \tilde{C}_{0}^r + \frac{1}{Re_j} \tilde{D}_0^t \right) + P_r(t) + \gamma(t) \left[ \tilde{H}_0^r + \frac{1}{Re_j} \tilde{L}_r^t + \sum_{i=1}^{N_V} \tilde{H}_{i,0}^r a_i(t) \right] + \gamma^2(t) \tilde{K}_0^r + \frac{d\gamma(t)}{dt} \tilde{C}_r^r + a_r(t) \left( \tilde{S}_r^V + \gamma(t) \tilde{S}_r^V \right)$$  \hspace{1cm} (5.32)

$$\dot{b}_s(t) = \sum_{i=1}^{N_V} \tilde{E}_{s,0}^i a_i(t) + \sum_{j=1}^{N_T} \left( \tilde{E}_{s,j}^0 + \frac{1}{Pr_j} \tilde{F}_{s,j}^t \right) b_j(t) + \sum_{i=1}^{N_V} \sum_{j=1}^{N_T} \tilde{E}_{s,j}^i a_i(t) b_j(t) + \left( \tilde{E}_{s,0}^0 + \frac{1}{Pr_j} \tilde{F}_{s,0}^t \right) + \gamma(t) \left[ \tilde{N}_{s,0}^0 + \sum_{j=1}^{N_T} b_j \tilde{N}_{s,j}^0 \right] + b_a(t) \left( \tilde{S}_s^T + \gamma(t) \tilde{S}_s^T \right)$$ \hspace{1cm} (5.33)

Quite logically, the terms $\tilde{S}_r^V$ and $\tilde{S}_s^T$ are taken to be the stabilization terms obtained when the control function is set to zero and are therefore identical to the ones obtained in the partial stabilization method described previously. The $\gamma$ dependent terms are then obtained by assuming the value of $\gamma$ to be a non-zero constant. If $\gamma$ is constant, then the rate of change of the energy content in the $s^{th}$ velocity mode and $s^{th}$ temperature mode must be zero and for a constant non-zero value of $\gamma$:

$$\left\langle \frac{1}{2} \frac{d}{dt} (a_r^2) \right\rangle = \left( C_{r,0}^r + \frac{1}{Re_j} D_r^r \right) \lambda_r + \sum_{i=1}^{N_V} \sum_{j=1}^{N_V} C_{i,j}^r \langle a_i(t) a_j(t) a_r(t) \rangle + \lambda_r \tilde{S}_r^V + \gamma \left( \lambda_r \tilde{H}_r^r + \lambda_r \tilde{S}_r^V \right)$$ \hspace{1cm} (5.34)

$$= 0$$

$$\left\langle \frac{1}{2} \frac{d}{dt} (b_s^2) \right\rangle = \sum_{i=1}^{N_V} \tilde{E}_{s,0}^i \langle a_i(t) b_s(t) \rangle + \left( E_{s,j}^0 + \left( \frac{1}{Pr_j} + \alpha_r^2 \right) \frac{1}{Re_j} F_{s,j}^t \right) \sigma_s + \sum_{i=1}^{N_V} \sum_{j=1}^{N_T} E_{s,j}^i \langle a_i(t) b_j(t) b_s(t) \rangle + \sigma_s \tilde{S}_s^T + \gamma \left( \sigma_s \tilde{N}_{s,s}^0 + \sigma_s \tilde{S}_s^T \right) \hspace{1cm} (5.35)$$

$$= 0$$

By design, the terms $\tilde{S}_r^V$ and $\tilde{S}_s^T$ will cancel out the first part of equations 5.34 and 5.35 so that the only terms left are:

$$\tilde{S}_r^V = -\tilde{H}_r^r \hspace{1cm} \tilde{S}_s^T = -\tilde{N}_{s,s}^0$$

Because this stabilization method is supposed to damp the system for $\gamma(t)$ asymptotically tending towards 0 and towards a non-zero constant, it will be referred to as asymptotic linear stabilization method.

**Reduced Order Models**

Based on the preliminary POD results, it was decided that forced cases at rather high forcing frequencies would constitute a good starting point to obtain a first reduced order model. In addition, it is believed that
these flows could benefit more from controlled based tuning than lower forcing frequency ones, where the quasi-unforced parts of the cycle play too significant of a role and the error arising from the mode segregation evidenced previously could affect the stability and accuracy of the ROMs.

Forced jet simulations are carried out using the ROM grid described in Chapter 3 at forcing conditions corresponding to Case IV at $St_{\infty} = 0.159$. The jet inlet velocity is modulated using a square wave signal between $BR_h = 0.50$ and $BR_l = 0.150$. Snapshots are collected at 50 equally spaced phase-locked positions (equivalent sampling frequency 500Hz or $St_s = 7.9$) over four cycles for a total of 200 snapshots. In the following section, the transition from $BR_l$ to $BR_h$ occurs at $t_c = 0$.

The POD metrics are presented in figure 5.53 for both velocity and temperature decompositions. While the temperature metrics evidence the stair-like distribution of the energy levels mentioned in the POD preliminary study with the first two modes at 22% of the total pseudo-thermal energy each, the velocity distribution shows a clearly dominant first mode with 36% of the total kinetic energy while the second mode only carries 11%. This is due to the modified velocity decomposition including the control signal $\gamma(t)\vec{u}_{\text{ref}}$, which prevents from direct comparisons with the preliminary study. Overall, 65 velocity POD modes are required to reconstruct 99% of the total kinetic energy from $\vec{u}'$, and 70 temperature modes to reconstruct 99% of the total pseudo-thermal energy of $T'$.

The velocity POD modes $\lambda_2$ iso-surfaces are presented in figure 5.54, also including for reference the $\lambda_2$ iso-surfaces for the fields $\vec{u}$ and $\vec{u}_{\text{ref}}$. The first two velocity POD modes capture predominantly large scale features at the jet exit likely corresponding to the starting structures, whereas higher order modes such as modes 3, 4, 5 and 6 also capture hairpin vortices in the mid and far-fields with increasing frequency. Interestingly, modes 7 to 12 appear to capture wall structures, predominantly in the far-field. The POD temporal coefficients in figure 5.58 show dynamics much different from the one of the preliminary POD at equivalent forcing conditions, which was again expected due to the different decomposition used in this section. The temperature POD modes in figure 5.55 are however much more similar to the ones observed in figure 5.24 in the preliminary study. The first modes exhibit large scale temperature fluctuation areas, associated with the convection of the starting vortex, while higher order modes correspond to higher harmonics of the first two modes, although the last temperature modes appear to capture anti-symmetric dynamics in the far-field region.

Reduced order models using the POD-Galerkin method are derived according to the method described in the previous section. Once again, the pressure term in equation 5.22 is neglected since ROMs integrated using empirical pressure correction term did not provide any noticeable improvement. In this case, it is possible to compute coefficients for all the velocity and temperature modes so that the complete set of
Figure 5.54: First 12 velocity POD modes along with $\tilde{u}_m$ and $\tilde{u}_{ref}$ $\lambda_2$ iso-surfaces for POD decomposition for Case IV at $St_\infty = 0.159$. Mode energy provided in percent of the total energy.
Figure 5.55: First 14 temperature POD modes iso-surfaces for POD decomposition for Case IV at $St_\infty = 0.159$. Pink and cyan contours are for negative values, red and blue for positive ones. Mode energy provided in percent of the total energy.
Figure 5.56: Estimates of $\log(E_{V1})$ and $\log(E_{V2})$ over one (left), two (center) and three (right) forcing cycles for Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$.

Figure 5.57: Estimates of $\log(E_{T1})$ and $\log(E_{T2})$ over one (left), two (center) and three (right) forcing cycles for Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$. 
Figure 5.58: Case IV at $St_{\infty} = 0.159$ ROM temporal coefficients for the velocity ($N_V = 25$) and temperature ($\etasymenom{N_V, N_T} = \etasymenom{15, 45}$) fields (solid lines) along with corresponding POD temporal coefficients (symbols).
Figure 5.59: Estimates of $\log(E_V^1)$ and $\log(E_V^2)$ over one (left), two (center) and three (right) forcing cycles for Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using partial local linear stabilization.

Figure 5.60: Estimates of $\log(E_T^1)$ and $\log(E_T^2)$ over one (left), two (center) and three (right) forcing cycles for Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using partial local linear stabilization.
Figure 5.61: Estimates of $\log(E^V_1)$ and $\log(E^V_2)$ over one (left), two (center) and three (right) forcing cycles for Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using partial broad linear stabilization.

Figure 5.62: Estimates of $\log(E^T_1)$ and $\log(E^T_2)$ over one (left), two (center) and three (right) forcing cycles for Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using partial broad linear stabilization.
Figure 5.63: Estimates of $\log(E_1^V)$ and $\log(E_2^V)$ over one (left), two (center) and three (right) forcing cycles for Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using asymptotic linear stabilization.

Figure 5.64: Estimates of $\log(E_1^T)$ and $\log(E_2^T)$ over one (left), two (center) and three (right) forcing cycles for Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using asymptotic linear stabilization.
Figure 5.65: Case IV at $St_\infty = 0.159$ ROM temporal coefficients for the velocity ($N_V = 35$) and temperature ($N_V, N_T = (10, 30)$) fields using partial local (solid) partial broad (dashed) and asymptotic (dash dot dot) linear damping along with corresponding POD temporal coefficients (symbols).
ODEs representing 100% of the flow energy could be integrated. Error estimates $E^V_1$, $E^V_2$, $E^T_1$ and $E^T_2$ are computed for $(N_V, N_T)$ in $[10, 200] \times [10, 200]$ over one, two and three cycle periods and are presented in figures 5.56 and 5.57. For both velocity and temperature fields, it appears rather clearly that the error with respect to both the modeled flow and the total flow increases consistently with increasing number of velocity POD modes. However, the error levels for the best performing velocity and temperature ROMs are rather low, around 8% and 16% over the first cycle period, especially when compared to non-stabilized unforced jets in the previous sections. It also seems that none of the ROMs including more than 125 velocity POD modes could be integrated over a full cycle, while ROMs including more than 90 velocity modes diverged before the end of the second cycle and those with more than 70 modes diverged before the end of the third one. Based on the total error $E^V_2$, an optimum number of velocity modes of $N_V = 25$ for the velocity ROM is identified and based on $E^T_2$ an optimum pair $(N_V, N_T) = (10, 45)$ for the temperature ROM corresponding to minimum total error is found. Both solutions are presented in figure 5.58. Overall, the agreement between both ROMs and the projected LES data is reasonably good. The most striking discrepancy resides in the first velocity equation, where the model diverges from the projected data systematically at the end of each part of the cycle, but is most of the time brought back into agreement at the next transition. The velocity model is performing reasonably well over two forcing cycles, although the agreement with the projected LES data clearly starts to be affected towards the end of the second cycle with significant discrepancy in amplitude and phase, even more so for the higher order modes. Similarly, rather good agreement between the temperature ROM and the projected data can be observed over at least the first cycle but phase and amplitude start diverging towards the end of the second cycle, in particular for higher order temperature modes.

Following the method used in the unforced jet configuration, the velocity and temperature reduced order models are integrated using partial local, broad and asymptotic stabilization methods described in respectively equations 5.24, 5.25, 5.28, 5.29 and 5.32, 5.33. Error surfaces corresponding to the estimates of $E^V_1$, $E^V_2$, $E^T_1$ and $E^T_2$ are computed for each method and presented in figures 5.59, 5.60, 5.61, 5.62 and 5.63, 5.64 respectively. All three methods exhibit virtually identical minimum error level values, and stabilized ROMs integrations with 200 velocity modes over a full period (compared to a maximum of $N_V = 125$ for the non-stabilized ROMs). While the local and broad methods only permits to integrate ROMs with $N_V \leq 150$ over more than two periods (compared to $N_V \leq 75$ for the non-stabilized ROMs), the asymptotic method permits to integrate ROMs with $N_V \leq 190$ over two periods, and $N_V \leq 180$ over three periods, therefore showing the latter stabilizes systems with large number of velocity modes better. Minimum error levels for all methods are consistently lower than the non-stabilized ROMs by approximately a factor 4 for the velocity ROMs and by a factor 3 in the short term and a factor 60 in the long term for the temperature ROMs. Optimum values for the number of velocity and temperature POD modes are evaluated using $E^V_2$ and $E^T_2$ and correspond to $N_V = 35$ for the velocity ROM and $(N_V, N_T) = (10, 30)$ for the temperature ROM. The temporal coefficients for those integrated reduced order models are presented in figure 5.65.
While the behavior of the first velocity mode is only marginally improved, most of the amplitude and phase discrepancies growing during the second forcing cycle are significantly reduced when using any of the stabilization methods. Overall, the dynamics of the stabilized ROMs using any of the damping methods are quite satisfying over at least 2 cycles.

Finally, the values for the turbulent viscosity and turbulent thermal diffusivity for the stabilized ROMs are presented in figure 5.66. Interestingly, while most of the velocity modes have a positive turbulent viscosity, the second one actually requires a negative value. Similarly, the first temperature mode requires a negative turbulent thermal diffusivity which is quite different from the unforced jets results. At this point, it should be kept in mind that first, the decomposed (and therefore integrated) signal does not correspond to the overall velocity fluctuation, but rather to the deviation from the reference signal $\gamma(t)\tilde{u}_{ref}$ and therefore cannot be directly interpreted as a physical field. Then, it is reminded that the stabilization method adopted here is only an asymptotic (partial) method and only takes care of the under-diffusive character of the uncontrolled flow. Because of this, it is believed that a more complete formulation of the energy conservation constraints (including conservation for $\frac{d}{dt} \neq 0$) would greatly improve the ROM dynamics and in particular the deviations of the first mode.

**Phase Averaged Flow**

Based on the results from the preliminary POD, study showing only little difference between instantaneous and phase averaged flow POD decompositions, the question of whether or not it is possible to obtain a reduced order model for the phase averaged flow is posed. In other terms, if most of the turbulent fluctuation part of the instantaneous velocity field is removed in the process of truncating the POD series, why not only model the dominant phase averaged features in the first place?

**Modified Formulation**

To obtain a reduced order model of the phase averaged flow, the model equations are once again derived since the governing equations for the phase averaged flow are different from the one of the instantaneous flow in equation 5.19. We define the phase averaged flow using the extended Reynolds velocity decomposition:

$$\bar{u} = \bar{u}_T + \bar{u}$$
$$T = \bar{T}_T + \bar{T}$$

Figure 5.67: POD decomposition metrics for velocity (red) and temperature (blue) fields with $N = 200$ for phase average Case IV at $St_\infty = 0.159$. 

where \( \widetilde{\nu}_T^N \) and \( \widetilde{T}_T^N \) are the phase averaged flow and temperature fields corresponding to a period \( T \) over \( N \) periods defined by:

\[
\begin{align*}
\widetilde{\nu}_T^N (\theta) &= (\widetilde{\nu}_T) = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\nu}(\theta + nT) \\
\widetilde{T}_T^N (\theta) &= (\widetilde{T}_T) = \frac{1}{N} \sum_{n=0}^{N-1} T(\theta + nT)
\end{align*}
\]

and \( \tilde{u}, \tilde{T} \) are the remaining fluctuations.

The Navier-Stokes equations for the flow are then:

\[
\begin{align*}
\nabla \left( \widetilde{\nu}_T^N + \tilde{u} \right) &= 0 \\
\frac{\partial}{\partial \theta} \left( \widetilde{\nu}_T^N + \tilde{u} \right) + \left( \widetilde{\nu}_T^N + \tilde{u} \right) \nabla \left( \widetilde{\nu}_T^N + \tilde{u} \right) &= -\nabla P + \frac{1}{Re_j} \nabla \left( \widetilde{\nu}_T^N + \tilde{u} \right) \\
\frac{\partial}{\partial \theta} \left( \widetilde{T}_T^N + \tilde{T} \right) + \left( \widetilde{\nu}_T^N + \tilde{u} \right) \nabla \left( \widetilde{T}_T^N + \tilde{T} \right) &= \frac{1}{Pr Re_j} \nabla \left( \widetilde{T}_T^N + \tilde{T} \right)
\end{align*}
\] (5.36)

Instead of applying the time average operator to the Navier-Stokes equation to fall back on the Reynolds average equations, the phase averaging operator \( (\cdot)^T \) is applied to obtain the governing equations for the phase averaged flow:

\[
\begin{align*}
\nabla \left( \widetilde{\nu}_T^N \right) &= 0 \\
\frac{\partial}{\partial \theta} \left( \widetilde{\nu}_T^N \right) + \left( \widetilde{\nu}_T^N \nabla \right) \left( \widetilde{\nu}_T^N \right) &= -\nabla P + \frac{1}{Re_j} \nabla \left( \widetilde{\nu}_T^N \right) - \left( \widetilde{\nu} \nabla \right) \left( \tilde{u} \right) \\
\frac{\partial}{\partial \theta} \left( \widetilde{T}_T^N \right) + \left( \widetilde{\nu}_T^N \nabla \right) \left( \widetilde{T}_T^N \right) &= \frac{1}{Pr Re_j} \nabla \left( \widetilde{T}_T^N - \tilde{T} \right) - \left( \widetilde{\nu} \nabla \right) \left( \widetilde{T} \right)
\end{align*}
\] (5.37)

These equations are not much different from the ones for the instantaneous flow, with exception of the inclusion of turbulent source/sink terms \( \left( \tilde{u} \nabla \right) (\tilde{u}) \) and \( \left( \tilde{u} \nabla \right) (\tilde{T}) \), corresponding to the Reynolds stress terms in the Reynolds Average Navier-Stokes equations. We can include these terms as turbulent viscosity and turbulent thermal diffusivity by making use of the divergence free character of the fluctuation field so that \( \left( \left( \tilde{u} \nabla \right) (\tilde{u}) \right) = \nabla \cdot \left( \widetilde{\nu} \tilde{u} \right) \) and \( \left( \left( \tilde{u} \nabla \right) (\tilde{T}) \right) = \nabla \cdot \left( \widetilde{\nu} \tilde{T} \right) \). These terms can then be included in the diffusive terms on the right-hand side of governing equations for the phase averaged flow:

\[
\begin{align*}
\nabla \left( \widetilde{\nu}_T^N \right) &= 0 \\
\frac{\partial}{\partial \theta} \left( \widetilde{\nu}_T^N \right) + \left( \widetilde{\nu}_T^N \nabla \right) \left( \widetilde{\nu}_T^N \right) &= -\nabla P + \nabla \cdot \left( \frac{1}{Re_j} \nabla \widetilde{\nu}_T^N - \widetilde{\nu} \tilde{u} \right) \\
\frac{\partial}{\partial \theta} \left( \widetilde{T}_T^N \right) + \left( \widetilde{\nu}_T^N \nabla \right) \left( \widetilde{T}_T^N \right) &= \nabla \cdot \left( \frac{1}{Pr Re_j} \nabla \widetilde{T}_T^N - \widetilde{\nu} \tilde{T} \right)
\end{align*}
\] (5.38)

From here on, two approaches can be adopted to treat these governing equations. The first one is to purely and simply neglect the phase averaged Reynolds stresses. The second one is to model those stresses following the steps of turbulence modeling as an additional viscosity and thermal diffusivity using Boussinesq’s eddy viscosity assumption so that \( \tilde{u} \tilde{u} \sim -\nu_T \left( \widetilde{\nu}_T^N \right) \nabla \tilde{u}_T \) and \( \tilde{u} \tilde{T} \sim -\alpha_T \left( \widetilde{\nu}_T^N \right) \nabla \widetilde{T}_T^N \).
The governing equations can then be written as:

\[ \nabla \left( \tilde{u}_T^N \right) = 0 \]

\[ \frac{\partial}{\partial \theta} \left( \tilde{u}_T^N \right) + \left( \tilde{u}_T^N, \nabla \right) \left( \tilde{u}_T^N \right) = -\nabla P + \nabla \cdot \left( \left( \frac{1}{Re} + \nu_T \tilde{u}_T^N \right) \nabla \tilde{u}_T^N \right) \]

\[ \frac{\partial}{\partial \theta} \left( \tilde{T}_T^N \right) + \left( \tilde{u}_T^N, \nabla \right) \left( \tilde{T}_T^N \right) = \nabla \cdot \left( \left( \frac{1}{Pr} + \alpha_T \tilde{u}_T^N \tilde{T}_T^N \right) \frac{1}{Re} \nabla \tilde{T}_T^N \right) \]  \hspace{1cm} (5.39)

It can be noticed that at this point, neglecting the turbulent stresses is equivalent to setting \( \nu_T \) and \( \alpha_T \) to zero. In turbulence modeling, all the “art” resides in the evaluation of the values of, or fields \( \nu_T \) and \( \alpha_T \). Here, these terms will be at first neglected and a reduced order model for the phase averaged flow will be derived following this assumption. However, it will be assumed that the inclusion and evaluation of these terms can be done a posteriori as part of the stabilization process. Effectively, the stabilization terms will be used not only compensate the inaccuracies arising from the simulations, the snapshots sampling, or the POD series truncation, but also the omission of the Reynolds stresses.

By assuming, a priori, that the turbulent viscosity and thermal diffusivity are null, the governing equations for the phase averaged velocity and temperature fields reduce down to the usual Navier-Stokes equations. A reduced order model is then derived by means of the POD-Galerkin projection method described for the instantaneous flow using the decomposition for the phase averaged velocity field introduced in the previous section:

\[ \tilde{u}_T^N = \bar{u} + \gamma(t)\tilde{u}_{ref} + \bar{u}^\prime \]

where the reference field is still taken to be:

\[ \tilde{u}_{ref} = \bar{u}(BR = BR_0) - \bar{u}(BR = BR_1) \]

Using this assumption, the procedure to obtain the reduced order model equations is then identical to the one described for the instantaneous fields and provided in equations 5.22 and 5.23.

**Stabilization Methods**

Since the ROM equations for the phase averaged flow field are left unchanged from the ones of the instantaneous fields, the damping methods that will be used in this section will also be identical. It should be kept in mind however that for the phase averaged field, damping terms will also be used to model a posteriori the Reynolds stresses neglected a priori when deriving the reduced order model equations.

**Reduced Order Models**

A phase averaged flow is obtained using the same parameters as for the instantaneous flow, for a solution integrated over 10 forcing cycles. The flow is then phase averaged at the same 50 discrete phase locations as for the previous decomposition, thus providing a set of 50 POD snapshots. The transition from \( BR_0 \) to \( BR_1 \) occurs at \( t_c = 0 \). The POD metrics are presented in figure 5.67. Overall, the energy distributions for both velocity and temperature POD modes are very comparable to the one in figure 5.53 for the instantaneous flow, in particular for the dominant mode. This is in agreement with the results from the preliminary POD analysis. The first velocity mode is dominant with 37% of the energy which is equivalent to the energy content of the first mode in the instantaneous decomposition. The second mode is significantly weaker with 17% of the total kinetic energy, against 11% for the second mode of the instantaneous decomposition. The temperature energy levels also exhibit a stair-like distribution, characteristic of flows dominated by strong convective mechanisms. The energy content from the phase averaged temperature field decomposition is globally higher than corresponding modes from the instantaneous one. The 99% total kinetic energy and pseudo-thermal cumulative energy thresholds are reached when including 12 velocity modes and 12 temperature modes respectively. In comparison, thresholds for their instantaneous fields counterparts were corresponding to 65 modes for the velocity field and 70 modes for the temperature field. The phase average values agree also quite well with the ones obtained in the preliminary study (respectively 12 and 15). The POD shape functions for the velocity and temperature are presented in figure 5.68 and 5.69. The first 6 velocity modes
and 10 temperature modes bear strong resemblance with the instantaneous decompositions in the previous section. However, higher order modes, which here correspond to higher harmonics of the lower order ones, differ quite significantly from their counterparts in the instantaneous decomposition which are capturing either wall structures or anti-symmetric behavior.

Reduced order models corresponding to the non-stabilized equations are integrated for multiple values of \((N_V, N_T)\) in the range \([5, 50] \times [5, 50]\) and corresponding error surfaces for \(E_1^V\), \(E_2^V\), \(E_1^T\) and \(E_2^T\) are generated and presented in figures 5.70 and 5.71. First, all the ROMs are successfully integrated up to three forcing cycles including all 50 velocity and temperature modes. In general, increasing the number of velocity modes has a negative impact on both \(E_1^V\) and \(E_2^V\), as well as on \(E_1^T\) and \(E_2^T\) (with exception of the ROMs at \(N_T = 5\)), and increasing the number of temperature modes also has a rather negative impact on \(E_1^T\) and \(E_2^T\). However, the overall error levels are quite low when compared to the instantaneous ROMs with as much as an order of magnitude difference on \(E_2^T\) in the short term. Because the error levels are overall low, it was decided to present in figure 5.72 the best performing integrated ROMs, such that \(N_V > 12\) and \(N_T > 12\), corresponding to the 99% energy threshold values from the POD. For the velocity model, this corresponds to \(N_V = 14\) and for the temperature model to \((N_V, N_T) = (25, 15)\). The agreement between projected LES data and ROM integrated results is qualitatively comparable to the one obtained for the instantaneous flow field, with however only half the number of modes. Once again, a clear discrepancy in the first velocity mode is observed. Amplitude and phase are starting to diverge during the second cycle, although the amplitudes appear to grow less for the higher orders in the phase averaged field than the instantaneous one. While slowly deteriorating, the agreement between the reduced order model the projected LES data over two forcing cycles is found to be reasonably good.

The reduced order models are then integrated using local, broad and asymptotic stabilization methods and the corresponding computed error surfaces are presented in figures 5.73 and 5.74 for the local stabilization, figures 5.75 and 5.76 for the broad one, and figures 5.76 and 5.77 for the asymptotic one. All three stabilization methods exhibit comparable error levels, although in the long term, ROMs with \(N_T = 50\) stabilized using the local and asymptotic methods start diverging in the second cycle. The asymptotic formulation provides more stability to the systems integrated using large numbers of velocity modes and slightly better performance in the long term when compared to the other two. Minimum total error levels on the velocity fields for the stabilized ROMs are on average two time lower in the short term, and almost ten times lower in the long term than comparable non-stabilized ROMs. The effect of the stabilization is even greater on the temperature field with total error levels decreased by a factor 10 in the short term and a factor 100 in the long term. The stabilized reduced order models corresponding to the ones presented previously are shown in figure 5.79. Overall, the agreement between LES projected data and integrated ROMs is clearly improved over the non-stabilized models with exception of the first velocity POD mode which is left quasi-unchanged. The asymptotically stabilized velocity ROM seems to have slightly better performance towards the end of the second cycle, in particular when considering higher order modes, though the difference is not significant enough to draw any definitive conclusion. The temperature ROM is almost completely insensitive to the type of stabilization method used, yet shows clearly improved agreement with the projected data. Finally, the computed turbulent viscosity and thermal diffusivity are presented in figure 5.80. Although not many points are shown here because of the low values of \(N_V\) and \(N_T\), the trends appear to be similar to the ones obtained for the instantaneous fields, with globally increasing values of the turbulent viscosity and decreasing values of the turbulent thermal diffusivity for increasing mode ranks.

### 5.3 Conclusion

The preliminary POD analysis of the inclined film cooling jet at \(BR = 0.15\) described in this section showed that when considering only part of the physical domain and at such low blowing ratios, POD converged when using 200 snapshots. A total of 78 velocity modes and 76 temperature modes were necessary to gather 99% of the total fluctuation energy. The most energetic velocity POD modes were found to be associated with the convection of the shear layer vortices encountered at this blowing ratio while higher order modes accounted for their interaction with side vortices located near the wall. The temperature analysis led to a more convoluted decomposition, due to the non-optimality of the norm associated to standard POD. The reconstruction of both fields has shown that satisfactory results can be obtained by including a total of
Figure 5.68: First 12 velocity POD modes along with $\bar{u}_m$ and $\bar{u}_{ref}$ $\lambda_2$ iso-surfaces for POD decomposition for phase averaged Case IV at $St_\infty = 0.159$. Mode energy provided in percent of the total energy.
Figure 5.69: First 14 temperatu re POD modes iso-surfaces for POD decomposition for phase averaged Case IV at $St_\infty = 0.159$. Pink and cyan contours are for negative values, red and blue for positive ones. Mode energy provided in percent of the total energy.
Figure 5.70: Estimates of $\log(E^V_1)$ and $\log(E^V_2)$ over one (left), two (center) and three (right) forcing cycles for phase averaged Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$.

Figure 5.71: Estimates of $\log(E^T_1)$ and $\log(E^T_2)$ over one (left), two (center) and three (right) forcing cycles for phase averaged Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$. 
Figure 5.72: phase averaged Case IV at $St_{\infty} = 0.159$ ROM temporal coefficients for the velocity ($N_V = 14$) and temperature ($(N_V, N_T) = (25,15)$) fields (solid lines) along with corresponding POD temporal coefficients (symbols).
Figure 5.73: Estimates of $\log(E_{1})$ and $\log(E_{2})$ over one (left), two (center) and three (right) forcing cycles for phase averaged Case IV at $St_{\infty} = 0.159$ ROMs integrated at multiple values of $N_{V}$ and $N_{T}$ using partial local linear stabilization.

Figure 5.74: Estimates of $\log(E_{1}^{T})$ and $\log(E_{2}^{T})$ over one (left), two (center) and three (right) forcing cycles for phase averaged Case IV at $St_{\infty} = 0.159$ ROMs integrated at multiple values of $N_{V}$ and $N_{T}$ using partial local linear stabilization.
Figure 5.75: Estimates of $\log(E_{V1}^T)$ and $\log(E_{V2}^T)$ over one (left), two (center) and three (right) forcing cycles for phase averaged Case IV at $St_{\infty} = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using partial broad linear stabilization.

Figure 5.76: Estimates of $\log(E_{T1}^T)$ and $\log(E_{T2}^T)$ over one (left), two (center) and three (right) forcing cycles for phase averaged Case IV at $St_{\infty} = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using partial broad linear stabilization.
Figure 5.77: Estimates of $\log(E_{V1})$ and $\log(E_{V2})$ over one (left), two (center) and three (right) forcing cycles for phase averaged Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using asymptotic linear stabilization.

Figure 5.78: Estimates of $\log(E_{T1})$ and $\log(E_{T2})$ over one (left), two (center) and three (right) forcing cycles for phase averaged Case IV at $St_\infty = 0.159$ ROMs integrated at multiple values of $N_V$ and $N_T$ using asymptotic linear stabilization.
Figure 5.79: phase averaged Case IV at $St_\infty = 0.159$ ROM temporal coefficients for the velocity ($N_V = 14$) and temperature ($N_V, N_T = (25,15)$) fields using partial local (solid), partial broad (dashed) and asymptotic (dashed dot dot) linear damping along with corresponding POD temporal coefficients (symbols).
Figure 5.80: phase averaged Case IV ROM (a) normalized turbulent momentum diffusivity $\nu_T/\nu$ and (b) normalized turbulent thermal diffusivity $\alpha_T/\alpha$ calculated for $N_V = 14$ and $N_T = 15$.

20 POD modes. However, due to the nature of the proper orthogonal decomposition, the relative error in the vicinity of the jet exit remains high and could potentially affect the behavior of a reduced order model based on a truncated series. Preliminary proper orthogonal decomposition analysis was also carried on the pulsed jet at two distinct forcing frequencies. The analysis of the lowest frequency case ($St_\infty = 0.016$) included 50 phase averaged snapshots and showed that a total of 41 velocity and 33 temperature modes were required to gather more than 99% of the fluctuation energy. The different POD modes had a distinct support of action due to the presence of multiple flow regime within a single forcing cycle. Because of the segregation of the POD modes to specific cycle parts, the POD series truncation for the forced flow could lead to the suppression of the dynamics of part of the cycle. The higher forcing frequency ($St_\infty = 0.159$) analysis included 25 phase averaged modes and required 12 velocity and 15 temperature modes to capture more than 99% of the total fluctuation energy. The POD revealed the homogeneity of the flow in the streamwise direction due to the limited range of length and time scales present in the domain. Because of the significantly greater complexity of the problem at low forcing frequency, 40 POD modes were required to obtain less than 4% relative error on the velocity field, while the same error levels were reached with only 15 modes in the higher frequency case. The temperature field required fewer modes and respectively 10 and 6 modes were necessary at $St_\infty = 0.016$ and $St_\infty = 0.159$ to reach identical accuracy levels.

Reduced order modeling efforts were carried out using a refined grid and over the complete simulation domain, following the POD-Galerkin method described in the previous chapter. Stabilization methods, necessary in realistic three dimensional flows, were discussed and an extension of the model introduced by Cazemier et al. 1998 was proposed. Unforced jets were first considered in both attached and transitional regimes. In the attached jet regime, the effects of the POD domain selection were evidenced by the change of the dominant structures from hairpin vortices to wall vortices between the preliminary analysis and the full domain decomposition. Although decreasing with increasing number of modes, the error levels on the total velocity field of the non-stabilized reduced order models were found to be quite significant. In the short term ($\Delta t_c < 2$), minimum error levels were reached for $(N_V, N_T) = (250, 120)$ but tracking was found to be qualitatively poor beyond $t_c = 1$. A growing divergence in the amplitude and phase of the ROMs was attributed to the interruption of the energy cascade introduced by the POD series truncation, the turbulence model used in the simulations, and the sampling frequency used in the snapshot collection. Stabilized ROMs using either local or broad linear stabilization methods evidenced greatly improved performance over non stabilized ones, with error levels over 30 times lower for the velocity field and 200 times lower in the long term for the temperature ROM. Minimum error levels were reached for the stabilized ROMs using $(N_V, N_T) = (275, 100)$. Stabilized ROMs showed reasonable agreement with the projected LES data up to three convective time scale units. In the transitional jet regime, the POD reflected well the increased turbulent character of the flow, with fuller energy distributions and busier POD modes. Non stabilized reduced order models were derived and integrated for values of $(N_V, N_T)$ in the range $[30, 500] \times [30, 500]$. The equations corresponding to the higher order modes were found to diverge rather rapidly beyond $t_c = 1$ and cause the entire system, including the temperature ROM, to diverge around $t_c = 2.5$, preventing any
integration past this point. Consequently, error levels for the velocity ROMs in the short term were found to be extremely high, while corresponding temperature error levels were found to be surprisingly lower than comparable results in the attached configuration. Reduced order models including local and broad stabilization terms exhibited much improved error levels. It was found that the broad stabilization method was much more effective when integrating systems including large number of velocity POD modes and therefore permitted to further reduce error levels on the velocity field, even more so in the long term. The temperature ROMs were however much less sensitive to the type of stabilization method used than the velocity models. Qualitatively, both stabilization methods provided much improved ROMs dynamics with an advantage to the broadly stabilized reduced order models which showed very good agreement with the projected LES data at least up to $t_c = 4$.

Reduced order models for instantaneous and phase averaged forced flows at $BR_b = 0.5$, $BR_t = 0.15$, $DC = 50\%$ and $St_\infty = 0.159$ were obtained and integrated. Instantaneous flows were decomposed using Graham et al. (1999a) formulation. While the POD metrics and shape function of the temperature field were quite similar to the ones of the preliminary study, the POD for the velocity was found to be different due to the use of a different velocity field decomposition. Non-stabilized ROMs showed reasonable agreement with the projected LES data, although divergence started to grow during the second forcing cycle. Three stabilization methods were proposed (two partial and one complete asymptotic) and applied to the velocity and temperature models. All three methods provided significant improvement over the non-stabilized equations with yet a small advantage to the asymptotic stabilization when dealing with systems including larger numbers of velocity modes. Overall any of the stabilized model provided satisfying representation of the flow over at least two cycle periods. Reduced order model equations for the phase averaged flow were derived by neglecting a priori the Reynolds stresses and correcting for those a posteriori when using stabilization methods. Overall, reduced order models of the phase averaged flow were found to perform at least as well as the ones for the instantaneous flow using only half as many modes, thus reducing the computational cost associated with the integration of reduced order models. The stabilized equations provided significant improvement over the non stabilized ones, in particular for the temperature model. It appears clear that the introduction of discrete, strong vortical structures through jet actuation, provided a much better defined and more distinct POD basis responsible for the better accuracy of the forced ROMs over the unforced ones.

These results show that reasonably accurate reduced order models could be obtained for the attached and transient unforced regimes and quite accurate models could be obtained for both the instantaneous and phase averaged flows. Phase averaged models provided comparable error levels to those of instantaneous ones using only half the number of velocity and temperature modes. In all the investigated cases, stabilization methods were found to be beneficial, if not necessary. In unforced conditions, the local linear stabilization of Cazemier et al. (1998) and the proposed broad linear stabilization methods exhibited comparable performances, yet the broad formulation usually resulted in more stable equations allowing the integration of larger system and leading to lower error levels. This was particularly true for the more turbulent transient jet regime. In forced conditions, all three proposed methods, damping either only the base flow or both the base and controlled flows in an asymptotic way, provided comparable results in terms of the global error estimates and the agreement of the ROMs dynamics with the projected LES data. The asymptotic formulation seemed to provide some improved stabilization for models including large numbers of modes, compared to the other two. Further improvement of the reduced order models in both unforced and forced conditions will require more sophisticated stabilization methods, in particular with time-dependent capabilities. This is especially true for the forced jet. However, as it stands, it is believed the presented methods provide acceptable models in unforced conditions and relatively good ones in forced conditions, so that they can serve as an initial set of governing equations for the development of a flow controller.
Chapter 6
Concluding Remarks and Suggested Future Work

The current study had two principal objectives. The first one was to provide a description and an analysis of forced jets in cross-flow over a range of parameters applicable to film cooling systems.

To this end, a fundamental and well documented vertical jet setup was first investigated. A preliminary study at unforced low blowing ratio conditions was carried out to provide a baseline for the forced jet analysis, but also to fill in the knowledge gap on these particular jets, usually investigated at higher blowing ratios. It was found that at low blowing ratios the set of dominant vortical structures for the attached jet (hairpin, horseshoe, inner vortex and side vortices) did not directly correspond to the one usually described in detached conditions (shear layer ring vortices, horseshoe vortex, wake vortices and counter-rotating vortex pair), but rather to an evolution of it. Starting from the low end of the blowing ratio range, this evolution was described for increasing blowing ratios values, and the connections between the low blowing ratio and high blowing ratio structures were established. Forced jets using a square wave excitation were then investigated over a wide range of forcing parameters. Starting vortices generated at the transition from low to high blowing ratio were observed and the existence of three types of starting structures (single starting vortex ring, starting vortex ring and trailing column and starting hairpin vortex) established in past numerical studies was confirmed experimentally and extended by a fourth one (starting hairpin vortex with trailing structures). The scaling parameters used to classify the different types of vortical structures were also extended to non-zero low blowing ratio jets and verified both experimentally and numerically.

A more application-relevant setup implementing an inclined jet was also investigated in unforced and forced conditions. The unforced jet study evidenced the differences between the vertical and inclined configurations. In particular, it was found that the smoother transition from attached to detached regime in the inclined jet was to be attributed to the absence of a stable vortical structure inside the jet pipe (inner vortex), otherwise present in the vertical configuration. This set aside, strong similarities between the unforced inclined and vertical jet vortical systems were found. Although used in the past in forced film cooling systems but also in vortex jet generators for airfoil separation control, the fluidics of forced inclined jet in cross-flow had not been extensively investigated. The starting vortical structures in the inclined jet evidenced some similarities but also some profound differences with the ones in the vertical configuration. To start with the similarities, starting vortices of the ring and hairpin types were found in both configurations. However, the dynamics of those structures were found to be significantly different in terms of trajectory, evolution, but also interaction with the other parts of the cycle, due to their reduced vertical momentum and the slanted nature of the jet exit in the inclined jet configuration. Furthermore, an additional type of starting vortices (compound starting vortices) was identified for the inclined jet, raising to six the number of starting vortex regimes found in this configuration. Investigations on the proper scaling parameters were also conducted using experimental data and the results were found to be consistent with the one obtained in the vertical jet setup. The controversial results obtained in previous forced film cooling studies were put in perspective with the current findings, showing that both studies had in fact been investigating two very different systems at very different operating conditions.

The second objective of this study was to derive the first reduced order models of realistic film cooling flows under unforced but also forced conditions for both the velocity and the temperature fields. To this end, preliminary studies on canonical flows were conducted to develop the necessary numerical tools and understanding associated with the Proper Orthogonal Decomposition - Galerkin Projection model reduction method. The POD method was also used to obtain statistical information on film cooling flows energetically dominant features in both unforced and forced conditions. In particular it showed that POD directly applied to low forcing frequency film cooling jets could segregate the POD modes to specific parts of the cycle and truncated POD series could suffer from poor accuracy in the parts of the cycle with lower energy content. Using the POD-Galerkin method, unforced reduced order models for the velocity and temperature fields were
obtained in both the attached and transitional regimes based on high accuracy LES simulations. Though initially quite accurate, both models were found to diverge rather rapidly, due to growing instability in the higher order POD modes eventually propagating. A previously formulated local linear stabilization method intended to balance the energy content in each individual equation was implemented, along with a proposed broader formulation for this method. The stabilized models provided much improved dynamics over the non-stabilized ones, in particular in the transitional regime. The proposed broad formulation of the linear stabilization terms was found to perform better than the local one in systems involving large number of modes. Reduced order models for the forced flow were also obtained using the POD-Galerkin method. Non-stabilized equations for the instantaneous flow provided a good approximation of the LES data, though the agreement was slowly deteriorated due to non-conservation of the global energy from intrinsic diffusivity underestimation. Three linear stabilization methods (two partial and a complete asymptotic one) were proposed and evaluated. The stabilized models provided improved agreement regardless of the method used, though the asymptotic one allowed better stabilization of systems including larger numbers of modes. Finally, equations for reduced order models of the phase averaged forced velocity and temperature fields were derived and integrated in their stabilized and non-stabilized formulations. Models for the phase averaged flow were found to perform comparably or better than equivalent instantaneous models while using only half the number of modes, therefore reducing the computational requirements.

In the perspective of obtaining a reduced order model of either the unforced or forced jet, the understanding of the underlying flow mechanisms associated with the physical object may appear as unimportant. Indeed, the POD-Galerkin model reduction method can be considered as a purely mathematical tool and applied to any set of data readily available to try and obtain reduced order model equations at any operating conditions. Such strategy is unfortunately very often adopted. However, the accuracy of a model is only as good as the accuracy with which its domain of validity is known. Therefore, while the reduced order model for the attached jet could be carelessly integrated for blowing ratio values in the transitional, or even the detached jet regimes, it is unlikely these would provide a good approximation of the flow simply because the empirical POD modes used to build the model do not carry the information relevant to the flow physics in those regimes. In the same way, a mathematically correct set of reduced order model equations for a forced jet could be built entirely based on POD modes corresponding to selected unforced flows. It is, once more, unlikely the performance of such model will be satisfying since the unforced empirical POD functions do not contain the physics relevant to the transient phases (starting vortices and ingestion) introduced when forcing the jet. Along those lines, a forced jet reduced order model derived from POD empirical functions obtained in forcing conditions resulting in the generation of a starting hairpin vortex or even compound starting system will not be able to represent accurately forced flows beyond $BR_h = 0.68$ where starting ring vortices are expected to form. Because of this, it appears quite relevant, if not essential, to have a good understanding of the physics, instabilities and features of the flow that is intended to be modeled in order to carefully select empirical functions representative of those, but also to know over which range of parameters the resulting model can be considered as valid. This was the intent of the current study.

Future work based on the current study could include:

- **Further investigation of the ingestion mechanisms in forced conditions.** Investigations based on PIV measurements and/or numerical simulations could be carried out to quantify and provide scaling parameters for this transient regime as well.

- **Development of reduced order models for additional jet regimes.** While this study proved that reduced order models could be obtained for a forced jet in the compound starting vortex system regime, it also showed that multiple types of forced regimes existed. Reduced order models for additional regimes relevant to film cooling should be derived.

- **Evaluation of the robustness of the obtained reduced order models.** A parametric study comparing the results from integration of LES simulations with ROM integration at non-nominal operating conditions could be carried out to detect the range of validity of the models obtained in the current study in both unforced and forced conditions.

- **Improvement of the stabilization algorithm in unforced conditions.** A more accurate formulation for stabilizing the equations based on instantaneous energy conservation rather than time averaged...
considerations could provide much more accurate ROMs. Other estimation methods for the turbulent viscosity and thermal diffusivity could also be implemented.

- **Improvement of the stabilization algorithm in forced conditions.** Forced flows, more than unforced ones, necessitate a more accurate stabilization formulation in order to allow energy exchange between the individual modes. This could evidently lead to more accurate ROM and in particular alleviate the discrepancies observed between the most energetic velocity mode and the projected LES data.

- **Investigation of alternative projection basis.** Alternative projection basis capable of sorting modes based on their dynamical relevance rather than purely their energy content could provide improved reduced order models dynamics.

- **Controller design based on the obtained reduced order equations.** A controller based on the current reduced order models should be developed in order to define optimum forcing parameters in steady and especially unsteady cross-flow environment.
References


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Appendix
POD-Galerkin Reduced Order Model Derivation

Galerkin projection for the unforced flows

In this section the details of the derivation of the POD-Galerkin reduced order model equations are described for the unforced flows.

Recalling the notations from Chapter 4 concerning the momentum and energy equations:

\[
\frac{\partial \vec{u}'}{\partial t} = M(\vec{u}', \vec{u}) \quad (6.1)
\]

\[
\frac{\partial T'}{\partial t} = E(\vec{u}', \vec{u}, T, T') \quad (6.2)
\]

with

\[
M(\vec{u}', \vec{u}) = -\left(\left(\vec{u} + \vec{u}'\right) \cdot \nabla\right)\left(\vec{u} + \vec{u}'\right) - \nabla P + \frac{1}{Re} \Delta \left(\vec{u} + \vec{u}'\right)
\]

and

\[
E(\vec{u}', \vec{u}, T, T') = -\left(\left(\vec{u} + \vec{u}'\right) \cdot \nabla\right) (T + T') + \frac{1}{Pr} \frac{1}{Re} \Delta (T + T')
\]

And recalling the notations for the decompositions of the fluctuation fields \(\vec{u}'\) and \(T'\) so that \(\vec{u}'(x, t) = \sum_{n=1}^{NV} a_n(t) \vec{\varphi}_n(x)\) (\(\lambda_n\) are the associated eigenvalues) and \(T'(x, t) = \sum_{m=1}^{NT} b_m(t) \psi_m(x)\) (\(\sigma_m\) are the associated eigenvalues) and can be replaced in equations 6.1 and 6.2 to give:

\[
\frac{\partial}{\partial t} \left(\sum_{n=1}^{NV} a_n(t) \vec{\varphi}_n(x)\right) = M \left(\sum_{n=1}^{NV} a_n(t) \vec{\varphi}_n(x), \vec{u}\right) \quad (6.3)
\]

\[
\frac{\partial}{\partial t} \left(\sum_{m=1}^{NT} b_m(t) \psi_m(x)\right) = E \left(\sum_{n=1}^{NV} a_n(t) \vec{\varphi}_n(x), \vec{u}, \sum_{m=1}^{NT} b_m(t) \psi_m(x), T\right) \quad (6.4)
\]

Velocity Reduced Order Model

Projecting the momentum equation 6.3 onto the \(r^{th}\) velocity POD mode \(\vec{\varphi}_r\):

\[
\left(\frac{\partial}{\partial t} \left(\sum_{n=1}^{NV} a_n(t) \vec{\varphi}_n(x)\right), \vec{\varphi}_r(x)\right) = M \left(\sum_{n=1}^{NV} a_n(t) \vec{\varphi}_n(x), \vec{u}\right), \vec{\varphi}_r(x) \quad (6.5)
\]

The left-hand side of the projected momentum equation 6.5 can be expanded into:
\[
\left( \frac{\partial}{\partial t} \left( \sum_{n=1}^{N_V} a_n (t) \vec{\varphi}_n (x) \right), \vec{\varphi}_r (x) \right) = \sum_{n=1}^{N_V} \left( \frac{\partial}{\partial t} (a_n(t) \vec{\varphi}_n (x)), \vec{\varphi}_r (x) \right)
\]
\[
= \sum_{n=1}^{N_V} \frac{\partial}{\partial t} a_n(t) (\vec{\varphi}_n (x), \vec{\varphi}_r (x))
\]
\[
= \dot{a}_r (t)
\]

Thus equations 4.30:

\[
\dot{a}_r (t) = \left( M \left( \sum_{n=1}^{N_V} a_n(t) \vec{\varphi}_n (x), \vec{\bar{u}} \right), \vec{\varphi}_r (x) \right)
\]

(6.6)

Evaluating the right-hand side of equation 6.6 we get (dropping the independent variables \( t \) and \( x \)):

\[
\left( M \left( \sum_{n=1}^{N_V} a_n \vec{\varphi}_n, \vec{\bar{u}} \right), \vec{\varphi}_r \right) = - \left( \left( \vec{\bar{u}} + \sum_{n=1}^{N_V} a_n \vec{\varphi}_n \right) \cdot \nabla \right) \left( \vec{\bar{u}} + \sum_{m=1}^{N_V} a_m \vec{\varphi}_m \right), \vec{\varphi}_r \right)
\]
\[
= \left( \vec{\bar{u}} \cdot \nabla \right) \left( \sum_{n=1}^{N_V} a_n \vec{\varphi}_n \right), \vec{\varphi}_r \right)
\]
\[
+ \left( \vec{\bar{u}} \cdot \nabla \right) \vec{\bar{u}}, \vec{\varphi}_r \right)
\]
\[
= \sum_{n=1}^{N_V} a_n(t) \left[ (\vec{\varphi}_n \cdot \nabla) \vec{\bar{u}}, \vec{\varphi}_r \right) + (\nabla \cdot \vec{\varphi}_n, \vec{\bar{u}}), \vec{\varphi}_r \right)
\]
\[
+ \sum_{n=1}^{N_V} \sum_{m=1}^{N_V} a_n a_m \left( (\vec{\varphi}_n \cdot \nabla) \vec{\varphi}_m, \vec{\varphi}_r \right) \]

(6.7)

The first (convective) term of equation 6.7 can be decomposed into:

\[
\left( (\vec{\bar{u}} + \sum_{n=1}^{N_V} a_n \vec{\varphi}_n) \cdot \nabla \left( \vec{\bar{u}} + \sum_{m=1}^{N_V} a_m \vec{\varphi}_m \right), \vec{\varphi}_r \right) = \left( \left( \sum_{n=1}^{N_V} a_n \vec{\varphi}_n \right) \cdot \nabla \right) \vec{\bar{u}}, \vec{\varphi}_r \right)
\]
\[
+ \left( \vec{\bar{u}} \cdot \nabla \right) \left( \sum_{n=1}^{N_V} a_n \vec{\varphi}_n \right), \vec{\varphi}_r \right)
\]
\[
+ \left( \vec{\bar{u}} \cdot \nabla \right) \vec{\bar{u}}, \vec{\varphi}_r \right)
\]
\[
= \sum_{n=1}^{N_V} a_n(t) \left[ ((\vec{\varphi}_n \cdot \nabla) \vec{\bar{u}}, \vec{\varphi}_r \right) + ((\nabla \cdot \vec{\varphi}_n) \vec{\bar{u}}, \vec{\varphi}_r \right) \]
\]
\[
+ \left( \vec{\bar{u}} \cdot \nabla \right) \vec{\bar{u}}, \vec{\varphi}_r \right)
\]
\[
+ \sum_{n=1}^{N_V} \sum_{m=1}^{N_V} a_n a_m \left( (\vec{\varphi}_n \cdot \nabla) \vec{\varphi}_m, \vec{\varphi}_r \right) \]

The second (diffusive) term of equation 6.7 can be decomposed into:
\[
\left( \frac{1}{Re} \Delta \left( \overline{\mathbf{u}} + \sum_{n=1}^{N_V} a_n \varphi_n \right), \varphi_r \right) = \frac{1}{Re} \left( \Delta \left( \sum_{n=1}^{N_V} a_n \varphi_n \right), \varphi_r \right) \\
+ \frac{1}{Re} \left( \Delta \overline{\mathbf{u}}, \varphi_r \right) \\
= \frac{1}{Re} \sum_{n=1}^{N_V} a_n \left( \Delta \varphi_n, \varphi_r \right) \\
+ \frac{1}{Re} \left( \Delta \overline{\mathbf{u}}, \varphi_r \right)
\]

Finally we obtain the reduced order model for the velocity field:

\[
a_r(t) = \sum_{i=1}^{N_V} \left( C^r_{i,0} + \frac{1}{Re} D^r_i \right) a_i(t) + \sum_{i=1}^{N_V} \sum_{j=1}^{i} C^r_{i,j} a_i(t) a_j(t) + \left( C^r_0 + \frac{1}{Re} D^r_0 \right) + P_r(t) \tag{6.8}
\]

where

\[
\overline{\mathbf{u}} = \overline{\mathbf{u}} + \mathbf{u}^r = \overline{\mathbf{u}} + \sum_{n=1}^{N_V} a_n(t) \varphi_n(x) \\
C^r_0 = - \left( (\nabla \cdot \overline{\mathbf{u}}) \varphi_r, \varphi_r \right) = - \int_{\Omega_x} \frac{\partial}{\partial x_m} (\overline{\mathbf{u}}_l) \varphi_{r,l} dx \\
D^r_0 = (\Delta \overline{\mathbf{u}}, \varphi_r) = \int_{\Omega_x} \frac{\partial}{\partial x_m} (\overline{\mathbf{u}}_l) \varphi_{r,l} dx \\
C^r_{i,0} = - \left( (\nabla \cdot \varphi_i) \varphi_r, \varphi_r \right) = - \int_{\Omega_x} \left( \overline{\mathbf{u}}_m \frac{\partial}{\partial x_m} \varphi_{i,l} + \varphi_{i,m} \frac{\partial}{\partial x_m} \overline{\mathbf{u}}_l \right) \varphi_{r,l} dx \\
D^r_i = (\Delta \varphi_i, \varphi_r) = \int_{\Omega_x} \frac{\partial}{\partial x_m} (\varphi_{i,l}) \varphi_{r,l} dx \\
C^r_{i,j} = - \left( (\nabla \cdot \varphi_j) \varphi_r, \varphi_r \right) = - \int_{\Omega_x} \left( \varphi_{j,m} \frac{\partial}{\partial x_m} \varphi_{j,l} + \varphi_{j,m} \frac{\partial}{\partial x_m} \varphi_{i,l} \right) \varphi_{r,l} dx \\
P_r(t) = - \left( \nabla P(t), \varphi_r \right)
\]

and \(\overline{\mathbf{u}}_m\) corresponds to the \(m^{th}\) component of the vector \(\overline{\mathbf{u}}\) and \(\varphi_{r,l}\) corresponds to the \(l^{th}\) component of the vector \(\varphi_r\). Redundant indexes imply summation on those indexes according to Einstein notation.

**Temperature Reduced Order Model**

Projecting the energy equation 6.4 onto the \(s^{th}\) temperature POD mode \(\psi_s\) gives:

\[
\left( \frac{\partial}{\partial t} \left( \sum_{m=1}^{N_T} b_m(t) \psi_m(x) \right), \psi_s(x) \right) = \left( E \sum_{n=1}^{N_V} a_n(t) \varphi_n(x), \overline{\mathbf{u}}_m, \sum_{m=1}^{N_T} b_m(t) \psi_m(x), \overline{\mathbf{u}} \right), \psi_s(x) \tag{6.9}
\]

and the left-hand-side of the projected energy equation 6.9 into:

\[
\left( \frac{\partial}{\partial t} \left( \sum_{m=1}^{N_T} b_m(t) \psi_m(x) \right), \psi_s(x) \right) = \sum_{m=1}^{N_T} \left( \frac{\partial}{\partial t} (b_m(t) \psi_m(x)), \psi_s(x) \right) \\
= \sum_{m=1}^{N_T} \left( \frac{\partial}{\partial t} (b_m(t) \psi_m(x)) (\psi_m(x), \psi_s(x)) \right) \\
= \frac{\partial}{\partial t} \left( b_s(t) \right)
\]

Equation 6.9 becomes then:
\[ \dot{b}_s(t) = \left( E \left( \sum_{n=1}^{N_V} a_n(t) \varphi_n(x), \bar{\alpha}, \sum_{m=1}^{N_T} b_m(t) \psi_m(x), T \right), \psi_s(x) \right) \] (6.10)

Evaluating the right-hand side of equation 6.10 we get (dropping the independent variables \( t \) and \( x \):

\[ \left( E \left( \sum_{n=1}^{N_V} a_n \varphi_n, \bar{\alpha}, \sum_{m=1}^{N_T} b_m \psi_m, T \right), \psi_s \right) = -\left( \left( \left( \sum_{n=1}^{N_V} a_n \varphi_n \right) \cdot \nabla \right) \left( T + \sum_{m=1}^{N_T} b_m \psi_m \right), \psi_s \right) \]
\[ + \left( \frac{1}{Pr \cdot Re} \Delta \left( T + \sum_{m=1}^{N_T} b_m \psi_m \right), \psi_s \right) \] (6.11)

The first (convective) term of 6.11 can be decomposed into:

\[ \left( \left( \sum_{n=1}^{N_V} a_n \varphi_n \right) \cdot \nabla \right) \left( T + \sum_{m=1}^{N_T} b_m \psi_m \right), \psi_s \] = \[ \left( \left( \sum_{n=1}^{N_V} a_n \varphi_n \right) \cdot \nabla \right) T, \psi_s \]
\[ + \left( \sum_{n=1}^{N_V} a_n \varphi_n, \nabla \right) \sum_{m=1}^{N_T} b_m \psi_m, \psi_s \]
\[ + \left( \left( \sum_{n=1}^{N_V} a_n \varphi_n \right) \cdot \nabla \right) \left( \sum_{m=1}^{N_T} b_m \psi_m \right), \psi_s \]
\[ + \left( \left( \sum_{n=1}^{N_V} a_n \varphi_n \right) \cdot \nabla \right) \psi_s \]
\[ = \sum_{n=1}^{N_V} a_n \left( \left( \varphi_n \cdot \nabla \right) T, \psi_s \right) + \sum_{m=1}^{N_T} b_m \left( \left( \varphi_n \cdot \nabla \right) \psi_m, \psi_s \right) \]
\[ + \sum_{n=1}^{N_V} \sum_{m=1}^{N_T} b_m a_n \left( \left( \varphi_n \cdot \nabla \right) \psi_m, \psi_s \right) \]

The second (diffusive) term of 6.11 can be decomposed into:

\[ \left( \frac{1}{Pr \cdot Re} \Delta \left( T + \sum_{m=1}^{N_T} b_m \psi_m \right), \psi_s \right) = \frac{1}{Pr \cdot Re} \left( \Delta \left( \sum_{m=1}^{N_T} b_m \psi_m \right), \psi_s \right) \]
\[ + \left( \frac{1}{Pr \cdot Re} \Delta T, \psi_s \right) \]
\[ = \frac{1}{Pr \cdot Re} \sum_{m=1}^{N_T} b_m \left( \Delta \psi_m, \psi_s \right) \]
\[ + \frac{1}{Pr \cdot Re} \left( \Delta T, \psi_s \right) \]

The inner product is repeated for \( r = 1, 2, \ldots, N_V \) and \( s = 1, 2, \ldots, N_T \) providing \( N_V \) ODEs for the momentum equations and \( N_T \) equations for the energy equation. Finally, the ROM equations can be reduced to:

\[ \dot{b}_s(t) = \sum_{i=1}^{N_V} E_{s,0}^i a_i(t) + \sum_{j=1}^{N_T} \left( E_{s,j}^0 + \frac{1}{Pr \cdot Re} E_{s,j}^1 \right) b_j(t) + \sum_{i=1}^{N_V} \sum_{j=1}^{N_T} E_{s,i,j}^0 a_i(t) b_j(t) + \left( E_{s,0}^0 + \frac{1}{Pr \cdot Re} E_{s,0}^1 \right) \] (6.12)
where

\[ T = T + T' = T + \sum_{m=1}^{N_T} b_m(t) \psi_m(x) \]

\[ E_{s,0}^0 = - ((\bar{u} \nabla) \bar{T}, \psi_s) = - \int_{\Gamma} \bar{u}_m \frac{\partial}{\partial x_m} (\bar{T}) \psi_s \, dx \]

\[ F_{s,0} = (\nabla T, \psi_s) = \int_{\Gamma} \frac{\partial}{\partial x_m} (\bar{T}) \psi_s \, dx \]

\[ E_{s,0}^i = - ((\bar{\varphi}_i \nabla) \bar{T}, \psi_s) = - \int_{\Gamma} \bar{\varphi}_i_m \frac{\partial}{\partial x_m} (\bar{T}) \psi_s \, dx \]

\[ E_{s,j}^0 = - ((\bar{u} \nabla) \varphi_j, \psi_s) = - \int_{\Gamma} \bar{u}_m \frac{\partial}{\partial x_m} (\varphi_j) \psi_s \, dx \]

\[ F_{s,j} = (\Delta \varphi_j, \psi_s) = \int_{\Gamma} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_m} (\varphi_j) \psi_s \, dx \]

\[ E_{s,j}^i = - ((\bar{\varphi}_i \nabla) \varphi_j, \psi_s) = - \int_{\Gamma} \bar{\varphi}_i_m \frac{\partial}{\partial x_m} (\varphi_j) \psi_s \, dx \]

and \( u_m \) corresponds to the \( m \)th component of the vector \( \bar{u} \) and \( \varphi_{r,l} \) corresponds to the \( l \)th component of the vector \( \bar{\varphi}_r \). Redundant indexes imply summation on those indexes according to Einstein notation.

**Galerkin projection for the forced flows**

Recalling the notations of Chapter 5:

\[ \bar{u} = \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \]  

(6.13)

where \( \gamma(t) \) is the control function, \( \bar{u}_{ref} \) a reference velocity field, \( \bar{u} \) the average field and \( \bar{u}' \) the fluctuation and:

\[ \bar{u}_{ref} = \bar{u}(BR = BR_h) - \bar{u}(BR = BR_l) \]

\[ \bar{u} = \bar{u} - \gamma(t) \bar{u}_{ref} \]

\[ \bar{u}' = \bar{u} - \gamma(t) \bar{u}_{ref} - \bar{u} \]

The fluctuation part that is decomposed using POD so that:

\[ \bar{u}' = \sum_{m=1}^{N_T} a_m(t) \bar{\varphi}_m(x) \]  

(6.14)

\[ \bar{u} = \bar{u} + \gamma(t) \bar{u}_{ref} + \sum_{m=1}^{N_T} a_m(t) \bar{\varphi}_m(x) \]  

(6.15)

The temperature field stays homogeneous at the jet inlet therefore:

\[ T = \bar{T} + T' = \bar{T} + \sum_{n=1}^{N_T} b_n(t) \psi_n(x) \]

Effectively, using this velocity decomposition changes the general form of the ROM equations 4.32 and 4.33.

Rewriting the Navier Stokes equations using the previously introduced velocity decomposition:

\[ T = \bar{T} + T' = \bar{T} + \sum_{n=1}^{N_T} b_n(t) \psi_n(x) \]
\[ \nabla \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) = 0 \]

\[
\frac{\partial}{\partial t} \left( \gamma(t) \bar{u}_{ref} + \bar{u}' \right) + \left( \nabla \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) \cdot \nabla \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) = -\nabla P + \frac{1}{Re} \Delta \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) \\
\frac{\partial T'}{\partial t} + \left( \nabla + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) \cdot \nabla \left( T + T' \right) = \frac{1}{Pr Re} \Delta \left( T + T' \right) \quad (6.16)
\]

Using similar notation as the one used in Chapter 4:

\[
\frac{\partial \bar{u}'}{\partial t} = \bar{M} \left( \bar{u}', \bar{u}_{ref}, \gamma \right) \quad (6.17)
\]

\[
\frac{\partial T'}{\partial t} = \bar{E} \left( \bar{u}', \bar{u}_{ref}, \gamma, T, T' \right) \quad (6.18)
\]

where

\[
\bar{M}(\bar{u}', \bar{u}_{ref}, \gamma, \gamma) = -\left( \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) \cdot \nabla \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) - \nabla P + \frac{1}{Re} \Delta \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) - \frac{\partial}{\partial t} \left( \gamma(t) \bar{u}_{ref} \right) \right)
\]

and

\[
\bar{E}(\bar{u}', \bar{u}_{ref}, \gamma, T, T') = -\left( \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \bar{u}' \right) \cdot \nabla \left( T + T' \right) + \frac{1}{Pr Re} \Delta \left( T + T' \right) \right)
\]

**Velocity Reduced Order Model**

Projecting the momentum equation 6.17 onto the \( r \)-th velocity mode \( \bar{\varphi}_{r} \), and replacing the velocity fluctuation by the POD decomposition we get:

\[
\left( \frac{\partial}{\partial t} \left( \sum_{m=1}^{N_{V}} a_{m}(t) \bar{\varphi}_{m}(x) \right), \bar{\varphi}_{r}(x) \right) = \left( \bar{M} \left( \sum_{m=1}^{N_{V}} a_{m}(t) \bar{\varphi}_{m}(x), \bar{u}_{ref}, \gamma, \bar{\varphi}_{r}(x) \right) \right) \quad (6.19)
\]

Evaluating the left-hand side of equation 6.19 gives:

\[
\left( \frac{\partial}{\partial t} \left( \sum_{m=1}^{N_{V}} a_{m}(t) \bar{\varphi}_{m}(x) \right), \bar{\varphi}_{r}(x) \right) = \sum_{m=1}^{N_{V}} \left( \frac{\partial}{\partial t} \left( a_{m}(t) \bar{\varphi}_{m}(x) \right), \bar{\varphi}_{r}(x) \right) = \dot{a}_{r}(t)
\]

The right-hand side of equation 6.19 can be expanded (dropping the independent variables \( t \) and \( x \)):

\[
\left( \bar{M} \left( \sum_{m=1}^{N_{V}} a_{m} \bar{\varphi}_{m}, \bar{u}_{ref}, \gamma, \bar{\varphi}_{r} \right) \right) = \\
- \left( \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \sum_{m=1}^{N_{V}} a_{m} \bar{\varphi}_{m} \right) \cdot \nabla \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \sum_{m=1}^{N_{V}} a_{m} \bar{\varphi}_{m} \right), \bar{\varphi}_{r} \right) \\
+ \left( \frac{1}{Re} \Delta \left( \bar{u} + \gamma(t) \bar{u}_{ref} + \sum_{m=1}^{N_{V}} a_{m} \bar{\varphi}_{m} \right), \bar{\varphi}_{r} \right) - \left( \nabla P, \bar{\varphi}_{r} \right) \\
- \left( \frac{\partial}{\partial t} \left( \gamma(t) \bar{u}_{ref} \right), \bar{\varphi}_{r} \right) \quad (6.20)
\]

The first (convective) term of equation 6.20 can be decomposed into:
The second (diffusive) term of equation 6.20 can be decomposed into:

$$\left( \left[ \nabla \left( \pi + \gamma(t) \bar{u}_{ref} + \sum_{m=1}^{N_V} a_m \bar{\varphi}_m \right) \right], \bar{\varphi}_r \right) = \left( \nabla \right) \left( \pi + \gamma(t) \bar{u}_{ref} + \sum_{m=1}^{N_V} a_m \bar{\varphi}_m \right), \bar{\varphi}_r$$

$$+ \gamma(t) \left( \left( \bar{u}_{ref} \cdot \nabla \right) \sum_{m=1}^{N_V} a_m \bar{\varphi}_m, \bar{\varphi}_r \right) + \left( \left( \nabla \right) \nabla \left( \pi \right), \bar{\varphi}_r \right)$$

Finally, we obtain the reduced order model for the controlled velocity field: 

$$a_r(t) = \sum_{i=1}^{N_V} \left( \tilde{C}_{i,0}^T + \frac{1}{Re} \tilde{D}_{i}^T \right) a_i(t) + \sum_{i=1}^{N_V} \sum_{j=1}^{i} \tilde{C}_{i,j}^T a_i(t) a_j(t) + \left( \tilde{C}_{0}^T + \frac{1}{Re} \tilde{D}_{0}^T \right) P_r(t) + \gamma(t) \left[ \tilde{H}_0^T + \tilde{L}_0^T + \frac{1}{Re} \tilde{H}_{i,0}^T a_i(t) \right] + \gamma^2(t) \tilde{K}_0^T + \frac{d\gamma(t)}{dt} \tilde{G}_0^T$$
where
\[
\hat{u} = \bar{u} + \gamma(t)\bar{u}_{ref} + \hat{u}' = \bar{u} + \gamma(t)\bar{u}_{ref} + \sum_{n=1}^{N_T} a_n(t)\varphi_n(x)
\]
\[
\hat{C}_0^r = - ((\bar{u}, \nabla) \bar{u}, \varphi) = - \int_{\Omega_s} \bar{u} \frac{\partial}{\partial x_l} (\bar{u}_m) \varphi_{r,l} dx
\]
\[
\hat{D}_0^r = (\Delta \bar{u}, \varphi) = \int_{\Omega_s} \frac{\partial}{\partial x_m \partial x_m} (\bar{u}_m) \varphi_{r,l} dx
\]
\[
\hat{C}_{i,0}^r = - ((\bar{u}, \nabla) \varphi_i, \varphi_r) = - \int_{\Omega_s} \left( \bar{u}_m \frac{\partial}{\partial x_m} \varphi_{i,l} + \varphi_{i,m} \frac{\partial}{\partial x_m} \bar{u}_l \right) \varphi_{r,l} dx
\]
\[
\hat{D}_{i}^r = (\Delta \varphi_i, \varphi_r) = \int_{\Omega_s} \frac{\partial}{\partial x_m \partial x_m} (\varphi_{i,l}) \varphi_{r,l} dx
\]
\[
\hat{C}_{i,j}^r = - ((\varphi_i, \nabla) \varphi_j, \varphi_r) = - \int_{\Omega_s} \left( \varphi_{i,m} \frac{\partial}{\partial x_m} \varphi_{j,l} + \varphi_{j,m} \frac{\partial}{\partial x_m} \varphi_{i,l} \right) \varphi_{r,l} dx
\]
\[
\hat{P}_r(t) = - (\nabla P(t), \varphi_r)
\]
\[
\hat{H}_0^r = - ((\bar{u}, \nabla) \bar{u}_{ref}, \varphi_r) = - \int_{\Omega_s} \left( \bar{u}_m \frac{\partial}{\partial x_m} u_{ref,l} + u_{ref,m} \frac{\partial}{\partial x_m} \bar{u}_l \right) \varphi_{r,l} dx
\]
\[
\hat{L}_0^r = (\Delta \bar{u}_{ref}, \varphi_r) = \int_{\Omega_s} \frac{\partial}{\partial x_m \partial x_m} (u_{ref,l}) \varphi_{r,l} dx
\]
\[
\hat{H}_{i,0}^r = - ((\bar{u}_{ref}, \nabla) \varphi_i, \varphi_r) = - \int_{\Omega_s} \left( u_{ref,m} \frac{\partial}{\partial x_m} \varphi_{i,l} + \varphi_{i,m} \frac{\partial}{\partial x_m} u_{ref,l} \right) \varphi_{r,l} dx
\]
\[
\hat{K}_0^r = - ((\bar{u}_{ref}, \nabla) \bar{u}_{ref}, \varphi_r) = - \int_{\Omega_s} u_{ref,l} \frac{\partial}{\partial x_l} (u_{ref,m}) \varphi_{r,l} dx
\]
\[
\hat{G}_0^r = - (\bar{u}_{ref}, \varphi_r) = - \int_{\Omega_s} u_{ref,l} \varphi_{r,l} dx
\]
and \(\bar{u}_m\) corresponds to the \(m^{th}\) component of the vector \(\bar{u}\) and \(\varphi_{r,l}\) corresponds to the \(l^{th}\) component of the vector \(\varphi_r\). Redundant indexes imply summation on those indexes according to Einstein notation.

**Temperature Reduced Order Model**

Similarly, projecting the energy equation 6.18 onto the \(s^{th}\) temperature mode \(\psi_s\), and replacing the temperature fluctuation by the POD decomposition we get:

\[
\left( \frac{\partial}{\partial t} \left( \sum_{n=1}^{N_T} b_n(t) \psi_n(x) \right), \psi_s \right) = \left( \hat{E} \left( \sum_{m=1}^{N_T} a_m(t) \varphi_m(x), \bar{u}, \bar{u}_{ref}, \gamma, T, \sum_{n=1}^{N_T} b_n(t) \psi_n(x) \right), \psi_s \right)
\]  
(6.23)

The same can be done with the energy equations 6.23 gives:

\[
\left( \frac{\partial}{\partial t} \left( \sum_{n=1}^{N_T} b_n(t) \psi_n(x) \right), \psi_s \right) = \sum_{n=1}^{N_T} \left( \frac{\partial}{\partial t} \left( b_n(t) \psi_n(x) \right), \psi_s \right) = \hat{b}_s(t)
\]

The right-hand side of equation 6.23 is expanded (dropping the independant variables \(t\) and \(x\)):  

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\[
\left( \hat{E} \left( \sum_{m=1}^{N_V} a_m \varphi_m, \vec{u}, \vec{u}_{ref}, \gamma, T, \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right) = \\
- \left( \left( \vec{a} + \gamma(t) \vec{u}_{ref} + \sum_{m=1}^{N_V} a_m \varphi_m \right) \cdot \nabla \right) \left( T + \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right)
\]
\[
\left( \frac{1}{Pr} \frac{1}{Re} \Delta \left( T + \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right) \tag{6.24}
\]

The first (convective) term of equation 6.24 can be expanded into:

\[
\left( \left( \vec{a} + \gamma(t) \vec{u}_{ref} + \sum_{m=1}^{N_V} a_m \varphi_m \right) \cdot \nabla \right) \left( T + \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right) = \\
\left( \vec{a} \cdot \nabla \right) \left( \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right) + \left( \left( \sum_{m=1}^{N_V} a_m \varphi_m \right) \cdot \nabla \right) T, \psi_s \right) \\
+ \gamma(t) \left( (\vec{u}_{ref} \cdot \nabla) \left( \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right) + \left( \vec{u}_{ref} \cdot \nabla \right) T, \psi_s \right) \\
+ \left( \vec{a} \cdot \nabla \right) T, \psi_s \right) + \left( \left( \sum_{m=1}^{N_V} a_m \varphi_m \right) \cdot \nabla \right) \left( \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right)
\]

\[
\left( \left( \vec{a} + \gamma(t) \vec{u}_{ref} + \sum_{m=1}^{N_V} a_m \varphi_m \right) \cdot \nabla \right) \left( T + \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right) = \\
\sum_{n=1}^{N_T} \left( \vec{a} \cdot \nabla \right) \left( \psi_n \right), \psi_s \right) + \sum_{m=1}^{N_V} \left( \varphi_m \cdot \nabla \right) T, \psi_s \right) \\
+ \gamma(t) \left( \sum_{n=1}^{N_T} b_n \left( \vec{u}_{ref} \cdot \nabla \right) \psi_n, \psi_s \right) + \left( \vec{u}_{ref} \cdot \nabla \right) T, \psi_s \right) \\
+ \left( \vec{a} \cdot \nabla \right) T, \psi_s \right) + \sum_{n=1}^{N_T} \sum_{m=1}^{N_V} b_n a_m \left( \varphi_m \cdot \nabla \right) \psi_n, \psi_s \right)
\]

The second (diffusive) term of 6.24 can be expanded into:

\[
\left( \frac{1}{Pr} \frac{1}{Re} \Delta \left( T + \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right) = \left( \frac{1}{Pr} \frac{1}{Re} \Delta \left( \sum_{n=1}^{N_T} b_n \psi_n \right), \psi_s \right) \\
+ \frac{1}{Pr} \frac{1}{Re} \left( \Delta T, \psi_s \right)
\]
\[
= \frac{1}{Pr} \frac{1}{Re} \sum_{n=1}^{N_T} b_n \left( \Delta \psi_n, \psi_s \right) \\
+ \frac{1}{Pr} \frac{1}{Re} \left( \Delta T, \psi_s \right)
\]

Finally, the temperature reduced order model equations can be reduced down to:
\[ b_s(t) = \sum_{i=1}^{N_v} \tilde{E}_{s,0}^i a_i(t) + \sum_{j=1}^{N_T} \left( \tilde{E}_{s,j}^0 + \frac{1}{Pr} \frac{1}{Re} \tilde{F}_{s,j} \right) b_j(t) + \sum_{i=1}^{N_v} \sum_{j=1}^{N_T} \tilde{E}_{s,j}^i a_i(t)b_j(t) + \left( \tilde{E}_{s,0}^0 + \frac{1}{Pr} \frac{1}{Re} \tilde{F}_{s,0} \right) + \gamma(t) \left[ \tilde{N}_{s,0}^0 + \sum_{j=1}^{N_T} b_j \tilde{N}_{s,j}^0 \right] \] (6.25)

where

\[ T = T + T' = T + \sum_{m=1}^{N_T} b_m(t) \psi_m(x) \]

\[ \tilde{E}_{s,0}^0 = -((\vec{\pi} \cdot \nabla) T, \psi_s) = -\int_{\Omega_x} \vec{\pi}_m \frac{\partial}{\partial x_m} (T) \psi_s dx \]

\[ \tilde{F}_{s,0} = (\triangle T, \psi_s) = \int_{\Omega_x} \frac{\partial}{\partial x_m \partial x_m} (T) \psi_s dx \]

\[ \tilde{E}_{s,0}^i = -((\vec{\varphi}_i \cdot \nabla) T, \psi_s) = -\int_{\Omega_x} \vec{\varphi}_{i,m} \frac{\partial}{\partial x_m} (T) \psi_s dx \]

\[ \tilde{E}_{s,j}^0 = -((\vec{\pi} \cdot \nabla) \psi_j, \psi_s) = -\int_{\Omega_x} \vec{\pi}_m \frac{\partial}{\partial x_m} (\psi_j) \psi_s dx \]

\[ \tilde{F}_{s,j} = (\triangle \psi_j, \psi_s) = \int_{\Omega_x} \frac{\partial}{\partial x_m \partial x_m} (\psi_j) \psi_s dx \]

\[ \tilde{E}_{s,j}^j = -((\vec{\varphi}_j \cdot \nabla) \psi_j, \psi_s) = -\int_{\Omega_x} \vec{\varphi}_{j,m} \frac{\partial}{\partial x_m} (\psi_j) \psi_s dx \]

\[ \tilde{F}_{s,0} = (\triangle \psi_s, \psi_s) = \int_{\Omega_x} \frac{\partial}{\partial x_m \partial x_m} (\psi_s) \psi_s dx \]

\[ \tilde{N}_{s,j}^0 = -((\vec{u}_{ref} \cdot \nabla) T, \psi_s) = -\int_{\Omega_x} \vec{u}_{ref,m} \frac{\partial}{\partial x_m} (T) \psi_s dx \]

\[ \tilde{N}_{s,j}^0 = -((\vec{u}_{ref} \cdot \nabla) \psi_j, \psi_s) = -\int_{\Omega_x} \vec{u}_{ref,m} \frac{\partial}{\partial x_m} (\psi_j) \psi_s dx \]

and \( \vec{\pi}_m \) corresponds to the \( m^{th} \) component of the vector \( \vec{\pi} \) and \( \vec{\varphi}_{r,l} \) corresponds to the \( l^{th} \) component of the vector \( \vec{\varphi}_r \). Redundant indexes imply summation on those indexes according to Einstein notation.
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